Recent advances in equilibrium problems*

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1 Introduction: the equilibrium problem

The problem of interest is the so-called *equilibrium problem* $EP(f, C)$, which is defined as follows:

$$\text{find } \bar{x} \in C \text{ such that } f(\bar{x}, y) \geq 0 \text{ for all } y \in C,$$

where $C \subseteq \mathbb{R}^n$ is any nonempty closed set and the equilibrium bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in C$. Starting from the seminal papers [10, 11, 16] and following the Stampacchia tradition (see [4]), many researchers have worked on this topic. The purpose of these works was to study the interdependence between some particular problems as optimization, variational inequality, fixed point, complementarity, and noncooperative games. Different concepts of monotonicity for bifunctions provide key tools both for the theoretical investigations and the development of solution algorithms in this unified framework. The first and most exploited one is indeed called monotonicity, namely a bifunction f is said to be *monotone* on the set C if

$$f(x, y) + f(y, x) \leq 0, \qquad \forall x, y \in C.$$

We briefly review the formulation of the above problems as equilibria underlying the connection of monotonicity with suitable assumptions on the original problems.

1. Considering $\varphi : C \to \mathbb{R}$, the *minimization problem* asks for finding $\bar{x} \in C$ such that $\varphi(\bar{x}) \leq \varphi(y)$, for all $y \in C$, and it can be formulated as an equilibrium problem setting $f(x, y) = \varphi(y) - \varphi(x)$. The point $\bar{x}$ solves the optimization problem previously stated if and only if it solves $EP(f, C)$. Note that this bifunction f satisfies $f(x, y) + f(y, x) = 0$, hence it is monotone.

2. A special class of equilibrium problems, widely applied for studying e.g. economic equilibria, are *variational inequalities*. Given $F : C \to \mathbb{R}^n$, the variational inequality consists in finding a vector $\bar{x} \in C$ such that

$$\langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \qquad \forall y \in C.$$

This problem can be formulated as an equilibrium problem $EP(f, C)$ where $f(x, y) = \langle F(x), y - x \rangle$. In this case, the bifunction f results to be monotone if and only if F is monotone, that is

$$\langle F(x) - F(y), x - y \rangle \geq 0, \qquad \forall x, y \in C.$$

When the set C is a cone (i.e. $x \in C \Rightarrow \tau x \in C$ for all $\tau \geq 0$) the variational inequality problem is equivalent to the *complementarity problem*, i.e. finding $\bar{x} \in C$ such that $F(\bar{x}) \in C^*$ and $\langle F(\bar{x}), \bar{x} \rangle = 0$,

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where C^* is the dual cone of C:

$$C^* = \{d \in \mathbb{R}^n \ : \ \langle d, v \rangle \geq 0, \quad \forall v \in C\}.$$

Therefore, a complementarity problem can be viewed as an equilibrium problem.

3. We can also formulate *saddle point* problems in terms of equilibria. Recall that, given $\varphi : C_1 \times C_2 \to \mathbb{R}$ where $C_1 \subseteq \mathbb{R}^{n_1}$ and $C_2 \subseteq \mathbb{R}^{n_2}$ are convex sets, a point $(\bar{x}_1, \bar{x}_2) \in C_1 \times C_2$ is said to be a saddle point for φ if and only if

$$\varphi(\bar{x}_1, y_2) \leq \varphi(\bar{x}_1, \bar{x}_2) \leq \varphi(y_1, \bar{x}_2), \qquad \forall (y_1, y_2) \in C_1 \times C_2.$$

This problem can be equivalently formulated as a suitable $EP(f, C)$ choosing $C = C_1 \times C_2$ and $f((x_1, x_2), (y_1, y_2)) = \varphi(y_1, x_2) - \varphi(x_1, y_2)$. Note that this bifunction is monotone since

$$f((x_1, x_2), (y_1, y_2)) + f((y_1, y_2), (x_1, x_2)) = 0.$$

4. Other important problems that can be viewed in terms of equilibria are the *fixed point* problems. Let $\varphi : C \to C$ be given. A point $\bar{x} \in C$ is a fixed point for φ if and only if $\varphi(\bar{x}) = \bar{x}$. This problem can be equivalently formulated as an $EP(f, C)$ where $f(x, y) = \langle x - \varphi(x), y - x \rangle$. Note that f is monotone if and only if φ is non-expansive, i.e.

$$\langle \varphi(x) - \varphi(y), x - y \rangle \leq \|x - y\|^2, \qquad \forall x, y \in C.$$

5. Let us consider the *multiobjective optimization* problem

$$\min_{\text{int}\mathbb{R}^\ell_+} \{g(x) \ : \ x \in C\},$$

where $g = (g_1, g_2, \ldots, g_\ell) : \mathbb{R}^n \to \mathbb{R}^\ell$ and $C \subseteq \mathbb{R}^n$. The notation $\min_{\text{int}\mathbb{R}^\ell_+}$ marks optimality with respect to the cone $\text{int}\mathbb{R}^\ell_+$: $\bar{x} \in C$ is said to be a weak vector minimum point of the multiobjective problem if and only if there is no $x \in C$ such that $g(\bar{x}) - g(x) \in \text{int}\mathbb{R}^\ell_+$ It is easy to show that $\bar{x}$ is a weak vector minimum point if and only if it solves $EP(f, C)$ with $f(x, y) = \max_{i=1,\ldots,\ell}[g_i(y) - g_i(x)]$.

6. Also the Nash equilibrium, one of the most used solution concept in game theory, can be be viewed in terms of the equilibrium problem $EP(f, C)$. Suppose we have a finite set of N players, each labeled by an integer $i = 1, \ldots, N$. Player i has a real valued payoff function θ_i that depends on all players strategies $x = (x_1, \ldots, x_N)$, where each component $x_i \in \mathbb{R}^{n_i}$ represents the strategy of the i-th player. The vector of strategies $x \in \mathbb{R}^n$ with $n = \sum_{i=1}^{N} n_i$ is denoted by $x = (x_i, x_{-i})$, where x_{-i} denotes the strategy vector of all the players different from player i. Let C_i denote the strategy set of the i-th player and $C = \prod_{i=1}^{N} C_i$. In this setting, each player faces the problem of choosing a strategy that minimizes its payoff, fixed the strategies of the other players, that is player i solves the following optimization problem:

$$\min_{x_i \in C_i} \theta_i(x_i, x_{-i}).$$

A vector of strategies $\bar{x} \in C$ is called a *Nash equilibrium point* if, for all $i = 1, \ldots, N$, one has

$$\theta_i(\bar{x}_i, \bar{x}_{-i}) \leq \theta_i(x_i, \bar{x}_{-i}), \qquad \forall x_i \in C_i.$$

In other words, $\bar{x}$ is a Nash equilibrium if no player can make any profit with a unilateral change of strategy. Now consider the so called Nikaido-Isoda bifunction

$$f(x, y) = \sum_{i=1}^{n} \left[\theta_i(y_i, x_{-i}) - \theta_i(x_i, x_{-i})\right].$$

The vector $\bar{x}$ is a Nash equilibrium if and only if it is a solution of the equilibrium problem $EP(f, C)$.

2 Existence results

It is well known that the famous Fan-Knaster-Kuratowski-Mazurkiewicz Theorem [15, 22] played a very important role in the study of nonlinear analysis. Many of the existence results for $EP(f, C)$ are based on this theorem. In this section we recall some of the most important ones.

Theorem 2.1 ([6]). *Let C be convex and compact. Suppose that*

(i) f is properly quasimonotone, i.e.

$$\min_{i=1,\ldots,k} f(x_i, y) \leq 0$$

for all $x_1, \ldots, x_k \in C$ and $y \in \operatorname{conv}\{x_1, \ldots, x_k\}$;

(ii) $f(\cdot, y)$ is upper sign continuous for every $y \in C$, i.e.

$$f((1-t)x + ty, y) \geq 0 \quad \Rightarrow \quad f(x, y) \geq 0$$

for every $x \in C$ and for every $t \in (0, 1)$;

(iii) $f(x, \cdot)$ is quasiconvex for every $x \in C$, i.e the level sets

$$\{y \in C \ : \ f(x, y) \leq \alpha\}$$

are convex for every $\alpha \in \mathbb{R}$;

(iv) the level set $\{y \in C \ : \ f(x, y) \leq 0\}$ is closed, for every $x \in C$;

(v) if $f(x, y) = 0$ and $f(x, z) < 0$, then $f(x, (1-t)y + tz) < 0$, for every $t \in (0, 1)$.

Then $EP(f, C)$ has a solution.

The next results deal with the case of an unbounded feasible set. In order to study this case, the following coercivity condition, introduced in [6], has been widely used:

$$\exists r > 0 \text{ s.t. } \forall x \in C \text{ with } \|x\| > r, \ \exists y \in C \text{ with } \|y\| < \|x\| \text{ s.t. } f(x, y) \leq 0. \tag{1}$$

Theorem 2.2 ([6]). *Let C be convex. Suppose that f satisfies the assumptions (ii), (iii) (iv) and (v) of Theorem 2.1 and the coercivity condition (1). Moreover suppose that*

(i') f is pseudomonotone on the set C, i.e.

$$f(x, y) \geq 0 \quad \Rightarrow \quad f(y, x) \leq 0$$

for all $x, y \in C$.

Then $EP(f, C)$ has a solution.

If we weaken the assumption of pseudomonotonicity on f and we deal with quasimonotone bifunctions, we need a stronger notion of upper sign continuity in order to obtain an existence result with an unbounded feasible set.

Theorem 2.3 ([6]). *Let C be convex. Suppose that f satisfies the assumptions (ii), (iii), (iv) and (v) of Theorem 2.1 and the coercivity condition (1). Moreover suppose that* *(i") f is quasimonotone on the set C, i.e.*

$$f(x,y) > 0 \quad \Rightarrow \quad f(y,x) \le 0$$

for all $x, y \in C$;

(ii') for every $x, y, z \in C$ and for every $t \in (0,1)$

$$f((1-t)x + ty, z) \le 0 \quad \Rightarrow \quad f(x,z) \le 0;$$

(v') if $f(x,y) = 0$ and $f(x,z) > 0$, then $f(x,(1-t)y + tz) > 0$, for every $t \in (0,1)$.

Then $EP(f,C)$ has a solution and the solution set is bounded.

In order to avoid any assumption of convexity both for the domain C and for the bifunction, some authors (see for instance [1, 7, 10, 28] and references therein) proposed a different approach which allows to obtain existence results for $EP(f,C)$ assuming the following triangle inequality property:

$$f(x,y) \le f(x,z) + f(z,y), \qquad \forall x, y, z \in C. \tag{2}$$

Now we recall some recent existence results for $EP(f,C)$ when the bifunction f satisfies property (2). First we recall the following continuity definitions introduced in [14] that are useful in the sequel.

Definition 2.1. *A function $\varphi : \mathbb{R}^n \to \mathbb{R}$ is said to be*

- lower semicontinuous from above at $x_0 \in \mathbb{R}^n$ *if, for any sequence $\{x_k\}$ converging to x_0 and satisfying $\varphi(x_{k+1}) \le \varphi(x_k)$, for all $k \in \mathbb{N}$, we have $\varphi(x_0) \le \lim_{k\to+\infty} \varphi(x_k)$;*
- upper semicontinuous from below at $x_0 \in \mathbb{R}^n$ *if, for any sequence $\{x_k\}$ converging to x_0 and satisfying $\varphi(x_{k+1}) \ge \varphi(x_k)$, for all $k \in \mathbb{N}$, we have $\varphi(x_0) \ge \lim_{k\to+\infty} \varphi(x_k)$.*

The function φ is said to be lower semicontinuous from above (resp. upper semicontinuous from below) on $\mathbb{R}^n$ if it is lower semicontinuous from above (resp. upper semicontinuous from below) at every point of $\mathbb{R}^n$.

It is clear that lower (resp. upper) semicontinuity implies lower semicontinuity from above (resp. upper semicontinuity from below) but the reverse implications do not hold. Nevertheless it was proved in [14] that these two conditions are sufficient to establish a generalization of the Weierstrass Theorem which is the main tool to prove the following result.

Theorem 2.4 ([12])**.** *Suppose that f satisfies property (2) and C is compact.*

(a) If there exists $\bar{z} \in C$ such that $f(\bar{z}, \cdot)$ is lower semicontinuous from above, then $EP(f,C)$ has a solution.

(b) If there exists $\bar{z} \in C$ such that $f(\cdot, \bar{z})$ is upper semicontinuous from below, then $EP(f,C)$ has a solution.

The last result concerns the case when C is unbounded.

Theorem 2.5 ([12])**.** *Suppose that f satisfies property (2). If the coercivity condition (1) holds and $f(x,\cdot)$ is lower semicontinuous from above for all $x \in C$, then $EP(f,C)$ has a solution.*

3 Equivalent reformulations

We investigate when two equilibrium problems are equivalent, i.e. they have the same solution set. This kind of equivalence may be exploited to transform the initial equilibrium problem in a new one with some additional nice properties, which can be profitable to obtain existence results or to formulate iterative solution methods. First, we give a sufficient condition which guarantees the equivalence between two equilibrium problems in which the bifunctions are pseudoconvex and Lipschitz with respect to the second variable. Afterwards, we consider pseudomonotone equilibrium problems and we show how this additional assumption allows to characterize problems that are equivalent for every convex set C. In the next section the first equivalence result is used to obtain a gap function for the original equilibrium problem with better regularity properties than the simplest one.

Given a locally Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we denote by $\partial^\circ \varphi(x)$ the Clarke subdifferential at x and φ is said to be *∂°–pseudoconvex* if for every $x, y \in \mathbb{R}^n$ and $x^* \in \partial^\circ \varphi(x)$ such that $\langle x^*, y - x\rangle \geq 0$ we have $\varphi(y) \geq \varphi(x)$.

In this section we suppose that the bifunction f is extended-valued and $f(x, \cdot)$ is locally Lipschitz for all $x \in \mathbb{R}^n$. We denote by $\partial^\circ_y f(x, y)$ the Clarke subdifferential of $f(x, \cdot)$ at y. Given a multivalued map $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ we denote by $D(T)$ the domain of T and by $Z(T)$ the set of zeros of T, i.e.

$$Z(T) = \{x \in \mathbb{R}^n \ : \ 0 \in T(x)\}.$$

Definition 3.1. *Two maps T_1, T_2 are called* equivalent*, shortly $T_1 \sim T_2$, if $D(T_1) = D(T_2)$, $Z(T_1) = Z(T_2)$, and for every $x \in \mathbb{R}^n \setminus Z(T_1)$,*

$$\bigcup_{r>0} rT_1(x) = \bigcup_{s>0} sT_2(x).$$

The class of multivalued maps used in the sequel is defined by $x \rightrightarrows \partial^\circ_y f(x, x)$ where f is any bifunction.

Theorem 3.1. *Let f and g be two equilibrium bifunctions with the same domain and assume that $g(x, \cdot)$ is ∂°–pseudoconvex for all $x \in C$. If*

(i) $Z(\partial^\circ_y f(\cdot, \cdot)) \cap C \subseteq Z(\partial^\circ_y g(\cdot, \cdot)) \cap C$;

(ii) for every $x \in C \setminus Z(\partial^\circ_y f(\cdot, \cdot))$

$$\partial^\circ_y f(x, x) \subseteq \bigcup_{s>0} s\partial^\circ_y g(x, x);$$

then every solution of $EP(f, C)$ is a solution of $EP(g, C)$.

Proof. If $\bar{x}$ solves $EP(f, C)$ then it minimizes $f(\bar{x}, \cdot)$ on the feasible set C. ¿From the first order optimality condition there exists $\bar{x}^*_f \in \partial^\circ_y f(\bar{x}, \bar{x})$ such that $-\bar{x}^*_f \in N(C, \bar{x})$, where $N(C, \bar{x})$ is the normal cone of C at $\bar{x}$. If $0 \in \partial^\circ_y f(\bar{x}, \bar{x})$, from (i) and pseudoconvexity, $g(\bar{x}, \cdot)$ has a global minimum at $\bar{x}$. Otherwise, from (ii) we deduce that there exist $s > 0$ and $\bar{x}^*_g \in \partial^\circ_y g(\bar{x}, \bar{x})$ such that $\bar{x}^*_f = s\bar{x}^*_g$; hence, dividing by s and using the fact that $N(C, \bar{x})$ is a cone, from the optimality condition we have $-\bar{x}^*_g \in N(C, \bar{x})$. But $g(\bar{x}, \cdot)$ is pseudoconvex and the previous condition guarantees that $\bar{x}$ is a global minimizer for $g(\bar{x}, \cdot)$ on C. Hence $g(\bar{x}, y) \geq g(\bar{x}, \bar{x}) = 0$ for all $y \in C$. □

As immediate consequence of Theorem 3.1 we deduce the following sufficient condition for the equivalence between two pseudoconvex equilibrium problems.

Corollary 3.1. *Let f and g be two equilibrium bifunctions with the same domain and assume that $f(x, \cdot)$, $g(x, \cdot)$ are ∂°–pseudoconvex for all $x \in C$. If*

$$\partial^\circ_y f(\cdot, \cdot) \sim \partial^\circ_y g(\cdot, \cdot) \tag{3}$$

then $EP(f,C)$ and $EP(g,C)$ are equivalent.

It is clear that the converse is not true. Indeed, the differentiable bifunctions $f,g:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}$ defined by

$$f(x,y)=y_1^2+2y_2^2-x_1^2-2x_2^2 \quad \text{and} \quad g(x,y)=2y_1^2+y_2^2-2x_1^2-x_2^2$$

have the same unique solution $(\bar{x}_1,\bar{x}_2)=(0,0)$ on the set $C=[-1,1]\times[-1,1]$; nevertheless, for every $(x_1,x_2)\in C$ with $x_1x_2\neq 0$ there is no positive constant α such that $\alpha(2x_1,4x_2)=(4x_1,2x_2)$.

Now we show that the reformulation of suitable equilibrium problems as generalized variational inequalities (see [25]) can be viewed as a particular application of our result. We recall that if $K\subseteq\mathbb{R}^n$ is a nonempty convex and closed set then $\partial^\circ\sigma(0|K)=K$, where

$$\sigma(x|K)=\sup\{\langle x^*,x\rangle \;:\; x^*\in K\}$$

is the support function associated to K. ¿From Corollary 3.1 we deduce that the solution set of any $EP(f,C)$ coincides with the set of $\bar{x}\in C$ such that

$$\sigma(y-\bar{x}|\partial_y^\circ f(\bar{x},\bar{x}))\geq 0, \qquad \forall y\in C. \tag{4}$$

Moreover, since $F(x)=\partial_y^\circ f(x,x)$ is compact, problem (4) collapses into the generalized variational inequality $GVI(F,C)$, which consists in finding $\bar{x}\in C$ and $\bar{y}^*\in\partial_y^\circ f(\bar{x},\bar{x})$ such that

$$\langle\bar{y}^*,y-\bar{x}\rangle\geq 0, \qquad \forall y\in C.$$

In the particular case when $f(x,\cdot)$ is a convex and differentiable function, the equilibrium problem $EP(f,C)$ is equivalent to the variational inequality:

$$\text{find } \bar{x}\in C \text{ such that } \langle\nabla_y f(\bar{x},\bar{x}),y-\bar{x}\rangle\geq 0 \text{ for all } y\in C. \tag{5}$$

Some preliminary ideas on this result can be found in [36, 40]. Though condition (3) does not characterize equivalent equilibrium problems, some conclusions can still be drawn if we assume that f and g are pseudomonotone. In order to obtain such a result we recall that a multivalued map $T:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is said to be

- *pseudomonotone* if for all $x,y\in\mathbb{R}^n$ and all $x^*\in T(x)$ satisfying the inequality $\langle x^*,y-x\rangle\geq 0$ then $\langle y^*,x-y\rangle\leq 0$ for all $y^*\in T(y)$;
- *cyclically pseudomonotone* if for all $x_1,x_2,\ldots,x_k\in\mathbb{R}^n$ and $x_i^*\in T(x_i)$ with $i=1,2,\ldots,k-1$ such that $\langle x_i^*,x_{i+1}-x_i\rangle\geq 0$ then $\langle x_k^*,x_1-x_k\rangle\leq 0$ for all $x_k^*\in T(x_k)$.

Lemma 3.1. *Let $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ be (cyclically) pseudomonotone, Lipschitz and ∂°–pseudoconvex in the second variable, then the associated operator $\partial_y^\circ f(\cdot,\cdot)$ is (cyclically) pseudomonotone.*

Proof. Let's prove only the cyclically case. Fix $x_1,x_2,\ldots,x_k\in\mathbb{R}^n$ and $x_i^*\in\partial_y^\circ f(x_i,x_i)$ with $i=1,\ldots,k-1$ such that $\langle x_i^*,x_{i+1}-x_i\rangle\geq 0$. Hence, from the pseudoconvexity of $f(x_i,\cdot)$ we deduce

$$f(x_i,x_{i+1})\geq f(x_i,x_i)=0, \qquad \forall i=1,2,\ldots,k-1.$$

Since f is cyclically pseudomonotone then $f(x_k,x_1)\leq 0$. Applying the Lebourg mean value theorem to the functions $f(x_k,\cdot)$ on the segment (x_1,x_k) we deduce the existence of $z=tx_1+(1-t)x_k$ with $t\in(0,1)$ and $z^*\in\partial_y^\circ f(x_k,z)$ such that

$$0\leq f(x_k,x_k)-f(x_k,x_1)=\langle z^*,x_k-x_1\rangle=t^{-1}\langle z^*,x_k-z\rangle.$$

¿From ([35, Theorem 4.1]) the multivalued map $\partial_y^\circ f(x_k, \cdot)$ is pseudomonotone and then

$$0 \geq \langle x_k^*, z - x_k \rangle = t \langle x_k^*, x_1 - x_k \rangle, \quad \forall x_k^* \in \partial_y^\circ f(x_k, x_k)$$

which completes the proof. □

The last result proves, in a sense, the converse of Corollary 3.1 under the further assumption of pseudomonotonicity.

Theorem 3.2. *Let $f, g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be pseudomonotone, Lipschitz and ∂°–pseudoconvex in the second variable. If $EP(f, C)$ and $EP(g, C)$ are equivalent for every convex subset $C \subseteq \mathbb{R}^n$, then*

$$\partial_y^\circ f(\cdot, \cdot) \sim \partial_y^\circ g(\cdot, \cdot).$$

Proof. Set $F(x) = \partial_y^\circ f(x, x)$ and $G(x) = \partial_y^\circ g(x, x)$. Since $EP(f, C)$ and $EP(g, C)$ are equivalent to the generalized variational inequalities $GVI(F, C)$ and $GVI(G, C)$ respectively, then $GVI(F, C)$ and $GVI(G, C)$ have the same solution sets. Moreover the maps F and G have compact convex values and, from the previous lemma, they are pseudomonotone. Hence, from [18, Proposition 3.3] $F \sim G$ as required. □

4 Solution methods

Different approaches to solve equilibrium problems have been proposed in literature, often extending solution methods originally conceived for optimization problems or variational inequalities to the framework of more general equilibrium problems.

A class of methods is based on a fixed-point formulation via the auxiliary problem principle. These methods are based on iterative convex minimization procedure and extend the classical projection-type algorithms for variational inequalities. They ask for the solution of one appropriate convex minimization problem at each iteration and converge under strong monotonicity and Lipschitz-type conditions on f [29, 31]. Bundle techniques can be exploited to extend the approach to nonsmooth equilibrium problems by approximating the function $f(x, \cdot)$ with a convex piecewise linear function [32].

Another approach is based on proximal-like techniques [2, 17, 24, 31, 33, 37]. Convergence is achieved under pseudomonotonicity assumptions, but generally two convex minimization problems have to be solved at each iteration as in the extragradient-type methods developed in [2, 17, 33, 37]. The optimization problems to be solved at each iteration may be constrained [2, 17, 37]. In [33] an interior-quadratic proximal term replaces the usual quadratic proximal term. An inexact proximal point method for solving nonmonotone equilibrium problems is developed in [24], replacing the original problem with a sequence of regularized ones. In [23] a method in which two convex minimization problems are solved at each iteration is developed as well for smooth monotone equilibrium problems, and it relies on hyperplane projection techniques.

In [21] another approach has been considered: the equilibrium problem is reformulated as a convex feasibility problem and the solution methods developed for this latter problem (see [5]) are applied. Each iteration requires the approximate solution of one minimization problem (generally not convex) and a projection, and convergence holds under pseudomonotonicity assumptions.

In this section we focus on descent methods, which are based on the reformulation of the equilibrium problem as a global optimization problem through appropriate gap functions [8, 9, 13, 26, 27, 30, 38, 39, 40]. Solution methods for optimization problems generally converge to a stationary point, which may be not a global optimum unless strong assumptions are made. Therefore assumptions, which guarantee that all the stationary points of the gap function are actually solutions of the equilibrium problem, are a key point to most of the methods which have been developed up to now [13, 26, 27, 30, 38, 39]. In order to devise solution methods which do not require the above "stationarity property", a parametric family of auxiliary equilibrium problems and the corresponding family of gap function can be exploited [8, 9]. Descent algorithms have also been developed for specific classes of equilibrium problems such as Nash ones [19, 20, 34].

Based on the equivalence between equilibrium problems and variational inequalities (see (5)), equilibrium problems could be solved just exploiting the available algorithms for variational inequalities. On the other hand, the computation of the operator of the equivalent variational inequality (5) requires the explicit evaluation of the partial derivatives of f and algorithms may require further smoothness assumptions on the operator and eventually the computation of the second order derivatives of f. On the contrary, devising solution methods directly for equilibrium problems via optimization requires weaker smoothness assumptions and the evaluation of derivatives may be not needed at all (if, for instance, the computation of the gap function is made via derivative-free methods). In fact, in what follows the equilibrium bifunction f is supposed to be just continuously differentiable.

A function $g : C \to \mathbb{R}$ is said to be a *gap function* for $EP(f, C)$ if g is nonnegative on C and $\bar{x}$ solves $EP(f, C)$ if and only if $\bar{x} \in C$ and $g(\bar{x}) = 0$. Thus the solution set of $EP(f, C)$ coincides with the set of global minima of g on C. The simplest gap function is given by

$$\varphi(x) = -\inf\{f(x, y) \ : \ y \in C\}, \tag{6}$$

provided that the infimum is attained for every $x \in C$. Such function, which in general is not differentiable, coincides with the one introduced in [3] when $EP(f, C)$ is a variational inequality. The exploitation of auxiliary equilibrium problems allows to obtain continuously differentiable gap functions [30]. Given $\alpha > 0$ and a continuously differentiable bifunction $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$h(x, y) \geq 0 \text{ for all } x, y \in C \text{ and } h(z, z) = 0 \text{ for all } z \in C, \tag{7}$$

$$h(x, \cdot) \text{ is strongly convex for all } x \in C, \tag{8}$$

$$\nabla_y h(z, z) = 0 \text{ for all } z \in C, \tag{9}$$

$$\langle \nabla_x h(x, y) + \nabla_y h(x, y), y - x \rangle \geq 0 \text{ for all } x, y \in C, \tag{10}$$

the original problem $EP(f, C)$ is equivalent to the auxiliary problem $EP(f_\alpha, C)$, where $f_\alpha = f + \alpha h$ satisfies $f_\alpha(x, x) = 0$ for all $x \in C$. In fact, condition (9) guarantees the equality $\nabla_y f(x, x) = \nabla_y f_\alpha(x, x)$ and therefore the equivalence follows from Corollary 3.1. Thus the gap function (6) associated to the auxiliary problem, i.e.

$$\varphi_\alpha(x) = -\min\{f_\alpha(x, y) \ : \ y \in C\},$$

provides a gap function also for $EP(f, C)$. Moreover, the continuity and the strong convexity of $f_\alpha(x, \cdot)$ imply the existence of a unique minimizer $y_\alpha(x)$. Since $f(x, x) = 0$, the solution set of $EP(f, C)$ coincides with the set of the fixed points of the function y_α. Furthermore, classical results guarantee that φ_α is continuously differentiable with

$$\nabla \varphi_\alpha(x) = -\nabla_x f_\alpha(x, y_\alpha(x)).$$

Notice that the most used regularizing bifunction $h(x, y) = \|y - x\|^2$ satisfies all the above assumptions. Though condition (10) is not needed for the equivalence between $EP(f, C)$ and the auxiliary problem $EP(f_\alpha, C)$, it is useful in the development of solution methods. Indeed, assumptions of this kind are the key tools to devise descent methods.

Definition 4.1. *A differentiable bifunction f is called*

- ∇–monotone *on C if $\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x \rangle \geq 0$ for all $x, y \in C$;*
- strictly ∇–monotone *on C if $\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x \rangle > 0$ for all $x, y \in C$ with $x \neq y$;*

- strongly ∇–monotone *on* C *if there exists* $\tau > 0$ *such that*

$$\langle \nabla_x f(x,y) + \nabla_y f(x,y), y - x \rangle \geq \tau \|y - x\|^2, \qquad \forall x, y \in C.$$

Obviously, strong ∇-monotonicity implies strict ∇-monotonicity, which implies ∇-monotonicity. When $EP(f,C)$ is a variational inequality, the (strict, strong) ∇-monotonicity collapses to the (strict, strong) monotonicity of the operator F. Moreover, ∇-monotonicity and monotonicity of bifunctions are not related (see Example 2.2 in [8]).

Another useful assumption is the following [8]:

$$f(x,y) + \langle \nabla_x f(x,y), y - x \rangle \geq 0, \qquad \forall x, y \in C. \tag{11}$$

All the bifunctions for which $f(\cdot, y)$ is concave for all $y \in C$ satisfy (11). When $EP(f,C)$ is a variational inequality, condition (11) collapses to the monotonicity of the operator F. Moreover, it is stronger than ∇-monotonicity, but it is not related to strict ∇-monotonicity.

The standard descent method for φ_α works fine for $EP(f,C)$ under suitable assumptions: though the gap function is not convex on C, and therefore finding its global minima is not an easy task, the strict ∇-monotonicity of f guarantees that its stationary points coincide with the global minima; moreover, the same assumption provides a natural descent direction.

Theorem 4.1 ([8, 30])**.** *Suppose that f is strictly ∇–monotone on C.*

(a) If $\bar{x}$ is a stationary point of φ_α over C, i.e. $\langle \nabla \varphi_\alpha(\bar{x}), y - \bar{x} \rangle \geq 0$ for all $y \in C$, then $\bar{x}$ solves $EP(f,C)$.

(b) If $x \in C$ is not a solution of $EP(f,C)$, then $\langle \nabla \varphi_\alpha(x), y_\alpha(x) - x \rangle < 0$.

Therefore, a descent algorithm follows immediately from the above result.

Algorithm 1.

0. Choose $\alpha > 0$, $x^0 \in C$ and set $k = 0$.
1. Compute $y^k = \arg\min\{f(x^k, y) + \alpha\, h(x^k, y) \;:\; y \in C\}$, set $d^k = y^k - x^k$.
2. If $d^k = 0$, then *STOP*.
3. Compute $t_k = \arg\min\{\varphi_\alpha(x^k + t\, d^k) \;:\; t \in [0,1]\}$.
4. Set $x^{k+1} = x^k + t_k\, d^k$, $k = k + 1$ and goto step 1.

Theorem 4.2 ([30])**.** *If C is bounded and f is strictly ∇–monotone on C, then Algorithm 1 either stops at a solution of $EP(f,C)$ after a finite number of iterations or it produces a sequence $\{x^k\}$ such that any of its cluster points solves $EP(f,C)$.*

Remark 4.1. *In [13, 26, 30] descent methods with Armijo-type inexact line search have been proposed under assumptions which guarantee the strong ∇–monotonicity of f. However, it is possible to achieve the convergence just requiring strict ∇–monotonicity if the line search procedure is slightly modified setting $t_k = \gamma^s$, where s is the smallest non-negative integer such that*

$$\varphi_\alpha(x^k + \gamma^s\, d^k) \leq \varphi_\alpha(x^k) - \beta\, \gamma^{2s}\, \|d^k\|,$$

being $\beta, \gamma \in (0,1)$ fixed parameters.

Remark 4.2. *When C is not bounded, some additional assumptions are needed to obtain a global error bound for the solution of $EP(f,C)$ and hence to guarantee the convergence of Algorithm 1. Such assumptions could be the strong monotonicity of f [30] or other monotonicity type conditions for f and h [13].*

The strict ∇–monotonicity assumption cannot be relaxed to ∇–monotonicity in order to guarantee that $y_\alpha(x) - x$ is a descent direction for φ_α. In fact, under ∇–monotonicity there can be stationary points of φ_α which are not solution of $EP(f, C)$, and Algorithm 1 may get trapped into one of them (see Example 2.5 in [8]). In order to overcome this drawback, condition (11) can be exploited to devise a solution method which does not look for the stationary points of a given gap function. The main idea is to rely on the whole family of the auxiliary equilibrium problems $EP(f_\alpha, C)$ and the corresponding family of gap functions: a descent direction is searched exploiting also the parameter α.

Theorem 4.3 ([8])**.** *If condition (11) holds, then:*

(a) for any $\alpha > 0$ and $x \in C$

$$\langle \nabla\varphi_\alpha(x), y_\alpha(x) - x\rangle \le -\varphi_\alpha(x) - \alpha\left[h(x, y_\alpha(x)) + \langle \nabla_x h(x, y_\alpha(x)), y_\alpha(x) - x\rangle\right];$$

(b) if $x \in C$ is not a solution of $EP(f, C)$ and $\eta \in (0, 1)$, then for any α sufficiently small

$$-\varphi_\alpha(x) - \alpha\left[h(x, y_\alpha(x)) + \langle \nabla_x h(x, y_\alpha(x)), y_\alpha(x) - x\rangle\right] < -\eta\, \varphi_\alpha(x), \tag{12}$$

and therefore $y_\alpha(x) - x$ is a descent direction for φ_α at x.

This result provides the key idea to formulate the method: given a value for α, the corresponding gap function φ_α is exploited as long as inequality (12) holds and $y_\alpha(x)-x$ provides a descent direction; when this is no longer the case, the value of α is decreased and a new search is performed with the new gap function.

Algorithm 2

0. Choose $\eta, \gamma \in (0, 1)$, $\beta \in (0, \eta)$, a sequence $\alpha_k \downarrow 0$, $x^0 \in C$ and set $k = 1$.
1. Set $z^0 = x^{k-1}$ and $j = 0$.
2. Compute $y^j = \arg\min\{f(z^j, y) + \alpha_k h(z^j, y) \ : \ y \in C\}$, set $d^j = y^j - z^j$.
3. If $d^j = 0$, then *STOP*.
4. If inequality (12) holds,

   then compute the smallest non-negative integer s such that

   $$\varphi_{\alpha_k}(z^j + \gamma^s d^j) \le \varphi_{\alpha_k}(z^j) - \beta\gamma^s \varphi_{\alpha_k}(z^j),$$

   set $t_j = \gamma^s$, $z^{j+1} = z^j + t_j\, d^j$, $j = j + 1$ and goto step 2

   else set $x^k = z^j$, $k = k + 1$ and goto step 1.

Theorem 4.4 ([8])**.** *If C is bounded and (11) holds, then Algorithm 2 either stops at a solution of $EP(f, C)$ after a finite number of iterations or it produces either an infinite sequence $\{x^k\}$ or an infinite sequence $\{z^j\}$ such that any of its cluster points solves $EP(f, C)$.*

When the set C is given by a set of possibly nonlinear convex constraints, the evaluation of φ_α could be computationally expensive. For such a reason in [9] a new gap function has been introduced replacing C by a polyhedral approximation. Suppose that C is the intersection of a bounded polyhedron

$$D = \{y \in \mathbb{R}^n \ : \ \langle a_j, y\rangle \le b_j, \quad j = 1, \ldots, r\}$$

where $a_j \in \mathbb{R}^n$ and $b_j \in \mathbb{R}$, and a convex set given by

$$\widetilde{C} = \{x \in \mathbb{R}^n \ : \ c_i(x) \le 0, \quad i = 1, \ldots, m\},$$

where $c_i : \mathbb{R}^n \to \mathbb{R}$ are twice continuously differentiable convex functions, and there exists $\hat{x} \in D$ such that $c_i(\hat{x}) < 0$ for all $i = 1, \ldots, m$.

The function

$$\widetilde{\varphi}_\alpha(x) = -\min\{f_\alpha(x, y) \; : \; y \in P(x)\},$$

where

$$P(x) = \{y \in D \; : \; c_i(x) + \langle \nabla c_i(x), y - x \rangle \leq 0, \quad i = 1, \ldots, m\},$$

is a polyhedral approximation of the feasible region C, is a gap function for $EP(f, C)$ [9]. Denote by $\widetilde{y}_\alpha(x)$ the unique minimizer of $f_\alpha(x, \cdot)$ over $P(x)$ and by $\Lambda_\alpha(x)$ the corresponding set of Lagrange multipliers $(\lambda, \mu) \in \mathbb{R}^m_+ \times \mathbb{R}^r_+$, where the multipliers λ are associated to the linearized constraints and the multipliers μ to the linear constraints.

The underlying idea is the same as in Algorithm 2: exploiting $\widetilde{y}_\alpha(x) - x$ as a search direction. However, $\widetilde{y}_\alpha(x) \in P(x)$ while in general it does not lie in C, and thus the new point generated could be unfeasible. A penalization of the gap function is required: the exact penalty function

$$\psi_{\alpha,\varepsilon}(x) = \widetilde{\varphi}_\alpha(x) + \frac{1}{\varepsilon} \, \|(c_1^+(x), \ldots, c_m^+(x))\|, \tag{13}$$

where $c_i^+(x) = \max\{0, c_i(x)\}$ and $\varepsilon > 0$, is considered.

When the assumption (11) holds on the polyhedron D, results similar to Theorem 4.3 can be obtained. In particular, $\widetilde{y}_\alpha(x) - x$ is a descent direction for $\psi_{\alpha,\varepsilon}$ at x either if x is feasible and the regularization parameter α is small enough or if x is infeasible and the penalization parameter ε is small enough.

Theorem 4.5 ([9])**.** *If condition (11) holds on D, then:*

(a) for any $x \in D$, $\alpha > 0$, $\varepsilon > 0$

$$\psi^\circ_{\alpha,\varepsilon}(x; \widetilde{y}_\alpha(x) - x) \leq -\psi_{\alpha,\varepsilon}(x) - \alpha \left[h(x, \widetilde{y}_\alpha(x)) + \langle \nabla_x h(x, \widetilde{y}_\alpha(x)), \widetilde{y}_\alpha(x) - x \rangle \right],$$

where $\psi^\circ_{\alpha,\varepsilon}(x; \widetilde{y}_\alpha(x) - x)$ denotes the generalized directional derivative of $\psi_{\alpha,\varepsilon}$ at x in the direction $\widetilde{y}_\alpha(x) - x$;

(b) if $x \in C$ does not solve $EP(f, C)$ and $\eta \in (0, 1)$, then for any $\varepsilon > 0$ and α sufficiently small

$$-\psi_{\alpha,\varepsilon}(x) - \alpha \left[h(x, \widetilde{y}_\alpha(x)) + \langle \nabla_x h(x, \widetilde{y}_\alpha(x)), \widetilde{y}_\alpha(x) - x \rangle \right] \leq -\eta \, \psi_{\alpha,\varepsilon}(x) < 0,$$

and therefore $\widetilde{y}_\alpha(x) - x$ is a descent direction for $\psi_{\alpha,\varepsilon}$ at x;

(c) if $x \in D \setminus C$ and $(\lambda, \mu) \in \Lambda_\alpha(x)$, then $\psi^\circ_{\alpha,\varepsilon}(x; \widetilde{y}_\alpha(x) - x) < 0$ holds for any $\alpha > 0$ and ε such that $1/\varepsilon > \|\lambda^+\|$, where

$$\lambda_i^+ = \begin{cases} \lambda_i & \text{if } c_i(x) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

The method exploits $\psi_{\alpha,\varepsilon}$ as long as $\widetilde{y}_\alpha(x) - x$ provides a descent direction, otherwise both the value for α and ε are decreased.

Algorithm 3

0. Choose $\beta, \gamma, \delta, \eta \in (0, 1)$, sequences $\alpha_k, \varepsilon_k \downarrow 0$, $x^0 \in D$ and set $k = 1$.
1. Set $z^0 = x^{k-1}$ and $j = 0$.
2. Compute $y^j = \arg\min\{f(z^j, y) + \alpha_k h(z^j, y) \; : \; y \in D \cap P(z^j)\}$ and λ^j any Lagrange multiplier vector corresponding to the linearized constraints. Set $d^j = y^j - z^j$.

3. If $d^j = 0$, then *STOP*.

4. If the following relations hold

   - $1/\varepsilon_k \geq \|(\lambda^j)^+\| + \delta$,
   - $\psi_{\alpha_k,\varepsilon_k}(z^j) > 0$,
   - $-\psi_{\alpha_k,\varepsilon_k}(z^j) - \alpha_k \left[h(z^j, y^j) + \langle \nabla_x h(z^j, y^j), y^j - z^j \rangle\right] \leq -\eta\, \psi_{\alpha_k,\varepsilon_k}(z^j)$

   then compute the smallest non-negative integer s such that

$$\psi_{\alpha_k,\varepsilon_k}(z^j + \gamma^s\, d^j) \leq \psi_{\alpha_k,\varepsilon_k}(z^j) - \beta\, \gamma^{2s}\, \|d^j\|$$

   set $t_j = \gamma^s$, $z^{j+1} = z^j + t_j\, d^j$, $j = j + 1$ and goto step 2

   else set $x^k = z^j$, $k = k + 1$ and goto step 1.

Theorem 4.6 ([9])**.** *If C is bounded and (11) holds on D, then Algorithm 3 either stops at a solution of $EP(f, C)$ after a finite number of iterations or it produces either an infinite sequence $\{x^k\}$ or an infinite sequence $\{z^j\}$ such that any of its cluster points solves $EP(f, C)$.*

Another well-known optimization approach for solving $EP(f, C)$ is based on the so-called D-gap functions, which are the difference between two gap functions and allow to reformulate $EP(f, C)$ as an unconstrained minimization problem. In [27, 38] the authors introduce the D-gap functions, give conditions for the equivalence between stationary points of a D-gap function and the solutions of $EP(f, C)$, and conditions under which a D-gap function provides a global error bound for $EP(f, C)$. Descent algorithm for minimizing the equivalent optimization problem have been proposed in [27, 39].

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