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On critical double phase Kirchhoff problems with singular nonlinearity

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Abstract
The paper deals with the following double phase problem

$$\begin{aligned} & -m\left[\int_\Omega \left(\frac{|\nabla u|^p}{p} + a(x)\frac{|\nabla u|^q}{q}\right) \mathrm{d}x\right] \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) \\ & \quad = \lambda u^{-\gamma} + u^{p^*-1} && \text{in } \Omega, \\ & u > 0 && \text{in } \Omega, \\ & u = 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, $N \geq 2$, m represents a Kirchhoff coefficient, $1 < p < q < p^*$ with $p^* = Np/(N-p)$ being the critical Sobolev exponent to p, a bounded weight $a(\cdot) \geq 0$, $\lambda > 0$ and $\gamma \in (0,1)$. By the Nehari manifold approach, we establish the existence of at least one weak solution.

Keywords Critical growth · Double phase operator · Fibering method · Nehari manifold · Nonlocal Kirchhoff term · Singular problem

Mathematics Subject Classification 35A15 · 35J15 · 35J60 · 35J62 · 35J75

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1 Introduction

In this paper, we combine the effects of a nonlocal Kirchhoff coefficient and a double phase operator with a singular term and a critical Sobolev nonlinearity. Precisely, we study the problem

$$\begin{aligned} -m\left[\int_\Omega\left(\frac{|\nabla u|^p}{p}+a(x)\frac{|\nabla u|^q}{q}\right)\mathrm{d}x\right]\mathcal{L}^a_{p,q}(u)&=\lambda u^{-\gamma}+u^{p^*-1} &&\text{in } \Omega,\\ u&>0 &&\text{in } \Omega,\\ u&=0 &&\text{on } \partial\Omega, \end{aligned} \tag{P_λ}$$

where along the paper, and without further mentioning, $\Omega \subset \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, dimension $N \geq 2$, $\lambda > 0$ is a real parameter and exponent $\gamma \in (0,1)$. The main operator $\mathcal{L}^a_{p,q}$ is the so-called double phase operator given by

$$\mathcal{L}^a_{p,q}(u) := \operatorname{div}\big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big), \quad u \in W^{1,\mathcal{H}}_0(\Omega), \tag{1.1}$$

with $W^{1,\mathcal{H}}_0(\Omega)$ being the homogeneous Musielak-Orlicz Sobolev space where we assume that

(h_1) $1 < p < N,\ p < q < p^*$ and $0 \leq a(\cdot) \in L^\infty(\Omega)$ with p^* being the critical Sobolev exponent to p given by

$$p^* = \frac{Np}{N-p}. \tag{1.2}$$

While the nonlocal term m in (P_λ) denotes a Kirchhoff coefficient satisfying

(h_2) $m : [0,\infty) \to [0,\infty)$ is a continuous function defined by

$$m(t) = a_0 + b_0 t^{\theta-1} \quad \text{for all } t \geq 0,$$

where $a_0 \geq 0$, $b_0 > 0$ with $\theta \in \big[1, p^*/q\big)$.

Problem (P_λ) is said to be of double phase type because of the presence of two different elliptic growths p and q. The study of double phase problems and related functionals originates from the seminal paper by Zhikov [25], where he introduced for the first time in literature the related energy functional to (1.1) defined by

$$\omega \mapsto \int_\Omega \big(|\nabla\omega|^p + a(x)|\nabla\omega|^q\big)\,\mathrm{d}x. \tag{1.3}$$

This kind of functional has been used to describe models for strongly anisotropic materials in the context of homogenization and elasticity. Indeed, the modulating coefficient $a(\cdot)$ dictates the geometry of composites made of two different materials with distinct power hardening exponents p and q. From the mathematical point of view, the behavior of (1.3) is related to the sets on which the weight function $a(\cdot)$ vanishes or not. In this direction, Zhikov found other mathematical applications for (1.3) in the study of duality theory and of the Lavrentiev gap phenomenon, as shown in [26, 27]. Also, (1.3) belongs to the class of the integral functionals with nonstandard growth condition, according to Marcellini's terminology [22, 23]. Following this line of research, Mingione et al. provide famous results in the regularity theory of local minimizers of (1.3), see, for example, the works of Baroni-Colombo-Mingione [4, 5] and Colombo-Mingione [9, 10].

Starting from [25], several authors studied existence and multiplicity results for nonlinear problems driven by (1.1) with the help of different variational techniques. In particular, Fiscella-Pinamonti [18] introduced two different double phase problems of Kirchhoff type, with the same variational structure set in $W_0^{1,\mathcal{H}}(\Omega)$. By the mountain pass and fountain theorems, existence and multiplicity results are provided in [18]. Following this direction, in [17] Fiscella-Marino-Pinamonti-Verzellesi consider some classes of Kirchhoff type problems on a double phase setting but with nonlinear boundary conditions. Combining variational methods, truncation arguments and topological tools, different multiplicity results are established. Recently, the authors [2] were able to study a Kirchhoff problem like (P_λ), but involving a subcritical term. By a suitable Nehari manifold decomposition, the existence of two different solutions are provided in [2]. We also mention the works of Cammaroto-Vilasi [7], Isernia-Repovš [20] and Ambrosio-Isernia [1] for Kirchhoff type problems driven by the $p(\cdot)$-Laplacian or the (p, q)-Laplacian.

The main novelty, as well as the main difficulty, of problem (P_λ) is the presence of a critical Sobolev nonlinearity. Indeed, in order to overcome the lack of compactness at critical levels arising from the presence of the critical term in (P_λ), the same fibering analysis used in [2] cannot work. For this, we exploit other variational tools inspired by more recent situations as in [14]. For this, Farkas-Fiscella-Winkert [14] used a suitable convergence analysis of gradients in order to handle the critical Sobolev nonlinearity of problem

$$\begin{aligned} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big) &= \lambda|u|^{\vartheta-2}u + |u|^{p^*-2}u && \text{in } \Omega,\\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Following this direction, we mention [15, 16] concerning existence results for critical double phase problems involving a singular term and defined on Minkowski spaces in terms of Finsler manifolds, that is driven by the Finsler double phase operator

$$\mathcal{L}_{p,q}^{F,a}(u) := \operatorname{div}\big(F^{p-1}(\nabla u)\nabla F(\nabla u) + a(x)F^{q-1}(\nabla u)\nabla F(\nabla u)\big),$$

where $(\mathbb{R}^N, F)$ stands for a Minkowski space. While, Crespo-Blanco-Papageorgiou-Winkert [12] consider a nonhomogeneous singular Neumann double phase problem with critical growth on the boundary, given by

$$\begin{aligned} -\operatorname{div}\big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big) + \alpha(x)u^{p-1} &= \zeta(x)u^{-\gamma} + \lambda u^{q_1-1} && \text{in } \Omega,\\ \big(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\big)\cdot\nu &= -\beta(x)u^{p_*-1} && \text{on } \partial\Omega. \end{aligned} \tag{1.4}$$

By the fibering approach introduced by Drábek-Pohozaev [13] along with a Nehari manifold decomposition, the existence of at least two solutions of (1.4) is obtained in [12].

Inspired by the above papers, we solve problem (P_λ) by a variational approach. Indeed, a function $u \in W_0^{1,\mathcal{H}}(\Omega)$ is said to be a weak solution of problem (P_λ) if $u^{-\gamma}\varphi \in L^1(\Omega)$, $u > 0$ a.e. in Ω and

$$m(\phi_{\mathcal{H}}(\nabla u))\Big\langle \mathcal{L}_{p,q}^{a}(u), \varphi\Big\rangle = \lambda\int_\Omega u^{-\gamma}\varphi\,\mathrm{d}x + \int_\Omega u^{p^*-1}\varphi\,\mathrm{d}x$$

is satisfied for all $\varphi \in W_0^{1,\mathcal{H}}(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W_0^{1,\mathcal{H}}(\Omega)$ and its dual space $W_0^{1,\mathcal{H}}(\Omega)^*$ In particular, the weak solutions of (P_λ) are the critical points of the energy functional $J_\lambda : W_0^{1,\mathcal{H}}(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \left[a_0 \phi_\mathcal{H}(\nabla u) + \frac{b_0}{\theta}\phi^\theta_\mathcal{H}(\nabla u)\right] - \frac{\lambda}{1-\gamma}\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega |u|^{p^*}\,\mathrm{d}x,$$

for any $u \in W^{1,\mathcal{H}}_0(\Omega)$, where

$$\phi_\mathcal{H}(u) = \int_\Omega \left(\frac{|u|^p}{p} + a(x)\frac{|u|^q}{q}\right)\mathrm{d}x.$$

Hence, the main result reads as follows.

Theorem 1.1 *Let hypotheses* (h_1)-(h_2) *be satisfied. Then there exists* $\lambda^* > 0$ *such that for all* $\lambda \in (0, \lambda^*]$ *problem* (P_λ) *has at least one weak solution* u_λ *such that* $J_\lambda(u_\lambda) < 0$.

The proof of Theorem 1.1 is based on a suitable minimization argument on the Nehari manifold. For this, we extract a minimizing sequence whose energy values converge to a negative number. However, in order to verify that the sequence actually converges to a solution of (P_λ) we need a truncation argument combined with a delicate gradient analysis, inspired by [14].

The paper is organized as follows. In Sect. 2, we recall the main properties of Musielak-Orlicz Sobolev spaces $W^{1,\mathcal{H}}_0(\Omega)$ and state the main embeddings concerning these spaces. Section 3 gives a detailed analysis of the fibering map, presents the main properties of suitable subsets of the Nehari manifold and finally shows the existence of a weak solution of problem (P_λ).

2 Preliminaries

In this section, we will present the main properties and embedding results for Musielak-Orlicz Sobolev spaces. First, we denote by $L^r(\Omega) = L^r(\Omega;\mathbb{R})$ and $L^r(\Omega;\mathbb{R}^N)$ the usual Lebesgue spaces with the norm $\|\cdot\|_r$ and the corresponding Sobolev space $W^{1,r}_0(\Omega)$ is equipped with the norm $\|\nabla\cdot\|_r$, for $1 \le r \le \infty$.

Suppose hypothesis (h_1) and consider the nonlinear function $\mathcal{H} : \Omega \times [0,\infty) \to [0,\infty)$ defined by

$$\mathcal{H}(x,t) = t^p + a(x)t^q.$$

The Musielak-Orlicz Lebesgue space $L^\mathcal{H}(\Omega)$ is given by

$$L^\mathcal{H}(\Omega) = \left\{u : \Omega \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \varrho_\mathcal{H}(u) < \infty\right\}$$

equipped with the Luxemburg norm

$$\|u\|_\mathcal{H} = \inf\left\{\tau > 0 \,\middle|\, \varrho_\mathcal{H}\left(\frac{u}{\tau}\right) \le 1\right\},$$

where the modular function is given by

$$\varrho_\mathcal{H}(u) := \int_\Omega \mathcal{H}(x,|u|)\,\mathrm{d}x = \int_\Omega \left(|u|^p + a(x)|u|^q\right)\mathrm{d}x.$$

Next, we recall the relation between the norm $\|\cdot\|_{\mathcal{H}}$ and the modular function $\varrho_{\mathcal{H}}$, see Liu-Dai [21, Proposition 2.1] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Proposition 2.13].

Proposition 2.1 *Let* (h_1) *be satisfied,* $u \in L^{\mathcal{H}}(\Omega)$ *and* $c > 0$. *Then the following hold:*

(i) *If* $u \neq 0$, *then* $\|u\|_{\mathcal{H}} = c$ *if and only if* $\varrho_{\mathcal{H}}(\frac{u}{c}) = 1$;
(ii) $\|u\|_{\mathcal{H}} < 1$ (*resp.* $> 1, = 1$) *if and only if* $\varrho_{\mathcal{H}}(u) < 1$ (*resp.* $> 1, = 1$);
(iii) *If* $\|u\|_{\mathcal{H}} < 1$, *then* $\|u\|_{\mathcal{H}}^q \leq \varrho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^p$;
(iv) *If* $\|u\|_{\mathcal{H}} > 1$, *then* $\|u\|_{\mathcal{H}}^p \leq \varrho_{\mathcal{H}}(u) \leq \|u\|_{\mathcal{H}}^q$;
(v) $\|u\|_{\mathcal{H}} \to 0$ *if and only if* $\varrho_{\mathcal{H}}(u) \to 0$;
(vi) $\|u\|_{\mathcal{H}} \to \infty$ *if and only if* $\varrho_{\mathcal{H}}(u) \to \infty$.

Moreover, we define the weighted space

$$L^q_a(\Omega) = \left\{ u : \Omega \to \mathbb{R} \,\middle|\, u \text{ is measurable and } \int_\Omega a(x)|u|^q \,\mathrm{d}x < \infty \right\}$$

endowed with the seminorm

$$\|u\|_{q,a} = \left(\int_\Omega a(x)|u|^q \,\mathrm{d}x \right)^{\frac{1}{q}}.$$

The corresponding Musielak-Orlicz Sobolev space $W^{1,\mathcal{H}}(\Omega)$ is defined by

$$W^{1,\mathcal{H}}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) \ : \ |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

equipped with the norm

$$\|u\|_{1,\mathcal{H}} = \|\nabla u\|_{\mathcal{H}} + \|u\|_{\mathcal{H}},$$

where $\|\nabla u\|_{\mathcal{H}} = \|\,|\nabla u|\,\|_{\mathcal{H}}$. In addition, we denote by $W^{1,\mathcal{H}}_0(\Omega)$ the completion of $C^\infty_0(\Omega)$ in $W^{1,\mathcal{H}}(\Omega)$. Thanks to hypothesis ($\mathrm{h}_1$), we know that

$$\|u\| = \|\nabla u\|_{\mathcal{H}},$$

is an equivalent norm in $W^{1,\mathcal{H}}_0(\Omega)$, see Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Proposition 2.16(ii)]. Furthermore, it is known that $L^{\mathcal{H}}(\Omega)$, $W^{1,\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$ are uniformly convex and so reflexive Banach spaces, see Colasuonno-Squassina [8, Proposition 2.14] or Harjulehto-Hästö [19, Theorem 6.1.4].

Finally, we recall some useful embedding results for the spaces $L^{\mathcal{H}}(\Omega)$ and $W^{1,\mathcal{H}}_0(\Omega)$, see Colasuonno-Squassina [8, Proposition 2.15] or Crespo-Blanco-Gasiński-Harjulehto-Winkert [11, Propositions 2.17 and 2.19].

Proposition 2.2 *Let* (h_1) *be satisfied and let* p^* *be the critical exponent to* p *given in* (1.2). *Then the following embeddings hold:*

(i) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *and* $W^{1,\mathcal{H}}_0(\Omega) \hookrightarrow W^{1,r}_0(\Omega)$ *are continuous for all* $r \in [1, p]$;

(ii) $W_0^{1,\mathcal{H}}(\Omega) \hookrightarrow L^r(\Omega)$ *is continuous for all* $r \in [1, p^*]$ *and compact for all* $r \in [1, p^*)$;
(iii) $L^{\mathcal{H}}(\Omega) \hookrightarrow L^q_a(\Omega)$ *is continuous;*
(iv) $L^q(\Omega) \hookrightarrow L^{\mathcal{H}}(\Omega)$ *is continuous.*

3 Proof the main result

In order to solve problem (P_λ), we apply a minimization argument for J_λ on a suitable subset of $W_0^{1,\mathcal{H}}(\Omega)$. For this, we define the fibering function $\psi_u : [0,\infty) \to \mathbb{R}$ defined by

$$\psi_u(t) = J_\lambda(tu) \quad \text{for all } t \geq 0,$$

which gives

$$\psi_u(t) = \left[a_0 \phi_{\mathcal{H}}(t\nabla u) + \frac{b_0}{\theta} \phi_{\mathcal{H}}^\theta(t\nabla u)\right] - \lambda \frac{t^{1-\gamma}}{1-\gamma} \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - \frac{t^{p^*}}{p^*} \int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

It is easy to see that $\psi_u \in C^\infty((0,\infty))$. In particular, we have for $t > 0$

$$\begin{aligned} \psi_u'(t) = {} & \left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right) \\ & - \lambda t^{-\gamma} \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - t^{p^*-1} \int_\Omega |u|^{p^*}\,\mathrm{d}x \end{aligned}$$

and

$$\begin{aligned} \psi_u''(t) = {} & \left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1)t^{p-2}\|\nabla u\|_p^p + (q-1)t^{q-2}\|\nabla u\|_{q,a}^q\right] \\ & + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right)^2 \\ & + \lambda\gamma t^{-\gamma-1} \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x - (p^*-1)t^{p^*-2} \int_\Omega |u|^{p^*}\,\mathrm{d}x. \end{aligned}$$

Thus, we can introduce the Nehari manifold related to our problem which is defined by

$$\mathcal{N}_\lambda = \left\{u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\} \ : \ \psi_u'(1) = 0\right\}.$$

In particular, we have $u \in \mathcal{N}_\lambda$ if and only if

$$\left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u)\right]\left(\|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q\right) = \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x + \int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

Also $tu \in \mathcal{N}_\lambda$ if and only if $\psi_{tu}'(1) = 0$. Observe that $\mathcal{N}_\lambda$ contains all weak solutions of (P_λ). Moreover, we define the following subsets of $\mathcal{N}_\lambda$

$$\mathcal{N}_\lambda^+ = \left\{u \in \mathcal{N}_\lambda \ : \ \psi_u''(1) > 0\right\} \quad \text{and} \quad \mathcal{N}_\lambda^\circ = \left\{u \in \mathcal{N}_\lambda \ : \ \psi_u''(1) = 0\right\}.$$

In contrast to [2] we are not going to study the set $\mathcal{N}_\lambda^- = \left\{u \in \mathcal{N}_\lambda \ : \ \psi_u''(1) < 0\right\}$. The next Lemma can be shown as in [2, Lemmas 3.1 and 3.2] replacing r by p^*.

Lemma 3.1 *Let hypotheses* (h_1)-(h_2) *be satisfied.*

(i) *The functional* $J_\lambda\big|_{\mathcal{N}_\lambda}$ *is coercive and bounded from below for any* $\lambda > 0$.
(ii) *There exists* $\Lambda_1 > 0$ *such that* $\mathcal{N}_\lambda^\circ = \emptyset$ *for all* $\lambda \in (0, \Lambda_1)$.

Let S be the best Sobolev constant in $W_0^{1,p}(\Omega)$ defined as

$$S = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_p^p}{\|u\|_{p^*}^p}. \tag{3.1}$$

Note that we can write $\psi_u'(t)$ in the form

$$\psi_u'(t) = t^{-\gamma}\left(\sigma_u(t) - \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x\right), \quad t > 0, \tag{3.2}$$

where

$$\sigma_u(t) = \left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left(t^{p-1+\gamma}\|\nabla u\|_p^p + t^{q-1+\gamma}\|\nabla u\|_{q,a}^q\right) - t^{p^*-1+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

From this definition we see that $tu \in \mathcal{N}_\lambda$ if and only if

$$\sigma_u(t) = \lambda \int_\Omega |u|^{1-\gamma}\,\mathrm{d}x. \tag{3.3}$$

The next Lemma shows that $\mathcal{N}_\lambda^+$ is nonempty whenever λ is sufficiently small.

Lemma 3.2 *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied and let* $u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$. *Then there exist* $\Lambda_2 > 0$ *and unique* $t_1^u < t_{\max}^u < t_2^u$ *such that*

$$0 < \sigma_u'(t_1^u) = (t_1^u)^\gamma \psi_u''(t_1^u), \quad 0 > \sigma_u'(t_2^u) = (t_2^u)^\gamma \psi_u''(t_2^u) \quad \text{and} \quad \sigma_u(t_{\max}^u) = \max_{t>0} \sigma_u(t)$$

whenever $\lambda \in (0, \Lambda_2)$. *In particular,* $t_1^u u \in \mathcal{N}_\lambda^+$ *for* $\lambda \in (0, \Lambda_2)$.

Proof For $u \in W_0^{1,\mathcal{H}}(\Omega) \setminus \{0\}$ the equation

$$\begin{aligned}
0 = \sigma_u'(t) = &\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1+\gamma)t^{p-2+\gamma}\|\nabla u\|_p^p + (q-1+\gamma)t^{q-2+\gamma}\|\nabla u\|_{q,a}^q\right]\\
&+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-1+\gamma}\|\nabla u\|_p^p + t^{q-1+\gamma}\|\nabla u\|_{q,a}^q\right)\\
&\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right)\\
&- (p^*-1+\gamma)t^{p^*-2+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x
\end{aligned}$$

can be equivalently written as

$$\begin{aligned}
&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1+\gamma)t^{p-p^*}\|\nabla u\|_p^p + (q-1+\gamma)t^{q-p^*}\|\nabla u\|_{q,a}^q\right]\\
&\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-p^*+1}\|\nabla u\|_p^p + t^{q-p^*+1}\|\nabla u\|_{q,a}^q\right)\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right)\\
&\quad = (p^*-1+\gamma)\int_\Omega |u|^{p^*}\,\mathrm{d}x.
\end{aligned} \tag{3.4}$$

From $p^* > q\theta$ and $\theta \geq 1$ we see that

$$\begin{aligned} p(\theta-1)+p-p^* &< \min\{p(\theta-1)+q-p^*, q(\theta-1)+p-p^*\} \\ &\leq \max\{p(\theta-1)+q-p^*, q(\theta-1)+p-p^*\} \\ &< q(\theta-1)+q-p^* = q\theta - p^* < 0. \end{aligned} \tag{3.5}$$

We denote the left-hand side of (3.4) by

$$\begin{aligned} T_u(t) =& \left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(t\nabla u)\right]\left[(p-1+\gamma)t^{p-p^*}\|\nabla u\|_p^p + (q-1+\gamma)t^{q-p^*}\|\nabla u\|_{q,a}^q\right] \\ &+ b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(t\nabla u)\left(t^{p-p^*+1}\|\nabla u\|_p^p + t^{q-p^*+1}\|\nabla u\|_{q,a}^q\right) \\ &\left(t^{p-1}\|\nabla u\|_p^p + t^{q-1}\|\nabla u\|_{q,a}^q\right). \end{aligned}$$

Then, from (3.5) and $0 < \gamma < 1 < p < q < p^*$, we know that

$$\text{(i)} \quad \lim_{t\to 0^+} T_u(t) = \infty, \quad \text{(ii)} \quad \lim_{t\to\infty} T_u(t) = 0, \quad \text{(iii)} \quad T_u'(t) < 0 \ \text{ for all } t > 0.$$

From the intermediate value theorem along with (i) and (ii) we can find $t_{\max}^u > 0$ such that (3.4) holds. In addition, (iii) implies that $t_{\max}^u$ is unique due to the injectivity of T_u. Moreover, if we consider $\sigma_u'(t) > 0$, then in place of (3.4) we get

$$T_u(t) > (p^*-1+\gamma)\int_\Omega |u|^{p^*}\,\mathrm{d}x.$$

Since T_u is strictly decreasing, this holds for all $t < t_{\max}^u$. The same can be said for $\sigma_u'(t) < 0$ and $t > t_{\max}^u$. Hence, σ_u is injective in $(0, t_{\max}^u)$ and in $(t_{\max}^u, \infty)$. Furthermore,

$$\sigma_u(t_{\max}^u) = \max_{t>0} \sigma_u(t)$$

with the global maximum $t_{\max}^u > 0$ of σ_u. Moreover, we have

$$\lim_{t\to 0^+} \sigma_u(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \sigma_u(t) = -\infty.$$

Applying the estimate $p\phi_{\mathcal{H}}(\nabla u) \geq \|\nabla u\|_p^p$ we obtain

$$\sigma_u'(t) \geq \frac{b_0}{p^{\theta-1}}(p\theta-1+\gamma)t^{p\theta-2+\gamma}\|\nabla u\|_p^{p\theta} - (p^*-1+\gamma)t^{p^*-2+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x, \tag{3.6}$$

which by using Hölder's inequality and (3.1) results in

$$t_{\max}^u \geq \frac{1}{\|\nabla u\|_p}\left(\frac{b_0(p\theta-1+\gamma)S^{\frac{p^*}{p}}}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{1}{p^*-p\theta}} := t_0^u. \tag{3.7}$$

Note that σ_u is increasing on $(0, t_{\max}^u)$. Hence from $p\phi_{\mathcal{H}}(\nabla u) \geq \|\nabla u\|_p^p$, $p < q$, Hölder's inequality, (3.1) and the representation of t_0^u in (3.7) we have

$$
\begin{aligned}
\sigma_u(t^u_{\max}) &\geq \sigma_u(t^u_0) \geq \frac{b_0}{p^{\theta-1}}(t^u_0)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta} - (t^u_0)^{p^*-1+\gamma}\int_\Omega |u|^{p^*}\,\mathrm{d}x\\
&\geq (t^u_0)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta}\left(\frac{b_0}{p^{\theta-1}} - (t^u_0)^{p^*-p\theta}S^{\frac{-p^*}{p}}\|\nabla u\|_p^{p^*-p\theta}\right)\\
&\geq \left(\frac{p^*-p\theta}{p^*-1+\gamma}\right)\frac{b_0}{p^{\theta-1}}(t^u_0)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta}\\
&> \left(\frac{p^*-q\theta}{p^*-1+\gamma}\right)\frac{b_0}{p^{\theta-1}}(t^u_0)^{p\theta-1+\gamma}\|\nabla u\|_p^{p\theta}\\
&= \left(\frac{p^*-q\theta}{p^*-1+\gamma}\right)\|\nabla u\|_p^{1-\gamma}\frac{b_0}{p^{\theta-1}}\left(\frac{b_0(p\theta-1+\gamma)S^{\frac{p^*}{p}}}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{p\theta-1+\gamma}{p^*-p\theta}}\\
&\geq \Lambda_2\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x,
\end{aligned}
$$

where Λ_2 is given by

$$
\Lambda_2 = \frac{b_0}{p^{\theta-1}}\left(\frac{p^*-q\theta}{p^*-1+\gamma}\right)\left(\frac{b_0(p\theta-1+\gamma)S}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{p\theta-1+\gamma}{p^*-p\theta}}\frac{S^{\frac{1-\gamma}{p}}}{|\Omega|^{\frac{p^*+\gamma-1}{p^*}}}.
$$

From the considerations above we conclude that

$$
\sigma_u(t^u_{\max}) > \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x
$$

whenever $\lambda \in (0,\Lambda_2)$. Since σ_u is injective in $(0,t^u_{\max})$ and in $(t^u_{\max},\infty)$, we can find unique $t^u_1, t^u_2 > 0$ such that

$$
\sigma_u(t^u_1) = \lambda\int_\Omega |u|^{1-\gamma}\,\mathrm{d}x = \sigma_u(t^u_2) \quad \text{with} \quad \sigma_u'(t^u_2) < 0 < \sigma_u'(t^u_1).
$$

Due to (3.3) we have $t^u_1 u \in \mathcal{N}_\lambda$. Then, from the representation in (3.2), we observe that

$$
\sigma_u'(t) = t^\gamma\psi_u''(t) + \gamma t^{\gamma-1}\psi_u'(t).
$$

Finally, since $\psi_u'(t^u_1) = \psi_u'(t^u_2) = 0$ and $\sigma_u'(t^u_2) < 0 < \sigma_u'(t^u_1)$ we derive that

$$
0 < \sigma_u'(t^u_1) = (t^u_1)^\gamma\psi_u''(t^u_1) \quad \text{and} \quad 0 > \sigma_u'(t^u_2) = (t^u_2)^\gamma\psi_u''(t^u_2).
$$

This shows, in particular, that $t^u_1 u \in \mathcal{N}_\lambda^+$ for $\lambda \in (0,\Lambda_2)$. □

Next we show that the modular $\varrho_{\mathcal{H}}(\nabla\cdot)$ is upper bounded with respect to the elements of $\mathcal{N}_\lambda^+$. The proof is similar to that in [2, Proposition 3.4] and so we omitted it.

Lemma 3.3 *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied. Then there exist* $\Lambda_3 > 0$ *and constant* $D_1 = D_1(\lambda) > 0$ *such that*

$$
\varrho_{\mathcal{H}}(\nabla u) = \|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q < D_1
$$

for every $u \in \mathcal{N}_\lambda^+$ *and for every* $\lambda \in (0,\Lambda_3)$.

By Lemma 3.1(ii), we observe that $\mathcal{N}_\lambda^+$ is closed in $W_0^{1,\mathcal{H}}(\Omega)$ for $\lambda > 0$ small enough. We define

$$\Theta_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u).$$

The next proposition shows that $\Theta_\lambda^+ < 0$. We refer to [2, Proposition 4.1] for its proof.

Proposition 3.4 *Let hypotheses* (h_1)-(h_2) *be satisfied and let* $\lambda \in (0, \min\{\Lambda_1, \Lambda_2\})$, *with* Λ_1, Λ_2 *given in Lemmas* 3.1*(ii) and* 3.2. *Then* $\Theta_\lambda^+ < 0$.

Based on the implicit function theorem in its version stated in Berger [6, p. 115] we can proof the following Lemma which proof is similar to the one in [2, Lemma 4.2].

Lemma 3.5 *Let hypotheses* (h_1)-(h_2) *be satisfied and let* $\lambda > 0$. *Let us consider* $u \in \mathcal{N}_\lambda^+$. *Then there exist* $\varepsilon > 0$ *and a continuous function* $\zeta : B_\varepsilon(0) \to (0,\infty)$ *such that*

$$\zeta(0) = 1 \quad \text{and} \quad \zeta(v)(u+v) \in \mathcal{N}_\lambda^+ \quad \text{for all } v \in B_\varepsilon(0),$$

where $B_\varepsilon(0) := \{v \in W_0^{1,\mathcal{H}}(\Omega) : \|v\| < \varepsilon\}$.

Now, we set $\Lambda^* := \min\{\Lambda_1, \Lambda_2, \Lambda_3\}$ with Λ_1, Λ_2 and $\Lambda_3 > 0$ given in Lemmas 3.1(ii), 3.2 and 3.3 . Let $\lambda \in (0, \Lambda^*)$. Applying Ekeland's variational principle, we obtain a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ satisfying

$$\theta_\lambda^+ < J_\lambda(u_n) < \theta_\lambda^+ + \frac{1}{n}, \tag{3.8}$$

$$J_\lambda(u) \geq J_\lambda(u_n) + \frac{\|u - u_n\|}{n} \tag{3.9}$$

for any $u \in \mathcal{N}_\lambda^+$. By Lemma 3.1(i), we know that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Hence, by Proposition 2.2(ii) along with the reflexivity of $W_0^{1,\mathcal{H}}(\Omega)$, there exist a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and an element $u_\lambda \in W_0^{1,\mathcal{H}}(\Omega)$ such that

$$u_n \rightharpoonup u_\lambda \quad \text{in } W_0^{1,\mathcal{H}}(\Omega), \quad u_n \to u_\lambda \quad \text{in } L^s(\Omega) \quad \text{and} \quad u_n \to u_\lambda \quad \text{a.e. in } \Omega \tag{3.10}$$

for any $s \in [1, p^*)$. By the coercivity given in Lemma 3.1(i), we can assume that there exist $E_1,\ E_2 \geq 0$ such that

$$\lim_{n\to\infty} \|u_n\|_p^p = E_1 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|_{q,a}^q = E_2. \tag{3.11}$$

We get the following technical results.

Lemma 3.6 *Let hypotheses* (h_1)-(h_2) *be satisfied, let* $\lambda \in (0, \Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)–(3.9). *Then* $u_\lambda \neq 0$.

Proof Let us assume by contradiction that $u_\lambda = 0$. Then $\psi'_{u_n}(1) = 0$ implies

$$\left[a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right](\|u_n\|_p^p + \|u_n\|_{q,a}^q) - \lambda \int_\Omega |u_n|^{1-\gamma}\,dx - \int_\Omega |u_n|^{p^*}\,dx = 0.$$

Using (3.10), (3.11) and letting $n \to \infty$, we get

$$\left[a_0 + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\right](E_1 + E_2) - d^{p^*} = 0, \tag{3.12}$$

where we set

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p^*} \,\mathrm{d}x =: d^{p^*} \geq 0.$$

Moreover by (3.8) we have

$$\lim_{n\to\infty} J_\lambda(u_n) = \Theta_\lambda^+ < 0,$$

which implies that

$$\left[a_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right) + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta}\right] - \frac{d^{p^*}}{p^*} < 0. \tag{3.13}$$

Recall that $E_1, E_2 \geq 0$. Then, taking the value of d^{p^*} from (3.12) into (3.13), we derive that

$$\left[a_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right) + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta}\right] - \left[a_0 + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\right]\frac{E_1 + E_2}{p^*} < 0.$$

This implies

$$a_0\left[\frac{E_1}{p} + \frac{E_2}{q} - \frac{E_1 + E_2}{p^*}\right] + b_0\left[\frac{1}{\theta}\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta} - \left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1}\frac{E_1 + E_2}{p^*}\right] < 0$$

and so

$$\begin{aligned} a_0\left[E_1\left(\frac{1}{p} - \frac{1}{p^*}\right) + E_2\left(\frac{1}{q} - \frac{1}{p^*}\right)\right] + b_0\left(\frac{E_1}{p} + \frac{E_2}{q}\right)^{\theta-1} \\ \left[E_1\left(\frac{1}{p\theta} - \frac{1}{p^*}\right) + E_2\left(\frac{1}{q\theta} - \frac{1}{p^*}\right)\right] < 0, \end{aligned}$$

which is a contradiction because of $p < q \leq q\theta < p^*$. □

Lemma 3.7 *Let hypotheses* (h$_1$)–(h$_2$) *be satisfied, let* $\lambda \in (0, \Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)–(3.9). *Then* $\liminf_{n\to\infty} \psi_{u_n}''(1) > 0$, *that is,*

$$\begin{aligned} \liminf_{n\to\infty} \Big\{ & \left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-1+\gamma)\|\nabla u_n\|_p^p + (q-1+\gamma)\|\nabla u_n\|_{q,a}^q\right] \\ & + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 - (p^*-1+\gamma)\int_\Omega |u_n|^{p^*}\,\mathrm{d}x \Big\} > 0. \end{aligned}$$

Proof Since $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$, we have $\psi_{u_n}'(1) = 0$ and $\psi_{u_n}''(1) > 0$, that is,

$$\left[a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-1+\gamma)\|\nabla u_n\|_p^p+(q-1+\gamma)\|\nabla u_n\|_{q,a}^q\right] + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2-(p^*-1+\gamma)\int_\Omega |u_n|^{p^*}\,\mathrm{d}x>0$$

and

$$\left[a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p+(q-p^*)\|\nabla u_n\|_{q,a}^q\right] + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2+\lambda(p^*-1+\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x>0. \tag{3.14}$$

Thus, in order to prove the lemma, it is enough to show that

$$\liminf_{n\to\infty}\left\{\left[a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p+(q-p^*)\|\nabla u_n\|_{q,a}^q\right] + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2 + \lambda(p^*-1+\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x\right\}>0.$$

By contradicting (3.14), let us assume that

$$\liminf_{n\to\infty}\left\{\left[a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p+(q-p^*)\|\nabla u_n\|_{q,a}^q\right] + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2+\lambda(p^*-1+\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x\right\}=0. \tag{3.15}$$

By Lebesgue dominated convergence theorem, we obtain

$$\lim_{n\to\infty}\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x=\int_\Omega |u_\lambda|^{1-\gamma}\mathrm{d}x. \tag{3.16}$$

Using (3.16) in (3.15), we get

$$\liminf_{n\to\infty}\left\{\left[a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left[(p-p^*)\|\nabla u_n\|_p^p+(q-p^*)\|\nabla u_n\|_{q,a}^q\right] + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2\right\} = -\lambda(p^*-1+\gamma)\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x,$$

which yields, by applying (3.11),

$$-\lambda\int_\Omega |u_\lambda|^{1-\gamma}\,\mathrm{d}x=\left[a_0+b_0\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}\right]\frac{\left[(p-p^*)E_1+(q-p^*)E_2\right]}{(p^*-1+\gamma)} + \frac{b_0(\theta-1)}{(p^*-1+\gamma)}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-2}(E_1+E_2)^2. \tag{3.17}$$

From this, due to $p < q < p^*$, we have

$$-\lambda \int_\Omega |u_\lambda|^{1-\gamma}\,dx \le b_0\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}\left[\frac{(q-p^*)(E_1+E_2)}{(p^*-1+\gamma)}+\frac{b_0(\theta-1)q(E_1+E_2)}{(p^*+\gamma-1)}\right]$$
$$=\frac{b_0(q\theta-p^*)(E_1+E_2)}{(p^*+\gamma-1)}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}. \tag{3.18}$$

Considering $\psi'_{u_n}(1)=0$ and (3.16), we have

$$\lim_{n\to\infty}\int_\Omega |u_n|^{p^*}\,dx = \left[a_0+b_0\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}\right]\left[E_1+E_2\right]-\lambda\int_\Omega |u_\lambda|^{1-\gamma}\,dx.$$

From this and (3.17), we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\int_\Omega |u_n|^{p^*}\,dx\\
&=\left[a_0+b_0\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}\right]\left[\left(\frac{p+\gamma-1}{p^*+\gamma-1}\right)E_1+\left(\frac{q+\gamma-1}{p^*+\gamma-1}\right)E_2\right]\\
&\quad+\frac{b_0(\theta-1)}{p^*-1+\gamma}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-2}(E_1+E_2)^2\\
&\ge \frac{b_0(p+\gamma-1)}{p^*+\gamma-1}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}(E_1+E_2)+\frac{b_0(p\theta-p)}{p^*-1+\gamma}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}(E_1+E_2)\\
&=\frac{b_0(p\theta+\gamma-1)}{p^*+\gamma-1}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1}(E_1+E_2)\\
&\ge \frac{b_0(p\theta+\gamma-1)}{p^{\theta-1}(p^*+\gamma-1)}E_1^\theta.
\end{aligned} \tag{3.19}$$

For any fixed $w \in W_0^{1,\mathcal{H}}(\Omega)\setminus\{0\}$, we know that there exists a unique $t_{\max} > 0$ such that $\sigma'_w(t_{\max}) = 0$. From this and (3.6), we conclude that

$$t_{\max} \ge \left(\frac{b_0(p\theta+\gamma-1)\|\nabla w\|_p^{p\theta}}{p^{\theta-1}(p^*-1+\gamma)\int_\Omega |w|^{p^*}\,dx}\right)^{\frac{1}{p^*-p\theta}} := t_{00}. \tag{3.20}$$

It is easy to verify that $t_{\max} \ge t_{00} \ge t_0^w$ as defined in (3.7) and from the proof of Lemma 3.2, we know that $\Lambda_2 > 0$ is chosen in such a way that

$$\frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}(t_0^w)^{p\theta+\gamma-1}\|\nabla w\|_p^{p\theta} \ge \Lambda_2\int_\Omega |w|^{1-\gamma}dx.$$

We define

$$S(w) := \frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}(t_{00})^{p\theta+\gamma-1}\|\nabla w\|_p^{p\theta}-\Lambda_2\int_\Omega |w|^{1-\gamma}dx \ge 0 \quad \text{for all } w \in W_0^{1,\mathcal{H}}(\Omega), \tag{3.21}$$

with t_{00} given in (3.20). Taking $w = u_n$ in (3.21) and then passing to the limit as $n \to \infty$ we get

$$\lim_{n\to\infty} S(u_n) \geq 0.$$

On the other hand, by Lemma 3.6 and (3.11), we have that at least one of E_1 and E_2 is not zero. Let us assume, without any loss of generality, that $E_1 > 0$, $E_2 \geq 0$. Then by (3.18), (3.19), (3.20) along with $q\theta < p^*$ and $\lambda \in (0, \Lambda_2)$, we obtain

$$\begin{aligned}\lim_{n\to\infty} S(u_n) &\leq \frac{\frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}\left(\frac{b_0(p\theta+\gamma-1)E_1^\theta}{p^{\theta-1}(p^*-1+\gamma)}\right)^{\frac{(p\theta-1+\gamma)}{p^*-p\theta}} E_1^\theta}{\left(\frac{b_0(p\theta+\gamma-1)}{p^{\theta-1}(p^*+\gamma-1)}E_1^\theta\right)^{\frac{p\theta+\gamma-1}{p^*-p\theta}}} \\ &\quad + \frac{\Lambda_2}{\lambda}\frac{b_0(q\theta-p^*)(E_1+E_2)}{(p^*+\gamma-1)}\left(\frac{E_1}{p}+\frac{E_2}{q}\right)^{\theta-1} \\ &< \frac{b_0(p^*-q\theta)}{p^{\theta-1}(p^*+\gamma-1)}E_1^\theta + \frac{b_0(q\theta-p^*)E_1^\theta}{p^{\theta-1}(p^*+\gamma-1)} = 0.\end{aligned}$$

This proves the assertion of the lemma. □

Let $h \in W_0^{1,\mathcal{H}}(\Omega)$ be nonnegative. From Lemma 3.5 there exists a sequence of maps $\{\zeta_n\}_{n\in\mathbb{N}}$ such that $\zeta_n(0) = 1$ and $\zeta_n(th)(u_n + th) \in \mathcal{N}_\lambda^+$ for sufficiently small $t > 0$ and for each $n \in \mathbb{N}$. From this and $u_n \in \mathcal{N}_\lambda$, we have the equations

$$\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\right]\left(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q\right) - \lambda\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - \int_\Omega |u_n|^{p^*}\,\mathrm{d}x = 0 \tag{3.22}$$

and

$$\begin{aligned}&\left[a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n)\right]\left(\zeta_n^p(th)\|\nabla w_n\|_p^p + \zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\right) \\ &\quad - \lambda\zeta_n^{1-\gamma}(th)\int_\Omega |w_n|^{1-\gamma}\,\mathrm{d}x - \zeta_n^{p^*}(th)\int_\Omega |w_n|^{p^*}\,\mathrm{d}x = 0\end{aligned} \tag{3.23}$$

where $w_n = u_n + th$.

Lemma 3.8 *Let hypotheses* (h$_1$)-(h$_2$) *be satisfied, let* $\lambda \in (0, \Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)–(3.9)*. For any nonnegative function* $h \in W_0^{1,\mathcal{H}}(\Omega)$*, the sequence* $\{\langle\zeta_n'(0), h\rangle\}_{n\in\mathbb{N}}$ *is uniformly bounded.*

Proof Subtracting (3.22) from (3.23), we get

$$
\begin{aligned}
&(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\Big[(\|\nabla w_n\|_p^p - \|\nabla u_n\|_p^p) + (\|\nabla w_n\|_{q,a}^q - \|\nabla u_n\|_{q,a}^q)\\
&\quad + (\zeta_n^p(th) - 1)\|\nabla w_n\|_p^p + (\zeta_n^q(th) - 1)\|\nabla w_n\|_{q,a}^q\Big]\\
&\quad + b_0\big[\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n) - \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\big]\Big(\zeta_n^p(th)\|\nabla w_n\|_p^p + \zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\Big)\\
&\quad - \lambda\big(\zeta_n^{1-\gamma}(th) - 1\big)\int_\Omega |w_n|^{1-\gamma}\,\mathrm{d}x - \lambda\int_\Omega \big(|w_n|^{1-\gamma} - |u_n|^{1-\gamma}\big)\,\mathrm{d}x\\
&\quad - \big(\zeta_n^{p^*}(th) - 1\big)\int_\Omega |w_n|^{p^*}\,\mathrm{d}x - \int_\Omega \big(|w_n|^{p^*} - |u_n|^{p^*}\big)\,\mathrm{d}x = 0.
\end{aligned}
\tag{3.24}
$$

For notational convenience, we set

$$
\langle u_n, h\rangle_p = \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x \quad\text{and}\quad \langle u_n, h\rangle_{q,a} = \int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x.
$$

We have the following limits

$$
\begin{aligned}
\lim_{t\to 0}\frac{\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n) - \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)}{t} &= (\zeta_n'(0), h)(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)\\
&\quad + (\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a}),\\
\lim_{t\to 0}\frac{\|\nabla w_n\|_p^p - \|\nabla u_n\|_p^p}{t} &= p\langle u_n, h\rangle_p,\\
\lim_{t\to 0}\frac{\|\nabla w_n\|_{q,a}^q - \|\nabla u_n\|_{q,a}^q}{t} &= q\langle u_n, h\rangle_{q,a},\\
\lim_{t\to 0}\big(|w_n|^{p^*} - |u_n|^{p^*}\big)\,\mathrm{d}x &= p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x,\\
\lim_{t\to 0}\frac{\zeta^s(th) - 1}{t} &= s\langle\zeta_n'(0), h\rangle \quad\text{for any } s > 1.
\end{aligned}
\tag{3.25}
$$

Taking into account

$$
\int_\Omega \big(|w_n|^{1-\gamma} - |u_n|^{1-\gamma}\big)\,\mathrm{d}x \geq 0
$$

since h is nonnegative, dividing both sides of (3.24) by $t > 0$ and then passing the limit as $t \to 0^+$, we obtain

$$
\begin{aligned}
0 &\leq (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\Big(p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla h\,\mathrm{d}x + q\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\nabla h\,\mathrm{d}x\\
&\quad + p\,\langle\zeta_n'(0), h\rangle\|\nabla u_n\|_p^p + q\langle\zeta_n'(0), h\rangle\|\nabla u_n\|_{q,a}^q\Big)\\
&\quad + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)\langle\zeta_n'(0), h\rangle\Big(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q\Big)^2\\
&\quad - \lambda(1-\gamma)\langle\zeta_n'(0), h\rangle\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - p^*\langle\zeta_n'(0), h\rangle\int_\Omega |u_n|^{p^*}\,\mathrm{d}x - p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x.
\end{aligned}
$$

This implies

$$
\begin{aligned}
0 \leq \langle \zeta_n'(0), h\rangle & \Big[(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\Big[p\|\nabla u_n\|_p^p + q\|\nabla u_n\|_{q,a}^q\Big] \\
& + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)\Big(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q\Big)^2 \\
& - \lambda(1-\gamma)\int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - p^*\int_\Omega |u_n|^{p^*}\mathrm{d}x\Big] + (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \Big(p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + q\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\Big) \\
& - p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x.
\end{aligned}
$$

Therefore, using the fact that $u_n \in \mathcal{N}_\lambda$, we have

$$
\begin{aligned}
0 \leq \langle \zeta_n'(0), h\rangle & \Big\{(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\Big[(p+\gamma-1)\|\nabla u_n\|_p^p + (q+\gamma-1)\|\nabla u_n\|_{q,a}^q\Big] \\
& + b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p + \|\nabla u_n\|_{q,a}^q)^2 - (p^*+\gamma-1)\int_\Omega |u_n|^{p^*}\mathrm{d}x\Big\} \\
& + \Big[(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\Big(p\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + q\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\Big) \\
& - p^*\int_\Omega |u_n|^{p^*-2}u_n h\,\mathrm{d}x\Big].
\end{aligned}
$$

Now using Lemma 3.7 and taking into account the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$, we infer that $\{\langle\zeta_n'(0), h\rangle\}_{n\in\mathbb{N}}$ is bounded below for any nonnegative $h \in W_0^{1,\mathcal{H}}(\Omega)$.

It remains to show that $\{\langle\zeta_n'(0), h\rangle\}_{n\in\mathbb{N}}$ is bounded above for any nonnegative $h \in W_0^{1,\mathcal{H}}(\Omega)$. Assume by contradiction that $\limsup_{n\to\infty}\langle\zeta_n'(0), h\rangle = \infty$. Thus, without loss of generality, we can consider $\zeta_n(th) > \zeta_n(0) = 1$ for $n \in \mathbb{N}$ large enough . It is easy to see that

$$
|\zeta_n(th) - 1|\|u_n\| + \zeta_n(th)\|th\| \geq \|(\zeta_n(th)-1)u_n + th\zeta_n(th)\| = \|\zeta_n(th)w_n - u_n\|.
$$

Applying this in (3.9) with $u = \zeta_n(th)w_n$, we get

$$
\begin{aligned}
& |\zeta_n(th) - 1|\frac{\|u_n\|}{n} + \zeta_n(th)\frac{\|th\|}{n} \\
& \geq J_\lambda(u_n) - J_\lambda(\zeta_n(th)w_n) \\
& = a_0\big[\phi_{\mathcal{H}}(\nabla u_n) - \phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n)\big] + \frac{b_0}{\theta}\big[\phi_{\mathcal{H}}^\theta(\nabla u_n) - \phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n)\big] \\
& \quad - \frac{\lambda}{1-\gamma}\int_\Omega \big[|u_n|^{1-\gamma} - |\zeta_n(th)w_n|^{1-\gamma}\big]\,\mathrm{d}x - \frac{1}{p^*}\int_\Omega \big[|u_n|^{p^*} - |\zeta_n(th)w_n|^{p^*}\big]\,\mathrm{d}x.
\end{aligned}
$$

Using (3.22) and (3.23) in the inequality above, we obtain

$$
\begin{aligned}
&|\zeta_n(th)-1|\frac{\|u_n\|}{n}+\zeta_n(th)\frac{\|th\|}{n}\\
&=a_0\Bigg[\phi_{\mathcal{H}}(\nabla u_n)-\phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n)-\frac{1}{1-\gamma}\\
&\quad\Big(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q-\zeta_n^p(th)\|\nabla w_n\|_p^p-\zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\Big)\Bigg]\\
&\quad+b_0\Bigg[\frac{\phi_{\mathcal{H}}^\theta(\nabla u_n)-\phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n)}{\theta}-\frac{\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)}{1-\gamma}\Big(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q\Big)\\
&\quad+\frac{\phi_{\mathcal{H}}^{\theta-1}(\zeta_n(th)\nabla w_n)}{1-\gamma}\Big(\zeta_n^p(th)\|\nabla w_n\|_p^p+\zeta_n^q(th)\|\nabla w_n\|_{q,a}^q\Big)\Bigg]\\
&\quad-\left(\frac{1}{1-\gamma}-\frac{1}{p^*}\right)\int_\Omega\left[|\zeta_n(th)w_n|^{p^*}-|u_n|^{p^*}\right]\mathrm{d}x.
\end{aligned}
$$

Now dividing the above inequality by $t>0$, passing to the limit as $t\to 0^+$ and using (3.25), we have

$$
\begin{aligned}
\frac{\|h\|}{n}&\geq a_0\Bigg[\langle u_n,h\rangle_p+\langle u_n,h\rangle_{q,a}-\langle\zeta_n'(0),h\rangle\Big(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q\Big)\\
&\quad+\frac{1}{1-\gamma}\Big\{\langle\zeta_n'(0),h\rangle\Big(p\|\nabla u_n\|_p^p+q\|\nabla u_n\|_{q,a}^q\Big)+p\langle u_n,h\rangle_p+q\langle u_n,h\rangle_{q,a}\Big\}\Bigg]\\
&\quad+b_0\Bigg[\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\langle\zeta_n'(0),h\rangle\Big(p\|\nabla u_n\|_p^p+q\|\nabla u_n\|_{q,a}^q\Big)\\
&\quad+\frac{1}{1-\gamma}\Big\{\langle\zeta_n'(0),h\rangle(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2\\
&\quad+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\langle\zeta_n'(0),h\rangle\Big(p\|\nabla u_n\|_p^p+q\|\nabla u_n\|_{q,a}^q\Big)+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\\
&\quad\big(p\langle u_n,h\rangle_p+q\langle u_n,h\rangle_{q,a}\big)\Big\}\Bigg]\\
&\quad-\left(\frac{p^*-1+\gamma}{1-\gamma}\right)\left[\langle\zeta_n'(0),h\rangle\int_\Omega|u_n|^{p^*}\,\mathrm{d}x+\int_\Omega|u_n|^{p^*-2}u_nh\,\mathrm{d}x\right]\\
&=\frac{\langle\zeta_n'(0),h\rangle}{1-\gamma}\Bigg[\Big(a_0+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\Big)\Big\{(p-1+\gamma)\|\nabla u_n\|_p^p+(q-1+\gamma)\|\nabla u_n\|_{q,a}^q\Big\}\\
&\quad+b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2\\
&\quad-(p^*-1+\gamma)\int_\Omega|u_n|^{p^*}\,\mathrm{d}x-\frac{(1-\gamma)\|u_n\|}{n}\Bigg]\\
&\quad+\frac{a_0}{1-\gamma}\Big[(p-\gamma+1)\langle u_n,h\rangle_p+(q-\gamma+1)\langle u_n,h\rangle_{q,a}\Big]\\
&\quad+\frac{b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)}{1-\gamma}\Big[p\langle u_n,h\rangle_p+q\langle u_n,h\rangle_{q,a}\Big]\\
&\quad-\left(\frac{p^*-1+\gamma}{1-\gamma}\right)\int_\Omega|u_n|^{p^*-2}u_nh\,\mathrm{d}x,
\end{aligned}
$$

which gives a contradiction if we take the limits $n \to \infty$ on both sides, considering $\limsup_{n\to\infty}\langle \zeta_n'(0), h\rangle = \infty$, since by Lemma 3.7 and the boundedness of $\{u_n\}_{n\in\mathbb{N}}$, there exists some $M_1 > 0$ such that

$$\begin{aligned}&\Bigg[\big(a_0+\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)\big)\Big\{(p-1+\gamma)\|\nabla u_n\|_p^p+(q-1+\gamma)\|\nabla u_n\|_{q,a}^q\Big\}\\&\quad+b_0(\theta-1)\phi_{\mathcal{H}}^{\theta-2}(\nabla u_n)(\|\nabla u_n\|_p^p+\|\nabla u_n\|_{q,a}^q)^2\\&\quad-(p^*-1+\gamma)\int_\Omega |u_n|^{p^*}\,\mathrm{d}x-\frac{(1-\gamma)\|u_n\|}{n}\Bigg]>M_1\end{aligned}$$

for $n\in\mathbb{N}$ large enough. Thus $\{\langle\zeta_n'(0),h\rangle\}_{n\in\mathbb{N}}$ must be bounded above. □

Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, without loss of generality, we may assume that $u_n \geq 0$ a. e. in Ω and so, $u_\lambda \geq 0$ a. e. in Ω. With this assumption, we state our next result.

Lemma 3.9 *Let hypotheses* (h_1)-(h_2) *be satisfied, let* $\lambda \in (0,\Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)–(3.9). *For any* $h \in W_0^{1,\mathcal{H}}(\Omega)$ *and* $n\in\mathbb{N}$, $u_n^{-\gamma}h \in L^1(\Omega)$ *and as* $n\to\infty$

$$\begin{aligned}&(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\int_\Omega|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x+\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right]\\&\quad-\lambda\int_\Omega u_n^{-\gamma}h\,\mathrm{d}x-\int_\Omega u_n^{p^*-1}h\,\mathrm{d}x=o_n(1).\end{aligned}\tag{3.26}$$

Proof Let $h \in W_0^{1,\mathcal{H}}(\Omega)$ be nonnegative and recall the following estimate from the proof of Lemma 3.8

$$\begin{aligned}&|\zeta_n(th)-1|\frac{\|u_n\|}{n}+\zeta_n(th)\frac{\|th\|}{n}\\&\geq a_0\big[\phi_{\mathcal{H}}(\nabla u_n)-\phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n)\big]+\frac{b_0}{\theta}\big[\phi_{\mathcal{H}}^\theta(\nabla u_n)-\phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n)\big]\\&\quad-\frac{\lambda}{1-\gamma}\int_\Omega\big[|u_n|^{1-\gamma}-|\zeta_n(th)w_n|^{1-\gamma}\big]\,\mathrm{d}x-\frac{1}{p^*}\int_\Omega\big[|u_n|^{p^*}-|\zeta_n(th)w_n|^{p^*}\big]\,\mathrm{d}x\\&=a_0\big[(\phi_{\mathcal{H}}(\nabla u_n)-\phi_{\mathcal{H}}(\nabla w_n))+(\phi_{\mathcal{H}}(\nabla w_n)-\phi_{\mathcal{H}}(\zeta_n(th)\nabla w_n))\big]\\&\quad+\frac{b_0}{\theta}\big[(\phi_{\mathcal{H}}^\theta(\nabla u_n)-\phi_{\mathcal{H}}^\theta(\nabla w_n))+(\phi_{\mathcal{H}}^\theta(\nabla w_n)-\phi_{\mathcal{H}}^\theta(\zeta_n(th)\nabla w_n))\big]\\&\quad-\frac{\lambda}{1-\gamma}\int_\Omega\big[|u_n|^{1-\gamma}-|w_n|^{1-\gamma}\big]\,\mathrm{d}x-\frac{\lambda}{1-\gamma}\int_\Omega\big[|w_n|^{1-\gamma}-|\zeta_n(th)w_n|^{1-\gamma}\big]\,\mathrm{d}x\\&\quad-\frac{1}{p^*}\int_\Omega\big[|u_n|^{p^*}-|w_n|^{p^*}\big]\,\mathrm{d}x-\frac{1}{p^*}\int_\Omega\big[|w_n|^{p^*}-|\zeta_n(th)w_n|^{p^*}\big]\,\mathrm{d}x.\end{aligned}$$

Dividing the above equation with $t > 0$ and then passing to limit as $t\to 0^+$, we get

$$
\begin{aligned}
&|\langle \zeta_n'(0), h\rangle| \frac{\|u_n\|}{n} + \frac{\|h\|}{n}\\
&\geq -(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\Big[\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a} + \langle \zeta_n'(0), h\rangle(\|u_n\|_p^p + \|u_n\|_{q,a}^q)\Big]\\
&\quad - \frac{\lambda}{1-\gamma} \liminf_{t\to 0^+} \int_\Omega \frac{\left[|u_n|^{1-\gamma} - |w_n|^{1-\gamma}\right]}{t}\,\mathrm{d}x + \lambda\langle \zeta_n'(0), h\rangle \int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x\\
&\quad + \langle \zeta_n'(0), h\rangle \int_\Omega |u_n|^{p^*}\,\mathrm{d}x + \int_\Omega u_n^{p^*-1} h\,\mathrm{d}x\\
&= -\langle \zeta_n'(0), h\rangle \Bigg[(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\Big[(\|u_n\|_p^p + \|u_n\|_{q,a}^q)\Big]\\
&\quad - \lambda \int_\Omega |u_n|^{1-\gamma}\,\mathrm{d}x - \int_\Omega |u_n|^{p^*}\,\mathrm{d}x\Bigg]\\
&\quad - (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a}\right]\\
&\quad - \frac{\lambda}{1-\gamma} \liminf_{t\to 0^+} \int_\Omega \frac{\left[|u_n|^{1-\gamma} - |w_n|^{1-\gamma}\right]}{t}\,\mathrm{d}x + \int_\Omega u_n^{p^*-1} h\,\mathrm{d}x\\
&= -(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a}\right]\\
&\quad - \frac{\lambda}{1-\gamma} \liminf_{t\to 0^+} \int_\Omega \frac{\left[|u_n|^{1-\gamma} - |w_n|^{1-\gamma}\right]}{t}\,\mathrm{d}x + \int_\Omega u_n^{p^*-1} h\,\mathrm{d}x,
\end{aligned}
$$

where we used $u_n \in \mathcal{N}_\lambda$ that is $\psi_{u_n}'(1) = 0$. This implies

$$
\begin{aligned}
&\frac{\lambda}{1-\gamma} \liminf_{t\to 0^+} \int_\Omega \frac{\left[|u_n + th|^{1-\gamma} - |u_n|^{1-\gamma}\right]}{t}\,\mathrm{d}x\\
&\quad \leq (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a}\right]\\
&\qquad - \int_\Omega u_n^{p^*-1} h\,\mathrm{d}x + |\langle \zeta_n'(0), h\rangle| \frac{\|u_n\|}{n} + \frac{\|h\|}{n}.
\end{aligned}
\tag{3.27}
$$

Observe that $|u_n + th|^{1-\gamma} - |u_n|^{1-\gamma} \geq 0$, so we can use Fatou's lemma in (3.27) to obtain

$$
\begin{aligned}
\lambda \int_\Omega u_n^{-\gamma} h\,\mathrm{d}x &\leq (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\langle u_n, h\rangle_p + \langle u_n, h\rangle_{q,a}\right]\\
&\quad - \int_\Omega u_n^{p^*-1} h\,\mathrm{d}x + |\langle \zeta_n'(0), h\rangle| \frac{\|u_n\|}{n} + \frac{\|h\|}{n}.
\end{aligned}
$$

Recall that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W_0^{1,\mathcal{H}}(\Omega)$. Then, passing to the limit as $n \to \infty$ in the above estimate, we obtain

$$
\begin{aligned}
&(a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla h\,\mathrm{d}x + \int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla h\,\mathrm{d}x\right]\\
&\quad - \lambda \int_\Omega u_n^{-\gamma} h\,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h\,\mathrm{d}x \geq o_n(1),
\end{aligned}
\tag{3.28}
$$

for each nonnegative $h \in W_0^{1,\mathcal{H}}(\Omega)$, where we used the uniform boundedness from Lemma 3.8.

We aim to establish that (3.28) holds true for any arbitrary $h \in W_0^{1,\mathcal{H}}(\Omega)$. For this, we replace h in (3.28) by $(u_n + \varepsilon h)^+$ with $\varepsilon > 0$ and $h \in W_0^{1,\mathcal{H}}(\Omega)$. Renaming as $h_\varepsilon = u_n + \varepsilon h$ and using (3.28), we get

$$
\begin{aligned}
o_n(1) \le\ & (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon^+ \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon^+ \,\mathrm{d}x \right] \\
& - \lambda \int_\Omega u_n^{-\gamma} h_\varepsilon^+ \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h_\varepsilon^+ \,\mathrm{d}x \\
=\ & (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x \right] \\
& + (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x \right] \\
& - \lambda \int_\Omega u_n^{-\gamma} (h_\varepsilon + h_\varepsilon^-) \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} (h_\varepsilon + h_\varepsilon^-) \,\mathrm{d}x \\
=\ & \left[(a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \left[(\|u_n\|_p^p + \|u_n\|_{q,a}^q) \right] - \lambda \int_\Omega |u_n|^{1-\gamma} \,\mathrm{d}x - \int_\Omega |u_n|^{p^*} \,\mathrm{d}x \right] \\
& + \varepsilon \Bigg\{ (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \Bigg[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \\
& + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \Bigg] \\
& - \lambda \int_\Omega u_n^{-\gamma} h \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h \,\mathrm{d}x \Bigg\} - \lambda \int_\Omega u_n^{-\gamma} h_\varepsilon^- \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h_\varepsilon^- \,\mathrm{d}x \\
& + (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon^- \,\mathrm{d}x \right].
\end{aligned}
$$

We define $\Omega_\varepsilon = \{x \in \Omega : u_n + \varepsilon h \le 0\}$. Using $u_n \in \mathcal{N}_\lambda$ and $\int_\Omega u_n^{-\gamma} h_\varepsilon^- \,\mathrm{d}x \ge 0$ in the above estimate, we get

$$
\begin{aligned}
o_n(1) \le\ & \varepsilon \Bigg\{ (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \left[\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x + \int_\Omega a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h \,\mathrm{d}x \right] \\
& - \lambda \int_\Omega u_n^{-\gamma} h \,\mathrm{d}x - \int_\Omega u_n^{p^*-1} h \,\mathrm{d}x \Bigg\} + \int_{\Omega_\varepsilon} u_n^{p^*-1} h_\varepsilon \,\mathrm{d}x - (a_0 + b_0 \phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
& \left[\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x + \int_{\Omega_\varepsilon} a(x) |\nabla u_n|^{q-2} \nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x \right].
\end{aligned}
\tag{3.29}
$$

Note that

$$
\begin{aligned}
-\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x &= -\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n+\varepsilon h)\,\mathrm{d}x \\
&= -\int_{\Omega_\varepsilon} |\nabla u_n|^{p}\,\mathrm{d}x - \varepsilon\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla h\,\mathrm{d}x \\
&\le -\varepsilon\int_{\Omega_\varepsilon} |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla h\,\mathrm{d}x
\end{aligned}
$$

and similarly,

$$
-\int_{\Omega_\varepsilon} a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla h_\varepsilon \,\mathrm{d}x \le -\varepsilon\int_{\Omega_\varepsilon} a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla h\,\mathrm{d}x.
$$

Moreover, applying Hölder's inequality and $u_n \le -\varepsilon h$ in Ω_ε, we have

$$
\begin{aligned}
\left|\int_{\Omega_\varepsilon} u_n^{p^*-1}h_\varepsilon\,\mathrm{d}x\right| &\le \left|\int_{\Omega_\varepsilon} u_n^{p^*}\,\mathrm{d}x\right| + \epsilon\left|\int_{\Omega_\varepsilon} u_n^{p^*-1}|h|\,\mathrm{d}x\right| \\
&\le \varepsilon^{p^*}\int_{\Omega_\varepsilon}|h|^{p^*}\,\mathrm{d}x + \varepsilon\left(\int_{\Omega_\varepsilon} u_n^{p^*}\,\mathrm{d}x\right)^{\frac{p^*-1}{p^*}}\left(\int_{\Omega_\varepsilon}|h|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}}.
\end{aligned}
$$

Putting all these in (3.29), we infer that

$$
\begin{aligned}
o_n(1) \le \varepsilon\Bigg\{&(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + \int_{\Omega} a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right] \\
&-\lambda\int_{\Omega} u_n^{-\gamma}h\,\mathrm{d}x - \int_{\Omega} u_n^{p^*-1}h\,\mathrm{d}x\Bigg\} + \varepsilon^{p^*}\int_{\Omega_\varepsilon}|h|^{p^*}\,\mathrm{d}x \\
&+\varepsilon\left(\int_{\Omega_\varepsilon} u_n^{p^*}\,\mathrm{d}x\right)^{\frac{p^*-1}{p^*}}\left(\int_{\Omega_\varepsilon}|h|^{p^*}\,\mathrm{d}x\right)^{\frac{1}{p^*}} - \varepsilon(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\
&\left[\int_{\Omega_\varepsilon}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + \int_{\Omega_\varepsilon} a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right].
\end{aligned} \tag{3.30}
$$

Since $|\Omega_\varepsilon| \to 0$ as $\varepsilon \to 0^+$ and by the boundedness of $\{u_n\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$, if we divide (3.30) by $\varepsilon > 0$ and then pass to the limit as $\varepsilon \to 0^+$, we obtain

$$
\begin{aligned}
&(a_0+b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\left[\int_{\Omega}|\nabla u_n|^{p-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x + \int_{\Omega} a(x)|\nabla u_n|^{q-2}\nabla u_n\cdot\nabla h\,\mathrm{d}x\right] \\
&-\lambda\int_{\Omega} u_n^{-\gamma}h\,\mathrm{d}x - \int_{\Omega} u_n^{p^*-1}h\,\mathrm{d}x \ge o_n(1),
\end{aligned} \tag{3.31}
$$

as $n \to \infty$. By the arbitrariness of $h \in W_0^{1,\mathcal{H}}(\Omega)$, (3.31) actually implies (3.26) which completes the proof. □

Now, we prove the compactness property of the energy functional J_λ in a suitable range of λ. For this purpose, we set for any $\lambda > 0$

$$
c_\lambda := \alpha_2 - \alpha_1\lambda^{\frac{p^*}{p^*-1+\gamma}}
$$

where

$$\alpha_0 := \left(\frac{1}{q\theta} - \frac{1}{p^*}\right), \qquad \alpha_1 := \frac{(p^*-1+\gamma)|\Omega|}{p^*}\left(\frac{q\theta-1+\gamma}{q\theta(1-\gamma)}\right)^{\frac{p^*}{p^*-1+\gamma}}\left(\frac{1-\gamma}{p^*\alpha_0}\right)^{\frac{1-\gamma}{p^*-1+\gamma}} \tag{3.32}$$

and

$$\alpha_2 := \alpha_0\left(\frac{Sb_0}{p^{\theta-1}}\right)^{\frac{p^*}{p^*-p}}\left(S^{p^*}\left(\frac{b_0}{p^{\theta-1}}\right)^{p}\right)^{\frac{(\theta-1)p^*}{(p^*-p\theta)(p^*-p)}}. \tag{3.33}$$

Also, for any $k \in \mathbb{N}$, let T_k be the truncation defined by

$$T_k(t) := \begin{cases} t & \text{if } |t| \le k, \\ k\dfrac{t}{|t|} & \text{if } |t| > k. \end{cases}$$

Proposition 3.10 *Let hypotheses* (h_1)-(h_2) *be satisfied, let* $\lambda \in (0, \Lambda^*)$ *and let* $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{N}_\lambda^+$ *be a sequence satisfying* (3.8)–(3.9) *and*

$$J_\lambda(u_n) \to c < c_\lambda \quad \text{as } n \to \infty. \tag{3.34}$$

Then $\{u_n\}_{n\in\mathbb{N}}$ *possesses a strongly convergent subsequence in* $W_0^{1,\mathcal{H}}(\Omega)$.

Proof Fixing $k \in \mathbb{N}$ and taking $h = T_k(u_n - u_\lambda) \in W_0^{1,\mathcal{H}}(\Omega)$ as a test function in (3.26), we get

$$\begin{aligned} o_n(1) = (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\bigg[\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla T_k(u_n - u_\lambda)\,\mathrm{d}x \\ + \int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n \nabla T_k(u_n - u_\lambda)\,\mathrm{d}x\bigg] \\ - \lambda \int_\Omega u_n^{-\gamma}\, T_k(u_n - u_\lambda)\,\mathrm{d}x - \int_\Omega u_n^{p^*-1} T_k(u_n - u_\lambda)\,\mathrm{d}x := I - J - K \quad \text{as } n \to \infty. \end{aligned} \tag{3.35}$$

Using Young's inequality, Propositions 2.1(iii)–(iv), 2.2(ii) and boundedness of the sequences $\{u_n\}_{n\in\mathbb{N}}, \{T_k(u_n - u_\lambda)\}_{n\in\mathbb{N}}$ in $W_0^{1,\mathcal{H}}(\Omega)$, we obtain

$$\begin{aligned} |J| &\le |I| + |K| + o_n(1) \\ &\le (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \int_\Omega |\nabla u_n|^{p-1}|\nabla T_k(u_n - u_\lambda)|\,\mathrm{d}x \\ &\quad + \int_\Omega a(x)|\nabla u_n|^{q-1}|\nabla T_k(u_n - u_\lambda)|\,\mathrm{d}x \\ &\quad + \int_\Omega |u_n|^{p^*-1}|T_k(u_n - u_\lambda)|\,\mathrm{d}x + o_n(1) \\ &\le (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n))\big(\rho_{\mathcal{H}}(\nabla u_n) + \rho_{\mathcal{H}}(\nabla T_k(u_n - u_\lambda))\big) \\ &\quad + k\int_\Omega u_n^{p^*-1}\,\mathrm{d}x + o_n(1) \le C(1+k) \end{aligned} \tag{3.36}$$

with a constant C independent of n and k, that is, the sequence $\{u_n^{-\gamma}\ T_k(u_n-u_\lambda)\}_{n\in\mathbb{N}}$ is uniformly integrable. Then, using (3.10) and Vitali's convergence theorem, we get

$$\int_\Omega u_n^{-\gamma}\ T_k(u_n-u_\lambda)\,\mathrm{d}x \to 0.$$

By Hölder's inequality, we observe that

$$[L^{\mathcal{H}}(\Omega)]^N \ni g \longmapsto \int_\Omega \left(|\nabla u_\lambda|^{p-2}+a(x)|\nabla u_\lambda|^{q-2}\right)\nabla u_\lambda\cdot g\,\mathrm{d}x$$

is a bounded linear functional. From (3.10), we see that $\nabla T_k(u_n-u_\lambda)\rightharpoonup 0$ in $[L^{\mathcal{H}}(\Omega)]^N$, so we can get

$$\lim_{n\to\infty}\int_\Omega \left(|\nabla u_\lambda|^{p-2}+a(x)|\nabla u_\lambda|^{q-2}\right)\nabla u_n\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x=0. \tag{3.37}$$

Let $\phi_{\mathcal{H}}(\nabla u_n)\to\beta:=\frac{E_1}{p}+\frac{E_2}{q}$ as $n\to\infty$, where E_1 and E_2 are defined in (3.11). Thus, by using (3.36)–(3.37) in (3.35) and the fact that $a_0\geq 0$, $b_0>0$, $\beta>0$, we get

$$\begin{aligned}(a_0+b_0\beta^{\theta-1})\limsup_{n\to\infty}\Bigg[&\int_\Omega\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x\\
&+\int_\Omega a(x)\left(|\nabla u_n|^{q-2}\nabla u_n-|\nabla u_\lambda|^{q-2}\nabla u_\lambda\right)\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x\Bigg]\\
&=\limsup_{n\to\infty}\int_\Omega u_n^{p^*-1}T_k(u_n-u_\lambda)\,\mathrm{d}x\leq Ck.\end{aligned}$$

By Simon's inequalities, see [24, formula (2.2)], we rewrite the above estimate as

$$\begin{aligned}&\limsup_{n\to\infty}\left[\int_\Omega\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot\nabla T_k(u_n-u_\lambda)\,\mathrm{d}x\right]\\
&\leq\frac{Ck}{(a_0+b_0\beta^{\theta-1})}.\end{aligned} \tag{3.38}$$

Set

$$s_n(x)=\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u_\lambda|^{p-2}\nabla u_\lambda\right)\cdot\nabla(u_n-u_\lambda).$$

Note that $s_n(x)\geq 0$ a. e. in Ω. We divide the set Ω by

$$E_n^k=\{x\in\Omega\,:\,|u_n(x)-u_\lambda(x)|\leq k\}\ \text{ and }\ F_n^k=\{x\in\Omega\,:\,|u_n(x)-u_\lambda(x)|>k\},$$

where $k,n\in\mathbb{N}$ are fixed. Let $\eta\in(0,1)$. Then, from the definition of T_k, Hölder's inequality, (3.38) and the fact that $\lim_{n\to\infty}|F_n^k|=0$, we get

$$\begin{aligned}\limsup_{n\to\infty}\int_\Omega s_n^\eta\,\mathrm{d}x&\leq\limsup_{n\to\infty}\left(\int_{E_n^k}s_n\,\mathrm{d}x\right)^\eta|E_n^k|^{1-\eta}+\limsup_{n\to\infty}\left(\int_{F_n^k}s_n\,\mathrm{d}x\right)^\eta|F_n^k|^{1-\eta}\,\mathrm{d}x.\\
&\leq\left(\frac{Ck}{(a_0+b_0\beta^{\theta-1})}\right)^\eta|\Omega|^{1-\eta}.\end{aligned}$$

Letting $k \to 0^+$, we obtain that $s_n^\eta \to 0$ in $L^1(\Omega)$. Thus, we may assume that $s_n \to 0$ a. e. in Ω (up to a subsequence) which along with Simon's inequalities [24, formula (2.2)] gives that

$$\nabla u_n \to \nabla u_\lambda \quad \text{a. e. in } \Omega. \tag{3.39}$$

Let M be the nodal set of the weight function $a(\cdot)$ given by

$$M := \{x \in \Omega : a(x) = 0\}.$$

Since, the sequences $\{|\nabla u_n|^{p-2}\nabla u_n\}_{n\in\mathbb{N}}$ and $\{|\nabla u_n|^{q-2}\nabla u_n\}_{n\in\mathbb{N}}$ are bounded in $L^{p'}(\Omega)$ and $L^{q'}(\Omega \setminus M, a(x)\,\mathrm{d}x)$, respectively, then by using (3.39) and [3, Proposition A.8], we conclude that

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u_\lambda = \|\nabla u_\lambda\|_p^p$$

and

$$\int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla u_\lambda = \int_{\Omega\setminus M} a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla u_\lambda = \|\nabla u_\lambda\|_{q,a}^q.$$

Furthermore, using (3.10), (3.39) and the Brezis-Lieb Lemma, we obtain

$$\begin{aligned} &\rho_{\mathcal{H}}(\nabla u_n) - \rho_{\mathcal{H}}(\nabla u_n - \nabla u_\lambda) = \rho_{\mathcal{H}}(\nabla u_\lambda) + o_n(1), \\ &\|u_n\|_{p^*}^{p^*} - \|u_n - u_\lambda\|_{p^*}^{p^*} = \|u_\lambda\|_{p^*}^{p^*} + o_n(1) \end{aligned} \tag{3.40}$$

as $n \to \infty$. Let $\|u_n - u_\lambda\|_{p^*} \to \ell$ for some $\ell \geq 0$. Now, by taking $u_n - u_\lambda$ as a test function in (3.26), we get

$$\begin{aligned} o_n(1) &= (a_0 + b_0\phi_{\mathcal{H}}^{\theta-1}(\nabla u_n)) \\ &\quad \left[\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla(u_n - u_\lambda)\,\mathrm{d}x + \int_\Omega a(x)|\nabla u_n|^{q-2}\nabla u_n \cdot \nabla(u_n - u_\lambda)\,\mathrm{d}x\right] \\ &\quad - \lambda \int_\Omega u_n^{-\gamma}\,(u_n - u_\lambda)\,\mathrm{d}x - \int_\Omega u_n^{p^*-1}(u_n - u_\lambda)\,\mathrm{d}x \\ &= (a_0 + b_0\beta^{\theta-1})\big[\rho_{\mathcal{H}}(\nabla u_n) - \rho_{\mathcal{H}}(\nabla u_\lambda) + o_n(1)\big] - \|u_n\|_{p^*}^{p^*} + \|u_\lambda\|_{p^*}^{p^*} + o_n(1) \end{aligned}$$

as $n \to \infty$. Hence, by (3.10) and (3.40) it follows that

$$(a_0 + b_0\beta^{\theta-1})\big[\rho_{\mathcal{H}}(\nabla u_n) - \rho_{\mathcal{H}}(\nabla u_\lambda)\big] = \ell^{p^*} + o_n(1) \quad \text{as } n \to \infty, \tag{3.41}$$

which further gives

$$(a_0 + b_0\beta^{\theta-1}) \lim_{n\to\infty} \left(\|\nabla u_n - \nabla u_\lambda\|_p^p + \|\nabla u_n - \nabla u_\lambda\|_{q,a}^q\right) \leq \ell^{p^*}. \tag{3.42}$$

Now, we claim that $\ell = 0$. Assume by contradiction that $\ell > 0$. By (3.1) and (3.42), we have

$$Sa_0\ell^p \leq S(a_0 + b_0\beta^{\theta-1})\ell^p \leq (a_0 + b_0\beta^{\theta-1}) \lim_{n\to\infty} \|\nabla u_n - \nabla u_\lambda\|_p^p \leq \ell^{p^*}. \tag{3.43}$$

Note that (3.42) implies that

$$(a_0 + b_0\beta^{\theta-1})(E_1 + E_2 - \|\nabla u_\lambda\|_p^p - \|\nabla u_\lambda\|_{q,a}^q) \le \ell^{p^*}. \tag{3.44}$$

Using (3.43) in (3.44), we get

$$\begin{aligned}
\left(\ell^{p^*}\right)^{\frac{p^*-p}{p}} &\ge (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} (E_1 + E_2 - \|\nabla u_\lambda\|_p^p - \|\nabla u_\lambda\|_{q,a}^q)^{\frac{p^*-p}{p}} \\
&= (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \lim_{n\to\infty} \left(\|\nabla u_n - \nabla u_\lambda\|_p^p + \|\nabla u_n - \nabla u_\lambda\|_{q,a}^q\right)^{\frac{p^*-p}{p}} \\
&\ge (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \lim_{n\to\infty} \left(\|\nabla u_n - \nabla u_\lambda\|_p^p\right)^{\frac{p^*-p}{p}} \\
&\ge S^{\frac{p^*-p}{p}} (a_0 + b_0\beta^{\theta-1})^{\frac{p^*-p}{p}} \ell^{p^*-p} \\
&\ge S^{\frac{p^*}{p}} (a_0 + b_0\beta^{\theta-1})^{\frac{p^*}{p}}.
\end{aligned} \tag{3.45}$$

From (3.45) and (3.1), we obtain

$$\begin{aligned}
E_1^{\frac{p^*-p}{p}} &\ge \left(E_1 - \|\nabla u_\lambda\|_p^p\right)^{\frac{p^*-p}{p}} \\
&= \left(\lim_{n\to\infty} \|\nabla u_n - \nabla u_\lambda\|_p^p\right)^{\frac{p^*-p}{p}} \ge S^{\frac{p^*-p}{p}} \ell^{p^*-p} \\
&\ge S^{\frac{p^*}{p}} (a_0 + b_0\beta^{\theta-1}).
\end{aligned}$$

This gives

$$E_1 \ge S^{\frac{p^*}{p^*-p}} (a_0 + b_0\beta^{\theta-1})^{\frac{p}{p^*-p}} \ge S^{\frac{p^*}{p^*-p}} \left(\frac{b_0}{p^{\theta-1}}\right)^{\frac{p}{p^*-p}} E_1^{\frac{(\theta-1)p}{p^*-p}}$$

and so we have

$$E_1 \ge \left[S^{p^*} \left(\frac{b_0}{p^{\theta-1}}\right)^p\right]^{\frac{1}{p^*-p\theta}}. \tag{3.46}$$

Combining (3.45) and (3.46), we obtain

$$\begin{aligned}
\ell^{p^*} \ge S^{\frac{p^*}{p^*-p}} (a_0 + b_0\beta^{\theta-1})^{\frac{p^*}{p^*-p}} &\ge \left(\frac{Sb_0}{p^{\theta-1}}\right)^{\frac{p^*}{p^*-p}} E_1^{\frac{(\theta-1)p^*}{p^*-p}} \\
&\ge \left(\frac{Sb_0}{p^{\theta-1}}\right)^{\frac{p^*}{p^*-p}} \left[S^{p^*} \left(\frac{b_0}{p^{\theta-1}}\right)^p\right]^{\frac{(\theta-1)p^*}{(p^*-p\theta)(p^*-p)}}.
\end{aligned} \tag{3.47}$$

For any $n \in \mathbb{N}$, we have

$$J_\lambda(u_n) - \frac{1}{q\theta}\langle J'_\lambda(u_n), u_n\rangle = a_0\phi_{\mathcal{H}}(\nabla u_n) + \frac{b_0}{\theta}\phi^\theta_{\mathcal{H}}(\nabla u_n) - \frac{1}{q\theta}m(\phi_{\mathcal{H}}(\nabla u_n))\left\langle \mathcal{L}^a_{p,q}(u_n), u_n\right\rangle$$
$$- \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q\theta}\right)\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x + \left(\frac{1}{q\theta} - \frac{1}{p^*}\right)\int_\Omega u_n^{p^*}\,\mathrm{d}x$$
$$\geq \left(\frac{1}{q\theta} - \frac{1}{p^*}\right)\|u_n\|^{p^*}_{p^*} - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q\theta}\right)\int_\Omega u_n^{1-\gamma}\,\mathrm{d}x.$$

From this, as $n \to \infty$, by (3.47), (3.40), Hölder's and Young's inequality, we derive

$$c = \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{q\theta}\langle J'_\lambda(u_n), u_n\rangle\right)$$
$$\geq \alpha_0\left(\ell^{p^*} + \|u_\lambda\|^{p^*}_{p^*}\right) - \lambda\left(\frac{1}{1-\gamma} - \frac{1}{q\theta}\right)|\Omega|^{\frac{p^*-1+\gamma}{p^*}}\|u_\lambda\|^{1-\gamma}_{p^*}$$
$$\geq \alpha_0\ell^{p^*} - \alpha_1\lambda^{\frac{p^*}{p^*-1+\gamma}}$$
$$\geq \alpha_0\left(\frac{Sb_0}{p^{\theta-1}}\right)^{\frac{p^*}{p^*-p}}\left[S^{p^*}\left(\frac{b_0}{p^{\theta-1}}\right)^p\right]^{\frac{(\theta-1)p^*}{(p^*-p\theta)(p^*-p)}} - \alpha_1\lambda^{\frac{p^*}{p^*-1+\gamma}} = c_\lambda,$$

where α_0, α_1 are defined in (3.32). The above estimates gives a contradiction to (3.34). Hence $\ell = 0$ and using (3.41) and Proposition 2.1(v), we conclude the proof. □

Remark 3.11 By taking $\lambda \in (0, \Lambda_*)$ with $\Lambda_* := \left(\alpha_2\alpha_1^{-1}\right)^{\frac{p^*-1+\gamma}{p^*}}$ and α_1, α_2 are defined in (3.32) and (3.33) respectively, we have $c_\lambda > 0$.

Proof (Proof of Theorem 1.1) Fix $\lambda < \lambda^* := \min\{\Lambda^*, \Lambda_*\}$. From Lemma 3.1(ii) and Ekeland's variational principle there exists a minimizing sequence $\{u_n\}_{n\in\mathbb{N}} \in \mathcal{N}^+_\lambda \setminus \{0\}$ verifying (3.8), (3.9), (3.10) and (3.34) with $c = \Theta^+_\lambda$. Hence, by combining Propositions 3.4 and 3.10 , we obtain $u_n \to u_\lambda$ strongly in $W^{1,\mathcal{H}}_0(\Omega)$ (up to a subsequence). This further implies that $u_\lambda \in \mathcal{N}_\lambda$ and by Lemma 3.7, we get $u_\lambda \in \mathcal{N}^+_\lambda$ with u_λ achieving Θ^+_λ since J_λ is continuous on $W^{1,\mathcal{H}}_0(\Omega)$. Since $0 \notin \mathcal{N}^+_\lambda$ and $u_n \geq 0$ we have $u_\lambda \not\equiv 0$ and $u_\lambda \geq 0$. Letting $n \to \infty$ in (3.26), we obtain that u satisfies $u_\lambda^{-\gamma}\varphi \in L^1(\Omega)$ and

$$m(\phi_{\mathcal{H}}(\nabla u_\lambda))\left\langle \mathcal{L}^a_{p,q}(u_\lambda), \varphi\right\rangle = \lambda\int_\Omega u_\lambda^{-\gamma}\varphi\,\mathrm{d}x + \int_\Omega u_\lambda^{r-1}\varphi\,\mathrm{d}x$$

for all $\varphi \in W^{1,\mathcal{H}}_0(\Omega)$. Finally, by using Proposition 3.4, Lemma 3.5 and by repeating the proof of [2, Proposition 4.3 and Proposition 4.4, Step 1], we obtain $u_\lambda > 0$ a. e. in Ω. □

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