



The boundary dual of the bulk symplectic form

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ABSTRACT

In this paper, we study the overlaps of wavefunctionals prepared by turning on sources in the Euclidean path integral. For nearby states, these overlaps give rise to a Kähler structure on the space of sources, which is naturally induced by the Fubini–Study metric. The Kähler form obtained this way can also be thought of as a Berry curvature and, for holographic field theories, we show that it is identical to the gravitational symplectic form in the bulk. We discuss some possible applications of this observation, in particular a boundary prescription to calculate the variation of the volume of a maximal slice.

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1. Introduction

The Ryu–Takayanagi formula [1] opened an intriguing connection between quantum gravity and quantum information. Since then, there have been many advances in understanding the holographic duals of quantum information measures in AdS/CFT, see [2] for a review. Most of these quantities are tied to subregions or mixed states and there has not been much progress in understanding measures associated with pure states. While there have been some proposals for the duals of complexity or fidelity [3–5], none of these have been derived. It is therefore highly desirable to understand better the bulk duals of boundary quantities that only depend on wave functionals, hoping that they will teach us about the structure of the Hilbert space in quantum gravity. In this paper, we are going to give a precise mapping between the antisymmetrized overlap of states (naturally understood as a Kähler form in the space of state deformations) and the bulk symplectic form. This generalizes the work of [5] and identifies the precise bulk dual. The states in consideration are prepared by turning on arbitrary (possibly complex) Euclidean sources, which correspond to classical geometries. This new entry in the dictionary has similarities with the relative entropy for subregions: it compares two neighbouring states and the bulk and boundary quantities are equal [6]. We will show that our result has implications for the dual of the

volume of the extremal time slice, which we will explore further in [7].

2. Symplectic form from Fubini–Study metric

Suppose we have a map from a complex manifold \mathcal{M} to the Hilbert space \mathcal{H} . Let the manifold have complex coordinates α_i, α_i^* and the associated state be $|\alpha\rangle$. We assume that the map is “holomorphic” (i.e. the complex structure on the manifold and the Hilbert space is compatible) in the sense that

$$\partial_{\alpha_i^*} |\alpha\rangle = 0, \quad \partial_{\alpha_i} \langle\alpha| = 0. \quad (1)$$

Note that the $|\alpha\rangle$ states are thus necessarily un-normalized. We can pull back the Fubini–Study metric to obtain a line element on the manifold \mathcal{M}

$$\begin{aligned} ds^2 &= \frac{\langle\delta\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle} - \frac{|\langle\psi|\delta\psi\rangle|^2}{\langle\psi|\psi\rangle^2} \\ &= \left(\frac{\partial_{\alpha_i^*} \langle\alpha|\partial_{\alpha_j}|\alpha\rangle}{\langle\alpha|\alpha\rangle} - \frac{\partial_{\alpha_i^*} \langle\alpha|\alpha\rangle \langle\alpha|\partial_{\alpha_j}|\alpha\rangle}{\langle\alpha|\alpha\rangle^2} \right) d\alpha_i^* d\alpha_j \\ &= \partial_{\alpha_i} \partial_{\alpha_j^*} \log \langle\alpha|\alpha\rangle d\alpha_i d\alpha_j^*. \end{aligned} \quad (2)$$

We see that the conditions (1) ensure that the manifold \mathcal{M} endowed with the pullback of the FS metric is Kähler, with Kähler potential

$$\mathcal{K} = \log \langle\alpha|\alpha\rangle. \quad (3)$$

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Therefore, there also exists a closed 2-form, the Kähler form on \mathcal{M}

$$\Omega = i \partial_{\alpha_i} \partial_{\alpha_j^*} \log \langle \alpha | \alpha \rangle d\alpha_i \wedge d\alpha_j^*. \quad (4)$$

Since a Kähler form is always a symplectic form, we deduce that \mathcal{M} is symplectic. In the following, we denote globally the coordinates of \mathcal{M} by $\tilde{\alpha} = (\alpha, \alpha^*)$ and most of the time, we will plug explicit variations into the symplectic form

$$\Omega(\delta_1 \tilde{\alpha}, \delta_2 \tilde{\alpha}) = i \partial_{\alpha_i} \partial_{\alpha_j^*} \log \langle \alpha | \alpha \rangle (\delta_1 \alpha_i \delta_2 \alpha_j^* - 1 \leftrightarrow 2). \quad (5)$$

The above symplectic form can be interpreted as the Berry curvature 2-form associated to the Berry connection [8] on the parameter space \mathcal{M}

$$\mathcal{A} = i \langle \Psi_\alpha | d | \Psi_\alpha \rangle, \quad | \Psi_\alpha \rangle = \frac{|\alpha\rangle}{\sqrt{\langle \alpha | \alpha \rangle}}. \quad (6)$$

Indeed, in terms of the Kähler potential (3)

$$\mathcal{A} = \frac{i}{2} \partial_{\alpha_i} \mathcal{K} d\alpha_i - \frac{i}{2} \partial_{\alpha_i^*} \mathcal{K} d\alpha_i^* = \frac{i}{2} (\partial - \partial^*) \mathcal{K}, \quad (7)$$

therefore $\Omega = d\mathcal{A}$.

As a warm up example, let us apply this machinery to the simple harmonic oscillator. Conditions (1) are satisfied by the unnormalized coherent states

$$|\alpha\rangle = e^{\alpha a^\dagger} |0\rangle, \quad \langle \alpha | \alpha \rangle = e^{|\alpha|^2}. \quad (8)$$

This results in the Kähler potential $\mathcal{K} = |\alpha|^2$ and symplectic form

$$\Omega = i d\alpha \wedge d\alpha^* = -2dp \wedge dq, \quad (9)$$

where we have written the last line in terms of real coordinates $d\alpha = dq + idp$.

3. Quantum field theory

This construction automatically gives a symplectic form on the space of sources in any QFT. Define

$$|\lambda\rangle = T e^{-\int_{t_E < 0} dt E d^{d-1} \tilde{x} \lambda(t_E, \tilde{x}) O(t_E, \tilde{x})} |0\rangle, \quad (10)$$

$$\langle \lambda | = \langle 0 | T e^{-\int_{t_E > 0} dt E d^{d-1} \tilde{x} \lambda^*(-t_E, \tilde{x}) O^\dagger(t_E, \tilde{x})}.$$

Here, t_E is Euclidean time and \tilde{x} are the spatial coordinates. These states should be thought of as being prepared by a path integral. In a CFT, the wavefunctional of the vacuum is obtained by path integrating over half of a S^d . The states $|\lambda\rangle$ are obtained by inserting additional sources on this manifold. Note that these two states only depend on $\lambda(t_E), \lambda^*(t_E)$ for $t_E < 0$ and we treat these two functions independently. The symbol T denotes Euclidean time ordering. When $\lambda(t_E, \tilde{x})$ is independent of t_E , one can think of these states as the ground states of the deformed Hamiltonian

$$H = H_{CFT} + \int d^{d-1} \tilde{x} \lambda(\tilde{x}) O(0, \tilde{x}). \quad (11)$$

More generally, in holographic theories we expect states corresponding to classical geometries to be of the form $|\lambda\rangle$ and they are completely determined by the value of the sources [9]. Around any such state, we expect that for small variations of λ (compared to the classical background), these states correspond to bulk coherent states [10–12]. Note that using this basis of Euclidean sources to parametrise states is quite redundant and we expect these states to have non-zero overlap.

The $|\lambda\rangle$ states satisfy (1). They give rise to the Kähler potential $\mathcal{K} = \log Z[\tilde{\lambda}]$, where $Z[\tilde{\lambda}]$ is the partition function with sources

$$\tilde{\lambda}(x) = \begin{cases} \lambda(x) & t_E^x < 0 \\ \lambda^*(x^T) & t_E^x > 0, \end{cases} \quad (12)$$

where we use the shorthand $x^T = (-t_E^x, \tilde{x})$ for coordinates reflected in Euclidean time. We think about the Kähler potential as a functional of half-sided sources (λ, λ^*) and obtain the Kähler form via (4)

$$\begin{aligned} \Omega(\delta_1 \tilde{\lambda}, \delta_2 \tilde{\lambda}) &= i(\delta_1^* \delta_2 - \delta_2^* \delta_1) \log Z[\tilde{\lambda}] \\ &= i \int_{\substack{t_E^x > 0 \\ t_E^y < 0}} dx dy G_\lambda^c(x, y) \delta \lambda_{[1}^* \delta \lambda_{2]}(x^T, y), \end{aligned} \quad (13)$$

$$\delta \lambda_{[1}^* \delta \lambda_{2]}(x^T, y) = \delta \lambda_1^*(x^T) \delta \lambda_2(y) - \delta \lambda_1(y) \delta \lambda_2^*(x^T),$$

where we have defined the connected two point function $G_\lambda^c(x, y) = \langle O^\dagger(x) O(y) \rangle_{\tilde{\lambda}} - \langle O^\dagger(x) \rangle_{\tilde{\lambda}} \langle O(y) \rangle_{\tilde{\lambda}}$, and the $\langle \cdot \rangle_{\tilde{\lambda}}$ denotes (normalized) expectation value with the sources $\tilde{\lambda}$ turned on. Alternatively, we may write this in terms of the change in the one point function $\delta \langle O(x) \rangle = \int dy G_\lambda^c(x, y) \delta \lambda(y)$ as

$$\Omega(\delta_1 \tilde{\lambda}, \delta_2 \tilde{\lambda}) = i \int_{t_E > 0} dx (\delta \lambda_1^* \delta_2 \langle O \rangle - \delta \lambda_2^* \delta_1 \langle O \rangle). \quad (14)$$

Besides the Kähler form, a Kähler manifold has a complex structure and a Kähler metric. The complex structure J is naturally inherited from the property of the CFT inner product $\langle \lambda_1 | \lambda_2 \rangle = -\langle i \lambda_1 | \lambda_2 \rangle = i \langle \lambda_1 | \lambda_2 \rangle$, and acts on variations as

$$J: \delta \lambda \mapsto i \delta \lambda, \quad \delta \lambda^* \mapsto -i \delta \lambda^*. \quad (15)$$

The Kähler metric is given by the symmetric part of the double variation

$$\begin{aligned} g(\delta_1 \tilde{\lambda}, \delta_2 \tilde{\lambda}) &= \frac{1}{2} \Omega(\delta_1 \tilde{\lambda}, J(\delta_2 \tilde{\lambda})) \\ &= \frac{1}{2} (\delta_1 \delta_2^* + \delta_2 \delta_1^*) \log Z[\tilde{\lambda}]. \end{aligned} \quad (16)$$

We note that when we only source marginal operators and take the sources to be constant, the metric (16) is proportional to the usual Zamolodchikov metric on the conformal manifold [13,14].

4. Holographic theories

Here we show that (13) is the bulk symplectic form for holographic theories. We assume O is a single trace operator dual to a bulk field ϕ , which can have spin but we will omit the indices for simplicity. The standard dictionary [9] tells us that to leading order in N , the overlap between states is just computed by the gravitational action in the presence of sources

$$\langle \lambda | \lambda \rangle \equiv Z_{CFT}[\tilde{\lambda}] = e^{-S_{\text{grav}}^{\text{on-shell}}[\tilde{\lambda}]}, \quad (17)$$

where $\tilde{\lambda}$, defined in (12), sets the boundary condition for the bulk fields. Therefore the Kähler potential is $\mathcal{K} = -S_{\text{grav}}^{\text{on-shell}}[\tilde{\lambda}]$. We now want to calculate (13). Let us study the variations of the on-shell gravitational action with respect to the boundary conditions for this. The variation of the Lagrangian density satisfies

$$\delta(\mathcal{L} d^{d+1}x) = -E_\phi \delta \phi d^{d+1}x + d\theta(\phi, \delta \phi), \quad (18)$$

where θ is the symplectic one-form density [15,16], which is a one-form in the space of field variations and a d form in spacetime, and E_ϕ is the equation of motion. Since we want to work with

on-shell configurations, when we integrate this formula over the Euclidean asymptotically AdS bulk X , the whole contribution will be a boundary term. In this way the only physical variations are those of the boundary condition $\delta\tilde{\lambda}$. This gives

$$\tilde{\delta}S_{\text{grav}}^{\text{on-shell}}[\tilde{\lambda}] = \int_{\partial X=S^d} \theta(\tilde{\lambda}, \delta\tilde{\lambda}). \quad (19)$$

Now because of (12) and linearity of θ in the variation, if we do a strictly (anti)holomorphic variation, the integral localizes on the (upper)lower hemisphere

$$\delta_1 S = \int_{t_E < 0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_1), \quad \delta_1^* S = \int_{t_E > 0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_1). \quad (20)$$

We then obtain the Kähler form using (13) and the fact that variations commute

$$\begin{aligned} \Omega(\delta_1 \tilde{\lambda}, \delta_2 \tilde{\lambda}) &= i([\delta_1 + \delta_1^*]\delta_2^* - [\delta_2 + \delta_2^*]\delta_1^*)S_{\text{grav}}^{\text{on-shell}}[\tilde{\lambda}] \\ &= i \left[\tilde{\delta}_1 \int_{t_E > 0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_2) - \tilde{\delta}_2 \int_{t_E > 0} \theta(\tilde{\lambda}, \delta\tilde{\lambda}_1) \right]. \end{aligned} \quad (21)$$

Now we use the extrapolate dictionary, relating sources $\tilde{\lambda}$ to the boundary values of the dual bulk fields ϕ . Note that ϕ can be the metric when we are sourcing the stress tensor. Using that the bulk symplectic 2-form density is

$$\omega_{\text{bulk}}(\phi; \delta\phi_1, \delta\phi_2) = \delta_1 \theta(\phi, \delta_2 \phi) - \delta_2 \theta(\phi, \delta_1 \phi), \quad (22)$$

we arrive at

$$\Omega = i \int_{(\partial X)^+} \omega_{\text{bulk}} = i \int_{t_E > 0} dx (\delta\lambda_1^* \delta_2 \langle O \rangle - \delta\lambda_2^* \delta_1 \langle O \rangle) \quad (23)$$

where $(\partial X)^+$ denotes the $t_E > 0$ part of the boundary of X . One could write an equivalent formula using only the lower part $(\partial X)^-$. In (23), using the usual dictionary we think of $\delta\lambda, \delta\langle O \rangle$ (which are canonical conjugates) in terms of the asymptotic values of $\delta\phi$.

4.1. Pushing into the bulk

The expression derived above is integrated on part of the boundary of the Euclidean manifold, while the gravitational symplectic form should be integrated on a Cauchy slice. When the fields satisfy the equations of motion, the symplectic form is conserved $d\omega_{\text{bulk}}(\phi, \delta\phi_1, \delta\phi_2) = 0$ and can be pushed to other codimension 1 surfaces. The relation between $\delta\phi$ and the variations of the sources is formally given by the Euclidean boundary to bulk propagator $G_E(Y|y) = \langle \Phi(Y) O(y) \rangle_{\tilde{\lambda}}$:

$$\delta\phi(Y) = \delta\phi^+(Y) + \delta\phi^-(Y) \quad (24)$$

with

$$\delta\phi^\pm(Y) = \int_{(\partial X)^\pm} dy G_E(Y|y) \delta\tilde{\lambda}(y) \quad (25)$$

with $\tilde{\lambda}$ defined in (12). Here, Y are bulk coordinates and we are using the standard dictionary which relates bulk classical configurations to boundary sources.

We can now push the surface $(\partial X)^+$ in (23) to the Euclidean bulk. When ∂X is a sphere, $(\partial X)^+$ is simply the northern hemisphere such that we can push it to any surface Σ that is anchored

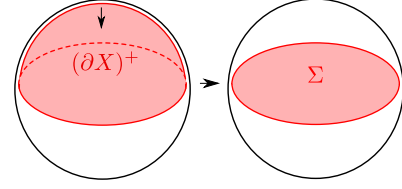


Fig. 1. Using conservation we can push the symplectic form from the Euclidean upper hemisphere $(\partial X)^+$ to an arbitrary slice Σ anchored at $t_E = 0$.

at $t_E = 0$, see Fig. 1. However, to relate it to the bulk symplectic structure, we need to integrate the symplectic flux on a Lorentzian initial value surface. Consider the case when the background λ is real, the boundary condition $\tilde{\lambda}$ (and hence the bulk configuration) is Z_2 symmetric and there is a Z_2 invariant slice Σ_0 that is special. This slice continues nicely to Lorentzian since all fields are real there and their time derivatives are all zero because of the Z_2 symmetry. Therefore, we can pick variations in the complexified tangent space that correspond to real Lorentzian initial data. Reality of this data is precisely the requirement that $\delta\lambda^*$ is the complex conjugate of $\delta\lambda$. One can see this by looking at (24) and using that Z_2 symmetry of the slice implies $G_E(Y_0|y) = G_E(Y_0|y^T)$ and $\partial_n G_E(Y_0|y) = -\partial_n G_E(Y_0|y^T)$ (here ∂_n is the normal derivative to Σ_0) for any Y_0 lying on Σ_0 , from which it follows that

$$\delta\varphi(Y_0) = \text{Re}[\delta\phi^+(Y_0)], \quad \delta\pi(Y_0) = \text{Im}[\partial_n \delta\phi^+(Y_0)]. \quad (26)$$

Here, π are the conjugate momenta to the initial data φ , as defined through the bulk Lagrangian. Note that we use ϕ for time dependent field configurations and φ for initial data. In this way, the main result of the paper is:

$$\Omega(\delta\tilde{\lambda}_1, \delta\tilde{\lambda}_2) = \int_{\Sigma_0} \omega_{\text{Lor}}(\phi, \delta\phi_1, \delta\phi_2), \quad (27)$$

that is, the boundary symplectic form is given in the bulk by the symplectic form of the initial data in the Lorentzian $t = 0$ time slice Σ_0 and the position and momenta are determined by $\delta\phi$ through (26).

To create more general states, we need to consider complex λ . In this case, the boundary $t = 0$ surface has $Z_2 + \mathcal{C}$ symmetry, where \mathcal{C} is complex conjugation. If we assume that this symmetry extends to the bulk such that it acts pointwise, we can consider surfaces Σ which are fixed by this symmetry. By construction, one has $\text{Im}\varphi = 0$ and $\text{Re}\pi = 0$ on such surfaces. Similarly, the extrinsic curvature must be imaginary, while the induced metric must be real on the surface. These therefore correspond to real Lorentzian initial data. The Lorentzian variations (24) are real again if $\delta\lambda^*$ is the complex conjugate of $\delta\lambda$, provided we assume that G_E is invariant under the action of $Z_2 + \mathcal{C}$. Given that these $Z_2 + \mathcal{C}$ symmetric geometries are complex, there is in principle a family of such geometries, which correspond to different choices of bulk Lorentzian slices, since Σ will not have vanishing extrinsic curvature tensor. From the requirement of having well defined Lorentzian data, it seems like the maximal volume slice should be preferred [17]. We will analyze these situations in more detail in [7].

4.2. Complex structure and Kähler metric

From (24), the action of the complex structure (15) is nonlocal on the bulk variations. Regardless, it still has a nice interpretation, as hinted from calculating the norm of a variation in the Kähler metric (16)

$$g(\delta_1 \tilde{\lambda}, \delta_1 \tilde{\lambda}) = \frac{1}{2} \Omega(\delta_1 \tilde{\lambda}, J(\delta_1 \tilde{\lambda})) \\ = \int_{\Sigma} \omega_{\text{Lor}}(\phi, \delta_1 \phi^-(Y), [\delta_1 \phi^-(Y^T)]^*), \quad (28)$$

where in the last line we have assumed that the slice Σ is $Z_2 + \mathcal{C}$ symmetric, which implies that when Y is close to Σ , we have $G_E^*(Y^T|y^T) = G_E(Y|y)$. We used the action of J on the sources (15) and we dropped the diagonal terms $\delta_1 \lambda \delta_1 \lambda$ and $\delta_1 \lambda^* \delta_1 \lambda^*$ since we know from the field theory expression (13) that they must cancel. Notice that this expression, when continued to Lorentzian, is the Klein–Gordon norm for the projected variation $\delta_1 \phi^-$, defined in (25), which is sourced only from the lower hemisphere. By construction, the Kähler metric is positive definite, so we propose to interpret $\delta \phi^- = \frac{(1+J)}{2} \delta \phi$ as the generalization (in the absence of time translation symmetry) of the positive frequency part of the solution $\delta \phi$, which is important for defining the quantization of field theory in curved spacetime [18]. The relation between positive frequencies and negative Euclidean times is manifest for perturbations around empty AdS, where one can go to Fourier space [12]. Note that we do not have an explicit bulk alternative expression for the variation in the Kähler metric, other than projecting the field variation onto holomorphic sources, but it is natural to expect this to be the subspace on which the Klein–Gordon norm is positive.

5. Special variations

In this section, we will explore some particular examples of variations which have nice interpretations. We will consider the equation (27) directly in Lorentzian, for arbitrary states (not necessarily time reflection symmetric).

5.1. Time translations

We can run a basic sanity check on (27) by recovering Wald's conserved charges [19]. When the variation is a diffeomorphism, the symplectic form becomes a boundary term, which is interpreted as the boundary conserved energy. Let us take the variation of the bulk initial data to be a small translation along the normal vector field ξ to our Cauchy slice, i.e. $\delta \phi = \mathcal{L}_\xi \phi$. This is a pure diffeomorphism, but because it is non-trivial at the boundary, it acts physically on the Hilbert space: it is a time derivative. We can regard this variation as a purely imaginary complex source, since $|\psi(t)\rangle = e^{-iHt} |\psi\rangle$.¹ In this way, we have:

$$\Omega(\delta \tilde{\lambda}, \mathcal{L}_\xi \tilde{\lambda}) = (\delta_\lambda + \delta_{\lambda^*}) \frac{\langle \lambda | H | \lambda \rangle}{\langle \lambda | \lambda \rangle} \equiv \delta \langle H \rangle, \quad (29)$$

since the time derivative gets opposite sign contributions when acting on the holomorphic/antiholomorphic sources. Comparing with (27) gives

$$\delta \langle H \rangle = \int_{\Sigma} \omega_{\text{Lor}}(\phi, \delta \phi, \mathcal{L}_\xi \phi), \quad (30)$$

which is just the usual covariant phase space definition of the Hamiltonian.

¹ This is a boundary diffeomorphism at positive times, which is equivalent to inserting the Hamiltonian: $i \int_{t>0} dy \nabla^a \xi^b T_{ab} = i \int_{t=0} d\Sigma^b \xi^a T_{ab} = i H[\xi]$.

5.2. Volume of an extremal slice

In Einstein gravity, the symplectic form reads as

$$\Omega(\delta_1 \tilde{\lambda}, \delta_2 \tilde{\lambda}) = \int_{\Sigma} (\delta_1 h_{ab} \delta_2 p^{ab} - \delta_2 h_{ab} \delta_1 p^{ab}) \\ p_{ab} = \sqrt{h}(K_{ab} - h_{ab} K), \quad (31)$$

where h_{ab} is the induced metric on Σ and $K_{ab} = -\frac{1}{2}(\nabla_a n_b + \nabla_b n_a)$, is its extrinsic curvature (n_a is the outward pointing normal from Σ). Consider the Weyl-like transformation²

$$\delta_w h_{ab} = 0, \quad \delta_w K_{ab} = \alpha h_{ab}, \quad (32)$$

where α is a small, dimensionful parameter. Denoting $\delta \tilde{\lambda}_w$ the corresponding tangent space vector we have

$$\Omega(\delta_w \tilde{\lambda}, \delta \tilde{\lambda}) = \alpha (d-1) \int_{\Sigma} \sqrt{h} h^{ab} \delta h_{ab} \\ = 2\alpha (d-1) \delta V, \quad (33)$$

where V is the volume of the slice Σ . In order for (32) to be an allowed deformation of initial data, the perturbed data must satisfy the momentum and Hamiltonian constraints

$$\delta_w (\nabla^j K_{jk} - \nabla_k K_j^j) = \alpha (\nabla^j h_{jk} - \nabla_k h_j^j) = 0 \\ \delta_w (K_{ij} K^{ij} - (K)^2 - R_d - d(d-1)) = 2\alpha (d-1) K. \quad (34)$$

Note that in the first equation, $\delta_w \nabla_k = 0$ since these derivatives depend only on h_{ij} and $\delta_w h_{ij} = 0$. The second equation is not satisfied automatically, it requires $K = K_i^i = 0$, which is equivalent with the surface Σ being extremal. Therefore, we can only access the variation of the volume of extremal slices. In order to have a complete CFT description of this volume, we need to understand the variation of the boundary background metric ($\delta_w \lambda_{ab}, \delta_w \lambda_{ab}^*$) that induces (32). In the case where the geometry dual to the state $|\lambda\rangle$ is known, these are in principle obtainable from the standard dictionary (24). However, it is clearly desirable to have a more general understanding of these variations and why they are special. We will explore this and its relations with complexity [3] in more detail in [7].

This boundary expression for the volume is purely Euclidean and it is different from what one would get using HKLL [21] in Lorentzian: while the usual HKLL expression would be very complicated in the boundary – it depends on the change of the boundary vev's through a very non-trivial kernel – the present expression is simple in terms of the boundary sources and vevs. It seems unlikely that such a simplification could be obtained using Lorentzian techniques since it heavily relies on the use of the symplectic form.

6. Conclusions

In this work, we have studied the overlaps of nearby path integral states in holographic quantum field theories. We have found that in the field theory side, this gives rise to a Kähler form on the space of Euclidean sources which on the bulk side is dual to the gravitational symplectic form. This gives a precise duality between overlaps in the boundary and codimension one surfaces in the bulk. We have considered the case when the boundary has the topology of a ball, but it is clear that we can push the symplectic form for more complicated topologies, as long as the topology

² A similar transformation arises as a diffeomorphism for spheres in flat space in [20].

of the $t = 0$ slice is the same as that of the boundary for positive Euclidean times (see [22] for a set of coordinates which foliate geometries with arbitrary boundary topologies in $d = 2$). Of course, since our formalism is completely covariant, it applies to any theory of gravity using Wald's formalism [19], so this matching is true for general theories of gravity.

There are many immediate questions that we have not touched upon. In an upcoming work [7], we will discuss some of these, such as the JKM ambiguities in the bulk symplectic flux [23], bulk quantum corrections and a more detailed discussion of the dual description for the volume of the maximal slice. It would be interesting to investigate the relation of our work to the holographic renormalization group [24], where the relevant Euclidean evolution does not keep the $t = 0$ slice fixed.

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