CONCENTRATION PHENOMENA FOR THE SCHRÖDINGER-POISSON SYSTEM IN $\mathbb{R}^2$

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Abstract. We perform a semiclassical analysis for the planar Schrödinger-Poisson system

$$(SP_\varepsilon) \qquad \begin{cases} -\varepsilon^2 \Delta\psi + V(x)\psi = E(x)\psi & \text{in } \mathbb{R}^2, \\ -\Delta E = |\psi|^2 & \text{in } \mathbb{R}^2, \end{cases}$$

where ε is a positive parameter corresponding to the Planck constant and V is a bounded external potential. We detect solution pairs $(u_\varepsilon, E_\varepsilon)$ of the system (SP_ε) as $\varepsilon \to 0$, leaning on a nongeneracy result in [3].

1. Introduction

We are concerned with the planar Schrödinger-Poisson system

$$\begin{cases} -\varepsilon^2 \Delta\psi + V(x)\psi = E(x)\psi & \text{in } \mathbb{R}^2, \\ -\Delta E = |\psi|^2 & \text{in } \mathbb{R}^2, \end{cases} \tag{1.1}$$

which presents some special features, because of the different nature of the Newtonian potential in two-dimensional space. This system has been derived in $\mathbb{R}^3$ by R. Penrose in [21] in his description of the self-gravitational collapse of a quantum mechanical system (see also [20, 22, 19, 18]). The rigorous mathematical study of the nonlinear Schrödinger equation with nonlocal nonlinearity, involving a Coulomb type convolution potential, dates back to the seminal papers by Lieb [14] and Lions [15]. Successively in [24] Wei and Winter studied the semiclassical limit for the Schrödinger-Poisson system, after showing the nondegeneracy of the least energy solutions of a related limiting system (see also [13]). We also mention the papers [6, 8, 9, 17] where variational and topological methods have been employed to derive concentration phenomena for generalized NLS equations with more general nonlocal nonlinearity in dimensional $d \geq 3$, where the nondegeneracy properties of the linearized operators do not hold.

The rigorous study of the Schrödinger-Poisson system in $\mathbb{R}^2$ remained open for long time, since it appears more delicate. Differently from the Coulomb potential, the Newton potential in $\mathbb{R}^2$ is sign-changing and it presents singularities at zero and infinity. Moreover we recall that the Poisson equation $-\Delta E = |\psi|^2$ determines the solution $E \colon \mathbb{R}^2 \to \mathbb{R}$ only up to harmonic functions, and every semibounded harmonic function is costant in $\mathbb{R}^2$. Therefore if $\psi \in L^\infty(\mathbb{R}^2)$ and E solves the Poisson equation under suitable additional assumption at infinity, such as $E(x) \to -\infty$ as $|x| \to +\infty$, then we have $E(x) = \Phi_\psi(x) + c$, where c is a constant and Φ_ψ is the convolution of fundamental solution of $-\Delta$ in $\mathbb{R}^2$ with $|\psi|^2$, namely $\Phi_\psi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log \frac{1}{|x-y|} |\psi(y)|^2 dy$.

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In literature, apart from some numerical results in [12], existence and uniqueness results of spherically symmetric solutions of (1.1) were proved by Stubbe and Vuffray [5], for $V \equiv 1$, using shooting methods for the associated ODE system (see also [4] for the one-dimensional case).
In [16] Masaki proved a global well-posedness of the Cauchy problem for (1.1) in a subspace of $H^1(\mathbb{R}^2)$, where $E(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log \frac{|y|}{|x-y|} |\psi(y)|^2 dy$, which means $E(0) = 0$.

In the more natural case, E coincides with the Newtonian potential Φ of $|\psi|^2$, the Schrödinger-Poisson system with a constant potential can be written as the following Schrödinger equation with a nonlocal nonlinearity:

$$-\Delta u + u = \frac{1}{2\pi} \left[\log \frac{1}{|\cdot|} \star |u|^2 \right] u, \quad x \in \mathbb{R}^2. \tag{1.2}$$

For such an integro-differential equation, unlike the 3D case, the applicability of variational tools is not straightforward, because the usual Sobolev spaces do not provide a good environment to work in. In [23] Stubbe tackled this problem by setting a suitable variational framework for (1.2) within the space

$$X = \left\{ u \in H^1(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} \log(1+|x|)|u(x)|^2 \, dx < \infty \right\},$$

endowed with the norm

$$\|u\|_X^2 = \int_{\mathbb{R}^2} \left(|\nabla u|^2 + |u|^2 \right) \, dx + \int_{\mathbb{R}^2} \log(1+|x|)|u(x)|^2 \, dx.$$

The space X provides a reasonable variational framework, but its norm does not detect the invariance of the problem under translations; furthermore the quadratic part of the energy functional associated to (1.2) is not coercive on X. These difficulties enforced the implementation of new variational ideas and estimates to treat nonlinear Schrödinger equation with nonlocal nonlinearities involving logarithmic type convolution potential [7, 10, 11]. In particular in [10], the authors proved the existence result of an unique positive ground state solution U to (1.2). Sharp asymptotics and the nondegeneracy of the ground state solution U has been proved in [3].
In the present paper we study the existence of solution pairs of the Schrödinger-Poisson system as the parameter $\varepsilon \to 0^+$. This study presents some new aspects with respect to the 3D case, since the Newtonian potential in $\mathbb{R}^2$ does not scale algebraically.
The semiclassical analysis remained in the background until very recent years and, to the best of our knowledge, it has only been treated by Masaki in [16] via WKB approximation.
Here we adapt some pertubation method developed in [1, 2] in the variational framework X where the norm depends on the weight $x \mapsto \log(1+|x|)$. This makes it more involved to apply a finite dimensional reduction.

In the rest of the paper we will consider a potential function $V \colon \mathbb{R}^2 \to \mathbb{R}$ satisfying the following condition:

(V) $V \in \mathrm{C}^2(\mathbb{R}^2)$, $\inf_{x \in \mathbb{R}^2} V(x) > 0$ and

$$\sup_{x \in \mathbb{R}^2} \left[|V(x)| + \sum_{j=1}^{2} |\partial_j V(x)| + \sum_{i,j=1}^{2} \left| \partial_{ij}^2 V(x) \right| \right] < +\infty.$$

Setting $v(x) = \varepsilon\psi(x)$, the system (1.1) can be written

$$\begin{cases} -\varepsilon^2\Delta v + V(x)v = Ev & \text{in } \mathbb{R}^2, \\ -\varepsilon^2\Delta E = |v|^2 & \text{in } \mathbb{R}^2. \end{cases} \tag{1.3}$$

Our main existence result can be summarized as follows.

Theorem 1.1. *Suppose that V satisfies (V) and has a non-degenerate critical point x_0, i.e. $\nabla V(x_0) = 0$ and $D^2V(x_0)$ is either positive- or negative-definite. Then, for every $\varepsilon > 0$ sufficiently small, the system (1.3) possesses a solution $(v_\varepsilon, E_\varepsilon)$ such that*

$$v_\varepsilon(x) \simeq U\left(\frac{x-x_0}{\varepsilon}\right), \quad E_\varepsilon(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \log\frac{\varepsilon}{|x-z|}|v_\varepsilon(z)|^2\,dz$$

where U is the unique (up to translations) positive ground state solution of the limiting equation

$$-\Delta u + V(x_0)u = \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star |u|^2\right]u, \quad x \in \mathbb{R}^2. \tag{1.4}$$

Remark 1.2. In Theorem (1.1) we have $E_\varepsilon(x) = \varepsilon^{-2}\Phi_{v_\varepsilon}(x) + c_\varepsilon$ where $\Phi_{v_\varepsilon}(x) = \log\frac{1}{|\cdot|} \star v_\varepsilon^2$ and $c_\varepsilon = \varepsilon^{-2}\log\varepsilon\|v_\varepsilon\|_2^2$. Coming back to the system (1.1), we derive the existence of the solution pair $(\varepsilon^{-1}v_\varepsilon, E_\varepsilon)$ for $\varepsilon > 0$ small.

2. Functional setting

Without loss of generality, we will assume that $x_0 = 0$ and $V(0) = 1$. Setting $u(x) = v(\varepsilon x)$ and $\omega(x) = E(\varepsilon x)$, the system (1.3) becomes

$$\begin{cases} -\Delta u + V(\varepsilon x)u = \omega(x)u & \text{in } \mathbb{R}^2, \\ -\Delta\omega = |u|^2 & \text{in } \mathbb{R}^2. \end{cases} \tag{2.1}$$

The second equation in (2.1) can be explicitly solved with respect to ω. Choosing ω as the convolution of the fundamental solution of $-\Delta$ in $\mathbb{R}^2$ with $|u|^2$, this system can be written as the single nonlocal equation

$$-\Delta u + V(\varepsilon x)u = \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star |u|^2\right]u, \quad x \in \mathbb{R}^2. \tag{2.2}$$

We consider the functional space

$$X = \left\{u \in H^1(\mathbb{R}^2) \mid |u|_* < +\infty\right\},$$

where

$$|u|_*^2 = \int_{\mathbb{R}^2} \log(1+|x|)\,|u(x)|^2\,dx.$$

We endow X with the norm

$$\|u\|_X^2 = \|u\|_{H^1}^2 + |u|_*^2$$

and the associated scalar product

$$\langle u \mid v\rangle_X = \int_{\mathbb{R}^2} [\nabla u\cdot\nabla v + uv]\,dx + \int_{\mathbb{R}^2} \log(1+|x|)u(x)v(x)\,dx.$$

The norms in $H^1(\mathbb{R}^2)$ and $L^q(\mathbb{R}^2)$ will be denoted by $\|\cdot\|_{H^1}$ and $|\cdot|_q$, respectively.

Solutions to (2.2) correspond to critical points of the energy functional $I_\varepsilon \colon X \to \mathbb{R}$ defined by

$$I_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + V_\varepsilon |u|^2\, dx - \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log\left(\frac{1}{|x-y|}\right) |u(x)|^2 |u(y)|^2\, dx\, dy,$$

where we set $V_\varepsilon(x) = V(\varepsilon x)$.
We observe that

$$\|u\|^2 = \int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\varepsilon |u|^2\right] dx + |u|_*^2 \tag{2.3}$$

can be considered as an equivalent norm on X by virtue of assumption (V). The functional I_ε fails to be continuous on the Sobolev space $H^1(\mathbb{R}^2)$. On the contrary, arguing as in [10, Lemma 2.2], we can infer the following regularity result on X.

Proposition 2.1. *If V satisfies (V), then I_ε is a functional of class C^2 on X.*

3. Limiting Equation

We consider the planar integro-differential equation

$$-\Delta u + u = \frac{1}{2\pi}\left[\log\frac{1}{|\cdot|} \star |u|^2\right] u, \qquad \text{in } \mathbb{R}^2, \tag{3.1}$$

which has the rôle of a *limiting problem* for (2.2). We define the energy functional $I \colon X \to \mathbb{R}$ associated to (3.1):

$$I(u) = \frac{1}{2}\|u\|_{H^1}^2 + \frac{1}{8\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log(|x-y|)|u(x)|^2|u(y)|^2\, dx\, dy.$$

For future reference, we introduce some shorthand: let us set

$$B(f,g) = -\frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y|\ f(x)g(y)\, dx\, dy,$$

so that

$$I(u) = \|u\|_{H^1}^2 - \frac{1}{4}B(u^2,u^2).$$

It follows from [10, Lemma 2.2] that I is of class C^2 and that

$$I'(u)[\varphi] = \int_{\mathbb{R}^2} [\nabla u \cdot \nabla\varphi + u\varphi] - B(u^2, u\varphi)$$
$$I''(u)[\varphi,\psi] = \int_{\mathbb{R}^2} [\nabla\varphi \cdot \nabla\psi + \varphi\psi] - B(u^2,\varphi\psi) - 2B(u\varphi, u\psi).$$

It has been proved in [10, Theorem 1.1] that the restriction of I to the associated Nehari manifold

$$\mathcal{N} = \{u \in X \setminus \{0\} \mid I'(u)[u] = 0\}$$

attains a global minimum. Moreover, every minimizer $u \in \mathcal{N}$ of $I_{|\mathcal{N}}$ is a solution of(3.1) which does not change sign and obeys the variational characterization

$$I(u) = \inf_{u\in X}\sup_{t\in\mathbb{R}} I(tu).$$

From [10, Theorem 1.3] we have the following result.

Theorem 3.1. *Every positive solution $u \in X$ of (3.1) is radially symmetric up to translation and strictly decreasing in the distance from the symmetry center. Moreover u is unique, up to translation in $\mathbb{R}^2$.*

Moreover, from [3, Theorem 1], the sharp asymptotics of the radially symmetric positive solution of (3.1) are known.

Theorem 3.2. *If $u \in X$ is a radially symmetric positive solution of (3.1), there exists $\mu > 0$ such that, as $|x| \to +\infty$,*

$$u(x) = \frac{\mu + o(1)}{\sqrt{|x|}(\log|x|)^{1/4}} \exp\left(-\sqrt{M}\mathrm{e}^{-1/M} \int_1^{|x|\mathrm{e}^{1/M}} \sqrt{\log s}\, ds\right),$$

where $M = (2\pi)^{-1} \int_{\mathbb{R}^2} |u|^2\, dx$.

We consider the linearization on a positive solution u of (3.1). Let $\mathcal{L}(u) \colon \tilde{X} \to L^2(\mathbb{R}^2)$ be the linear operator defined by

$$\mathcal{L}(u) \colon \varphi \mapsto -\Delta\varphi + (1-w)\varphi + 2u\left(\frac{\log}{2\pi} \star (u\varphi)\right),$$

where

$$w \colon \mathbb{R}^2 \to \mathbb{R}, \quad x \mapsto \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}|u(y)|^2 dy$$

and

$$\tilde{X} = \left\{\varphi \in X \mid \text{for every } \psi \in C_c^\infty(\mathbb{R}^2)\colon \int_{\mathbb{R}^2} \varphi\mathcal{L}(u)\psi = \int_{\mathbb{R}^2} f\psi\right\} \tag{3.2}$$

By standard arguments, one easily shows that $\mathcal{L}(u)$ is a self adjoint operator acting on $L^2(\mathbb{R}^2)$ with domain $\tilde{X}$. Also, differentiating the equation (3.1), it is clear that $\alpha_1\partial_{x_1}u + \alpha_2\partial_{x_2}u \in \ker\mathcal{L}(u)$ for every $\alpha_1, \alpha_2 \in \mathbb{R}$.
The following result has been proved in [3, Theorem 3].

Theorem 3.3. *Let $u \in X$ be a positive solution of (3.1). Then*

$$\ker\mathcal{L}(u) = \left\{\gamma\cdot\nabla u \mid \gamma \in \mathbb{R}^2\right\}.$$

The functional-analytic properties of the second derivative of I will play a crucial rôle in our analysis.

Lemma 3.4. *Let $u \in X$ be a positive solution of (3.1). The operator $I''(u)$ is a Fredholm operator of index zero from X to its dual space X^*.*

Proof. We will actually prove that $I''(u) = A + K$, where A is a bounded invertible operator and K is a compact operator on X.

Set $c^2 = \frac{1}{2\pi}\int_{\mathbb{R}^2} |u(y)|^2\,dy$. For any $\varphi \in X$ and $\psi \in X$, we have

$$\begin{aligned} I''(u)[\varphi,\psi] &= \int_{\mathbb{R}^2} [\nabla\varphi(x)\nabla\psi(x) + \varphi(x)\psi(x)]\;dx \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y||u(y)|^2\varphi(x)\psi(x)\,dx\,dy \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|u(y)\varphi(y)u(x)\psi(x)\,dy\,dx \\ &= \int_{\mathbb{R}^2} (\nabla\varphi(x)\nabla\psi(x) + \varphi(x)\psi(x) + c^2\log(1+|x|)\varphi(x)\psi(x))\,dx \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\log|x-y| - \log(1+|x|)]\,|u(y)|^2\varphi(x)\psi(x)\,dx\,dy \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|u(y)\varphi(y)u(x)\psi(x)\,dx\,dy. \end{aligned}$$

We have deduced the decomposition $I''(u) = A + K$, where the operators A and K act as follows:

$$\langle A\varphi,\psi\rangle = \int_{\mathbb{R}^2} \Big(\nabla\varphi\cdot\nabla\psi + \varphi\psi + c^2\log(1+|x|)\varphi(x)\psi(x)\Big)\,dx \tag{3.3}$$

and

$$\begin{aligned} \langle K\varphi,\psi\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}[\log|x-y| - \log(1+|x|)]|u(y)|^2\varphi(x)\psi(x)\,dx\,dy \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y|u(y)\varphi(y)u(x)\psi(x)\,dx\,dy. \end{aligned} \tag{3.4}$$

Equation (3.3) implies that the correspondence

$$u \in X \mapsto \langle Au, u\rangle$$

is an equivalent norm on X. It follows that the operator A is invertible from X to X^*.

We claim that K is compact from X to X^*. Indeed, let $\{\varphi_n\}_n \subset X$ be a sequence such that $\varphi_n \rightharpoonup 0$ as $n \to +\infty$. It follows that $\|\varphi_n\|_X \leq D$ for any $n \in \mathbb{N}$.
We prove that

$$\lim_{n\to+\infty} \sup_{\substack{\psi\in X \\ \|\psi\|_X=1}} |\langle K\varphi_n,\psi\rangle| = 0. \tag{3.5}$$

Fix $\varepsilon > 0$ and $\psi \in X$ such that $\|\psi\|_X = 1$. Since $u \in X$, there exists $M > 0$ such that

$$\frac{D}{2\pi}\int_{|y|>M} \log(1+|y|)|u(y)|^2\,dy < \frac{\varepsilon}{4} \quad\text{and}\quad \frac{D}{\pi}\int_{|y|>M} |u(y)|^2\,dy < \frac{\varepsilon}{4}.$$

We evaluate

$$\begin{aligned}\langle K\varphi_n,\psi\rangle &= \frac{1}{2\pi}\int_{|y|>M}\int_{\mathbb{R}^2}\big[\log(1+|x-y|)-\log(1+|x|)\big]|u(y)|^2\varphi_n(x)\psi(x)\,dx\,dy\\ &+\frac{1}{2\pi}\int_{|y|\le M}\int_{\mathbb{R}^2}\big[\log(1+|x-y|)-\log(1+|x|)\big]|u(y)|^2\varphi_n(x)\psi(x)\,dx\,dy\\ &-\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)|u(y)|^2\varphi_n(x)\psi(x)\,dx\,dy\\ &+\frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1+|x-y|)u(y)\varphi_n(y)u(x)\psi(x)\,dx\,dy\\ &-\frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)u(y)\varphi_n(y)u(x)\psi(x)\,dx\,dy.\end{aligned}$$

Recalling the elementary inequality $\log(1+|x-y|)\le\log(1+|x|)+\log(1+|y|)$ for $x\in\mathbb{R}^2$, $y\in\mathbb{R}^2$, we have that

$$\begin{aligned}|\langle K\varphi_n,\psi\rangle| &\le \frac{1}{2\pi}\int_{|y|>M}|u(y)|^2\,dy\int_{\mathbb{R}^2}\big[2\log(1+|x|)+\log(1+|y|)\big]|\varphi_n(x)||\psi(x)|\,dx\\ &+\frac{1}{2\pi}\int_{|y|\le M}|u(y)|^2\,dy\int_{\mathbb{R}^2}\left|\log\left(\frac{1+|x-y|}{1+|x|}\right)\right||\varphi_n(x)||\psi(x)|\,dx\\ &+\frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)|u(y)|^2|\varphi_n(x)||\psi(x)|\,dx\,dy\\ &+\frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(1+|x-y|)|u(y)||\varphi_n(y)||u(x)|\psi(x)|\,dy\\ &+\frac{1}{\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log\left(1+\frac{1}{|x-y|}\right)|u(y)||\varphi_n(y)||u(x)||\psi(x)|\,dx\,dy.\end{aligned}$$

Firstly, we estimate

$$\begin{aligned}&\frac{1}{2\pi}\int_{|y|>M}|u(y)|^2\,dy\int_{\mathbb{R}^2}\big[2\log(1+|x|)+\log(1+|y|)\big]|\varphi_n(x)||\psi(x)|\,dx\\ &\le\left(\frac{1}{\pi}\int_{|y|>M}|u(y)|^2dy\right)\|\varphi_n\|_X\|\psi\|_X+\frac{1}{2\pi}\left(\int_{|y|>M}\log(1+|y|)|u(y)|^2dy\right)\|\varphi_n\|_2\|\psi\|_2\\ &\qquad\le\frac{D}{\pi}\left(\int_{|y|>M}|u(y)|^2dy\right)+\frac{D}{2\pi}\left(\int_{|y|>M}\log(1+|y|)|u(y)|^2dy\right)\le\frac{\varepsilon}{2}.\end{aligned}$$

We claim that for every $M>0$, there exists $L>0$ such that for any $y\in\mathbb{R}^2$ with $|y|\le M$ and for any $x\in\mathbb{R}^2$ we have

$$\left|\log\frac{1+|x-y|}{1+|x|}\right|<L. \tag{3.6}$$

Indeed for any $x\in\mathbb{R}^2$ and $y\in\mathbb{R}^2$, $|y|\le M$ we have

$$\frac{1+|x-y|}{1+|x|}\le 1+M.$$

Now take $R=2M-1>0$, we have that $\frac{M}{1+|x|}<1/2$ for any $x\in\mathbb{R}^2$, and $|x|\ge R$.

It follows that for any $x \in \mathbb{R}^2$, $y \in \mathbb{R}^2$ with $|x| \geq |y|$, $|x| \geq R$ and $|y| \leq M$:

$$\frac{1+|x-y|}{1+|x|} \geq \frac{1+||x|-|y||}{1+|x|} \geq 1 - \frac{|y|}{1+|x|} \geq 1 - \frac{M}{1+|x|} > \frac{1}{2}.$$

On the other hand, if $|x| \leq R$:

$$\frac{1+|x-y|}{1+|x|} \geq \frac{1}{1+R} = \frac{1}{2M}.$$

Conversely if $|x| \leq |y|$, we infer that $|x| \leq M$ and

$$\frac{1+|x-y|}{1+|x|} \geq \frac{1}{1+M}.$$

We conclude that there exists $L > 0$ such that (3.6) holds.
It follows that

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}^2} \int_{|y| \leq M} & \left| \log \left(\frac{1+|x-y|}{1+|x|} \right) \right| |u(y)|^2 |\varphi_n(x)| |\psi(x)| \, dx \, dy \\
&\leq \frac{L}{2\pi} \int_{|y| \leq M} |u(y)|^2 \, dy \int_{\mathbb{R}^2} |\varphi_n(x)| |\psi(x)| \, dx \leq \frac{L}{2\pi} \left(\int_{|y| \leq M} |u(y)|^2 \, dy \right) \|\varphi_n\|_2 \|\psi\|_2 \\
&\leq \frac{\Gamma L}{2\pi} \|\varphi_n\|_2 \|\psi\|_X \leq \frac{\Gamma L}{2\pi} \|\varphi_n\|_2,
\end{aligned}$$

where $\Gamma = \int_{|y| \leq M} |u(y)|^2 dy$.

By Hardy-Sobolev-Littlewood inequality we have

$$\begin{aligned}
\frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} & \log \left(1 + \frac{1}{|x-y|} \right) |u(y)|^2 |\varphi_n(x)| |\psi(x)| \, dx \, dy \\
&\leq \frac{1}{2\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{1}{|x-y|} |u(y)|^2 |\varphi_n(x)| |\psi(x)| \, dx \, dy \leq \frac{1}{2\pi} \|u\|_{8/3}^2 \|\varphi_n\|_{8/3} \|\psi\|_{8/3} \\
&\leq \frac{c_3}{2\pi} \|u\|_{8/3}^2 \|\varphi_n\|_{8/3} \|\psi\|_X \leq \frac{c_3}{2\pi} \|u\|_{8/3}^2 \|\varphi_n\|_{8/3}.
\end{aligned}$$

where $c_3 > 0$ is a suitable constant. Moreover we can take $R > 0$ such that

$$\frac{D}{\pi} \left(\int_{|y| > R} \log(1+|y|) |u(y)|^2 dy \right)^{\frac{1}{2}} \|u\|_2 < \frac{\varepsilon}{4}.$$

We have

$$
\begin{aligned}
&\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log(1+|x-y|)|u(y)||\varphi_n(y)||u(x)||\psi(x)|\,dx\,dy\\
&\le \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log(1+|x|)|u(y)||\varphi_n(y)||u(x)||\psi(x)|\,dx\,dy\\
&\quad+\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\log(1+|y|)|u(y)||\varphi_n(y)||u(x)||\psi(x)|\,dx\,dy\\
&\le \frac{1}{\pi}\|u\|_2\|u\|_X\|\varphi_n\|_2\|\psi\|_X\\
&\quad+\frac{1}{\pi}\int_{|y|\le R}\log(1+|y|)u(y)|\varphi_n(y)|dy\int_{\mathbb{R}^2}|u(x)||\psi(x)|dx\\
&\quad+\frac{1}{\pi}\int_{|y|>R}\log(1+|y|)u(y)|\varphi_n(y)|dy\int_{\mathbb{R}^2}|u(x)||\psi(x)|dx\\
&\le \frac{1}{\pi}\|u\|_X^2\|\varphi_n\|_2+\frac{1}{\pi}\log(1+R)\|u\|_2^2\|\varphi_n\|_2\|\psi\|_2\\
&\quad+\frac{D}{\pi}\left(\int_{|y|>R}\log(1+|y|)|u(y)|^2\,dy\right)^{1/2}\|u\|_2\|\psi\|_X\\
&\le \frac{1}{\pi}\|u\|_X^2\|\varphi_n\|_2+\frac{1}{\pi}\log(1+R)\|u\|_2^2\|\varphi_n\|_2\\
&\quad+\frac{D}{\pi}\left(\int_{|y|>R}\log(1+|y|)|u(y)|^2\,dy\right)^{1/2}\|u\|_2\\
&\le \frac{1}{\pi}\left(1+\log(1+R)\right)\|u\|_X^2\|\varphi_n\|_2+\frac{\varepsilon}{4}.
\end{aligned}
$$

By the Hardy-Sobolev-Littlewood inequality we have

$$
\begin{aligned}
\frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}&\log\left(1+\frac{1}{|x-y|}\right)|\varphi_n(y)|u(y)u(x)|\psi(x)|\,dx\,dy\\
&\le \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2}\frac{1}{|x-y|}u(y)|\varphi_n(y)|u(x)|\psi(x)|\,dx\,dy\\
&\le \frac{1}{\pi}\|u\|_{8/3}^2\|\varphi_n\|_{8/3}\|\psi\|_X.
\end{aligned}
$$

Finally we conclude that

$$
\begin{aligned}
|\langle K\varphi_n,\psi\rangle| \le \frac{3\varepsilon}{4}&+\frac{c_3}{2\pi}\|u\|_{8/3}^2\|\varphi_n\|_{8/3}+\frac{\Gamma L}{2\pi}\|\varphi_n\|_2\\
&+\frac{1}{\pi}(1+\log(1+R))\|u\|_X^2\|\varphi_n\|_2+\frac{1}{\pi}\|u\|_{8/3}^2\|\varphi_n\|_{8/3}.
\end{aligned}
$$

Taking into account that X is compactly embedded into $L^s(\mathbb{R}^2)$ for any $s\in[2,+\infty)$, we derive that $\|\varphi_n\|_2\to 0$ and $\|\varphi_n\|_{8/3}\to 0$ as $n\to+\infty$. Therefore there exists $n_0\in\mathbb{N}$ such that for any $n\ge n_0$

$$
\frac{c_3}{2\pi}\|u\|_{8/3}^2\|\varphi_n\|_{8/3}+\frac{\Gamma L}{2\pi}\|\varphi_n\|_2+\frac{1}{\pi}(1+\log(1+R))\|u\|_X^2\|\varphi_n\|_2+\frac{1}{\pi}\|u\|_{8/3}^2\|\varphi_n\|_{8/3}<\frac{\varepsilon}{4}.
$$

We derive that $\lim_{n\to+\infty} |\langle K\varphi_n, \psi\rangle| = 0$, uniformly with respect to ψ. Therefore K is compact and the proof is complete. □

Definition 3.5. In the sequel, we will denote by U the unique positive solution of (3.1) such that

$$U(0) = \max_{x\in\mathbb{R}^2} U(x).$$

From the non-degeneracy result, we can infer the following convexity property of $I''(U)$.

Proposition 3.6. *The operator $I''(U)$ has only one negative eigenvalue, and therefore there exists $\delta > 0$ such that*

$$I''(U)[v,v] \geq \delta \|v\|_X^2 \tag{3.7}$$

for every $v \perp_X \operatorname{span}\left\{U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}\right\}$, where $\perp_X$ means orthogonality with respect to the inner product $\langle \cdot \mid \cdot \rangle_X$

Proof. Since

$$-\Delta U + U + \frac{1}{2\pi}\left[\log\star|U|^2\right] U = 0,$$

we find that

$$I''(U)[U,U] = \langle \mathcal{L}(U)U, U\rangle = -2\left(\int_{\mathbb{R}^2} |\nabla U|^2 + \int_{\mathbb{R}^2} |U|^2\right) < 0.$$

Let now $\varphi \in \ker I''(U)$, namely $\varphi \in X$ and $I''(U)\varphi = 0$ in X^*. It follows that $I''(U)\varphi = 0$ also in $\widetilde{X}^*$, but $\varphi \in \widetilde{X}$, so that $\mathcal{L}(U)\varphi = 0$. Hence $\varphi \in \operatorname{span}\{\partial_1 U, \partial_2 U\}$.
On the other hand, if $\varphi \in \operatorname{span}\{\partial_1 U, \partial_2 U\}$, then $\mathcal{L}(U)\varphi = 0$ in $\widetilde{X}^*$. Let $\psi \in X$. By density, ψ is the limit in X of a sequence $g_n \in C_0^\infty(\mathbb{R}^2)$. It follows that

$$I''(U)[\varphi,\psi] = \lim_{n\to+\infty} I''(U)[\varphi, g_n] = \lim_{n\to+\infty} \langle \mathcal{L}(U)\varphi, g_n\rangle = 0$$

and thus $\varphi \in \ker I''(U)$. This shows that $\ker I''(U) = \operatorname{span}\{\partial_1 U, \partial_2 U\}$.
Taking into account that U is a Mountain Pass solution, by Proposition 3.1, we deduce that there exists $\delta > 0$ such that (3.7) holds. □

4. The perturbation technique

We will look for solutions to (2.2) near the embedded submanifold $Z = \{z_\xi \mid \xi \in \mathbb{R}^2\}$, where we set $z_\xi(x) = U(x-\xi)$. Although the norm of X is not invariant under the group of translations defined on X by

$$\tau_\xi u \colon x \in \mathbb{R}^2 \mapsto u(x-\xi),$$

the elementary inequality

$$\log(1+|x-y|) \leq \log(1+|x|+|y|) \leq \log(1+|x|) + \log(1+|y|)$$

yields that $u \in X$ and $\xi \in \mathbb{R}^2$ implies $\tau_\xi u \in X$. It follows that $U(\cdot - \xi) = \tau_\xi U \in X$ for every $\xi \in \mathbb{R}^2$. The invariance under translation of I then implies that Z is a manifold of critical points of I.

We will show that each point of Z is an approximate critical point of I_ε, and that there exists a true critical point of I_ε located in a tubular neighborhood of Z, provided ε is small enough.

Lemma 4.1. *Let assumption (V) be satisfied. Then there exists a constant $C > 0$ such that, for every $\xi \in \mathbb{R}^2$ and every $\varepsilon > 0$ sufficiently small, we have*

$$\|I_\varepsilon'(z_\xi)\| \leq C\left(\varepsilon|\nabla V(0)| + \varepsilon^2\right).$$

Proof. Since z_ξ is a critical point of I, it follows easily that

$$|I_\varepsilon'(z_\xi)[v]|^2 \leq \|v\|_2^2 \int_{\mathbb{R}^2} |V(\varepsilon x) - 1|^2 |z_\xi|^2 \, dx$$

for any $v \in X$. Using the boundedness of D^2V and the exponential decay of z_ξ at infinity, we can prove easily that

$$\int_{\mathbb{R}^2} |V(\varepsilon x) - 1|^2 |z_\xi|^2 \, dx \leq C\varepsilon^2 |\nabla V(0)|^2 + C\varepsilon^4.$$

□

Proposition 4.2. *There exist a constant $\widetilde{C} > 0$ and a constant $M > 0$ such that for every $\xi \in \mathbb{R}^2$, $|\xi| \leq M$, we have*

$$I''(z_\xi)[\varphi, \varphi] \geq \widetilde{C}\|\varphi\|_X^2 \tag{4.1}$$

for every $\varphi \perp_X \left(\operatorname{span}\left\{z_\xi, \frac{\partial z_\xi}{\partial x}, \frac{\partial z_\xi}{\partial y}\right\}\right)$, where $\perp_X$ means orthogonality with respect to the inner product $\langle \cdot \mid \cdot \rangle_X$.

Proof. For the sake of simplicity we denote here $\perp_X$ by $\perp$. In order to get a contradiction, we suppose that there exists a sequence $\{\xi_n\}_n$ in $\mathbb{R}^2$ such that $\xi_n \to 0$ and there exists a sequence $\{\varphi_n\}_n \subset X$ such that $\varphi_n \in \left(\operatorname{span}\left\{z_{\xi_n}, \frac{\partial z_{\xi_n}}{\partial x}, \frac{\partial z_{\xi_n}}{\partial y}\right\}\right)^\perp$,

$$\begin{aligned}
&\varphi_n \rightharpoonup \bar{\varphi} \quad \text{in } X \text{ and in } H^1(\mathbb{R}^2)\\
&\varphi_n \to \bar{\varphi} \quad \text{in } L^2(\mathbb{R}^2),\\
&\|\varphi_n\|_X = 1 \quad \text{for every } n \in \mathbb{N},
\end{aligned}$$

and

$$I''(z_{\xi_n})[\varphi_n, \varphi_n] \leq \frac{1}{n}.$$

Assume that $\bar{\varphi} \neq 0$. Then,

$$\begin{aligned}
\frac{1}{n} \geq I''(z_{\xi_n})[\varphi_n, \varphi_n] &= I''(U)[\varphi_n, \varphi_n] + I''(z_{\xi_n})[\varphi_n, \varphi_n] - I''(U)[\varphi_n, \varphi_n]\\
&\geq I''(U)[\varphi_n, \varphi_n] - \|I''(z_{\xi_n}) - I''(U)\| \, \|\varphi_n\|_X^2 = I''(U)[\varphi_n, \varphi_n] - o(1)
\end{aligned}$$

as $n \to +\infty$. Indeed, the functional I'' is continuous at the point U, and the exponential decay of U at infinity (see Theorem 3.2) immediately yields that $z_{\xi_n} \to U$ strongly in X.

We claim that $\bar{\varphi} \perp U$, $\bar{\varphi} \perp \frac{\partial U}{\partial x}$ and $\bar{\varphi} \perp \frac{\partial U}{\partial y}$ in X. We only prove the first orthogonality property, the other two being similar. By assumption, we have that $\varphi_n \perp z_{\xi_n}$, $\varphi_n \perp \frac{\partial z_{\xi_n}}{\partial x}$, $\varphi_n \perp \frac{\partial z_{\xi_n}}{\partial y}$ for every $n \in \mathbb{N}$. Now,

$$\langle \varphi_n \mid U \rangle_X = -\langle \varphi_n \mid z_{\xi_n} - U \rangle_X.$$

The right-hand side converges to zero because $z_{\xi_n} \to U$ and $\{\varphi_n\}_n$ is a bounded sequence; the left-hand side converges to $\langle \bar\varphi \mid U \rangle_X$. We conclude that $\bar\varphi \perp U$ in X. In a similar way we can prove that $\bar\varphi \perp \frac{\partial U}{\partial x}$ and $\bar\varphi \perp \frac{\partial U}{\partial y}$.
As a consequence,

$$0 \geq \liminf_{n \to +\infty} I''(z_{\xi_n})[\varphi_n, \varphi_n] \geq \liminf_{n \to +\infty} I''(U)[\varphi_n, \varphi_n] \geq I''(U)[\bar\varphi, \bar\varphi] \geq \delta \, \|\bar\varphi\|_X^2 \, .$$

Here we have used Theorem 3.6 and the fact that the linear operator $I''(U)$ is the sum of a lower semicontinuous operator A and of a compact operator K introduced in (3.3) and (3.4). This shows that $\varphi = 0$.
But now, exactly as before,

$$\frac{1}{n} \geq I''(U)[\varphi_n, \varphi_n] - o(1) = \langle A\varphi_n, \varphi_n \rangle + \langle K\varphi_n, \varphi_n \rangle - o(1) \geq C \|\varphi_n\|_X^2 - o(1) \\ \geq C - o(1),$$

a contradiction. □

In what follows, for each $z_\xi \in Z$, we denote by P_ξ^ε the orthogonal projection of X onto $\left(T_{z_\xi} Z\right)^\perp$, where X is endowed with the norm (2.3) (depending on ε) and $\perp$ is the orthogonality with respect the associated inner product. We aim to construct, for every $z_\xi \in Z$, an element $w = w(\varepsilon, \xi) \in \left(T_{z_\xi} Z\right)^\perp$ such that

$$P_\xi^\varepsilon I_\varepsilon'(z_\xi + w) = 0 \qquad (4.2)$$

and

$$(\mathrm{Id} - P_\xi^\varepsilon) I_\varepsilon'(z_\xi + w) = 0.$$

Clearly, the point $u_\varepsilon = z_\xi + w(\varepsilon, z_\xi)$ will be a critical point of I_ε, i.e. a solution to (2.2).
To solve the auxiliary equation (4.2) we first write

$$P_\xi^\varepsilon I_\varepsilon'(z_\xi + w) = P_\xi^\varepsilon I_\varepsilon'(z_\xi) + P_\xi^\varepsilon I_\varepsilon''(z_\xi)[w] + R(z_\xi, w).$$

We will show that $R(z_\xi, w) = o(\|w\|)$ uniformly with respect to $z_\xi \in Z$ for $|\xi|$ bounded. Then we will show that the linear operator

$$B_{\varepsilon,\xi} = -\left(P_\xi^\varepsilon I_\varepsilon''(z_\xi)\right)^{-1}$$

exists and is continuous, so that the equation $P_\xi^\varepsilon I_\varepsilon'(z_\xi + w) = 0$ is equivalent to

$$w = B_{\varepsilon,\xi}\left(P_\xi^\varepsilon I_\varepsilon'(z_\xi) + R(z_\xi, w)\right),$$

a fixed-point problem in the unknown $w \in \left(T_{z_\xi} Z\right)^\perp$.

Lemma 4.3. *Let M be the constant introduced in Proposition 4.2. For ε sufficiently small, the operator $L_\xi = P_\xi^\varepsilon \circ I_\varepsilon''(z_\xi) \circ P_\xi^\varepsilon$ is invertible, and there exists a constant $C > 0$ such that*

$$\left\| L_\xi^{-1} \right\| \leq C.$$

for every $\xi \in \mathbb{R}^2$ with $|\xi| \leq M$.

Proof. Let $\xi \in \mathbb{R}^2$, $|\xi| \le M$. For simplicity we denote here P_ξ^ε by P_ξ. We write $\left(T_{z_\xi} Z\right)^\perp = V_1 \oplus V_2$, where

$$V_1 = \operatorname{span}\{P_\xi z_\xi\}$$
$$V_2 = \left(\operatorname{span}\{z_\xi\} \oplus T_{z_\xi} Z\right)^\perp,$$

so that $V_1 \perp V_2$. We claim that for $\varepsilon \to 0^+$

$$\|z_\xi - P_\xi z_\xi\| = o(1), \quad I_\varepsilon''(z_\xi)[z_\xi, \cdot] = \left(\frac{1}{\pi} \log \star |z_\xi|^2\right) z_\xi + o(1). \tag{4.3}$$

It follows from (4.3) that

$$\begin{aligned} L_\xi(z_\xi) &= P_\xi \circ I_\varepsilon''(z_\xi)[P_\xi z_\xi] = P_\xi \left(I_\varepsilon''(z_\xi)[z_\xi, \cdot] + o(1)\right) \\ &= P_\xi \left(- \left(\frac{1}{\pi} \log \frac{1}{|\cdot|} \star |z_\xi|^2 \right) z_\xi + o(1) \right) \\ &= \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log|x-y| |z_\xi(x)|^2 |z_\xi(y)|^2 \, dx \, dy \right) z_\xi + o(1). \end{aligned}$$

As a consequence, the operator L_ξ, in matrix form with respect to the decomposition $\left(T_{z_\xi} Z\right)^\perp = V_1 \oplus V_2$, can be written as

$$L_\xi = \begin{bmatrix} \left(\int_{\mathbb{R}^2 \times \mathbb{R}^2} \log|x-y| |z_\xi(x)|^2 |z_\xi(y)|^2 \, dx \, dy \right) \operatorname{Id} + o(1) & o(1) \\ o(1) & A_\xi \end{bmatrix}$$

where the operator A_ξ satisfies $A_\xi \ge C^{-1} \operatorname{Id}$ according to (4.1) in Proposition 4.2.
It now follows from (3.5) that L_ξ is negative definite on V_1 and thus globally invertible on $\left(T_{z_\xi} Z\right)^\perp$. It remains to prove the previous claim.
Recalling the definition of $z_\xi(x) = U(x - \xi)$ and the exponential decay of U at infinity, we see that

$$\begin{aligned} \langle z_\xi \mid \partial_{\xi_j} z_\xi \rangle &= -\langle z_\xi \mid \partial_{x_j} z_\xi \rangle = -\langle z_\xi \mid \partial_{x_j} z_\xi \rangle_X + \int_{\mathbb{R}^2} \left(V(\varepsilon x) - 1\right) z_\xi \partial_{x_j} z_\xi \, dx \\ &= o(1) \quad \text{as } \varepsilon \to 0 \end{aligned}$$

for every $i \in \{1, \dots, n\}$. Therefore, $\|z_\xi - P_\xi z_\xi\| = o(1)$ as $\varepsilon \to 0$. This proves the first part of (4.3). The second identity is proved as follows: we compute

$$I_\varepsilon''(z_\xi)[z_\xi, v] = I''(z_\xi)[z_\xi, v] + \int_{\mathbb{R}^2} \left(V_\varepsilon - 1\right) z_\xi v$$

and recall that z_ξ solves

$$-\Delta z_\xi + z_\xi = \frac{1}{2\pi} \left[\log \frac{1}{|\cdot|} \star |z_\xi|^2 \right] z_\xi.$$

Since $\int_{\mathbb{R}^2} \left(V_\varepsilon - 1\right) z_\xi v = o(1)\|v\|$ for ε small, we conclude that, for any $v \in X$, we have

$$I_\varepsilon''(z_\xi)[z_\xi, v] = I''(z_\xi)[z_\xi, v] + \int_{\mathbb{R}^2} \left(V(\varepsilon x) - 1\right) z_\xi v \, dx = \left\langle \left(\frac{1}{\pi} \log|\cdot| \star |z_\xi|^2 \right) z_\xi \mid v \right\rangle + o(1)\|v\|.$$

□

Proposition 4.4. *Let assumption (V) be satisfied. Then for every ε small, there exists a unique $w = w(\varepsilon,\xi) \in (T_{z_\xi}Z)^\perp$ with $|\xi| \le M$ such that $I'_\varepsilon(z_\xi + w(\varepsilon,\xi)) \in T_{z_\xi}Z$. The function $(\varepsilon,\xi) \mapsto w(\varepsilon,\xi)$ is of class C^1 with respect to ξ, and there holds*

$$\|w(\varepsilon,\xi)\| \le C\left(\varepsilon|\nabla V(0)| + \varepsilon^2\right) \tag{4.4}$$

$$\|\partial_\xi w\| \le C\left(\varepsilon|\nabla V(0)| + \varepsilon^2\right) + o(\varepsilon^2). \tag{4.5}$$

Moreover, the function $\Theta_\varepsilon(\xi) = I_\varepsilon(z_\xi + w(\varepsilon,\xi))$ is of class C^1 and the condition $\Theta'_\varepsilon(\xi_0) = 0$ implies $I'_\varepsilon(z_{\xi_0} + w(\varepsilon,\xi_0)) = 0$.

Proof. Let us recall that our aim is to construct a solution $w \in (T_{z_\xi}Z)^\perp$ to (4.2). We write

$$I'_\varepsilon(z_\xi + w) = I'_\varepsilon(z_\xi) + I''_\varepsilon(z_\xi)[w] + R(z_\xi, w),$$

where

$$R(z_\xi, w) = I'_\varepsilon(z_\xi + w) - I'_\varepsilon(z_\xi) - I''_\varepsilon(z_\xi)[w].$$

By the invertibility of $L_\xi = P^\varepsilon_\xi \circ I''_\varepsilon(z_\xi) \circ P^\varepsilon_\xi$ (see Lemma 4.3), the function w solves (4.2) if and only if

$$w = N_{\varepsilon,\xi}(w), \tag{4.6}$$

where

$$N_{\varepsilon,\xi}(w) = -L_\xi^{-1}\left(P^\varepsilon_\xi \circ I'_\varepsilon(z_\xi) + P^\varepsilon_\xi R(z_\xi, w)\right).$$

We can now show that, for ε sufficiently small, equation (4.6) can be solved by means of the Contraction Mapping Theorem.
First of all, understanding the L^2-duality, we have

$$I'_\varepsilon(z_\xi + w) = -\Delta z_\xi + V_\varepsilon z_\xi - \Delta w + V_\varepsilon w + \frac{1}{2\pi}\left[\log\star(z_\xi + w)^2\right](z_\xi + w),$$

$$I'_\varepsilon(z_\xi) = -\Delta z_\xi + V_\varepsilon z_\xi + \frac{1}{2\pi}\left[\log\star|z_\xi|^2\right] z_\xi$$

and

$$I''_\varepsilon(z_\xi)[w] = -\Delta w + V_\varepsilon w + \frac{1}{2\pi}\left[\log\star|z_\xi|^2\right] w + \frac{1}{\pi}\left[\log\star(z_\xi w)\right] z_\xi.$$

Therefore, again with respect to the L^2-duality,

$$\begin{aligned} R(z_\xi, w) &= I'_\varepsilon(z_\xi + w) - I'_\varepsilon(z_\xi) - I''_\varepsilon(z_\xi)[w] \\ &= \frac{1}{\pi}\left[\log\star(z_\xi w)\right] w + \frac{1}{2\pi}\left[\log\star|w|^2\right] z_\xi + \frac{1}{2\pi}\left[\log\star|w|^2\right] w. \end{aligned}$$

We have

$$\|R(z_\xi, w)\| \le C\left(\|w\|^2 + o(\|w\|^2)\right) \tag{4.7}$$

as $\|w\| \to 0$. Indeed we have for any $\phi \in X$

$$\begin{aligned}
|\langle R(z_\xi, w), \phi\rangle| \le{}& \frac{1}{\pi}\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y| z_\xi(x) w(x) w(y)\phi(y)\,dx\,dy\right| \\
&+ \frac{1}{\pi}\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y||w(x)|^2 z_\xi(y)\phi(y)\,dx\,dy\right| \\
&+ \frac{1}{\pi}\left|\int_{\mathbb{R}^2\times\mathbb{R}^2} \log|x-y||w(x)|^2 w(y)\phi(y)\,dx\,dy\right| \\
\le{}& \frac{1}{\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|)+\log(1+|y|)]|z_\xi(x)||w(x)||w(y)||\phi(y)|\,dx\,dy \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|)+\log(1+|y|)]|w(x)|^2|z_\xi(y)||\phi(y)|\,dx\,dy \\
&+ \frac{1}{2\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} [\log(1+|x|)+\log(1+|y|)]|w(x)|^2|w(y)||\phi(y)|\,dx\,dy \\
\le{}& \|w\|_2\|\phi\|_2\|z_\xi\|_X\|w\|_X + \|z_\xi\|_2\|w\|_2\|w\|_X\|\phi\|_X + \|z\|_2\|w\|_2\|w\|_X\|\phi\|_X \\
&+ \|w\|_X^2\|z_\xi\|_2\|\phi\|_2 + \|w\|_2\|w\|_X^2\|\phi\|_2 + \|w\|_2^2\|w\|_X\|\phi\|_X.
\end{aligned}$$

Since $\phi \in X$ is arbitrary, we have

$$\|R(z_\xi, w)\| \le C_1\|z_\xi\|\|w\|^2 + C_2\|w\|^3 \tag{4.8}$$

and thus we infer (4.7). In a similar way we can deduce that

$$\|R(z_\xi, w_1) - R(z_\xi, w_2)\| \le C\left(\|w_1\| + \|w_2\| + o(\|w_1 - w_2\|)\right)\|w_1 - w_2\| \tag{4.9}$$

Using Lemma 4.1, (4.7) and (4.9), we find that

$$\begin{aligned}
\|N_{\varepsilon,\xi}(w)\| &\le C\left(\varepsilon|\nabla V(0)| + \varepsilon^2 + \|w\|^2 + o(\|w\|^2)\right) \\
\|N_{\varepsilon,\xi}(w_1) - N_{\varepsilon,\xi}(w_2)\| &\le C\left(\|w_1\| + \|w_2\| + o(\|w_1 - w_2\|)\right)\|w_1 - w_2\|.
\end{aligned}$$

As a consequence, the operator $N_{\varepsilon,\xi}$ is a contraction on the closed subset

$$W_C = \left\{ w \in \left(T_{z_\xi} Z\right)^\perp \mid \|w\| \le C\left(\varepsilon|\nabla V(0)| + \varepsilon^2\right)\right\},$$

provided that $C > 0$ is sufficiently large, and $\varepsilon > 0$ is sufficiently small. The Contraction Mapping Theorem yields a unique fixed point $w = w(\varepsilon, \xi)$ of $N_{\varepsilon,\xi}$ in W_C such that (4.4) holds. The last statements of the Proposition are proved by a straightforward modification of the arguments contained in [2, pp. 129–130], so we present only a sketch of the ideas.
Let us define the map $H\colon \mathbb{R}^2 \times X \times \mathbb{R}^2 \times \mathbb{R} \to X \times \mathbb{R}^2$,

$$H(\xi, w, \alpha, \varepsilon) = \begin{pmatrix} I'_\varepsilon(z_\xi + w) - \sum_{i=1}^2 \alpha_i \partial_{x_i} z_\xi \\ (\langle w \mid \partial_{x_1} z_\xi\rangle, \langle w \mid \partial_{x_2} z_\xi\rangle) \end{pmatrix}.$$

In particular, $w \in (T_{z_\xi} Z)^\perp$ solves the equation $P_\xi I'_\varepsilon(z_\xi + w) = 0$ if and only if $H(\xi, w, \alpha, \varepsilon) = 0$. With estimates similar to those we have shown above, we can prove that $\frac{\partial H}{\partial(w,\alpha)}(\xi, 0, 0, \varepsilon)$ is uniformly invertible in ξ for ε small enough. By the Implicit Function Theorem, the map $\xi \mapsto (w_\xi, \alpha_\xi)$ is of class C^1.

Differentiating the identity $H(\xi, w_\xi, \alpha_\xi, \varepsilon) = 0$ with respect to ξ, we obtain

$$\frac{\partial H}{\partial \xi}(\xi, w, \alpha, \varepsilon) + \frac{\partial H}{\partial(w,\alpha)}(\xi, w, \alpha, \varepsilon)\frac{\partial(w_\xi, \alpha_\xi)}{\partial \xi} = 0,$$

hence

$$\begin{aligned}\|\partial_\xi w\| &\leq C \left\| \frac{\partial H}{\partial(w,\alpha)}(\xi, w, \alpha, \varepsilon)[\partial_\xi z_\xi, \alpha] \right\| \\ &\leq C \left(\|I_\varepsilon''(z_\xi + w)[\partial_\xi z_\xi]\| + |\alpha| + \|w\| \right).\end{aligned}$$

It now follows easily that (4.4) holds. □

5. The reduced functional

Following [2], the manifold

$$Z^\varepsilon = \left\{ z_\xi + w(\varepsilon, \xi) \mid \xi \in \mathbb{R}^2, \ |\xi| \leq M, \quad \varepsilon \ll 1 \right\}$$

is a natural constraint for I_ε, in the sense that any critical point of I_ε constrained to Z^ε is a free critical point of I_ε. To prove the existence of a critical point of the functional I_ε, it is therefore sufficient to show that the constrained functional $\Theta_\varepsilon \colon \overline{B(0,M)} \subset \mathbb{R}^2 \to \mathbb{R}$ defined by

$$\Theta_\varepsilon(\xi) = I_\varepsilon(z_\xi + w)$$

possesses a critical point. To this aim, we evaluate

$$\begin{aligned}\Theta_\varepsilon(\xi) &= I(z_\xi + w) + \frac{1}{2}\int_{\mathbb{R}^2} (V_\varepsilon - 1)\,|z_\xi + w|^2\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^2} |\nabla(z_\xi + w)|^2 + |z_\xi + w|^2\,dx \\ &\quad + \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y||z_\xi(x) + w(x)|^2|z_\xi(y) + w(y)|^2 \;dx\,dy \\ &\quad + \frac{1}{2}\int_{\mathbb{R}^2} (V_\varepsilon - 1)\,|z_\xi + w|^2\,dx \\ &= I(z_\xi) + \frac{1}{2}\int_{\mathbb{R}^2} (V_\varepsilon - 1)\,|z_\xi + w|^2 + R_\varepsilon(w),\end{aligned}$$

where

$$\begin{aligned}R_\varepsilon(w) &= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla w|^2 + w^2\right)\,dx + \frac{1}{8\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y||w(x)|^2|w(y)|^2\,dx\,dy \\ &\quad + \int_{\mathbb{R}^2} \left(\nabla z_\xi \cdot \nabla w + z_\xi w\right)\,dx \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| z_\xi(x) w(x) |z_\xi(y)|^2\,dx\,dy \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| z_\xi(x) w(x) z_\xi(y) w(y)\,dx\,dy \\ &\quad + \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log|x-y| z_\xi(x) w(x) |w(y)|^2\,dx\,dy.\end{aligned}$$

According to Proposition 4.4, the function Θ_ε can be expanded as

$$\Theta_\varepsilon(\xi) = b_0 + \frac{1}{2}\int_{\mathbb{R}^2} (V(\varepsilon x) - 1)\,|z_\xi + w|^2\,dx + o(\varepsilon^2), \tag{5.1}$$

where $b_0 = I(z_\xi) = I(U)$. Let us define $Q_2 = D^2V(0)$ and the function $\Gamma\colon \mathbb{R}^2 \to \mathbb{R}$,

$$\Gamma(\xi) = \int_{\mathbb{R}^2} Q_2(x)|z_\xi(x)|^2\,dx.$$

From now on, we will suppose for the sake of definiteness that $x_0 = 0$ is a proper local minimum of V, so that $D^2V(0)$ is a positive-definite quadratic form. The case of a proper local maximum can be treated analogously.

Lemma 5.1. *The point $\xi = 0$ is a strict local minimum for Γ.*

Proof. By oddness, $\partial_1\partial_2\Gamma(0) = 0$. Since $\nabla Q_2(x)\cdot x = 2Q_2(x) > 0$, we conclude that $D^2\Gamma(0)$ is positive-definite. □

We fix a number $\bar\xi > 0$ in such a way that $\bar\xi < M$ and

$$\Gamma(\xi) > \Gamma(0)$$

for every $\xi \in \overline{B}\setminus\{0\}$, where $B = B(0,\bar\xi)$.

Lemma 5.2. *For $\varepsilon > 0$ sufficiently small, there results $\Theta_\varepsilon(0) < \inf_{|\xi|=\bar\xi}\Theta_\varepsilon(\xi)$.*

Proof. We recall the asymptotic expansion (5.1) and observe that

$$\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon^2}\int_{\mathbb{R}^2}(V_\varepsilon - 1)\,|z_\xi + w|^2\,dx = \frac{1}{2}\int_{\mathbb{R}^2} Q_2|z_\xi|^2\,dx = \frac{1}{2}\Gamma(\xi).$$

Hence

$$\Theta_\varepsilon(\xi) - \Theta_\varepsilon(0) = \frac{1}{2}\varepsilon^2\left(\Gamma(\xi) - \Gamma(0)\right) + o(\varepsilon^2).$$

It now follows from the choice of $\bar\xi$ that $\Theta_\varepsilon(\xi) - \Theta_\varepsilon(0) > 0$ if $|\xi| = \bar\xi$ and $\varepsilon > 0$ is small enough. The proof is complete. □

Proof of Theorem 1.1. We have just shown that the function Θ_ε must have a minimum at some $\xi = \xi(\varepsilon)$ in the ball $B \subset B(0,M)$. This gives rise to a critical point $u_\varepsilon = z_\xi + w(\varepsilon,\xi) \in Z^\varepsilon$ of the functional I_ε with $\varepsilon \sim 0$. Now, for every $\xi \in \overline{B}$,

$$0 \le \Theta_\varepsilon(\xi) - \Theta_\varepsilon(\xi(\varepsilon)) = \frac{1}{2}\varepsilon^2\left(\Gamma(\xi) - \Gamma(\xi(\varepsilon)\right) + o(\varepsilon^2);$$

as $\varepsilon \to 0$, we may assume that $\xi(\varepsilon) \to \xi_0$ and we obtain $\Gamma(\xi) - \Gamma(\xi_0) \ge 0$ for every $\xi \in \overline{B}$. Our choice of $\bar\xi$ forces $\xi_0 = 0$, so that $\xi(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence $u_\varepsilon = z_{\xi(\varepsilon)} + w(\varepsilon,\xi(\varepsilon)) \to U$.
Coming back to the system (1.3) we obtain the existence of pairs of solution $(v_\varepsilon, E_\varepsilon)$ where

$$v_\varepsilon(x) = u_\varepsilon\left(\frac{x}{\varepsilon}\right) \simeq U\left(\frac{x}{\varepsilon}\right)$$

and

$$\begin{aligned}E_\varepsilon(x) = \omega\left(\frac{x}{\varepsilon}\right) &= -\int_{\mathbb{R}^2} \log\left|\frac{x}{\varepsilon} - y\right| |u_\varepsilon(y)|^2\, dy \\ &= -\frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \log\frac{|x-z|}{\varepsilon}\left|u_\varepsilon\left(\frac{z}{\varepsilon}\right)\right|^2 dz \\ &= \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \log\frac{\varepsilon}{|x-z|}|v_\varepsilon(z)|^2\, dz.\end{aligned}$$

Therefore we have $E_\varepsilon(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} \log\frac{1}{|x-z|}|v_\varepsilon(z)|^2\, dz + c_\varepsilon$, with $c_\varepsilon = \frac{\log\varepsilon}{\varepsilon^2}\|v_\varepsilon\|_2^2$. ☐

Remark 5.3. Our Theorem 1.1 can be slightly generalized. Indeed, we can assume that the potential V has a non-degenerate critical point at some x_0, in the sense $\nabla V(x_0) = 0$ and there exists an integer $m \geq 1$ such that $D^{2m}V(x_0)$ is either positive- or negative-definite. The proof then requires only a higher-order expansion of $I_\varepsilon(z+w)$ in ε. We omit the details for brevity.

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