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NONLINEAR EFFECTS WITHIN INVARIANCE PRINCIPLES

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Abstract. We study invariance principles in the rough path topology for stationary discrete and continuous time processes. Simple second moment computations give explicit Green-Kubo-type formulas for both the covariance and the linear correction, called the area anomaly, of the iterated integral of the limiting Brownian motion. A key observation is that the area anomaly admits a structural description: it is the antisymmetric part of the relevant Green–Kubo expression. This provides a unified explanation of area corrections that appear in several different frameworks, including Markovian, regenerative, and deterministic fast-slow models, and highlights a common mechanism underlying results that previously arose from model-specific arguments.

1. **Introduction.**

Random walks are stochastic processes in which a particle undergoes successive, independent, and identically distributed steps in various directions. When these steps are small and frequent, the cumulative effect over time resembles the continuous and erratic motion observed in Brownian motion in rather general settings. The (weak) convergence of random walks whose jumps have finite variance and vanishing mean, to Brownian motion, also known as the functional central limit theorem, is formalized by Donsker's invariance principle. This principle is well understood, and, in particular, the conclusion remains valid even when the assumption of independent steps is relaxed to suitable mixing conditions that quantify correlation. Classical references include Hall-Heyde [16] and Theorem 1 in Doukhan et al. [6].

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From Donsker's perspective, no matter if the random walk (rescaled) is a piecewise constant process or a piecewise linear process, the resulting (weak) limit and Brownian motion remain identical. However, if employed as a driving noise of a dynamical system, this is no longer the case, and hence, from a modeling point of view, it requires (or permits) a choice: The first one, common in financial mathematics, leads to stochastic recursions that converge weakly to Itô stochastic differential equations (ISDEs).[1] The second one, commonly used by physicists and geometers, leads to random ODEs with weak "Wong-Zakai" limits described by Stratonovich stochastic differential equations (SSDEs).

Related work on limit SDEs. Classical stochastic analysis treatises such as Ikeda-Watanabe [17] consider SSDEs as drift-perturbed ISDEs; a geometric "Wong-Zakai" approach is advocated by Stroock [27]. The latter has been given a profound understanding through Lyons' theory of rough paths which reduces the Wong-Zakai results to establishing convergence of piecewise linear approximations at the level of process and Lévy area; an application to support theory was discussed in [12].

Interestingly, other approximations, like those induced by considering charged Brownian particles in a magnetic field [9] lead to limit SDEs which are in general of neither Itô nor Stratonovich type. In the (somewhat restricted) case of exact coefficient fields (think additive noise or diffeomorphic transformations thereof), the same structure appears for an intriguing class of fast processes that play the role of our rescaled random walks. This was explicitly pointed out by Melbourne and Stuart [23].

In a celebrated series of works, starting with [18] and [19], exactness was removed, thanks to rough path theory. In a later joint work with one of the authors [4], we obtained validity of these results under optimal moment assumptions. We also draw attention to the recent work by Gottwald and Melbourne [15], where the role of Lévy area and possible corrections are discussed, fully aligned with the rough path view taken in this work.

We would also like to point to more results, in the Markovian setting, by Deuschel-Orenshtein-Perkowski [5] and the one in the regenerative/i.i.d.-like setting, by Orenshtein [25] where the focus is on the relevant rough paths results; for the non-Markovian situation and almost-sure results see Friz-Kifer [11].

All these cited works yield, in different settings and with explicit formulae, that the limiting dynamics are, in general, of neither Itô nor Stratonovich type. This can be understood through the fact that a recurring feature in these works is the appearance of a linear correction in the second level of the limiting rough path, often referred to as the area anomaly. Although this phenomenon has been identified in several concrete settings, including dynamical systems, Markovian models, and regenerative structures, it typically appears through model-dependent and technically involved arguments. One of the main observations of this work is that the diverse instances of this phenomenon can all be traced back to the same structural mechanism: the area anomaly is represented by the antisymmetric Green-Kubo expression reflecting the correlations of the underlying stationary process. In this sense, the "anomaly" is not anomalous at all, but the natural antisymmetric contribution that arises in the Brownian rough-path limit.

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[1] Within stochastic analysis, the so-called UT/UCV theory will cover this situation nicely.

Main insights and the structure of the paper. The purpose of this short paper, deliberately non-technical, at least in comparison to most of the aforementioned works, is to point out rather easy second moment computations that, for all of the aforementioned situations, lead to the correct formula of Green-Kubo type; such formulas are familiar to people with knowledge in homogenization of stochastic processes, as discussed for instance in the book by Pavliotis and Stuart [26], which provides a uniform way to write down the limiting Brownian rough-path characteristics in all these settings.

Although covariance and area-correction formulas are known in the individual settings we consider, their derivations in the literature rely on arguments that are specific to each model. The present work highlights that elementary second moment computations recover all these expressions in a unified way. This conceptual unification does not seem to be stated explicitly in the existing literature, and clarifies the structural origin of the area anomaly across the various discrete and continuous time frameworks treated below.

To avoid misunderstandings, we do not offer to newly (re-)prove results in [5, 9, 11, 14, 18, 19, 25], and in particular shall not give a general proof of weak convergence, nor focus on comparing particular conditions in various settings in the literature. Our intention is to point out that, across a range of settings where a sufficiently nice convergence is known, the same structure of the limiting Brownian rough paths appears.

In the following, after introducing the appropriate general set-up in which our results will be formulated, we discuss iterated invariance principles for processes with stationary increments in different situations; each section contains a main theorem followed by various relevant examples. Section 2.1 deals with the discrete time case, with results on problems of Itô type in Section 2.1.1 and those of Wong-Zakai type in Section 2.1.2. Continuous time iterated invariance principles are discussed in Section 2.2: we deal with stationary continuous time processes in Section 2.2.1, and discuss continuous time suspension flows, which are built from discrete settings, in Section 2.2.2.

Finally, we remark that some of the examples in the literature are non-stationary (to be understood in the appropriate sense). Hence, a recurring feature of our paper is the adjustment of the various models to the stationary settings. This requires an additional argument which may be of interest beyond the problem of iterated invariance principles; see, for instance, Theorem 2.15 and Example 2.8.

2. **Iterated invariance principles in different settings.**

Let us present the terminology that shall be used throughout the paper. A random process $t \mapsto (S(t), \mathbb{S}(t))$ on $\mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2}$ is any random element of $C([0,T], \mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2})$ or $D([0,T], \mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2})$ for some $T > 0$, where $C([0,T], \mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2})$ (respectively $D([0,T], \mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2})$) is the space of continuous (respectively càdlàg) functions $[0,T] \to \mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2}$ with the uniform (respectively Skorohod) topology. A d-dimensional continuous-time process $\Xi = \Xi(t)$ is a random element of the space $C([0,T], \mathbb{R}^d)$ or $D([0,T], \mathbb{R}^d)$ for some $T > 0$.

With a mild abuse of notation, we also write $\Xi = \Xi(t) = \Xi(t; \omega)$ (and correspondingly to $\Xi = \Xi(k) = \Xi(k; \omega)$), referring to the states ω from the appropriate probability space, so that for any fixed $t \in [0,T]$ (and correspondingly for any $k \in \mathbb{N}$), $\Xi(t) = \Xi(t; \cdot)$ (and correspondingly, $\Xi(k) = \Xi(k; \cdot)$) is a $\mathbb{R}^d$-valued random variable.

We shall consistently use the notation $\mathbf{P}$ and $\mathbf{E}$ for the corresponding probability measure and expectation, respectively.

A Brownian rough path over $\mathbb{R}^d$ with characteristics $(\Sigma, \Gamma) \in (\mathbb{R}^d)^{\otimes 2} \times (\mathbb{R}^d)^{\otimes 2}$ is any process defined by $t \mapsto \mathbf{B}(t) = (B(t), \mathbb{B}(t))$, where B is a d-dimensional Brownian motion with covariance

$$\Sigma = \mathbf{E}(B(1) \otimes B(1)) \in (\mathbb{R}^d)^{\otimes 2}$$

and $\mathbb{B}$ is of the form

$$\mathbb{B}(t) = \int_0^t B \otimes \mathrm{d}B + t\Gamma \in (\mathbb{R}^d)^{\otimes 2},$$

that is,

$$\mathbb{B}^{i,j}(t) = \int_0^t B^i(s)\mathrm{d}B^j(s) + t\Gamma^{i,j},\ i, j = 1, ..., d.$$

The integral above is in the sense of Itô, and $\mathbf{E}(X)$ is defined component-wise, whenever X is a random vector or a random matrix. In particular, $\Gamma = \mathbf{E}(\mathbb{B}(1))$. We recall some well-known strong limit theorems. The Itô Brownian rough path ($\Gamma = 0$) arises from considering lifted (càdlàg) piecewise constant approximations, which is tantamount to taking left-point Riemann-Stieltjes approximations of $\int B \otimes \mathrm{d}B$. The Stratonovich Brownian rough path ($\Gamma = \frac{1}{2}\Sigma$) arises, e.g., from piecewise approximations, corresponding with mid-point Riemann-Stieltjes approximations of $\int B \otimes \mathrm{d}B$. All mentioned approximations are *pointwise*, i.e., for fixed t, and in $\mathbf{P}$-probability, but can be upgraded to convergence in appropriate rough path metrics and $L^{\infty-}(\mathbf{P})$. Below we will be interested in weak, a.k.a. in law, limit theorems.

We shall consider the following definition.

Definition 2.1. Let $t \mapsto (S_N(t), \mathbb{S}_N(t))$ be a random process on $\mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2}$. We say that the iterated invariance principle holds for $(S_N, \mathbb{S}_N)$ with a Brownian rough path limit $(B, \mathbb{B})$ over $\mathbb{R}^d$ if the following two conditions hold.

1. For all t, the $\mathbb{R}^d \times (\mathbb{R}^d)^{\otimes 2}$-valued sequence $(S_N(t), \mathbb{S}_N(t))$ converges in law to $(B(t), \mathbb{B}(t))$.
2. $\mathbf{E}(S_N(1) \otimes S_N(1))$ converges to $\mathbf{E}(B(1) \otimes B(1))$ and $\mathbf{E}(\mathbb{S}_N(1))$ converges to $\mathbf{E}(\mathbb{B}(1))$.

We refer to Ekisheva-Houdré [7] (and the references therein), which contains various conditions to the central limit theorem together with convergence of moments.

2.1. Discrete iterated invariance principles.

2.1.1. *Itô-type iterated invariance principles.* Consider a d-dimensional discrete-time process $\xi = \xi(k; \omega)$ which is strictly stationary in the variable k and with rescaled partial sums

$$S_N(t) := \frac{1}{N^{1/2}} \sum_{0 \leqslant k < [Nt]} \xi(k). \tag{2.1}$$

Assume $\mathbf{E}\xi(0) = 0$, and consider S_N with the canonical enhancement of iterated sums $\mathbb{S}_N$ defined by

$$\mathbb{S}_N(t) := \frac{1}{N} \sum_{0 \leqslant k < \ell < [Nt]} (\xi(k) \otimes \xi(\ell)). \tag{2.2}$$

Theorem 2.2. *Let $\xi = \xi(k;\omega)$ be a discrete random process which is strictly stationary in the variable k, with finite second moments and summable correlations, that is,*

$$\Delta(0) \quad \textit{and} \quad \sum_{n=1}^{\infty} \Delta(n) \textit{ are finite,}$$

where $\Delta(n) := \mathbf{E}(\xi(0) \otimes \xi(n))$ for $n = 0, 1, \ldots$. Consider the rescaled piecewise constant approximations S_N (2.1) with the iterated sum process $\mathbb{S}_N$ (2.2). If the iterated invariance principle holds for $(S_N, \mathbb{S}_N)$, then the Brownian rough path limit $(B, \mathbb{B})$ is necessarily with characteristics

$$\Sigma = \Delta(0) + 2\,\mathrm{Sym}(\Gamma) \quad \textit{and} \quad \Gamma = \sum_{n=1}^{\infty} \Delta(n),$$

where $\mathrm{Sym}(\Gamma)$ denotes the symmetric part of the matrix Γ.

The expression for the covariance matrix Σ is sometimes called the *Green-Kubo formula.*

Proof. Let $\sigma_n := \sum_{k=1}^{n} \Delta(k)$. Using stationarity, we see that

$$\sum_{0 \leqslant k < \ell < N} \mathbf{E}(\xi(k) \otimes \xi(\ell)) = \sum_{0 \leqslant k < \ell < N} \Delta(\ell - k) = \sum_{1 \leq k < N} (N-k)\Delta(k) = \sum_{n=1}^{N} \sigma_n.$$

The summability of the correlations $\Delta(n)$ implies that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sigma_n = \lim_{N\to\infty} \sigma_N = \sum_{k=1}^{\infty} \Delta(k).$$

Hence,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leqslant k < \ell < N} \mathbf{E}(\xi(k) \otimes \xi(\ell)) = \sum_{k=1}^{\infty} \Delta(k)$$

and, analogously,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{0 \leqslant k < \ell < N} \mathbf{E}(\xi(\ell) \otimes \xi(k)) = \sum_{k=1}^{\infty} \Delta(k)^T,$$

where $\Delta(n)^T = \mathbf{E}(\xi(n) \otimes \xi(0))$. Therefore,

$$\begin{aligned}\Sigma &\approx \mathbf{E}(S_N(1) \otimes S_N(1)) = \frac{1}{N} \sum_{0 \leqslant k, \ell < N-1} \mathbf{E}(\xi(k) \otimes \xi(\ell)) \\ &\approx \mathbf{E}(\xi(0) \otimes \xi(0)) + \sum_{n=1}^{\infty} \mathbf{E}(\xi(0) \otimes \xi(n) + \xi(n) \otimes \xi(0)),\end{aligned}$$

where here and hereafter we use the notation $a_N \approx b_N$ if $a_N - b_N \to 0$ as N tends to infinity.

In terms of $\Delta(n)$, this becomes $\Sigma = \Delta(0) + 2\sum_{n=1}^{\infty} \mathrm{Sym}(\Delta(n))$. On the other hand, $\mathbf{E}(\mathbb{S}_N(1)) \approx \mathbf{E}(\mathbb{B}(1)) = \Gamma$. The computation above gives

$$\Gamma \approx \frac{1}{N} \sum_{0 \leqslant k < \ell < N} \mathbf{E}(\xi(k) \otimes \xi(\ell)) \approx \sum_{n=1}^{\infty} \Delta(n).$$

$\square$

Example 2.3. Kelly-Melbourne [18, 19] and (8) and (19) in Chevyrev et al. [3]: Considering maps $T : M \to M$ on a compact manifold M with invariant and ergodic measure μ on a hyperbolic invariant set $\Lambda \subset M$, Theorem 2.2 applies with

$$\Delta(n) := \mathbf{E}(\xi(0) \otimes \xi(n)) = \int (v \otimes (v \circ T^n))\mathrm{d}\mu,$$

for a μ-centered Hölder observable $v : \Lambda \to \mathbb{R}^d$, where the assumption of Theorem 2.2 corresponds with summable decay of correlations, i.e. sufficiently fast mixing. The probability space in this case is thus (Λ, μ), and the stationary process is $\xi(n; \omega) = (v \circ T^n)(\omega)$. Notable examples are uniformly expanding maps T and non-uniformly expanding maps T with appropriate Young towers.

Example 2.4. Kurtz-Protter [20]. The ξ are taken i.i.d., centred with finite second moments. Then, $\Gamma = 0$, and one gets the expected Itô limit. (The S_N are martingales, and satisfy the UCV condition.) This already implies tightness in rough paths metrics, cf. [2], and also in càdlàg rough path spaces [13].

Example 2.5. Theorem 2.2 in Friz-Kifer [11]. Direct mixing conditions are given. With

$$\Delta(n) := \mathbf{E}(\xi(0) \otimes \xi(n))$$

an almost-sure iterated invariance principle is shown, with Σ, Γ as described by Theorem 2.10.

2.1.2. *Wong-Zakai type iterated invariance principles.* We maintain the discrete setup, but do not work with the piecewise constant partial sums S_N, but instead, à la Wong-Zakai, with their piecewise-linear counterpart $\hat{S}_N$. Write $(\hat{S}_N, \hat{\mathbb{S}}_N)$ for the canonical (geometric) rough path lift. More explicitly,

$$\hat{\mathbb{S}}_N(t) = \int_0^t \hat{S}_N(s) \otimes \mathrm{d}\hat{S}_N(s), \tag{2.3}$$

where the integration is in the Riemann–Stieltjes sense.

Theorem 2.6. *In the assumptions of Theorem 2.2, if the iterated invariance principle holds for $(\hat{S}_N, \hat{\mathbb{S}}_N)$ (defined above, see (2.3)), then the (Stratonovich) Brownian rough path limit $(B, \hat{\mathbb{B}})$ has characteristics $(\Sigma, \hat{\Gamma})$,*

$$\Sigma = \Delta(0) + 2\,\mathrm{Sym}\left(\sum_{n=1}^{\infty} \Delta(n)\right), \qquad \hat{\Gamma} = \frac{1}{2}\Delta(0) + \sum_{n=1}^{\infty} \Delta(n).$$

As a consequence,

$$\hat{\mathbb{B}}(1) = \int_0^1 B \otimes \circ\mathrm{d}B + \mathrm{Anti}\left(\sum_{n=1}^{\infty} \Delta(n)\right),$$

where we used the notation Anti(Γ) *for the antisymmetric part of the matrix Γ, and the last integration is in the sense of Stratonovich.*

Proof. The covariance Σ is not affected by the problem modification, as before. We have a discrete Itô-Stratonovich correction of the form

$$\hat{\mathbb{S}}_N(1) = \mathbb{S}_N(1) + \frac{1}{2N} \sum_{0 \leqslant k < N} \xi(k) \otimes \xi(k)$$

and, hence,

$$\mathbf{E}(\hat{\mathbb{S}}_N(1)) = \mathbf{E}(\mathbb{S}_N(1)) + \frac{1}{2}\Delta(0).$$

The argument in the proof of Theorem 2.2 gives

$$\hat{\Gamma} = \mathbf{E}(\hat{\mathbb{B}}(1)) = \mathbf{E}(\mathbb{B}(1)) + \frac{1}{2}\Delta(0) = \Gamma + \frac{1}{2}\Delta(0).$$

Note that

$$\hat{\mathbb{B}}(1) = \int_0^1 B \otimes \mathrm{d}B + \hat{\Gamma} = \left(\int_0^1 B \otimes \circ \mathrm{d}B - \frac{1}{2}\Sigma\right) + \hat{\Gamma}$$

and compute, using the previously obtained expression for $\hat{\Gamma}$ and Σ,

$$\hat{\Gamma} - \frac{1}{2}\Sigma = \mathrm{Anti}\left(\sum_{n=1}^{\infty} \Delta(n)\right).$$

This finishes the proof. □

Example 2.7. Breuillard-Friz-Huesmann [1]. The ξ are taken i.i.d., so that $\Gamma = 0$, and one gets the classical Wong-Zakai/Stratonovich limit.

Example 2.8. Orenshtein [25], Theorem 1.5 (see also the main results of [21] and [22]): Hereinafter, for a process X and a pair of time indices $s < t$, we denote by $X_{s,t} := X_t - X_s$ its increments. Here, $\xi(n) = X_{n-1,n}$ for $n \in \mathbb{N}$, where X_n is a delayed regenerative process on $\mathbb{R}^d$ with respect to the random regeneration times $0 = \tau_0 < \tau_1 < \tau_2 < ... < \infty$ a.s. That is, the sequence of epochs

$$\mathcal{E}_k := \left(T_{k+1}, (X_{\tau_k, \tau_k+\ell})_{\ell=0,1,\dots,T_{k+1}},\right)$$

where $T_0 = 0$ and $T_{k+1} = \tau_{k+1} - \tau_k$, is independent and identically distributed for $k \geq 1$ and independent of the delay epoch $\mathcal{E}_0$.

A stationary version $\bar{X}$ of X is constructed by letting $\bar{X}_{V,V+n} = \tilde{X}_{\tau_1,\tau_1+n}$ for all $n \geq 0$, where V is distributed uniformly on $\{0, 1, ..., T^* - 1\}$, T^* is an independent size-biased version of $T_2 = \tau_2 - \tau_1$, and $(\tilde{X}_n, n = 0, 1, ..., T^* - V)$ is an independent copy of $(X_n, n = 0, 1, ..., T^* - V)$, c.f. [24, 28, 29]. In particular, we have that the increments of the stationary process coincide with the ones of a delay-free version of the original process after a time shift of $U = T^* - V$.

In more detail, let T^* be the size-biased version of T_2 taken independently of X. This is a positive integer-valued random variable whose mass function is

$$\mathbb{P}(T^* = m) := \frac{m\mathbb{P}(T_2 = m)}{\mathbb{E}[T_2]}, m \in \mathbb{N}.$$

Let U be uniformly distributed on $\{1, 2, ..., T^*\}$, and set $V = T^* - U$ (note that it is uniform on $\{0, 1, ..., T^* - 1\}$). Let $\tilde{X}$ be a copy of X independent of anything else, and let $\tilde{\tau}_1, \tilde{\tau}_2, ...$ be the corresponding regeneration times. Finally, a stationary version of X is defined by

$$\bar{X}_n := \begin{cases} \tilde{X}_{\tilde{\tau}_1+V+n}, & \text{if } n \in \{0, 1, .., U\}, \\ \tilde{X}_{\tilde{\tau}_1+T^*} + X_{\tau_1, \tau_1+n-U}, & \text{if } n \in \{U, U+1, ...\}. \end{cases}$$

By construction, $\bar{X} = (\bar{X}_k)_{k \geq 0}$ is a delayed regenerative process. Let

$$\Xi(X)(m,n) = \sup\left\{|X_{k,\ell}| : m \leq k \leq \ell \leq n\right\}.$$

Assume the process X satisfies the conditions of [25, Theorem 1.5], and assume also the moment condition $\mathbb{E}[T_2^2 \Xi(X)(\tau_1, \tau_2)^2] < \infty$. Then, the iterated invariance principle holds for $\bar{X}$ with the same characteristics as X. This is an immediate application of [25, Theorem 1.5] to the process $(\bar{X}_k)_{k \geq 0}$, after verifying the moment assumption for the delay epoch of $\bar{X}$.

To obtain the formulae of [25], note that in this case $\bar{\xi}(n) = \bar{X}_{n,n+1}$ and

$$\bar{\Delta}(n) := \mathbf{E}[\bar{\xi}(0) \otimes \bar{\xi}(n)].$$

For $\xi_{\tau_1}(n) = X_{\tau_1+n,\tau_1+n+1}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \bar{\Delta}(n) &= \lim_{N\to\infty} \mathbb{E}[\bar{\xi}(0) \otimes \bar{X}_{1,N}] \\ &= \mathbb{E}[\bar{\xi}(0) \otimes \bar{X}_{1,U}] \\ &= \frac{1}{\mathbb{E}[T_2]} \sum_{m=1}^{\infty} \mathbb{E}\left[\sum_{k=0}^{m-1} \sum_{\ell=k+1}^{m-1} \xi_{\tau_1}(k) \otimes \xi_{\tau_1}(\ell)\right] \mathbb{P}(T_2 = m) \\ &= \frac{1}{\mathbb{E}[T_2]} \mathbb{E}\left[\sum_{\tau_1 \le k < \ell < \tau_2} \xi(k) \otimes \xi(\ell)\right].\end{aligned}$$

Similarly, we find

$$\mathrm{Anti}\left(\sum_{n=1}^{\infty} \bar{\Delta}(n)\right) = \frac{1}{\mathbb{E}[T_2]} \mathbb{E}\left[\mathrm{Anti}\left(\sum_{\tau_1 \le k < \ell < \tau_2} \xi(k) \otimes \xi(\ell)\right)\right]$$

and

$$\begin{aligned}\bar{\Delta}(0) + 2\sum_{n=1}^{\infty} \mathrm{Sym}(\bar{\Delta}(n)) &= \frac{1}{\mathbb{E}[T_2]} \mathbb{E}\Big[\sum_{\tau_1 \le k, \ell < \tau_2} \xi(k) \otimes \xi(\ell)\Big] \\ &= \frac{1}{\mathbb{E}[T_2]} \mathbb{E}\big[X_{\tau_1,\tau_2} \otimes X_{\tau_1,\tau_2}\big].\end{aligned}$$

Remark 2.9. In [25], the moment condition for X is $\mathbb{E}[T_{k+1} \Xi(X)(\tau_k, \tau_{k+1})^{2p}] < \infty$ for $k = 0, 1$ and $p = 0, 1$. However, the moment condition for the delay (that is, for the case $k = 0$) can be reduced to $\mathbb{E}[\tau_1^{\epsilon} \Xi(X)(0, \tau_1)^{\epsilon p}] < \infty$ for some $\epsilon > 0$ without an essential change in the proof. In particular, in the stationary version, the assumption $\mathbb{E}[T_2^2 \Xi(X)(\tau_1, \tau_2)^{2p}] < \infty$ for $p \in \{0, 1\}$ can be reduced to $\mathbb{E}[T_2^{1+\epsilon} \Xi(X)(\tau_1, \tau_2)^{2p}] < \infty$ for some $\epsilon > 0$.

2.2. **Continuous iterated invariance principles.**

2.2.1. *Invariance principles under continuous time assumptions.* Consider a d-dimensional continuous-time process $\Xi = \Xi(t) = \Xi(t; \omega)$ with rescaled integrals

$$\bar{S}_N(t) = \frac{1}{N^{1/2}} \int_0^{Nt} \Xi(s) \mathrm{d}s \tag{2.4}$$

canonically enhanced with the iterated integrals (that is, the integration is taken in the sense of Riemann-Stieltjes), denoted by $\bar{\mathbb{S}}_N$.

Theorem 2.10. *Assume that $\Xi = \Xi(t) = \Xi(t; \omega)$ is strictly stationary in t and that the iterated invariance principle holds for $(\bar{S}_N, \bar{\mathbb{S}}_N)$ (see (2.4)) with a Brownian rough path limit $(\bar{B}, \bar{\mathbb{B}})$ with characteristics $(\bar{\Sigma}, \bar{\Gamma})$. Assume that $\int_0^{\infty} \bar{\Delta}(s)\mathrm{d}s$ is well-defined and finite, where $\bar{\Delta}(s) := \mathbf{E}(\Xi(0) \otimes \Xi(s))$. Then, necessarily we have the characteristics*

$$\bar{\Sigma} = 2\,\mathrm{Sym}\left(\int_0^{\infty} \bar{\Delta}(s)\mathrm{d}s\right), \qquad \bar{\Gamma} = \int_0^{\infty} \bar{\Delta}(s)\mathrm{d}s.$$

As a consequence, we have the area correction

$$\bar{\mathbb{B}}(1) = \int_0^1 B \otimes \circ \mathrm{d}B + \mathrm{Anti}\left(\int_0^\infty \bar{\Delta}(s)\mathrm{d}s\right).$$

Proof. As in the proof of Theorem 2.2 for the discrete case (e.g., by considering time integrals instead of sums), we use stationarity and integrable correlations and the fact that $\mathbf{E}(\bar{S}_N(1) \otimes \bar{S}_N(1)) \approx \mathbf{E}(\bar{B}(1) \otimes \bar{B}(1))$ to obtain that

$$\begin{aligned}\bar{\Sigma} = \mathbf{E}(\bar{B}(1) \otimes \bar{B}(1)) &\approx \frac{1}{N}\int_{0\leqslant s,s'\leqslant N} \mathbf{E}(\Xi(s) \otimes \Xi(s'))\mathrm{d}s\mathrm{d}s' \\ &\approx \int_0^\infty \mathbf{E}(\Xi(0) \otimes \Xi(s) + \Xi(s) \otimes \Xi(0))\mathrm{d}s.\end{aligned}$$

Hence, in terms of the correlations, this becomes

$$\bar{\Sigma} = 2\int_0^\infty \mathrm{Sym}(\bar{\Delta}(s))\mathrm{d}s,$$

the Green-Kubo formula. On the other hand, since $\bar{\Gamma} = \mathbf{E}(\bar{\mathbb{B}}(1)) \approx \mathbf{E}(\bar{\mathbb{S}}_N(1))$, where

$$\bar{\mathbb{S}}_N(1) = \frac{1}{N}\int_{0\leqslant s<s'\leqslant N} \Xi(s) \otimes \Xi(s')\mathrm{d}s\mathrm{d}s',$$

we have

$$\bar{\Gamma} \approx \frac{1}{N}\int_{0\leqslant s<s'\leqslant N} \mathbf{E}(\Xi(s) \otimes \Xi(s'))\mathrm{d}s\mathrm{d}s' \approx \int_0^\infty \mathbf{E}(\Xi(0) \otimes \Xi(s))\mathrm{d}s = \int_0^\infty \bar{\Delta}(s)\mathrm{d}s.$$

Next, since

$$\bar{\mathbb{B}}(1) \equiv \int_0^1 B \otimes \mathrm{d}B + \bar{\Gamma} = \left(\int_0^1 B \otimes \circ\mathrm{d}B - \frac{1}{2}\bar{\Sigma}\right) + \bar{\Gamma}$$

and

$$\bar{\Gamma} - \frac{1}{2}\bar{\Sigma} = \int_0^\infty (\bar{\Delta}(s) - \mathrm{Sym}(\bar{\Delta}(s)))\mathrm{d}s = \int_0^\infty \mathrm{Anti}(\bar{\Delta}(s))\mathrm{d}s,$$

the proof is finished. □

Example 2.11. Friz-Gassiat-Lyons [9], or Ch. 3 in Friz-Hairer [10]. Here, $\Xi = \Xi(t;\omega)$ is a multidimensional OU process (denoted $\tilde{Y}$ in [10]) with dynamics

$$\mathrm{d}\Xi = -M\Xi\mathrm{d}t + \mathrm{d}B;$$

a stationary solution exists under a spectral gap condition, and is explicitly given by

$$\Xi(s) = \int_{-\infty}^s e^{-M(s-r)}\mathrm{d}B(r).$$

This allows one to compute $\bar{\Delta}(s) := \mathbf{E}(\Xi(0) \otimes \Xi(s))$. By Itô isometry,

$$\bar{\Delta}(s) = \int_{-\infty}^0 e^{-M(0-r)}e^{-M^\star(s-r)}\mathrm{d}r =: \Sigma_{\mathrm{OU}}e^{-M^\star s}$$

and, since Σ_{OU} is symmetric,

$$(\star) : \int_0^\infty \bar{\Delta}(s)\mathrm{d}s = \Sigma_{\mathrm{OU}}(M^\star)^{-1}$$

Recall that $\bar{S}_N(t) = \frac{1}{N^{1/2}}\int_0^{Nt}\Xi(s)\mathrm{d}s$ represents physical Brownian motion with mass $1/N$.

By Theorem 2.10, the limit Brownian covariance equals

$$2\,\mathrm{Sym}\left(\int_0^\infty \bar{\Delta}(s)\mathrm{d}s\right) = \Sigma_{\mathrm{OU}}(M^\star)^{-1} + (M)^{-1}\Sigma_{\mathrm{OU}}$$

while we have area correction given by

$$\mathrm{Anti}\left(\int_0^\infty \bar{\Delta}(s)\mathrm{d}s\right) = \frac{1}{2}[\Sigma_{\mathrm{OU}}(M^\star)^{-1} - (M)^{-1}\Sigma_{\mathrm{OU}}]$$

which vanishes iff $M{=}M^\star$.

Example 2.12. Deuschel-Orenshtein-Perkowski [5] considered

$$\Xi(s) = F(X(s)),$$

for some Markov process X on some Polish space with generator $\mathcal{L}$ on some space M and a probability measure π which is invariant and ergodic for both $\mathcal{L}$ and $\mathcal{L}^*$, see the precise formulation in [5, Chapter 3]. Assume that $\mathbf{E}F(X(0)) = 0$ and that there exists a solution in L^2 to the Poisson equation $-\mathcal{L}\phi = F$, they first recall (cf. p.7) the validity of the functional CLT, with "half-covariance" given by

$$\frac{1}{2}\bar{\Sigma} = \mathbf{E}(\phi(X(0)) \otimes ((-\mathcal{L}_S)\phi)(X(0)))$$

and then establish (p.6, Theorem 3.3) an area correction of the form

$$\mathbf{E}(\phi(X(0)) \otimes (\mathcal{L}_A\phi)(X(0)).$$

Let us illustrate how this can be derived from Theorem 2.10. We start by computing

$$\bar{\Gamma} \quad \equiv \int_0^\infty \bar{\Delta}(s)\mathrm{d}s \equiv \int_0^\infty \mathbf{E}(\Xi(0) \otimes \Xi(s))\mathrm{d}s$$

using the Markov structure at hand. With $\Xi(s) = F(X(s))$, we have

$$\begin{aligned}
\bar{\Gamma} = \int_0^\infty \left(\int_M F \otimes e^{s\mathcal{L}} F \mathrm{d}\pi\right)\mathrm{d}s \quad &= \quad \int_M \left(F \otimes \left(\int_0^\infty e^{s\mathcal{L}} F \mathrm{d}s\right)\mathrm{d}\pi\right) \\
&= \quad \int_M (F \otimes (-\mathcal{L}^{-1}F)\mathrm{d}\pi) \\
&= \quad \int_M (-\mathcal{L}\phi \otimes \phi)\mathrm{d}\pi.
\end{aligned}$$

The last two steps are justified when $\mathcal{L}$ has a spectral gap, and $-\mathcal{L}\phi = F$ is uniquely solvable. Decomposing $\mathcal{L} = \mathcal{L}_S + \mathcal{L}_A = \frac{1}{2}(\mathcal{L} + \mathcal{L}^\star) + \frac{1}{2}(\mathcal{L} - \mathcal{L}^\star)$ into symmetric and antisymmetric operators, we see

$$\mathrm{Sym}(\bar{\Gamma}) = \mathrm{Sym}\left(\int_M (-\mathcal{L}\phi \otimes \phi)\mathrm{d}\pi\right) = \int_M (-\mathcal{L}_S\phi \otimes \phi)\mathrm{d}\pi = \int_M (\phi \otimes -\mathcal{L}_S\phi)\mathrm{d}\pi$$

and

$$\mathrm{Anti}(\bar{\Gamma}) = \mathrm{Anti}\left(\int_M (-\mathcal{L}\phi \otimes \phi)\mathrm{d}\pi\right) = \int_M (-\mathcal{L}_A\phi \otimes \phi)\mathrm{d}\pi = \int_M (\phi \otimes \mathcal{L}_A\phi)\mathrm{d}\pi.$$

These expressions agree exactly with the half-covariance and area correction given by Deuschel-Orenshtein-Perkowski [5]. The more general case, Theorem 3.3 of [5], is a Kipnis-Varadhan type invariance principle, where the assumption of existence of solution to the Poisson equation is weakened to a condition on the behavior of the solution π_λ to the resolvent equation $(\lambda - \mathcal{L})\phi_\lambda = F$ for small λ, known as the H^{-1} condition (for the precise formulation, see (3.1) of [5]). In those settings, $\bar{\Gamma}$ is obtained by the limit as $\lambda \downarrow 0$ of the Laplace transform of the correlations. More precisely,

$$\begin{aligned}\bar{\Gamma} &= \lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda s}\mathbf{E}(\Xi(0)\otimes\Xi(s))\mathrm{d}s\\ &= \lim_{\lambda\downarrow 0}\int_0^\infty e^{-\lambda s}\int \mathbf{E}(\Xi(0)\otimes\Xi(s)|X(0)=x)\mathrm{d}\pi(x)\mathrm{d}s\\ &= \lim_{\lambda\downarrow 0}\int F(x)\otimes\left(\int_0^\infty e^{-\lambda s}\mathbf{E}(F(X(s))|X(0)=x)\mathrm{d}s\right)\mathrm{d}\pi(x)\\ &= \lim_{\lambda\downarrow 0}\int F\otimes\phi_\lambda\mathrm{d}\pi\\ &= \lim_{\lambda\downarrow 0}\int(-\mathcal{L}\phi_\lambda\otimes\phi_\lambda)\mathrm{d}\pi,\end{aligned}$$

where the existence of the limits is justified as in [5], in particular the last equality follows as the one in pp. 1463, second line in that paper.

Example 2.13. (Example 2.11 revisited) We can view Example 2.11 from a Markov /Deuschel-Orenshtein-Perkowski [5] perspective by considering the generator

$$\mathcal{L}f(x) = -Mx\cdot\nabla f(x) + \frac{1}{2}\Delta f(x)$$

for $f \in C^2(\mathbb{R}^d,\mathbb{R}^d)$, where the operator acts in the usual way in each component. The Poisson equation

$$-\mathcal{L}\phi = x \in \mathbb{R}^d$$

is solved by $\phi(x) = M^{-1}x$ centred w.r.t. the invariant measure $\pi = N(0,\Sigma_{\mathrm{OU}})$. Note that $\mathcal{L}$ is self-adjoint if and only if M is symmetric and

$$\textstyle\int(-\mathcal{L}\phi\otimes\phi)\mathrm{d}\pi \displaystyle = \int(x\otimes M^{-1}x)\mathrm{d}\pi = \int(xx^\star)\mathrm{d}\pi(M^\star)^{-1} = \Sigma_{\mathrm{OU}}(M^\star)^{-1}$$

which is precisely $(\star)$ above, and in particular

$$\mathrm{Anti}(\bar{\Gamma}) = \mathrm{Anti}\left(\int_M(-\mathcal{L}\phi\otimes\phi)\mathrm{d}\pi\right) = \int_M(\phi\otimes\mathcal{L}_A\phi)\mathrm{d}\pi$$

vanishes if $\mathcal{L}$ is self-adjoint, which holds if and only if $M = M^\star$.

2.2.2. *Invariance principles for suspension flows under discrete time assumptions.* We consider the following setting (cf. [11]) as specification of the previous section, allowing for weaker assumptions on the integrability of correlations $\Delta(t)$.

Take a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with $\mathbb{P}$-preserving transformation $\theta:\Omega\to\Omega$ and a *roof function* $\tau:\Omega\to(0,\infty)$ which is bounded from below and above, i.e.

$$L^{-1}\le\tau\le L,\qquad\text{for some } L>0.$$

We consider the probability space $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$ with

$$\bar{\Omega} = \{(\omega,t)\,:\,\omega\in\Omega,\,0\le t\le\tau(\omega),\,(\omega,\tau(\omega))=(\theta\omega,0)\},$$

where $\bar{\mathcal{F}}$ is the restriction to $\bar{\Omega}$ of $\mathcal{F}\times\mathcal{B}([0,L])$, and $B([0,L])$ denotes the Borel σ-algebra. Denoting $\bar{\tau} = \int\tau\mathrm{d}\mathbb{P}$, the measure $\bar{\mathbb{P}}$ is given, for any $A\in\bar{\mathcal{F}}$, by

$$\bar{\mathbb{P}}(A) = \frac{1}{\bar{\tau}}\int_{\bar{\Omega}}\mathbb{1}_A(\omega,t)\mathrm{d}\mathbb{P}(\omega)\mathrm{d}t.$$

Now we can introduce the d-dimensional continuous-time process Ξ on $\bar{\Omega}$ as

$$\Xi(t;(\omega,s)) = \Xi(t+s;(\omega,0)) = \Xi(0;(\omega,t+s)) \ \text{ if } 0 \le t+s < \tau(\omega),$$

$$\Xi(t;(\omega,s)) = \Xi(0;(\theta^k\omega,u)), \ \text{if } t+s = u + \sum_{j=0}^{k}\tau(\theta^j\omega),\ 0 \le u < \tau(\theta^k\omega).$$

Note that the *suspension* Ξ with respect to the map θ is a stationary process on $(\bar{\Omega},\bar{\mathcal{F}},\bar{\mathbb{P}})$, writing $\Xi(t;\omega)$ for $\Xi(t;(\omega,0))$.

We can now introduce $\xi(0;\omega) = \int_0^{\tau(\omega)} \Xi(s;\omega)\mathrm{d}s$ and $\xi(k;\omega) = \xi(0;\theta^k(\omega))$ for $k \ge 0$, and take the partial sums $(S_N,\mathbb{S}_N)$ for ξ as in (2.1) and the partial sums $(\bar{S}_N,\bar{\mathbb{S}}_N)$ for Ξ as in (2.4). We consider the following situation (which is, for example, satisfied in [18, Theorem 6.1] or [11, Theorem 2.5]), again denoting $\Delta(n) = \mathbf{E}(\xi(0)\otimes\xi(n))$.

Assumption 2.14. (a) The iterated invariance principle holds for $(S_N,\mathbb{S}_N)$ with Brownian rough path limit $(B,\mathbb{B})$ and characteristics

$$\Sigma = \Delta(0) + 2\,\mathrm{Sym}(\Gamma), \qquad \Gamma = \sum_{n=1}^{\infty}\Delta(n)$$

according to Theorem 2.2, i.e. the summability assumption is satisfied.

(b) The iterated invariance principle holds for $(\bar{S}_N,\bar{\mathbb{S}}_N)$ with Brownian rough path limit $(\bar{B},\bar{\mathbb{B}})$ such that

$$\bar{B} = (\bar{\tau})^{-1/2}B,$$

$$\bar{\mathbb{B}}(t) = \int_0^t B\otimes \mathrm{d}B + t(\bar{\tau})^{-1}\Gamma + t\int_{\bar{\Omega}}\left(\int_0^u \Xi(s;\omega)\mathrm{d}s\right)\otimes\Xi(0;\omega)\,\mathrm{d}\bar{\mathbb{P}}.$$

We can now state our final theorem.

Theorem 2.15. *Under Assumption 2.14, the Brownian rough path limit $(\bar{B},\bar{\mathbb{B}})$ has the characteristics*

$$\bar{\Sigma} = (\bar{\tau})^{-1}\Sigma, \qquad \bar{\Gamma} = (\bar{\tau})^{-1}\Gamma + \int_{\bar{\Omega}}\left(\int_0^u \Xi(s;\omega)\mathrm{d}s\right)\otimes\Xi(0;\omega)\,\mathrm{d}\bar{\mathbb{P}},$$

and the area correction

$$\bar{\mathbb{B}}(1) = \int_0^1 B\otimes\circ\mathrm{d}B + \mathrm{Anti}\left(\bar{\Gamma}\right).$$

Proof. The expression for the covariance matrix follows directly from the assumption. Concerning the area correction, as before, we need to compute $\bar{\Gamma} - \frac{1}{2}\bar{\Sigma}$. We have

$$\begin{aligned}\bar{\Gamma} - \frac{1}{2}\bar{\Sigma} &= (\bar{\tau})^{-1}\left(\Gamma - \mathrm{Sym}(\Gamma) - \frac{1}{2}\Delta(0)\right)\\ &\quad + \int_{\bar{\Omega}}\left(\int_0^u \Xi(s;\omega)\mathrm{d}s\right)\otimes\Xi(0;\omega)\,\mathrm{d}\bar{\mathbb{P}}(\omega,u)\\ &= (\bar{\tau})^{-1}\,\mathrm{Anti}\,(\Gamma)\\ &\quad + \left(\int_{\bar{\Omega}}\left(\int_0^u \Xi(s;\omega)\mathrm{d}s\right)\otimes\Xi(0;\omega)\,\mathrm{d}\bar{\mathbb{P}}(\omega,u) - \frac{1}{2}(\bar{\tau})^{-1}\Delta(0)\right).\end{aligned}$$

It remains to show that

$$(\bar{\tau})^{-1}\Delta(0) = 2\,\mathrm{Sym}\left(\int_{\bar{\Omega}}\left(\int_0^u \Xi(s;\omega)\mathrm{d}s\right)\otimes\Xi(0;\omega)\,\mathrm{d}\bar{\mathbb{P}}(\omega,u)\right).$$

We observe this as follows:

$$
\begin{aligned}
\Delta(0) = \mathbf{E}(\xi(0)\otimes\xi(0)) &= \int_\Omega \left(\int_0^{\tau(\omega)} \Xi(u;\omega)\mathrm{d}u\right) \otimes \left(\int_0^{\tau(\omega)} \Xi(s;\omega)\mathrm{d}s\right) \mathrm{d}\mathbb{P} \\
&= \int_\Omega \left(\int_0^{\tau(\omega)} \Xi(u;\omega)\mathrm{d}u\right) \otimes \left(\int_0^{u} \Xi(s;\omega)\mathrm{d}s + \int_u^{\tau(\omega)} \Xi(s;\omega)\mathrm{d}s\right) \mathrm{d}\mathbb{P} \\
&= \int_\Omega \int_0^{\tau(\omega)} \Xi(u;\omega) \otimes \left(\int_0^{u} \Xi(s;\omega)\mathrm{d}s\right) \mathrm{d}u\, \mathrm{d}\mathbb{P} \\
&\quad + \int_\Omega \int_0^{\tau(\omega)} \left(\int_0^{s} \Xi(u;\omega)\mathrm{d}u\right) \otimes \Xi(s;\omega)\, \mathrm{d}s\, \mathrm{d}\mathbb{P} \\
&= \int_\Omega \Xi(0;\omega) \otimes \left(\int_0^{u} \Xi(s;\omega)\mathrm{d}s\right) \mathrm{d}\bar{\mathbb{P}} + \int_\Omega \left(\int_0^{s} \Xi(u;\omega)\mathrm{d}u\right) \otimes \Xi(0;\omega)\, \mathrm{d}\bar{\mathbb{P}}.
\end{aligned}
$$

This concludes the proof. ☐

Example 2.16. Again, we refer to Kelly-Melbourne [18, 19] and Chevyrev et al. [3]. Theorem 2.10 applies with

$$\bar{\Delta}(s) := \mathbf{E}(\Xi(0)\otimes\Xi(s)) = \int (v \otimes (v\circ\phi_s))\mathrm{d}\nu,$$

for a flow ϕ on some compact manifold M with invariant and ergodic measure ν on a hyperbolic invariant set Ω, and for a ν-centered Hölder observable $v : \Omega \to \mathbb{R}^d$. The probability space is thus (Ω, ν) and $\Xi(s;\omega) = (v\circ\phi_s)(\omega)$. In particular, the covariance and area corrections from Theorem 2.10 agree with those given in [18, Theorem 1.1 (b) and Corollary 8.3], whenever these integrals are finite. The key examples are semi-flows which are suspensions of the uniformly and non-uniformly expanding maps mentioned in Example 2.3, and flows that are suspensions of uniformly and non-uniformly hyperbolic diffeomorphisms. This includes Axiom A flows, which are typically mixing for $v \in C^\infty$ [8], and by that satisfy the assumptions for Theorem 2.10, and non-uniformly hyperbolic flows constructed as suspensions over Young towers with exponential tails.

However, note that the formulas in Theorem 2.10 are only valid if the (semi-) flows are (sufficiently fast) mixing themselves; otherwise, the covariance and area correction can be given in terms of iterations of the associated Poincaré map $T : \Lambda \to \Lambda$, with ergodic, invariant measure μ on Λ, precisely in terms of Theorem 2.15, assuming that T is mixing such that Theorem 2.2 holds (see also [18, Corollary 8.1]); in more detail,

$$\bar{\Omega} = \{(\omega, t) : \omega \in \Lambda,\ 0 \le t \le \tau(\omega),\ (\omega, \tau(\omega)) = (T\omega, 0)\},$$

such that $\nu = (\bar{\tau})^{-1}(\mu \times \text{Lebesgue})$ is an ergodic invariant probability measure for $\phi_s(\omega, u) = (\omega, u+s)$, where $\bar{\tau} = \int_\Lambda \tau \mathrm{d}\mu$. Then the summability condition applies to

$$\Delta(n) := \mathbf{E}(\xi(0)\otimes\xi(n)) = \int (\bar{v} \otimes (\bar{v}\circ T^n))\mathrm{d}\mu,$$

where

$$\bar{v}(\omega) = \int_0^{\tau(\omega)} v(\omega, s)\mathrm{d}s = \int_0^{\tau(\omega)} (v\circ\phi_0)(\omega, s)\mathrm{d}s = \int_0^{\tau(\omega)} \Xi(s;\omega)\mathrm{d}s$$

and the stationary processes are $\xi(n;\omega) = (\bar{v}\circ T^n)(\omega)$ and $\Xi(s;\omega) = (v\circ\phi_s)(\omega, 0) = (v\circ\phi_0)(\omega, s)$.

Example 2.17. Friz-Kifer 2021 [11]. Rescaling time via $t \to \bar{\tau} t$, the process V^ε therein fulfills Assumption 2.14 according to [11, Theorem 2.5]; note that our notation ξ corresponds with their η and our notation Ξ with their ξ. Theorem 2.15 can be applied to obtain the area correction. Note that in [11], the term

$$\int_{\bar{\Omega}} \left(\int_0^u \Xi(s;\omega)\mathrm{d}s \right) \otimes \Xi(0;\omega)\, \mathrm{d}\bar{\mathbb{P}}$$

as a summand in $\bar{\mathbb{B}}(1)$, and thereby also in $\bar{\Gamma}$, is expressed as

$$\mathbf{E}\left(\int_0^{\tau(\omega)} \Xi(s;\omega)\mathrm{d}s \otimes \int_0^s \Xi(u;\omega)\,\mathrm{d}u \right),$$

i.e., in terms of the invariant measure $\mathbb{P}$ for the discrete time system.

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