Heterogeneous expectations and transitional dynamics in an evolutionary overlapping generations model

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Abstract

How agents form their expectations, and which equilibria are consequently selected, is an issue that comes over macroeconomic growth models that study the transition between states characterized by different levels of capital. Existing theoretical works in this area have been mostly dominated by perfect foresight models although alternative expectations schemes have been proposed in the last decades.

In the present paper we propose an OLG model with capital accumulation assuming that agents employ simple yet heterogeneous rules to forecast the future course of the interest rate and the real wage. The forecasting rules are selected in accordance with an evolutionary mechanism which evaluates the performance of the rules agents are currently adopting on the basis of a fitness measure. We analytically show that the introduction of heterogeneous expectations gives rise to multiple equilibria and the presence of the evolutionary selection among rules results in stabilizing the transitional dynamics and prevents the possibility that the economy might be locked into a poverty trap.

Keywords: Heterogeneous expectations; evolutionary switching; overlapping generations; transitional dynamics; equilibrium selection.

JEL codes: D84, E32, E70, C69.

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1 Introduction

Overlapping generations (OLG) models have been constituting a widespread tool for the analysis of several economic topics, ranging from economic growth (de la Croix and Michel, 2002), international economics and business cycle dynamics (Grandmont, 1985) to monetary policy (Blanchard and Fischer, 1989), inequality (Böhm and Vachadze, 2008, 2010), allocation of resources across generations and factors that may trigger the fertility transition (Fanti and Gori, 2011, 2013). When capital accumulation is included, OLG models turn out to be suitable to formalize the evolution of an economic system since they relate the resulting growth path to the individual behavior of the agents that populate the different generations. Since agents have to make inter-temporal decisions, how they form expectations about their second period of life is of utmost importance, as this ultimately affects the transitional dynamics of the underlying economy. In fact, agents' expectations about future variables influence their realizations, and thus the types of prediction rules that are adopted play a major effect on the consequent dynamics. This is the reason why, during the last decades, several papers devoted their focus to the role that expectations exert on the long-run macroeconomic phenomena emerging from the OLG setup. Traditional models have been assuming that agents have perfect foresight and, under this hypothesis, Galor and Ryder (1989), among others, showed that in a standard OLG model with capital accumulation, multiple equilibria may exist. When multiple equilibria are present, the selected one does not only depend on the economic framework but also on the expectations that agents adopt in forecasting the relevant variables (see Benhabib and Farmer, 1999).

However, the hypothesis of perfect foresight is very demanding, as it requires agents to own a certain degree of knowledge and computational capabilities that in reality do not have. In order to face this shortcoming, other expectation mechanisms that are based on a reduced level of rationality have been proposed. Agents are more likely to form their expectations relying only on current or past information, or have some scheme of adaptive learning. A huge number of papers have been studying the economic consequences of the different expectation designs on the resulting equilibria and their stability, with the ultimate aim of stabilizing aggregate fluctuations. The literature that deals with OLG models under different expectation mechanisms is vast and considers both pure exchange models (see e.g. Grandmont 1985, Bullard 1994, Duffy 1994, Arifovic 1995, Wenzelburger 2002 and Tuinstra 2003) and frameworks with capital accumulation (see e.g. Böhm and Wenzelburger 1999, Michel and de la Croix 2000, Chen et al. 2008, Chen and Li 2008, Cavalli and Naimzada 2016). All these works constitute a relevant contribution for the understanding of the properties that characterise OLG models when moving beyond the assumption of perfect foresight.

The OLG model we propose can be instrumental to focus not only on the stability properties of the equilibrium, but also on the transitional dynamics that turns out to be relevant when multiple equilibria are present, and that may be used to explain the progression from poverty to prosperity for several economic contexts. Additionally, another assumption that we try to overcome in this work is that agents are generally homogeneous with respect to the adopted expectations, and often only one forecasting rule at a time has been considered without allowing agents to hold different expectation mechanisms and to switch among them.

A few papers have recently considered the presence of heterogeneous expectations in OLG models. We would like to mention the works by Heemeijer et al. (2012) and Arifovic et al. (2019), that, using an experimental approach, focus on how boundedly rational agents form expectations and how they coordinate on a certain long-run outcome when multiple equilibria are present. Instead, the paper by Brazier et al. (2008) employs a monetary OLG setup where agents switch between different heuristics, to discuss the cause and the durability of the fall in the observed inflation volatility. Depending on the heuristics agents adopt, their simulations can result in reduced inflation volatility as well as in time-series characterized by large fluctuations. The model by Boone and Quaghebeur (2018), lately, explores through numerical simulations the transitional dynamics in an OLG framework with heuristic switching following policy shocks. In addition, micro-level empirical evidence, including Attanasio and Kaufmann (2009), also indicates a significant role for expectations in the determination of human capital investment. Thus, both the influence of stochastic shocks and the role of expectations suggest that the transition dynamics out of poverty traps is a natural application for the study of bounded rationality and, more broadly, for the role of heterogeneous expectations.

Following the literature of expectations formation in OLG models, in the present paper we consider an OLG model with capital accumulation, in line with Chen et al. (2008) and Cavalli and Naimzada (2016), assuming that agents employ simple yet heterogeneous rules to forecast the future course of the interest rate and the real wage. In fact, the paper by Chen et al. (2008) separately analyses three different types of real interest rate expectations, namely perfect foresight, myopic expectations and adaptive expectations, showing that economic dynamics is simple under perfect foresight while can be rather complex under the other two types of expectation formations, with the occurrence of chaotic dynamics. Cavalli and Naimzada (2016) retrieve the paper by Chen et al. (2008) and consider an OLG model with capital accumulation where agents work in both periods of life and may employ perfect foresight, myopic foresight or adaptive expectations. They show that under myopic or adaptive expectations, if the share of time devoted to labor in the second period of life is large enough, periodic and complex dynamics can occur,as well as scenarios with coexistence between the stable steady state and a periodic or chaotic attractor, giving rise to multistability. However, the two aforementioned papers do not allow for the possibility that agents may switch among different forecasting rules, being these available to their choice according to a certain performance measure. Therefore, in modelling the expectations selection mechanism, we adhere to the heuristic switching mechanism proposed by Brock and Hommes (1997, 1998), assuming that agents, in general, do not possess the cognitive abilities as assumed by the rational expectations literature nor to act as econometricians. Agents evaluate the performance of the forecasting rule they are currently adopting on the basis of a fitness measure: if it performs well, the probability that agents will use the same rule in the next period will be higher, while if it poorly performs, there is a higher probability that they switch to another rule. This form of learning has become standard in the last decades within the literature that deals with heterogeneous agents models (see Hommes, 2021 and the references therein) thanks to its capability to generate endogenous dynamics that reproduces real macro-financial outcomes. In this regard, the paper by Anufriev and Hommes (2012) proposed a heuristic switching model which is able to capture both individual and aggregate dynamics in various economic settings, while a model with homogeneous expectations would leave open the question on why different patterns in aggregate behavior may arise. Moreover, empirical evidence for heuristic switching learning models can be found in survey data (Branch, 2004), estimated financial models (Boswijk et al., 2007) and estimated macro-models (Cornea-Madeira et al., 2019). We ground on these arguments to adopt the same learning mechanism based on heterogeneous switching mechanism for our OLG framework, including parsimonious decision rules which also lead to an analytically tractable model.[1]

In the present paper, we carry on the analysis in two steps. In the first step we only introduce heterogeneous agents without allowing them to switch among the different rules, in order to focus on the role that the presence of different expectations play on the existence and the stability of the steady states. In particular, we consider two classes of agents, namely a type of agents that predicts the steady state values for the real wage and the interest rate and another type which is purely myopic. We analytically show that the introduction of these two classes of agents gives rise to multiple equilibria characterized by different level of capital and that the proportion of agents adopting a certain rule may or may not drive the transitional dynamics towards a sustained eco-

[1]We would like to recall that there also exist different learning mechanism, such as the adaptive learning (see Evans and Honkapohja, 2013) in which agents, acting as econometricians, revise their choice of forecasting model over time as new data become available. The evolutionary selection among different forecasting heuristics is also a key feature of the paper by Anufriev et al. (2019) in which agents use simple linear first order price forecasting rules, adapting them to the complex evolving market environment with a Genetic Algorithm optimization procedure.

nomic growth, thus escaping the poverty trap or being locked into it. This result is in line with that of Steiger (2011) which examines the effect of econometric learning on stability and transitional dynamics, showing that the expectation scheme, which is grounded on the bounded rationality of the agents, permits the economy to escape the poverty trap. But differently from Steiger (2011), our heterogeneous expectations framework gives rise to further steady states which are selected on the basis of the weight which is assigned to a certain class of agents. The issue of selecting among multiple equilibria is certainly of relevance in the OLG context and, in fact, the recent paper by Arifovic et al. (2019) has empirically addressed the equilibrium selection problem by showing how agents, by using simple rules, coordinate on certain form of beliefs.

The second step of our analysis is directly addressed to the issue of equilibria selection since some of them might be undesirable, for instance from the point of view of policy makers that aim to stabilize aggregate fluctuations or to ensure the convergence to a certain desirable equilibrium level. We thus allow agents to switch among the rules they employ, and show how the introduction of the switching mechanism is able to lead the dynamics towards the steady state which is desirable for a sustained growth. We show that the introduction of the heuristic switching mechanism results in reducing the number of equilibria (and thus it eases the issue of equilibria selection), in stabilizing the transitional dynamics and in ruling out the possibility that the economy might be locked into a poverty trap.

The remainder of the paper is organized as follows: Section 2 outlines the baseline overlapping generations model and the evolutionary mechanism for selecting the different forecasting rules; Section 3 contains the analytical results on the existence of the steady states, the stability conditions and the simulations for illustrating the transitional dynamics; Section 4 complements the results with analytical and numerical evidence on occurring complex dynamics; finally, Section 5 concludes. All the proofs of the Propositions are collected in the Appendix.

2 The baseline model

The model we develop is grounded on the paper by Cavalli and Naimzada (2016), which, in turn, takes inspiration from the work by Chen et al. (2008). In the first part, the model setup coincides with that in Cavalli and Naimzada (2016), but for the reader's sake we start summarizing it.

We consider an overlapping generations model with productive capital in which a normalized to 1 population of agents lives for two periods. We thus have a young and an old generation. Each agent works in both periods of life, supplying one unit of labor in the former period and $0 < h < 1$ unit in the latter one. This may represent a relevant argument to be included in an OLG context especially in developed economies affected by population ageing where mature individuals' consumption plans are based also on the provision of public pensions that depends on the amount of labor provided. There seems to be a consensus between politicians, economists and international organizations, in advocating an increase in the activity rate among individuals and many countries have also already significantly increased the compulsory age of retirement. These arguments motivate the inclusion of labour also in the second period of life, whose role described by the parameter h will be analyzed in Section 3.2. The present work, however, being mainly focused on the role of the heterogeneous expectations, considers an exogenous fraction of labour supply in the second period while the endogenization of the old age labour supply is left for future research.[2]

Young agents, at each discrete time t, assign part of their wage w_t to the consumption c_t and part to saving s_t, and this gives rise to the budget constraint

$$w_t \geq c_t + s_t,\ c_t \geq 0\, s_t \geq 0. \tag{1}$$

For the second period of life their income is made up by the return on the savings made at time t and the wage at time $t+1$, so that the budget constraint for consumption c_{t+1} of old agents at time $t+1$ is

$$c_{t+1} \leq s_t R^e_{t+1} + h w^e_{t+1},\ c_{t+1} \geq 0, \tag{2}$$

where R^e_{t+1} and w^e_{t+1} are respectively the expected real interest rate and the expected second period life wage.

Agents have identical Cobb-Douglas preferences, and are hence endowed with the logarithmic utility function (see e.g. de la Croix and Michel, 2002):

$$U = \log c_t + \delta \log c_{t+1}, \tag{3}$$

where $\delta \in (0,1)$ is the discount factor.

Each young generation maximizes the utility in (3) under the constraints (1) and (2), and this leads to

$$U = \log(w_t - s_t) + \delta \log(s_t R^e_{t+1} + h w^e_{t+1})$$

[2]Such assumption is in agreement with what outlined in the book by de la Croix and Michel (2002), Section 1.8.8, where the authors also suggest that $h < 1$ can be interpreted as part-time work or early retirement. Moreover, the idea that agents work in both periods of their life, and in particular only for a fraction of their second period, has been adopted by many others works, such as those by Fanti (2014, 2015), Kunze (2014), Miyazaki (2014) and Chen (2018).

from which, the optimal saving decision made by the young agent reads as

$$s_t = \max\left\{\frac{\delta}{1+\delta}w_t - \frac{h}{1+\delta}\frac{w^e_{t+1}}{R^e_{t+1}}, 0\right\}, \tag{4}$$

for given R^e_{t+1} and w^e_{t+1}.

Following Chen et al. (2008), a Cobb-Douglas production function is considered, that is $Y_t = AK_t^{\alpha}L_t^{1-\alpha}$, where K_t represents the capital employed in the production process while L_t is the labor used for output in period t. The parameters $A > 0$ and $\alpha \in (0, 1)$ represent the total factor productivity and the capital share, respectively. After taking the intensive form, the factor prices are given by

$$w_t = A(1-\alpha)k_t^{\alpha} \tag{5}$$

$$R_t = A\alpha k_t^{\alpha-1}, \tag{6}$$

where R_t is the real interest rate and k_t is the capital per capita. Imposing the equilibrium condition in the capital market $k_{t+1} = s_t$, recalling (4), we immediately obtain the equilibrium trajectory $\{k_t\}_{t\geq 0}$

$$k_{t+1} = \max\left\{\frac{\delta}{1+\delta}w_t - \frac{h}{1+\delta}\frac{w^e_{t+1}}{R^e_{t+1}}, 0\right\}, \tag{7}$$

which depends on the initial condition k_0. In order to obtain the explicit dynamics for the capital per capita k_t in (7), we need to specify the expected values for w^e_{t+1} and R^e_{t+1}. In this regard, differently from Chen et al. (2008) and Cavalli and Naimzada (2016) that separately examined different expectations schemes in the same OLG setup, we consider a heterogeneous expectations framework where agents can choose and switch between different forecasting rules, which are updated over time based upon a publicly available evolutionary fitness or performance measure, as for example the squared forecast error associated with each predictor. In particular, we shall consider the case of two types of agents, which we name "steady-state forecaster" and "naive".

Steady-state forecasters, even if not perfectly rational, are assumed to have a deeper knowledge on the economic environment they operate in. We assume that they actually know all the fundamental elements characterizing the market (i.e. they have structural knowledge of the supply and demand curves and they know the preferences of the other agents), but they are not able to predict how the other agents will behave (i.e. their expectations about R and w) nor, acting in a stochastically perturbed environment, to disentangle temporary shocks on wages and interest rates in order to be able to have a perfect knowledge of the current market configuration. Hence, they act as if all the agents had their same information endowment and if the economic system were at (or quickly reached in a near future) the steady state[3]

$$k^* = \left(\frac{\alpha\delta(1-\alpha)A}{\alpha(1+\delta) + h(1-\alpha)} \right)^{\frac{1}{1-\alpha}}. \tag{8}$$

Given their information endowment, their are able to determine, in each period, the amount of savings offered by a homogeneous set of rational agents and, through the equilibrium condition, to achieve the capital law of motion of the corresponding model, for which they can study its asymptotic behaviour and obtain the steady state capital value. Accordingly, they formulate a forecast for the wage and the interest rate which is consistent with the hypothesis of the system converging to the steady state k^*, namely their expectations are

$$w^e_{1,t+1} = w^* = A(1-\alpha)(k^*)^\alpha \qquad R^e_{1,t+1} = R^* = A\alpha(k^*)^{\alpha-1} \tag{9}$$

Naturally, in an environment with heterogeneity, the behaviour of these agents is not completely rational because it does not consider that in the market there are non-rational agents such as the naive. A strategy of this type, however, requires a certain degree of information about the economic fundamentals, and therefore we shall assume positive information-gathering costs for this class of agents.[4]

On the other hand, naive agents are myopic and expect that both the second period interest rate and wage will remain the same as in the current period. Accordingly, their expectations specializes as:

$$w^e_{2,t+1} = w_t \qquad R^e_{2,t+1} = R_t \tag{10}$$

In so doing, the average wage and interest rate expectations can be written as

$$w^e_{t+1} = n_{1,t+1}w^* + (1-n_{1,t+1})w_t \tag{11}$$

$$R^e_{t+1} = n_{1,t+1}R^* + (1-n_{1,t+1})R_t, \tag{12}$$

where $n_{1,t}$ represents the fraction of steady-state forecasters.

Fractions are updated according to an evolutionary fitness or performance measure. Rules with a better performance will attract more followers. The fitness measure is given by the squared forecast

[3]As it has been shown in Cavalli and Naimzada (2016), in a setting in which all the agents are rational, i.e. in the setting assumed by steady-state forecasters, the unique steady state toward which dynamics can converge is k^*. Indeed, also $k = 0$ is a steady state of map (7) under rational expectations, but it is easy to show that it is locally asymptotically unstable and attract no trajectory starting from $k_0 > 0$. The information endowment of steady-state forecasters allow them to be aware of these facts and to consequently form expectations as in (9).

[4]See also Hommes (2013).

error, i.e.

$$V_{i,t} = -\sum_x (x^e_{i,t} - x_t)^2 - C_i \tag{13}$$

where $x \in \{w, R\}$, $i \in \{1, 2\}$ and $C_i > 0$ represents the cost per time period for obtaining predictor i. For sophisticated predictors, such as rational expectations or those requiring a deeper knowledge and higher computational capabilities, the costs C_i may be positive (see e.g. Sethi and Franke (1995), Brock and Hommes (1997) and Hommes (2013) and also the literature on rational inattention among which we mention Sims (2003), Dewan and Neligh (2020) and Oprea (2020)), while for a simple habitual rule of thumb predictor, such as naive or adaptive expectations, these costs C_i can be considered as null. Therefore, in the rest of the analysis we shall assume $C_1 - C_2 = C > 0$.

The fraction of agents choosing a strategy i is given by the discrete choice model (see Brock and Hommes (1997)):

$$n_{i,t+1} = \frac{e^{\beta V_{i,t}}}{\sum_{i=1}^2 e^{\beta V_{i,t}}}, \tag{14}$$

where the parameter $\beta > 0$ is the intensity of choice, measuring how sensitive agents are in selecting the optimal prediction strategy. In the extreme case $\beta = 0$, differences in fitness can not be observed and all fractions will be fixed over time and equal to $1/2$. The other extreme case, $\beta \to +\infty$, corresponds to the case in which, in each period, all traders choose the optimal forecast. An increase in the intensity of choice β represents an increase in the degree of rationality with respect to the evolutionary selection of expectations.

Making explicit the factor prices through (5) and (6), equation (7) can be re-written as

$$k_{t+1} = \max\left\{\frac{\delta}{1+\delta}A(1-\alpha)k_t^\alpha - \frac{h}{1+\delta}\frac{n_{1,t+1}w^* + (1-n_{1,t+1})A(1-\alpha)k_t^\alpha}{n_{1,t+1}R^* + (1-n_{1,t+1})A\alpha k_t^{\alpha-1}}, 0\right\} \tag{15}$$

where

$$n_{1,t+1} = \frac{e^{-\beta\left[(R^*-R_t)^2+(w^*-w_t)^2+C\right]}}{e^{-\beta[(R^*-R_t)^2+(w^*-w_t)^2+C]} + e^{-\beta[(R_t-R_{t-1})^2+(w_t-w_{t-1})^2]}}$$

and, taking into account factor prices, the previous equation reads as

$$n_{1,t+1} = \frac{e^{-\beta\left[(R^*-A\alpha k_t^{\alpha-1})^2+(w^*-A(1-\alpha)k_t^\alpha)^2+C\right]}}{e^{-\beta\left[(R^*-A\alpha k_t^{\alpha-1})^2+(w^*-A(1-\alpha)k_t^\alpha)^2+C\right]} + e^{-\beta\left[(A\alpha k_t^{\alpha-1}-A\alpha k_{t-1}^{\alpha-1})^2+(A(1-\alpha)k_t^\alpha-A(1-\alpha)k_{t-1}^\alpha)^2\right]}}. \tag{16}$$

We stress that the map defining the right hand side of (15) is not defined for $k_t = 0$, but it has a continuous extension to $[0, +\infty)$. Therefore, let $g : [0, +\infty) \times [0, 1] \to \mathbb{R}$, $(k, n) \mapsto g(k, n)$ be defined by the right hand side of (15) on $(0, +\infty) \times [0, 1]$ and by $g(0, n) = 0$ for $n \in [0, 1]$. The model can thus be written as a two-dimensional dynamical system, that is

$$F:\begin{cases} k_{t+1}=g(k_t,n_{1,t+1}(k_t,z_t))=\max\left\{\frac{\delta}{1+\delta}A(1-\alpha)k_t^{\alpha}-\frac{h}{1+\delta}\frac{n_{1,t+1}w^*+(1-n_{1,t+1})A(1-\alpha)k_t^{\alpha}}{n_{1,t+1}R^*+(1-n_{1,t+1})A\alpha k_t^{\alpha-1}},0\right\} \\ z_{t+1}=k_t \end{cases} \tag{17}$$

where $n_{1,t+1}$ has been defined in (16) and the equilibrium values for the wage and the interest rate are respectively given by $w^*=A(1-\alpha)(k^*)^{\alpha}$ and $R^*=A\alpha(k^*)^{\alpha-1}$ where k^* is given by (8).[5]

3 Model functioning: analytical results and numerical simulations

3.1 The case of fixed fractions

We start by analyzing the case in which there is no endogenous switching among forecasting strategies. If we do not allow agents to switch between the two rules, we can think about the unit mass of agents populating the economy that splits into two groups. In particular, we can denote by n_1 the fixed fraction of steady-state forecasters, while $n_2=1-n_1$ represents the fraction of agents adopting the naive forecasting strategy. Accordingly, the map in (17) reduces to the one-dimensional system

$$k_{t+1}=M(k_t) \tag{18}$$

where $M:[0,+\infty)\to\mathbb{R}$ is defined by

$$M(k)=\begin{cases} 0 & \text{if } k=0 \\ \max\{f(k),0\} & \text{if } k>0 \end{cases}$$

and function $f:(0,+\infty)\to\mathbb{R}$ is specified as

$$f(k)=\frac{A(1-\alpha)\delta k^{\alpha}}{\delta+1}-\frac{h\left(A(1-\alpha)k^{\alpha}(n_1-1)-A(1-\alpha)\left(\frac{A\alpha\delta(1-\alpha)}{\alpha(\delta+1)+h(1-\alpha)}\right)^{\frac{\alpha}{1-\alpha}}n_1\right)}{A\alpha\left(k^{\alpha-1}(n_1-1)-\frac{(\alpha(\delta+1)+h(1-\alpha))n_1}{A\alpha\delta(1-\alpha)}\right)(\delta+1)}.$$

It is worth to notice that for $n_1=0$ the model in (18) reduces to the one in Cavalli and Naimzada (2016), Eq. (13).

As concerns the map M and the fixed points that it may feature, we can state the following:

[5]We stress that, given the information endowment of steady-state forecasters, they are able to obtain the steady configuration of the economic system assuming that all the agents are rational. They are not able to know (17) and its possible steady states, their expectations can not be affected by the agent's heterogeneity and by the presence of naive agents, and, as a consequence, they are not affected by the steady states of map (17). Their expectations are those given by (9).

Proposition 1. *Concerning the possible steady states of* (18)*, we have the following scenarios*

a) If $0 < \alpha < 1/2$ *there exists* $\bar{n}_1 \in (1/2, 1)$ *such that we have 2 steady states consisting of* $k_0^* = 0$ *and* k^* *if* $n_1 \in (0, \bar{n}_1)$ *and 4 steady states consisting of* $k_0^* = 0 < k_1^* < k_2^* < k^*$ *if* $n_1 \in (\bar{n}_1, 1)$

b) If $\alpha = 1/2$ *and* $n_1 \leq \frac{1+\delta+h}{1+\delta+2h}$ *we have 2 steady states consisting of* $k_0^* = 0$ *and* k^**, while if* $n_1 > \frac{1+\delta+h}{1+\delta+2h}$*, we have 3 steady states, consisting of* $k_0^* = 0, k_1^* \in (0, k^*)$ *and* k^*

c) If $1/2 < \alpha < 1$ *we have 3 steady states, consisting of* $k_0^* = 0, k_1^* \in (0, k^*)$ *and* k^*

Moreover, we have

$$\lim_{k \to +\infty} f(x) = \begin{cases} +\infty & \text{if} \quad n_1 > \frac{h(1-\alpha)}{\alpha(1+\delta)+2h(1-\alpha)} \\ +\infty & \text{if} \quad n_1 = \frac{h(1-\alpha)}{\alpha(1+\delta)+2h(1-\alpha)} \text{ and } \alpha > 1/2 \\ \frac{A^2\delta^2(\delta+2h+1)}{4h(\delta+h+1)^2} & \text{if} \quad n_1 = \frac{h(1-\alpha)}{\alpha(1+\delta)+2h(1-\alpha)} \text{ and } \alpha = 1/2 \\ -\frac{h(1-\alpha)\left(\frac{A\alpha\delta(1-\alpha)}{\alpha(\delta+1)+h(1-\alpha)}\right)^{\frac{1}{1-\alpha}}}{\alpha(\delta+1)} & \text{if} \quad n_1 = \frac{h(1-\alpha)}{\alpha(1+\delta)+2h(1-\alpha)} \text{ and } \alpha < 1/2 \\ -\infty & \text{if} \quad n_1 < \frac{h(1-\alpha)}{\alpha(1+\delta)+2h(1-\alpha)} \end{cases}$$

Proof. See Appendix. □

The stability of k^* is studied in the following proposition.

Proposition 2. *The steady state* k^* *is locally asymptotically stable provided that*

$$n_1 > 1 - \alpha \left(\frac{(\alpha + 1)(1 + \delta)}{h(1 - \alpha)} + 1 \right) \tag{19}$$

Proof. See Appendix. □

As a preliminary inspection, it is useful to observe that the right hand side of (19) is always strictly smaller than 1, so for each parameters configuration there is a suitably large share of steady-state forecasters for which the steady state k^* becomes stable.

Secondly, in order to provide insights into the results of the previous two propositions, in Figure 1 we report the shape of the map in (18) for different values of the parameter n_1, i.e. the fraction of steady-state forecasters. The first row of Figure 1 highlights the case in which only a small fraction of steady-state forecasters is present ($n_1 = 0.295$) and two steady states exists, namely $k_0^* = 0$ and k^*. In this configuration, where the majority of agents is of the naive type, the steady state k^* is locally asymptotically stable. In fact, when the system is not in equilibrium, agents mostly expect that the future wage and interest rate are consistent with the actual realizations of such variables and, accordingly, savings are such that the subsequent level of the capital monotonically increase or decreases depending on whether the initial value of k is lower or higher than the steady state value

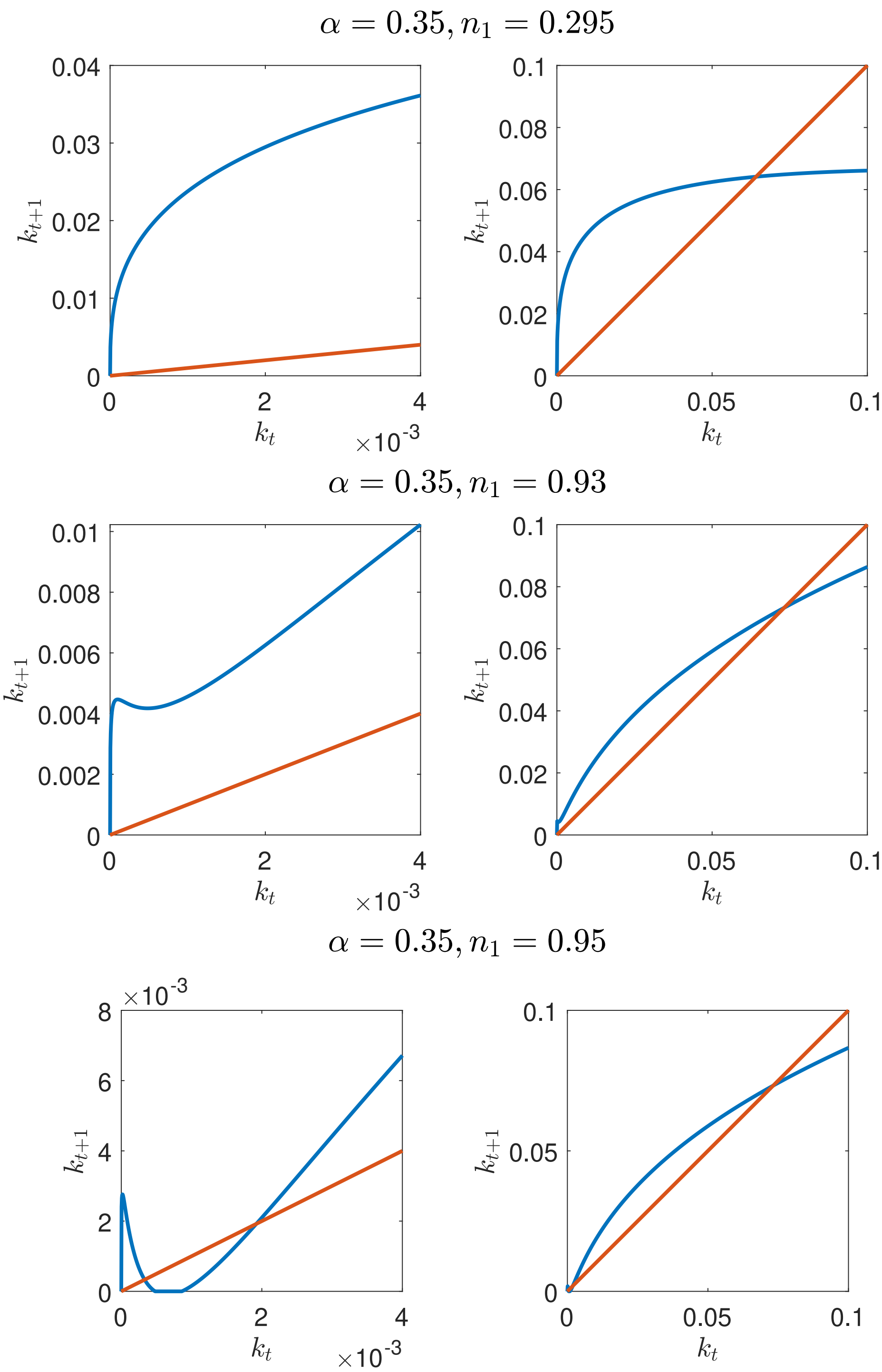


Figure 1: The graph of the map M (in blue) for different values of the fraction of steady-state forecasters. The panels on the right column report enlargements of the corresponding left panels in order to show the birth of the additional steady states. Other parameters are: $A = 1$, $\delta = 0.9$, $h = 0.7$.

(note that in this case equilibrium dynamics are very similar to the basic Solow model, see Solow 1956). If the role of steady-state forecasters becomes more relevant and their proportion reach a sufficiently large value (e.g. $n_1 = 0.93$), the shape of the map in (18) starts folding, as highlighted in the second row of Figure 1, and thus signaling the occurrence of a qualitative change in the dynamic behavior, which is also associated with a different configuration of the steady states. In fact, as the fraction of steady-state forecasters increases further ($n_1 = 0.95$), two further steady states arise as the effect of an occurred fold bifurcation, as the third row in Figure 1 denotes. We thus have a situation of multistability in which, besides the steady state k^*, two further steady states have appeared, namely k_1^* and k_2^*. We can provide an economic rationale behind the emergence of these two further steady states. Let us consider a situation in which the economy is characterized by an initial low level of capital, far from the level k^*. Since the majority of agents are steady-state forecasters, they expect values of w and R consistent with their equilibrium values as well as with a high level of k. Based on their expectations, they are prone to consume more and save less, and this generates a reduction in the capital accumulation which, in turn, gives rise to a contraction in the growth path which settles down to a steady state characterized by a lower level of capital with respect to k^*. Instead, if we consider an initial state characterized by a capital level which is closer to k^*, the expectations of steady-state forecasters are more consistent with high level of k. Accordingly, the transition dynamics is able to settle down to the steady state k^*, thanks to the decreasing returns in capital implying that the larger the capital is, the smaller the gain from additional capital is. The case reported in the third row of Figure 1 is an example of dynamics leading to a poverty trap where the long run outcome depends on the initial condition. Specifically, there are two locally stable attractors, whose basins of attraction are bounded by the unstable steady state k_2^*. Of course there exists a variety of reasons that advocate the emergence of poverty traps (see e.g. Matsuyama, 1997 and Azariadis and Stachurski, 2005) such as market failures, bad domestic policies, etc. In our setting the emergence of a poverty trap can be ascribed to the nature of agents' expectations and to their forecast error which is responsible for the deterioration in the capital accumulation. When agents are mostly steady-state forecasters, the way in which they form expectations together with their amount, may lead the economy to an attractor characterized by a low level of capital.

The results discussed above can be also confirmed by looking at the time series of Figure 2 which reports the transitional dynamics occurring when the fraction of steady-state forecasters is varied. In particular, the left panel of Figure 2 depicts two different dynamics corresponding to the two

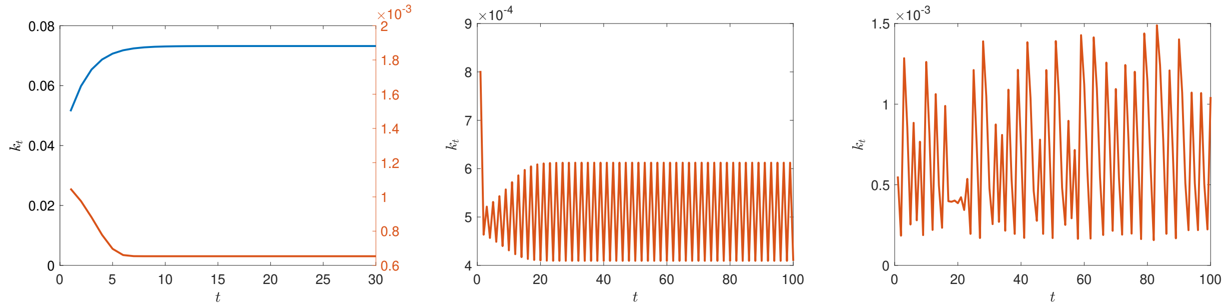


Figure 2: Transitional dynamics on varying the fraction of steady-state forecasters. The left panel, obtained for $n_1 = 0.93$, shows the convergence to the low capital steady state (orange line) and to the steady state k^* characterized by a high level of capital (blue line). The central panel, obtained for $n_1 = 0.948$, reports periodic dynamics around the steady state characterized by a low level of capital. Finally, the right panel, obtained for $n_1 = 0.949$ depicts aperiodic fluctuations. Other parameters are: $A = 1$, $\alpha = 0.35$, $\delta = 0.9$, $h = 0.7$.

steady states. The orange trajectory refers to the steady state associated with a low level of capital, i.e. the poverty trap, while the blue trajectory is associated with the steady state characterized by a higher level of capital. The emergence of this multistability situation has to be ascribed to the role of steady-state forecasters and to their expectation formation mechanism which may generate consumption and savings patterns that induce the economy towards a steady configuration in which the capital shrinks. On the other hand, if the initial capital level is sufficiently large, the poverty trap can be escaped even in the presence of a majority of steady-state forecasters, and the economy is able to reach a substantial gain in the capital level. Nonetheless the steady state characterized by a lower level of capital can also undergo a qualitative change in its behavior, i.e. a bifurcation, and exhibit periodic or aperiodic fluctuations, as long as the fraction of steady-state forecasters further increases. The central panel of Figure 2, indeed, refers to a configuration in which the dynamics of capital periodically oscillates around the steady state while the right panel depicts irregular oscillations which endogenously arise.

The latter dynamic result is further supported by the bifurcation diagrams of Figure 3 showing the values visited or approached asymptotically by the variable k as a function of the bifurcation parameter n_1. The left panel of such figure depicts, in black, the dynamics related to the steady state k^* which is always locally asymptotically stable as long as the fraction of steady-state forecasters n_1 increases. The blue color is employed to highlight the basin of attraction of the steady state k^*. On the other hand, the same panel reveals, in its bottom right corner, also the onset of instability for the steady state characterized by a lower level of capital. This occurrence can be better appraised by looking at the enlargement in the right panel of Figure 3. Such a steady state, depicted in red and whose basin of attraction is coloured in yellow, loses stability through a period doubling bifurcation which gives rise to period dynamics that in turn result into cycles of higher periodicity

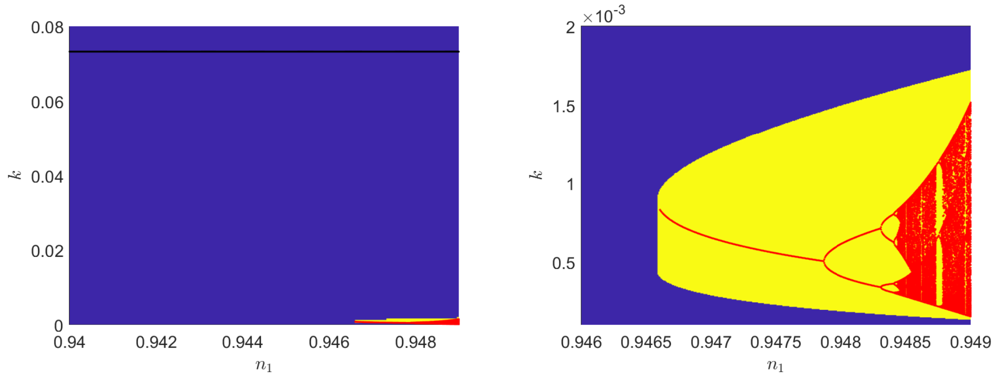


Figure 3: Bifurcation diagram on varying the fraction n_1 . The right panel shows the dynamics around the low capital steady state (bifurcation diagram in red) together with the steady state k^* (black line) which is always locally stable. The yellow color refers to the basin of attraction of the low capital steady state while the blue color is used for the basin of k^*. The right panel is an enlargement around the low capital steady state in order to show its destabilization through a period-doubling bifurcation on growing n_1. Other parameters are: $A = 1$, $\alpha = 0.35$, $\delta = 0.9$, $h = 0.7$.

and, finally, into chaos, following the traditional cascade of period doubling.

3.2 The case of endogenous switching

In this part we analyze the dynamics of the model when agents are allowed to revise their expectations over time on the basis of a performance measure as outlined in Section 2. The system is now described by the map in (17), for which we can state the following:

Proposition 3. *The steady states of the map in* (17) *are* $(0, 0)$ *and* (k^*, k^*), *to which corresponds the fraction* $n_1^* = 1/(1 + e^{\beta C})$ *of steady-state forecasters.*[6]

Proof. See Appendix. □

The first striking difference that can be appraised from the previous proposition is that, in the case of endogenous switching between the two rules, the number of equilibria is reduced, namely the two intermediate fixed points k_1^* and k_2^* are no more present. Moreover, it is also possible to study the local stability conditions of the steady state (k^*, k^*), as the next Proposition highlights.

Proposition 4. *The steady state* (k^*, k^*) *is locally asymptotically stable for* (17) *provided that* $h \leq \alpha(\alpha + 1)(1 + \delta)/(1 - \alpha)^2$ *or, otherwise, if*

$$\beta C < \ln\left(-\frac{\alpha + \alpha\delta + \alpha h + \alpha^2\delta - \alpha^2 h + \alpha^2}{\alpha - h + \alpha\delta + 2\alpha h + \alpha^2\delta - \alpha^2 h + \alpha^2}\right) \tag{20}$$

[6]As evident from the proof, for $\alpha \in (1/2, 1]$ there exists a set of parameters' configurations for which an additional steady state exists. Such set has null measure, so its economical relevance is negligible.

Proof. See Appendix. □

The result of the latter Proposition clarifies the role of the intensity of choice parameter β, associated to the introduction of the switching mechanism, on the local stability. If the parameter h is sufficiently small, then β does not influence the stability of the steady state, as prescribed by the first condition of Proposition 4. On the other hand, if h increases, the role of β becomes relevant as the first condition in Proposition 4 may be no longer satisfied. Thus, if the information costs are positive, an increase in the intensity of choice β may turn the steady state (k^*, k^*) unstable. And this result is consistent with that of Brock and Hommes (1997). Additionally, it can be the case that the steady state (k^*, k^*) is always unstable when the right side of the inequality in 20 is negative, regardless the value of β.

In order to portray the dynamic behavior of the model, we complement our analytical results with a series of simulations. In particular, in what follows we will consider the same parameter set that has been employed in the previous Section, when no heuristic switching was considered. In the next Section, instead, we shall provide evidence of the presence of persistent periodic and quasi-periodic fluctuations in the capital dynamics.

Figure 4 shows the transitional dynamics obtained with different initial conditions when the heuristic switching mechanism is considered. Comparing the dynamics emerging after the introduction of the heuristic switching with that obtained in the case of exogenous fractions (cf. Figure 2, left panel), we observe the dynamics is not affected by the initial distribution of the capital and it monotonically converges quite quickly towards the steady state k^*. The right panel of Figure 4 plots the corresponding agents' fractions dynamics obtained with the same initial conditions and shows that, in the long run, the surviving fraction is that of naive type. This can be easily explained by considering the fact that when the dynamics is into its transition towards the steady state, there exists a substantial difference between the performance of the two forecasting rules that, in turn, may lead to phases in which the dominance of the steady-state forecasting strategy emerges. On the other hand, when the dynamics reaches the steady state, the two forecasting strategies perform almost equally and thus it is reasonable to believe that the majority of the agents will adopt the cheapest rule, and hence leading to a predominance of naive agents in the steady state.

To sum up, in the present setting, agents switch between the different heuristics based upon their relative past forecasting performance and this leads the economy to a sustained economic growth. In fact, thanks to the introduction of the evolutionary mechanism, the multiplicity of equilibria is avoided and the possibility that the economy remains locked in the poverty trap is ruled out. The introduction of this form of adaptive learning thus allows the economy to escape from the low-capital

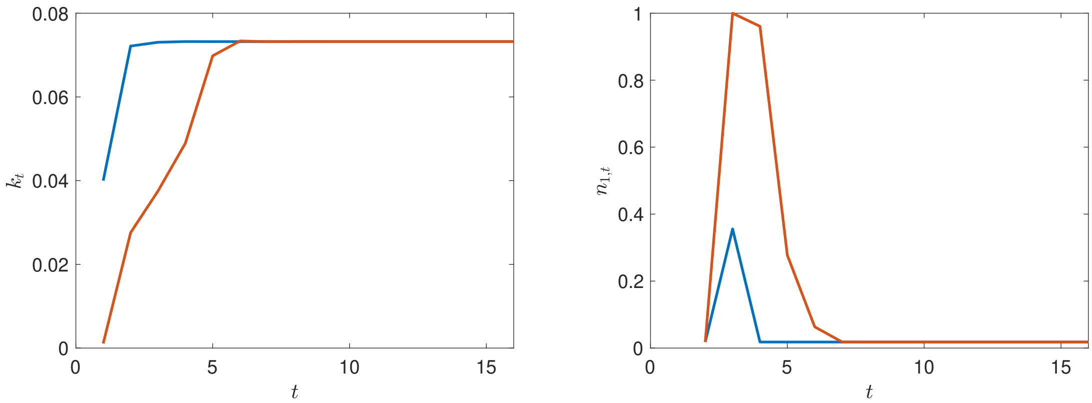


Figure 4: Transitional dynamics under heterogeneous expectations and heuristic switching. The left panel reports the evolution of the capital while the right panel refers to the fractions. Other parameters are: $A = 1$, $\alpha = 0.35$, $\delta = 0.9$, $h = 0.7$, $C = 1$, $\beta = 4$. Initial conditions are selected as in Figure 2.

equilibrium and to make a transition to a neighborhood of the upper steady state characterized by a sustained economic growth.

4 The onset of complex dynamics

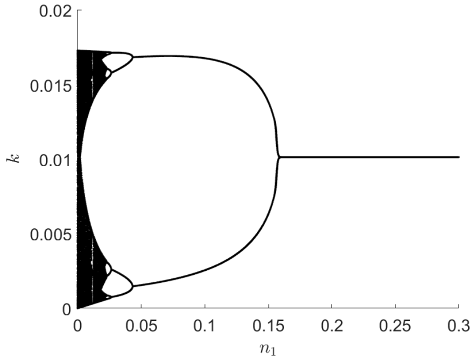
Several studies have shown that complex dynamics, such as permanent periodic or quasi-periodic, and even chaotic, oscillations can arise in overlapping generations models with heterogeneous expectations (see e.g. Tuinstra and Wagener 2007, Chen et al. 2008, Cavalli and Naimzada, 2016). In this section, we illustrate that this eventuality may occur also in our setting both in the case of no switching between forecasting rules and in the endogenous switching case. Firstly, we proceed by stating a proposition that analytically proves the occurrence of entropic chaos, referring to [19, Definition 2], for the dynamical system without heuristic switching and, secondly, we show how the same result still holds also when the system is augmented to consider the possibility of switching among forecasting rules.

In Section 3.1, we have already shown the occurrence of complex and chaotic dynamics for the steady state characterized by a lower level of capital. Thus, we can prove that entropic chaos arises in our system according to the following:

Proposition 5. *The entropic chaos occurs for the map in (18) when h is near $\tilde{h} = \dfrac{1+\delta}{(1-\alpha)^2}\alpha^{\frac{1-2\alpha}{1-\alpha}}$ and n_1 is near* 0.

Proof. See Appendix $\square$

Thanks to the result contained in the previous proposition, we can show some further evidence that highlights complex dynamics related to the steady state k^*. In the left panel of Figure 5, we



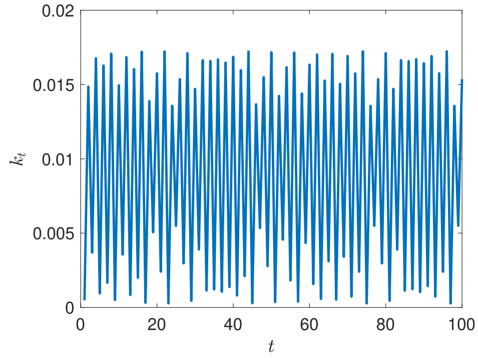


Figure 5: The left panel reports the bifurcation diagram on varying n_1 for $h = \tilde{h}$ while the right panel shows the corresponding time series obtained for $n_1 = 0.01$. Other parameters are: $A = 1$, $\alpha = 0.2$, $\delta = 0.1$.

report a bifurcation diagram on varying n_1 and for $h = \tilde{h} \approx 0.514$ as stated in the Proposition 5. The plot clearly shows that when n_1 is near zero or, at least, small enough, the dynamics is characterized by a chaotic motion while, as n_1 increases, we observe a local stabilizing effect played by the increasing fraction of steady-state forecasters. In particular, through a series of period-halving bifurcations, the dynamics turn into periodic with a smaller amplitude, and finally are fully locally stabilized when n_1 grows further. The right panel of Figure 5 confirms the chaotic dynamics by portraying a time series obtained for $n_1 = 0.01$ and $h = \tilde{h}$ in which the course of the capital evolves without a predictable patterns, yet being characterized by continuous upswings and downswing around the unstable steady state.

The occurrence of complex dynamics and chaos is a feature that characterizes also the system when the endogenous switching mechanism between expectations is introduced. On this regard, we can state the following:

Proposition 6. *The entropic chaos occurs for the map in* (17) *when h is near* $\tilde{h} = \dfrac{1+\delta}{(1-\alpha)^2}\alpha^{\frac{1-2\alpha}{1-\alpha}}$ *and large β.*

Proof. See Appendix. □

The previous result is confirmed by Figure 6 in which two bifurcation diagrams with respect to the intensity of choice β are reported, showing the bifurcation route from a stable steady state for low values of β to the complicated dynamical behavior for high values of β. The left panel depicts the dynamics of the capital on varying the intensity of choice β, whose increase plays a destabilizing effect through a cascade of period-doubling bifurcations that opens the onset for chaotic dynamics. The destabilizing role of β is also evident from the right panel of Figure 6 in which the bifurcation diagram with respect to the fraction n_1 is reported. As long as β increases, we first observe a

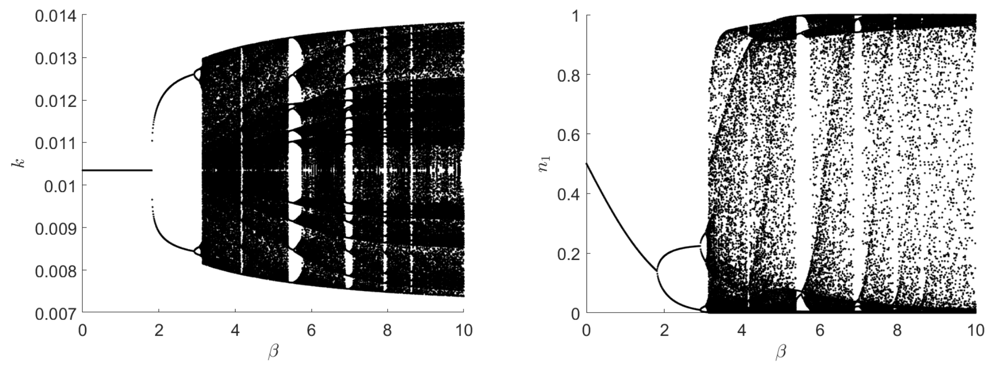


Figure 6: Bifurcation diagrams on varying the intensity of choice β for k (left) and n_1 (right). Other parameters are: $A = 1$, $\alpha = 0.2$, $\delta = 0.1$, $h = \tilde{h} \approx 0.514$, $C = 1$.

reduction in the fraction of steady-state forecasters and secondly, due to a period doubling route-to-chaos, a rebound and a large increase in the amplitude of the fluctuations in the fractions dynamics. We can provide a simple economic intuition for this occurrence. Suppose that, starting from an initial state close to the steady state, most of the agents are using the cheaper myopic forecasting rule, because they do not have to pay for a costly and more sophisticated forecasting rule that would provide an almost identical forecast. When the steady state turns unstable and the dynamics diverge from the steady state, it might be the case that the performance of the myopic rule decreases and, at some point, it may becomes advantageous to buy the more sophisticated steady-state forecasting rule. If the intensity of choice to switch between the two types is high, most agents will then switch towards the steady-state forecasting rule. As a result, the dynamics of the capital are led back towards to the steady state. The process then repeats. Chaotic capital fluctuations thus arise from a (boundedly) rational choice between the two types of expectations.

Finally, the chaotic time series of Figure 7 show the capital (left panel) and fractions (right panel) dynamics which are characterized by persistent fluctuations featuring an irregular and continuous switching between stable phases, with values of the capital close to the steady state, and unstable phases characterized by fluctuating values of the capital with a large amplitude and fractions that exhibit huge spikes.

5 Concluding remarks

In this paper we have developed and analyzed an overlapping generations model with capital accumulation in a setup populated by agents that own heterogeneous expectations, which are selected on the basis of an evolutionary mechanism. Agents use simple rules, heuristics, to forecast the future value of the interest rate and the real wage. The use of these heuristics is instrumental to form

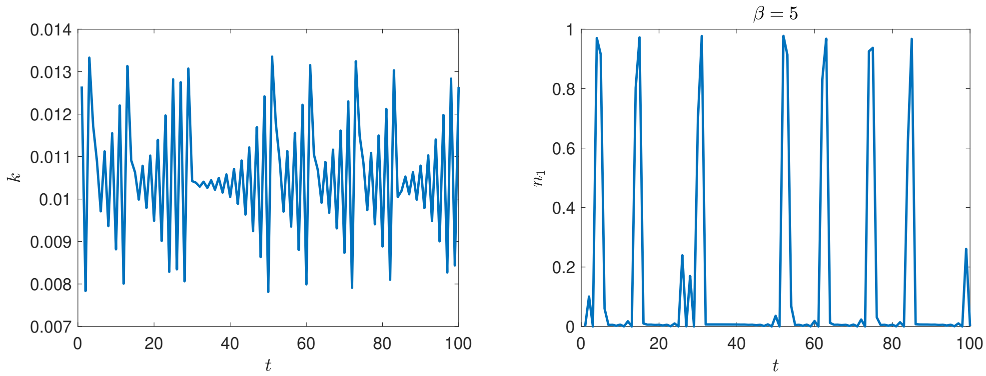


Figure 7: Chaotic time series of the capital k_t and fractions n_t of steady-state forecasters. Other parameters are: $A = 1$, $\alpha = 0.2$, $\delta = 0.1$, $h = \tilde{h} \approx 0.514$, $\beta = 5$, $C = 1$.

their expectations because, in general, they do not possess perfect cognitive abilities as traditionally assumed. Instead, agents have a certain number of different heuristics at their disposal and, on a regular basis, they evaluate the predictive power of the rule they are currently employing. If it performs well, the probability that agents will use the same heuristic in the next period will be higher; on the contrary, if the performance is poor, there is a higher probability that agents switch to another rule.

The present contribution builds on the literature that goes beyond the rational expectations paradigm by exploiting the approach based on the evolutionary switching mechanism among heuristics within an overlapping generations model. In so doing, we have studied the transitional dynamics of the model for two different settings, namely the case in which the fraction of agents employing each type of heuristic is fixed as well as the case in which agents are allowed to switch among the different forecasting rules. Our analysis has lead to a series of findings. In a context without heuristic switching, the system that describes the evolution of the capital accumulation features multiple equilibria, characterized by different levels of capital. The resulting dynamics depends on the heuristic that is used the most. Accordingly, the system can be locked into the poverty trap due to the expectation formation mechanism of agents that may induce consumption and savings patterns which leads the economy towards a steady configuration in which the capital accumulation shrinks. Instead, when the evolutionary selection among heuristics is considered, the economy is able to rule out the poverty trap thanks to the learning mechanism that allows agents to select the best performing rule. Therefore, the evolutionary process of heuristic selection has a stabilising effect on the transitional dynamics, thus enhancing the stability of the model and the changeover towards a sustained economic growth.

Further developments of the present work could be the introduction of different forecasting rules and the evaluation of their performances within an experimental environment with human subjects similar to that of Arifovic et al. (2019).

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Compliance with ethical standards

The authors have no relevant financial or non-financial interests to disclose.

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Appendix

Proof of Proposition 1. Introducing $\rho_1 = \frac{A\alpha\delta(1-\alpha)}{\alpha(\delta+1)+h(1-\alpha)}$, $\rho_2 = A(1-\alpha)$ and $\rho_3 = A\alpha$ we can rewrite function f as

$$f(k) = \frac{\rho_2 \delta k^\alpha}{\delta+1} - \frac{h\left(\rho_2 k^\alpha (n_1-1) - \frac{\rho_2 n_1}{\rho_1^{\frac{\alpha}{\alpha-1}}}\right)}{\left(\rho_3 k^{\alpha-1}(n_1-1) - \frac{\rho_3 n_1}{\rho_1}\right)(\delta+1)}.$$

Note that

$$f'(k) = \frac{\rho_2 k^\alpha (q_A k^2 + q_B k + q_C k^{\alpha+1} + q_D k^{2\alpha})}{\rho_1^{\frac{\alpha}{\alpha-1}} \rho_3 k^{1+2\alpha}(\delta+1)(k^{1-\alpha} n_1 + \rho_1(1-n_1))^2}, \tag{21}$$

where we set

$$\begin{aligned}
q_A &= \alpha \rho_1^{\frac{\alpha}{\alpha-1}} n_1(\rho_3 \delta n_1 - \rho_1 h + \rho_1 h n_1),\\
q_B &= -\rho_1^2 h n_1 (1-\alpha)(1-n_1),\\
q_C &= \rho_1 \rho_1^{\frac{\alpha}{\alpha-1}} (1-n_1)(\rho_1 h n_1 - \rho_1 h + 2\alpha \rho_3 \delta n_1),\\
q_D &= \alpha \rho_1^{\frac{3\alpha-2}{\alpha-1}} \rho_3 \delta (1-n_1)^2.
\end{aligned}$$

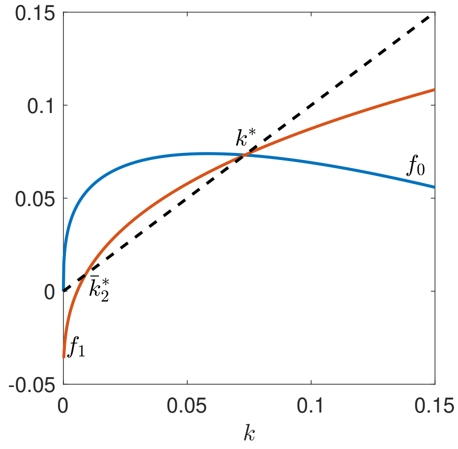

Figure 8: Maps $f_0(k)$ (blue) and $f_1(k)$ (red) for $A = 1, \alpha = 0.2, \delta = 0.9$ and $h = 0.7$.

In what follows, to put in evidence the dependency of f on n_1 we write f_{n_1}. In particular, we introduce functions $f_i : (0, +\infty) \to \mathbb{R}$ for $i = 0, 1$ defined by

$$f_0(k) = \frac{\rho_2 \delta k^\alpha}{\delta + 1} - \frac{\rho_2 h k^\alpha}{\rho_3 k^{\alpha-1}(\delta + 1)} = \frac{\rho_2(\rho_3 \delta k^\alpha - hk)}{\rho_3(\delta + 1)} \quad f_1(k) = \frac{\rho_2 \delta k^\alpha}{\delta + 1} - \frac{\rho_1 \rho_2 h}{\rho_1^{\frac{\alpha}{\alpha-1}} \rho_3(\delta + 1)}$$

which are actually the functions obtained by respectively setting $n_1 = 0$ and $n_1 = 1$ in function f. We recall that map f_0 has been studied in Cavalli and Naimzada (2016).

We start noting that $k^* = \rho_1^{\frac{1}{1-\alpha}}$ is a solution to $f(k) = k$. Moreover,

$$f'(k^*) = \alpha + \frac{h(1-\alpha)(n_1 - 1 + \alpha)}{\alpha(1+\delta)}$$

Moreover, it is easy to see that f_0 is concave, unimodal, $\lim_{k \to 0^+} f_0(k) = 0$ and $\lim_{k \to 0^+} f_0'(k) = +\infty$ and its unique steady state is k^*. Similarly, it is easy too see that f_1 is concave, strictly increasing, $\lim_{k \to 0^+} f_1(k) < 0$, $f_1(k) < 0$ on $\left(0, \left(\frac{h\rho_1^{\frac{1}{1-\alpha}}}{\rho_3 \delta}\right)^{\frac{1}{\alpha}}\right)$ and $\lim_{k \to 0^+} f_1'(k) = +\infty$. Moreover, in addition to k^*, function f_1 has an additional fixed point $\bar{k}_2^* \in (0, k^*)$. In fact, since $f_1(k) - k$ is a strictly concave function, it can have at most two zeros. One of them is indeed k^*. Since

$$f_1'(k^*) = \alpha + \frac{h(1-\alpha)}{\delta + 1} < 1$$

there is a left neighborhood of k^* on which $f_1(k) > k$. Since $\lim_{k \to 0^+} f_1(k) < 0$, by the intermediate values theorem there must be another solution to $f_1(k) - k$ on $(0, k^*)$.

Two examples for functions f_0 and f_1 are reported in Figure 8.

To study the possible solutions to $f(k)=k$, we start noting that

$$\partial_{n_1} f(k) = \frac{\rho_2 h k^{\alpha+1}\left(\rho_1^{\frac{\alpha}{\alpha-1}} k - \rho_1\right)}{\rho_1^{\frac{1}{\alpha-1}}\rho_3(\delta+1)(kn_1+\rho_1 k^\alpha - \rho_1 k^\alpha n_1)^2} > 0 \Leftrightarrow \rho_1^{\frac{\alpha}{\alpha-1}} k - \rho_1 > 0 \Leftrightarrow k > \rho_1^{\frac{1}{1-\alpha}} = k^* \tag{22}$$

This means that if we set $k \in (0,k^*)$ (respectively $n_1 \in (k^*,+\infty)$), function $f_{n_1}(k)$ is strictly decreasing (respectively strictly increasing), and f lies between f_1 (lower bound) and f_0 (upper bound) for $k \in (0,k^*)$ and lies between f_0 (lower bound) and f_1 (upper bound) for $k>0$. This also implies that $f_{n_1}(k) > f_1(k) \geq k$ on $[k_2^*, k^*)$ and $f_{n_1}(k) < f_1(k) < k$ on $(k^*,+\infty)$, so equation $f_{n_1}(k)=k$ has the unique solution k^* on $[\bar{k}_2^*,+\infty)$ for any $n_1 \in (0,1)$

To complete the proof, we need to find the remaining solutions to $f(k)-k=0$ on $(0,k^*)$. We have

$$f(k)-k = \frac{a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k}{\rho_1^{\frac{\alpha}{\alpha-1}}\rho_3(\delta+1)(kn_1+\rho_1 k^\alpha(1-n_1))} \tag{23}$$

where we set

$$\begin{aligned}
a_1 &= \rho_1^{\frac{2\alpha-1}{\alpha-1}}\rho_2\rho_3\delta(1-n_1)\\
a_2 &= \rho_1^{\frac{\alpha}{\alpha-1}}(-\rho_1\rho_3(1-n_1)(1+\delta) - \rho_1\rho_2 h(1-n_1) + \rho_2\rho_3\delta n_1)\\
a_3 &= -\rho_1^{\frac{\alpha}{\alpha-1}}\rho_3 n_1(\delta+1)\\
a_4 &= -\rho_1\rho_2 h n_1
\end{aligned}$$

We distinguish three situations

- $0<\alpha<1/2$

From (23), steady states of $f(k)$ are the zeros of function $a_1k^{2\alpha}+a_2k^{\alpha+1}+a_3k^2+a_4k$ on $(0,k^*)$. Let's study its number of inflection points, which depends on the solutions to $2a_1\alpha(2\alpha-1)k^{2\alpha-2} + a_2\alpha(\alpha+1)k^{\alpha-1}+2a_3>0$. Setting $t=k^{\alpha-1}$, we can rewrite the previous inequality as

$$z(t) = 2a_1\alpha(2\alpha-1)t^2 + a_2\alpha(\alpha+1)t + 2a_3 > 0 \tag{24}$$

in which $z:(0,+\infty)\to\mathbb{R}$ is a parabola.

If $\alpha<1/2$, $z(t)$ is a concave parabola and $z(t)=0$ has either no or two solutions. If

$$a_2 \leq 0 \Leftrightarrow n_1 \leq \frac{\rho_1(\rho_3+\rho_3\delta+\rho_2 h)}{\rho_1\rho_3+\rho_1\rho_3\delta+\rho_2\rho_3\delta+\rho_1\rho_2 h} = \frac{1}{2}$$

we have that $z(t)$ is decreasing for $t>0$, and so we have $z(t)<0$ and $a_1k^{2\alpha}+a_2k^{\alpha+1}+a_3k^2+a_4k$ has no inflection points. Since $\lim_{k\to 0^+}(a_1k^{2\alpha}+a_2k^{\alpha+1}+a_3k^2+a_4k)' = \lim_{k\to 0^+} a_1k^{2\alpha-1} = +\infty$, it is convex on $(0,+\infty)$. A convex function has at most 2 solutions: since $a_1k^{2\alpha}+a_2k^{\alpha+1}+a_3k^2+a_4k=0$

for $k = 0$ and $k = k^*$, no other solutions are possible on $(0, k^*)$ and hence f has no other fixed points for $\alpha < 1/2$ and $n_1 \leq 1/2$.

Conversely, if $a_2 > 0$, $z(t) = 0$ can have two solutions t_{f_1} and t_{f_2}. In such case, we have that $a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k$ is strictly convex $(0, t_{f_1}^{1/(\alpha-1)})$ and on $(t_{f_2}^{1/(\alpha-1)}, +\infty)$, and it is strictly concave $(t_{f_1}^{1/(\alpha-1)}, t_{f_2}^{1/(\alpha-1)})$. So, it has at most three extrema and, consequently, at most four zeros for $k > 0$, which actually reduces to at most two since $a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k = 0$ for $k = 0$ and $k = k^*$. This means that $f_{n_1}(k)$ has either zero or 2 fixed points on $(0, k^*)$ for $n_1 > 1/2$.

Let us consider a $k_B \in \left(\left(\frac{h \rho_1^{\frac{1}{1-\alpha}}}{\rho_3 \delta} \right)^{\frac{1}{\alpha}}, \bar{k}_2^* \right)$, for which we know $0 < f_1(k_B) < k_B$. Thanks to (22), we know that there exists $\bar{n}_1$ such that $0 < f_{n_1}(k_B) < k_B$ for $n_1 \in (\bar{n}_1, 1)$

If $\alpha < 1/2$, we have

$$\lim_{k \to 0^+} f'(k) = \lim_{k \to 0^+} \frac{\rho_2 k^{3\alpha} q_D}{\rho_1^{\frac{\alpha}{\alpha-1}} \rho_3 k^{1+2\alpha} (\delta + 1) \rho_1^2 (1 - n_1)^2} = \lim_{k \to 0^+} \frac{\alpha \rho_2 \delta}{(\delta + 1)} \frac{1}{k^{1-\alpha}} = +\infty \tag{25}$$

and this guarantees that for any $n_1 \in (0, 1)$ there exists a right neighborhood $I_+(0)$ of $k = 0$ on which $f_{n_1}(k) > k$. Let $k_A \in I_+(0)$. We have that graph of f connects point $(k_A, f(k_A)))$ (that lies above the 45-degree line) and point $(k_B, f(k_B)))$ (that lies below the 45-degree line), so thanks to the intermediate values theorem f has at least a fixed point on (k_A, k_B). Moreover, for any $n_1 \in (0, 1)$, we have that $f_{n_1}(k) > k$ on a left neighborhood $I_-(k^*)$ of k^*. Let $k_C \in I_-(k^*)$. We have that graph of f connects point $(k_B, f(k_B)))$ (that lies below the 45-degree line) and point $(k_C, f(k_C)))$ (that lies above the 45-degree line), so thanks to the intermediate values theorem f has at least a fixed point on (k_B, k_C). So f has at least two fixed points on $(0, k^*)$ for any $n_1 \in (\bar{n}_1, 1)$. Since we already showed that f has at most two fixed points on $(0, k^*)$, we can conclude that for any $n_1 \in (\bar{n}_1, 1)$ f has exactly two fixed points on $(0, k^*)$. Thanks to the regularity of f, the transition from having no fixed points to having two fixed points two fixed points on $(0, k^*)$ occurs by means of the tangency of f with the 45-degree line. At the corresponding n_f we then have a fold bifurcation.

- $\alpha = 1/2$

In this case we can rewrite (21) as

$$f'(k) = \frac{\rho_2 k^{1/2} (q_A k^2 + (q_B + q_D) k + q_C k^{3/2})}{\rho_1^{-1} \rho_3 k^2 (\delta + 1)(k^{1/2} n_1 + \rho_1 (1 - n_1))^2}$$

If

$$q_B + q_D = \frac{\rho_1 (1 - n_1)(-\rho_3 \delta n_1 + \rho_3 \delta - \rho_1 h n_1)}{2} > 0$$

i.e. if and only if

$$n_1 < \frac{\rho_3\delta}{\rho_3\delta + \rho_1 h} = \frac{1+\delta+h}{1+\delta+2h}$$

we have

$$\lim_{k\to 0^+} f'(k) = \lim_{k\to 0^+} \frac{\rho_2 k^{3/2}(q_B + q_D)}{\rho_1^{-1}\rho_3 k^2(\delta+1)\rho_1^2(1-n_1)^2} = +\infty$$

If $n_1 = \frac{1+\delta+h}{1+\delta+2h}$, we have that (23) becomes

$$f(k) - k = \frac{k(\delta+h+1)\left(2(\delta+h+1)\sqrt{k} - A\delta\right)}{2(2\delta+2h+1+\delta^2+h^2+2\delta h)\sqrt{k} + A\delta h},$$

which is indeed strictly positive on $(0, k^*)$. Recalling (22), this means that $f_{n_1}(k)$ has no additional fixed points on $(0, k^*)$ for $n_1 \in \left(0, \frac{1+\delta+h}{1+\delta+2h}\right]$.

Conversely, if $q_B + q_D < 0$, i.e. if $n_1 > \frac{1+\delta+h}{1+\delta+2h}$ we have $\lim_{k\to 0^+} f'(k) = -\infty$, so $f < 0$ on a right neighborhood $I_+(0)$ of 0. We already noted that for any $n_1 \in (0,1)$ we have that $f_{n_1}(k) > k$ on a left neighborhood $I_-(k^*)$ of k^*. Hence, the intermediate values theorem guarantees that f has at least a fixed point on $(0, k^*)$. However, it is easy to see that for $\alpha = 1/2$, since $a_2 > 0$ for $n_1 > \frac{1+\delta+h}{1+\delta+2h} > 1/2$, the right hand side in inequality (24) is an increasing line, leading to the existence of an inflection point for $f(k) - k$. This means that $a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k = 0$ has at most three solutions, which become at most one since $k = 0$ and $k = k^*$ are always solutions. So f has exactly one fixed point on $(0, k^*)$.

- $1/2 < \alpha < 1$

As for $\alpha < 1/2$, steady states of $f(k)$ are the zeros of function $a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k$ on $(0, k^*)$ and we study its inflection points solving $2a_1\alpha(2\alpha-1)k^{2\alpha-2} + a_2\alpha(\alpha+1)k^{\alpha-1} + 2a_3 > 0$. In this case, the right hand side of (24) is a convex parabola that intersects the vertical axis at $a_3 < 0$. Hence, $a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k$ is strictly convex $(0, t_f^{1/(\alpha-1)})$ and it is strictly concave on $(t_{f_1}^{1/(\alpha-1)}, t_{f_2}^{1/(\alpha-1)})$. So, it has at most two extrema and, consequently, at most three zeros for $k > 0$, which actually reduces to at most one since $a_1 k^{2\alpha} + a_2 k^{\alpha+1} + a_3 k^2 + a_4 k = 0$ for $k = 0$ and $k = k^*$. This means that $f_{n_1}(k)$ has at most one fixed point on $(0, k^*)$.

If $\alpha > 1/2$, we have

$$\lim_{k\to 0^+} f'(k) = \lim_{k\to 0^+} \frac{\rho_2 k^{\alpha+1} q_B}{\rho_1^{\frac{\alpha}{\alpha-1}}\rho_3 k^{1+2\alpha}(\delta+1)\rho_1^2(1-n_1)^2} = \lim_{k\to 0^+} -\frac{\rho_2 n_1 h(1-\alpha)}{\rho_1^{\frac{\alpha}{\alpha-1}}(\delta+1)\rho_3(1-n_1)}\frac{1}{k^\alpha} = -\infty$$

Proceeding as in the case of $\alpha = 1/2$, this guarantees that f has at least a fixed point on $(0, k^*)$. This allows concluding that f has exactly a fixed point on $(0, k^*)$. □

Proof of Proposition 2. A straightforward computation shows that

$$f'(k^*) = \alpha + \frac{h(1-\alpha)(n_1 - 1 + \alpha)}{\alpha(1+\delta)} < 1,$$

so stability is guaranteed by $f'(k^*) > -1$, i.e. by

$$n_1 > \frac{h - \alpha - \alpha\delta - 2\alpha h - \alpha^2\delta + \alpha^2 h - \alpha^2}{h(1-\alpha)}$$

which immediately provides (19). □

Proof of Proposition 3. It is straightforward to see that if $(\tilde{k}, \tilde{k})$ is a steady state for (17), then $\tilde{k}$ is a steady state for (18) for a suitable value of n_1. Then the steady states of (17) are a subset of those for (18) and we can start the proof from the results of Proposition 1. If we set $k_t = k_{t-1} = 0$ in (17) we indeed find an identity. Setting $k_t = k_{t-1} = k^*$, we find $n_{1,t} = n_1^*$ from (16). We have already shown in the proof of Proposition 1 that for any exogenous value of n_1, k^* is a steady state of (18), and this allows concluding.

Now we discuss the possible existence of additional steady states (i.e. a steady state different from $(0,0)$ and (k^*, k^*)). Let $\alpha \in (0, 1/2)$, which corresponds to case a) in Proposition 1. We know that in this there are two additional steady states provided that n_1 is suitably large, but necessarily we need $n_1 > 1/2$. Let $k_t = k_{t-1} = k_i^*$, for $i = 1$ or $i = 2$. In both cases, from (16), we have $n_{1,i} = 1/(1 + \exp(\beta(\varepsilon_i + C))$, where $\varepsilon_i = (R^* - A\alpha(k_i^*)^{\alpha-1})^2 + (w^* - A(1-\alpha)(k_i^*)^{\alpha})^2$. Since $\varepsilon_i > 0$, we indeed have $n_{1,i} < 1/2$, which is conflicting with the necessary condition $n_1 > 1/2$. So when $\alpha \in (0, 1/2)$ the only steady states are $(0,0)$ and (k^*, k^*).

A similar situation holds for $\alpha = 1/2$, i.e. in case b) of Proposition 1. If $n_1 \leq (1+\delta+h)/(1+\delta+2h)$ we already now that we do not have additional steady states. Since $(1+\delta+h)/(1+\delta+2h) > 1/2$, we again have that setting $k_t = k_{t-1} = k_1^*$ would lead to a requirement on n_1 that is conflicting with $n_1 > 1/2$.

Let $\alpha > 1/2$. In case c) of Proposition 1 we do not have any constraint on n_1 for the existence of the additional steady state k_1^*. This means that setting $k_t = k_{t-1} = k_1^*$ we obtain $n_{1,1}^* = 1/(1 + \exp(\beta(\varepsilon_1 + C))$, which however is possible for a unique value of C for each given configuration of the remaining parameters. The subset of all parameters' vectors for which (k_1^*, k_1^*) is a steady state has then null Lebesgue measure with respect to the set of all vectors of feasible parameters. This concludes the proof. □

Proof of Proposition 4. We start computing the Jacobian matrix $J(k, z)$ related to System (17), whose elements are

$$
\begin{aligned}
j_{11} \quad = \quad & \frac{h\left(k^{\alpha}\rho_2(n_1(k,z)-1)-\frac{\rho_2 n_1(k,z)}{\rho_1^{\frac{\alpha}{\alpha-1}}}\right)\left(k^{\alpha-1}\rho_3\frac{\partial}{\partial k}n_1(k,z)-\frac{\rho_3\frac{\partial}{\partial k}n_1(k,z)}{\rho_1}+k^{\alpha-2}\rho_3(n_1(k,z)-1)(\alpha-1)\right)}{\left(k^{\alpha-1}\rho_3(n_1(k,z)-1)-\frac{\rho_3 n_1(k,z)}{\rho_1}\right)^2(\delta+1)} \\
& -\frac{h\left(k^{\alpha}\rho_2\frac{\partial}{\partial k}n_1(k,z)-\frac{\rho_2\frac{\partial}{\partial k}n_1(k,z)}{\rho_1^{\frac{\alpha}{\alpha-1}}}+\alpha k^{\alpha-1}\rho_2(n_1(k,z)-1)\right)}{\left(k^{\alpha-1}\rho_3(n_1(k,z)-1)-\frac{\rho_3 n_1(k,z)}{\rho_1}\right)(\delta+1)}+\frac{\alpha\delta k^{\alpha}\rho_2}{k(\delta+1)}
\end{aligned}
$$

$$
j_{12} = \frac{hk^{\alpha+1}\rho_1^{\frac{\alpha}{\alpha-1}}\rho_2\left(k\rho_1^{\frac{1}{\alpha-1}}-1\right)\frac{\partial}{\partial z}n_1(k,z)}{\rho_3(\delta+1)\left(k^{\alpha}\rho_1^{\frac{\alpha}{\alpha-1}}+k\rho_1^{\frac{1}{\alpha-1}}n_1(k,z)-k^{\alpha}\rho_1^{\frac{\alpha}{\alpha-1}}n_1(k,z)\right)^2}
$$

and $j_{21}=1$ and $j_{22}=0$, in which we set

$$
n_1(k,z) = \frac{e^{-\beta[(R^*-A\alpha k^{\alpha-1})^2+(w^*-A(1-\alpha)k^{\alpha})^2+C]}}{e^{-\beta[(R^*-A\alpha k^{\alpha-1})^2+(w^*-A(1-\alpha)k^{\alpha})^2+C]}+e^{-\beta[(A\alpha k^{\alpha-1}-A\alpha z^{\alpha-1})^2+(A(1-\alpha)k^{\alpha}-A(1-\alpha)k^{\alpha})^2]}}
$$

To study stability of (k^*,k^*) we set $k=z=k^*$, obtaining

$$
J(k^*,k^*) = \begin{pmatrix} \frac{\alpha+h+\alpha\delta-\alpha h}{\delta+1}-\frac{e^{C\beta}}{e^{C\beta}+1}\left[\frac{h(1-\alpha)}{(\delta+1)}+\frac{h(1-\alpha)^2}{\alpha(\delta+1)}\right] & 0 \\ 1 & 0 \end{pmatrix}
$$

whose eigenvalues are indeed $\lambda_1=(J(k^*,k^*))_{1,1}$ and $\lambda_2=0$. Since λ_1 is strictly decreasing with respect to β and C, we can conclude that they have a destabilizing effect. Stability is guaranteed provided that $\lambda_1<1$, which after some algebraic manipulations leads to the unconditionally true inequality $-\alpha(1-h)-\alpha\delta-e^{\beta C}(\alpha(1+\delta)+h(1-\alpha))<0$, and by $\lambda_1>-1$, which provides

$$
(\alpha-h+\alpha\delta+2\alpha h+\alpha^2\delta-\alpha^2 h+\alpha^2)e^{\beta C}+\alpha+\alpha\delta+\alpha h+\alpha^2\delta-\alpha^2 h+\alpha^2>0
$$

If the coefficient of term $e^{\beta C}$ is non-negative, i.e. if $h\le\alpha(\alpha+1)(\delta+1)/(1-\alpha)^2$, then the previous condition is always fulfilled and (k^*,k^*) is unconditionally stable, otherwise stability is guaranteed provided that (20) holds. This concludes the proof. □

Proof of Proposition 5. Consider h and n_1 together as a vector-valued parameter (h,n_1). For the unperturbed system with $(h,n_1)=(h_2,0)$, following immediately Proposition 3 in Cavalli and Naimzada (2016), we get that map M is unimodal and concave. Moreover, if k_m is the per capital for which map M attains its maximum value, then $M(M(k_m))=0$, that is, the trajectory of k_m remains at the steady state $k_0^*=0$ after the second iterates. This shows the topological entropy of the unperturbed system is $\log 2$ so the entropic chaos occurs.

For parameter (h, n_1) slightly perturbed from $(h_2, 0)$, we consider the difference equation of the form $k_{t+1} - f(k_t) = 0$. At the unperturbed parameter $(h_2, 0)$, the difference equation reduces to $k_{t+1} - f_0(k_t) = 0$ with $h = h_2$, which is just shown to have positive topological entropy. By applying Theorem 1 of [35], the topological entropies of the slightly unperturbed systems remains positive, so that the entropic chaos occurs.

□

Proof of Proposition 6. Similar to the proof of Proposition 5, the results following Theorem 2 and Theorem 1 of [35]. □