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SOLUTION METHODS FOR MIXED VARIATIONAL INEQUALITIES

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Abstract: In this paper two descent methods with respect to a gap function for solving a class of monotone mixed variational inequalities are proposed. We show that the two algorithms, based on an exact and an Armijo-type line search procedure, respectively, are globally convergent.

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1. Introduction

Let X be a nonempty, closed, and convex subset of $\mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$ a map, and $f : \mathbb{R}^n \to \mathbb{R}$ a real valued function. The Mixed Variational Inequality (MVI, for short) problem is to find a point $x^* \in X$ such that

$$\langle F(x^*), y - x^* \rangle + f(y) - f(x^*) \geq 0 \qquad \forall\, y \in X, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$.

This problem was originally considered by Lescarret [6] and Browder [2] for its applications in mathematical physics. Afterwards, it has been shown that the MVI problem has a large variety of applications in various fields such as mechanics, economics and operation research, see [1], [5], [7], [10] and references

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therein. When $f \equiv 0$, the MVI problem reduces to the classical variational inequality problem.

One of the main approaches for solving the classical variational inequality or the MVI consists in minimizing a gap function associated to the problem. Plenty of corresponding descent type methods have been developed for VIs (see e.g. [3] and references therein). Descent type methods have been also presented for the MVI problem (see [4] and [9]). In [4] the author proposed a descent method which utilizes an inexact Armijo type line search procedure. The convergence was proved by assuming the operator F to be strongly monotone.

In this paper, we devise two global convergent descent algorithms (with an exact and an inexact line search procedure, respectively) with respect to a gap function, for solving MVIs with monotone (not necessarily strongly monotone) operator.

In the rest of the paper we consider the following assumptions.

(A1) The set $X \subseteq \mathbb{R}^n$ is nonempty, closed, and convex; the map $F : Y \to \mathbb{R}^n$ is continuously differentiable, where Y is an open convex set such that $X \subset Y$; the function $f : \mathbb{R}^n \to \mathbb{R}$ is convex.

(A2) The map F is monotone on Y, i.e. $\langle F(x) - F(y), x - y\rangle \geq 0$ for all $x, y \in Y$.

(A3) The set X is bounded.

2. Gap Functions

In this paper we consider the following function (see [4]):

$$\varphi_\alpha(x) = \max_{y \in X} \left[\langle F(x), x - y\rangle - \frac{\alpha}{2}\|x - y\|_G^2 + f(x) - f(y) \right], \tag{2}$$

where α is a positive parameter, G is a symmetric positive definite matrix, and $\|\cdot\|_G$ denotes the norm in $\mathbb{R}^n$ defined by $\|x\|_G = \sqrt{\langle x, G\,x\rangle}$. It easy to check that for each $x \in X$ the optimization problem (2) has a unique solution which will be denoted by $y_\alpha(x)$.

Under assumption (A1), the function defined in (2) is a gap function for the problem (1), i.e. it is nonnegative on X and the set of zeros coincide with the set of solutions of (1), see [4]. Therefore, if the problem (1) has a solution, it is equivalent to the following constrained optimization problem:

$$\min_{x \in X} \ \varphi_\alpha(x). \tag{3}$$

We remark that, by definition, the gap function φ_α is nondifferentiable since the function f is so. However, under the assumption (A1), the function φ_α is locally Lipschitz continuous on X and it is directionally derivable with respect to any direction (see [10, Proposition 4.19]). Moreover, the Clarke subdifferential and the directional derivative of φ_α at any $x \in X$ can be determined explicitly.

Note also that problem (3) may have local minima which differ from the global one. Therefore in order to solve (1) with a descent method with respect to φ_α, we need some monotonicity properties of the operator F. In [4] it has been proved that, if F is strongly monotone on Y (i.e. $\langle F(x) - F(y), x - y\rangle \geq \tau \|x - y\|^2$, for all $x, y \in Y$, for some $\tau > 0$) and x is not a solution of (1), then $y_\alpha(x) - x$ is a descent direction at any point x for the gap function φ_α. When F is only monotone, the vector $y_\alpha(x) - x$ is not necessarily a descent direction for φ_α, but it satisfies a condition which will be exploited in the following and that allows constructing descent methods for the gap function φ_α.

Theorem 1. *Let assumptions (A1) − (A2) be fulfilled and let $\alpha > 0$. Then, for each $x \in X$, the vector $y_\alpha(x) - x$ satisfies the following condition:*

$$\varphi'_\alpha(x; y_\alpha(x) - x) \leq -\varphi_\alpha(x) + \frac{\alpha}{2}\|x - y_\alpha(x)\|_G^2 \leq 0, \tag{4}$$

where $\varphi'_\alpha(x; y_\alpha(x) - x)$ denotes the directional derivative of φ_α at x with respect to the direction $y_\alpha(x) - x$.

Proof. Let $x \in X$. From [10, Proposition 4.19] we have

$$\begin{aligned}\varphi'_\alpha(x; y_\alpha(x) - x) &= \langle F(x) - (\nabla F(x)^\mathsf{T} - \alpha G)(y_\alpha(x) - x), y_\alpha(x) - x\rangle \\ &\quad + f'(x; y_\alpha(x) - x) \\ &= \langle F(x), y_\alpha(x) - x\rangle + \alpha\|y_\alpha(x) - x\|_G^2 \\ &\quad - \langle y_\alpha(x) - x, \nabla F(x)(y_\alpha(x) - x)\rangle + f'(x; y_\alpha(x) - x).\end{aligned} \tag{5}$$

By assumption (A2), it follows that the matrix $\nabla F(x)$ is positive semidefinite (see [3, Proposition 2.3.2]) and hence

$$\langle y_\alpha(x) - x, \nabla F(x)(y_\alpha(x) - x)\rangle \geq 0. \tag{6}$$

Moreover, since the function f is convex we have

$$f'(x; y_\alpha(x) - x) \leq f(y_\alpha(x)) - f(x). \tag{7}$$

Therefore, taking into account of (5), (6), (7), and that $\varphi_\alpha(x) \geq \frac{\alpha}{2}\|x - y_\alpha(x)\|_G^2$ (see [4]), we have

$$\begin{aligned}\varphi'_\alpha(x; y_\alpha(x) - x) &\leq \langle F(x), y_\alpha(x) - x\rangle + \alpha\|y_\alpha(x) - x\|_G^2 + f(y_\alpha(x)) - f(x) \\ &= -\varphi_\alpha(x) + \frac{\alpha}{2}\|x - y_\alpha(x)\|_G^2 \leq 0. \quad \square\end{aligned}$$

Algorithm 1

0. (Initial step)
   Let G be a symmetric positive definite matrix and $\eta \in (0,1)$.
   Let $\{\alpha_k\}$ be a sequence strictly decreasing to 0.
   Choose any $x^0 \in X$ and set $k = 0$.
1. (Stopping criterion)
   If $\varphi_{\alpha_k}(x^k) = 0$

      then STOP,

   else set $k = k + 1$.
2. (Minimization of φ_{α_k})

   2a. (Initialization)
   Set $i = 0$ and $z^0 = x^{k-1}$.

   2b. **If** $-\varphi_{\alpha_k}(z^i) + \dfrac{\alpha_k}{2}\,\|z^i - y_{\alpha_k}(z^i)\|_G^2 < -\eta\,\varphi_{\alpha_k}(z^i)$

      then (line search)
         set $d^i = y_{\alpha_k}(z^i) - z^i$
         compute $t_i \in \arg\min_{t\in[0,1]} \varphi_{\alpha_k}(z^i + t\,d^i)$

   else (update of x^k)
      set $x^k = z^i$ and return to step 1.

   2c. (Update of z^i)
   Set $z^{i+1} = z^i + t_i\,d^i$, $i = i + 1$, and return to step 2b.

This result is useful to derive a modified descent method with exact line search procedure (Section 3) and one with inexact Armijo-type line search procedure (Section 4) for solving the MVI problem (1). The basic idea is to use (4) to obtain, if possible, a descent direction. Indeed, if $x \in X$ satisfies the condition

$$-\varphi_\alpha(x) + \frac{\alpha}{2}\,\|x - y_\alpha(x)\|_G^2 < -\eta\,\varphi_\alpha(x), \tag{8}$$

where $\eta \in (0,1)$, then from (4) and (8) we get

$$\varphi'_\alpha(x; y_\alpha(x)) < -\eta\,\varphi_\alpha(x).$$

Hence the vector $d = y_\alpha(x) - x$ is a descent direction for φ_α at x and we can perform a line search procedure with respect to the direction d. Otherwise, if x does not solve the problem (1) and does not satisfy (8), we reduce the parameter α.

3. Descent Method with Exact Line Search

In this section we describe the descent method with an exact line search and we prove its global convergence to a solution of the problem (1).

Theorem 2. *If the assumptions $(A1)-(A3)$ are fulfilled, then Algorithm 1 either stops at a solution of the problem (1) after a finite number of iterations, or generates a sequence $\{x^k\}$ such that any of its cluster points solves (1), or generates a sequence $\{z^i\}$, for some fixed α_k, such that any of its cluster points solves (1).*

Proof. There are three possible cases.

Case 1. The algorithm stops at x^k after a finite number of iterations. From the stopping criterion at step 1 it follows that $\varphi_{\alpha_k}(x^k) = 0$, thus x^k solves the problem (1).

Case 2. The algorithm generates an infinite sequence $\{x^k\}$. From condition at step 2b we have

$$\varphi_{\alpha_k}(x^k) \leq \frac{\alpha_k}{2\,(1-\eta)}\, \|x^k - y_{\alpha_k}(x^k)\|_G^2 \qquad \forall\ k \in \mathbb{N}.$$

Since x^k and $y_{\alpha_k}(x^k)$ belong to X which is bounded, the norm $\|x^k - y_{\alpha_k}(x^k)\|_G^2$ is bounded above. Moreover $\lim\limits_{k\to\infty} \alpha_k = 0$, thus

$$\lim_{k\to\infty} \varphi_{\alpha_k}(x^k) = 0. \tag{9}$$

The sequence $\{x^k\}$ has cluster points because it is bounded. Let x^* be any cluster point of $\{x^k\}$ and x^{k_s} a subsequence converging to x^*. From the definition of φ_{α_k} it follows that for each $y \in X$ we have

$$\varphi_{\alpha_{k_s}}(x^{k_s}) \geq \langle F(x^{k_s}), x^{k_s} - y\rangle - \frac{\alpha_{k_s}}{2}\, \|x^{k_s} - y\|_G^2 + f(x^{k_s}) - f(y) \qquad \forall\ s \in \mathbb{N}.$$

Taking into account the continuity of F and f, $\lim\limits_{k\to\infty} \alpha_k = 0$, and (9), then passing to the limit we obtain

$$0 \geq \langle F(x^*), x^* - y\rangle + f(x^*) - f(y).$$

Since y is arbitrary, we have proved that x^* solves the problem (1).

Case 3. The algorithm generates an infinite sequence $\{z^i\}$ for a fixed $\alpha_k = \alpha$. Let z^* be any cluster point of $\{z^i\}$ and z^{i_s} a subsequence converging to z^*. Assume by contradiction that z^* does not solve (1), thus $\varphi_\alpha(z^*) > 0$. Moreover, for all $s \in \mathbb{N}$ we have:

$$-\,\varphi_\alpha(z^{i_s}) + \frac{\alpha}{2}\, \|z^{i_s} - y_\alpha(z^{i_s})\|_G^2 < -\,\eta\, \varphi_\alpha(z^{i_s}).$$

Hence passing to the limit, since φ_α and y_α are continuous (see [4]), we obtain

$$-\varphi_\alpha(z^*) + \frac{\alpha}{2}\,\|z^* - y_\alpha(z^*)\|_G^2 \le -\eta\,\varphi_\alpha(z^*) < 0.$$

Using Theorem 1 we have:

$$\varphi'_\alpha(z^*; y_\alpha(z^*) - z^*) < 0,$$

thus $d^* = y_\alpha(z^*) - z^*$ is a descent direction for φ_α at z^* and

$$\min_{t\in[0,1]} \varphi_\alpha(z^* + t\,d^*) < \varphi_\alpha(z^*). \tag{10}$$

On the other hand the sequence $\{\varphi_\alpha(z^i)\}$ is monotone decreasing and from the step length rule it follows that for each $t \in [0,1]$ we have:

$$\varphi_\alpha(z^{i_s+1}) \le \varphi_\alpha(z^{i_s} + t\,(y_\alpha(z^{i_s}) - z^{i_s})), \qquad \forall\ s \in \mathbb{N}.$$

Passing to the limit we obtain:

$$\varphi_\alpha(z^*) \le \varphi_\alpha(z^* + t\,(y_\alpha(z^*) - z^*))) \qquad \forall\ t \in [0,1],$$

that is

$$\varphi_\alpha(z^*) = \min_{t\in[0,1]} \varphi_\alpha(z^* + t\,d^*)$$

which is impossible because it contradicts (10). Thus z^* is a solution of the problem (1). □

4. Descent Method with Armijo-Type Line Search

In this section we describe the descent method with an Armijo-type line search and we prove it is globally convergent to a solution of (1).

Theorem 3. *If the assumptions* $(A1)-(A3)$ *are fulfilled, then Algorithm 2 either stops at a solution of the problem (1) after a finite number of iterations, or generates a bounded sequence* $\{x^k\}$ *such that any of its cluster points solves (1), or generates a bounded sequence* $\{z^i\}$*, for some fixed* α_k*, such that any of its cluster points solves (1).*

Proof. First, we show that the algorithm is well defined, i.e. that the line search procedure is always finite. Assume, by contradiction, that there are $i, k \ge 0$ such that the inequality

$$\varphi_{\alpha_k}(z^i + \gamma^m\,d^i) - \varphi_{\alpha_k}(z^i) > -\beta\,\gamma^m\,\varphi_{\alpha_k}(z^i),$$

holds for all $m \in \mathbb{N}$. Then we have:

$$\varphi'_{\alpha_k}(z^i; d^i) = \lim_{m\to+\infty} \frac{\varphi_{\alpha_k}(z^i + \gamma^m\,d^i) - \varphi_{\alpha_k}(z^i)}{\gamma^m} \ge -\beta\,\varphi_{\alpha_k}(z^i).$$

Algorithm 2

0. (Initial step)
   Let G be a symmetric positive definite matrix, $\eta, \gamma \in (0,1)$, and $\beta \in (0,\eta)$.
   Let $\{\alpha_k\}$ be a sequence strictly decreasing to 0.
   Choose any $x^0 \in X$ and set $k = 0$.
1. (Stopping criterion)
   If $\varphi_{\alpha_k}(x^k) = 0$

   then STOP,

   else set $k = k + 1$.
2. (Minimization of φ_{α_k})

   2a. (Initialization)
   Set $i = 0$ and $z^0 = x^{k-1}$.

   2b. **If** $-\varphi_{\alpha_k}(z^i) + \dfrac{\alpha_k}{2}\, \|z^i - y_{\alpha_k}(z^i)\|_G^2 < -\eta\, \varphi_{\alpha_k}(z^i)$

   then (line search)
   set $d^i = y_{\alpha_k}(z^i) - z^i$
   compute the smallest nonnegative integer m such that:

$$\varphi_{\alpha_k}(z^i + \gamma^m\, d^i) - \varphi_{\alpha_k}(z^i) \le -\beta\, \gamma^m\, \varphi_{\alpha_k}(z^i)$$

   set $t_i = \gamma^m$,

   else (update of x^k)
   set $x^k = z^i$ and return to step 1.

   2c. (Update of z^i)
   Set $z^{i+1} = z^i + t_i\, d^i$, $i = i + 1$, and return to step 2b.

Combining (4) and step 2b, we get:

$$\varphi'_{\alpha_k}(z^i; d^i) \le -\varphi_{\alpha_k}(z^i) + \frac{\alpha_k}{2}\, \|d^i\|_G^2 < -\,\eta\, \varphi_{\alpha_k}(z^i),$$

therefore

$$(\eta - \beta)\, \varphi_{\alpha_k}(z^i) < 0,$$

which is impossible because $\eta > \beta$ and $\varphi_{\alpha_k}(z^i) \ge 0$. So the line search procedure is always finite.

There are three possible cases.

Case 1. The algorithm stops at x^k after a finite number of iterations. From the stopping criterion it follows that x^k solves (1).

Case 2. The algorithm generates an infinite sequence $\{x^k\}$. As in the Case 2 of Theorem 2, it can be proved that any cluster point of $\{x^k\}$ solves (1).

Case 3. The algorithm generates an infinite sequence $\{z^i\}$ for a fixed $\alpha_k = \alpha$. Let us consider two possible subcases: either $\limsup_{i\to\infty} t_i > 0$, or $\limsup_{i\to\infty} t_i = 0$.

Subcase 3a. If $\limsup_{i\to\infty} t_i > 0$, then there exists $t^* > 0$ and a subsequence $\{t_{i_s}\}$ such that $t_{i_s} \geq t^* > 0$ for all $s \in \mathbb{N}$. Since the sequence $\{z^i\}$ is infinite, we have:

$$\varphi_\alpha(z^{i_s}) - \varphi_\alpha(z^{i_s+1}) \geq \beta\, t_{i_s}\, \varphi_\alpha(z^{i_s}) \geq \beta\, t^*\, \varphi_\alpha(z^{i_s}) \geq 0. \tag{11}$$

The sequence $\{\varphi_\alpha(z^i)\}$ is monotone decreasing and bounded below, hence

$$\lim_{i\to\infty} [\varphi_\alpha(z^i) - \varphi_\alpha(z^{i+1})] = 0,$$

and in particular

$$\lim_{s\to\infty} [\varphi_\alpha(z^{i_s}) - \varphi_\alpha(z^{i_s+1})] = 0. \tag{12}$$

Using (11) and (12), we obtain $\lim_{s\to\infty} \varphi_\alpha(z^{i_s}) = 0$ and thus $\lim_{i\to\infty} \varphi_\alpha(z^i) = 0$. If z^* is any cluster point of $\{z^i\}$, then from the continuity of φ_α it follows that $\lim_{i\to\infty} \varphi_\alpha(z^i) = \varphi_\alpha(z^*)$, hence $\varphi_\alpha(z^*) = 0$, i.e. z^* is a solution of the problem (1).

Subcase 3b. If $\limsup_{i\to\infty} t_i = 0$, then $\lim_{i\to\infty} t_i = 0$. From the step length rule it follows that for all $i \in \mathbb{N}$,

$$\varphi_\alpha(z^i + \gamma^{-1}\, t_i\, d^i) - \varphi_\alpha(z^i) > -\,\beta\, \gamma^{-1}\, t_i\, \varphi_\alpha(z^i).$$

By the mean value theorem we have

$$\varphi_\alpha(z^i + \gamma^{-1}\, t_i\, d^i) - \varphi_\alpha(z^i) = \langle \xi^i, \gamma^{-1}\, t_i\, d^i \rangle,$$

where $\xi^i \in \partial\varphi_\alpha(z^i + \theta_i\, \gamma^{-1}\, t_i\, d^i)$ for some $\theta_i \in (0,1)$. We set $w^i = z^i + \theta_i\, \gamma^{-1}\, t_i\, d^i$. From [10, Proposition 4.19] it follows that

$$\xi^i = F(w^i) - (\nabla F(w^i)^{\mathsf{T}} - \alpha\, G)\, (y_\alpha(w^i) - w^i) + g^i,$$

for some $g^i \in \partial f(w^i)$. Therefore, for all $i \in \mathbb{N}$, we have:

$$\langle F(w^i) - (\nabla F(w^i)^{\mathsf{T}} - \alpha\, G)\, (y_\alpha(w^i) - w^i), d^i \rangle + \langle g^i, d^i \rangle > -\,\beta\, \varphi_\alpha(z^i).$$

The sequences $\{z^i\}$ and $\{d^i\}$ are bounded, thus also $\{g^i\}$ is bounded. Let z^* be any cluster point of $\{z^i\}$. Since $\lim_{i\to\infty} t_i = 0$ and the set-valued map ∂f is closed, passing to the limit and taking a subsequence if necessary, we get:

$$\langle F(z^*) - (\nabla F(z^*)^{\mathsf{T}} - \alpha\, G)\, (y_\alpha(z^*) - z^*), d^* \rangle + \langle g^*, d^* \rangle \geq -\,\beta\, \varphi_\alpha(z^*), \tag{13}$$

where $d^* = y_\alpha(z^*) - z^*$ and $g^* \in \partial f(z^*)$. Since

$$f'(z^*; d^*) = \max_{g \in \partial f(z^*)} \langle g, d^* \rangle, \tag{14}$$

from [10, Proposition 4.19], (13) and (14) it follows that:

$$\varphi'_\alpha(z^*;d^*) = \langle F(z^*) - (\nabla F(z^*)^\mathsf{T} - \alpha\, G)\,(y_\alpha(z^*) - z^*), d^*\rangle + f'(z^*;d^*)$$
$$\geq \langle F(z^*) - (\nabla F(z^*)^\mathsf{T} - \alpha\, G)\,(y_\alpha(z^*) - z^*), d^*\rangle + \langle g^*, d^*\rangle \geq -\,\beta\,\varphi_\alpha(z^*). \quad (15)$$

Moreover, for all $i \in \mathbb{N}$, we have:

$$-\,\varphi_\alpha(z^i) + \frac{\alpha}{2}\,\|z^i - y_\alpha(z^i)\|_G^2 < -\,\eta\,\varphi_\alpha(z^i),$$

hence passing to the limit and taking a subsequence if necessary, and using Theorem 1 we obtain:

$$\varphi'_\alpha(z^*;d^*) \leq -\varphi_\alpha(z^*) + \frac{\alpha}{2}\,\|d^*\|_G^2 \leq -\,\eta\,\varphi_\alpha(z^*). \quad (16)$$

From (15) and (16) we get

$$(\eta - \beta)\,\varphi_\alpha(z^*) \leq 0.$$

Since $\eta > \beta$ and $\varphi_\alpha(z^*) \geq 0$, it follows that $\varphi_\alpha(z^*) = 0$. i.e. z^* solves the problem (1). □

Remark 4. In Algorithms 1 and 2 the sequence $\{\alpha_k\}$ can be chosen adaptively, for example (see also [11]) such as:

$$\alpha_k = \begin{cases} \alpha_{k-1} & \text{if } \varphi_{\alpha_{k-1}}(x^{k-1}) \leq \nu_{k-1}, \\ \mu\,\alpha_{k-1} & \text{otherwise,} \end{cases} \quad (17)$$

where $0 < \mu < 1$ and $\{\nu_k\}$ is a sequence decreasing to 0. Indeed, if the algorithm generates an infinite sequence $\{x^k\}$ with $\{\alpha_k\}$ chosen by (17), then either $\lim_{k\to\infty} \alpha_k = 0$, which can be treated as in the Case 2 of Theorem 2 or Theorem 3, or one has

$$\alpha_k = \bar{\alpha} \quad \text{and} \quad \varphi_{\bar{\alpha}}(x^k) \leq \nu_k \qquad \forall\ k > \bar{k},$$

hence $\lim_{k\to\infty} \varphi_{\bar{\alpha}}(x^k) = 0$. Then for each cluster point x^* of $\{x^k\}$ we have $\varphi_{\bar{\alpha}}(x^*) = 0$, that is x^* solves the problem (1).

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