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MATHEMATISCHE NACHRICHTEN

ORIGINAL ARTICLE

Some surfaces with canonical maps of degrees 10, 11, and 14

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Abstract

In this note, we present examples of complex algebraic surfaces of general type with canonical maps of degrees 10, 11, and 14. They are constructed as quotients of a product of two Fermat septics using certain free actions of the group $\mathbb{Z}_7^2$.

KEYWORDS

Beauville surface, canonical map, product–quotient surface, surface of general type

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1 | INTRODUCTION

Beauville has shown in [5] that if the image of the canonical map Φ_{K_S} of a surface S of a general type is a surface, then the following inequality holds:

$$d := \deg(\Phi_{K_S}) \leq 9 + \frac{27 - 9q}{p_g - 2} \leq 36.$$

Here, q is the irregularity and p_g is the geometric genus of S. In particular, $28 \leq d$ is only possible if $q = 0$ and $p_g = 3$. Motivated by this observation, the construction of surfaces with $p_g = 3$ and canonical map of degree d for every value $2 \leq d \leq 36$ is an interesting, but still widely open problem [12, Question 5.2]. In particular for most values $10 \leq d$, it is not known if a surface realising that degree exists. Indeed, for a long time, the only examples were the surfaces of Persson [16], with canonical map of degree 16 and Tan [22], with degree 12. In recent years, this problem attracted the attention of many authors, putting an increased effort in the construction of new examples. As a result, previously unknown surfaces with degrees $d = 12, 16, 20, 24, 32$, and 36 have been discovered, by Rito [17–20], Gleissner, Pignatelli and Rito [10] and Nguyen [13, 14]. In this work, we present surfaces with canonical maps of degrees $d = 10, 11$, and 14. According to our knowledge, they are the first surfaces for those degrees. They can be described using Pardini's theory of branched abelian covers [15], which is one of the standard techniques in this subject, cf. [12]. However, we decided to present them in an elementary way using plane curves and basic algebraic geometry at graduate textbook level [21]. Our construction is completely self-contained, basically reference free and easily accessible. It can be sketched as follows: the surfaces S, which all have $p_g = 3$, arise as quotients of a product of two Fermat septics

$$F = \{x_0^7 + x_1^7 + x_2^7 = 0\} \subset \mathbb{P}^2$$

modulo certain free and diagonal actions of the group $\mathbb{Z}_7^2$. Their explicit construction allows us to write the canonical system of each of them in terms of three $\mathbb{Z}_7^2$-invariant holomorphic two-forms on the product $F \times F$. It turns out that for none of them $|K_S|$ is base-point free, that is, the canonical map $\Phi_{K_S} : S \dashrightarrow \mathbb{P}^2$ is just a rational map. To compute its degree, we resolve the indeterminacy by a sequence of blowups and compute the degree of the resulting morphism via elementary intersection theory.

We point out that our surfaces are particular examples of surfaces isogenous to a product, that is, quotients of a product of two curves modulo a free action of a finite group. This construction goes back to an exercise in Beauville's book on *Complex Algebraic Surfaces* [6, Exercises X.13 4], where a free action of $\mathbb{Z}_5^2$ on a product of two Fermat quintics is used to construct a surface with $p_g = q = 0$. This example served as an important inspiration for our work. By now, many more surfaces isogenous to a product have been constructed. Apart from other works, that mainly deal with irregular surfaces, we want to mention the complete classification of surfaces isogenous to a product with $p_g = q = 0$ [1] and the classification for $p_g = 1$ and $q = 0$ under the assumption that the action is diagonal [9]. However, for higher values of p_g, a classification of regular surfaces isogenous to a product is much more involved and is not yet established. Recently, similar constructions involving non-free actions on a product of Fermat curves have been used to provide other interesting projective manifolds that helped us to understand some important geometric phenomena. Most notably are the rigid but not infinitesimally rigid manifolds [4] of Bauer and Pignatelli that gave a negative answer to a question of Kodaira and Morrow [11, p. 45] and, to a lesser degree, also the infinite series of n-dimensional infinitesimally rigid manifolds of general type with non-contractible universal cover for each $n \geq 3$, provided by Frapporti and the second author of this paper [8].

2 THE SURFACES

Let F be the Fermat septic curve

$$F = \{x_0^7 + x_1^7 + x_2^7 = 0\} \subset \mathbb{P}^2.$$

In this section, we construct a series of surfaces S, as quotients of a product of two copies of F, modulo a suitable diagonal action of the group $\mathbb{Z}_7^2$. For any surface S, we determine the canonical map Φ_{K_S} and compute its degree.

On the first copy of F, we define the action of $\mathbb{Z}_7^2$ as

$$\phi : \mathbb{Z}_7^2 \to \mathrm{Aut}(F), \quad (a,b) \mapsto [(x_0 : x_1 : x_2) \mapsto (x_0 : \zeta_7^a x_1 : \zeta_7^b x_2)], \quad \text{where} \quad \zeta_7 := \exp\left(\frac{2\pi\sqrt{-1}}{7}\right).$$

This action has 21 points with nontrivial stabilizer. They form three orbits of length 7. A representative of each orbit and a generator of the stabilizer is given by:

Point	$(\mathbf{-1} : \mathbf{0} : \zeta_7)$	$(\mathbf{-1} : \zeta_7 : \mathbf{0})$	$(\mathbf{0} : \mathbf{-1} : \zeta_7)$
Generator	$(1,0)$	$(0,1)$	$(6,6)$

Note that the automorphisms $\phi(a,b)$ are precisely the deck transformations of the cover

$$\pi : F \to \mathbb{P}^1, \qquad (x_0 : x_1 : x_2) \mapsto (x_1^7 : x_2^7).$$

The cover has degree 49 and is branched along $(0 : 1)$, $(1 : 0)$, and $(-1 : 1)$. In particular, $F/\mathbb{Z}_7^2 \simeq \mathbb{P}^1$ and π is the quotient map. On the second copy of F, for which we use the homogenous variables $y = (y_0 : y_1 : y_2)$, the group acts by $\phi \circ A$, where $A \in \mathrm{Aut}(\mathbb{Z}_7^2)$ is an automorphism depending on the specific example. The explicit choices for A are stated in the table below. Let S be the quotient

$$S := (F \times F)/\mathbb{Z}_7^2 \qquad \text{modulo the diagonal action} \qquad \phi \times (\phi \circ A).$$

To write the canonical system of the corresponding unmixed quotient, we need to fix a suitable basis of the space $H^0(F, \Omega^1_F)$ of *global holomorphic 1-forms* on F. In affine coordinates, such a basis is given by

$$\{\omega_{jk} := u^j v^{k-6} du \mid j + k \leq 4\}, \qquad \text{where} \qquad u := \frac{x_1}{x_0} \quad \text{and} \quad v := \frac{x_2}{x_0}.$$

Note that:

I) The action of $\mathbb{Z}_7^2$ on $H^0(F, \Omega^1_F)$ under pullback with ϕ is

$$\phi(a,b)^*(\omega_{jk}) = \zeta_7^{a(j+1)+b(k-6)} \omega_{jk}.$$

II) The space of canonical sections $H^0(K_S)$ is isomorphic to $\left(H^0(\Omega^1_F) \otimes H^0(\Omega^1_F)\right)^{\mathbb{Z}_7^2}$, where the action on the tensor product is diagonal, that is, $(a,b) \in \mathbb{Z}_7^2$ acts via

$$\phi(a,b)^* \otimes \phi(A(a,b))^*.$$

The observations I) and II) imply:

Lemma 1. *A basis of $H^0(K_S)$ is given by the $\mathbb{Z}_7^2$-invariant tensors $\omega_{jklm} := \omega_{jk} \otimes \omega_{lm}$. A tensor ω_{jklm} is invariant if and only if for all $(a,b) \in \mathbb{Z}_7^2$ it holds:*

$$a(j+1) + b(k-6) + a'(l+1) + b'(m-6) \equiv 0 \bmod 7, \quad \textit{where} \quad \begin{pmatrix} a' \\ b' \end{pmatrix} := A \begin{pmatrix} a \\ b \end{pmatrix}.$$

We can now state and prove our main result:

Theorem 1. *For all $A \in \mathrm{Aut}(\mathbb{Z}_7^2)$ in the table below, the diagonal action $\phi \times (\phi \circ A)$ of $\mathbb{Z}_7^2$ on the product of two Fermat septics is free. The quotient is a regular smooth projective surface S of general type with $p_g = 3$. A basis of $H^0(K_S)$, the canonical map Φ_{K_S} in projective coordinates and its degree are stated in the table:*

No	A	Basis of $H^0(K_S)$	$\Phi_{K_S}(x,y)$	$\deg(\Phi_{K_S})$
1.	$\begin{pmatrix} 3 & 3 \\ 6 & 2 \end{pmatrix}$	$\{\omega_{0203}, \omega_{1004}, \omega_{3112}\}$	$(x_0^2x_2^2y_0y_2^3 : x_0^3x_1y_2^4 : x_1^3x_2y_0y_1y_2^2)$	5
2.	$\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$	$\{\omega_{1022}, \omega_{2131}, \omega_{4010}\}$	$(x_0^3x_1y_1^2y_2^2 : x_0x_1^2x_2y_1^3y_2 : x_1^4y_0^3y_1)$	7
3.	$\begin{pmatrix} 4 & 5 \\ 3 & 1 \end{pmatrix}$	$\{\omega_{1304}, \omega_{2210}, \omega_{3012}\}$	$(x_1x_2^3y_2^4 : x_1^2x_2^2y_0^3y_1 : x_0x_1^3y_0y_1y_2^2)$	10
4.	$\begin{pmatrix} 2 & 6 \\ 1 & 4 \end{pmatrix}$	$\{\omega_{0011}, \omega_{1202}, \omega_{2040}\}$	$(x_0^4y_0^2y_1y_2 : x_0x_1x_2^2y_0^2y_2^2 : x_0^2x_1^2y_1^4)$	11
5.	$\begin{pmatrix} 3 & 3 \\ 6 & 4 \end{pmatrix}$	$\{\omega_{0103}, \omega_{1310}, \omega_{3031}\}$	$(x_0^3x_2y_0y_2^3 : x_1x_2^3y_0^3y_1 : x_0x_1^3y_1^3y_2)$	14
6.	$\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$	$\{\omega_{0101}, \omega_{1313}, \omega_{3030}\}$	$(x_0^3x_2y_0^3y_2 : x_1x_2^3y_1y_2^3 : x_0x_1^3y_0y_1^3)$	14
7.	$\begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}$	$\{\omega_{0202}, \omega_{2121}, \omega_{4040}\}$	$(x_0^2x_2^2y_0^2y_2^2 : x_0x_1^2x_2y_0y_1^2y_2 : x_1^4y_1^4)$	Φ_{K_S} is composed with a pencil

The image of the canonical map of the last surface is the conic $\{z_1^2 = z_0z_2\} \subset \mathbb{P}^2$. The surfaces nos. 3, 4, 5, and 6 of the table are the first known examples of surfaces with $\deg(\Phi_{K_S}) = 10$, 11, and 14.

The degrees 5 and 7 of the first and second surfaces have also been realized by a different construction [12, Example 4.5].

Remark 1. The surfaces S in the table from above are examples of *unmixed surfaces isogenous to a product of curves*, that is, surfaces isomorphic to a quotient of a product of two curves $C_1 \times C_2$ of genus $g(C_i) \geq 2$ modulo a free and diagonal action of a finite group G. More precisely, they are *Beauville surfaces*, which are by definition the *rigid surfaces* isogenous to a product. Rigidity means that they do not admit nontrivial deformations. This is equivalent to the fact that the quotient curves C_i/G are isomorphic to $\mathbb{P}^1$ and the quotient maps $C_i \to C_i/G \simeq \mathbb{P}^1$ are branched in three points. Such surfaces can be described purely in terms of group theory, cf. [3]. Using the MAGMA [7] algorithm from the paper [10], one can classify all regular unmixed surfaces isogenous to a product of curves with $p_g = 3$ and abelian group G. Among them are the unmixed Beauville surfaces with $p_g = 3$ and abelian group. The latter form seven biholomorphism classes, which are exactly the surfaces in the table of our theorem.

To compute the degree of the canonical map of each of our surfaces, we use the following lemma.

Lemma 2. *Let $|M|$ be a two-dimensional linear system on a surface S spanned by D_1, D_2, and D_3. Assume that S has at most isolated base-points. Given a point p of S, suppose we can write the divisors D_i around p as*

$$D_1 = aH, \quad D_2 = bK \quad \textit{and} \quad D_3 = cH + dK,$$

where H and K are reduced, smooth, and intersect transversally at p and a, b, c, d are non-negative integers, $b \leq a$. Assume that

- *$d \geq b$ or*
- *$b \neq 0$ and $c + md \geq a$, where $a = mb + q$ with $0 \leq q < b$.*

Then after blowing up at most (ab)-times we obtain a new linear system $|\hat{M}|$ such that no infinitely near point of p is a base-point of $|\hat{M}|$. Moreover, $\hat{M}^2 = M^2 - ab$.

Proof. We prove the lemma by induction on (a, b) with $b \leq a$. Here, we are considering the lexicographic order $\leq$ defined on the lower diagonal $\Delta^{\geq} := \{(a, b) : a \geq b\} \subseteq \mathbb{N} \times \mathbb{N}$ as follows:

$$(a', b') \leq (a, b) \text{ if and only if } a' < a \text{ or } a' = a \text{ and } b' \leq b.$$

Suppose that $(a, b) = 0$. Then, p is not a base-point of $|M|$ and so $\hat{M} = M^2 = M^2 - ab$. Now suppose that the statement is true for $(a', b') < (a, b)$. Our aim is to prove it for (a, b). We blow up the point p, take the pullback of the divisors D_i and remove the fixed part, which is the exceptional divisor bE of the blowup. In fact, the pullback of D_3 contains $c + d$ times E and $c + d \geq b$, thanks to the assumptions $c + md \geq a$ or $d \geq b$: if $d \geq b$, then $c + d \geq b$, otherwise if $d < b$ and $c + md \geq a$, then

$$c + d - b \geq c + m(d - b) \geq c + md - a \geq 0.$$

Restricted to the preimage of our neighborhood of p, these divisors are:

$$a\hat{H} + (a - b)E, \qquad b\hat{K} \qquad \text{and} \qquad c\hat{H} + d\hat{K} + (c + d - b)E.$$

Here, $\hat{H}$ and $\hat{K}$ are the strict transforms of H and K. Let $|\hat{M}|$ be the linear system generated by these three divisors, then $\hat{M}^2 = M^2 - b^2$. If $a = b$ or $b = 0$, then $|\hat{M}|$ is base-point free and we are done. Otherwise, on the preimage, the linear system $|\hat{M}|$ has precisely one new base-point: the intersection point of $\hat{K}$ and E. Locally near this point the three divisors spanning $|\hat{M}|$ are:

$$(a - b)E, \qquad b\hat{K} \qquad \text{and} \qquad d\hat{K} + (c + d - b)E.$$

We need to distinguish two cases, when $m = 1$ or when $m > 1$. In the first case $a - b = q < b$, so that $(b, q) < (a, b)$. We can write $b = m'q + q'$, with $0 \leq q' < q$ and define new coefficients $a' := b$, $b' := q$, $c' := d$ and $d' := c + d - b$. Then, they fulfill the inductive hypothesis, because:

If $c + d \geq a$, then

$$d' = c + d - b \geq a - b = q = b',$$

else if $d \geq b$, then

$$c' + m'd' \geq c' = d \geq b = a'.$$

By induction, the self-intersection of the new linear system $\hat{M}$ is equal to

$$\hat{M}^2 = (M^2 - b^2) - qb = M^2 - ab.$$

In the case $m > 1$, then $b \leq a - b$ and $(a - b, b) < (a, b)$. We define new variables $a' := a - b$, $b' := b$, $c' := c + d - b$, and $d' := d$. Observe that $a' = a - b = (m-1)b' + q$ and we can define $m' := m - 1$. They satisfy the inductive hypothesis, because of the estimations:

If $c + md \geq a$, then

$$c' + m'd' = c + d - b + (m-1)d = c + md - b \geq a - b = a',$$

else if $d \geq b$, then $d' \geq b'$. Hence, the self-intersection of the new linear system $\hat{M}$ is equal to

$$\hat{M}^2 = (M^2 - b^2) - (a-b)b = M^2 - ab. \qquad \square$$

Proof of Theorem 1. First, we show that the three diagonal actions $\phi \times (\phi \circ A)$ on $F \times F$ are free. Indeed, as remarked above, the non-trivial stabilizers of the points on the first copy of F are generated by (1,0), (0,1), and (6,6). However, none of these elements have a fixed point on the second copy of F under the twisted actions $\phi \circ A$. Thus, the actions are free and the quotient surfaces S are smooth, projective, and of general type. The latter holds because the genus of the Fermat septic is $g(F) = 15 \geq 2$. Moreover, they are regular surfaces, because

$$H^0(\Omega^1_S) \simeq \left(H^0(\Omega^1_F) \oplus H^0(\Omega^1_F)\right)^{\mathbb{Z}_7^2} = H^0(\Omega^1_F)^{\mathbb{Z}_7^2} \oplus H^0(\Omega^1_F)^{\mathbb{Z}_7^2} = 0,$$

as $F/\mathbb{Z}_7^2$ is biholomorphic to $\mathbb{P}^1$. The geometric genus of each S is therefore equal to

$$p_g = \chi(\mathcal{O}_S) - 1 = \frac{(g(F)-1)^2}{|\mathbb{Z}_7^2|} - 1 = \frac{14^2}{49} - 1 = 3.$$

Using Lemma 1, we compute a basis of $H^0(K_S)$ for each surface S. Replacing the affine variables by $\frac{x_i}{x_0}$ and $\frac{y_j}{y_0}$ and multiplying by $x_0^4 y_0^4$ we obtain the bi-quartics that define the canonical map.

It remains to determine the degree of Φ_{K_S} for each S. For this purpose, we *resolve the indeterminacy* of these maps by a sequence of blowups, as explained in the textbook [6, Theorem II.7]:

$$\begin{array}{ccc} \hat{S} & \longrightarrow & S \\ & \searrow^{\Phi_{\hat{M}}} & \downarrow{\scriptstyle \Phi_{K_S}} \\ & & \mathbb{P}^2. \end{array}$$

Here, $|\hat{M}|$ is a base-point free linear system. The self-intersection $\hat{M}^2$ is positive if and only if $\Phi_{\hat{M}}$ is not composed with a pencil. In this case, $\Phi_{\hat{M}}$ is onto and it holds:

$$\deg(\Phi_{K_S}) = \deg(\Phi_{\hat{M}}) = \hat{M}^2.$$

For the computation of the resolution, it is convenient to write the divisors of the bi-quartics defining Φ_{K_S} as linear combinations of the reduced curves $F_j := \{x_j = 0\}$ and $G_k := \{y_k = 0\}$ on S. Note that F_j and G_k intersect transversally in only one point and $(F_j, F_k) = (G_j, G_k) = 0$, for all j, k. Thus, these curves can be illustrated as a grid of three vertical and three horizontal lines.

Consider the third surface in the table. Here, the divisors of the three bi-quartics spanning the canonical system $|K_S|$ are:

$$F_1 + 3F_2 + 4G_2, \qquad 2F_1 + 2F_2 + 3G_0 + G_1, \qquad \text{and} \qquad F_0 + 3F_1 + G_0 + G_1 + 2G_2.$$

The fixed part of $|K_S|$ is F_1 and the mobile part $|M|$ has precisely four base-points:

$$F_1 \cap G_2 = \{p_{12}\}, \quad F_2 \cap G_0 = \{p_{20}\}, \quad F_2 \cap G_1 = \{p_{21}\}, \quad \text{and} \quad F_2 \cap G_2 = \{p_{22}\}.$$

In this case, observe that $M^2 = (3F_2 + 4G_2)^2 = 24$. We can perform the computation of the difference $M^2 - \hat{M}^2$ by applying Lemma 2 recursively to each base-point of $|M|$:

- Around p_{12}, the divisors D_i are given by $4G_2, F_1$ and $2F_1 + 2G_2$. In the notation of the lemma, $a = 4, b = 1$ and $c = d = 2$. This implies $c + md = 10 \geq 4$ and $d \geq q - 1 = -1$. The correction term is $ab = 4$.
- Around p_{20} the divisors are $3F_2$, $2F_2 + 3G_0$ and G_0. In this case, $a = 3, b = 1, c = 2, d = 3$ and the correction term is $ab = 3$.
- Around p_{21}, we have $3F_2, 2F_2 + G_1$ and G_1, which yields 3 as correction term.
- Around p_{22} we have $3F_2 + 4G_2, 2F_2$ and $2G_2$, thus the correction term is 4.

The degree of the canonical map is therefore given by

$$\deg(\Phi_{K_S}) = M^2 - (M^2 - \hat{M}^2) = 24 - 4 - 3 - 3 - 4 = 10.$$

The degree of the canonical map of all of the other surfaces can be computed in the same way, except for the second one that we treat separately. Here, the fixed part of $|K_S|$ is $F_1 + G_1$ and the mobile part is spanned by three divisors

$$3F_0 + G_1 + 2G_2, \qquad F_0 + F_1 + F_2 + 2G_1 + G_2, \qquad \text{and} \qquad 3F_1 + 3G_0.$$

There are three base-points $F_0 \cap G_0 = \{p_{00}\}$, $F_1 \cap G_1 = \{p_{11}\}$, and $F_1 \cap G_2 = \{p_{12}\}$. Lemma 2 applies to p_{00} and p_{11}, and yields 3 as the correction term for both points. However, it cannot be applied to the third point p_{12}. Near this point, the configuration of the three divisors spanning $|M|$ is

$$3F_1, \quad 2G_2, \qquad \text{and} \qquad F_1 + G_2.$$

The corresponding correction term of $M^2 - \hat{M}^2$ is 5 and we conclude

$$\deg(\Phi_{K_S}) = M^2 - (M^2 - \hat{M}^2) = (3F_1 + 3G_0)^2 - 3 - 3 - 5 = 7. \qquad \square$$

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REFERENCES

[1] I. Bauer, F. Catanese, and F. Grunewald, *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves*, Pure Appl. Math. Q. **4** (2008), 547–586.

[2] I. Bauer and F. Catanese, *On rigid compact complex surfaces and manifolds*, Adv. Math. **333** (2018), 620–669.

[3] I. Bauer, S. Garion, and A. Vdovina (eds.), *Beauville surfaces and groups*, Springer proceedings in mathematics & statistics, Proceedings of the conference held at the University of Newcastle, Newcastle-upon-Tyne, June 7–9, 2012, vol. 123, 2015.

[4] I. Bauer and R. Pignatelli, *Rigid but not infinitesimally rigid compact complex manifolds*, Duke Math. J. **170** (2021), 1757–1780.

[5] A. Beauville, *L'application canonique pour les surfaces de type ǵeneral*, Invent. Math. **55** (1979), 121–140.

[6] A. Beauville, *Complex algebraic surfaces*, Pondon Math. Soc. Student Texts. **34** (1996).

[7] W. Bosma and J. Cannon, *The Magma algebra system. I. The user language*, Computational algebra and number theory (London, 1993), J. Symb. Comput. **24** (1997), 235–265.

[8] D. Frapporti and C. Gleissner, *Rigid manifolds of general type with non-contractible universal cover*, 2021, arXiv:2104.06775v2.

[9] C. Gleissner, *The classification of regular surfaces isogenous to a product of curves with* $\chi(\mathcal{O}_S) = 2$, Beauville surfaces and groups, Springer Proc. Math. Stat., vol. 123, Springer, 2015, pp. 79–95.

[10] C. Gleissner, R. Pignatelli, and C. Rito, *New surfaces with canonical map of high degree*, Commun. Anal. Geom. (2018), in press. arXiv:1807.11854.

[11] J. Morrow and K. Kodaira, *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York, 1971.

[12] M. M. Lopes and R. Pardini, *On the degree of the canonical map of a surface of general type*, 2021, arXiv:2103.01912v1.

[13] B. Nguyen, *A new example of an algebraic surface with canonical map of degree 16*, Arch. Math. (Basel) **113** (2019), 385–390.

[14] B. Nguyen, *Some examples of algebraic surfaces with canonical map of degree 20*, C. R. Math. Acad. Sci. **359** (2021), 1145–1153.

[15] R. Pardini, *Abelian covers of algebraic varieties*, J. Reine Angew. Math. **417** (1991), 191–213.

[16] U. Persson, *Double coverings and surfaces of general type*, Algebraic geometry, Lecture notes in math., vol. 687, Springer, 1978, pp. 168–195.

[17] C. Rito, *New canonical triple covers of surfaces*, Proc. Amer. Math. Soc. **143** (2015), 4647–4653.

[18] C. Rito, *A surface with canonical map of degree 24*, Int. J. Math. **28** (2017).

[19] C. Rito, *A surface with $q = 2$ and canonical map of degree 16*, Michigan Math. J. **66** (2017), 99–105.

[20] C. Rito, *Surfaces with canonical map of maximum degree*, J. Algebr. Geom. **31** (2022), 127–135.

[21] I. Shafarevich, *Basic algebraic geometry 1, Varieties in projective space*, Springer, Heidelberg, 2013.

[22] S. L. Tan, *Cusps on some algebraic surfaces and plane curves*, Complex analysis, complex geometry and related topics-Namba, vol. 60, 2003, pp. 106–121.

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