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Killing spinors, extensions and metric diagonalization in pseudo-Riemannian geometry

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Abstract

In the thesis, I will prove new extension results to obtain pseudo-Riemannian manifolds of dimension n endowed with a Killing spinor, starting from a manifold of dimension n-1 or n-3 with appropriate additional structure. I will also prove a generalization to the smooth, indefinite setting of a known result on the diagonalization of metrics on a 3-dimensional manifold.

I will present a construction that revolves around $\mathfrak{z}$-standard Lie algebras, which are standard Lie algebras $\mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ endowed with a Sasaki structure (g, ξ, η, φ) such that $\varphi(e_0)$ lies in the center $\mathfrak{z}$ of $\mathfrak{g}$. There are two main results. The first one guarantees that a suitable central extension $\mathfrak{g}$ of a nilpotent pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$, admitting a derivation $\check{D} \in \operatorname{Der}(\check{\mathfrak{g}})$ commuting with the complex structure and satisfying an additional technical condition, extends to a $\mathfrak{z}$-standard Sasaki Lie algebra of the form $\mathfrak{g} \rtimes_D \mathbb{R}$, where D is a derivation of $\mathfrak{g}$ extending $\check{D}$. The second result specializes the first one to obtain $\mathfrak{z}$-standard pseudo-Sasaki-Einstein Lie algebras. I will also classify $\mathfrak{z}$-standard Sasaki Lie algebras up to dimension 7 obtained extending abelian pseudo-Kähler Lie algebras, Einstein-Sasaki $\mathfrak{z}$-standard Lie algebras up to dimension 7, and present examples of the construction in dimension 9.

Moreover, I prove an extension result in the non-invariant, analytic setting. I obtain a pseudo-Riemannian spin manifold carrying a Killing spinor by extending a manifold endowed with a real or imaginary harmful structure, i.e., a pair of spinors (ψ, φ) satisfying a coupled PDE system involving a symmetric endomorphism A which is additionally required to satisfy $d \operatorname{tr} A + \delta A = 0$. I prove that the metric of the manifold extends to an Einstein metric, in a space-like or time-like direction, depending on the harmful structure, whether it is real or imaginary, respectively. I point out that, in the definite setting, the condition on the endomorphism can be dropped. I then define the Killing spinor by extending the harmful structure by parallel transport.

Finally, I prove that it is always possible to diagonalize the metric of a smooth, Lorentzian, 3-dimensional manifold by applying the technique of moving frames. I show that the existence of coordinates that diagonalize the metric is equivalent to the existence of a coframe satisfying a specific PDE system, hence I prove that the system is diagonal hyperbolic and that the associated Cauchy problem admits non-characteristic initial data.

Introduction

An Einstein manifold is a (pseudo)-Riemannian manifold (Z, h) such that the metric tensor satisfies the Einstein equation

$$\mathrm{ric}(h) = \lambda h,$$

where $\mathrm{ric}(h)$ is the Ricci tensor of h. A Killing spinor is a section Ψ of the spinor bundle of a smooth pseudo-Riemannian spin manifold (Z, h) satisfying

$$\nabla_X \Psi = \lambda X \cdot \Psi, \quad X \in \Gamma(TZ),\, \lambda \in \mathbb{C}.$$

The two, Einstein metrics and Killing spinors, have been of interest both from a mathematical and a physical point of view since at least the second half of the 20th century. While there is no need for further introduction on Einstein metrics, I will spend a few words on the geometry of manifolds carrying a Killing spinor.

From the standpoint of physics, Killing spinors appear in the literature in general relativity since [76] and later in supergravity in [39]. From a mathematical point of view, on a compact Riemannian manifold with positive curvature, Killing spinors realize the lowest possible eigenvalue as eigenvectors of the Dirac operator (see [46]). Furthermore, the geometry of a manifold (M, g) carrying a Killing spinor is particularly rigid (see [5, 7]), the scalar curvature is forced to be $4n(n-1)\lambda^2$ whenever Ψ is not identically zero, hence λ is real or purely imaginary; in the case of a Riemannian manifold, the metric is forced to be Einstein, and Ricci-flat for $\lambda = 0$, in which case the spinor is said to be parallel. Moreover, the topology of the manifold changes if λ is real or purely imaginary. In the former case, the manifold is compact, in the latter the manifold M can be either the hyperbolic space or a warped product (see [7]). In the indefinite setting, the constraint on the metric does not always hold. For instance, it was shown in [13] that Lorentzian manifolds carrying a Killing spinor are forced to be Einstein only if the Killing number λ is real; in the same paper, examples of Lorentzian manifolds carrying a Killing spinor with $\lambda \in i\mathbb{R}$ which are not Einstein are presented. In both the Riemannian and indefinite case, Killing and parallel spinors are studied in connection with holonomy (see [77, 5, 10]). By [5], a Riemannian manifold carrying a Killing spinor is either Einstein-Sasaki, 3-Einstein Sasaki, nearly-parallel G_2, 6-dimensional nearly Kähler or a round sphere.

One way to approach the classification of manifolds with special geometries, such as an Einstein metric, is the study of necessary and sufficient conditions on lower-dimensional (pseudo)-Riemannian manifolds (M, g) such that the consequent metric extension (Z, h) is endowed with the specific geometric structure.

The extension problem related to the Einstein equation has been studied by D. DeTurck in [35], both in its physical and in mathematical flavor. In particular, in the latter he characterizes smooth Riemannian manifolds that embed as hypersurfaces into Lorentzian ones endowed with an Einstein metric, thanks to the hyperbolicity of the PDE system that arises. On the other hand, when both the starting manifold and the extension have a Riemannian metric, extension results generating Einstein manifolds were obtained by N. Koiso in [56], where, in order to solve the elliptic PDE system, the necessity to add the condition of real analyticity on the initial data arose. In the invariant setting, Einstein metrics have been studied extensively in various papers in more recent years. It was proved by J. Heber in [52] that for any completely solvable metric Lie algebra, any Einstein metric is standard, i.e., the Lie algebra decomposes as $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{a}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{a}$ is abelian, and their sum is orthogonal. When talking about the metric on a Lie algebra, what is actually meant is the left-invariant metric on the Lie group considered at the identity and extended by left translation. Furthermore, in the same paper, Heber introduces Iwasawa standard Lie algebras, i.e., standard Lie algebras where for any $A \in \mathfrak{a}$ the adjoint $\operatorname{ad} A$ is symmetric, and proves that any standard Einstein solvmanifold is isometric to a solvmanifold admitting an Iwasawa decomposition, where a solvmanifold is intended to be a simply connected solvable Lie group with a left-invariant metric. In 2008 Y. Nikolayevsky studied in [65] sufficient conditions on Riemannian nilpotent Lie algebras so that they embed as nilradicals in Riemannian solvable Einstein Lie algebras. In particular, it was proven that every nilpotent Lie algebra $\mathfrak{g}$ carries a non-zero, semisimple derivation N, unique up to automorphism of the Lie algebra, such that

$$\operatorname{tr}(N\varphi) = \operatorname{tr}(\varphi), \quad \varphi \in \operatorname{Der} \mathfrak{g},$$

where tr is the trace and $\operatorname{Der}(\mathfrak{g})$ the space of derivations of $\mathfrak{g}$, called *pre-Einstein* or *Nikolayevsky derivation*, which allows to variationally characterize the Einstein solvable extension. In the same paper, the author points out that it should be possible to classify all Riemannian Einstein solvmanifolds depending on which nilpotent Lie algebra is actually an Einstein nilradical. In [59], J. Lauret improved on the result from Heber, showing that any Einstein solvmanifold is actually standard, while in [58] he proved that the study of all invariant Ricci solitons on solvmanifolds can be reduced to the study of nilsolitons. More recently, C. Böhm and R. A. Lafuente proved in [18] the Alekseevskii conjecture: every connected homogeneous Einstein manifold of negative scalar curvature is diffeomorphic to $\mathbb{R}^n$. A subsequent approach to the study of (pseudo)-Riemannian Einstein extensions of solvmanifolds is the study of nilsolitons, i.e. nilpotent Lie algebras such that $\operatorname{Ric} = \lambda \operatorname{Id} + D$, where D is a derivation. In [28] the authors show that in the indefinite setting, the classification of Einstein metrics is not reducible to the classification of nilsolitons, and extending a nilsoliton can induce four different geometries depending on λ and D. In later works, D. Conti, V. Del Barco e F. A. Rossi introduced a variation of the Nikolayevsky derivation, called *metric Nikolayevsky*, in order to study the uniqueness of ad-invariant metrics on nilmanifolds up to automorphism (see [24]).

The extension problem can also be related to manifolds carrying Killing spinors. It is known that for parallel spinors, the restriction gives rise to generalized Killing spinors, i.e., spinors that satisfy the equation

$$\nabla_X \Psi = \frac{1}{2} W(X) \cdot \Psi,$$

where W is the Weingarten operator of the embedding. In the Riemannian setting, it was proved in [1] that this is also a characterization, which means that any real analytic Riemannian manifold endowed with a generalized Killing spinor extends to a Riemannian manifold of one dimension higher that is endowed with a parallel spinor. In this case, the classification of the holonomy groups of a manifold with a parallel spinor in [77] allows one to recast the problem in terms of G-structures and differential forms. Extending the metric amounts to solving appropriate evolution equations in the sense of [54] (see also [32, 25]); for some instances of G, the existence of a solution can then be proved using the integrability of an exterior differential system associated to the G-structure (see [32, 17]). In the case of time-like normal vector field, the restriction of the parallel spinor gives rise to an imaginary generalized Killing spinor, i.e. one which satisfies

$$\nabla_X \Psi = \frac{i}{2} W(X) \cdot \Psi.$$

The real analytic assumption in both cases cannot be eschewed as counterexamples where non-real analytic Riemannian manifolds with generalized Killing spinors which cannot be extended were constructed in [17]. The extension problem is not fully understood for general signature of the hypersurface, the first extension result having been obtained in [6], where is assumed that the normal field is space-like and ∇W totally symmetric. In the special case of Lorentzian extensions of Riemannian hypersurfaces, a proof of existence was given in [11] for real analytic data, and [61] for smooth data, under the condition

$$U_\psi \cdot \psi = i u_\psi \psi,$$

with U_ψ denoting the Riemannian Dirac current and u_ψ its norm. This algebraic condition on the spinor corresponds to imposing that the parallel spinor on Z is null. For the 4-dimensional case, an alternative proof using the polyform associated to the square of the spinor was given in [64]. In these results, the metric on Z is not automatically Ricci-flat. In order to obtain a Ricci-flat metric, one needs to impose additional constraints involving the tensor A and the scalar curvature s, namely

$$s = \operatorname{tr} A^2 - (\operatorname{tr} A)^2, \quad d \operatorname{tr} A + \delta A = 0.$$

It was shown in [63] that a Riemannian metric on a 3-manifold with a generalized Killing spinor such that the previous equation holds can be extended to a Ricci-flat Lorentzian manifold with a parallel spinor. The case where Z has dimension three has been studied in [62]. Hypersurfaces inside a nearly-Kähler 6-manifold have been studied in [42], where the corresponding evolution equations are also introduced. Also in this context, solving the evolution equations can be used effectively to produce explicit metrics; this approach has been instrumental in the construction of inhomogeneous nearly-Kähler manifolds in [45]. For real analytic data, the existence of an extension for the geometries corresponding to nearly-Kähler, nearly-parallel G_2 and Einstein-Sasaki structures on Z has been proved in [23].

Another interesting topic in pseudo-Riemannian geometry is the diagonalizability of the metric of a manifold. Let (M, g) be a smooth n-dimensional (pseudo)-Riemannian manifold and let $(x^1, \ldots, x^n)$ be a set of coordinates of a chart around a point $p \in M$, such that the metric assumes the form

$$g = \sum_{i=1}^{n} f_i(x^1, \ldots, x^n) dx^i \otimes dx^i,$$

which will be called *orthogonal chart.* The question is if it is always possible to find an atlas of orthogonal charts. In the 2-dimensional case, the answer is always affirmative, as it was shown in [36]. In higher dimension, the problem was first tackled by D. DeTurck and D. Yang in [37], where they were able to prove that smooth Riemannian 3-manifolds admit a set of global coordinates that diagonalize the metric applying the technique of moving frames. The problem was then extended in two directions: higher dimension, which was briefly mentioned in the aforementioned paper, and different signature. The first direction was investigated by P. Tod in [75], where he worked out the precise conditions on the Riemann tensor, and more specifically on the Weyl tensor, that allow the metric to be diagonalized. In the case of indefinite signature, O. Kowalski and M. Sekizawa in [57] proved that real analytic Lorentzian 3-manifolds admit a global set of coordinates that diagonalize the metric, proving the result with a more analytical approach. More recently, P. Gauduchon and A. Moroianu proved in [49] that both the complex and the quaternionic projective spaces $\mathbb{CP}^n$ and $\mathbb{HP}^q$ do not admit orthogonal coordinates for $n, q \geq 2$.

In this thesis, I will focus primarily on metric extensions in the indefinite setting that lead to an Einstein manifold carrying a Killing spinor. I will also show how the diagonalization of a 3-dimensional Lorentzian manifold is possible in the smooth category.

In Chapter 1, I will recall some definitions and results regarding Lie theory, spin geometry and Kähler and Sasaki structures both on Lie groups and on general manifolds, and take the chance to introduce the notation used for the remainder of the manuscript. In particular, I will recall a result for standard Lie algebras $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$ which allows one to obtain an isometric Lie algebra $\tilde{\mathfrak{g}}^*$ by projecting $\operatorname{ad} X$ on its symmetric part, first obtained in [4] and generalize it to the indefinite setting and extend it to non necessarily standard Lie algebras (Proposition 1.1.5). At the end of the chapter, I will also prove that pseudo-Sasaki Lie algebras never admit pseudo-Iwasawa decompositions (Proposition 1.4.6). This obstruction motivates the choice to consider more general standard decompositions of Lie algebras.

In Chapter 2, I will present new results concerning pseudo-Sasaki extensions of pseudo-Kähler Lie algebras. I will first prove a number of useful formulas relating a Lie algebra and a Sasaki structure, as well as some properties of the Ricci tensor of a nilmanifold endowed with an Einstein metric. Next, I will introduce a new decomposition of (pseudo)-Sasaki Lie algebra denoted $\mathfrak{z}$-*standard*, i.e., a standard decomposition $\mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ together with a compatibility condition relating the center $\mathfrak{z}$ of $\mathfrak{g}$ to the Sasaki structure, and characterize the geometry of a quotient, showing that it is endowed with a pseudo-Kähler structure, which will be called *Kähler reduction*. $\mathfrak{z}$-standard Lie algebras were introduced in [30] in order to circumvent the obstruction mentioned in the first chapter. Subsequently, I explain how to invert the reduction process and prove two extension results, Theorem 2.4.1 and Theorem 2.4.9, in order to construct Sasaki and Sasaki-Einstein Lie algebras $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ by extending suitable pseudo-Kähler nilpotent Lie algebras in three dimensions less. This procedure differs from the double extension procedure considered in [14], in that the two "extra" dimensions span a definite two-plane, rather than neutral. Finally, I will give a classification of $\mathfrak{z}$-standard Sasaki and Sasaki-Einstein Lie algebras in dimension 3, 5 and 7, and provide some examples in dimension 9. This extension by three dimension can actually be seen in the context of hypersurfaces embedded in manifolds carrying Killing spinors, where the role of hypersurface is played by the Lie algebra $\mathfrak{g}$. These results appear in two papers ([30, 31]) in collaboration with my supervisor D. Conti and F. A. Rossi.

In Chapter 3, I will provide a new embedding result in the more general context of Killing spinor in any dimension. In particular, I will show that, in the analytic category and under suitable assumptions, it is possible to isometrically embed a pseudo-Riemannian manifold (M, g) of dimension n and signature (r, s) in a pseudo-Riemannian manifold (Z, h) as a hypersurface, such that Z admits a Killing spinor. The first step will be to recall some results obtained by N. Koiso in [56] and adapt them to the indefinite setting. Subsequently, I will characterize the geometry of a hypersurface embedded in a pseudo-Riemannian manifold endowed with a Killing spinor and, in Theorem 3.2.1, prove that it admits two spinors (ψ, φ) satisfying the system

$$\begin{cases} \nabla^{\Sigma M}_X \psi = \frac{1}{2} A(X) \odot \psi + \lambda X \odot \varphi \\ \nabla^{\Sigma M}_X \varphi = \lambda X \odot \psi - \frac{1}{2} A(X) \odot \varphi \end{cases}$$

if the normal vector field to the hypersurface is space-like, or a similar system if the normal is time-like. I will denote a manifold *weakly harmful* when it admits a pair of spinors satisfying such a system, while it will be dubbed simply *harmful* with the addition of a technical condition, namely $d \operatorname{tr} A + \delta A = 0$, meaning to suggest the fact that such a structure potentially leads to a Killing spinor on the extension. Thereafter, the extension process begins. I first prove that it is possible to extend the metric of a manifold endowed with a harmful structure, in order for it to be Einstein. Finally, I show that the harmful structure extends to a Killing spinor. This result appears in [33], written in collaboration with my supervisor.

In Chapter 4, I will extend the result first obtained by D. DeTurck and D. Yang in the smooth Riemannian category to the smooth indefinite one. In particular, I will prove that any smooth 3-dimensional Lorentzian manifold (M, g) admits an atlas such that the metric written in coordinates assumes a diagonal form. I first recall some useful definitions and existence results concerning systems of partial differential equations, namely *symmetric* and *diagonal hyperbolic* systems. I then prove, by applying the technique of moving frames, that it is possible to rewrite the system of PDE's defining the constraints on the diagonalizability of the metric in the form of a diagonal hyperbolic system. Finally, I show that it is possible to construct non-characteristic initial data for the Cauchy problem associated to the system, proving the initial statement.

Contents

Chapter 1

Preliminary notions

In this chapter, I will recall known facts for future reference and to fix the notations and conventions. The first section concerns Lie theory, where I will give some definitions and classical results, as well as more recent theorems, more specific to the thesis, which will prove useful in later chapters. In the second section, I will recall the definition of Clifford algebras and the construction of the spin group, as well as present the classification of Clifford algebras in terms of matrix algebras given in [60] and recall some properties of the spinor representation. In the third section, I will recall the definition of a spin structure over a manifold and the associated spinor bundle, as well as the construction of the spin connection, ending the section with some examples. The last section provides definitions of pseudo-Kähler and pseudo-Sasaki structures, and how they relate to each other. I also prove that a (pseudo)-Sasaki solvmanifold never admits a (pseudo)-Iwasawa decomposition (Proposition 1.4.6)

1.1 Lie theory

In this section, I will recall some basic notions to fix the notation and some relevant results to the following discussion. Unless otherwise stated, definitions and results can be found in [41].

A *Lie group* is a smooth manifold G with a group operation $\cdot$ compatible with the differentiable structure. Given $g \in G$, one defines the *left* (resp. *right*) *translation* $L_g\colon G \to G$ such that $L_g(h) = g \cdot h$ (resp. $R_g\colon G \to G$, $R_g(h) = h \cdot g$). A vector field $X \in \Gamma(TG)$ is called *left-invariant* if

$$(L_g)_{*|_h}(X) = X_{gh} \text{ for any } g, h \in G,$$

while the set $\mathfrak{g} = \{X \in \Gamma(TG) \,|\, X \text{ is left invariant}\}$ will be the Lie algebra of G with the Lie bracket $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{g}$. Thanks to the left and right translations, it is possible to define the *adjoint map* $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$, defined as follows: $\mathrm{Ad}_g = (A_g)_*$ where

$$\mathrm{A}_g(h) = \left(L_g \circ R_{g^{-1}}\right)(h) = ghg^{-1}.$$

By proceeding one step further in the same direction, it is possible to define $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, defined as $\mathrm{ad} = \mathrm{Ad}_*$, which, for $X, Y \in \mathfrak{g}$, becomes $\mathrm{ad}(X)(Y) = [X, Y]$. It is now possible to define the *Killing form* as

$$B(X,Y) = \mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)), \quad X, Y \in \mathfrak{g}.$$

Furthermore, if (G, g) is a pseudo-Riemannian Lie group, the metric g is called *left invariant* if L_g is an isometry for every $g \in G$. Recall that if (G, g) is connected and simply-connected, and g is left-invariant, the metric structure on the Lie group is entirely determined by the scalar product on $T_eG \cong \mathfrak{g}$. Thus, it is reasonable to study the metric structure of a Lie algebra (pseudo-Riemannian, Einstein, Ricci-flat...), with the understanding that the terms refer to the underlying connected, simply-connected Lie group with its left-invariant metric. For this reason, the section will be focused on Lie algebras. A Lie algebra $\mathfrak{g}$ is a vector space together with a Lie bracket. Hence, a Lie algebra is completely determined by the choice of a basis of the vector space and the bracket relations. In the following, keeping this in mind, I will describe the structure of a Lie algebra by the Chevalley-Eilenberg differentials, which are defined in the following manner. Let $(e_1, \dots, e_n)$ be a basis for the vector space, which I will identify with $\mathbb{R}^n$, and $(e^1, \dots, e^n)$ its dual basis. Then the Lie algebra structure is determined if one sets the differentials $de^i = c^i_{jk} e^j \wedge e^k = c^i_{jk} e^{jk}$ for each $i = 1, \dots, n$ by applying the differential formula:

$$c^i_{jk} = de^i(e_j, e_k) = -e^i([e_j, e_k]),$$

thus $[e_j, e_k] = -c^i_{jk} e_i$, where the sum over repeated indices is implied. In general, a Lie algebra $\mathfrak{g}$ will be described by

$$\mathfrak{g} = \left(c^1_{jk} e^{jk}, \dots, c^n_{jk} e^{jk} \right).$$

This notation is a variation of the one introduced by S. Salamon in [70].

Example 1.1.1. Recall that the 3-dimensional Heisenberg group $H_3 \subset \mathrm{GL}(3, \mathbb{R})$ is the Lie group generated by the matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbb{R},$$

while its Lie algebra is the subalgebra $\mathfrak{h}_3 \subset \mathfrak{gl}(3, \mathbb{R})$ generated by the matrices

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let e_1, e_2, e_3 be a basis for $\mathfrak{h}_3$ such that $de^1 = de^2 = 0$, $de^3 = e^{12}$. Then the only non-zero bracket is $[e_1, e_2] = e_3$, which is precisely the one defining the 3-dimensional Heisenberg Lie algebra $\mathfrak{h}_3$. In the notation introduced, one can write $\mathfrak{h}_3 = (0, 0, e^{12})$.

Recall that the *lower central series* of a Lie algebra $\mathfrak{g}$ is the sequence

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_1 \supseteq [\mathfrak{g}_1, \mathfrak{g}] \supseteq \dots$$

while the *derived series* is a sequence

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}^{(1)} \supseteq [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \supseteq \dots$$

If for some $n \in \mathbb{N}$ the lower central series terminates, i.e. $\mathfrak{g}_n = 0$, then $\mathfrak{g}$ is called *nilpotent*, while, if the derived series terminates, $\mathfrak{g}$ is said to be *solvable*. As $\mathfrak{g}^{(k)} \subseteq \mathfrak{g}_k$, any nilpotent Lie algebra is automatically solvable, but the converse is not true in general. Given a Lie algebra $\mathfrak{g}$, its largest solvable ideal $\mathfrak{r} \subset \mathfrak{g}$ is called the *radical* of $\mathfrak{g}$. A non-zero finite-dimensional Lie algebra $\mathfrak{g}$ is called *semisimple* if it has no non-zero solvable ideals, that is, if $\mathfrak{r} = 0$.

Lemma 1.1.2 ([41, Lemma 4.7]). *If $\mathfrak{g}$ is a Lie algebra, then $\mathfrak{g}/\mathfrak{r}$ is semisimple.*

Now, let $\mathfrak{g}$ be a Lie algebra endowed with a pseudo-Riemannian metric g. If $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{n} \rtimes \mathfrak{a}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{a}$ is abelian and the decomposition is orthogonal, then $\mathfrak{g}$ will be called *standard*. A standard decomposition is *pseudo-Iwasawa* if $\operatorname{ad} X$ is symmetric for all $X \in \mathfrak{a}$. These definitions mimic and generalize analogous definitions for Riemannian metrics (see [52]), and they have proved useful in the study of Einstein metrics ([28]).

For the remainder of the section, I will present some general facts on the metric of a Lie algebra which admits a semidirect decomposition. In particular, I will prove some formulas about the Ricci tensor of a Lie algebra obtained by central extension, an expression of the Levi-Civita connection on a standard Lie algebra and an isometry result. Recall that a Lie algebra $\mathfrak{g}$ is a *central extension* of a nilpotent Lie algebra $\check{\mathfrak{g}}$ if they satisfy

$$0 \to \mathbb{R}^k \to \mathfrak{g} \to \check{\mathfrak{g}} \to 0;$$

where, as vector spaces, $\mathfrak{g} = \check{\mathfrak{g}} \oplus \mathbb{R}^k$. Let $\{e_s\}$ be a basis of $\mathbb{R}^k$; the elements $\{e^s\}$ of the dual basis can be viewed as elements of $\mathfrak{g}^*$, and the Lie algebra structure of $\mathfrak{g}$ is entirely determined by $\check{\mathfrak{g}}$ and the exterior derivatives $\{de^s\}$. Explicitly,

$$[v, w]_{\mathfrak{g}} = [v, w]_{\check{\mathfrak{g}}} - \sum_s de^s(v, w) e_s, \quad v, w \in \mathfrak{g}.$$

Lemma 1.1.3 ([31, Lemma 3.1]). *Let $\check{\mathfrak{g}}$ be a nilpotent Lie algebra with a metric $\check{g}$; on the central extension $\mathfrak{g} = \check{\mathfrak{g}} \oplus \mathbb{R}^k$, fix a metric of the form*

$$g = \check{g} + \sum_s \varepsilon_s e^s \otimes e^s, \quad \varepsilon_s = \pm 1.$$

Then, for $v, w \in \check{\mathfrak{g}}$, the Ricci tensors of g and $\check{g}$ are related by

$$\operatorname{ric}(v, w) = \widecheck{\operatorname{ric}}(v, w) - \frac{1}{2} \sum_s \varepsilon_s g(v \lrcorner\, de^s, w \lrcorner\, de^s),$$

$$\operatorname{ric}(v, e_s) = \frac{1}{2} \varepsilon_s g(dv^\flat, de^s), \qquad \operatorname{ric}(e_s, e_t) = \frac{1}{2} \varepsilon_s \varepsilon_t g(de^s, de^t).$$

Proof. By construction, $\operatorname{ad} v = \breve{\operatorname{ad}} v - \sum_s v \lrcorner de^s \otimes e_s$. For one-forms α on $\check{\mathfrak{g}}$, zero-extended to $\mathfrak{g}$, one has $d\alpha = \check{d}\alpha$. Since the musical isomorphisms relative to g and $\check{g}$ are compatible, using [26, Proposition 2.1] one obtains

$$\begin{aligned}
\operatorname{ric}(v, w) &= \frac{1}{2} g(dv^\flat, dw^\flat) - \frac{1}{2} g(\operatorname{ad} v, \operatorname{ad} w) \\
&= \frac{1}{2} g(\check{d}v^\flat, \check{d}w^\flat) - \frac{1}{2} g(\breve{\operatorname{ad}} v, \breve{\operatorname{ad}} w) - \frac{1}{2} g\Big(\sum_s v \lrcorner de^s \otimes e_s, \sum_\ell w \lrcorner de^\ell \otimes e_\ell\Big) \\
&= \breve{\operatorname{ric}}(v, w) - \frac{1}{2} \sum_s \varepsilon_s g(v \lrcorner de^s, w \lrcorner de^s).
\end{aligned}$$

□

In general, for a metric Lie algebra, the Levi-Civita connection assumes the following form

$$\nabla_w v = -\operatorname{ad}(v)^s w - \frac{1}{2} (\operatorname{ad} w)^* v. \tag{1.1}$$

The formula follows immediately from the Koszul formula. In order to specialize to the standard case, I will need to fix an orthogonal basis $\{e_s\}$ on the abelian factor $\mathfrak{a}$ such that $\tilde{g}(e_s, e_s) = \varepsilon_s$.

Lemma 1.1.4 ([30, Lemma 2.5]). *Let $\tilde{\mathfrak{g}}$ be a Lie algebra with a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$. Then*

$$\widetilde{\nabla}_H X = \widetilde{\operatorname{ad}}(H)^a (X), \qquad \widetilde{\nabla}_X H = -\widetilde{\operatorname{ad}}(H)^s (X),$$

for all $H \in \mathfrak{a}$, $X \in \tilde{\mathfrak{g}}$. In addition, if $\{e_i\}$ is an orthogonal basis of $\mathfrak{a}$ and $v, w \in \mathfrak{g}$, I have

$$\widetilde{\nabla}_w v = -\operatorname{ad}(v)^s w - \frac{1}{2} (\operatorname{ad} w)^* v + \sum_s \varepsilon_s \tilde{g}(\widetilde{\operatorname{ad}}(e_s)^s v, w) e_s, \quad v, w \in \mathfrak{g}.$$

Proof. By applying (1.1) to $\widetilde{\nabla}$, one gets

$$\begin{aligned}
\widetilde{\nabla}_H X &= -\widetilde{\operatorname{ad}}(X)^s H - \frac{1}{2} (\widetilde{\operatorname{ad}} H)^* X = -\frac{1}{2} \widetilde{\operatorname{ad}}(X) H - \frac{1}{2} \widetilde{\operatorname{ad}}(X)^* H - \frac{1}{2} \widetilde{\operatorname{ad}}(H)^* X \\
&= \widetilde{\operatorname{ad}}(H)^a (X)
\end{aligned}$$

and

$$\widetilde{\nabla}_X H = -\widetilde{\operatorname{ad}}(H)^s X - \frac{1}{2} (\widetilde{\operatorname{ad}} X)^* H = -\widetilde{\operatorname{ad}}(H)^s X.$$

Now observe that $\widetilde{\operatorname{ad}}(v)^* w = \operatorname{ad}(v)^* w + \sum_s \varepsilon_s \tilde{g}([v, e_s], w) e_s$. Therefore,

$$\begin{aligned}
\widetilde{\nabla}_w v &= -\frac{1}{2} \widetilde{\operatorname{ad}}(v) w - \frac{1}{2} \widetilde{\operatorname{ad}}(v)^* w - \frac{1}{2} \widetilde{\operatorname{ad}}(w)^* v \\
&= -\frac{1}{2} \operatorname{ad}(v) w - \frac{1}{2} \operatorname{ad}(v)^* w - \frac{1}{2} \operatorname{ad}(w)^* v - \frac{1}{2} \sum_s \varepsilon_s \big(\tilde{g}([v, e_s], w) - \tilde{g}([w, e_s], v)\big) e_s \\
&= -\operatorname{ad}(v)^s w - \frac{1}{2} \operatorname{ad}(w)^* v + \frac{1}{2} \sum_s \varepsilon_s \big(\tilde{g}(\operatorname{ad}(e_s) v, w) + \tilde{g}(\operatorname{ad}(e_s)^* v, w)\big) e_s.
\end{aligned}$$

□

For the remainder of the thesis, given a Lie algebra $\mathfrak{g}$ with a metric g, for any endomorphism $f\colon \mathfrak{g} \to \mathfrak{g}$, I will write $f = f^s + f^a$, where f^s is symmetric and f^a is skew-symmetric relative to the metric, i.e.,

$$f^s = \frac{1}{2}(f + f^*), \qquad f^a = \frac{1}{2}(f - f^*).$$

Next, consider a semidirect product $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$, with $\mathfrak{a}$ abelian, and fix any metric. In [40, Section 1.8], [28, Proposition 1.19] and [4] it was shown that under certain conditions one can obtain an isometric Lie algebra by projecting $\operatorname{ad} X$ on its symmetric part, for each $X \in \mathfrak{a}$. These results assume that the decomposition is standard; however, the proof holds more generally, without assuming that the metric is standard and taking more general projections:

Proposition 1.1.5 ([30, Proposition 2.2])**.** *Let $\tilde{\mathfrak{g}}$ be a pseudo-Riemannian Lie algebra (not necessarily standard) of the form $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$; let $\chi\colon \mathfrak{a} \to \operatorname{Der}(\mathfrak{g})$ be a Lie algebra homomorphism such that, extending $\chi(X)$ to $\tilde{\mathfrak{g}}$ by declaring it to be zero on $\mathfrak{a}$,*

$$\chi(X)^s = (\operatorname{ad} X)^s, \qquad [\chi(X), \operatorname{ad} Y] = 0, \;\; X, Y \in \mathfrak{a}. \tag{1.2}$$

Let $\tilde{\mathfrak{g}}^*$ be the Lie algebra $\mathfrak{g} \rtimes_\chi \mathfrak{a}$. Then there is an isometry between the connected, simply connected Lie groups with Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$, with the corresponding left-invariant metrics, whose differential at e is the identity of $\mathfrak{g} \oplus \mathfrak{a}$ as a vector space.

Proof. Observe that for every X in $\mathfrak{a}$, $\chi(X)$ is a derivation of $\mathfrak{g}$ that commutes with $\operatorname{ad} \mathfrak{a}$ by (1.2), and therefore a derivation of $\tilde{\mathfrak{g}}$. For X in $\mathfrak{a}$, write $\operatorname{ad} X = A(X) + \chi(X)$, where $A(X)$ is an antisymmetric derivation of $\tilde{\mathfrak{g}}$. By construction, $A(X)$ is zero on $\mathfrak{a}$.

The rest of the proof is identical to [28, Proposition 1.19], except that one replaces $(\operatorname{ad} X)^a$ with $A(X)$, and one cannot assume that $\exp \mathfrak{g} \exp \mathfrak{a}$ equals the whole connected, simply-connected group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$; however, it is clear that $\exp A(X)$ fixes the connected subgroup with Lie algebra $\mathfrak{a}$, which is what is needed. □

As a consequence, one has a result analogous to [28, Proposition 1.19] for nonstandard metrics ([30, Corollary 2.3]):

Corollary 1.1.6. *Let $\tilde{\mathfrak{g}}$ be a pseudo-Riemannian Lie algebra of the form $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$; suppose that, for every X in $\mathfrak{a}$, $(\operatorname{ad} X)^*$ is a derivation of $\tilde{\mathfrak{g}}$ vanishing on $\mathfrak{a}$, and furthermore*

$$[(\operatorname{ad} X)^*, \operatorname{ad} Y] = 0, \quad X, Y \in \mathfrak{a}. \tag{1.3}$$

Define $\chi\colon \mathfrak{a} \to \operatorname{Der}(\mathfrak{g})$ as $\chi(X) = (\operatorname{ad} X)^s$. Let $\tilde{\mathfrak{g}}^*$ be the solvable Lie algebra $\mathfrak{g} \rtimes_\chi \mathfrak{a}$.

Then there is an isometry between the connected, simply connected Lie groups with Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}^*$, with the corresponding left-invariant metrics, whose differential at e is the identity of $\mathfrak{g} \oplus \mathfrak{a}$ as a vector space.

Example 1.1.7. Consider the 5-dimensional Lie algebra

$$\tilde{\mathfrak{g}} = (0, -2e^{12} - 2e^{34}, -3e^{45} - e^{13} + 3e^{24}, 3e^{35} - 3e^{23} - e^{14}, 2e^{12} + 2e^{34}),$$

with the metric

$$\tilde{g} = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5.$$

To apply Proposition 1.1.5 let $\mathrm{ad}\colon \tilde{\mathfrak{g}} \to \mathrm{Der}(\tilde{\mathfrak{g}})$ be the adjoint representation for $\tilde{\mathfrak{g}}$ and consider the decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathfrak{a}$ where

$$\mathfrak{g} = \mathrm{Span}\{e_1, e_2 - e_5, e_3, e_4\} \text{ and } \mathfrak{a} = \mathrm{Span}\{e_5\}.$$

To obtain an isometric Lie algebra $\tilde{\mathfrak{g}}^*$, let $\bar{\mathrm{ad}}$ be its adjoint representation defined as

$$\tilde{\mathrm{ad}} e_1 = \mathrm{ad}\, e_1, \quad \tilde{\mathrm{ad}} e_2 = \mathrm{ad}\, e_2 - \mathrm{ad}\, e_5, \quad \tilde{\mathrm{ad}} e_3 = \mathrm{ad}\, e_3, \quad \tilde{\mathrm{ad}} e_4 = \mathrm{ad}\, e_4, \quad \tilde{\mathrm{ad}} e_5 = (\mathrm{ad}\, e_5)^s.$$

Some easy computations show that

$$\tilde{\mathfrak{g}}^* = (0, -2e^{12} - 2e^{34}, -e^{13}, -e^{14}, 2e^{12} + 2e^{34}),$$
$$\tilde{g} = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5.$$

This Lie algebra admits a standard decomposition $\mathrm{Span}\{e_2, e_3, e_4, e_5\} \rtimes \mathrm{Span}\{e_1\}$, with

$$\mathrm{Span}\{e_2, e_3, e_4, e_5\} \cong (-2E^{23}, 0, 0, 2E^{23})$$

and

$$\mathrm{ad}\, e_1 = 2e^2 \otimes (e_2 - e_5) + e^3 \otimes e_3 + e^4 \otimes e_4.$$

1.2 Clifford algebras and the spin group

Throughout this section, unless otherwise stated, all references to [60] refer to the first chapter.

Let V be a vector-space on some field $\mathbb{F}$, which will be assumed here and in the following to have characteristic different from 2, and $q\colon V \to \mathbb{F}$ a quadratic form, possibly degenerate. Consider

(a) the tensor algebra

$$\mathcal{F}(V) = \sum_{r=0}^{\infty} \bigotimes^r V,$$

(b) the ideal $\mathfrak{I}_q(V) = \langle v \otimes v + q(v) \cdot 1, v \in V \rangle \subset \mathcal{F}(V)$.

It is called *Clifford algebra associated to* V *and* q the algebra defined as

$$\mathrm{Cl}(V, q) = \mathcal{F}(V)/\mathfrak{I}_q(V).$$

Clearly $V = \bigotimes^1 V \hookrightarrow \mathrm{Cl}(V, q)$, furthermore, $\mathrm{Cl}(V, q)$ is generated by V and the identity 1 subject to the relation $v \cdot v = -q(v) \cdot 1$, where $\cdot$ is the multiplication of the algebra. Furthermore, if $v, w \in V$, then it also holds that

$$v \cdot w + w \cdot v = -2q(v, w),$$

where $2q(v, w) = q(v + w) - q(v) - q(w)$.

Proposition 1.2.1 ([60, Proposition 1.1]). *Let $f\colon V \to \mathcal{A}$ be a linear map between V and an $\mathbb{F}$-associative algebra such that $f(v) \cdot f(v) = -q(v) \cdot 1$. Then f extends uniquely to a map*

$$\tilde{f}\colon \operatorname{Cl}(V, q) \to \mathcal{A}.$$

Remark 1.2.1. Using the previous result, it is easy to see that Clifford algebras are functorial in the sense that if one considers

$$(V, q) \xrightarrow{f} (V', q') \xrightarrow{g} (V'', q'')$$

then one can find a unique

$$\operatorname{Cl}(V, q) \xrightarrow{\tilde{f}} \operatorname{Cl}(V', q') \xrightarrow{\tilde{g}} \operatorname{Cl}(V'', q'')$$

and $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$.

It follows then that

$$\mathrm{O}(V, q) = \{\, f \in \mathrm{GL}(V, q) \mid f^* q = q \,\} \subseteq \operatorname{Aut}(\operatorname{Cl}(V, q))$$

extends canonically. In particular, the reflection

$$\alpha\colon V \to V, \quad \alpha(v) = -v$$

extends linearly to a map $\alpha\colon \operatorname{Cl}(V, q) \to \operatorname{Cl}(V, q)$. Since it is an involution, there is a splitting in the $+1$ and -1 eigenspaces of α

$$\operatorname{Cl}(V, q) = \operatorname{Cl}^0(V, q) \oplus \operatorname{Cl}^1(V, q),$$

where $\operatorname{Cl}^i(V, q) = \{\varphi \in \operatorname{Cl}(V, q) \,|\, \alpha(\varphi) = (-1)^i \varphi\}$. In particular $\operatorname{Cl}^0(V, q)$ is a subalgebra of $\operatorname{Cl}(V, q)$, called the *even part*, which contains the identity.

Proposition 1.2.2 ([60, Proposition 1.2]). *For any quadratic form $q\colon V \to \mathbb{F}$, there is a linear isomorphism $\operatorname{Cl}(V, q) \cong \Lambda^* V$.*

Hence, Clifford algebras are more subtle than the exterior algebra of a vector space. Indeed, the two coincide only if $q \equiv 0$ and $v \wedge v = v \cdot v$.

Another consequence of this result is that if $V_1 \oplus_q V_2 = V$, where $\oplus_q$ is the orthogonal decomposition with respect to (with respect to) q, then there exists a natural isomorphism

$$\operatorname{Cl}(V, q) \to \operatorname{Cl}(V_1, q_1) \oplus \operatorname{Cl}(V_2, q_2), \quad q_i = q_{|V_i}.$$

Given the tensor algebra $\mathcal{F}(V)$, it is possible to define the *transpose* of an element $s \in \mathcal{F}(V)$, denoted by s^t, by setting for $v_1, \dots, v_r$ elements of a basis of V

$$^t\colon v_1 \otimes \cdots \otimes v_r \to v_r \otimes \cdots \otimes v_1,$$

that descends to $\mathrm{Cl}(V,q)$ since it preserves $\mathfrak{I}_q(V)$. Next, I will recall the construction of the group $\mathrm{Spin}(V,q)$ as the two-sheeted cover of $\mathrm{SO}(V,q)$. In order to do so, consider

$$\mathrm{Cl}^\times(V,q) = \{\varphi \in \mathrm{Cl}(V,q) \mid \exists\, \varphi^{-1}, \varphi^{-1}\varphi = \varphi\varphi^{-1} = 1\},$$

the *multiplicative group of units* in the Clifford algebra, containing $\{ v \in V \mid q(v) \neq 0 \} \subset V$. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then $\mathrm{Cl}^\times(V,q)$ is a Lie group where one can define the *adjoint representation*

$$\mathrm{Ad}\colon \mathrm{Cl}^\times(V,q) \to \mathrm{Aut}(\mathrm{Cl}(V,q)), \quad \mathrm{Ad}_\varphi(x) = \varphi x \varphi^{-1}.$$

Proposition 1.2.3 ([60, Proposition 2.2])**.** *If $v \in V$ is such that $q(v) \neq 0$, then $\mathrm{Ad}_v(V) = V$; indeed*

$$-\mathrm{Ad}_v(w) = w - 2\frac{q(v,w)}{q(v)}v, \quad \forall\, w \in V. \tag{1.4}$$

Next, one needs to consider not only elements of V such that $\mathrm{Ad}_\varphi(V) = V$, but take into account also elements from $\mathrm{Cl}^\times(V,q)$. Notice that

$$(\mathrm{Ad}_v^* q)(w) = q(w) \quad \text{if} \quad q(v) \neq 0,$$

hence $\mathrm{Ad}_v \in \mathrm{O}(V,q)$; furthermore define $P(V,q) \subseteq \mathrm{Cl}^\times(V,q)$ to be the group generated by $v \in V$ such that $q(v) \neq 0$ and observe that the map $\mathrm{Ad}\colon P(V,q) \to \mathrm{O}(V,q)$ is a representation. Set

$$\mathrm{Pin}(V,q) = \langle v \in V \mid q(v) = \pm 1 \rangle \quad \text{and} \quad \mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap \mathrm{Cl}^0(V,q).$$

Notice that the two groups are generated by the generalized unit sphere of (V,q) as

$$\mathrm{Pin}(V,q) = \{v_1 \cdots v_r \in P(V,q) \colon q(v_j) = \pm 1 \text{ for all } j\}$$

and

$$\mathrm{Spin}(V,q) = \{v_1 \cdots v_r \in \mathrm{Pin}(V,q) \colon r \text{ is even}\}.$$

The right side of (1.4) is the reflection $\rho_v\colon V \to V$, with respect to $v^\perp = \mathrm{Span}\,\{v\}^\perp$, that maps $v \to -v$. But on the left side there is a minus sign, so one considers the *twisted adjoint representation*

$$\widetilde{\mathrm{Ad}}_\varphi(x) = \alpha(\varphi) x \varphi^{-1},$$

that is still an automorphism and that coincides with Ad in the case of φ even. In this way

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{q(v,w)}{q(v)}v$$

is exactly the reflection.

Proposition 1.2.4. *If $\dim V < \infty$ and q is non-degenerate then, if*

$$\tilde{P}(V,q) = \{ \varphi \in \mathrm{Cl}^\times(V,q) \mid \widetilde{\mathrm{Ad}}_\varphi(V) = V \} \supseteq P(V,q),$$

the map $\widetilde{\mathrm{Ad}}\colon \tilde{P}(V,q) \to \mathrm{GL}(V,q)$ has $\ker(\widetilde{\mathrm{Ad}}) = \mathbb{F}^\times$.

This characterization of the kernel is possible only for the twisted adjoint representation and only if the quadratic form is non-degenerate. Next, to prove that $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ are the two-sheeted covers of $\mathrm{O}(V,q)$ and $\mathrm{SO}(V,q)$, one needs to prove that the map $\widetilde{\mathrm{Ad}}\colon \tilde{P}(V,q) \to \mathrm{O}(V,q)$ is a homomorphism, and is able to do so thanks to the *norm map* $N\colon\ \mathrm{Cl}(V,q) \to \mathrm{Cl}(V,q)$ defined as

$$N(\varphi) = \varphi \cdot \alpha(\varphi^t),$$

which, if restricted to $\tilde{P}(V,q)$, becomes a homomorphism to $\mathbb{F}$. In this case, considering $\varphi = v_1 \cdot \cdots \cdot v_r \in P(V,q)$, one has that

$$\widetilde{\mathrm{Ad}}_\varphi = \rho_{v_1} \circ \cdots \circ \varphi_{v_r}, \quad \rho_{v_i} \text{ reflection with respect to } v_i^\perp.$$

Next, recall the classical result

Theorem 1.2.5 (Cartan-Dieudonné)**.** *Let V be a vector space of dimension n over a field $\mathbb{F}$ endowed with a non-singular quadratic form q. Then every isometry $\sigma\colon V \to V$ is the product of at most n symmetries with respect to non-singular hyperplanes.*

For a proof, see for example Artin's book [3]. A simple application allows one to deduce that $\widetilde{\mathrm{Ad}}$ is surjective on $\mathrm{O}(V,q)$ and that its restriction

$$\widetilde{\mathrm{Ad}}\colon SP(V,q) = P(V,q) \cap \mathrm{Cl}^0(V,q) \to \mathrm{SO}(V,q)$$

is also onto. The next step is to prove that the same holds for $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ as well. Note that $\rho_{tv} = \rho_v$ for every $t \neq 0$, so when is always possible to rescale v to a v' such that $q(v') = \pm 1$ the surjectivity passes from $P(V,q)$ and $SP(V,q)$ to $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ respectively. So if $\exists\, t \in \mathbb{F}$ such that $\pm 1 = q(v') = q(tv) = t^2 q(v)$ is solvable for t, i.e., the equation

$$t^2 = \pm a \tag{1.5}$$

always admits a solution in $\mathbb{F}$, then the surjectivity would pass down. If $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ then this is true, if $\mathbb{F} = \mathbb{Q}$ then it is not. If (1.5) is solvable, then $\widetilde{\mathrm{Ad}}$ is surjective and $\mathbb{F}$ is said to be *spin*.

Proposition 1.2.6. $\widetilde{\mathrm{Ad}}(\mathrm{Pin}(V,q))$ *and* $\widetilde{\mathrm{Ad}}(\mathrm{Spin}(V,q))$ *are normal subgroups of* $\mathrm{O}(V,q)$.

What follows is a preliminary result that guarantees that at least Pin and Spin are covers of the respective groups:

Theorem 1.2.7 ([60, Theorem 2.9])**.** *Let V be a finite-dimensional vector-space over a spin field $\mathbb{F}$, q a quadratic form non degenerate over V. Then the following sequences are exact*

$$0 \longrightarrow F \longrightarrow \mathrm{Pin}(V,q) \longrightarrow \mathrm{O}(V,q) \longrightarrow 1$$

$$0 \longrightarrow F \longrightarrow \mathrm{Spin}(V,q) \longrightarrow \mathrm{SO}(V,q) \longrightarrow 1$$

where

$$F = \begin{cases} \{\, 1, -1 \,\} = \mathbb{Z}_2 & \sqrt{-1} \notin \mathbb{F} \\ \{\, \pm 1, \pm\sqrt{-1} \,\} = \mathbb{Z}_4 & \textit{otherwise.} \end{cases}$$

Remark 1.2.2. The theorem still holds if one drops the hypothesis that $\mathbb{F}$ is a spin field, but in place of $\mathrm{SO}(V,q)$ and $\mathrm{O}(V,q)$ one needs to place an appropriate normal subgroup of the latter.

Sketch of the proof. Let $\varphi = v_1 \cdot \cdots \cdot v_r$ and assume $\varphi \in \ker \tilde{\mathrm{Ad}}$. By Proposition 1.2.4 it is known that $\varphi \in \mathbb{F}^\times$, hence

$$\varphi^2 = N(\varphi) = N(v_1) \cdot \cdots \cdot N(v_r) = \pm 1,$$

which establishes the kernel. Regarding surjectivity it suffices to notice that $\mathbb{F}$ is spin, therefore any $v \in V^\times$ may be renormalized to have q-length 1. □

It is interesting to notice that if $\mathbb{F}$ is spin, then either

$$\tilde{P}(V,q) = P(V,q) \quad \text{or} \quad \tilde{P}(V,q)/P(V,q) = \mathbb{Z}_2.$$

Real case

Suppose $\mathbb{F} = \mathbb{R}$ and write $q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$ where $r, s \geq 0$ and $r \leq n$. One then says that q has signature (r,s). To emphasize the signature of q I will write

$$\begin{aligned} \mathrm{O}(V,q) &= \mathrm{O}_{r,s} & \mathrm{Pin}(V,q) &= \mathrm{Pin}_{r,s} & \tilde{P}(V,q) &= \tilde{P}_{r,s} \\ \mathrm{SO}(V,q) &= \mathrm{SO}_{r,s} & \mathrm{Spin}(V,q) &= \mathrm{Spin}_{r,s} & P(V,q) &= P_{r,s}. \end{aligned}$$

Recall that SO_n is connected but not simply connected and that $\mathrm{SO}_{r,s}$ has exactly two connected components when $r, s > 0$. Moreover $\pi_1(\mathrm{SO}_n) = \mathbb{Z}_2$ and

$$\pi(\mathrm{SO}_{r,s}^0) = \pi(\mathrm{SO}_r) \times \pi(\mathrm{SO}_s)$$

whenever $n \geq 3$, thus

$$\pi(\mathrm{SO}_{r,s}^0) = \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Finally, one gets the full version of the partial result, Theorem 1.2.7:

Theorem 1.2.8 ([60, Theorem 2.10]). *For any (r,s) the following sequences are exact*

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_{r,s} \xrightarrow{\xi} \mathrm{SO}_{r,s} \longrightarrow 1$$

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}_{r,s} \longrightarrow \mathrm{O}_{r,s} \longrightarrow 1.$$

Furthermore, if $(r,s) \neq (1,1)$, the two-sheeted coverings are not trivial over each connected component of $\mathrm{O}_{r,s}$. In the particular case of $r = n$, one has

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \xrightarrow{\xi_0} \mathrm{SO}_n \longrightarrow 1$$

where the map $\xi_0 = \widetilde{\mathrm{Ad}}$ is the universal covering of SO_n.

1.2.1 Classification of Clifford algebras and spinor representation

In the following, I will restrict the attention to the case where $V = \mathbb{R}^n$ endowed with a quadratic form

$$q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2, \quad r + s = n$$

of signature (r, s). In this case, I will write $(\mathbb{R}^n, q) = \mathbb{R}^{r,s}$ and $\mathrm{Cl}(\mathbb{R}^n, q) = \mathrm{Cl}_{r,s}$. A simple way to present Clifford algebras in this setting is the following

Proposition 1.2.9. *Let $\{e_i\}$ be a q-orthonormal basis of $\mathbb{R}^{r+s} \subset \mathrm{Cl}_{r,s}$. Then $\mathrm{Cl}_{r,s}$ can be presented as generated by $e_1, \ldots, e_{r+s}$ such that*

$$e_ie_j + e_je_i = \begin{cases} -2\delta_{ij} & i \leq r \\ 2\delta_{ij} & i > r. \end{cases}$$

A useful object in the study of the spin representations, which will be carried out subsequently, is the volume element. Let $e_1, \ldots, e_{r+s}$ be a positively-oriented orthonormal base for $\mathbb{R}^{r,s}$. Then the *volume element* of $\mathrm{Cl}_{r,s}$ is

$$\omega_{r,s} = e_1 \cdots e_{r+s}.$$

It is easy to see that the definition is independent of the choice of orthonormal base, but not of the orientation. Next, I will recall some useful properties of the volume element.

Proposition 1.2.10 ([60, Proposition 3.3])**.** *If $r + s = n$, the volume element for $\mathrm{Cl}_{r,s}$ satisfies*

$$\begin{aligned} \omega_{r,s}^2 &= (-1)^{\frac{n(n+1)}{2}+s}, \\ v\omega_{r,s} &= (-1)^{n-1}\omega_{r,s}v, \quad v \in \mathbb{R}^{r,s}. \end{aligned} \tag{1.6}$$

In particular, for n odd ω is central, if n is even then

$$\varphi\omega_{r,s} = \omega_{r,s}\alpha(\varphi), \quad \varphi \in \mathrm{Cl}_{r,s}$$

Equation (1.6) is equivalent to saying that

$$\omega_{r,s}^2 = \begin{cases} 1 & r - s = 0, 3 \mod 4 \\ -1 & r - s = 1, 2 \mod 4 \end{cases}.$$

The following two results concern decompositions of $\mathrm{Cl}_{r,s}$ depending on the evenness of the dimension.

Proposition 1.2.11. *Assume $r+s$ is odd and the volume element satisfies $\omega_{r,s}^2 = 1$, and let $\pi^\pm = \frac{1}{2}(1 \pm \omega_{r,s})$. Then $\mathrm{Cl}_{r,s}$ decomposes as*

$$\mathrm{Cl}_{r,s} = \mathrm{Cl}_{r,s}^+ \oplus \mathrm{Cl}_{r,s}^-,$$

where $\mathrm{Cl}_{r,s}^\pm = \pi^\pm \cdot \mathrm{Cl}_{r,s}$ and $\alpha(\mathrm{Cl}_{r,s}^\pm) = \mathrm{Cl}_{r,s}^\mp$.

A similar result holds for $\mathrm{Cl}_{r,s}$-modules when $r+s$ is even. Indeed

Proposition 1.2.12. *Assume $r+s$ is even and $\omega_{r,s}^2 = 1$. If V is a $\mathrm{Cl}_{r,s}$-module, then it decomposes as*

$$V = V^+ \oplus V^-$$

in the $+1$ and -1 eigenspaces for the multiplication by $\omega_{r,s}$. In particular

$$V^+ = \pi^+ \cdot V \quad \text{and} \quad V^- = \pi^- \cdot V.$$

Furthermore, multiplication by a non-light-like vector interchanges the modules, i.e.

$$e\colon V^\pm \longrightarrow V^\mp, \quad e \in \mathbb{R}^{r,s},\ \langle e, e\rangle \neq 0.$$

I will now present a description of the Clifford algebras $\mathrm{Cl}_{r,s}$, summarizing everything in a table which can be also found in [60, Chapter 1]. I will first recall the first few Clifford algebras $\mathrm{Cl}_{r,s}$ for $r+s \leq 2$. Let e_1, e_2 be a basis for $\mathbb{R}^{r,s}$.

- $\mathrm{Cl}_{0,0} = \mathrm{Span}\,\{1\} = \mathbb{R}$ as there are no non-zero vectors.
- $\mathrm{Cl}_{1,0} = \mathrm{Span}\,\{1, e_1\} \cong \mathbb{C}$ since $e_1 \cdot e_1 = -1$, hence e_1 acts as i.
- $\mathrm{Cl}_{0,1} = \mathrm{Span}\,\{1, e_1\} \cong \mathbb{R} \oplus \mathbb{R}$ since $e_1 \cdot e_1 = 1$, and e_1 and 1 are linearly independent, so $e_1 = (1, -1)$.
- $\mathrm{Cl}_{2,0} = \mathrm{Span}\,\{1, e_1, e_2, e_1e_2\} \cong \mathbb{H}$ since $e_i \cdot e_i = -1 = (e_1 \cdot e_2) \cdot (e_1 \cdot e_2)$, and all three anti-commute, hence $e_1 = i$, $e_2 = j$ and $e_1e_2 = k$.
- $\mathrm{Cl}_{1,1} = \mathrm{Span}\,\{1, e_1, e_2, e_1 \cdot e_2\} \cong M(2, \mathbb{R})$ by setting

  $$e_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  and noticing that $e_1 \cdot e_1 = -Id = -e_2 \cdot e_2$.
- $\mathrm{Cl}_{0,2} = \mathrm{Span}\,\{1, e_1, e_2, e_1 \cdot e_2\} \cong M(2, \mathbb{R})$ by setting

  $$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

  and noticing that $e_1 \cdot e_1 = Id = e_2 \cdot e_2$.

Remark 1.2.3. Notice that by letting $\mathbb{H}$ act on $a + jb \in \mathbb{C}^2$ by left multiplication, one can represent $\mathrm{Cl}_{2,0}$ as complex matrices, i.e.,

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.$$

The next result shows that there exist some relations between Clifford algebras of different dimension.

Proposition 1.2.13 ([60, Theorem 4.1]). *Let $e_1, \dots, e_{r+s}$ be an orthonormal basis of $\mathbb{R}^{r,s} \subset \mathrm{Cl}_{r,s}$, with*

$$\langle e_i, e_i \rangle = \begin{cases} 1 & i \le r \\ -1 & i > r \end{cases}.$$

There are isomorphisms

1. $\mathrm{Cl}_{r+1,s+1} = \mathrm{Cl}_{1,1} \otimes \mathrm{Cl}_{r,s} = M(2, \mathrm{Cl}_{r,s})$, *with a positively-oriented orthonormal basis of $\mathbb{R}^{r+1,s+1}$ given by*

$$\begin{pmatrix} e_i & 0 \\ 0 & -e_i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2. $\mathrm{Cl}_{s+2,0} = \mathrm{Cl}_{2,0} \otimes \mathrm{Cl}_{0,s} = \mathrm{Cl}_{0,s} \otimes \mathbb{H} \subset M(2, \mathrm{Cl}_{0,s} \otimes \mathbb{C})$, *with a positively-oriented orthonormal basis of $\mathbb{R}^{s+2,0}$ given by*

$$\begin{pmatrix} 0 & -ie_i \\ -ie_i & 0 \end{pmatrix}, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

3. $\mathrm{Cl}_{0,r+2} = \mathrm{Cl}_{0,2} \otimes \mathrm{Cl}_{r,0} = M(2, \mathrm{Cl}_{r,0})$, *with a positively-oriented orthonormal basis of $\mathbb{R}^{0,r+2}$ given by*

$$\begin{pmatrix} 0 & e_i \\ -e_i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Sketch of the proof. Let s_1, s_2 be the generators of $\mathrm{Cl}_{1,1}$, and notice that

$$\mathcal{G} = \{e_1 \otimes s_1 s_2, \dots, e_{r+s} \otimes s_1 s_2, 1 \otimes s_1, 1 \otimes s_2\}$$

is a set of generators for $\mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{1,1}$. One then defines a map between an orthonormal basis $\mathcal{B} = \{b_1, \dots, b_{n+2}\}$ of $\mathbb{R}^{r+1,s+1}$ and $\mathcal{G}$ as

$$f(b_i) = \begin{cases} e_i \otimes s_1 s_2 & i \le n \\ 1 \otimes s_1 & i = n+1 \\ 1 \otimes s_2 & i = n+2. \end{cases}$$

It is easy to verify that the map satisfies the hypothesis of Proposition 1.2.1, hence it extends to a map $\tilde{f} \colon \mathrm{Cl}_{r+1,s+1} \to \mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{1,1}$, which is onto $\mathcal{G}$, hence surjective on the algebra. As $\dim \mathrm{Cl}_{r+1,s+1} = \dim(\mathrm{Cl}_{r,s} \otimes \mathrm{Cl}_{1,1})$, the map is actually an isomorphism. The same strategy is applied to prove the other two cases, defining a similar map f. □

Furthermore, one can deduce some more relations, indeed

Proposition 1.2.14 ([60, Theorem 4.5]). *For all $n \in \mathbb{N}$ the following holds:*

$$\mathrm{Cl}_{n+8,0} \cong \mathrm{Cl}_{n,0} \otimes \mathrm{Cl}_{8,0}, \quad \mathrm{Cl}_{0,n+8} \cong \mathrm{Cl}_{0,n} \otimes \mathrm{Cl}_{0,8},$$

where

$$\mathrm{Cl}_{8,0} = \mathrm{Cl}_{0,8} = \mathbb{R}(16) = M(16, \mathbb{R}).$$

Thanks to the preceding results, table 1.1 at the end of the chapter encodes all the Clifford algebras realizations as matrix algebras. Next, I will present the construction of the spin representations. Recall that a *representation* for the algebra $\mathrm{Cl}_{r,s}$ is an $\mathbb{R}$-algebra homomorphism

$$\rho\colon \mathrm{Cl}_{r,s} \to \mathrm{Hom}_{\mathbb{K}}(W, W),$$

where $\mathbb{K} \supseteq \mathbb{R}$ is a field and W is a finite dimensional vector space over $\mathbb{K}$. W will be called a $\mathrm{Cl}_{r,s}$*-module* over $\mathbb{K}$. As usual, I will write $\rho(\varphi)(w) = \varphi \cdot w$ for $\varphi \in \mathrm{Cl}_{r,s}$ and $w \in W$. In the following, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The $\mathbb{K}$-representation ρ is *reducible* if W decomposes as $W_1 \oplus W_2$, with $\rho(\varphi)(W_j) \subseteq W_j$ for any $\varphi \in \mathrm{Cl}_{r,s}$. If a representation is not reducible, it is called *irreducible*. Any representation of a Clifford algebra can be decomposed into the sum of irreducible representations. Two representations

$$\rho_j\colon \mathrm{Cl}_{r,s} \to \mathrm{Hom}_{\mathbb{K}}(W_j, W_j)$$

are *equivalent* if there exists a $\mathbb{K}$-linear map $F\colon W_1 \to W_2$ such that $F \circ \rho_1(\varphi) = \rho_2(\varphi) \circ F$ for any $\varphi \in \mathrm{Cl}_{r,s}$. It is possible to count the number of irreducible, inequivalent representations of $\mathrm{Cl}_{r,s}$ thanks to the following

Theorem 1.2.15 ([60, Theorem 5.7]). *Let $\nu_{r,s}$ denote the number of inequivalent irreducible representations of $\mathrm{Cl}_{r,s}$. Then*

$$\nu_{r,s} = \begin{cases} 2 & r - s + 1 \equiv 0 \mod 4 \\ 1 & \textit{otherwise.} \end{cases}$$

Now recall that $\mathrm{Cl}_{r,s} \cong \mathrm{Cl}^0_{r+1,s}$ and hence

$$\mathrm{Spin}_{r,s} \subset \mathrm{Cl}^0_{r,s} \cong \mathrm{Cl}_{r-1,s}$$

for any r, s. Thus, to obtain the representations of $\mathrm{Spin}_{r,s}$ one can restrict the attention to $\mathrm{Cl}_{r-1,s}$. Recall also that, by Propositions 1.2.11 and 1.2.12,

$$\mathrm{Cl}^{\pm}_{r,s} = (1 \pm \omega_{r,s})\,\mathrm{Cl}_{r,s}\,.$$

When $\omega^2_{r,s} = 1$, the representation of the Clifford algebra can be described based on the action of the volume element. The following two results describe the situation for real Clifford algebras.

Proposition 1.2.16 ([60, Proposition 5.9]). *Let $r - s = 3 \mod 4$ and consider any irreducible real representation $\rho\colon \mathrm{Cl}_{r,s} \to \mathrm{Hom}_{\mathbb{R}}(W, W)$. Then either*

$$\rho(\omega_{r,s}) = \mathrm{Id} \quad \textit{or} \quad \rho(\omega_{r,s}) = -\mathrm{Id},$$

which can both occur. The corresponding representations are inequivalent.

Proposition 1.2.17 ([60, Proposition 5.10]). *Let* $\rho\colon \mathrm{Cl}_{r,s} \to \mathrm{Hom}_{\mathbb{R}}(W, W)$ *be an irreducible real representation and* $r + s = 4m$. *Furthermore, consider the splitting*

$$W = W^+ \oplus W^-, \quad W^\pm = (1 \pm \rho(\omega_{r,s})) \cdot W.$$

Then the subspaces $W^\pm$ *are invariant under the action of* $\mathrm{Cl}^0_{r,s}$ *and, by the isomorphism* $\mathrm{Cl}^0_{r,s} \cong \mathrm{Cl}_{r-1,s}$, *these correspond to the irreducible real representations obtained in the previous proposition.*

The *real spinor representation* of $\mathrm{Spin}_{r,s}$ is the homomorphism

$$\Delta_{r,s}\colon \mathrm{Spin}_{r,s} \to \mathrm{GL}(S)$$

defined as the restriction of an irreducible real representation of $\mathrm{Cl}_{r,s} \to \mathrm{Hom}(S, S)$ to $\mathrm{Spin}_{r,s} \subset \mathrm{Cl}^0_{r,s} \subset \mathrm{Cl}_{r,s}$. The representation depends on $n = r + s$ as described in the following

Proposition 1.2.18. *Let* $n = r + s$ *and consider the real spinor representation*

$$\Delta_n\colon \mathrm{Spin}_{r,s} \to \mathrm{GL}(S).$$

One has that:

- *if* $n \equiv 3 \mod 4$, Δ_n *does not depend on the choice of the irreducible representation of* $\mathrm{Cl}_{r,s}$ *obtained in Proposition 1.2.16;*
- *if* $n \equiv 1, 2 \mod 8$, *then* Δ_n *is the sum of two equivalent irreducible representations of* $\mathrm{Cl}_{r,s}$*;*
- *if* $n \equiv 5, 7 \mod 8$, *then* Δ_n *is irreducible;*
- *if* $n \equiv 0 \mod 4$, *then it decomposes as*

$$\Delta_{4m} = \Delta^+_{4m} \oplus \Delta^-_{4m}$$

where $\Delta^\pm_{4m}$ *are inequivalent irreducible representations of* Spin_{4m}.

The representation of $\mathrm{Spin}_{r,s}$ induces a representation at the Lie algebra level. Recall that $\mathrm{Cl}^\times_{r,s}$ is a Lie group with Lie algebra $\mathfrak{cl}^\times_{r,s} = (\mathrm{Cl}_{r,s}, [\cdot,\cdot])$, where $[\varphi, \psi] = \varphi \cdot \psi - \psi \cdot \varphi$, and has an exponential mapping

$$\exp\colon \mathfrak{cl}^\times_{r,s} \to \mathrm{Cl}^\times_{r,s} \quad \exp(\psi) = \sum_{k=0}^{\infty} \frac{1}{k!}\psi^k.$$

One is then able to study the Lie algebra $\mathfrak{spin}_{r,s}$ as a subalgebra of $\mathfrak{cl}^\times_{r,s}$. In order to do so, one first defines for an orthonormal basis $(e_1, \dots, e_r, e_{r+1}, \dots, e_{r+s})$ of $\mathbb{R}^n$ such that

$$\langle e_i, e_i \rangle = \varepsilon_i = \begin{cases} 1 & i \le r \\ -1 & i > r, \end{cases}$$

and the correspondence

$$(v \wedge w)(x) \equiv (v^\flat \otimes w - w^\flat \otimes v)(x), \tag{1.7}$$

where $v^\flat = g(v, \cdot\,)$ is the usual musical isomorphism, and notices that, under this identification, $v \wedge w$ generate $\mathfrak{so}_{r,s}$ as a vector space, which corresponds to the space of skew-self-adjoint linear maps. It holds that

Proposition 1.2.19 ([60, Corollary 6.3])**.** *Let $\Delta\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}(W)$ be a representation obtained by restricting a representation of the Clifford algebra $\mathrm{Cl}_{r,s} \to \mathrm{Hom}(W, W)$. Let $\Delta_*\colon \mathfrak{so}_{r,s} \to \mathfrak{so}(W)$ be the corresponding associated representation of the Lie algebras obtained via pullback. Then under the correspondence $\mathfrak{so}_{r,s} \cong \Lambda^2\mathbb{R}^n$ one has*

$$\Delta_*(e_i \wedge e_j) = \frac{1}{4}[e_i, e_j]\,\cdot$$

where the dot denotes the Clifford multiplication on W.

To see why this is true, recall that the adjoint representation gives a surjective homomorphism

$$\mathrm{Spin}_{r,s} \xrightarrow{\xi} \mathrm{SO}_{r,s}\,.$$

This, in turn, induces a Lie algebra isomorphism $\Xi\colon \mathfrak{spin}_{r,s} \longrightarrow \mathfrak{so}_{r,s}$ and $\mathfrak{spin}_{r,s} \cong \Lambda^2\mathbb{R}^n$. Proposition 1.2.19 then follows, since this isomorphism is given explicitly on a basis $\{e_ie_j\}_{i<j}$ of $\mathfrak{spin}_{r,s}$ by

$$\Xi(e_ie_j) = 2e_i \wedge e_j$$

and

$$\Xi^{-1}(e_i \wedge e_j) = \frac{1}{4}[e_i, e_j]$$

by [60, Proposition 6.2].

1.3 Spin structures and spin bundles

In this section, I will recall the theory behind the existence of a spin structure over a manifold and the construction of the spinor bundle. Unless otherwise stated, references to [60] refer to the second chapter.

Let $E \xrightarrow{\pi} X$ be an oriented n-dimensional pseudo-Riemannian fiber bundle with signature (r, s). One can define the *orientation bundle* of E as

$$\mathrm{Or}(E) = P_O(E)/\,\mathrm{SO}_{r,s}$$

where $P_O(E)$ is the principal $\mathrm{O}_{r,s}$-bundle of orthonormal frames, in which the fiber in x is the set of possible oriented orthonormal bases of E_x. Note that $\mathrm{Or}(E)$ is a 2-sheeted covering of X and that E is orientable if and only if $\mathrm{Or}(E)$ is the trivial covering. There is a natural isomorphism

$$\mathrm{Cov}_2(X) \overset{w_1}{\cong} H^1(X, \mathbb{Z}_2)$$

where $\mathrm{Cov}_2(X)$ is the set of equivalence classes of 2-sheeted coverings of X, hence $\mathrm{Or}(E) \in \mathrm{Cov}_2(X)$ and one calls $w_1(\mathrm{Or}(E)) = w_1(E)$ the *first class of Stiefel-Whitney* of E. By the previous considerations

$$E \text{ orientable} \iff w_1(E) = 0 \in H^1(X, \mathbb{Z}_2)$$

and clearly there are only two possible orientations. This definition

(i) is natural: $w_1(f^*E) = f^*w_1(E)$ for any $E \xrightarrow{\pi} X$ and $f\colon X' \to X$ continuous;

(ii) satisfies the following: if $\mathbb{E}$ is the n-Grassmanian bundle on $BO_{r,s}$, classifying space, then $w_1(\mathbb{E}) \neq 0 \in H^1(BO_n, \mathbb{Z}_2)$.

Digression on classifying spaces

If G is a Lie group, let $P_G(X)$ be a principal G-bundle over some space X and denote by $\overline{P_G(X)}$ to be its isomorphism class. Furthermore, given any two topological spaces X, Y and a continuous map $f\colon X \to Y$, denote by $\bar{f}^{Htp.}$ the homotopy class of f. Then a *classifying space* for G is a connected topological space BG together with a principal G-bundle $EG \to BG$ such that

$$\{\, \overline{P_G(X)} \mid X \text{ cpt. and Hausdorff} \,\} \overset{1:1}{\iff} \{\, \bar{f}^{Htp.}\colon X \to BG \mid X \text{ cpt. and Hausdorff} \,\},$$

where the one-to-one correspondence is induced by the pull-back.

Example 1.3.1.
- S^1 is the classifying space for the infinite cyclic group $\mathbb{Z}$ and the principal $\mathbb{Z}$-bundle is $E\mathbb{Z} = \mathbb{R}$;
- $\mathbb{RP}^\infty$ is the classifying space for $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ where the principal $\mathbb{Z}_2$-bundle is $E\mathbb{Z}_2 = S^\infty$ seen as direct limit of S^n over n;
- $Gr(N, \mathbb{R}^\infty)$, the Grassmanian of n-planes in $\mathbb{R}^\infty$, is the classifying space for $\mathrm{O}_{r,s}$ and $E\,\mathrm{O}_{r,s} = V(n, \mathbb{R}^\infty)$ is the Stiefel manifold of n ordered, orthonormal vectors in $\mathbb{R}^\infty$.

Going back to the main discussion, one can obtain w_1 in another way. If X is connected, starting from the fibration $\mathrm{O}_{r,s} \to P_O(E) \to X$, one obtains

$$0 \longrightarrow H^0(X, \mathbb{Z}_2) \longrightarrow H^0(P_O(E), \mathbb{Z}_2) \longrightarrow H^0(\mathrm{O}_{r,s}, \mathbb{Z}_2) \xrightarrow{w_E} H^1(X, \mathbb{Z}_2).$$

Setting $w_1(E) = w_E(g_1)$, where g_1 is the generator of $H^0(\mathrm{O}_{r,s}, \mathbb{Z}_2)$, gives again the first Stiefel-Whitney class.

Remark 1.3.1. Choosing an orientation on E is equivalent to choosing a principal SO_n-bundle.

Remark 1.3.2. In the construction of $P_O(E)$ there is no requirement that E is orientable, that is only needed when one wants to replace $\mathrm{O}_{r,s}$ with the special subgroup SO_n. In this way, one can guarantee the existence of a continuously defined, positive vector field over X. It is then natural to ask if the structure group can be simplified further.

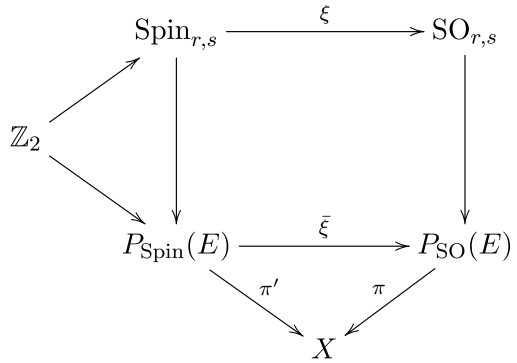
Now, for a spin structure, the idea is the same. Let $E \xrightarrow{\pi} X$ be an oriented pseudo-Riemannian bundle of dimension $r+s=n \geq 3$ and $P_{\mathrm{SO}}(E)$ its principal $\mathrm{SO}_{r,s}$-bundle given by its orientation. Recall from the previous section that

$$\xi\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}_{r,s}, \quad \ker \xi = \{-1, 1\} \cong \mathbb{Z}_2$$

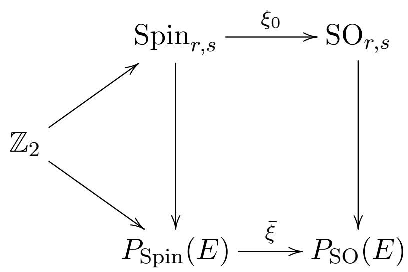
is the 2-sheeted covering. Then a *spin structure* over the bundle E is a principal $\mathrm{Spin}_{r,s}$-bundle $P_{\mathrm{Spin}}(E)$ together with a 2-sheeted covering

$$P_{\mathrm{Spin}}(E) \xrightarrow{\bar{\xi}} P_{\mathrm{SO}}(E),$$

such that $\bar{\xi}(pg) = \bar{\xi}(p)\xi(g)$ for $p \in P_{\mathrm{Spin}}(E)$ and $g \in \mathrm{Spin}_{r,s}$. The diagram of the fibration is the following



On the other hand if $P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$ is a 2-sheeted covering not trivial on the fibers of X, i.e., the diagram



commutes, then $P_{\mathrm{Spin}}(E)$ is a fiber bundle on X setting $\pi' = \xi \circ \pi$. Lifting the action $\mathrm{SO}_{r,s} \curvearrowright P_{\mathrm{SO}}(E)$ to a compatible action $\mathrm{Spin}_{r,s} \curvearrowright P_{\mathrm{Spin}}(E)$, one obtains the desired principal Spin_n-bundle.

Theorem 1.3.2 ([60, Theorem 1.4])**.** *The spin structures on E are in 1:1 correspondence with the 2-sheeted coverings of E that are not trivial over the fibers of* $P_{\mathrm{SO}}(E)$*.*

As for the orientation of a fiber bundle, one can trace back the existence of a spin structure to a fibration at the cohomology level. Consider the fibration

$$0 \longrightarrow H^1(X, \mathbb{Z}_2) \xrightarrow{\pi^*} H^1(P_{\mathrm{SO}}(E), \mathbb{Z}_2) \xrightarrow{i^*} H^1(\mathrm{SO}_n, \mathbb{Z}_2) \xrightarrow{w_E} H^2(X, \mathbb{Z}_2)$$

and define the *second Stiefel-Whitney class* of E as $w_2(E) = w_E(g_2)$ where g_2 is the generator of $H^1(\mathrm{SO}_n, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Following a similar reasoning to w_1's case, one obtains the following

Theorem 1.3.3 ([60, Theorem 1.7])**.** *If E is an oriented vector bundle over X then E admits a spin structure if and only if $w_2(E) = 0$.*

Summarizing, if $\rightsquigarrow$ denotes the reduction of the structural group, then

$$\begin{cases} P_O(E) \rightsquigarrow P_{\mathrm{SO}}(E) \implies E \text{ orientable} \implies \exists V \in \Gamma(E), V_p \neq 0, \forall\, p \in X \\ P_{\mathrm{SO}}(E) \rightsquigarrow P_{\mathrm{Spin}}(E) \implies \text{spin structure} \implies \exists \text{ spinor field.} \end{cases}$$

Then a *spin manifold* is a pseudo-Riemannian manifold endowed with a spin structure on its tangent bundle.

1.3.1 The associated spinor bundle and spin connection

I will now recall the construction of the associated fiber bundle to a principal bundle by a continuous map. Even though the construction can be given in a more general setting, I will present only the smooth case, as this one will be the only one needed. Let $\pi\colon P \to X$ be a smooth principal G-bundle, G a Lie group, and let $\mathrm{Diff}(F)$ be the group of diffeomorphisms of a manifold F. For any $\rho\colon G \to \mathrm{Diff}(F)$, one can define the *associated fiber bundle* to P by ρ as follows. Let G act on $P \times F$ freely as

$$\varphi_g(p, f) = (pg^{-1}, \rho(g)f), \quad g \in G,\ (p, f) \in P \times F.$$

The associated bundle to the representation ρ is

$$\pi_\rho\colon P \times_\rho F \to X$$

where $P \times_\rho F = P \times F/\varphi$ is the quotient by the action of G.

Remark 1.3.3. Recall that a bundle is totally described by the transition functions. So if $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ are the transition functions for P then

$$\rho \circ g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{Diff}(F)$$

are the transition functions for $P \times_\rho F$ and hence different ρ's give different associated bundles. An example is given shortly, when it will be discussed the difference between the Clifford and the spinor bundles.

Example 1.3.4. Let X be a manifold and consider $P = P_{GL}(X)$ the $\mathrm{GL}_n(\mathbb{R})$-bundle over X. Then, if $\rho_n\colon\ \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}(\mathbb{R}^n)$ is the standard representation, one obtains

$$\begin{aligned} TX &= P_{GL}(X) \times_\rho \mathbb{R}^n; \\ \Lambda^k TX &= P_{GL}(X) \times_{\wedge^k \rho} \Lambda^k \mathbb{R}^n; \\ \otimes^r_s TX &= P_{GL}(X) \times_{\otimes^r_s \rho} \otimes^r_s \mathbb{R}^n. \end{aligned}$$

In general, when E is an oriented vector bundle over X then

$$E = P_{\mathrm{SO}}(E) \times_\rho \mathbb{R}^n;$$
$$\Lambda^k E = P_{\mathrm{SO}}(E) \times_{\wedge^k \rho} \Lambda^k \mathbb{R}^n;$$
$$\otimes^r E = P_{\mathrm{SO}}(E) \times_{\otimes^r \rho} \otimes^r \mathbb{R}^n;$$

where $\wedge^k \rho$ and $\otimes^p_q \rho$ are the induced exterior power and tensor product. This also holds in general signature r, s.

Similarly, since an orthogonal transformation of $\mathbb{R}^{r,s}$ induces one on $\mathrm{Cl}_{r,s}$ that preserves the multiplication, the map

$$cl(\rho_{r,s})\colon \mathrm{SO}_{r,s} \to \mathrm{Aut}(\mathrm{Cl}_{r,s})$$

is a representation. Then, the *Clifford bundle* over a pseudo-Riemannian vector bundle E is

$$\mathrm{Cl}(E) = P_{\mathrm{SO}}(E) \times_{cl(\rho_{r,s})} \mathrm{Cl}_{r,s}\,.$$

Notice that the Clifford bundle $\mathrm{Cl}(E)$ could be also thought of as a bundle of Clifford algebras over X, where the fiberwise multiplication in $\mathrm{Cl}(E)$ induces an algebra structure on the space of sections $\Gamma(\mathrm{Cl}(E))$. Furthermore, the decomposition of Clifford algebras translates to Clifford bundles, that is

$$\mathrm{Cl}(E) = \mathrm{Cl}^0(E) \oplus \mathrm{Cl}^1(E)$$

corresponding to the eigenvalues of the automorphism $\alpha\colon \mathrm{Cl}(E) \to \mathrm{Cl}(E)$, obtained by extending the map $E \to E$, which sends $v \to -v$.

Now, let E be a pseudo-Riemannian vector bundle with $w_2(E) = 0$ and a spin structure $\xi\colon P_{\mathrm{Spin}_{r,s}}(E) \to P_{\mathrm{SO}_{r,s}}(E)$. A *real spinor bundle* over E is the bundle

$$S(E) = P_{\mathrm{Spin}_{r,s}}(E) \times_\mu M,$$

where M is a left module for $\mathrm{Cl}_{r,s}$ and $\mu\colon \mathrm{Spin}_{r,s} \to \mathrm{SO}(M)$ is the representation given by left multiplication by elements of $\mathrm{Spin}_{r,s} \subset \mathrm{Cl}^0_{r,s}$.

Next, in order to define the spin connection on a spin bundle, I will recall a different definition of connection specifically for principal bundles, which will prove useful in the following. Let $\pi\colon P \to X$ be a principal G-bundle, where G is a Lie group. An element $v \in \mathfrak{g}$ defines a vector field $\tilde{v}$ on P by setting

$$\tilde{v}_p = \frac{d}{dt}(p \cdot exp(tv))_{|t=0}.$$

Using this association one obtains an isomorphism $\mathfrak{g} \cong \mathcal{V}_p$, where the latter is the tangent plane to the orbit thorough p. The orbit are the fibers of π, while $\mathcal{V}_p$ is the "vertical" space with respect to π, so a connection is a choice of invariant n-spaces complementary to $\mathcal{V}_p$ and "horizontal". A *connection* on P is a G-invariant n-distribution τ on P such that the linear map

$$\pi_*\colon \tau_p \to T_{\pi p}(X)$$

is an isomorphism for all $p \in P$. The distribution that defines the connection is associated to a $\mathfrak{g}$-valued 1-form ϑ that can be defined in the following way. At every $p \in P$, τ_p defines a projection $T_p(P) \to \mathcal{V}_p$ that combined with the isomorphism $\mathfrak{g} \cong \mathcal{V}_p$ gives a linear map

$$\vartheta_p \colon T_pP \to \mathfrak{g}. \tag{1.8}$$

This defines the *connection 1-form* ϑ whose kernel is the distribution τ. This form has the following properties:

$$\begin{aligned} &\vartheta(\tilde{v}) \equiv v && \forall\, v \in \mathfrak{g} \\ &g^*(\vartheta) = \mathrm{Ad}_{g^{-1}} \vartheta && \forall\, g \in G \curvearrowright P. \end{aligned}$$

Given a connection is now possible to define the *curvature*, that is, the $\mathfrak{g}$-valued 2-form Θ given by

$$\Theta = d\vartheta + [\vartheta, \vartheta].$$

The curvature has the following properties

$$\begin{aligned} \Theta(\tilde{v}, \,.\,) &\equiv 0, \quad \forall\, v \in \mathfrak{g} \\ g^*(\Theta) &= \mathrm{Ad}_{g^{-1}} \Theta, \quad \forall\, g \in G \curvearrowright P. \end{aligned}$$

Example 1.3.5. Consider a principal $\mathrm{SO}_{r,s}$-bundle $P = P_{\mathrm{SO}}(E)$, where E is a smooth, oriented, pseudo-Riemannian vector bundle. A connection 1-form on P is a $n \times n$-matrix of 1-forms (ϑ_{ij}) such that $\vartheta_{ij} = -\varepsilon_i \varepsilon_j \vartheta_{ji}$. Clearly, when $r = n$, this means that (ϑ_{ij}) is skew-symmetric and, under the identification (1.7), may be written as

$$\vartheta = -\sum_{i<j} \vartheta_{ij} e_i \wedge e_j. \tag{1.9}$$

As usual, one defines the covariant derivative of a smooth vector bundle $\pi \colon E \to X$ as a linear map

$$\nabla \colon \Gamma(E) \to \Gamma(T^*X \otimes E)$$

such that $\nabla(fs) = df \otimes s + f\nabla s$ for all smooth functions f and sections s of E.

Proposition 1.3.6 ([60, Proposition 4.4])**.** *Consider the principal bundle $P = P_{\mathrm{SO}}(E)$ as in Example 1.3.5 and let $\mathcal{E} = (e_1, \ldots, e_n)$ be an orthonormal frame for E. There is a one-to-one correspondence between smooth connection 1-forms ϑ and covariant derivatives on E, related by*

$$\nabla e_i = \sum_{j=1}^{n} \tilde{\vartheta}_{ji} \otimes e_j, \tag{1.10}$$

where $\tilde{\vartheta} = \mathcal{E}^*\vartheta$, whenever, for any $v \in TX$ and $s_1, s_2 \in \Gamma(E)$, the covariant derivative satisfies

$$v\langle s_1, s_2 \rangle = \langle \nabla_v s_1, s_2 \rangle + \langle s_1, \nabla_v s_2 \rangle.$$

The same can be done for the curvature 2-form and $R = \tilde{\nabla} \circ \nabla$, where

$$\tilde{\nabla} \colon \Gamma(T^*X \otimes E) \to \Gamma(\Lambda^* X \otimes E),$$

obtaining the relation

$$Re_i = \sum_{j=1}^{n} \tilde{\Theta}_{ji} \otimes e_j, \quad \tilde{\Theta} = \mathcal{E}^*\Theta.$$

Now consider $\pi \colon P \to X$ a principal G-bundle, $\rho \colon G \to \mathrm{SO}_{r,s}$ a representation of G and $E_\rho = P \times_\rho \mathbb{R}^n$ the associated pseudo-Riemannian vector bundle. A connection τ on P induces in a canonical way τ_ρ on $P(E_\rho) = P \times_\rho \mathrm{SO}_{r,s}$ by extending it trivially to $P \times \mathrm{SO}_n$ and pushing it to $P(E_\rho)$. If the representation ρ is faithful, the map

$$i \colon P \to P(E_\rho), \quad i(p) = [(p, e)]$$

is an embedding. The following relates the connection 1-forms and curvature 2-forms of P and $P(E_\rho)$

Proposition 1.3.7 ([60, Proposition 4.7])**.** *Let ϑ be the connection 1-form on P and ϑ_ρ the one induced on $P(E_\rho)$. Then, for $P \subset P(E_\rho)$, it holds that*

$$(\vartheta_\rho)_{|P} = \rho_*\vartheta.$$

where $\rho_* \colon \mathfrak{g} \to \mathfrak{so}_n$ is the Lie algebra homomorphism associated to ρ. The same equation holds for the curvature 2-forms Θ and Θ_ρ.

Now let E be a pseudo-Riemannian vector bundle of dimension n endowed with a connection τ on $P_{\mathrm{SO}}(E)$ and a spin structure $\xi \colon P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$. The connection τ lifts to $\tilde{\tau}$ on $P_{\mathrm{Spin}}(E)$ via the covering map, which induces on the associated real spinor bundle $S(E) = \mathrm{Spin}_{r,s} \times_\mu M$ a connection and a covariant derivative. One then obtains

Proposition 1.3.8 ([60, Proposition 4.11])**.** *The covariant derivative ∇ on $S(E)$ acts as a derivative with respect to the module structure over* $\mathrm{Cl}(E)$:

$$\nabla(\varphi \cdot \sigma) = (\nabla\varphi) \cdot \sigma + \varphi \cdot (\nabla\sigma)$$

for φ, σ sections of $\mathrm{Cl}(E)$ *and $S(E)$ respectively.*

Now, one wants to get an equation similar to (1.10) for the spinorial covariant derivative. In order to do so, first consider the representation $\mu \colon \mathrm{Spin}_{r,s} \to \mathrm{SO}(M)$. In particular, it is known from Proposition 1.2.19 that $\mu_* \colon \mathfrak{spin}_{r,s} \to \mathfrak{so}(M)$ has the form $\mu_*(e_i \wedge e_j) = \frac{1}{4}[e_i, e_j]$, that is

$$\mu_*(e_i \wedge e_j)(\sigma) = \frac{1}{4}[e_i, e_j] \cdot \sigma, \tag{1.11}$$

for any $\sigma \in M$, where the dot is the Clifford multiplication. Next, fix an open and contractible $U \subset X$ and an oriented orthonormal frame $\mathcal{E} = (e_1, \ldots, e_n)$ over U, i.e., a section of $P_{\mathrm{SO}}(E)$. The section $\mathcal{E}$ can be lifted to a section $\tilde{\mathcal{E}}$ of $P_{\mathrm{Spin}}(E)$, again over U, in two different ways, both sections satisfying $\xi \circ \tilde{\mathcal{E}} = \mathcal{E}$, having $\xi\colon \mathrm{Spin} \to \mathrm{SO}$ as the double cover defined in the previous section. The connection 1-form on $P_{\mathrm{Spin}_{r,s}}(E)$ is then defined by simply lifting the one on $P_{\mathrm{SO}_{r,s}}(E)$ as $\tilde{\vartheta} = \xi^*\vartheta$, then, using the section $\tilde{\mathcal{E}}$, it is brought down to U, i.e.,

$$\tilde{\vartheta} = \tilde{\mathcal{E}}^*(\xi^*\vartheta) = (\xi \circ \tilde{\mathcal{E}})^*\vartheta = \mathcal{E}^*\vartheta.$$

A similar process is repeated with $S(E)$ in place of E. As there is a canonical embedding $P_{\mathrm{Spin}}(E) \subset P_{\mathrm{SO}}(S(E))$, $\tilde{\mathcal{E}}$ can be seen as an oriented orthonormal frame of $S(E)$. Hence, if ϑ^s is the connection form on $P_{\mathrm{SO}}(S(E))$, the one sought is $\tilde{\vartheta}^s = \tilde{\mathcal{E}}^*\vartheta^s$, that is, the restriction to $P_{\mathrm{Spin}}(E)$ of ϑ^s. Now, thanks to Proposition 1.3.7, ϑ^s restricted to $P_{\mathrm{Spin}}(E) \subset P_{\mathrm{SO}}(S(E))$ is also $\mu_*(\tilde{\vartheta})$. Hence

$$\tilde{\vartheta}^s = \mu_*\tilde{\vartheta},$$

and, by lowering the appropriate index, writing $\tilde{\vartheta}$ as in (1.9) and using (1.11), the connection 1-form becomes

$$\tilde{\vartheta}^s = -\frac{1}{2}\sum_{i<j} \varepsilon_i \tilde{\vartheta}_{ij} e_i e_j. \tag{1.12}$$

Finally, as mentioned before, the embedding $P_{\mathrm{Spin}}(E) \subset P_{\mathrm{SO}}(S(E))$ implies that the section $\tilde{\mathcal{E}}$ induces a section $\mathcal{L}$ of $P_{\mathrm{SO}}(S(E))$. To summarize, an oriented orthonormal frame $\mathcal{E}$ for E induces two possible sections $\mathcal{L} = (\sigma_1, \ldots, \sigma_N)$ and $-\mathcal{L} = (-\sigma_1, \ldots, -\sigma_N)$ of $P_{\mathrm{SO}}(S(E))$, depending on the choice of the lift of $\mathcal{E}$ to $P_{\mathrm{Spin}}(E)$.

Theorem 1.3.9 ([60, Theorem 4.14])**.** *Let ϑ be the connection 1-form on $P_{\mathrm{Spin}}(E)$ and let $S(E)$ be a spinor bundle. Then for a local orthonormal frame $\mathcal{E} = (e_1, \ldots, e_n)$ of E and an induced orthonormal frame $\mathcal{L} = (\sigma_1, \ldots, \sigma_N)$ of $S(E)$, the covariant derivative ∇^s is*

$$\nabla^s \sigma_\alpha = -\frac{1}{2}\sum_{i<j} \varepsilon_i \tilde{\vartheta}_{ij} \otimes e_i e_j \cdot \sigma_\alpha, \tag{1.13}$$

where $\tilde{\vartheta} = \mathcal{E}^*(\vartheta)$.

Given a pseudo-Riemannian spin manifold (M, g) with a spinor connection ∇^s, a complex number $\lambda \in \mathbb{C}$ and a (real) symmetric section $A \in \Gamma(\mathrm{End}(TM))$, there are four special classes of spinors that one can define. A spinor Ψ is said to be:

- a *parallel spinor* if
$$\nabla^s_X \Psi = 0;$$
- a *Killing spinor* if
$$\nabla^s_X \Psi = \lambda X \cdot \Psi;$$
- a *generalized Killing spinor* if
$$\nabla^s_X \Psi = \frac{1}{2} A(X) \cdot \Psi;$$

- an *imaginary A-Killing spinor* (see for example [11]) if

$$\nabla^s_X \Psi = \frac{i}{2} A(X) \cdot \Psi.$$

The importance of these classes of spinors is given by the geometry that they define on the manifold over which they are defined. In particular, it is possible to prove (see for instance [12]) that if (M, g) is a pseudo-Riemannian spin manifold and Ψ is a Killing spinor satisfying $\nabla^s_X \Psi = \lambda X \cdot \Psi$ then

$$\mathrm{Ric}(X) \cdot \Psi = 4n\lambda^2 X \cdot \Psi.$$

At least in the Riemannian case, this implies that the scalar curvature of the manifold satisfies $\mathrm{scal}(g) = 4n(n-1)\lambda^2$, hence the Killing number λ must be real or purely imaginary and the metric is Einstein. If Ψ is parallel then the manifold is Ricci-flat. Generalized Killing spinors, on the other hand, arise naturally as restrictions of parallel spinors to a hypersurface of the manifold and the symmetric tensor A takes the role of the second fundamental form. If the normal to the hypersurface is space-like, then the equation satisfied by the restricted spinor is $\nabla^s_X \Psi = \frac{1}{2} A(X) \cdot \Psi$; if the normal is time-like then the restriction satisfies $\nabla^s_X \Psi = \frac{i}{2} A(X) \cdot \Psi$. In the Riemannian setting, it was shown by [62] that the restriction of a Killing spinor to a surface satisfies

$$\nabla_X \psi = \frac{1}{2} A(X) \odot \psi + \lambda X \odot \omega \odot \psi,$$

and I will present the case in general dimension and signature in Chapter 3.

1.3.2 Examples of spin representations and spin connections

Finally, I present some explicit spin representations and an example of spin connection. It will be useful to prove a spinor analogue to Proposition 1.2.13, as it allows one to construct the higher-dimensional spin representations starting from the basic ones. These are $\mathrm{Cl}_{0,0} = \mathbb{R}$, with $\Sigma_{0,0} = \mathbb{C}$; $\mathrm{Cl}_{1,0} = \mathbb{C}$, with $e_1 = i$ acting as multiplication on $\Sigma_{1,0} = \mathbb{C}$; $\mathrm{Cl}_{0,1} = \mathbb{R} \oplus \mathbb{R}$, with $e_1 = (1, -1)$, acting on $\Sigma_{0,1} = \mathbb{C}$ as the identity.

Proposition 1.3.10. *In the setting of Proposition 1.2.13, one has that in each case the matrix realizations determine the $r+s+2$-dimensional spinor representations in terms of $\Sigma_{r,s} \otimes_{\mathbb{C}} \mathbb{C}^2 = \Sigma_{r,s} \oplus \Sigma_{r,s}$. In the even case, the half-spin representations are given by*

$$\Sigma^+_{r+1,s+1} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s}, \quad \Sigma^-_{r+1,s+1} = \Sigma^-_{r,s} \oplus \Sigma^+_{r,s}.$$

$$\Sigma^+_{s+2,0} = \left\{ \begin{pmatrix} v_+ + v_- \\ -v_+ + v_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{0,s} \right\}, \quad \Sigma^-_{s+2,0} = \left\{ \begin{pmatrix} v_+ + v_- \\ v_+ - v_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{0,s} \} \right\}$$

$$\Sigma^+_{0,r+2} = \left\{ \begin{pmatrix} v_+ + v_- \\ iv_+ - iv_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{r,0} \right\}, \quad \Sigma^-_{0,r+2} = \left\{ \begin{pmatrix} v_+ + v_- \\ -iv_+ + iv_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{r,0} \} \right\}$$

Clifford multiplication by a vector interchanges the $\Sigma^+_{r,s}$ *with* $\Sigma^-_{r,s}$ *eigenspaces, and* $\mathrm{Cl}^0_{r,s}$ *preserves them.*

In the odd case, the volume form acts on $\Sigma_{r,s}$ *as*

$$\rho_{r,s}\psi = i^{\frac{r-s+1}{2}}\psi, \tag{1.14}$$

for $\psi \in \Sigma_{r,s}$.

Proof. Assume $r+s$ is even. To see the half-spin representations, one writes down the volume form $\rho_{r+1,s+1}$ in each case, in terms of the volume form $\rho_{r,s}$ of $\mathbb{R}^{r,s}$. For $\mathrm{Cl}_{r+1,s+1}$, one gets

$$\rho_{r+1,s+1} = \begin{pmatrix} \rho_{r,s} & 0 \\ 0 & -\rho_{r,s} \end{pmatrix}.$$

Since $r-s = (r+1)-(s+1)$, the eigenvalue of each $\Sigma^\pm_{r+1,s+1}$ is the same as the eigenvalue of $\Sigma^\pm_{r,s}$. Thus, the half-spin representations are the subspaces of $\Sigma_{r,s} \oplus \Sigma_{r,s}$ given by

$$\Sigma^+_{r+1,s+1} = \Sigma^+_{r,s} \oplus \Sigma^-_{r,s}, \quad \Sigma^-_{r+1,s+1} = \Sigma^-_{r,s} \oplus \Sigma^+_{r,s}.$$

For $\mathrm{Cl}_{s+2,0}$, assuming s to be even, one gets that the volume form is

$$\rho_{s+2,0} = -i^{s+1}\begin{pmatrix} 0 & \rho_{0,s} \\ \rho_{0,s} & 0 \end{pmatrix}.$$

Therefore the eigenspaces are

$$\Sigma^+_{s+2,0} = \left\{ \begin{pmatrix} v_+ + v_- \\ -v_+ + v_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{0,s} \right\}, \quad \Sigma^-_{s+2,0} = \left\{ \begin{pmatrix} v_+ + v_- \\ v_+ - v_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{0,s}\} \right\}$$

For $\mathrm{Cl}_{0,r+2}$ with r even, the volume form is

$$\rho_{0,r+2} = (-1)^{r/2}\begin{pmatrix} 0 & \rho_{r,0} \\ -\rho_{r,0} & 0 \end{pmatrix}.$$

Therefore the eigenspaces are

$$\Sigma^+_{0,r+2} = \left\{ \begin{pmatrix} v_+ + v_- \\ iv_+ - iv_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{r,0} \right\},$$
$$\Sigma^-_{0,r+2} = \left\{ \begin{pmatrix} v_+ + v_- \\ -iv_+ + iv_- \end{pmatrix} \mid v_\pm \in \Sigma^\pm_{r,0}\} \right\}$$

It is straightforward in each case to see that Clifford multiplication by a vector interchanges the eigenspaces. Therefore, even elements preserve the eigenspaces.

Now consider the case of odd n. For $\mathrm{Cl}_{r+1,s+1}$, one has

$$\rho_{r+1,s+1} = \begin{pmatrix} \rho_{r,s} & 0 \\ 0 & \rho_{r,s} \end{pmatrix}.$$

For $\mathrm{Cl}_{s+2,0}$,

$$\rho_{s+2,0} = i^{s+1} \begin{pmatrix} \rho_{0,s} & 0 \\ 0 & \rho_{0,s} \end{pmatrix}.$$

For $\mathrm{Cl}_{0,r+2}$,

$$\rho_{0,r+2} = (-1)^{(r+1)/2} \begin{pmatrix} \rho_{r,0} & 0 \\ 0 & \rho_{r,0} \end{pmatrix} = i^{r+1} \begin{pmatrix} \rho_{r,0} & 0 \\ 0 & \rho_{r,0} \end{pmatrix}.$$

Observing that $\rho_{1,0}$ acts on the spin representation as i and $\rho_{0,1}$ acts as 1, it is now easy to see by induction on $r+s$ that $\rho_{r,s}$ acts as $i^{\frac{r-s+1}{2}}$. □

Next, I will present some examples, in particular the spin representations of $\mathrm{Cl}_{3,1}$ and $\mathrm{Cl}_{4,1}$.

Example 1.3.11. By Proposition 1.2.13 one can write $\mathrm{Cl}_{3,1} = M(2, \mathrm{Cl}_{2,0}) = M(2, \mathbb{H})$, with an orthonormal basis of $\mathbb{R}^{3,1}$ given by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Notice that in this case

$$\rho = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix},$$

so

$$\Sigma_{3,1}^+ = \left\{ \begin{pmatrix} (1+j)a \\ (1-j)b \end{pmatrix}, a, b \in \mathbb{C} \right\}, \quad \Sigma_{3,1}^- = \left\{ \begin{pmatrix} (1-j)u \\ (1+j)v \end{pmatrix}, u, v \in \mathbb{C} \right\}.$$

Example 1.3.12. It is known that $\mathrm{Cl}_{4,1} = M(2, \mathrm{Cl}_{3,0})$, $\mathrm{Cl}_{3,0} = \mathrm{Cl}_{0,1} \otimes \mathbb{H}$, and $\mathrm{Cl}_{0,1} = \mathbb{R} \oplus \mathbb{R}$ generated by $\varepsilon = (1, -1)$ acting as the identity on $\Sigma_{0,1} = \mathbb{C}$. Hence, one has that $\mathrm{Cl}_{3,0} = \mathbb{H} \oplus \mathbb{H}$ with basis

$$\begin{pmatrix} 0 & -i\varepsilon \\ -i\varepsilon & 0 \end{pmatrix} = k\varepsilon = a_1, \quad \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i(1,1) = a_2, \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = j(1,1) = a_3,$$

which act as $k, i, j \in \mathbb{H}$ respectively, while $\mathrm{Cl}_{4,1} = M(2, \mathbb{H}) \oplus M(2, \mathbb{H})$ with basis

$$\begin{pmatrix} a_1 & 0 \\ 0 & -a_1 \end{pmatrix}, \quad \begin{pmatrix} a_2 & 0 \\ 0 & -a_2 \end{pmatrix}, \quad \begin{pmatrix} a_3 & 0 \\ 0 & -a_3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which will be called respectively E_1, E_2, E_3, E_4, E_5. Accordingly, $\Sigma_{0,1} = \mathbb{C}$ where ε acts as the identity, and $\Sigma_{3,0} = \mathbb{C} \oplus \mathbb{C}$ by Proposition 1.2.13, with

$$\Sigma_{3,0} = \{(a, b) \mid a, b \in \mathbb{C}\}.$$

Hence $\Sigma_{4,1} = \Sigma_{3,0} \oplus \Sigma_{3,0} = \mathbb{C}^2 \oplus \mathbb{C}^2$, where for a spinor $\Phi = (a, b; a', b')$ one has, for instance, $E_1\Phi = (a_1(a, b); -a_1(a', b'))$ since ε acts as the identity on the spinor representation.

Finally, I present an example of the spin covariant derivative. I take this opportunity to introduce a particular Lie algebra, which will be reoccurring in later chapters.

Example 1.3.13. Consider the Lie algebra

$$\mathfrak{g} = (-2e^{23}, 3e^{13} - 3e^{34}, -3e^{12} + 3e^{24}, 2e^{23})$$

and the metric on the connected, simply connected Lie group G, associated to the scalar product

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 - e^4 \otimes e^4.$$

The corresponding connection form on G is

$$\vartheta^G = \begin{pmatrix} 0 & -e^3 & e^2 & 0 \\ e^3 & 0 & -2e^1 - 2e^4 & e^3 \\ -e^2 & 2e^1 + 2e^4 & 0 & -e^2 \\ 0 & e^3 & -e^2 & 0 \end{pmatrix}.$$

Thanks to the classification Table 1.1, it is easy to see that $\mathrm{Cl}_{3,1} = M(2, \mathbb{H})$, generated by the orthonormal basis of $\mathbb{R}^{3,1}$ given by

$$E_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and notice that in this case

$$\omega_{3,1} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

Consider the correspondence $e_i \longleftrightarrow E_i$, then the spin covariant derivative is

$$\nabla^{\Sigma G} = \frac{1}{2}\Big[2e^1 \otimes E_2E_3 - e^2 \otimes (E_1E_3 + E_3E_4) + e^3 \otimes (E_1E_2 + E_2E_4) + 2e^4 \otimes E_2E_3\Big].$$

1.4 Pseudo-Kähler and pseudo-Sasaki manifolds

In this section, I will recall the basic definitions and properties of pseudo-Kähler and pseudo-Sasaki manifolds. A more thorough treatment may be found in [19, 74].

In the following, (M, g) is a pseudo-Riemannian manifold of dimension n. If n is even, let $J \colon TM \to TM$ be an isomorphism of the tangent bundle of M. The isomorphism J is an *almost complex structure* if $J \circ J = -Id$. It is said to be *integrable* if the Nijenhuis tensor

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[X, JY] - J[JX, Y], \quad X, Y \in TM$$

vanishes everywhere. A *Kähler structure* on a pseudo-Riemannian manifold (M, g) is the pair (g, J), where J is an integrable almost complex structure compatible with the metric, i.e.,

$$g(JX, JY) = g(X, Y), \quad X, Y \in TM,$$

and the *Kähler 2-form*, defined as

$$\omega(X,Y) = g(X,JY), \quad X,Y \in TM,$$

is closed. In the future, I will reference a Kähler structure on a manifold M by indicating the Kähler and the almost complex structure (J,ω).

If n is odd, one can construct a counterpart to the Kähler structure. An *almost contact structure* on a $(2n+1)$-dimensional manifold M is a triple (φ,ξ,η), where φ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form, such that

$$\eta(\xi) = 1, \qquad \eta\circ\varphi = 0, \qquad \varphi^2 = -\operatorname{Id} + \eta\otimes\xi.$$

Given a pseudo-Riemannian metric g on M, the quadruple (φ,ξ,η,g) is called *an almost contact metric structure* if (φ,ξ,η) is an almost contact structure and

$$g(\xi,\xi) = \varepsilon \in \{\pm 1\}, \qquad \eta = \varepsilon\xi^\flat, \qquad g(\varphi X,\varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y),$$

for any vector fields X,Y. Since if (φ,ξ,η,g) is an almost contact metric structure with $g(\xi,\xi) = \varepsilon = -1$, then defining $\bar g = -g$ one has that $(\varphi,\xi,\eta,\bar g)$ is another almost contact metric structure such that $\bar g(\xi,\xi) = \bar\varepsilon = 1$, I will assume $\varepsilon = 1$ as it does not entail any loss of generality. One then defines the *fundamental* 2*-form* associated to the almost contact metric structure (φ,ξ,η,g) as

$$\Phi = g(\cdot,\varphi\cdot).$$

In addition, in analogy with the Nijenhuis tensor field for complex manifolds, one can define

$$N_\varphi(X,Y) = \varphi^2[X,Y] + [\varphi X,\varphi Y] - \varphi[\varphi X,Y] - \varphi[X,\varphi Y].$$

An almost contact metric structure (φ,ξ,η,g) is said to be *Sasaki* if (φ,ξ,η,g) satisfies $N_\varphi + d\eta\otimes\xi = 0$ and $d\eta = 2\Phi$.

Remark 1.4.1. The endomorphism φ is always skew-symmetric: indeed,

$$g(\varphi(X),Y) = -g(\varphi X,\varphi^2 Y - \eta(Y)\xi) = -g(X,\varphi(Y)) + \eta(X)\eta(\varphi(Y)) = -g(X,\varphi(Y)).$$

In fact, if φ is assumed to be skew-symmetric, $g(\varphi X,\varphi Y) = g(X,Y) - \varepsilon\eta(X)\eta(Y)$ is equivalent to $\varphi^2 = -\operatorname{Id} + \eta\otimes\xi$.

Sasaki structures can be characterized in terms of the covariant derivative $\nabla\varphi$; as usual, I will denote by ∇ the Levi-Civita connection, by R its curvature tensor, by ric its Ricci tensor.

Lemma 1.4.1 ([74, Proposition 1])**.** *Given an almost contact metric structure* (φ,ξ,η,g) *on a manifold of dimension* $2n+1$ *such that*

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X,$$

the following hold:

1. $\nabla_X \xi = -\varphi(X)$;

2. ξ *is a Killing vector field;*

3. $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$;

4. $\mathrm{ric}(\xi, X) = 2n\eta(X)$.

Pseudo-Sasaki manifolds are related to pseudo-Kähler geometry in the following way. Arguing as in [16, Theorem 7.3.16], one obtains:

Proposition 1.4.2. *Let* (φ, ξ, η, g) *be an almost contact pseudo-Riemannian metric structure on* M. *The following are equivalent:*

1. (φ, ξ, η, g) *is Sasaki;*

2. *the cone* $(\mathbb{R}^+ \times M, J, \omega)$ *with metric* $h = t^2 g + dt^2$ *is pseudo-Kähler;*

3. $(\nabla_X \varphi)Y = g(X,Y)\xi - \eta(Y)X$;

4. $\nabla_X \Phi = \eta \wedge X^\flat$.

Furthermore, one has

Proposition 1.4.3 ([67])**.** *Let* M *have a pseudo-Riemannian Sasaki structure* (φ, ξ, η, g). *Then the space of leaves of the Reeb foliation has an induced pseudo-Kähler structure.*

Recall also that given a Sasaki structure (φ, ξ, η, g) and a positive constant a, one can define another Sasaki structure by

$$\hat{\varphi} = \varphi, \qquad \hat{\xi} = a^{-1}\xi, \qquad \hat{\eta} = a\eta, \qquad \hat{g} = ag + (a^2 - a)\eta \otimes \eta.$$

Such a transformation is called a *$\mathcal{D}$-homothety.* This defines an equivalence relation between Sasaki structures on a given manifold.

There is a strong relation between (pseudo)-Kähler and (pseudo)-Sasaki structures on the one hand, and parallel and Killing spinors on the other. In particular, in the Riemannian setting the following two results hold:

Theorem 1.4.4 ([53, 77]))**.** *Let* (M, g) *be a simply connected, Kähler, Ricci-flat spin manifold. Then* M *admits a parallel spinor.*

and

Theorem 1.4.5 ([5])**.** *Let* (M, g) *be a complete and simply connected* n*-dimensional spin manifold admitting a Killing spinor with Killing number* $\lambda = \pm\frac{1}{2}$. *If* $n \geq 5$ *and odd, then it admits a Sasaki-Einstein structure.*

Conversely, a complete and simply connected Sasaki-Einstein spin manifold, then it carries a Killing spinor.

Remark 1.4.2. The last result summarizes a more detailed classification obtained in [5], where additional information on the geometry of the manifold is given, such as the existence of 3-Sasaki structures, the number of linearly independent Killing spinors and the sign of their Killing number.

In the indefinite setting, the situation is different. Indeed, the first result does not hold anymore, in particular there are examples of parallel spinors on 3-dimensional Lorentzian manifolds that are not Ricci-flat. The same is true for Killing spinors, as there are examples of Lorentzian manifolds admitting imaginary Killing spinors that are not Einstein (see [13]).

In the next chapter, I will introduce a new type of standard decomposition for metric Lie algebras, called $\mathfrak{z}$-standard. The reason to explore a different type of decomposition is given by the following result:

Proposition 1.4.6 ([30, Proposition 2.6])**.** *Let $\tilde{\mathfrak{g}}$ be a solvable Lie algebra with a Sasaki pseudo-Riemannian metric g. Then there is no pseudo-Iwasawa decomposition.*

Proof. Assume by contradiction that $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{a}$ is a pseudo-Iwasawa decomposition. Then by Lemma 1.1.4 and Lemma 1.4.1 one has

$$0 = \widetilde{\nabla}_H \xi = -\varphi(H), \quad H \in \mathfrak{a}.$$

This implies that $\mathfrak{a}$ is one-dimensional and spanned by ξ. Hence,

$$-\varphi X = \widetilde{\nabla}_X \xi = -\widetilde{\mathrm{ad}}(\xi) X.$$

However φ is skew-symmetric, while $\widetilde{\mathrm{ad}}(\xi)$ is symmetric, giving a contradiction. □

Table 1.1: The isomorphism of the Clifford algebras $\mathrm{Cl}_{r,s}$. The table can be found in [60].

Values for s \ Values for r	0	1	2	3	4	5	6	7	8
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\oplus\mathbb{H}(32)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\oplus\mathbb{R}(128)$	$\mathbb{R}(256)$

Chapter 2

Pseudo-Sasaki Einstein extensions of pseudo-Kähler Lie algebras

The structures now known as Sasaki were introduced in [71] as an odd-dimensional counterpart to Kähler geometry; they are characterized by an almost contact structure which is both normal and contact. Beside the analogy, they bear a strong relation to Kähler geometry in that both the cone over a Sasaki manifold and the space of leaves of the Reeb foliation carry a Kähler structure. For pseudo-Riemannian metrics, a completely analogous definition of Sasaki structure can be given, which was first considered in [74]; the relation to pseudo-Kähler geometry is the same as in the definite setting. Arguably, the most interesting Sasaki metrics are those satisfying the Einstein condition $\mathrm{ric} = 2ng$, where the Einstein constant is fixed by the dimension. Both in the Riemannian and indefinite case, Einstein-Sasaki metrics are characterized by the existence of a Killing spinor (see [8, 47]). Since I will be interested in the construction of Einstein-Sasaki Lie algebras, I will consider standard Lie algebras as all Riemannian Einstein solvmanifolds are of this type (see [52, 59]), and even in the indefinite case the standard condition has proved quite effective to produce examples (see [28, 29]). In the latter situation, however, things are more complicated (see e.g. [27]), but it is still possible to construct Einstein solvmanifolds by extending a nilsoliton; indeed, there is a correspondence between nilsolitons and a class of Einstein solvmanifolds for which $\tilde{\mathfrak{g}}$ admits a pseudo-Iwasawa decomposition (see [28]).

In this chapter, I will present and prove two extension results that hold in any even dimension, which allow one to extend pseudo-Kähler solvmanifolds by three dimensions, landing on a particular class of Sasaki structures on Lie algebras called $\mathfrak{z}$-*standard*. A $\mathfrak{z}$-standard Lie algebra is a Lie algebra with a standard decomposition $\mathfrak{g} \rtimes \mathfrak{a}$ and a compatibility condition with the Sasaki structure, related to the center $\mathfrak{z}$ of $\mathfrak{g}$, that gives rise to a natural notion of reduction to a Kähler Lie algebra in three dimensions less. First, I will characterize the geometry of the reduction of a $\mathfrak{z}$-standard Sasaki Lie algebra by three dimensions, showing that it inherits a Kähler structure as well as a particular derivation. Then, given a $(2n-2)$-dimensional pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$, I take a suitable central extension $\mathfrak{g} = \check{\mathfrak{g}} \oplus \mathrm{Span}\{b, \xi\}$ and, using the appropriate derivation D

found in the previous step, I consider the semidirect extension $\mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$. The resulting Lie algebra will have a $\mathfrak{z}$-standard Sasaki structure that, depending on the assumptions on D, will have an Einstein metric (Theorem 2.4.1 and Proposition 2.4.6). I will also present a Kähler analogue of this construction, which follows similar steps, and results in the construction of an Einstein pseudo-Kähler Lie algebra $\bar{\mathfrak{g}}$ that is not of pseudo-Iwasawa type (Corollary 2.4.8).

In the first section, I will lay out some groundwork, useful formulas and equivalent conditions for a Lie algebra to admit a pseudo-Sasaki structure. Next, I will give a geometric interpretation of the new structure and characterize the reduction to a codimension 3 pseudo-Kähler quotient. In the following section, I will present a constructive way to obtain $\mathfrak{z}$-standard Sasaki Lie algebras and a classification result up to dimension 7. Subsequently, I will specialize the previous results to the Einstein setting. I will define a different version of the Nikolayevsky derivation introduced in [65], which will be denoted $\mathfrak{cu}(p,q)$-Nikolayevsky, which will play a role in the construction of $\mathfrak{z}$-standard Einstein-Sasaki Lie algebras. In the last two sections of the chapter, I will prove the aforementioned extension result, give a classification of $\mathfrak{z}$-standard Einstein Sasaki Lie algebras up to dimension 7 and give some examples in dimension 9. The material present in this chapter appears in two papers I wrote in collaboration with my supervisor and Federico A. Rossi [31, 30].

2.1 Sasaki structures on Lie algebras

In this section, I consider rank one standard decompositions of solvable Lie algebras, meaning that the abelian factor $\mathfrak{a}$ is one-dimensional. Accordingly, $\tilde{\mathfrak{g}}$ will be a solvable Lie algebra endowed with a standard decomposition $\mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$, with D a derivation of $\mathfrak{g}$ and $\operatorname{ad} e_0 = D$; I will denote by $[\,,\,]$ and d the Lie bracket and exterior derivative on $\mathfrak{g}$.

Lemma 2.1.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, let D be a derivation, and let $\tau = \pm 1$. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has an almost contact metric structure $(\varphi, \xi, \eta, \tilde{g})$ such that*

$$\tilde{g} = g + \tau e^0 \otimes e^0, \qquad \tilde{\nabla}\xi = -\varphi$$

if and only if $\xi \in \mathfrak{g}$ and, writing $b = D^a(\xi)$, for all $u, w \in \mathfrak{g}$

$$\varphi(w) = \frac{1}{2}(\operatorname{ad} w)^*(\xi) + \tau g(b, w)e_0, \qquad \varphi(e_0) = -b, \tag{2.1}$$

$$D(\xi) = 0, \qquad (\operatorname{ad}\xi)^s = 0, \qquad (\operatorname{ad} b)^*(\xi) = 0, \tag{2.2}$$

$$g(w,u) = g(\xi,w)g(\xi,u) + \tau g(b,w)g(b,u) + \frac{1}{4}g((\operatorname{ad} w)^*\xi, (\operatorname{ad} u)^*\xi). \tag{2.3}$$

Proof. Given $\tilde{g} = g + \tau e^0 \otimes e^0$ and $\xi \in \tilde{\mathfrak{g}}$, define $\eta = \xi^\flat$ and $\varphi = -\tilde{\nabla}\xi$.

Write

$$\xi = v + ae_0, \quad v \in \mathfrak{g}, a \in \mathbb{R}.$$

By Lemma 1.1.4, one has

$$\widetilde{\nabla}_w\xi = \widetilde{\nabla}_w v + a\widetilde{\nabla}_w e_0 = -\operatorname{ad}(v)^s w - \frac{1}{2}(\operatorname{ad} w)^* v + \tau\tilde{g}(D^s(w), v)e_0 - aD^s(w),$$
$$\widetilde{\nabla}_{e_0}\xi = D^a(v).$$

Since $\varphi(X) = -\widetilde{\nabla}_X\xi$, I can write

$$\varphi(w) = \operatorname{ad}(v)^s w + \frac{1}{2}(\operatorname{ad} w)^* v - \tau\tilde{g}(D^s(w), v)e_0 + aD^s(w),$$
$$\varphi(e_0) = -D^a(v).$$

This determines an almost-contact metric structure if and only if φ is skew-symmetric and

$$\tilde{g}(X, Y) - \eta(X)\eta(Y) = \tilde{g}(\varphi X, \varphi Y). \tag{2.4}$$

The skew-symmetric condition implies

$$\tilde{g}(\varphi(w), e_0) + \tilde{g}(\varphi(e_0), w) = -\tau^2\tilde{g}(D^s(w), v) - \tilde{g}(D^a(v), w) = -\tilde{g}(D(v), w),$$

giving $D(v) = 0$. In addition,

$$\begin{aligned}
0 =&\tilde{g}(\varphi(w), u) + \tilde{g}(\varphi(u), w)\\
=&g(\operatorname{ad}(v)^s w, u) + g(\operatorname{ad}(v)^s u, w) + \frac{1}{2}\Big(g((\operatorname{ad} w)^* v, u) + g((\operatorname{ad} u)^* v, w)\Big)\\
&\quad + a\big(g(D^s(w), u) + g(D^s(u), w)\big)\\
=&2g(\operatorname{ad}(v)^s w, u) + 2ag(D^s(w), u),
\end{aligned}$$

giving $\operatorname{ad}(v)^s + aD^s = 0$ and

$$\varphi(w) = \frac{1}{2}(\operatorname{ad} w)^*(v) - \tau g(D^s(v), w)e_0 = \frac{1}{2}(\operatorname{ad} w)^*(v) + \tau g(D^a(v), w)e_0.$$

Evaluating (2.4) on w, e_0 one gets

$$\begin{aligned}
-a\tau g(v, w) &= \tilde{g}(w, e_0) - \eta(w)\eta(e_0) = \tilde{g}(\varphi(w), \varphi(e_0))\\
&= \tilde{g}(\frac{1}{2}(\operatorname{ad} w)^*(v) + \tau g(D^a(v), w)e_0, -D^a(v))\\
&= g(\frac{1}{2}(\operatorname{ad} w)^* v + \tau g(D^a(v), w)e_0, -D^a(v))\\
&= -\frac{1}{2}g((\operatorname{ad} w)^* v, D^a(v)) = -\frac{1}{2}g(v, [w, D^a(v)])\\
&= \frac{1}{2}g(w, (\operatorname{ad} D^a(v))^* v).
\end{aligned}$$

This holds for all w if and only if $(\operatorname{ad} D^a(v))^* v = -2a\tau v$. Since $\mathfrak{g}$ is nilpotent, the operator $\operatorname{ad} D^a(v)$ and its transpose are nilpotent, so $a = 0$ and $(\operatorname{ad} D^a(v))^* v = 0$.

Therefore, $\xi = v$, $b = D^a(v)$ and $(\operatorname{ad} b)^* v = 0$, showing that φ takes the form (2.1) and ξ satisfies (2.2). Evaluating (2.4) on w, u gives

$$\begin{aligned} g(w,u) - g(w,\xi)g(u,\xi) &= \tilde{g}(\varphi(w), \varphi(u)) \\ &= g\left(\frac{1}{2}(\operatorname{ad} w)^*\xi + \tau g(b,w)e_0, \frac{1}{2}(\operatorname{ad} u)^*\xi + \tau g(b,u)e_0\right) \\ &= \frac{1}{4} g((\operatorname{ad} w)^*\xi, (\operatorname{ad} u)^*(\xi)) + \tau g(b,w)g(b,u), \end{aligned}$$

proving (2.3).

Lastly, evaluating (2.4) on e_0, e_0, I get

$$\tau = \tilde{g}(e_0, e_0) - \eta(e_0)\eta(e_0) = \tilde{g}(-b,-b) = g(b,b);$$

however, this is a redundant condition, for $g(b,\xi) = g(D^a(\xi), \xi) = 0$, so (2.3) and (2.2) imply $g(b,u) = \tau g(b,b)g(b,u)$ for all u, which is equivalent to $g(b,b) = \tau$.

The converse is proved in the same way. □

Now observe that one can write

$$g((\operatorname{ad} w)^*(v), u) = g(v, [w,u]) = -dv^\flat(w,u) = -g((w \lrcorner dv^\flat)^\sharp, u),$$

so $(\operatorname{ad} w)^*(\xi) = -(w \lrcorner d\eta)^\sharp$. Recall that d denotes the Chevalley-Eilenberg operator on $\mathfrak{g}$, not $\tilde{\mathfrak{g}}$.

Lemma 2.1.2. *Let g be a metric on a Lie algebra $\mathfrak{g}$. Let Φ be a 2-form. Then*

$$\nabla_x \Phi = \frac{1}{2}\mathcal{L}_x \Phi - \frac{1}{2}(\operatorname{ad} x)^*\Phi + \frac{1}{2}\alpha_x^\Phi,$$

where

$$\alpha_x^\Phi(u,w) = \Phi(\operatorname{ad}(u)^*(x), w) - \Phi(\operatorname{ad}(w)^*(x), u).$$

Proof. Using (1.1) I have:

$$\begin{aligned} \nabla_x \Phi(u,w) &= -\Phi(\nabla_x u, w) - \Phi(u, \nabla_x w) \\ &= \frac{1}{2}\big(\Phi((\operatorname{ad} x)^* u + (\operatorname{ad} u)x + (\operatorname{ad} u)^* x, w) \\ &\qquad - \Phi((\operatorname{ad} x)^* w + (\operatorname{ad} w)x + (\operatorname{ad} w)^* x, u)\big) \\ &= -\frac{1}{2}(\operatorname{ad} x)^*\Phi(u,w) - \frac{1}{2}\Phi(\mathcal{L}_x u, w) + \frac{1}{2}\Phi(\mathcal{L}_x w, u) + \frac{1}{2}\alpha_x^\Phi(u,w) \\ &= -\frac{1}{2}(\operatorname{ad} x)^*\Phi(u,w) + \frac{1}{2}\mathcal{L}_x\Phi(u,w) + \frac{1}{2}\alpha_x^\Phi(u,w). \end{aligned}$$ □

Proposition 2.1.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, let D be a derivation and $\tau = \pm 1$. Then $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has a Sasaki structure*

$(\varphi,\xi,\eta,\tilde g)$ such that $\tilde g = g+\tau e^0\otimes e^0$ if and only if for some $\xi\in\mathfrak{g}$, $b=D^a(\xi)$, $\eta=\xi^\flat$, writing

$$\alpha_x(u,w)=d\eta(\operatorname{ad}(u)^*(x),w)-d\eta(\operatorname{ad}(w)^*(x),u),$$

the following hold for $x,y\in\mathfrak{g}$:

$$D(\xi)=0,\qquad (\operatorname{ad}\xi)^s=0,\qquad (\operatorname{ad}b)^*(\xi)=0, \tag{2.5}$$

$$D^a(d\eta)=0,\qquad D^a(b)=-\tau\xi, \tag{2.6}$$

$$\eta\wedge x^\flat=\frac14\alpha_x-\frac14(\operatorname{ad}x)^*(d\eta)+\frac14 d(\mathcal{L}_x\eta)+\tau b^\flat\wedge D^s(x)^\flat, \tag{2.7}$$

$$D^s(x)\lrcorner\, d\eta+x\lrcorner\, db^\flat+b\lrcorner\, dx^\flat+[x,b]^\flat=0. \tag{2.8}$$

Then φ is given by

$$\varphi(w)=\frac12(\operatorname{ad}w)^*(\xi)+\tau g(b,w)e_0,\qquad \varphi(e_0)=-b,\quad w\in\mathfrak{g}.$$

Proof. Suppose $(\varphi,\xi,\eta,\tilde g)$ is a Sasaki structure as in the hypothesis. Since Sasaki structures satisfy $\widetilde\nabla_X\xi=-\varphi(X)$, by Lemma 2.1.1 equations (2.1), (2.2), (2.3) hold.

Then (φ,ξ,η,g) is an almost contact metric structure, and it is Sasaki if and only if

$$\eta\wedge X^\flat=\widetilde\nabla_X\Phi. \tag{2.9}$$

For $u,v\in\mathfrak{g}$ one has

$$\Phi(u,w)=\tilde g(u,\varphi(w))=\frac12 g(u,(\operatorname{ad}w)^*(\xi))=-\frac12 g([u,w],\xi), \tag{2.10}$$
$$\Phi(e_0,w)=\tilde g(e_0,\varphi(w))=g(b,w).$$

Hence, by Lemma 1.1.4, equation (2.9) for $X=e_0$ implies

$$\begin{aligned}0&=(\widetilde\nabla_{e_0}\Phi)(u,w)=-\Phi(\widetilde\nabla_{e_0}u,w)-\Phi(u,\widetilde\nabla_{e_0}w)=-\Phi(D^a(u),w)-\Phi(u,D^a(w))\\&=\frac12 g([D^a(u),w],\xi)+\frac12 g([u,D^a(w)],\xi)=-\frac12 d\eta(D^a(u),w)-\frac12 d\eta(u,D^a(w))\\&=\frac12(D^a d\eta)(u,w).\end{aligned}$$

Similarly,

$$\begin{aligned}\tau g(w,\xi)&=-(\widetilde\nabla_{e_0}\Phi)(e_0,w)=\Phi(e_0,\widetilde\nabla_{e_0}w)=\Phi(e_0,D^a(w))=g(b,D^a(w))\\&=-g(D^a(b),w),\end{aligned}$$

i.e. $D^a(b)=-\tau\xi$. On the other hand, equation (2.9) for $X=x\in\mathfrak{g}$, evaluated on $u,w\in\mathfrak{g}$, gives

$$\begin{aligned}\eta\wedge x^\flat(u,w)=&(\widetilde\nabla_x\Phi)(u,w)=-\Phi(\widetilde\nabla_x u,w)-\Phi(u,\widetilde\nabla_x w)\\=&\Phi(\operatorname{ad}(u)^s(x)+\frac12(\operatorname{ad}x)^*(u)-\tau g(D^s(u),x)e_0,w)\\&-\Phi(\operatorname{ad}(w)^s(x)+\frac12(\operatorname{ad}x)^*(w)-\tau g(D^s(w),x)e_0,u).\end{aligned}$$

Next, since $\varphi(e_0) = -b$, by equation (2.10), one gets

$$\begin{aligned}
\eta \wedge x^\flat(u,w) =& -\frac{1}{2} g\left([\operatorname{ad}(u)^s(x) + \frac{1}{2}(\operatorname{ad} x)^*(u), w] - [\operatorname{ad}(w)^s(x) + \frac{1}{2}(\operatorname{ad} x)^*(w), u], \xi \right) \\
& - \tau g(b,w) g(D^s(x), u) + \tau g(b,u) g(D^s(x), w) \\
=& -\frac{1}{4} g\left(\big[[u,x] + (\operatorname{ad} u)^* x + (\operatorname{ad} x)^* u, w\big] - \big[[w,x] + (\operatorname{ad} w)^* x + (\operatorname{ad} x)^* w, u\big], \xi \right) \\
& + \tau (b^\flat \wedge D^s(x)^\flat)(u,w) \\
=& -\frac{1}{4} g\left(\big[(\operatorname{ad} u)^* x + (\operatorname{ad} x)^* u, w\big] - \big[(\operatorname{ad} w)^* x + (\operatorname{ad} x)^* w, u\big] + [[u,w],x], \xi \right) \\
& + \tau (b^\flat \wedge D^s(x)^\flat)(u,w).
\end{aligned}$$

Finally, recalling that $d\eta(u,v) = 2\Phi(u,v)$, again by equation (2.10), one gets

$$\begin{aligned}
4\eta \wedge x^\flat(u,w) =& \eta(\operatorname{ad}(u)^* x + (\operatorname{ad} x)^* u, w) - d\eta(\operatorname{ad}(w)^* x + (\operatorname{ad} x)^* w, u) - d\eta(x, [u,w]) \\
& + 4\tau (b^\flat \wedge D^s(x)^\flat)(u,w) \\
=& \alpha_x(u,w) - (\operatorname{ad} x)^*(d\eta)(u,w) + d(\mathcal{L}_x \eta)(u,w) + 4\tau (b^\flat \wedge D^s(x)^\flat)(u,w),
\end{aligned}$$

so

$$\eta \wedge x^\flat = \frac{1}{4}\alpha_x - \frac{1}{4}(\operatorname{ad} x)^*(d\eta) + \frac{1}{4} d(\mathcal{L}_x \eta) + \tau (b^\flat \wedge D^s(x)^\flat).$$

Finally,

$$\begin{aligned}
0 &= (\widetilde{\nabla}_x \Phi)(e_0, w) = -\Phi(\widetilde{\nabla}_x e_0, w) - \Phi(e_0, \widetilde{\nabla}_x w) \\
&= \Phi(D^s(x), w) - \Phi(e_0, \nabla_x w) = \frac{1}{2} g([w, D^s(x)], \xi) - g(b, \nabla_x w) \\
&= \frac{1}{2} g(D^s(x), (\operatorname{ad} w)^*(\xi)) + g\left(b, \operatorname{ad}(w)^s(x) + \frac{1}{2}(\operatorname{ad} x)^*(w) \right) \\
&= -\frac{1}{2} d\eta(w, D^s(x)) + \frac{1}{2} g\big(b, \operatorname{ad}(w)(x) + (\operatorname{ad} w)^*(x) + (\operatorname{ad} x)^*(w)\big).
\end{aligned}$$

Equivalently,

$$\begin{aligned}
0 &= -d\eta(w, D^s(x)) + g(b, \operatorname{ad}(w)(x) + (\operatorname{ad} w)^*(x) + (\operatorname{ad} x)^*(w)) \\
&= -d\eta(w, D^s(x)) + db^\flat(x,w) + dx^\flat(b,w) + g([x,b], w) \\
&= (D^s(x) \lrcorner\, d\eta + x \lrcorner\, db^\flat + b \lrcorner\, dx^\flat + [x,b]^\flat)(w).
\end{aligned}$$

Conversely, define $(\varphi, \xi, \eta, \tilde{g})$ as in the statement, and assume that (2.5)–(2.8) hold. Since $\operatorname{ad}\xi$ is antisymmetric,

$$\operatorname{ad}\xi = -(\operatorname{ad}\xi)^*, \qquad \xi \lrcorner\, d\eta = -(\operatorname{ad}\xi)^*(\xi)^\flat = (\operatorname{ad}\xi)(\xi)^\flat = 0.$$

Evaluating (2.7) on u, ξ, one obtains

$$\begin{aligned} g(u,\xi)g(x,\xi) - g(x,u) &= \frac{1}{4}d\eta(\operatorname{ad}(u)^*x + (\operatorname{ad} x)^*u, \xi) - \frac{1}{4}d\eta(\operatorname{ad}(\xi)^*x + (\operatorname{ad} x)^*\xi, u) \\ &\quad - \frac{1}{4}d\eta(x, [u,\xi]) + \tau(b^\flat \wedge D^s(x)^\flat)(u,\xi) \\ &= -\frac{1}{4}d\eta(-[\xi,x], u) - \frac{1}{4}d\eta(x,[u,\xi]) \\ &\quad - \frac{1}{4}d\eta((\operatorname{ad} x)^*\xi, u) + \tau g(b,u)g(D^s(x),\xi) \\ &= -\frac{1}{4}\eta([\xi,[u,\xi]]) + \frac{1}{4}(u \lrcorner d\eta)((\operatorname{ad} x)^*\xi) + \tau g(b,u)g(x, D^s\xi) \\ &= -\frac{1}{4}g((\operatorname{ad} u)^*\xi, (\operatorname{ad} x)^*\xi) - \tau g(b,u)g(x,b), \end{aligned}$$

which is equivalent to (2.3). Since (2.5) is assumed to hold and φ is defined so as to satisfy (2.1), Lemma 2.1.1 implies that $(\varphi, \xi, \eta, \tilde{g})$ is an almost contact metric structure. In order to prove that it is Sasaki, one only needs to verify that (2.9) holds, which follows from the computations above. □

Remark 2.1.1. The 2-form α_x^Φ of Lemma 2.1.2 and the 2-form α_x of Proposition 2.1.3 are related by $\alpha_x^{d\eta} = \alpha_x$.

Remark 2.1.2. Using Lemma 2.1.2, one sees that (2.7) can be rewritten as

$$\eta \wedge x^\flat = \frac{1}{2}\nabla_x d\eta + \tau b^\flat \wedge D^s(x)^\flat. \tag{2.11}$$

Using equation (1.1), one can read condition (2.8) as:

$$D^s(x) \lrcorner d\eta = \nabla_x b.$$

Remark 2.1.3. It is well known that on a Sasaki Lie algebra $\tilde{\mathfrak{g}}$ the center is contained in Span $\{\xi\}$; indeed, any element of the center satisfies $v \lrcorner d\eta = 0$, so it is a multiple of ξ.

If $\tilde{\mathfrak{g}}$ has nontrivial center, then $\mathfrak{z}(\tilde{\mathfrak{g}}) = \operatorname{Span}\{\xi\}$ and the quotient $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{\xi\}$ has an induced pseudo-Kähler structure $(\check{g}, J, \omega)$ by Proposition 1.4.3.

Remark 2.1.4. The equations of Proposition 2.1.3 simplify if one assumes that the center is nontrivial, because then $\operatorname{ad}\xi = 0$. However, the center may be trivial on a Sasaki Lie algebra, as is the case for the first Lie algebra $\tilde{\mathfrak{g}}$ appearing in Example 1.1.7. It is noteworthy that $\tilde{\mathfrak{g}}$ is isometric to a standard Lie algebra with nontrivial center (see Example 1.1.7).

2.2 Sasaki structures on Einstein Lie algebras

I will now specialize the previous discussion to include the Einstein condition on standard Lie algebras $\mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$, without assuming the pseudo-Iwasawa condition as Sasaki Lie algebras never admit such decomposition as seen in the previous chapter. I will write down the conditions that the induced metric g and the derivation $D = \operatorname{ad} e_0$ must satisfy, generalizing the nilsoliton equation. In particular, the conditions are satisfied if g is Ricci-flat and the symmetric part of D is an appropriate multiple of the identity.

I will then recall and generalize the construction of the Nikolayevsky and metric Nikolayevsky derivation ([66, 24]). I will show that a nilpotent Lie algebra admits a standard Einstein extension with the symmetric part of D equal to a multiple of the identity if and only if it is Ricci-flat and the metric Nikolayevsky derivation is nonzero. In this case, the extension is unique up to isometry.

Recall that in the Riemannian setting J. Lauret proved in [58] that there is a one-to-one correspondence between Einstein solvmanifolds and nilsolitons, i.e., nilpotent solvmanifolds such that the metric satisfies

$$\operatorname{Ric} = \lambda \operatorname{Id} + D$$

for some derivation D. In the indefinite setting, the nilsoliton equation can still be used, although the situation becomes more complicated, as there are four different geometries that arise (see [26]), which are all of pseudo-Iwasawa type. Thus, due to Proposition 1.4.6, one needs a more general equation to study Einstein-Sasaki solvmanifolds.

Recall that given endomorphisms f_1, f_2 of $\mathfrak{g}$, one has

$$g(f_1, f_2) = \operatorname{tr}(f_1 f_2^*) = \operatorname{tr}(f_1(f_2^s - f_2^a)).$$

Proposition 2.2.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, D a derivation and $\tau = \pm 1$. Then the metric $\tilde{g} = g + \tau e^0 \otimes e^0$ on $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ is Einstein if and only if*

$$\begin{gathered} \operatorname{Ric} = \tau\big(-\operatorname{tr}((D^s)^2)\operatorname{Id} - \frac{1}{2}[D, D^*] + (\operatorname{tr} D) D^s\big), \\ \operatorname{tr}(\operatorname{ad} v \circ D^*) = 0, \ \ v \in \mathfrak{g}; \end{gathered} \tag{2.12}$$

in this case, $\widetilde{\operatorname{ric}} = -\tau \operatorname{tr}((D^s)^2)\tilde{g}$.

Proof. By [28, Proposition 1.10], one has

$$\begin{aligned} \widetilde{\operatorname{ric}}(v, w) &= \operatorname{ric}(v, w) + \tau \tilde{g}(\frac{1}{2}[D, D^*](v), w) - \tau(\operatorname{tr} D)\tilde{g}(D^s(v), w) \\ \widetilde{\operatorname{ric}}(v, e_0) &= \frac{1}{2}\tilde{g}(\operatorname{ad} v, D) \\ \widetilde{\operatorname{ric}}(e_0, e_0) &= -\frac{1}{2}\tilde{g}(D, D) - \frac{1}{2}\operatorname{tr} D^2 = -\frac{1}{2}\operatorname{tr} D(D^s - D^a) - \frac{1}{2}\operatorname{tr} D(D^s + D^a) \\ &= -\operatorname{tr} D D^s = -\operatorname{tr}(D^s)^2. \end{aligned}$$

Thus, the Einstein condition $\widetilde{\operatorname{Ric}} = \lambda \operatorname{Id}$ holds if and only if

$$\lambda \operatorname{Id} = \operatorname{Ric} + \frac{1}{2}\tau[D, D^*] - \tau(\operatorname{tr} D)D^s, \qquad \operatorname{tr}(\operatorname{ad} v \circ D^*) = 0, \qquad \lambda = -\tau \operatorname{tr}(D^s)^2. \quad \square$$

Remark 2.2.1. Notice that if the derivation D is symmetric then the equation (2.12) is the usual Nilsoliton equation. Indeed, $D = D^s$, $[D, D^*] = 0$, hence

$$\operatorname{Ric} = \tau\bigl(-\operatorname{tr}(D^2)\operatorname{Id} + (\operatorname{tr} D)D\bigr).$$

Remark 2.2.2. If $h = -g$, then h, g have the same Ricci tensor and opposite Ricci operators; the operators $D \mapsto D^*$ and $D \mapsto D^s$ are identical. Therefore, if g satisfies

$$\operatorname{Ric}^g = \tau\Bigl(-\operatorname{tr}((D^s)^2)\operatorname{Id} - \frac{1}{2}[D, D^*] + (\operatorname{tr} D)D^s\Bigr), \qquad \operatorname{tr}(\operatorname{ad} v \circ D^*) = 0,\ v \in \mathfrak{g},$$

then

$$\operatorname{Ric}^h = (-\tau)\Bigl(-\operatorname{tr}((D^s)^2)\operatorname{Id} - \frac{1}{2}[D, D^*] + (\operatorname{tr} D)D^s\Bigr), \qquad \operatorname{tr}(\operatorname{ad} v \circ D^*) = 0,\ v \in \mathfrak{g}.$$

This amounts to the fact that $g + \tau e^0 \otimes e^0$ is Einstein if and only if so is $h - \tau e^0 \otimes e^0$.

Remark 2.2.3. One can write

$$[D, D^*] = [D^a + D^s, -D^a + D^s] = 2[D^a, D^s].$$

Although I will not need it in the rest of the thesis, I show for completeness that the condition that $\operatorname{tr}(\operatorname{ad} v \circ D^*)$ vanish can be eschewed under a suitable assumption on the eigenvalues of D.

Corollary 2.2.2. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, D a derivation such that $-\operatorname{tr} D$ is not an eigenvalue of D and $\tau = \pm 1$. Then the metric $\tilde{g} = g + \tau e^0 \otimes e^0$ on $\tilde{\mathfrak{g}} = \mathfrak{g} \ltimes_D \operatorname{Span}\{e_0\}$ is Einstein if and only if equation* (2.12) *holds, and in this case $\widetilde{\operatorname{ric}} = -\tau \operatorname{tr}((D^s)^2)\tilde{g}$.*

Proof. One direction follows from Proposition 2.2.1. For the other direction, assume that $f = (\operatorname{tr} D)\operatorname{Id} + D$ is invertible and (2.12) holds. Since $\operatorname{ad} v$ is a derivation,

$$\begin{aligned}
0 = \operatorname{tr}(\operatorname{ad} v \circ \operatorname{Ric}) &= -\operatorname{tr}((D^s)^2)\operatorname{tr}\operatorname{ad} v - \frac{1}{2}\operatorname{tr}([D, D^*] \circ \operatorname{ad} v) + (\operatorname{tr} D)\operatorname{tr}(\operatorname{ad} v \circ D^s)\\
&= -\frac{1}{2}\operatorname{tr}([\operatorname{ad} v, D] \circ D^*) + \frac{1}{2}(\operatorname{tr} D)\operatorname{tr}(\operatorname{ad} v \circ (D + D^*))\\
&= \frac{1}{2}\operatorname{tr}(\operatorname{ad} Dv \circ D^*) + \frac{1}{2}(\operatorname{tr} D)\operatorname{tr}(\operatorname{ad} v \circ D^*)\\
&= \frac{1}{2}\operatorname{tr}(\operatorname{ad}(f(v)) \circ D^*),
\end{aligned}$$

where I have used $\operatorname{tr}(\operatorname{ad} v \circ D) = 0$ (see e.g. [15, Chapter 1, Section 5.5]). Since f is invertible, this implies that $\operatorname{tr}(\operatorname{ad} w \circ D^*) = 0$ for all w, so $\tilde{g}$ is Einstein by Proposition 2.2.1. □

Example 2.2.3. Fix the Lie algebra $\mathfrak{g} = (0, 0, e^{12}, 0)$, which is the direct sum of the Heisenberg Lie algebra and $\mathbb{R}$; the notation, inspired by [70], means that $\mathfrak{g}^*$ has a fixed basis of 1-forms e^1, e^2, e^3, e^4 with $de^3 = e^1 \wedge e^2$ and the other forms closed. The two-parameter family of metrics $g = ae^1 \odot e^2 + be^3 \odot e^4$ has Ricci operator equal to

$$\mathrm{Ric} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{b}{2a^2} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider the derivation

$$D = \begin{pmatrix} -\frac{\mu}{4} & \lambda & 0 & 0 \\ -\frac{\mu^2}{8\lambda} & -\frac{\mu}{4} & 0 & 0 \\ 0 & 0 & -\frac{\mu}{2} & -\frac{b}{3a^2\mu\tau} \\ 0 & 0 & 0 & \mu \end{pmatrix},$$

where λ and μ are nonzero parameters. Then equation (2.12) is satisfied for any $\tau = \pm 1$. In this case

$$D^s = \begin{pmatrix} -\frac{\mu}{4} & \lambda & 0 & 0 \\ -\frac{\mu^2}{8\lambda} & -\frac{\mu}{4} & 0 & 0 \\ 0 & 0 & \frac{\mu}{4} & -\frac{b}{3a^2\mu\tau} \\ 0 & 0 & 0 & \frac{\mu}{4} \end{pmatrix},$$

hence $\mathrm{tr}(D^s) = 0$ and $\mathrm{tr}((D^s)^2) = 0$.

In order to obtain a standard Einstein metric, it is sufficient, thanks to Corollary 2.2.2, to show that $\mathrm{tr}\, D = 0$ is not an eigenvalue. Since μ is assumed not to be zero, D is not singular and 0 cannot be an eigenvalue. Moreover, since τ can be positive or negative, we obtain two possible Einstein metrics, one of signature $(3, 2)$ and one with the opposite signature.

Notice that the resulting standard Einstein Lie algebra $\widetilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathrm{Span}\,\{e_0\}$ has derived algebra equal to $\mathfrak{g}$, because D is surjective. Therefore, the standard decomposition is unique. In addition, it is not possible to use Proposition 1.1.5 to obtain an isometric standard Lie algebra of pseudo-Iwasawa type because D and D^s do not commute.

As a particular case, consider solutions of (2.12) such that $D^s = a\,\mathrm{Id}$. The case $a = 0$ corresponds to a standard extension by a skew-symmetric derivation of a Ricci-flat metric, which by Proposition 1.1.5 yields a Ricci-flat metric isometric to a product with a line.

In the case $a \neq 0$, one has that D is a derivation in the Lie algebra

$$\mathfrak{co}(r, s) = \mathfrak{so}(r, s) \oplus \mathrm{Span}\,\{\mathrm{Id}\},$$

where (r, s) is the signature of g, and the inclusion $\mathfrak{co}(r, s) \subset \mathfrak{gl}(\mathfrak{g})$ is determined by fixing an orthonormal frame. Additionally, D has nonzero trace. This implies that the *metric Nikolayevsky* derivation N is nonzero. I will now proceed to recall the construction of N, giving a slight generalization for use in later sections. For the proof, I refer to [66] and [24, Theorem 4.9].

Proposition 2.2.4. *Let $\mathfrak{h}$ be an algebraic subalgebra of $\mathfrak{gl}(n,\mathbb{R})$. There exists a semisimple derivation N in $\mathfrak{h}\cap\operatorname{Der}\mathfrak{g}$ such that*

$$\operatorname{tr}(N\psi)=\operatorname{tr}\psi,\qquad \psi\in\mathfrak{h}\cap\operatorname{Der}\mathfrak{g}.$$

The derivation N is unique up to automorphisms of $\mathfrak{h}$.

Remark 2.2.4. The algebraic hypothesis is present in order to be able to apply Levi's decomposition of a Lie algebra $\mathfrak{g}$ in the semidirect product of the radical $\mathfrak{r}$ and a semisimple subalgebra $\mathfrak{s}$.

For $\mathfrak{h}=\mathfrak{gl}(n,\mathbb{R})$ the derivation N of Proposition 2.2.4 corresponds to the pre-Einstein or Nikolayevsky derivation introduced in [66]; accordingly, I will refer to the derivation N of Proposition 2.2.4 as the $\mathfrak{h}$-Nikolayevsky derivation. For $\mathfrak{h}=\mathfrak{co}(r,s)$, the $\mathfrak{h}$-Nikolayevsky derivation is the metric Nikolayevsky derivation introduced in [24].

Notice that the $\mathfrak{h}$-Nikolayevsky derivation is zero if and only if all derivations in $\mathfrak{h}$ are traceless (i.e. $\mathfrak{h}$ is contained in $\mathfrak{sl}(n,\mathbb{R})$). In particular, one sees that there is derivation with $D^s=\operatorname{Id}$ if and only if the metric Nikolayevsky is nonzero.

In later sections, I will consider Lie algebras with an almost pseudo-Hermitian structure and use the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation, where

$$\mathfrak{cu}(p,q)=\mathfrak{u}(p,q)\oplus\operatorname{Span}\{\operatorname{Id}\}.$$

Like the Nikolayevsky and the metric Nikolayevsky, the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation turns out to have rational eigenvalues:

Proposition 2.2.5. *Let $\mathfrak{g}$ be a Lie algebra with an almost pseudo-Hermitian structure. Then the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation of $\mathfrak{g}$ has rational eigenvalues.*

Proof. The proof follows from [66] and [24, Theorem 4.9]. One can characterize elements of $\mathfrak{cu}(p,q)$ as elements of $\mathfrak{co}(2p,2q)$ that commute with the complex structure J.

If N is the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation, let $\mathfrak{g}^{\mathbb{C}}=\bigoplus\mathfrak{b}_t$ be the decomposition into eigenspaces and let $\pi_t\colon\mathfrak{g}^{\mathbb{C}}\to\mathfrak{b}_t$ denote the projections. Define

$$\mathfrak{n}=\left\{\sum\nu_t\pi_t\mid\sum\nu_t\pi_t\in(\operatorname{Der}\mathfrak{g}\cap\mathfrak{co}(2p,2q))^{\mathbb{C}}\right\}.$$

Since N commutes with J, each $\mathfrak{b}_t$ is J-invariant. Therefore, J commutes with projections, and I can write

$$\mathfrak{n}=\left\{\sum\nu_t\pi_t\mid\sum\nu_t\pi_t\in(\operatorname{Der}\mathfrak{g}\cap\mathfrak{h})^{\mathbb{C}}\right\}.$$

One can now proceed as in [24, Theorem 4.9] and show that N is the unique element of $\mathfrak{n}$ such that $\operatorname{tr}N\psi=\psi$ for all $\psi\in\mathfrak{n}$, and its coefficients ν_t are rational numbers. □

Lemma 2.2.6. *Let H be an algebraic subgroup of $\mathrm{SO}(r,s)$ with Lie algebra $\mathfrak{h}$ and let $\mathfrak{g}$ be a nilpotent Lie algebra with a H-structure. If D,D' are two elements of $(\mathfrak{h}\oplus\operatorname{Span}\{\operatorname{Id}\})\cap\operatorname{Der}\mathfrak{g}$ with the same trace, then the H-structures on $\mathfrak{g}\rtimes_D\operatorname{Span}\{e_0\}$ and $\mathfrak{g}\rtimes_{D'}\operatorname{Span}\{e_0\}$ are equivalent.*

Proof. The Lie algebra $\mathfrak{k} = (\mathfrak{h} \oplus \operatorname{Span}\{\mathrm{Id}\}) \cap \operatorname{Der}\mathfrak{g}$ is algebraic. Observe that that two commuting derivations of $\mathfrak{k}$ with the same trace determine equivalent extensions by Proposition 1.1.5, as their difference is in $\mathfrak{h} \cap \mathfrak{so}(p,q)$. I will use this fact repeatedly.

Denote by $\mathfrak{r}$ the radical of $\mathfrak{k}$. By [22], the fact that $\mathfrak{k}$ is algebraic implies that $\mathfrak{r}$ is also algebraic, and I can write $\mathfrak{r} = \mathfrak{n} \rtimes \mathfrak{a}$, where $\mathfrak{a}$ is an abelian Lie algebra consisting of semisimple elements and $\mathfrak{n}$ is the nilradical. Since $\mathfrak{a}$ is abelian, any two derivations in $\mathfrak{a}$ with the same trace determine isometric extensions. Thus, I only need to show that for any $D \in \mathfrak{k}$ there is an element of $\mathfrak{a}$ determining an equivalent extension.

Since $\mathfrak{k}$ is algebraic, one can write $D = D_{ss} + D_n$, where D_{ss} is semisimple, D_n is nilpotent, and $[D_{ss}, D_n] = 0$. Since D_n has trace zero, D and D_{ss} determine isometric extensions. Since D_{ss} is semisimple, so are

$$\operatorname{ad} D_{ss} \colon \mathfrak{k} \to \mathfrak{k}, \qquad \operatorname{ad} D_{ss} \colon \mathfrak{k}_0 \to \mathfrak{k}_0,$$

where $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{so}(p,q)$. One can choose a decomposition

$$\mathfrak{k} = \mathfrak{r} \oplus W,$$

where W is contained in $\mathfrak{h}$ and $\operatorname{ad} D_{ss}$-invariant. Indeed, it suffices to choose for W an $\operatorname{ad} D_{ss}$-invariant complement of $\mathfrak{k}_0 \cap \mathfrak{r}$ in $\mathfrak{k}_0$.

Accordingly, write $D_{ss} = D_{\mathfrak{r}} + D_W$. Then

$$[D_{ss}, D_W] = [D_{\mathfrak{r}}, D_W];$$

the left-hand side belongs to the $\operatorname{ad} D_{ss}$-invariant space W, and the right-hand side to the ideal $\mathfrak{r}$, so both must vanish.

Therefore, D_{ss} and $D_{\mathfrak{r}}$ are commuting derivations with the same trace, and they determine equivalent extensions.

Using the Jordan decomposition in the algebraic Lie algebra $\mathfrak{r}$, one sees that $D_{\mathfrak{r}}$ determines, up to equivalence, the same standard extension as its semisimple part. On the other hand, the latter is conjugate in $\mathfrak{r}$ to an element of $\mathfrak{a}$ by [55, Section 19.3]. The conjugation is realized by an element of the Lie group with Lie algebra $\mathfrak{k}$ which can be assumed to have determinant one, and therefore by an element of H. □

Theorem 2.2.7. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g such that the metric Nikolayevsky derivation N is nonzero. Then g is Ricci-flat and $\mathfrak{g}$ has an Einstein standard extension $\mathfrak{g} \rtimes_N \operatorname{Span}\{e_0\}$.*

Conversely, suppose $\mathfrak{g}$ is a nilpotent Lie algebra with a pseudo-Riemannian metric g and an Einstein standard extension with $D^s = a\,\mathrm{Id}$. Then g is Ricci-flat and, up to a scaling factor, the extension is isometric to either $\mathfrak{g} \oplus \mathbb{R}$ or $\mathfrak{g} \rtimes_N \operatorname{Span}\{e_0\}$ according to whether a is zero or not.

Proof. Let D be a multiple of N such that $D^s = \mathrm{Id}$. Every metric of the form $e^t g$ can be written as $g(\exp(tD)\cdot, \exp(tD)\cdot)$, i.e. it is related to g by an isomorphism. The Ricci tensor transforms accordingly; however, the Ricci tensor of $e^t g$ coincides with that of

g, and this forces it to be zero. Then $[D, D^*] = [D, 2\,\mathrm{Id} - D] = 0$ and (2.12) holds. In addition,

$$\operatorname{tr}(\operatorname{ad} v \circ D^*) = \operatorname{tr}(\operatorname{ad} v \circ (2\,\mathrm{Id} - D)) = \operatorname{tr}(2 \operatorname{ad} v - \operatorname{ad} v \circ D) = 0,$$

where $\operatorname{ad} v$ and $\operatorname{ad} v \circ D$ are traceless because $\mathfrak{g}$ is nilpotent and D is a multiple of the Nikolayevsky derivation. Thus, Proposition 2.2.1 implies that $\mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ is Einstein.

I claim that replacing D with a nonzero multiple, say $D' = kD$, has the effect of giving the same standard extension up to isometry and rescaling. Indeed, observe that $\{\exp tD\}$ acts on the metric g by rescaling while leaving D unchanged. This means that the $\tilde{g} = g + e^0 \otimes e^0$ and $\tilde{g}' = k^2 g + e^0 \otimes e^0$ are isometric metrics on $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$. Setting $e_0' = ke_0$, one can write $\tilde{g}' = k^2(g + (e^0)' \otimes (e^0)')$, and $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_{D'} \operatorname{Span}\{e_0\}$.

Now suppose that $\mathfrak{g}$ has a standard Einstein extension with $D^s = a\,\mathrm{Id}$. In this case, if $\mathfrak{g}$ has dimension n and $D^s = a\,\mathrm{Id}$, then $[D, D^*] = 2[D^a, D^s] = 0$ and (2.12) becomes

$$\operatorname{Ric} = \tau(-a^2 n\,\mathrm{Id} + na^2\,\mathrm{Id}) = 0.$$

If $a = 0$, D is skew-symmetric; by Proposition 1.1.5, one can assume $D = 0$ up to isometry, obtaining a direct product $\mathfrak{g} \times \mathbb{R}$.

If $a \neq 0$, D has nonzero trace and the metric Nikolayevsky N is nonzero, so it too has nonzero trace. I already observed that rescaling N yields an isometric extension up to isometry. Therefore, one can assume that D and N have the same trace and conclude by Lemma 2.2.6. □

2.3 $\mathfrak{z}$-Standard Sasaki structures

In this section, I will study the particular case for standard Lie algebras where the vector b of Proposition 2.1.3 is central in $\mathfrak{g}$. More precisely, I say that a Sasaki structure $(\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ on a Lie algebra $\tilde{\mathfrak{g}}$ is *$\mathfrak{z}$-standard* if there is a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ with $b = -\tilde{\varphi}(e_0)$ in the center of $\mathfrak{g}$ and $\tilde{g} = g + \tau e^0 \otimes e^0$, with $\tau = \pm 1$.

I will start by giving a geometric interpretation of this condition; to that end, I will recall a well-known construction: the moment map. Let $\tilde{\mathfrak{g}}$ be a Lie algebra with a Sasaki structure $(\tilde{\xi}, \tilde{\eta}, \tilde{g}, \tilde{\varphi})$. Let X be a nonzero vector in $\tilde{\mathfrak{g}}$. The associated, left-invariant Sasaki structure on the connected, simply connected group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$ is invariant under the left action of the group $\{\exp tX\}$. The fundamental vector field X^* is defined by

$$X^*_g = \frac{d}{dt}(\exp tX)g,$$

so identifying $T_g\tilde{G}$ with $\tilde{\mathfrak{g}}$ by left-translation one gets

$$L_{g^{-1}*}X^*_g = \frac{d}{dt}g^{-1}(\exp tX)g = \operatorname{Ad}(g^{-1})X.$$

The moment map is by definition

$$\mu(g) = \eta(\operatorname{Ad}(g^{-1})X).$$

Therefore,

$$d\mu_g(L_{g*}v) = \frac{d}{dt}|_{t=0}\mu(g\exp tv) = \frac{d}{dt}|_{t=0}\eta(\mathrm{Ad}(\exp -tv)\,\mathrm{Ad}(g^{-1})X)$$
$$= -\eta([v, \mathrm{Ad}(g^{-1})X]).$$

Now, if $\mu(g) = 0$ one has that $\mathrm{Ad}(g^{-1})X \in \ker\eta$. This implies that $\mathrm{Ad}(g^{-1})X \lrcorner\, d\eta$ is nonzero, i.e. there is some v such that $\eta([v, \mathrm{Ad}(g^{-1})X]) \neq 0$. Thus, 0 is a regular value and $\mu^{-1}(0)$ is a hypersurface.

Since X^* is nowhere zero, the action of $\{\exp tX\}$ is well-defined on $\mu^{-1}(0)$. Therefore, the quotient

$$\tilde{G}//\{\exp tX\} = \mu^{-1}(0)/\{\exp tX\}$$

is well-defined (locally), and it has an induced Sasaki structure.

$\mathfrak{z}$-standard Sasaki structures can be characterized as follows:

Lemma 2.3.1. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra with a Sasaki structure $(\varphi, \xi, \eta, \tilde{g})$. The following are equivalent:*

(i) there is a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \mathrm{Span}\{e_0\}$ with $\varphi(e_0)$ in the center of $\mathfrak{g}$;

(ii) $\tilde{\mathfrak{g}}$ contains a vector X with $\tilde{g}(X,X) \neq 0$ such that $\mathfrak{z}(X)$ is a nilpotent ideal of codimension one;

(iii) the simply connected Lie group $\tilde{G}$ with Lie algebra $\tilde{\mathfrak{g}}$ has a one-parameter subgroup $\{\exp tX\}$ such that

- *$\tilde{g}(X,X) \neq 0$;*
- *$\tilde{g}(X,X) \neq 0$ such that the zero set of the moment map is a normal nilpotent subgroup G; and*
- *$\{\exp tX\}$ commutes with G.*

Proof. If (i) holds, observe that e_0 is not a multiple of ξ by Proposition 2.1.3; thus, $X = -\varphi(e_0)$ has centralizer equal to $\mathfrak{g}$. This implies (ii).

Now assume that (ii) holds; then $\tilde{\mathfrak{g}}$ is solvable, as it contains a codimension one ideal. The zero level set of the moment map $\{g \mid \eta(\mathrm{Ad}(g^{-1})(X) = 0\}$ is the connected subgroup with Lie algebra $\mathfrak{z}(X)$, giving (iii).

Finally, suppose that (iii) holds. Since $\mu^{-1}(0)$ is a normal nilpotent subgroup, its Lie algebra is the nilpotent ideal

$$\mathfrak{g} = \ker X \lrcorner\, d\eta.$$

In addition, $\mu^{-1}(0)$ contains the identity, so $\eta(X) = 0$. This implies that $\mathfrak{g}$ has codimension one. By construction, $e_0 = \varphi(X)$ is orthogonal to $\mathfrak{g}$. Since X is not lightlike, the restriction of the metric to $\mathfrak{g}$ is definite; hence one has a standard decomposition $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \mathrm{Span}\{e_0\}$. By construction, $\varphi(e_0) = -X$, so it is central in $\mathfrak{g}$, giving (i). □

Given a $\mathfrak{z}$-standard Sasaki structure, Lemma 2.3.1 implies that $\{\exp tX\}$ is central in G, so the right action of $\{\exp tX\}$ preserves the Sasaki structure and the quotient $G/\exp\{tX\}$ is a Lie group with Lie algebra $\mathfrak{z}(X)/\operatorname{Span}\{X\}$, which is Sasaki by construction. Conversely, one can express $\mathfrak{z}(X)$ as a central extension of X, and then express $\mathfrak{g}$ as a standard extension of $\mathfrak{z}(X)$.

Example 2.3.2. In Example 1.1.7, $\{\exp te_2\}$ satisfies the conditions of Lemma 2.3.1; the three-dimensional quotient in this case is the Heisenberg algebra, with its Sasaki structure.

In the language of Proposition 2.1.3, one can express this as follows:

Corollary 2.3.3. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, D a derivation and $\tau = \pm 1$. Assume $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has a $\mathfrak{z}$-standard Sasaki structure $(\varphi, \xi, \eta, \tilde{g})$. Then, if $h \in \mathbb{R}$ and $b, \xi \in \mathfrak{z}(\mathfrak{g})$, the following hold for $x \in \mathfrak{g}$:*

$$D(\xi) = 0, \qquad D(b) = -2\tau\xi + hb, \qquad D^a(d\eta) = 0, \qquad D(d\eta) = 2db^\flat,$$
$$\eta \wedge x^\flat = \frac{1}{2}\nabla_x d\eta + \tau b^\flat \wedge D^s(x)^\flat,$$
$$d\eta(D^s(x), y) = d\eta(x, D^s(y)).$$

Furthermore, φ is given by

$$\varphi(w) = \frac{1}{2}(\operatorname{ad} w)^*(\xi) + \tau g(b, w)e_0, \qquad \varphi(e_0) = -b, \quad w \in \mathfrak{g}.$$

In addition, $\mathfrak{g}/\operatorname{Span}\{b\}$ has a Sasaki structure $(\check{\varphi}, \check{\xi}, \check{\eta}, \check{g})$ induced by the identification $\operatorname{Span}\{e_0, b\}^\perp \cong \mathfrak{g}/\operatorname{Span}\{b\}$; at the level of the corresponding Lie groups, this amounts to taking the Sasaki reduction by the left action of the one-parameter subgroup $\{\exp tb\}$.

Proof. I will specialize Proposition 2.1.3 with $b = -\varphi(e_0)$ central. Then $(\operatorname{ad} b)^*$ and $b \lrcorner dx^\flat$ are zero. In particular, from (2.8), I get

$$D^s(x) \lrcorner d\eta + x \lrcorner db^\flat = 0. \tag{2.13}$$

For $x = b$, this implies $D^s(b) \lrcorner d\eta = 0$. Since $d\eta$ is non-degenerate on $\operatorname{Span}\{b, \xi\}^\perp$, this implies that $D^s(b) \in \operatorname{Span}\{b, \xi\}$. Furthermore, one has

$$g(D^s(b), \xi) = g(b, D^s(\xi)) = g(b, -b) = -\tau,$$

so $D^s(b) = -\tau\xi + hb$ for some real constant h. Therefore,

$$D(b) = -2\tau\xi + hb.$$

Since D is a derivation, one has

$$0 = D[b, x] = [D(b), x] + [b, D(x)] = -2\tau[\xi, x].$$

Therefore ξ is in the center of $\mathfrak{g}$. Next, by (2.6), $D^a(d\eta) = 0$, so I observe that

$$\begin{aligned} D^s d\eta(x,y) &= Dd\eta(x,y) = -d\eta(Dx,y) - d\eta(x,Dy) \\ &= \eta([Dx,y]+[x,Dy]) = \eta(D[x,y]) = -2g(b,[x,y]) = 2db^\flat(x,y). \end{aligned} \tag{2.14}$$

Therefore, $D(d\eta) = 2db^\flat$ and (2.13) becomes equivalent to

$$0 = d\eta(D^s(x),y) + \frac{1}{2}(D^s d\eta)(x,y) = \frac{1}{2}\big(d\eta(D^s(x),y) - d\eta(x,D^s(y))\big).$$

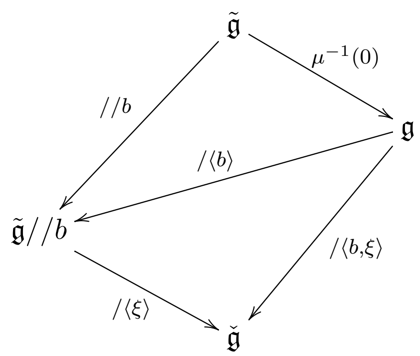
For the last part, observe that $\mathfrak{g}$ is the centralizer of b in $\tilde{\mathfrak{g}}$, and apply the observation before the statement. The fact that $(\check{\varphi},\check{\xi},\check{\eta},\check{g})$ is Sasaki descends from $\eta \wedge x^\flat = \frac{1}{2}\check{\nabla}_x d\eta$. □

One can describe the situation of the preceding corollary in terms of the Kähler quotient, as in the following theorem

Theorem 2.3.4. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, D a derivation and $\tau = \pm 1$. Assume $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \operatorname{Span}\{e_0\}$ has a $\mathfrak{z}$-standard Sasaki structure $(\varphi,\xi,\eta,\tilde{g})$. Then ξ is central in $\mathfrak{g}$ and there is $h \in \mathbb{R}$ such that*

1. $g(\xi,\xi) = 1$, $g(b,b) = \tau$, $g(b,\xi) = 0$;
2. *the quotient* $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{b,\xi\}$ *has a pseudo-Kähler structure* $(\check{g},J,\omega)$ *with* $(\mathfrak{g},g) \to (\check{\mathfrak{g}},\check{g})$ *a Riemannian submersion,* $\omega = \frac{1}{2}d\eta$ *and* $\check{D}(\omega) = db^\flat$;
3. *relative to the splitting* $\operatorname{Span}\{b,\xi\}^\perp \oplus \operatorname{Span}\{b\} \oplus \operatorname{Span}\{\xi\}$, D *takes the form*
$$D = \begin{pmatrix} \check{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix};$$
4. $[J,\check{D}] = 0$;
5. $\check{D}$ *is a derivation and* $[\check{D}^s,\check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$.

Remark 2.3.1. The situation of Theorem 2.3.4 can be visualized as follows



where the arrows denote a *determines*-type relation. Note that the arrows labeled "$\mu^{-1}(0)$" and "$//b$" refer to the reduction process mentioned at the start of the section, and the latter is the well-known *contact reduction*.

Proof. Defining $b = \varphi(e_0)$, since $g(\xi, \xi) = 1$ by definition of Sasaki structure and

$$g(b, \xi) = \tilde{g}(b, \xi) = -\tilde{g}(e_0, \varphi(\xi)) = 0, \qquad g(b, b) = \tilde{g}(e_0, e_0) = \tau$$

one gets the first condition.

Next, let $\check{\mathfrak{g}} = \mathfrak{g}/\operatorname{Span}\{b, \xi\}$. Then arguing as in Proposition 1.4.3 one sees that $\check{\nabla} d\eta$ is the projection of $\nabla d\eta$; projecting the equation (2.11), it can be seen that $d\eta$ is $\check{\nabla}$-parallel. Furthermore, for x orthogonal to b, ξ, by taking the interior product of (2.11) with ξ one gets that

$$x^\flat = \frac{1}{2}\xi \lrcorner \nabla_x d\eta - g(D^s(x), \xi)\tau b^\flat = \frac{1}{2}\xi \lrcorner \nabla_x d\eta;$$

and, by using Lemma 2.1.2, it becomes

$$x^\flat = \frac{1}{4}\xi \lrcorner (\alpha_x - (\operatorname{ad} x)^* d\eta + \mathcal{L}_x d\eta) = \frac{1}{4}(\operatorname{ad} x)^* \xi \lrcorner d\eta. \tag{2.15}$$

This implies that $d\eta$ is non-degenerate. Now set

$$J(x) = -\frac{1}{2}(x \lrcorner d\eta)^\sharp.$$

Then in $\operatorname{Span}\{b, \xi\}^\perp$ equation (2.15) reads

$$x^\flat = -\frac{1}{4}(x \lrcorner d\eta)^\sharp \lrcorner d\eta = \frac{1}{2}J(x) \lrcorner d\eta = -\big(J \circ J(x)\big)^\flat = -\big(J^2(x)\big)^\flat;$$

therefore, J is an almost complex structure, and $(\check{g}, J, d\eta)$ is a pseudo-Kähler structure. This proves the second statement.

In particular, I can write

$$d\eta(x, y) = 2g(x, Jy).$$

Now, from Corollary 2.3.3 write

$$d\eta(D^s(x), y) = d\eta(x, D^s(y))$$

as

$$g(JD^s(x), y) = g(Jx, D^s(y)) = -g(x, JD^s(y)),$$

i.e. $JD^s = -(JD^s)^* = D^s J$. In addition, $D^a d\eta = 0$ can be rewritten as

$$\begin{aligned} 0 &= D^a d\eta(x, y) = d\eta(D^a x, y) + d\eta(x, D^a y) = 2g(D^a x, JY) + 2g(x, JD^a y) \\ &= 2g(x, [J, D^a] y). \end{aligned}$$

This shows that J and D commute, which proves the fourth statement.

The Lie bracket on $\check{\mathfrak{g}}$ and the Lie bracket on $\mathfrak{g}$ are related by

$$[x,y]=[x,y]_{\check{\mathfrak{g}}}-\tau db^\flat(x,y)b-d\eta(x,y)\xi;$$

b,ξ are in the center for $\mathfrak{g}$. Relative to the splitting $\operatorname{Span}\{b,\xi\}^\perp\oplus\operatorname{Span}\{b\}\oplus\operatorname{Span}\{\xi\}$, D takes the form

$$D=\begin{pmatrix}\check{D} & 0 & 0\\ 0 & h & 0\\ 0 & -2\tau & 0\end{pmatrix}, \tag{2.16}$$

which proves the third statement.

To prove the fifth statement, notice that a linear map D of the form (2.16) automatically satisfies $D[x,y]=[Dx,y]+[x,Dy]$ when x lies in $\operatorname{Span}\{b,\xi\}$; therefore, D is a derivation if and only if for x,y in $\operatorname{Span}\{b,\xi\}^\perp$ one has

$$\begin{aligned}0=D[x,y]-[Dx,y]-[x,Dy]=&\check{D}[x,y]_{\check{\mathfrak{g}}}-\tau db^\flat(x,y)(hb-2\tau\xi)\\&-[\check{D}x,y]_{\check{\mathfrak{g}}}+\tau db^\flat(\check{D}x,y)b+d\eta(\check{D}x,y)\xi\\&-[x,\check{D}y]_{\check{\mathfrak{g}}}+\tau db^\flat(x,\check{D}y)b+d\eta(x,\check{D}y)\xi.\end{aligned}$$

Thus, D is a derivation if and only if $\check{D}$ is a derivation of $\check{\mathfrak{g}}$ and

$$\begin{aligned}hdb^\flat(x,y)&=db^\flat(\check{D}x,y)+db^\flat(x,\check{D}y),\\-2db^\flat(x,y)&=d\eta(\check{D}x,y)+d\eta(x,\check{D}y),\end{aligned}$$

where the latter is again $2db^\flat=\check{D}d\eta$. Then using $[J,D]=0$,

$$\begin{aligned}db^\flat(x,y)&=\frac{1}{2}\check{D}d\eta(x,y)=-\frac{1}{2}d\eta(\check{D}x,y)-\frac{1}{2}d\eta(x,\check{D}y)=-g(\check{D}x,Jy)-g(x,J\check{D}y)\\&=-g(x,(\check{D}^*J+J\check{D})y)=-2g(x,\check{D}^sJy).\end{aligned}$$

Thus

$$\begin{aligned}2hg(x,\check{D}^sJy)&=-hdb^\flat(x,y)=db^\flat(\check{D}x,y)+db^\flat(x,\check{D}y)\\&=2g(\check{D}x,\check{D}^sJy)+2g(x,\check{D}^sJ\check{D}y)\\&=2g(x,(\check{D}^s-\check{D}^a)\check{D}^sJy)+2g(x,\check{D}^s\check{D}Jy).\end{aligned}$$

Therefore,

$$h\check{D}^sJ=(\check{D}^s-\check{D}^a)\check{D}^sJ+\check{D}^s\check{D}J=2(\check{D}^s)^2J+[\check{D}^s,\check{D}^a]J,$$

i.e.

$$h\check{D}^s-2(\check{D}^s)^2=[\check{D}^s,\check{D}^a].$$

□

In the situation of Theorem 2.3.4, I will say that the pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$ is the *Kähler reduction* of the $\mathfrak{z}$-standard Sasaki structure of $\tilde{\mathfrak{g}}$. Notice that $\check{\mathfrak{g}}$ is indeed a Kähler reduction in the sense of symplectic geometry, arising from the action of $\{\exp tb\}$ on the pseudo-Kähler nilmanifold $\tilde{\mathfrak{g}}/\operatorname{Span}\{\xi\}$.

Example 2.3.5. In Example 1.1.7, one has

$$\check{\mathfrak{g}} = \operatorname{Span}\{e_3, e_4\}, \qquad \check{D} = I, \qquad b = -e_2, \qquad h = 2, \qquad \tau = -1,$$
$$\omega = e^{34}, \qquad db^\flat = de^2 = -2e^{34}, \qquad d\eta = 2e^{34}.$$

Corollary 2.3.3 has a Kähler analogue, which can be viewed as a consequence of Theorem 2.3.4, using the fact that any pseudo-Kähler Lie algebra yields a Sasaki Lie algebra by taking a central extension. Notice that this construction only works one way in general, i.e. it is not generally true that a Sasaki Lie algebra is a central extension of a pseudo-Kähler Lie algebra. This only occurs when ξ is central, which happens to be true in the situation of Theorem 2.3.4.

Proposition 2.3.6. *Let $\hat{\mathfrak{g}}$ be a nilpotent Lie algebra with a pseudo-Riemannian metric g, let D be a derivation and $\tau = \pm 1$. Suppose that $\bar{\mathfrak{g}} = \hat{\mathfrak{g}} \rtimes_D \operatorname{Span}\{e_0\}$ has a pseudo-Kähler structure $(\bar{J}, \bar{g}, \bar{\omega})$ such that $\bar{g} = \hat{g} + \tau e^0 \otimes e^0$, with $b = -\bar{J}e_0$ in the center of $\hat{\mathfrak{g}}$. Then*

1. *the quotient $\check{\mathfrak{g}} = \hat{\mathfrak{g}}/\operatorname{Span}\{b\}$ has a pseudo-Kähler structure $(\check{g}, \check{J}, \check{\omega})$ where the map $\pi\colon (\hat{\mathfrak{g}}, \hat{g}) \to (\check{\mathfrak{g}}, \check{g})$ is a Riemannian submersion, $\pi^*\check{\omega} = \tilde{\omega}|_{\mathfrak{g}}$ and $D(\omega) = db^\flat$;*
2. *relative to the splitting $\operatorname{Span}\{b\}^\perp \oplus \operatorname{Span}\{b\}$, D takes the form*
$$D = \begin{pmatrix} \check{D} & 0 \\ 0 & h \end{pmatrix};$$
3. *$[\check{J}, \check{D}] = 0$;*
4. *$\check{D}$ is a derivation and $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$.*

Proof. Write $\check{\mathfrak{g}} = \operatorname{Span}\{b\}^\perp$ in $\hat{\mathfrak{g}}$, and let ω be the restriction of $\tilde{\omega}$ to $\check{\mathfrak{g}}$. Then

$$\tilde{\omega} = \omega - \tau b \wedge e^0.$$

Let $\mathfrak{h} = \hat{\mathfrak{g}} \oplus \operatorname{Span}\{\xi\}$ be the central extension of $\hat{\mathfrak{g}}$ by the cocycle 2ω, $\check{\mathfrak{h}}$ the quotient $\mathfrak{h}/\operatorname{Span}\{b\}$, and $\tilde{\mathfrak{h}}$ the semidirect product $\mathfrak{h} \rtimes_{D'} \operatorname{Span}\{e_0\}$, where D' is defined by

$$D'v = Dv, \quad v \in \check{\mathfrak{g}}, \qquad D'\xi = 0, \qquad D'b = Db - 2\tau\xi.$$

One can summarize the situation as follows

$$\check{\mathfrak{h}} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{\xi\}, \qquad \mathfrak{h} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{b, \xi\}, \qquad \tilde{\mathfrak{h}} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{b, \xi, e_0\}.$$

One can view equivalently $\tilde{\mathfrak{h}}$ as the central extension of $\bar{\mathfrak{g}}$ by $2\bar{\omega}$. In particular, $\tilde{\mathfrak{h}}$ has a Sasaki metric $(\tilde{\varphi}, \xi, \tilde{h}, \tilde{\eta})$ induced by the pseudo-Kähler metric of $\bar{\mathfrak{g}}$ (see [51]). Explicitly, $\tilde{\eta}$ is the 1-form on $\tilde{\mathfrak{h}}$ that vanishes on $\bar{\mathfrak{g}}$, with $\tilde{\eta}(\xi) = 1$, so that $d\eta = 2\tilde{\omega}$, one has

$$\tilde{h} = \bar{g} + \tilde{\eta} \otimes \tilde{\eta}, \qquad \tilde{\varphi} = \bar{J}.$$

Since b is central in $\mathfrak{h}$, I can apply Theorem 2.3.4. Then $(\check{g}, \check{J}, \check{\omega})$ is pseudo-Kähler, and $\check{D}\omega = db^\flat$,

$$D' = \begin{pmatrix} \check{D} & 0 & 0 \\ 0 & h & 0 \\ 0 & -2\tau & 0 \end{pmatrix},$$

proving items 1 and 2. Items 3 and 4 follow directly from Theorem 2.3.4. $\square$

By joining Theorem 2.3.4 and Proposition 2.3.6, the diagram of Remark 2.3.1 can be expanded as in Figure 2.1, where the arrows labeled "$//b$" denote either the contact or the *symplectic* reduction.

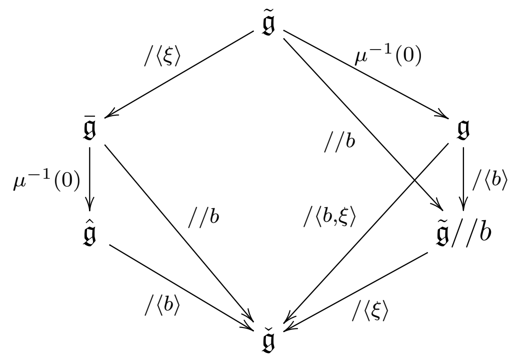


Figure 2.1: Diagram of reductions of a $\mathfrak{z}$-standard pseudo-Sasaki Lie algebra $(\tilde{\mathfrak{g}}, \tilde{g}, \xi, \eta, \varphi)$ of dimension $2n+3$, down to its Kähler reduction $\check{\mathfrak{g}}$ of dimension $2n$. Arrows denote a *determines*-type relation. In the diagram, $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}//b$ have a pseudo-Sasaki structure, $\bar{\mathfrak{g}}$ and $\check{\mathfrak{g}}$ have a pseudo-Kähler structure, $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ are subalgebra determined by the moment map.

2.4 Extension of pseudo-Kähler Lie algebras

In this section, I show how the reduction process of Theorem 2.3.4 can be inverted and describe a constructive way of obtaining $\mathfrak{z}$-standard Sasaki structures. Furthermore, I specialize the result in order to obtain an extension such that the metric is also Einstein.

Theorem 2.4.1. *Let $(\check{\mathfrak{g}}, J, \omega)$ be a pseudo-Kähler nilpotent Lie algebra. Let $\check{D}$ be a derivation of $\check{\mathfrak{g}}$, $\tau = \pm 1$, and $\mathfrak{g} = \check{\mathfrak{g}} \oplus \operatorname{Span}\{b, \xi\}$ a central extension of $\check{\mathfrak{g}}$ with a metric of the form:*

$$g(x,y) = \check{g}(x,y), \quad g(x,b) = 0 = g(x,\xi), \quad g(\xi,\xi) = 1, \quad g(b,b) = \tau, \quad g(b,\xi) = 0,$$

where $x, y \in \check{\mathfrak{g}}$. Assume furthermore

- *$d\xi^\flat = 2\omega$, where the right-hand-side is implicitly pulled back to $\tilde{\mathfrak{g}}$;*

- $db^\flat = \check{D}\omega$, *where the right-hand-side is implicitly pulled back to* $\tilde{\mathfrak{g}}$*;*
- $[J, \check{D}] = 0$*;*
- $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$ *for some constant* h.

Let $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$*, where*

$$[e_0, x] = \check{D}x, \qquad [e_0, b] = hb - 2\tau\xi, \qquad [e_0, \xi] = 0;$$

then $\tilde{\mathfrak{g}}$ *has a* $\mathfrak{z}$*-standard Sasaki structure* $(\varphi, \eta, \xi, \tilde{g})$ *given by*

$$\tilde{g} = g + \tau e^0 \otimes e^0, \qquad \varphi(x) = J(x) + \tau g(b, x)e_0, \qquad \varphi(e_0) = -b, \quad x \in \mathfrak{g}.$$

Conversely, every $\mathfrak{z}$*-standard Sasaki Lie algebra arises in this way.*

Proof. The fact that $D = \check{D} + \tau b^\flat \otimes (hb - 2\tau\xi)$ is a derivation is proved as in Theorem 2.3.4. Then one uses Proposition 2.1.3. To prove (2.8), write

$$\begin{aligned} db^\flat(y, x) &= \check{D}\omega(y, x) = -\omega(\check{D}y, x) - \omega(y, \check{D}x) = -g(\check{D}y, Jx) - g(y, J\check{D}x) \\ &= -g(y, (\check{D}^* J + J\check{D})x) = -g(y, J(\check{D} + \check{D}^*)x) = -2\omega(y, \check{D}^s x) = -d\eta(y, \check{D}^s x); \end{aligned}$$

then $D^s(x) \lrcorner\, d\eta + x \lrcorner\, db^\flat = 0$, which is equivalent to (2.8) since b is central.

To prove (2.11), notice that projecting this equation to $\Lambda^2 \check{\mathfrak{g}}$ simply says that ω is parallel on $\check{\mathfrak{g}}$. The interior product with ξ yields (2.15), which holds by construction. Finally, taking interior product of (2.11) with b and using the fact that $D^s(b) \in \operatorname{Span}\{b, \xi\}$, one computes

$$\begin{aligned} 0 &= \frac{1}{4} b \lrcorner (\alpha_x - (\operatorname{ad} x)^* d\eta + \mathcal{L}_x d\eta) + D^s(x)^\flat = \frac{1}{4}((\operatorname{ad} x)^* b \lrcorner\, d\eta) + D^s(x)^\flat \\ &= \left(\frac{1}{2} J((\operatorname{ad} x)^* b) + D^s(x)\right)^\flat. \end{aligned}$$

It also holds that $\operatorname{ad}(x)^* b = \operatorname{ad}(D^s(x))^* \xi = -2J(D^s(x))$. Therefore, this equation reduces to $J^2(D^s(x)) = -D^s(x)$, which is automatically satisfied.

The other hypotheses of Proposition 2.1.3 are trivially satisfied; therefore, $\tilde{\mathfrak{g}}$ has a Sasaki structure with

$$\varphi(w) = \frac{1}{2}(\operatorname{ad} w)^* \xi + \tau g(b, w)e_0 = -w \lrcorner\, \omega + \tau(g, b, w)e_0 = Jw + \tau(g, b, w)e_0. \qquad \square$$

Remark 2.4.1. It is no loss of generality to assume $h \geq 0$; indeed, changing the sign of $\check{D}$, e_0, b and h gives the same Sasaki Lie algebra up to isometric isomorphism.

Remark 2.4.2. The hypotheses of Theorem 2.4.1 are preserved if one rescales both h and $\check{D}$. This yields different metrics on $\tilde{\mathfrak{g}}$, which are however related by a $\mathcal{D}$-homothety (in particular, they have different curvature).

Accordingly, one can assume that either $h = 0$ or $h = 2$ up to $\mathcal{D}$-homothety. The condition $h = 0$ implies that $\operatorname{tr}(\check{D}^s)^2 = 0$. If $\check{\mathfrak{g}}$ is Riemannian, $\check{D}^s$ is diagonalizable, so $h = 0$ implies that $\check{D}$ is skew-symmetric.

Remark 2.4.3. One can always reverse the sign of the metric $\check{g}$ and the 2-form ω and obtain a different Sasaki metric on an isomorphic Lie algebra $\tilde{\mathfrak{g}}'$; the isomorphism is realized by the mapping $b \mapsto -b'$, $\xi \mapsto -\xi'$.

Let $(\check{\mathfrak{g}}_0, J_0, g_0, \omega_0)$, $(\check{\mathfrak{g}}_1, J_1, g_1, \omega_1)$ be pseudo-Kähler Lie algebras, with $\check{\mathfrak{g}}_1$ abelian. Let $\rho\colon \check{\mathfrak{g}}_0 \to \mathfrak{gl}(\check{\mathfrak{g}}_1)$ be a representation such that

$$\rho(X)\omega_1 = 0, \qquad [J_1, \rho(X)] + [\rho(J_0 X), J_1]J_1 = 0. \tag{2.17}$$

Then $\check{\mathfrak{g}}_0 \ltimes \check{\mathfrak{g}}_1$ has an almost Hermitian structure (g, J, ω), with $g = g_0 + g_1$, $\omega = \omega_0 + \omega_1$, and $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$. It is straightforward to check that ω is closed and J integrable, i.e. $\check{\mathfrak{g}}_0 \ltimes \check{\mathfrak{g}}_1$ is pseudo-Kähler. In addition, the projection π_1 on the factor $\check{\mathfrak{g}}_1$ is a derivation, giving a one-parameter family of derivations $\check{D} = \frac{h}{2}\pi_1$ that satisfy the hypotheses of Theorem 2.4.1. The resulting Sasaki extension $\tilde{\mathfrak{g}}$ takes the form

$$\begin{gathered} (\check{\mathfrak{g}}_0 \ltimes \check{\mathfrak{g}}_1 \oplus \operatorname{Span}\{b, \xi\}) \rtimes \operatorname{Span}\{e_0\}, \qquad d\xi^\flat = 2\omega, \qquad db^\flat = -h\omega, \\ [e_0, X_0] = 0, \qquad [e_0, X_1] = \frac{h}{2}X_1, \qquad [e_0, b] = hb - 2\tau\xi, \qquad [e_0, \xi] = 0, \end{gathered} \tag{2.18}$$

where X_0 denotes the generic element of $\check{\mathfrak{g}}_0$ and X_1 the generic element of $\check{\mathfrak{g}}_1$.

Proposition 2.4.2. *In the hypotheses of Theorem 2.4.1, if $\check{D}^s$ is a derivation and $[\check{D}^s, \check{D}^a] = 0$, one can assume up to isometry that $\check{\mathfrak{g}}$ is a semidirect product $\check{\mathfrak{g}} = \check{\mathfrak{g}}_0 \ltimes_\rho \check{\mathfrak{g}}_1$, where $\check{\mathfrak{g}}_0, \check{\mathfrak{g}}_1$ are pseudo-Kähler with $\check{\mathfrak{g}}_1$ abelian, $\check{D} = \frac{h}{2}\pi_1$ and $\tilde{\mathfrak{g}}$ takes the form* (2.18).

Proof. Write $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$, where $\operatorname{ad}(e_0) = \check{D} + hb^* \otimes (hb - 2\tau\xi)$. Then define

$$\chi\colon \operatorname{Span}\{e_0\} \to \operatorname{Der}\mathfrak{g}, \qquad \chi(e_0) = \check{D}^s + hb^* \otimes (hb - 2\tau\xi).$$

Then $\chi(e_0)^s = \operatorname{ad}(e_0)^s$ and $[\chi(e_0), \operatorname{ad} e_0] = 0$. Thus, the Lie algebra $\mathfrak{g} \rtimes_\chi \operatorname{Span}\{e_0\}$ is isometric to the Lie algebra $\tilde{\mathfrak{g}}$ constructed in Theorem 2.4.1. In other words, replacing $\check{D}$ with $\check{D}^s$ gives the same metric $\tilde{g}$ up to isometry. In addition, $\check{D}\omega = \check{D}^s\omega$, so $db^\flat$ is unchanged.

By Theorem 2.4.1, the minimal polynomial of $\check{D}$ divides $p(t) = ht - 2t^2$. Thus $\check{D}$ is diagonalizable over $\mathbb{R}$, and takes the form

$$\begin{pmatrix} 0 & 0 \\ 0 & \frac{h}{2}I \end{pmatrix}$$

in some basis; since $\check{D}$ commutes with J, its eigenspaces are J-invariant. Since it is symmetric, they are orthogonal. Since a diagonalizable derivation defines a grading, one has $\check{\mathfrak{g}} = \check{\mathfrak{g}}_0 \ltimes_\rho \check{\mathfrak{g}}_1$, hence the Kähler form splits as $\omega_0 + \omega_1$ and

$$J = \begin{pmatrix} J_0 & 0 \\ 0 & J_1 \end{pmatrix}.$$

Thus, $(\check{\mathfrak{g}}_0, J_0, \omega_0)$ is Kähler, $\check{\mathfrak{g}}_1$ is abelian, and (2.17) holds. □

Corollary 2.4.3. *In the hypotheses of Theorem 2.4.1, if* $\check{D}^s$ *is a derivation, and it is diagonalizable over* $\mathbb{C}$*, then one can assume up to isometry that* $\check{\mathfrak{g}}$ *is a semidirect product* $\check{\mathfrak{g}} = \check{\mathfrak{g}}_0 \ltimes_\rho \check{\mathfrak{g}}_1$*, where* $\check{\mathfrak{g}}_0, \check{\mathfrak{g}}_1$ *are pseudo-Kähler with* $\check{\mathfrak{g}}_1$ *abelian,* $\check{D} = \frac{h}{2}\pi_1$ *and* $\tilde{\mathfrak{g}}$ *takes the form* (2.18)*.*

Proof. Denote by $\check{\mathfrak{g}}^{\mathbb{C}}$ the complexification of $\check{\mathfrak{g}}$, with the scalar product obtained by complexifying the scalar product of $\check{\mathfrak{g}}$. The complexified endomorphisms $(\check{D}^s)^{\mathbb{C}} \colon \check{\mathfrak{g}}^{\mathbb{C}} \to \check{\mathfrak{g}}^{\mathbb{C}}$, $(\check{D}^a)^{\mathbb{C}} \colon \check{\mathfrak{g}}^{\mathbb{C}} \to \check{\mathfrak{g}}^{\mathbb{C}}$ are symmetric and antisymmetric, respectively. Furthermore, one gets

$$[(\check{D}^s)^{\mathbb{C}}, (\check{D}^a)^{\mathbb{C}}] = h(\check{D}^s)^{\mathbb{C}} - 2((\check{D}^s)^{\mathbb{C}})^2. \tag{2.19}$$

By hypothesis, there exists an orthonormal basis of eigenvectors of $(\check{D}^s)^{\mathbb{C}}$. Then $(\check{D}^s)^{\mathbb{C}}$ is diagonal in this basis, and $(\check{D}^a)^{\mathbb{C}}$ has zero on the diagonal. Therefore, $[(\check{D}^s)^{\mathbb{C}}, (\check{D}^a)^{\mathbb{C}}]$ has zero on the diagonal, so (2.19) implies that it vanishes, and one can apply Proposition 2.4.2. □

Next, in order to study Einstein metrics, I will present some results concerning the Ricci tensor of the metric constructed in Theorem 2.4.1.

Lemma 2.4.4. *The Ricci tensor of the metric on* $\mathfrak{g}$ *constructed in Theorem 2.4.1 is*

$$\operatorname{Ric}(v) = -2(\tau(\check{D}^s)^2 + \operatorname{Id})v, \quad v \in \operatorname{Span}\{b, \xi\}^{\perp},$$
$$\operatorname{Ric}(b) = \tau \operatorname{tr}((\check{D}^s)^2)b - (\operatorname{tr}\check{D})\xi, \qquad \operatorname{Ric}(\xi) = (2n-2)\xi - \tau(\operatorname{tr}\check{D})b.$$

where $\dim \mathfrak{g} = 2n$*.*

Proof. Since $\check{\mathfrak{g}}$ is pseudo-Kähler and nilpotent, $\breve{\operatorname{ric}}$ is zero by [43, Lemma 6.3]. By Lemma 1.1.3, one has

$$\begin{aligned}
\operatorname{ric}(v, w) &= -\frac{1}{2}\tau g(v \lrcorner\, db^\flat, w \lrcorner\, db^\flat) - \frac{1}{2} g(v \lrcorner\, d\eta, w \lrcorner\, d\eta) \\
&= -\frac{1}{2}\tau g(\check{D}^s(v) \lrcorner\, d\eta, \check{D}^s(w) \lrcorner\, d\eta) - \frac{1}{2} g(v \lrcorner\, d\eta, w \lrcorner\, d\eta) \\
&= -2\tau g(JD^s(v), JD^s(w)) - 2g(Jv, Jw) = -2\tau g(D^s(v), D^s(w)) - 2g(v, w).
\end{aligned}$$

Then

$$\begin{aligned}
\operatorname{ric}(v, b) &= \frac{1}{2}\tau g(dv^\flat, db^\flat) = \frac{1}{2}\tau g(dv^\flat, \check{D}\omega), & \operatorname{ric}(v, \xi) &= \frac{1}{2}\tau g(dv^\flat, d\eta) = \tau g(dv^\flat, \omega), \\
\operatorname{ric}(b, b) &= \frac{1}{2} g(db^\flat, db^\flat) = \frac{1}{2} g(\check{D}\omega, \check{D}\omega), & \operatorname{ric}(b, \xi) &= \frac{1}{2} g(db^\flat, d\eta) = g(\check{D}\omega, \omega), \\
\operatorname{ric}(\xi, \xi) &= \frac{1}{2}\tau g(d\eta, d\eta) = 2g(\omega, \omega).
\end{aligned}$$

It is possible to simplify these formulae by observing that

$$\begin{aligned}
\check{D}\omega(x, y) &= -\omega(\check{D}x, y) - \omega(x, \check{D}y) = -\check{g}(\check{D}x, Jy) - \check{g}(x, J\check{D}y) = -\check{g}(x, (J\check{D} + \check{D}^* J)y) \\
&= -\check{g}(x, (\check{D} + \check{D}^*)Jy),
\end{aligned}$$

and hence viewing $\check{D}\omega$ as a $(1,1)$ tensor $(\check{D}\omega)^\sharp = -(\check{D}+D^*)J$. Similarly, one has $\omega^\sharp = J$. Then

$$\begin{aligned} g(\omega,\omega) &= \frac{1}{2}g(J,J) = n-1,\\ g(\omega,\check{D}\omega) &= \frac{1}{2}g(J,-(\check{D}+\check{D}^*)J) = \frac{1}{2}\operatorname{tr}((\check{D}+\check{D}^*)J^2) = -\operatorname{tr}\check{D}^s = -\operatorname{tr}\check{D},\\ g(\check{D}\omega,\check{D}\omega) &= \frac{1}{2}g((\check{D}+\check{D}^*)J,(\check{D}+\check{D}^*)J) = \frac{1}{2}\operatorname{tr}((\check{D}+\check{D}^*)^2) = 2\operatorname{tr}(\check{D}^s)^2. \end{aligned}$$

Finally, observe that ω and $\check{D}\omega$ are d^*-closed, so (since $\check{\mathfrak{g}}$ is unimodular),

$$g(dv^\flat,\omega) = g(v^\flat,d^*\omega) = 0, \qquad g(dv^\flat,\check{D}\omega) = g(v^\flat,d^*\check{D}\omega) = 0.$$

Summing up,

$$\begin{aligned} &\operatorname{ric}(v,w) = -2\tau g(\check{D}^s(v),\check{D}^s(w)) - 2g(v,w), && \operatorname{ric}(v,b) = 0,\\ &\operatorname{ric}(v,\xi) = 0, && \operatorname{ric}(b,b) = \operatorname{tr}((\check{D}^s)^2),\\ &\operatorname{ric}(b,\xi) = -\operatorname{tr}\check{D}, && \operatorname{ric}(\xi,\xi) = (2n-2). \end{aligned}$$

□

Lemma 2.4.5. *With the hypothesis of Theorem 2.4.1, the metric* $\tilde{g} = g + \tau e^0\otimes e^0$ *on* $\tilde{\mathfrak{g}} = \mathfrak{g}\rtimes_D \operatorname{Span}\{e_0\}$ *is Einstein if and only if*

$$\tau = -1, \qquad \check{D}^s = \pm\operatorname{Id}, \qquad h = \pm 2.$$

Proof. By Proposition 2.2.1, $\tilde{g}$ is Einstein if and only if

$$\operatorname{Ric} = \tau\big(-\operatorname{tr}((D^s)^2)\operatorname{Id} + [D^s,D^a] + (\operatorname{tr}D)D^s\big), \qquad \operatorname{tr}(\operatorname{ad}v\circ D^*) = 0, \quad v\in\mathfrak{g}.$$

Than

$$D = \begin{pmatrix}\check{D} & 0 & 0\\ 0 & h & 0\\ 0 & -2\tau & 0\end{pmatrix}, \qquad D^* = \begin{pmatrix}\check{D}^* & 0 & 0\\ 0 & h & -2\\ 0 & 0 & 0\end{pmatrix},$$

$$D^s = \begin{pmatrix}\check{D}^s & 0 & 0\\ 0 & h & -1\\ 0 & -\tau & 0\end{pmatrix}, \qquad D^a = \begin{pmatrix}\check{D}^a & 0 & 0\\ 0 & 0 & 1\\ 0 & -\tau & 0\end{pmatrix}.$$

So

$$[D^s,D^a] = \begin{pmatrix} h\check{D}^s - 2(\check{D}^s)^2 & 0 & 0\\ 0 & 2\tau & h\\ 0 & h\tau & -2\tau\end{pmatrix}.$$

Multiplying by τ each side of (2.12) and using Lemma 2.4.4, yields

$$\begin{aligned} \begin{pmatrix} -2(\check{D}^s)^2 - 2\tau\operatorname{Id} & 0 & 0\\ 0 & \operatorname{tr}((\check{D}^s)^2) & -(\operatorname{tr}\check{D})\\ 0 & -\tau\operatorname{tr}\check{D} & \tau(2n-2)\end{pmatrix} &= -(\operatorname{tr}((\check{D}^s)^2) + h^2 + 2\tau)\begin{pmatrix}\operatorname{Id} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\\ &+ \begin{pmatrix} h\check{D}^s - 2(\check{D}^s)^2 & 0 & 0\\ 0 & 2\tau & h\\ 0 & h\tau & -2\tau\end{pmatrix} + (\operatorname{tr}\check{D}^s + h)\begin{pmatrix}\check{D}^s & 0 & 0\\ 0 & h & -1\\ 0 & -\tau & 0\end{pmatrix}, \end{aligned}$$

i.e.

$$\begin{aligned}(\operatorname{tr}((\check{D}^s)^2)+h^2)\operatorname{Id} &= (\operatorname{tr}\check{D}^s+2h)\check{D}^s,\\ 2\operatorname{tr}((\check{D}^s)^2) &= (\operatorname{tr}\check{D}^s)h,\\ \tau(2n+2) &= -(\operatorname{tr}((\check{D}^s)^2)+h^2).\end{aligned}$$

If this system of equations holds, $\check{D}^s$ is a multiple of the identity; setting $\operatorname{tr}\check{D}^s=\lambda$, so that $\operatorname{tr}((\check{D}^s)^2)=\frac{\lambda^2}{2n-2}$, one obtains

$$\tau=-1,\qquad h=\frac{2\lambda}{2n-2},\qquad \lambda=\pm(2n-2).$$

So the system holds if and only if $\check{D}^s=\pm\operatorname{Id}$ and $h=\pm 2$. This condition also implies $\operatorname{tr}(\operatorname{ad}v\circ D^*)=0$ because $\mathfrak{g}$ is unimodular and $\operatorname{tr}(\operatorname{ad}v\circ D)=0$ by [15, Chapter 1, Section 5.5], proving the equivalence in the statement. □

Remark 2.4.4. As observed in Remark 2.4.1, changing the sign of h, $\check{D}$, e_0 and b yields an isometric metric. Therefore, in the following I will only consider the case $h=2$ and $\check{D}^s=I$ for the Einstein setting.

The construction of Theorem 2.4.1 can be now specialized to the Sasaki-Einstein case as follows:

Proposition 2.4.6. *Let $(\check{\mathfrak{g}},J,\omega)$ be a pseudo-Kähler nilpotent Lie algebra and let $\check{D}$ be a derivation such that $\check{D}^s=\operatorname{Id}$ and commuting with J. If $\mathfrak{g}=\check{\mathfrak{g}}\oplus\operatorname{Span}\{b,\xi\}$ is the central extension of $\mathfrak{g}$ characterized by $d\xi^*=2\omega=db^*$, where $\{b^*,\xi^*\}$ is the basis dual to $\operatorname{Span}\{b,\xi\}$, with the metric $g=\check{g}-b^*\otimes b^*+\xi^*\otimes\xi^*$, then the semidirect product $\tilde{\mathfrak{g}}=\mathfrak{g}\ltimes\operatorname{Span}\{e_0\}$, where*

$$[e_0,x]=\check{D}x,\qquad [e_0,b]=2b+2\xi,\qquad [e_0,\xi]=0$$

has a Sasaki-Einstein structure $(\varphi,\eta,\xi,\tilde{g})$ given by

$$\tilde{g}=g-e^0\otimes e^0,\qquad \varphi(w)=J(w)-g(b,w)e_0,\qquad \varphi(e_0)=-b,\quad w\in\mathfrak{g}.$$

Proof. Since $\check{D}\omega=\check{D}^s\omega=-2\omega$, applying Theorem 2.4.1 with $h=2$ and $\tau=-1$ yields a Sasaki extension as in the statement, which is Einstein by Lemma 2.4.5. □

Example 2.4.7. Let $\check{\mathfrak{g}}=\mathbb{R}^{2n-2}$, with

$$Je_1=e_2,\ \dots,\ Je_{2n-3}=e_{2n-2},\qquad \omega=\varepsilon_1e^{12}+\dots+\varepsilon_{n-1}e^{2n-3,2n-2},\quad \varepsilon_i=\pm1,$$

and set $D=\operatorname{Id}$. Furthermore, set

$$d\xi^*=db^*=2\omega,\qquad \operatorname{ad}e_0=2b^*\otimes(b+\xi)+\sum e^i\otimes e_i.$$

The extension $\tilde{\mathfrak{g}}$ has a basis $\{e_0, e_1, \dots, e_{2n}\}$ such that

$$\begin{aligned} de^0 &= 0, \\ de^i &= e^{i,0}, \quad i = 1, \dots, 2n-2, \\ de^{2n} = de^{2n-1} &= 2\varepsilon_1 e^{12} + \cdots + 2\varepsilon_{n-1} e^{2n-3,2n-2} + 2e^{2n-1,0}, \end{aligned}$$

and the Einstein-Sasaki metric is

$$g = \sum_{i=1}^{n-1} \varepsilon_i (e^{2i-1} \otimes e^{2i-1} + e^{2i} \otimes e^{2i}) - e^{2n-1} \otimes e^{2n-1} + e^{2n} \otimes e^{2n} - e^0 \otimes e^0.$$

The quotient by $\xi = e_{2n}$ yields the Kähler Lie algebra

$$\begin{aligned} de^0 &= 0, \\ de^i &= e^{i,0}, \quad i = 1, \dots, 2n-2, \\ de^{2n-1} &= 2\varepsilon_1 e^{12} + \cdots + 2\varepsilon_{n-1} e^{2n-3,2n-2} + 2e^{2n-1,0}, \end{aligned}$$

with the pseudo-Kähler-Einstein metric

$$g = \sum_{i=1}^{n-1} \varepsilon_i (e^{2i-1} \otimes e^{2i-1} + e^{2i} \otimes e^{2i}) - e^{2n-1} \otimes e^{2n-1} - e^0 \otimes e^0.$$

When the ε_i are equal to -1, this is the negative definite symmetric metric on the Iwasawa subgroup of $\mathrm{SU}(1, n+2)$.

Proposition 2.4.6 has a Kähler analogue:

Corollary 2.4.8. *Let $(\check{\mathfrak{g}}, J, \omega)$ be a pseudo-Kähler nilpotent Lie algebra with nonzero metric Nikolayevsky derivation, and let $\check{D}$ be a derivation such that $\check{D}^s = \mathrm{Id}$. If $\mathfrak{g} = \check{\mathfrak{g}} \oplus \mathrm{Span}\{b\}$ is the central extension of $\mathfrak{g}$ characterized by $db^* = 2\omega$, where $\{b^*\}$ is the basis dual to $\mathrm{Span}\{b\}$, with the metric $g = \check{g} - b^* \otimes b^*$, then the semidirect product $\bar{\mathfrak{g}} = \mathfrak{g} \ltimes \mathrm{Span}\{e_0\}$, where*

$$[e_0, x] = \check{D}x, \qquad [e_0, b] = 2b$$

has a pseudo-Kähler-Einstein structure $(\bar{g}, \bar{J}, \bar{\omega})$ given by

$$\bar{g} = g - e^0 \otimes e^0, \qquad \bar{J}(w) = J(w) - g(b, w)e_0, \qquad \bar{J}(e_0) = -b, \quad w \in \mathfrak{g},$$

with $\overline{\mathrm{ric}} = (2n+2)\bar{g}$, with $2n$ the dimension of $\bar{\mathfrak{g}}$.

Proof. Take the Lie algebra constructed in Proposition 2.4.6 and take the quotient by ξ. Then by Proposition 1.4.3 it is Kähler-Einstein with $\overline{\mathrm{ric}} = (2n+2)\bar{\mathfrak{g}}$. □

Remark 2.4.5. If the Lie algebra $\check{\mathfrak{g}}$ is not abelian, then Corollary 2.4.8 produces pseudo-Kähler-Einstein rank-one extension which are not pseudo-Iwasawa, unlike the method presented in [69], where one constructs pseudo-Kähler-Einstein rank-one extensions of pseudo-Iwasawa-type.

Indeed, the derivation $\check{D} = \mathrm{ad}\, e_0$ of Corollary 2.4.8 is self-adjoint with respect to the metric if and only if $\check{D}^s = \frac{1}{2}(D + D^*)$ is a derivation, but since $\check{D}^s = \mathrm{Id}$, this happens only if the identity is a derivation, i.e. if $\check{\mathfrak{g}}$ is an abelian Lie algebra.

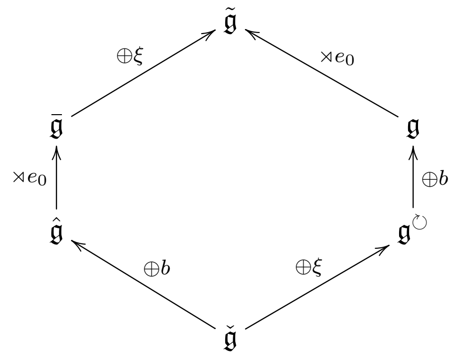


Figure 2.2: Diagram of extensions of a pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$. The arrows labeled "$\oplus b$" or "$\oplus\xi$" denote central extensions by $\operatorname{Span}\{b\}$ and $\operatorname{Span}\{\xi\}$ respectively, and $\mathfrak{g}^{\circlearrowright}$ denotes the S^1-bundle defined by ξ together with the appropriate Sasaki structure.

Remark 2.4.6. The pseudo-Kähler-Einstein quotient constructed in Example 2.4.7 is precisely the family of [69, Example 7.6], and since $\check{\mathfrak{g}}$ is abelian, this is consistent with the previous Remark 2.4.5.

In light of the preceding results, the arrows in the diagram of Figure 2.1 can be reversed as in figure 2.2.

The main theorem in the Einstein setting is the following

Theorem 2.4.9. *If $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ is a $\mathfrak{z}$-standard Sasaki-Einstein Lie algebra, the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation of its Kähler reduction is nonzero.*

Conversely, if $\check{\mathfrak{g}}$ is a pseudo-Kähler Lie algebra with nonzero $\mathfrak{cu}(p,q)$-Nikolayevsky derivation, it extends to a $\mathfrak{z}$-standard Sasaki-Einstein Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$, uniquely determined up to equivalence.

Proof. If $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ is a $\mathfrak{z}$-standard Sasaki-Einstein Lie algebra, Theorem 2.4.1 asserts that $\tilde{\mathfrak{g}}$ can be realized as an extension of its Kähler reduction $\check{\mathfrak{g}}$. By Proposition 2.4.6, $\check{D}$ is a derivation commuting with J such that $\check{D}^s = \operatorname{Id}$. This implies that $\check{D}$ is an element of

$$\mathfrak{co}(2p,2q) \cap \mathfrak{gl}(p+q,\mathbb{C}) = \mathfrak{cu}(p,q)$$

with nonzero trace; if such a $\check{D}$ exists, the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation is nonzero.

Now assume $\check{g}$ is pseudo-Kähler and $\mathfrak{cu}(p,q)$-Nikolayevsky derivation is nonzero. By rescaling, one obtains a derivation $\check{D}$ whose symmetric part is the identity; this yields a Sasaki-Einstein extension by Proposition 2.4.6.

To prove uniqueness, fix two derivations $\check{D}$, $\check{D}'$ commuting with J, $\check{D}^s = \operatorname{Id} = (\check{D}')^s$. The Lie algebras $\check{\mathfrak{g}} \rtimes_{\check{D}} \operatorname{Span}\{e_0\}$ and $\check{\mathfrak{g}} \rtimes_{\check{D}'} \operatorname{Span}\{e_0\}$ have a natural $\mathrm{U}(p,q)$-structure. By Lemma 2.2.6, they are equivalent.

One can view $\tilde{\mathfrak{g}}$ as a central extension $(\check{\mathfrak{g}} \rtimes_{\check{D}} \operatorname{Span}\{e_0\}) \oplus \operatorname{Span}\{b,\xi\}$, where db^* and $d\xi^*$ are determined by the $\mathrm{U}(p,q)$-invariant form ω. Therefore, $\tilde{\mathfrak{g}}$ and its counterpart obtained using $\check{D}'$ are equivalent. □

2.5 Classification results and examples

In this final section, I will present classification results for $\mathfrak{z}$-standard Sasaki Lie algebras whose Kähler reduction is abelian and Sasaki-Einstein Lie algebras, both up to dimension 7 and give some examples in dimension 9. I begin by noting that Proposition 2.4.2 of the previous section classifies $\mathfrak{z}$-standard Sasaki structures that reduce to an abelian Kähler Lie algebra, as positive-definiteness of the metric implies that $\check{D}^s$ is automatically a diagonalizable derivation in this case.

Theorem 2.5.1. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra of dimension 5 with a $\mathfrak{z}$-standard Sasaki structure. Then, up to isometry and $\mathcal{D}$-homothety, $\tilde{\mathfrak{g}}$ is one of*

$$(0,0,0,-2e^{12}-2\tau e^{35},0),$$
$$(0,0,2e^{35},-2e^{12}-2\tau e^{35},0),$$
$$(e^{15},e^{25},2\tau e^{12}+2e^{35},-2e^{12}-2\tau e^{35},0),$$

and the Sasaki structure is given by

$$\tilde{g}=\pm(e^1\otimes e^1+e^2\otimes e^2)+\tau e^3\otimes e^3+e^4\otimes e^4+\tau e^5\otimes e^5, \qquad \xi=e_4, \qquad \Phi=-e^{12}-\tau e^{35}.$$

Proof. The Kähler reduction $\check{\mathfrak{g}}$ is a nilpotent Lie algebra of dimension two, hence abelian. Assume first that $\check{\mathfrak{g}}$ has positive-definite signature. In some basis $\{e_1,e_2\}$, one can write

$$\check{g}=e^1\otimes e^1+e^2\otimes e^2, \quad \omega=-e^{12}, \quad J=e^1\otimes e_2-e^2\otimes e_1.$$

Derivations that commute with J lie in Span$\{I,J\}$. In particular, $\check{D}^s$ commutes with $\check{D}^a$, so Proposition 2.4.2 implies that up to isometry one can assume $\check{D}=0$ or $\check{D}=\frac{h}{2}I$.

Up to $\mathcal{D}$-homothety, it is possible to assume that either $h=0$ or $h=2$.

For $h=0$, (2.18) gives

$$\tilde{\mathfrak{g}}=(0,0,0,-2e^{12}-2\tau e^{35},0);$$

for $h=2$, either $\check{D}=0$ and

$$\tilde{\mathfrak{g}}=(0,0,2e^{35},-2e^{12}-2\tau e^{35},0),$$

or $\check{D}=I$ and

$$\tilde{\mathfrak{g}}=(e^{15},e^{25},2\tau e^{12}+2e^{35},-2e^{12}-2\tau e^{35},0).$$

In either case, the metric is

$$\tilde{g}=e^1\otimes e^1+e^2\otimes e^2+\tau e^3\otimes e^3+e^4\otimes e^4+\tau e^5\otimes e^5.$$

Taking into consideration the negative-definite metric on $\check{\mathfrak{g}}$ has the effect of adding the $\pm$ signs, as in Remark 2.4.3. □

Notice that the third Lie algebra appearing in Theorem 2.5.1 is Example 1.1.7.

I proceed to give a list of the 7-dimensional Lie algebras with a $\mathfrak{z}$-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$ up to isometry and $\mathcal{D}$-homothety. This list is given in Table 2.1, where I write the diagonal metric $\tilde{g}$ as a line vector with respect to the basis $\{e^1, \dots, e^7\}$, using the convention that $[1]_n$ is a vector of n elements, each equal to 1. For example $[1]_4 = (1,1,1,1)$ and $(\pm[1]_4, \tau, +1, \tau)$ represents the metric

$$\tilde{g} = \pm(e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 + e^4 \otimes e^4) + \tau e^5 \otimes e^5 + e^6 \otimes e^6 + \tau e^7 \otimes e^7.$$

Table 2.1: 7-dimensional Lie algebras with a $\mathfrak{z}$-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$ up to isometry and $\mathcal{D}$-homothety

n.	$\tilde{\mathfrak{g}}$	Metric $\tilde{g}$
1.	$0,0,0,0,0,-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
2.	$0,0,0,0,2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
3.	$0,0,e^{37},e^{47},2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
4.	$e^{17},e^{27},e^{37},e^{47},2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0$	$(\pm[1]_4,\tau,+1,\tau)$
5.	$0,0,0,0,0,-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
6.	$0,0,0,0,2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
7.	$0,0,e^{37},e^{47},-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
8.	$e^{17},e^{27},e^{37},e^{47},2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
9.	$\frac{1}{2}e^{17}+2\lambda e^{27}-\frac{1}{2}e^{37}-\lambda e^{47},-2\lambda e^{17}+\frac{1}{2}e^{27}+\lambda e^{37}-\frac{1}{2}e^{47},$ $\frac{1}{2}e^{17}+\lambda e^{27}-\frac{1}{2}e^{37},-\lambda e^{17}+\frac{1}{2}e^{27}-\frac{1}{2}e^{47},$ $\tau e^{12}-\tau e^{14}+\tau e^{23}+\tau e^{34},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
10.	$\frac{1}{2}e^{17}+2\lambda e^{27}-\frac{3}{2}e^{37}-\lambda e^{47},-2\lambda e^{17}+\frac{1}{2}e^{27}+\lambda e^{37}-\frac{3}{2}e^{47},$ $-\frac{1}{2}e^{17}+\lambda e^{27}-\frac{1}{2}e^{37},-\lambda e^{17}-\frac{1}{2}e^{27}-\frac{1}{2}e^{47},$ $\tau e^{12}-\tau e^{14}+\tau e^{23}+\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$
11.	$\frac{3}{2}e^{17}+2\lambda e^{27}+\frac{1}{2}e^{37}-\lambda e^{47},-2\lambda e^{17}+\frac{3}{2}e^{27}+\lambda e^{37}+\frac{1}{2}e^{47},$ $\frac{3}{2}e^{17}+\lambda e^{27}+\frac{1}{2}e^{37},-\lambda e^{17}+\frac{3}{2}e^{27}+\frac{1}{2}e^{47},$ $3\tau e^{12}-\tau e^{14}+\tau e^{23}-\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0$	$(\pm[1]_2,\mp[1]_2,\tau,+1,\tau)$

Theorem 2.5.2. *Let $\tilde{\mathfrak{g}}$ be a Lie algebra of dimension 7 with a $\mathfrak{z}$-standard Sasaki structure that reduces to an abelian pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$. Then, up to isometry and $\mathcal{D}$-homothety, the metric Lie algebra $(\tilde{\mathfrak{g}}, \tilde{g})$ is one of the Lie algebras appearing in Table 2.1 and the Sasaki structure is given by*

$$\xi = (e^6)^\flat = e_6, \qquad \eta = e^6, \qquad 2\Phi = d\eta = de^6$$

with respect to the basis $\{e^1,\dots,e^7\}$ *of Table 2.1.*

Proof. I will first consider the case where $\check{\mathfrak{g}}$ is positive definite, applying Corollary 2.4.3 and proceeding as in the proof of Theorem 2.5.1.

If $h=0$, one gets

$$(0,0,0,0,0,-2e^{12}-2e^{34}-2\tau e^{57},0);$$

for $h=2$, there are the three possibilities $\check{D}=0$, $\check{D}=e^3\otimes e_3+e^4\otimes e_4$, $\check{D}=I$, corresponding to

$$\begin{gathered}(0,0,0,0,2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0),\\ (0,0,e^{37},e^{47},2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0),\\ (e^{17},e^{27},e^{37},e^{47},2\tau e^{12}+2\tau e^{34}+2e^{57},-2e^{12}-2e^{34}-2\tau e^{57},0).\end{gathered}$$

The negative definite case gives rise to the same Lie algebras, with the restriction of the metric to $\check{\mathfrak{g}}$ of opposite sign.

In the neutral case, one can assume

$$\check{g}=e^1\otimes e^1+e^2\otimes e^2-e^3\otimes e^3-e^4\otimes e^4,\quad \omega=-e^{12}+e^{34},\quad J=e^1\otimes e_2-e^2\otimes e_1+e^3\otimes e_4-e^4\otimes e_3.$$

If $\check{D}^s$ is diagonalizable, Corollary 2.4.3 applies and computations as above yield

$$\begin{gathered}(0,0,0,0,0,-2e^{12}+2e^{34}-2\tau e^{57},0),\\ (0,0,0,0,2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0),\\ (0,0,e^{37},e^{47},-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0),\\ (e^{17},e^{27},e^{37},e^{47},2\tau e^{12}-2\tau e^{34}+2e^{57},-2e^{12}+2e^{34}-2\tau e^{57},0).\end{gathered}$$

If $\check{D}^s$ is not diagonalizable, one can exploit the $U(1,1)$ symmetry preserving the pseudo-Kähler structure of $\check{\mathfrak{g}}$. Indeed, a symmetric derivation commuting with J is effectively an element of $i\mathfrak{u}(1,1)$, with $U(1,1)$ acting on it by the adjoint action. Write $\check{D}^s=tI+\check{D}^s_0$, where $\check{D}^s_0$ is traceless. Then $\check{D}^s_0$ can be viewed as an element of $i\mathfrak{su}(1,1)$. Now $\mathrm{SU}(1,1)$ is isomorphic to $\mathrm{SL}(2,\mathbb{R})$ via the Cayley isomorphism

$$\mathrm{SL}(2,\mathbb{R})\ni g\mapsto CgC^{-1}\in\mathrm{SU}(1,1),\tag{2.20}$$

where $C=\begin{pmatrix}1&-i\\1&i\end{pmatrix}$. The action of $\mathrm{SL}(2,\mathbb{R})$ on its Lie algebra is conjugation, so any nondiagonalizable element of $\mathfrak{sl}(2,\mathbb{R})$ is in the $\mathrm{SL}(2,\mathbb{R})$-orbit of $\begin{pmatrix}0&1\\0&0\end{pmatrix}$. Reading this in $\mathfrak{su}(1,1)$ via (2.20) and multiplying by $-i$, one sees that $\check{D}^s_0$ corresponds to the complex matrix $\begin{pmatrix}1/2&-1/2\\1/2&-1/2\end{pmatrix}$; writing it as a real matrix, it reads

$$\check{D}^s=\begin{pmatrix}(t+\frac12)I&-\frac12 I\\ \frac12 I&(t-\frac12)I\end{pmatrix}.$$

A derivation $\check{D}$ that satisfies $[D, J] = 0$ and is not diagonalizable takes the form

$$\check{D} = \begin{pmatrix} x & \lambda_2 & \lambda_5 - 1 & -\lambda_6 \\ -\lambda_2 & x & \lambda_6 & \lambda_5 - 1 \\ \lambda_5 & \lambda_6 & x-1 & \lambda_8 \\ -\lambda_6 & \lambda_5 & -\lambda_8 & x-1 \end{pmatrix}.$$

Now, thanks to Proposition 1.1.5, one can consider any

$$\check{D}' = \begin{pmatrix} y & \mu_2 & \mu_5 - 1 & -\mu_6 \\ -\mu_2 & y & \mu_6 & \mu_5 - 1 \\ \mu_5 & \mu_6 & y-1 & \mu_8 \\ -\mu_6 & \mu_5 & -\mu_8 & y-1 \end{pmatrix}.$$

such that $[\check{D}', \check{D}] = 0$ and $\check{D}'^s = \check{D}^s$. This yields $y = x$, $\mu_5 = \lambda_5$, $\mu_6 = \lambda_6$ and $\mu_2 - \mu_8 = \lambda_2 - \lambda_8$, hence one can consider $\check{D}$ to be

$$\check{D} = \begin{pmatrix} x & \lambda_2 & \lambda_5 - 1 & -\lambda_6 \\ -\lambda_2 & x & \lambda_6 & \lambda_5 - 1 \\ \lambda_5 & \lambda_6 & x-1 & 0 \\ -\lambda_6 & \lambda_5 & 0 & x-1 \end{pmatrix}.$$

Again I distinguish two cases depending on h.

If $h = 0$ then equation $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$ yields

$$\check{D} = \begin{pmatrix} \frac{1}{2} & 2\lambda & -\frac{1}{2} & -\lambda \\ -2\lambda & \frac{1}{2} & \lambda & -\frac{1}{2} \\ \frac{1}{2} & \lambda & -\frac{1}{2} & 0 \\ -\lambda & \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}.$$

Hence $d\xi^\flat = -2e^{12} + 2e^{34}$, $db^\flat = \tau e^{12} - \tau e^{14} + \tau e^{23} + \tau e^{34}$, and the central Lie algebra extension is

$$\mathfrak{g} = (0, 0, 0, 0, \tau e^{12} - \tau e^{14} + \tau e^{23} + \tau e^{34}, -2e^{12} + 2e^{34}),$$

with metric

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 - e^4 \otimes e^4 + \tau b^\flat \otimes b^\flat + \xi^\flat \otimes \xi^\flat. \tag{2.21}$$

The Sasaki extension $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes \operatorname{Span}\{e_0\}$ is than determined by

$$d\xi^\flat = -2e^{12} + 2e^{34}, \qquad db^\flat = \tau e^{12} - \tau e^{14} + \tau e^{23} + \tau e^{34}$$
$$[e_0, x] = \check{D}x, \qquad [e_0, \xi] = 0, \qquad [e_0, b] = -2\tau\xi;$$

and the Lie algebra is

$$\tilde{\mathfrak{g}} = (\frac{1}{2}e^{17} + 2\lambda e^{27} - \frac{1}{2}e^{37} - \lambda e^{47}, -2\lambda e^{17} + \frac{1}{2}e^{27} + \lambda e^{37} - \frac{1}{2}e^{47}, \frac{1}{2}e^{17} + \lambda e^{27} - \frac{1}{2}e^{37},$$
$$-\lambda e^{17} + \frac{1}{2}e^{27} - \frac{1}{2}e^{47}, \tau e^{12} - \tau e^{14} + \tau e^{23} + \tau e^{34}, -2e^{12} + 2e^{34} - 2\tau e^{57}, 0).$$

If $h = 2$ then equation $[\check{D}^s, \check{D}^a] = h\check{D}^s - 2(\check{D}^s)^2$ yields two distinct solutions for $\check{D}$:

$$\check{D}_1 = \begin{pmatrix} \frac{1}{2} & 2\lambda & -\frac{3}{2} & -\lambda \\ -2\lambda & \frac{1}{2} & \lambda & -\frac{3}{2} \\ -\frac{1}{2} & \lambda & -\frac{1}{2} & 0 \\ -\lambda & -\frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \quad \text{or} \quad \check{D}_2 = \begin{pmatrix} \frac{3}{2} & 2\lambda & \frac{1}{2} & -\lambda \\ -2\lambda & \frac{3}{2} & \lambda & \frac{1}{2} \\ \frac{3}{2} & \lambda & \frac{1}{2} & 0 \\ -\lambda & \frac{3}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

$\check{D}_1$ yields $db^\flat = \tau e^{12} - \tau e^{14} + \tau e^{23} + \tau e^{34}$, hence

$$\mathfrak{g} = (0, 0, 0, 0, \tau e^{12} - \tau e^{14} + \tau e^{23} + \tau e^{34}, -2e^{12} + 2e^{34});$$

while for $\check{D}_2$ one gets $db^\flat = 3\tau e^{12} - \tau e^{14} + \tau e^{23} - \tau e^{34}$ and

$$\mathfrak{g} = (0, 0, 0, 0, 3\tau e^{12} - \tau e^{14} + \tau e^{23} - \tau e^{34}, -2e^{12} + 2e^{34}).$$

In both cases, the metric is given by (2.21). The resulting Lie algebras $\tilde{\mathfrak{g}}$ correspond to n. 10 and n. 11 in Table 2.1. □

I begin the classification in the Einstein case with a simple corollary of Theorem 2.4.9. In the case that $\check{\mathfrak{g}}$ is abelian, one obtains:

Corollary 2.5.3. *Every $\mathfrak{z}$-standard Sasaki-Einstein Lie algebra such that the Kähler reduction is an abelian Lie algebra is equivalent to one of those constructed in Example 2.4.7.*

Proof. If $\check{\mathfrak{g}}$ is an abelian Lie algebra, one can assume $\check{\mathfrak{g}} = \mathbb{R}^{2n-2}$, with

$$Je_1 = e_2, \ \ldots, \ Je_{2n-3} = e_{2n-2}, \qquad \omega = \varepsilon_1 e^{12} + \cdots + \varepsilon_{n-1} e^{2n-3,2n-2}, \quad \varepsilon_i = \pm 1;$$

the $\mathfrak{cu}(p, q)$-Nikolayevsky derivation is Id, so by Theorem 2.4.9 the extension is equivalent to one of those constructed in Example 2.4.7. □

In dimension 3, $\mathfrak{z}$-standard Sasaki-Einstein Lie algebras take the form $\mathbb{R}^2 \rtimes \mathrm{Span}\,\{e_3\}$, with $\operatorname{ad} e_3$ acting on $\mathbb{R}^2$ as the identity. In dimension 5, $\mathfrak{z}$-standard Sasaki-Einstein Lie algebras determine a reduction of dimension 2, which is abelian. Therefore, these metrics have the form given in Example 2.4.7, thus proving:

Proposition 2.5.4. *Let $\tilde{\mathfrak{g}}$ be a $\mathfrak{z}$-standard Sasaki-Einstein Lie algebra of dimension ≤ 5. Then $\tilde{\mathfrak{g}}$ is equivalent to one of*

$$(2e^{13}, 2e^{13}, 0), \qquad \tilde{g} = -e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3,$$
$$(e^{15}, e^{25}, 2e^{12} + 2e^{35}, 2e^{12} + 2e^{35}, 0), \qquad \tilde{g} = e^1 \otimes e^1 + e^2 \otimes e^2 - e^3 \otimes e^3 + e^4 \otimes e^4 - e^5 \otimes e^5,$$
$$(e^{15}, e^{25}, -2e^{12} + 2e^{35}, -2e^{12} + 2e^{35}, 0), \quad \tilde{g} = -e^1 \otimes e^1 - e^2 \otimes e^2 - e^3 \otimes e^3 + e^4 \otimes e^4 - e^5 \otimes e^5.$$

Note that the 5-dimensional solvable Lie algebras appearing in Proposition 2.5.4 are isomorphic; up to a sign, the metric of signature $(1, 4)$ is isometric to Example 1.1.7, which will appear again in the next chapter (Example 3.4.6).

In dimension 7, one can classify $\mathfrak{z}$-standard Sasaki-Einstein Lie algebras by using the classification of four-dimensional Lie algebras with a pseudo-Kähler metric in [68]:

Theorem 2.5.5. *Let $\tilde{\mathfrak{g}}$ be a $\mathfrak{z}$-standard Sasaki-Einstein Lie algebra of dimension* 7*. Then $\tilde{\mathfrak{g}}$ is equivalent to one of the following:*

1. *$\tilde{\mathfrak{g}}$ is the solvable Lie algebra*

$$(e^{17}, e^{27}, e^{37}, e^{47}, 2\varepsilon_1 e^{12} + 2\varepsilon_2 e^{34} + 2e^{57}, 2\varepsilon_1 e^{12} + 2\varepsilon_2 e^{34} + 2e^{57}, 0)$$

with metric

$$\tilde{g} = \varepsilon_1(e^1 \otimes e^1 + e^2 \otimes e^2) + \varepsilon_2(e^3 \otimes e^3 + e^4 \otimes e^4) + \gamma, \quad \varepsilon_1, \varepsilon_2 \in \{+1, -1\};$$

2. *$\tilde{\mathfrak{g}}$ is the solvable Lie algebra*

$$\Big(\frac{2}{3}e^{17}, \frac{2}{3}e^{27}, \frac{a}{3}e^{27} + \frac{4}{3}e^{37} + e^{12}, -\frac{a}{3}e^{17} + \frac{4}{3}e^{47},\\ 2(e^{13} + e^{24} + ae^{12} + e^{57}), 2(e^{13} + e^{24} + ae^{12} + e^{57}), 0\Big)$$

with metric

$$\tilde{g} = -a(e^1 \otimes e^1 + e^2 \otimes e^2) + e^1 \odot e^4 - e^2 \odot e^3 + \gamma, \quad a \in \mathbb{R};$$

3. *$\tilde{\mathfrak{g}}$ is the solvable Lie algebra*

$$\Big(\frac{2}{3}e^{17}, \frac{2}{3}e^{27}, \frac{b}{3}e^{17} + \frac{4}{3}e^{37} + e^{12}, \frac{b}{3}e^{27} + \frac{4}{3}e^{47},\\ 2a(e^{13} + e^{24}) + 2(e^{14} - e^{23} + be^{12} + e^{57}), 2a(e^{13} + e^{24}) + 2(e^{14} - e^{23} + be^{12} + e^{57}), 0\Big)$$

with metric

$$\tilde{g} = -b(e^1 \otimes e^1 + e^2 \otimes e^2) + a(e^1 \odot e^4 - e^2 \odot e^3) - e^1 \odot e^3 - e^2 \odot e^4 + \gamma, \quad a, b \in \mathbb{R};$$

where I have set $\gamma = -e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7$.

Proof. By Theorem 2.4.1, every $\mathfrak{z}$-standard Sasaki Lie algebra can be obtained by extending a four-dimensional pseudo-Kähler Lie algebra $\check{\mathfrak{g}}$. By the classification of [68], there are the following possibilities:

1. $\check{\mathfrak{g}}$ is abelian; one can assume that the metric is either positive-definite or neutral. Then one obtains the Lie algebras of Example 2.4.7, i.e.

$$\tilde{\mathfrak{g}} = (e^{17}, e^{27}, e^{37}, e^{47}, 2\varepsilon_1 e^{12} + 2\varepsilon_2 e^{34} + 2e^{57}, 2\varepsilon_1 e^{12} + 2\varepsilon_2 e^{34} + 2e^{57}, 0)$$

with metric

$$\tilde{g} = \varepsilon_1(e^1 \otimes e^1 + e^2 \otimes e^2) + \varepsilon_2(e^3 \otimes e^3 + e^4 \otimes e^4) - e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7,$$

where $\varepsilon_1, \varepsilon_2 = \pm 1$.

2. $\check{\mathfrak{g}} = (0, 0, e^{12}, 0)$, with $Je_1 = e_2, Je_3 = e_4$, and $\omega = e^{13} + e^{24} + ae^{12}$ for $a \in \mathbb{R}$. Then

$$\check{g} = -a(e^1 \otimes e^1 + e^2 \otimes e^2) + e^1 \odot e^4 - e^2 \odot e^3.$$

The generic $\check{D}$ satisfying the hypothesis of Proposition 2.4.6 is

$$\check{D} = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ \lambda & \frac{a}{3} & \frac{4}{3} & 0 \\ -\frac{a}{3} & \lambda & 0 & \frac{4}{3} \end{pmatrix}.$$

By Theorem 2.4.9, one can assume $\lambda = 0$. Therefore, the extension is

$$\tilde{\mathfrak{g}} = \Big(\frac{2}{3}e^{17}, \frac{2}{3}e^{27}, \frac{a}{3}e^{27} + \frac{4}{3}e^{37} + e^{12}, -\frac{a}{3}e^{17} + \frac{4}{3}e^{47},$$
$$2e^{13} + 2e^{24} + 2ae^{12} + 2e^{57}, 2e^{13} + 2e^{24} + 2ae^{12} + 2e^{57}, 0\Big)$$

with the metric

$$\tilde{g} = \check{g} - e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7.$$

3. $\check{\mathfrak{g}} = (0, 0, e^{12}, 0)$ with $Je_1 = e_2, Je_3 = e_4$, and $\omega = a(e^{13} + e^{24}) + e^{14} - e^{23} + be^{12}$ for $a, b \in \mathbb{R}$. Then

$$\check{g} = -b(e^1 \otimes e^1 + e^2 \otimes e^2) + a(e^1 \odot e^4 - e^2 \odot e^3) - e^1 \odot e^3 - e^2 \odot e^4.$$

The generic $\check{D}$ satisfying the hypothesis of Proposition 2.4.6 is

$$\check{D} = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ a\lambda + \frac{b}{3} & -\lambda & \frac{4}{3} & 0 \\ \lambda & a\lambda + \frac{b}{3} & 0 & \frac{4}{3} \end{pmatrix}.$$

Again, one may assume $\lambda = 0$ and obtain

$$\tilde{\mathfrak{g}} = \Big(\frac{2}{3}e^{17}, \frac{2}{3}e^{27}, \frac{b}{3}e^{17} + \frac{4}{3}e^{37} + e^{12}, \frac{b}{3}e^{27} + \frac{4}{3}e^{47},$$
$$2a(e^{13} + e^{24}) + 2e^{14} - 2e^{23} + 2be^{12} + 2e^{57}, 2a(e^{13} + e^{24}) + 2e^{14} - 2e^{23} + 2be^{12} + 2e^{57}, 0\Big)$$

with the metric

$$\tilde{g} = \check{g} - e^5 \otimes e^5 + e^6 \otimes e^6 - e^7 \otimes e^7. \qquad \square$$

I conclude the chapter by presenting some examples of pseudo-Kähler Lie algebras in dimension 6, which are candidates to extend to a 9-dimensional $\mathfrak{z}$-standard Sasaki Einstein Lie algebra.

Example 2.5.6. Consider the 6-dimensional Lie algebra $\mathfrak{g} = (0, 0, 0, e^{12}, e^{13}, e^{14} - e^{23})$, denoted by $\mathfrak{h}_{11}$ in [34]; by [70, 21], it admits a one-parameter family of complex structures. By the work of [34], it is known that it has a four-dimensional space of compatible pseudo-Kähler metrics.

Instead of fixing the complex structure, I use the explicit form of the two families of pseudo-Kähler structures given in [73].

The first one is $\omega_1 = e^{16} - \lambda e^{25} - (\lambda - 1)e^{34}$, which has as compatible canonical complex structure

$$J_1(e_2) = (1+b)ae_1, \quad J_1(e_4) = ae_3, \quad J_1(e_6) = \frac{(1+b)a}{b}e_5$$

and metric $g_1 = \omega_1 J_1$; while the second one is $\omega_2 = e^{16} + e^{24} - \frac{1}{2}(e^{25} - e^{34})$ which has as canonical complex structure compatible

$$J_2(e_2) = -ae_1, \quad J_2(e_3) = \frac{3}{2a}e_4 + \frac{3}{a}e_5, \quad J_2(e_4) = -\frac{2}{3}ae_3 - \frac{1}{a}e_6, \quad J_2(e_6) = -2ae_5$$

and metric $g_2 = \omega_2 J_2$.

In the first case, imposing $[D, J_1] = 0$ gives

$$D = \begin{pmatrix} \frac{\mu_1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_1}{3} & 0 & 0 & 0 & 0 \\ \frac{\mu_2}{b} & -a^2\frac{\mu_3}{b}(b+1) & \frac{2\mu_1}{3} & 0 & 0 & 0 \\ \frac{\mu_3}{b} & \mu_2 + \frac{\mu_2}{b} & 0 & \frac{2\mu_1}{3} & 0 & 0 \\ \frac{\mu_4}{b} & -a^2\frac{\mu_5}{b}(b+1)^2 & \mu_2 + \frac{\mu_2}{b} & -a^2\frac{\mu_3}{b}(b+1) & \mu_1 & 0 \\ \mu_5 & \mu_4 & \mu_3 & \mu_2 & 0 & \mu_1 \end{pmatrix}$$

and imposing $D^s = \mathrm{Id}$ gives $\mu_1 = \frac{3}{2}$ and $\mu_i = 0$ for $i = 2, \dots, 5$, that is

$$D = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{2} \end{pmatrix}.$$

On the other hand, $[D, J_2] = 0$ gives

$$D = \begin{pmatrix} \frac{\mu_1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\mu_1}{3} & 0 & 0 & 0 & 0 \\ 2\mu_2 & -\frac{2}{3}(2a^2\mu_3 + \mu_1) & \frac{2\mu_1}{3} & 0 & 0 & 0 \\ 2\mu_3 + \frac{\mu_1}{a^2} & 3\mu_2 & 0 & \frac{2\mu_1}{3} & 0 & 0 \\ 2(\mu_4 + 2\mu_3 + \frac{\mu_1}{a^2}) & -2(a^2\mu_5 + 3\mu_2) & 3\mu_2 & -\frac{2}{3}(2a^2\mu_3 + \mu_1) & \mu_1 & 0 \\ \mu_5 & \mu_4 & \mu_3 & \mu_2 & 0 & \mu_1 \end{pmatrix}$$

but imposing $D^s = \mathrm{Id}$ does not yield any solution for the μ_i.

Example 2.5.7. The following example shows a $\mathfrak{z}$-standard Sasaki-Einstein $\tilde{\mathfrak{g}}$ obtained by extending a 6-dimensional pseudo-Kähler Lie algebra with a derivation $\check{D}$ which is not a multiple of the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation. Consider the Lie algebra $\check{\mathfrak{g}} = (0,0,e^{12},0,0,0)$ with symplectic form $\omega = e^{13}+e^{24}+e^{56}$ and complex structure $J(e_1) = e_2$, $J(e_3) = e_4$ and $J(e_5) = e_6$. Then

$$\check{D} = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 \\ \mu & 0 & \frac{4}{3} & 0 & \lambda & -\nu \\ 0 & \mu & 0 & \frac{4}{3} & \nu & \lambda \\ \nu & -\lambda & 0 & 0 & 1 & -\rho \\ \lambda & \nu & 0 & 0 & \rho & 1 \end{pmatrix}$$

satisfies the hypothesis of Proposition 2.4.6, and therefore determines a $\mathfrak{z}$-standard Sasaki-Einstein $\tilde{\mathfrak{g}}$ Lie algebra of dimension 9. The derivation $\check{D}$ is not diagonalizable over $\mathbb{R}$, but has eigenvalues $(\frac{2}{3}, \frac{2}{3}, 1-i\rho, 1+i\rho, \frac{4}{3}, \frac{4}{3})$; therefore, $\check{D}$ is only a multiple of the $\mathfrak{cu}(p,q)$-Nikolayevsky derivation when ρ is zero. Note, however, that all the resulting extensions are isometric by Theorem 2.4.9.

Chapter 3

Embeddings with Killing spinors

In this chapter, I will prove an embedding result in the more general, non-invariant setting of pseudo-Riemannian spin manifolds admitting a Killing spinor. This result generalizes the known cases obtained in [1] and [6] as the one presented here holds in general signature and for $\lambda \neq 0$, although I require the extended metric to be Einstein.

The proof will be approached as follows: first, assuming (M, g) is a hypersurface of signature (r, s), with $r + s = n$, embedded in a pseudo-Riemannian manifold (Z, h) endowed with a Killing spinor, I will give a characterization of the geometry of (M, g), which will depend on the signature of (Z, h). The structure that arises will be called *weakly harmful*. Next, I will prove that a weakly harmful real analytic hypersurface that satisfies another technical condition embeds isometrically in an Einstein pseudo-Riemannian manifold one dimension higher and endowed with a Killing spinor. In the first section, I will recall and adapt to the indefinite setting some classical results obtained by Koiso in [56]. Subsequently, I will show how the hypersurface is characterized, giving the precise definition of the structure. In the remaining two sections, the embedding process will play out, first proving the isometric embedding, then extending the weakly harmful structure to a Killing spinor. I will conclude the chapter with a concrete example of a Lie algebra admitting a harmful structure. This result appears in a joint work with my supervisor in [33].

3.1 Hypersurfaces in Einstein manifolds

In this section, I recall Koiso's characterization of real analytic pseudo-Riemannian manifolds (M, g) which can be immersed as hypersurfaces in an Einstein manifold (see [56]). Whilst Koiso works in Riemannian signature, the proof works in the same way for arbitrary signature, though statements need to be adapted slightly.

The Einstein manifold will take the form of a generalized cylinder in the sense of [6], i.e. a product $Z = M \times (a, b)$ endowed with a metric of the form $g_t + dt^2$, with $\{g_t\}$ a one-parameter family of metrics on M. In the calculations, I will often drop the subscript t for simplicity.

The isometric embedding of (M, g) in the generalized cylinder will be obtained by imposing the initial condition $g_0 = g$. The Einstein condition is a PDE which can be expressed purely in terms of $\{g_t\}$; however, it will be convenient to write it in terms of both $\{g_t\}$ and the Weingarten operators $\{A_t\}$. Notice that the second fundamental form of the hypersurface $M \times \{t\}$ can be identified with $-\frac{1}{2}\dot{g}_t$, so

$$\dot{g}_t(X,Y) = -2g_t(A_t(X),Y). \tag{3.1}$$

I will need to consider the operator δ acting on tensors of type (k,h) as

$$\begin{cases} (\delta T)(v_1,\dots,v_k,\alpha^1,\dots,\alpha^{h-1}) = -\sum_{i=1}^n (\nabla_{e_i} T)(v_1,\dots,v_k,e^i,\alpha^1,\dots,\alpha^{h-1}), & h>0 \\ (\delta T)(v_1,\dots,v_{k-1}) = -\sum_{i=1}^n (\nabla_{e_i} T)((e^i)^\sharp, v_1,\dots,v_{k-1}), & h=0 \end{cases}$$

For the remainder of the chapter, $\{e_i\}$ denotes any frame, and $\{e^i\}$ its dual frame. In general, $\{e_i\}$ will be an orthonormal frame, so that the metric takes the form

$$\varepsilon_1 e^1 \otimes e^1 + \dots + \varepsilon_n e^n \otimes e^n,$$

where $\varepsilon_i = \pm 1$.

Notice that for vector fields, $\delta X = -\operatorname{Div} X$, and for 1-forms $\delta\alpha = d^*\alpha$; in particular, $\Delta f = \delta(df)$ for any function f.

Theorem 3.1.1 (Koiso [56]). *Let $\{g_t\}$ and $\{A_t\}$ be real analytic one-parameter families of metrics (resp. symmetric (1,1) tensors) on M defined on the interval (a,b), satisfying*

$$\begin{cases} \dot{g}_t(X,Y) = -2g_t(A(X),Y) \\ \dot{A} = -\operatorname{Ric}(g_t) + (\operatorname{tr} A)A + K\operatorname{Id} \end{cases}$$

Assume further that

$$s = (n-1)K - \operatorname{tr} A_0^2 + (\operatorname{tr} A_0)^2, \quad d\operatorname{tr} A_0 + \delta A_0 = 0. \tag{3.2}$$

Then $g_t + dt^2$ is an Einstein metric on $M \times (a,b)$ with Einstein constant K.

As an immediate consequence, one obtains:

Corollary 3.1.2. *A real analytic pseudo-Riemannian manifold (M,g) of signature (r,s) embeds isometrically as a hypersurface in an Einstein manifold of signature $(r+1,s)$ with $\operatorname{Ric}^Z = K\operatorname{Id}$ if and only if it admits a symmetric $(1,1)$ tensor A such that*

$$s = (n-1)K - \operatorname{tr} A^2 + (\operatorname{tr} A)^2, \quad d\operatorname{tr} A + \delta A = 0. \tag{3.3}$$

Proof. Apply the Cauchy-Kovaleskaya theorem. □

Corollary 3.1.3. *A real analytic pseudo-Riemannian manifold (M,g) of signature (r,s) embeds isometrically as a hypersurface in an Einstein manifold of signature $(r,s+1)$ with $\operatorname{Ric}^Z = K\operatorname{Id}$ if and only if it admits a symmetric $(1,1)$ tensor A such that*

$$s = (n-1)K + \operatorname{tr} A^2 - (\operatorname{tr} A)^2, \quad d\operatorname{tr} A + \delta A = 0. \tag{3.4}$$

Proof. Let $\tilde{g} = -g$ be the opposite metric, with signature (s, r). Then $\tilde{\mathrm{Ric}} = -\mathrm{Ric}$, $\tilde{s} = -s$, and $\tilde{\delta}A = \delta A$. Write also $\tilde{A} = -A$. Then

$$\tilde{s} = (n-1)(-K) - \operatorname{tr}\tilde{A}^2 + (\operatorname{tr}\tilde{A})^2, \quad d\operatorname{tr}\tilde{A} + \delta\tilde{A} = 0.$$

Therefore, one obtains a generalized cylinder $\tilde{g}_t + dt^2$ with $\mathrm{Ric}^Z = -K\,\mathrm{Id}$. By reversing the sign of the metric, one finds that $g_t - dt^2$ satisfies $\mathrm{Ric}^Z = K\,\mathrm{Id}$. □

3.2 Characterization of hypersurfaces

In this section, I study the geometry of a hypersurface embedded in a pseudo-Riemannian manifold with a (nonzero) Killing spinor. One can show that the hypersurface inherits two spinors which satisfy a coupled differential system involving a symmetric tensor A, which corresponds to the second fundamental form.

Recall from the first chapter that, if $\mathrm{Cl}_{r,s}$ is the Clifford algebra of signature (r, s) and $\Sigma_{r,s}$ the spinor representation, by definition, $\Sigma_{r,s}$ is a representation of $\mathrm{Cl}_{r,s}$; if $r+s$ is even and positive, $\Sigma_{r,s}$ splits into the sum of two representations of $\mathrm{Spin}_{r,s}$, denoted by $\Sigma^+_{r,s}$ and $\Sigma^-_{r,s}$, which can be identified as the ± 1-eigenspaces of Clifford multiplication by the volume form when $r - s$ is a multiple of 4, or the $\pm i$-eigenspaces if $r - s$ is not a multiple of 4.

Let N be a spin manifold of dimension n endowed with a pseudo-Riemannian metric of signature (r, s), and let ΣN denote the bundle of complex spinors; recall that ΣN splits as $\Sigma_+ N \oplus \Sigma_- N$ when n is even. Clifford multiplication gives a bundle map

$$TN \otimes \Sigma N \to \Sigma N, \quad v \otimes \psi \mapsto v \cdot \psi.$$

Let $e_1, \ldots, e_n$ be a positively-oriented orthonormal basis of TN. Recall from Proposition 1.2.10 that the volume element $\omega = e_1 \cdots e_{r+s}$ in $\mathrm{Cl}_{r,s}$, $r + s = n$ satisfies

$$\omega^2 = (-1)^{\frac{n(n+1)}{2}+s}, \quad e_i\omega = (-1)^{n-1}\omega e_i.$$

In other words,

$$\omega^2 = \begin{cases} 1 & r - s = 0, 3 \mod 4 \\ -1 & r - s = 1, 2 \mod 4 \end{cases}.$$

Now suppose (Z, h) is a pseudo-Riemannian spin manifold with a Killing spinor Ψ, i.e. $\nabla_X \Psi = \lambda X \cdot \Psi$ for any vector field X of Z, where λ is a complex constant. Since I am interested in hypersurfaces of Z, I will denote by $n + 1$ the dimension of Z. As the volume element is parallel, one has

$$\nabla_X(\omega \cdot \Psi) = \omega \cdot \nabla_X \Psi = \lambda\omega \cdot X \cdot \Psi = (-1)^n \lambda X \cdot (\omega \cdot \Psi).$$

Thus, $\omega \cdot \Psi$ is also Killing. Assume n is even. Then if $\lambda \neq 0$, Ψ and $\omega \cdot \Psi$ are necessarily independent, since they have opposite Killing numbers. In general, one can decompose Ψ as $\Psi^+ + \Psi^-$ and hence obtain

$$\nabla_X \Psi^+ = \lambda X \cdot \Psi^-, \nabla_X \Psi^- = \lambda X \cdot \Psi^+.$$

Let (M,g) be an oriented hypersurface, call $\iota\colon M\to Z$ the embedding, and let ν be a normal vector field, normalized so that $h(\nu,\nu)=1$ or $h(\nu,\nu)=-1$. Then M is also spin and one can define a bundle morphism from the complex Clifford bundle $\mathbb{C}\mathrm{l}\, M$ to $\iota^*\,\mathbb{C}\mathrm{l}\, Z$,

$$v\mapsto \nu\cdot v \quad (\text{resp. } v\mapsto i\nu\cdot v). \tag{3.5}$$

Recall that the Clifford algebra is graded over $\mathbb{Z}_2$ (see e.g. [60]); accordingly there is a splitting $\mathbb{C}\mathrm{l}\, Z=\mathbb{C}\mathrm{l}^0\, Z\oplus \mathbb{C}\mathrm{l}^1\, Z$. The bundle map (3.5) is an isomorphism onto $\iota^*\,\mathbb{C}\mathrm{l}^0\, Z$; indeed, it restricts to an algebra isomorphism on each fiber, realizing Clifford multiplication on M as

$$v\odot w=\nu\cdot v\cdot w \quad (\text{resp. } v\odot w\mapsto i\nu\cdot v\cdot w).$$

Recalling that n denotes the dimension of M, one obtains the identifications

$$\begin{aligned}\Sigma_+M\oplus\Sigma_-M&=\iota^*\Sigma Z, && n \text{ even},\\ \Sigma M&=\iota^*\Sigma_+Z, && n\text{ odd}.\end{aligned}\tag{3.6}$$

It was shown in [62] that a hypersurface inside a Riemannian manifold of dimension 3 with a Killing spinor inherits a spinor ψ satisfying

$$\nabla_X\psi=\frac{1}{2}A(X)\odot\psi+\lambda X\odot\omega\odot\psi;$$

this generalizes in a straightforward way to arbitrary hypersurfaces of signature (r,s) in manifolds with signature $(r+1,s)$, with $r+s$ even, as

$$\nabla_X\psi=\frac{1}{2}A(X)\odot\psi+\lambda i^{\frac{s-r+2}{2}}X\odot\omega\odot\psi. \tag{3.7}$$

For $r+s$ odd, the following holds:

Theorem 3.2.1. *Let Z be a pseudo-Riemannian spin manifold of dimension $n+1$ and signature $(r+1,s)$, with n odd, endowed with a Killing spinor Ψ such that*

$$\nabla^{\Sigma Z}_X\Psi=\lambda X\cdot\Psi,\quad \lambda\in\mathbb{C},$$

and let M be an oriented hypersurface of signature (r,s), with Weingarten operator $A(X)=-\nabla^{\Sigma Z}_X\nu$. Write $\Psi=\Psi_++\Psi_-$, and define spinors ψ and φ on M by restricting Ψ_+ and $\nu\cdot\Psi_-$ and applying the isomorphism (3.6)*. Then φ and ψ satisfy the coupled differential system*

$$\begin{cases}\nabla^{\Sigma M}_X\psi=\frac{1}{2}A(X)\odot\psi+\lambda X\odot\varphi\\ \nabla^{\Sigma M}_X\varphi=\lambda X\odot\psi-\frac{1}{2}A(X)\odot\varphi.\end{cases}\tag{3.8}$$

and the restriction of Ψ to M is given by $\psi-\nu\varphi$.

Proof. From equation (3.5) of [6] it is known that

$$\nabla_X^{\Sigma Z}\Psi^+ = \nabla_X^{\Sigma M}\psi - \frac{1}{2}\nu \cdot A(X)\cdot\psi, \quad X \in TM.$$

Using (3.5), $\nu\cdot\nu = -1$ and the fact that Clifford multiplication by a vector interchanges Σ_+ and Σ_-, one obtains

$$\nabla_X^{\Sigma M}\psi = \lambda\nu\cdot X\cdot(\nu\cdot\Psi^-) + \frac{1}{2}A(X)\odot\psi = \lambda X\odot\varphi + \frac{1}{2}A(X)\odot\psi,$$

and similarly,

$$\nabla_X^{\Sigma M}\varphi - \frac{1}{2}\nu\cdot A(X)\cdot\varphi = \nabla_X^{\Sigma Z}(\nu\cdot\Psi^-) = \nu\cdot\lambda X\cdot\Psi^+ - A(X)\cdot\Psi^-.$$

Thus

$$\nabla_X^{\Sigma M}\varphi = \frac{1}{2}\nu\cdot A(X)\cdot\varphi + \lambda X\odot\psi + A(X)\cdot\nu\cdot\varphi = \lambda X\odot\psi - \frac{1}{2}A(X)\odot\varphi.$$

□

In the same way one proves a similar result in the case that the normal is timelike.

Theorem 3.2.2. *Let Z be a pseudo-Riemannian spin manifold of dimension $n+1$ and signature $(r, s+1)$, with n odd, endowed with a Killing spinor Ψ, so that*

$$\nabla_X^{\Sigma Z}\Psi = \lambda X\cdot\Psi, \quad \lambda\in\mathbb{C},$$

and let M be an oriented hypersurface of signature (r,s) with $A(X) = \nabla_X^{\Sigma Z}\nu$ the Weingarten operator.

Write $\Psi = \Psi_+ + \Psi_-$, and define spinors ψ and φ on M by restricting Ψ_+ and $i\nu\cdot\Psi_-$ and applying the isomorphism (3.6)*. Then φ and ψ satisfy the coupled differential system*

$$\begin{cases}\nabla_X^{\Sigma M}\varphi = \frac{i}{2}A(X)\odot\varphi + \lambda X\odot\psi\\ \nabla_X^{\Sigma M}\psi = \lambda X\odot\varphi - \frac{i}{2}A(X)\odot\psi,\end{cases} \tag{3.9}$$

and the restriction of Ψ to M is given by $\psi - i\nu\varphi$.

Remark 3.2.1. In the even case, (3.8) and (3.9) still hold if one sets $\varphi = i^{\frac{s-r+2}{2}}\omega\odot\psi$, where ω is the volume form in M. In this case, φ is the restriction of $\nu\cdot\Psi$ under the isomorphism (3.6). Notice that this is simply a different way of writing (3.7) or its timelike analogue.

Recall that the constant λ appearing in the Killing spinor equation, and hence equations (3.8) and (3.9), is either real or purely imaginary.

These equations characterize a geometry that gives rise to a Killing spinor in one dimension higher, but only potentially; to indicate this, the structure will be called *harmful*. More precisely, given a pseudo-Riemannian spin manifold (M,g) of signature

(r, s), I will say that a *weakly harmful structure* on (M, g) is a pair of nowhere vanishing spinors (φ, ψ) satisfying either (3.8) or (3.9) for some symmetric tensor A and some constant λ, either real or purely imaginary; if $r + s$ is even, I further require that $\varphi = i^{\frac{s-r+2}{2}} \omega \cdot \psi$ where ω is the volume form. The weakly harmful structure will be called *real* if (3.8) holds and *imaginary* if (3.9) holds. If the symmetric tensor A additionally satisfies

$$d \operatorname{tr} A + \delta A = 0,$$

(φ, ψ) will be called a *harmful* structure.

Remark 3.2.2. In Corollary 3.3.4 I will show that, on a Riemannian manifold, a real weakly harmful structure is necessarily harmful.

Theorem 3.2.1 and its timelike counterpart, Theorem 3.2.2, show that any nondegenerate hypersurface inside an Einstein pseudo-Riemannian manifold (Z, h) endowed with a Killing spinor inherits a harmful structure. If (Z, h) is not assumed to be Einstein, one obtains a weakly harmful structure (see Corollary 3.1.2 and Corollary 3.1.3).

Notice that for $\lambda = 0$, ψ satisfies an equation analogous to the generalized Killing spinor equation of [6], with a factor of $-i$ if one takes the normal to be timelike, rather than spacelike.

3.3 Isometric embedding in an Einstein spin manifold

In this section I prove that a real analytic pseudo-Riemannian spin manifold of signature (r, s) with a harmful structure can be embedded isometrically in a pseudo-Riemannian Einstein manifold, of signature $(r+1, s)$ or $(r, s+1)$ accordingly to whether the harmful structure is real or imaginary. I will present the detailed proofs only for real harmful structures, as the imaginary case is entirely similar.

Since in this section all spinors are on the same manifold M, I will omit the symbol $\odot$ and indicate Clifford multiplication by juxtaposition.

Given a harmful structure satisfying (3.8) or (3.9), it will be convenient to introduce the tensor

$$F(X, Y) = (\nabla_X A)(Y) - (\nabla_Y A)(X).$$

I begin with the following Lemma regarding the spinorial Riemann tensor of (M, g):

Lemma 3.3.1. *Let (M, g) be a pseudo-Riemannian spin manifold with a real (weakly) harmful structure (φ, ψ), and let $X, Y \in TM$ be two vector fields. Then the curvature of M satisfies*

$$\begin{aligned}\mathcal{R}^M_{XY}\psi =& \frac{1}{2}\Big(F(X, Y) + A(Y)A(X) + g\big(A(Y), A(X)\big)\Big)\psi \\ &+ 2\lambda^2\big(YX + g(X, Y)\big)\psi.\end{aligned} \tag{3.10}$$

Proof. By applying twice (3.8), the curvature tensor $\mathcal{R}_{XY}\Psi$ becomes

$$\begin{aligned}\mathcal{R}^M_{XY}\psi =&\nabla_X\nabla_Y\psi - \nabla_Y\nabla_X\psi - \nabla_{[X,Y]}\psi = \\ =&\frac{1}{2}(\nabla_X(A(Y))\,\psi + A(Y)\nabla_X\psi) + \lambda(\nabla_X Y\varphi + Y\nabla_X\varphi) \\ &- \frac{1}{2}(\nabla_Y(A(X))\psi + A(X)\nabla_Y\psi) - \lambda(\nabla_Y X\varphi + X\nabla_Y\varphi) \\ &- \frac{1}{2}A([X,Y])\psi - \lambda[X,Y]\varphi \\ =&\lambda T^\nabla(X,Y)\varphi + \frac{1}{2}\Big((\nabla_X A)(Y) - (\nabla_Y A)(X) + A\big(T^\nabla(X,Y)\big)\Big)\psi \\ &+ \frac{1}{2}\left[A(Y)\left(\frac{1}{2}A(X)\psi + \lambda X\varphi\right) + \lambda Y\left(\lambda X\psi - \frac{1}{2}A(X)\varphi\right)\right] \\ &- \frac{1}{2}\left[A(X)\left(\frac{1}{2}A(Y)\psi + \lambda Y\varphi\right) + \lambda X\left(\lambda Y\psi - \frac{1}{2}A(Y)\varphi\right)\right]\end{aligned}$$

where $T^\nabla = 0$ is the torsion of the Levi Civita connection. One gets

$$\begin{aligned}\mathcal{R}^M_{XY}\psi =& -\frac{\lambda}{2}\big(A(Y)X - YA(X) - A(X)Y + XA(Y)\big)\varphi \\ &+\Big(\frac{1}{2}F(X,Y) + \frac{1}{4}\big(A(Y)A(X) - A(X)A(Y)\big) + \lambda^2\big(YX - XY\big)\Big)\psi\end{aligned}$$

Since for the Clifford product $vw + wv = -2g(v,w)$ and A is self-adjoint, the coefficient of φ equals zero and the statement follows. □

The following Lemma gives an expression for the Ricci tensor on a manifold for which the curvature satisfies (3.10).

Lemma 3.3.2. *Assume that (M,g) is an n-dimensional pseudo-Riemannian spin manifold with a real (weakly) harmful structure and fix an orthonormal frame $(e_1,\dots,e_n)$ for TM. Then the Ricci operator of M satisfies*

$$\begin{aligned}\mathrm{Ric}(X)\psi =&\Big(4(n-1)\lambda^2 X + (\mathrm{tr}\,A)A(X) - A^2(X)\Big)\psi \\ &+ \left(\nabla_X(\mathrm{tr}\,A) + \sum_{k=1}^n \varepsilon_k e_k(\nabla_{e_k}A)(X)\right)\psi.\end{aligned}$$

Proof. It is known, for example from equation (1.13) of [20], that

$$\mathrm{Ric}(X)\cdot\psi = -2\sum_{k=1}^n \varepsilon_k s_k \cdot \mathcal{R}_{Xs_k}\psi, \tag{3.11}$$

which holds for any orthonormal basis $(s_1,\dots,s_n)$, where $\langle s_i, s_j\rangle = \varepsilon_i\delta_{ij}$. Fix now an orthonormal frame $(e_1, \dots, e_n)$ for TM, so that $\langle e_i, e_j \rangle = g_{ij} = \varepsilon_i \delta_{ij}$.

$$\begin{aligned}\mathrm{Ric}(X)\psi =& -2\sum_{k=1}^{n} \varepsilon_k e_k \Big(2\lambda^2 (e_k X + g(X, e_k)) + \frac{1}{2} F(X, e_k) \\ &+ \frac{1}{2}\Big(A(e_k)A(X) + g\big(A(e_k), A(X)\big)\Big)\Big)\psi \\ =& \Big(4\lambda^2(n-1)X - \sum_{k=1}^{n} \varepsilon_k e_k \Big((\nabla_X A)(e_k) - (\nabla_{e_k} A)(X)\Big) \\ &- \sum_{k=1}^{n} \varepsilon_k e_k \Big(A(e_k)A(X) + g\big(A(e_k), A(X)\big)\Big)\Big)\psi.\end{aligned}$$

Recall that for any symmetric tensor W the following formula holds

$$\sum_{i=1}^{n} \varepsilon_i e_i \cdot W(e_i) = -\,\mathrm{tr}(W). \tag{3.12}$$

Then one gets

$$\begin{aligned}\mathrm{Ric}(X)\psi =& \Big(4(n-1)\lambda^2 X - \sum_{k=1}^{n} \varepsilon_k e_k \Big(A(e_k)A(X) + g\big(e_k, A^2(X)\big)\Big) \\ &+ \Big(\mathrm{tr}(\nabla_X A) + \sum_{k=1}^{n} \varepsilon_k e_k (\nabla_{e_k} A)(X)\Big)\Big)\psi \\ =& \Big(4(n-1)\lambda^2 X + (\mathrm{tr}\, A)A(X) - A^2(X) \\ &+ \Big(\nabla_X(\mathrm{tr}\, A) + \sum_{k=1}^{n} \varepsilon_k e_k (\nabla_{e_k} A)(X)\Big)\Big)\psi.\end{aligned}$$

□

The next lemma relates the scalar curvature of M to the tensor A.

Lemma 3.3.3. *Let (M, g) be pseudo-Riemannian spin manifold endowed with a real (weakly) harmful structure (φ, ψ). Then*

$$\mathrm{scal}^g \psi = (4n(n-1)\lambda^2 - \mathrm{tr}(A^2) + (\mathrm{tr}\, A)^2)\psi - 2(d\,\mathrm{tr}\, A + \delta^g A) \cdot \psi. \tag{3.13}$$

Proof. By (3.12), one can write

$$\begin{aligned}-\,\mathrm{scal}^g \psi = \sum_{j=1}^{n} \varepsilon_j e_j \,\mathrm{Ric}(e_j)\psi = \sum_{j=1}^{n} \varepsilon_j e_j \Big[& 4(n-1)\lambda^2 e_j + (\mathrm{tr}\, A)A(e_j) - A^2(e_j) \\ &+ \Big(\nabla_{e_j}(\mathrm{tr}\, A) + \sum_{k=1}^{n} \varepsilon_k e_k \,(\nabla_{e_k} A)(e_j)\Big)\Big]\psi.\end{aligned}$$

All terms are straightforward to compute, except the last one that yields

$$\begin{aligned}\sum_{j=1}^{n}\varepsilon_j e_j \sum_{k=1}^{n}\varepsilon_k e_k\,(\nabla_{e_k}A)(e_j) &= \sum_{j,k=1}^{n}\varepsilon_j\varepsilon_k(-e_k\,e_j - 2\langle e_j, e_k\rangle)\,(\nabla_{e_k}A)(e_j)\\ &= -\sum_{j,k=1}^{n}\varepsilon_k\Big(e_k\,\big(\varepsilon_j e_j\,(\nabla_{e_k}A)(e_j)\big) + 2\varepsilon_j^2\delta_{jk}\,(\nabla_{e_k}A)(e_j)\Big)\\ &=\sum_{k=1}^{n}\varepsilon_k e_k\,\mathrm{tr}\,\big(\nabla_{e_k}A\big) - 2\sum_{k=1}^{n}\varepsilon_k(\nabla_{e_k}A)(e_k)\\ &= d\,\mathrm{tr}\,A + 2\delta^g A.\end{aligned}$$

Putting everything together it follows that

$$\mathrm{scal}^g\,\psi = \big(4n(n-1)\lambda^2 + (\mathrm{tr}\,A)^2 - \mathrm{tr}(A^2)\big)\psi - 2\big(d\,\mathrm{tr}\,A + \delta^g A\big)\cdot\psi. \qquad \square$$

As an immediate consequence

Corollary 3.3.4. *On a Riemannian spin manifold, any real weakly harmful structure is harmful.*

Proof. Write (3.13) as

$$\big(\mathrm{scal}^g - 4n(n-1)\lambda^2 - (\mathrm{tr}\,A)^2 + \mathrm{tr}(A^2)\big)\psi = -2(d\,\mathrm{tr}\,A + \delta^g A)\cdot\psi;$$

this equation has the form $f\psi = X\cdot\psi$, which implies that $f = X = 0$ since

$$f^2\psi = fX\cdot\psi = Xf\Psi = X\cdot X\psi = -|X|^2\psi,$$

and ψ is nowhere zero. Thus

$$\mathrm{scal}^g = 4n(n-1)\lambda^2 - \mathrm{tr}(A^2) + (\mathrm{tr}\,A)^2, \quad d\,\mathrm{tr}\,A + \delta^g A = 0. \qquad \square$$

Remark 3.3.1. Notice that positive definiteness of g is essential in the proof of Corollary 3.3.4, as otherwise the vanishing of $|X|^2$ would not imply the vanishing of X. Notice also that considering an imaginary weakly harmful structure rather than a real one would make an imaginary unit appear, invalidating the argument.

Analogous results to Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.3.3 can be proved for imaginary harmful structures; the proofs are completely analogous. I summarize these results in the following:

Lemma 3.3.5. *Assume that (M, g) is an n-dimensional pseudo-Riemannian spin manifold with an imaginary harmful structure. Then:*

- *the curvature satisfies*

$$\begin{aligned}\mathcal{R}^M_{XY}\psi =& \frac{1}{2}\Big(A(X)A(Y) + g\big(A(Y), A(X)\big) - iF(X,Y)\Big)\psi\\ &+ 2\lambda^2(YX + g(X,Y))\psi;\end{aligned}$$

- *the Ricci operator satisfies*

$$\begin{aligned}\operatorname{Ric}(X)\psi =&\Big(4(n-1)\lambda^2 X - (\operatorname{tr} A)A(X) + A^2(X)\Big)\psi\\ &- i\left(\nabla_X(\operatorname{tr} A) + \sum_{k=1}^{n} \varepsilon_k e_k (\nabla_{e_k} A)(X)\right)\psi;\end{aligned}$$

- *the scalar curvature satisfies*

$$\operatorname{scal}^g \psi = (4n(n-1)\lambda^2 + \operatorname{tr}(A^2) - (\operatorname{tr} A)^2)\psi + 2i(d\operatorname{tr} A + \delta^g A)\cdot\psi.$$

I can now prove the main result of this section. It can be viewed as a generalization of a result of [1] for generalized Killing spinors in Riemannian manifolds; this results differs in that it allows nonzero λ, though the proof is similar.

Proposition 3.3.6. *Let (M, g) be a real analytic pseudo-Riemannian spin manifold of dimension n and signature (r, s) with a real (resp. imaginary) harmful structure (φ, ψ). Then (M, g) can be embedded isometrically in a pseudo-Riemannian Einstein manifold (Z, h) of signature $(r+1, s)$ (resp. $(r, s+1)$), with constant scalar curvature $4n(n+1)\lambda^2$.*

Proof. It is sufficient to apply Corollaries 3.1.2 or 3.1.3 appropriately; $d\operatorname{tr} A+\delta^g A$ is zero by assumption, and the scalar curvature satisfies (3.3) or (3.4) thanks to Lemma 3.3.3 and Lemma 3.3.5. □

3.4 Spinor extension

In this section, I improve the results of the previous section, showing that the spinors defining the harmful structure actually extend to Killing spinors on (Z, h).

As the arguments for the real and imaginary case are quite similar, I will give the complete proof of only the first one, hence $h(\nu, \nu) = 1$. Throughout this section, let (M, g) be a real analytic pseudo-Riemannian manifold with a real harmful structure (ψ, φ), and consider M to be embedded into an Einstein manifold (Z, h). Following [1], I will exploit the fact that a spinor on Z is Killing if and only if it is parallel relative to the modified connection

$$\tilde\nabla_X \Phi = \nabla_X \Phi - \lambda X \cdot \Phi.$$

Now define a spinor Ψ on Z by parallel transport of ψ (n even) or $\psi - \nu\varphi$ (n odd) relative to $\tilde\nabla$ along the geodesics tangent to ν. Clearly, since $\tilde\nabla_\nu \Psi = 0$, one gets that

$$\nabla_\nu \Psi = \lambda\nu\cdot\Psi. \tag{3.14}$$

Hence Ψ extends ψ and it satisfies the Killing equation for ν at least. The next part is not as trivial. One starts by computing $\tilde\nabla^Z_X \Psi_{|(M,0)}$, i.e. the restriction of $\tilde\nabla^Z_X \Psi$ to M

seen as a hypersurface embedded in Z. One needs to consider the even and odd case separately: the former gives

$$\begin{aligned}\tilde{\nabla}^Z_X\Psi_{|(M,0)} &= \nabla^Z_X\Psi_{|(M,0)} - \lambda X\cdot\Psi = \nabla^M_X\psi - \frac{1}{2}\nu\cdot A(X)\cdot\Psi - \lambda X\cdot\Psi\\ &= \lambda X\odot\varphi + \frac{1}{2}A(X)\odot\psi - \frac{1}{2}A(X)\odot\psi - \lambda X\cdot\Psi\\ &= \lambda\nu\cdot X\cdot(\nu\cdot\psi) - \lambda X\cdot\Psi_{|(M,0)} = \lambda X\cdot\psi - \lambda X\cdot\psi = 0,\end{aligned}$$

while the latter is

$$\begin{aligned}\tilde{\nabla}^Z_X\Psi_{|(M,0)} =&\nabla^Z_X\Psi_{|(M,0)} - \lambda X\cdot\Psi = \nabla^M_X(\psi-\nu\varphi) - \frac{1}{2}\nu\cdot A(X)\cdot\Psi - \lambda X\cdot\Psi\\ =&\frac{1}{2}A(X)\odot\psi + \lambda X\odot\varphi - \nabla^M_X\nu\odot\varphi - \nu\left(\lambda X\odot\psi - \frac{1}{2}A(X)\odot\varphi\right)\\ &-\frac{1}{2}A(X)\odot(\psi-\nu\varphi) - \lambda\nu\odot X\odot(\psi-\nu\varphi) = 0.\end{aligned}$$

Thus, the restriction of Ψ to M is parallel with respect to this connection both in the even and in the odd case. Following [6], I prove that Ψ is Killing by showing that $\tilde{\nabla}_X\Psi$ is zero for all vector fields X on Z obtained by extending a vector field on M by parallel transport along ν, meaning that $\nabla_\nu X = 0$; this condition implies

$$[X,\nu] = \nabla_X\nu = -A(X).$$

Throughout this section, the vector fields denoted by X or Y will be assumed to be of this type.

In order to show that $\tilde{\nabla}_X\Psi$ vanishes on Z, it will be sufficient to prove

$$\nabla_\nu\tilde{\nabla}_X\Psi = 0, \tag{3.15}$$

as $\tilde{\nabla}_X\Psi$ is identically zero on M.

Lemma 3.4.1. *Fix a spinor ψ on M and consider its extension Ψ to Z via $\tilde{\nabla}$-parallel transport along ν. Then*

$$\nabla_\nu\tilde{\nabla}_X\Psi = \mathcal{R}_{\nu X}\Psi + 2\lambda^2\nu X\Psi + \lambda\nu\tilde{\nabla}_X\Psi + \tilde{\nabla}_{A(X)}\Psi.$$

Proof. One has

$$\begin{aligned}\nabla_\nu\tilde{\nabla}_X\Psi &= \nabla_\nu\nabla_X\Psi - \lambda\big(\nabla_\nu X\Psi + X\nabla_\nu\Psi\big) = \nabla_\nu\nabla_X\Psi - \lambda^2 X\nu\Psi\\ 0 = \nabla_X\tilde{\nabla}_\nu\Psi &= \nabla_X\nabla_\nu\Psi - \lambda\big(\nabla_X\nu\Psi + \nu\nabla_X\Psi\big) = \nabla_X\nabla_\nu\Psi + \lambda\big(A(X)\Psi - \nu\nabla_X\Psi\big).\end{aligned}$$

Thus, subtracting the second one from the first one obtains

$$\begin{aligned}\nabla_\nu\tilde{\nabla}_X\Psi &= \mathcal{R}^Z_{\nu X}\Psi + \nabla_{A(X)}\Psi - \lambda^2 X\nu\Psi - \lambda\big(A(X)\Psi - \nu\nabla_X\Psi\big)\\ &= \mathcal{R}^Z_{\nu X}\Psi + \lambda\big(\lambda\nu X\Psi + \nu\nabla_X\Psi\big) + \tilde{\nabla}_{A(X)}\Psi\\ &= \mathcal{R}_{\nu X}\Psi + 2\lambda^2\nu X\Psi + \lambda\nu\tilde{\nabla}_X\Psi + \tilde{\nabla}_{A(X)}\Psi\end{aligned}$$

□

Recall that (Z,h) is an Einstein manifold, that is $\operatorname{Ric}^Z = ch$, where $c = 4n\lambda^2$. Following [1], I define the sections L, P of $(\nu^\perp)^* \otimes \Sigma Z$ and a section Q of $\bigwedge^2(\nu^\perp)^* \otimes \Sigma Z$ as

$$P(X) = \mathcal{R}^Z_{\nu X}\Psi + 2\lambda^2 \nu X\Psi, \qquad L(X) = \tilde{\nabla}_X \Psi,$$
$$Q(X,Y) = \mathcal{R}^Z_{XY}\Psi + 2\lambda^2\big(XY + \langle X,Y\rangle\big)\Psi,$$

and note that by Lemma 3.4.1

$$(\nabla_\nu L)(X) = \nabla_\nu \tilde{\nabla}_X \Psi = P(X) + \lambda\nu L(X) + L(A(X)). \tag{3.16}$$

The strategy is to show that L, P, Q satisfy a linear, homogeneous PDE; zero is a solution, so by uniqueness one deduces that L vanishes identically. It will simplify a bit the argument to observe that P can be obtained from Q by means of a contraction, so that the PDE can be expressed in terms of L and Q alone.

Lemma 3.4.2. *The sections P and Q are related by*

$$P(X) = \nu\varepsilon_j e_j Q(e_j, X).$$

Proof. Writing (3.11) as $\frac{1}{2}\operatorname{Ric}(Y)\psi = \sum_{k=1}^n \varepsilon_k e_k \cdot \mathcal{R}_{e_k Y}\psi + \nu R_{\nu Y}\psi$, one has

$$\begin{aligned}
\nu R_{\nu Y}\psi &= \frac{1}{2}\operatorname{Ric}(Y)\psi - \varepsilon_j e_j R_{e_j Y}\psi \\
&= 2\lambda^2 n Y\psi - \varepsilon_j e_j Q(e_j, Y) + 2\lambda^2 \varepsilon_j e_j e_j Y\psi + 2\lambda^2 \varepsilon_j e_j \langle e_j, Y\rangle\psi \\
&= 2\lambda^2 Y\psi - \varepsilon_j e_j Q(e_j, Y),
\end{aligned}$$

and by multiplying by ν one gets $P(X) = \nu\varepsilon_j e_j Q(e_j, X)$. □

The previous lemma shows that P is obtained from Q by a contraction, so the right-hand side of (3.16) can be expressed in terms of L and Q. The derivative of Q along ν is given by the following:

Proposition 3.4.3. *The section Q satisfies*

$$\nabla_\nu Q(X,Y) = \nu\varepsilon_j e_j\big((\nabla_X Q)(e_j, Y) - (\nabla_Y Q)(e_j, X)\big) + L_2(X,Y). \tag{3.17}$$

where L_2 depends linearly on L and Q.

Proof. By Lemma 3.4.2 one has

$$\begin{aligned}
\nabla_X(\mathcal{R}_{\nu Y}\Psi) =& \nabla_X(\nu\varepsilon_j e_j Q(e_j, Y) - 2\lambda^2 \nu Y\Psi) \\
=& \nu\varepsilon_j e_j\big((\nabla_X Q)(e_j, Y) + Q(\nabla_X e_j, Y) + Q(e_j, \nabla_X Y)\big) \\
&- A(X)\varepsilon_j e_j Q(e_j, Y) + \nu\varepsilon_j(\nabla_X e_j)Q(e_j, Y) \\
&+ 2\lambda^2\big(A(X)Y\Psi - \nu\nabla_X Y\Psi - \nu Y\nabla_X\Psi\big) \\
=& \nu\varepsilon_j e_j(\nabla_X Q)(e_j, Y) + 2\lambda^2\big(A(X)Y\Psi - \nu(\nabla_X Y)\Psi - \lambda\nu Y X\Psi\big) + U(X,Y),
\end{aligned}$$

where

$$\begin{aligned} U(X,Y) =& \nu\varepsilon_j\Big(e_j\big(Q(\nabla_X e_j, Y) + Q(e_j, \nabla_X Y)\big) + (\nabla_X e_j)Q(e_j, Y)\Big) \\ &- A(X)\varepsilon_j e_j Q(e_j, Y) - 2\lambda^2\nu Y L(X) \end{aligned}$$

depends linearly on L and Q; notice that U also depends on the connection form.

On the other hand,

$$\begin{aligned} \nabla_X(\mathcal{R}_{\nu Y}\Psi) =& (\nabla_X\mathcal{R})_{\nu Y}\Psi - \mathcal{R}_{A(X)Y}\Psi + \mathcal{R}_{\nu\nabla_X Y}\Psi + \mathcal{R}_{\nu Y}(L(X) + \lambda X\Psi) \\ =& (\nabla_X\mathcal{R})_{\nu Y}\Psi - Q(A(X), Y) + 2\lambda^2\big(A(X)Y + \langle A(X), Y\rangle\big)\Psi \\ &+ \nu\varepsilon_j e_j Q(e_j, \nabla_X Y) - 2\lambda^2\nu(\nabla_X Y)\Psi + \mathcal{R}_{\nu Y}L(X) \\ &+ \lambda\mathcal{R}_{\nu Y}X\cdot\Psi + \lambda X\big(\nu\varepsilon_j e_j Q(e_j, Y) - 2\lambda^2\nu Y\Psi) \\ =& (\nabla_X\mathcal{R})_{\nu Y}\Psi + 2\lambda^2\big(A(X)Y + \langle A(X), Y\rangle\big)\Psi - 2\lambda^2\nu(\nabla_X Y)\Psi \\ &+ \lambda\mathcal{R}_{\nu Y}X\cdot\Psi - 2\lambda^3 X\nu Y\Psi + V(X,Y), \end{aligned}$$

where

$$V(X,Y) = -Q(A(X), Y) + \nu\varepsilon_j e_j Q(e_j, \nabla_X Y) + \mathcal{R}_{\nu Y}L(X) + \lambda X\nu\varepsilon_j e_j Q(e_j, Y)$$

depends linearly on L and Q; notice that V also depends on the connection form and the curvature.

Equating the terms and isolating $(\nabla_X\mathcal{R})_{\nu Y}\Psi$ one obtains

$$\begin{aligned} (\nabla_X\mathcal{R})_{\nu Y}\Psi =& \nu\varepsilon_j e_j(\nabla_X Q)(e_j, Y) + 2\lambda^2\big(A(X)Y\Psi - \nu(\nabla_X Y)\Psi - \lambda\nu Y X\Psi\big) \\ &- 2\lambda^2\big(A(X)Y + \langle A(X), Y\rangle\big)\Psi + 2\lambda^2\nu(\nabla_X Y)\Psi - \lambda\mathcal{R}_{\nu Y}X\cdot\Psi \\ &+ 2\lambda^3 X\nu Y\Psi + U(X,Y) - V(X,Y) \\ =& \nu\varepsilon_j e_j(\nabla_X Q)(e_j, Y) - 2\lambda^2\langle A(X), Y\rangle\Psi + 4\lambda^3\nu\langle X, Y\rangle\Psi \\ &- \lambda\mathcal{R}_{\nu Y}X\cdot\Psi + S(X,Y), \end{aligned}$$

where $S(X,Y) = U(X,Y) - V(X,Y)$ depends linearly on L and Q.

Finally, one computes

$$\begin{aligned} \nabla_\nu Q(X,Y) =& (\nabla_\nu\mathcal{R})_{XY}\Psi + \lambda\mathcal{R}_{XY}\nu\cdot\Psi + \lambda\nu\mathcal{R}_{XY}\Psi + 2\lambda^3\nu\big(XY + \langle X, Y\rangle\big)\Psi \\ =& (\nabla_X\mathcal{R})_{\nu Y}\Psi - (\nabla_Y\mathcal{R})_{\nu X}\Psi + \lambda\mathcal{R}_{XY}\nu\cdot\Psi + \lambda\nu\mathcal{R}_{XY}\Psi \\ &+ 2\lambda^3\nu\big(XY + \langle X, Y\rangle\big)\Psi \\ =& \nu\varepsilon_j e_j(\nabla_X Q)(e_j, Y) - 2\lambda^2\langle A(X), Y\rangle\Psi + 4\lambda^3\nu\langle X, Y\rangle\Psi - \lambda\mathcal{R}_{\nu Y}X\cdot\Psi \\ &- \nu\varepsilon_j e_j(\nabla_Y Q)(e_j, X) + 2\lambda^2\langle A(Y), X\rangle\Psi - 4\lambda^3\nu\langle Y, X\rangle\Psi + \lambda\mathcal{R}_{\nu X}Y\cdot\Psi \\ &+ \lambda\mathcal{R}_{XY}\nu\cdot\Psi + \lambda\nu\mathcal{R}_{XY}\Psi + 2\lambda^3\nu\big(XY + \langle X, Y\rangle\big)\Psi + S(X,Y) - S(Y,X) \\ =& \nu\varepsilon_j e_j\big((\nabla_X Q)(e_j, Y) - (\nabla_Y Q)(e_j, X)\big) + L_2(X,Y), \end{aligned}$$

where $L_2(X,Y) = \lambda\nu Q(X,Y) + S(X,Y) - S(Y,X)$. □

One is now able to prove the main theorem, which improves Proposition 3.3.6.

Theorem 3.4.4. *Assume (M, g) is a real analytic pseudo-Riemannian spin manifold of signature (r, s) with a harmful structure (ψ, φ). Then:*

- *if (ψ, φ) is real, (M, g) embeds isometrically in a pseudo-Riemannian Einstein spin manifold (Z, h) with signature $(r + 1, s)$ and Weingarten operator A;*
- *if (ψ, φ) is imaginary, (M, g) embeds isometrically in a pseudo-Riemannian Einstein spin manifold (Z, h) with signature $(r, s + 1)$ and Weingarten operator A.*

In both cases ψ extends to a Killing spinor Ψ on Z satisfying $\nabla^Z_X \Psi = \lambda X\Psi$ for any $X \in TZ$.

Proof. The isometric embedding follows from Proposition 3.3.6; as explained at the beginning of this section, one can extend ψ to a spinor Ψ in such a way that (3.14) holds. One only needs to prove that Ψ satisfies the Killing equation; this is equivalent to showing that $L(X) \equiv 0$ on Z. It was shown above that $L(X)$ is zero on $M \times \{0\}$. To see that Q vanishes on $M \times \{0\}$, let X, Y be vector fields on M, and write

$$\begin{aligned}
Q(X, Y) &= \nabla_X \nabla_Y \Psi - \nabla_Y \nabla_X \Psi - \nabla_{[X,Y]}\Psi + 2\lambda^2(XY + \langle X, Y\rangle\Psi \\
&= \nabla_X(\lambda Y\Psi) - \nabla_Y(\lambda\nabla_X\Psi) - \lambda[X, Y]\Psi + 2\lambda^2(XY + \langle X, Y\rangle\Psi \\
&= \lambda(\nabla_X Y + \lambda Y X - \nabla_Y X - \lambda XY - [X, Y] + 2\lambda XY + 2\lambda\langle X, Y\rangle)\Psi = 0.
\end{aligned}$$

Using (3.16) and Proposition 3.4.3 one sees that L and Q satisfy the linear PDE system

$$\begin{cases}
(\nabla_\nu L)(X) = \lambda\nu L(X) + \nu\varepsilon_j e_j Q(e_j, X) + L(A(X)) = L_1(L, Q) \\
(\nabla_\nu Q)(X, Y) = \nu\varepsilon_j e_j\big((\nabla_X Q)(e_j, Y) - (\nabla_Y Q)(e_j, X)\big) = L_2(L, Q).
\end{cases}$$

By the Cauchy-Kowalewskaya Theorem it is known that the solution to the PDE system is unique and, since $L = 0 = Q$ is a solution, it is the only one. In particular $L = 0$ on Z and Ψ is a Killing spinor. □

Theorem 3.4.4 is not quite a generalization of the results of [1] for parallel spinors, in that it entails the extra hypothesis $d\,\mathrm{tr}\,A + \delta A = 0$. However, if one restricts to the Riemannian case, Corollary 3.3.4 can be applied to remove this extra hypothesis:

Corollary 3.4.5. *Assume (M, g) is a real analytic Riemannian spin manifold with a real weakly harmful structure (ψ, φ). Then (M, g) embeds isometrically in a Riemannian spin manifold (Z, h) with Weingarten operator A, and ψ extends to a Killing spinor Ψ on Z satisfying $\nabla^Z_X \Psi = \lambda X\Psi$ for any $X \in TZ$.*

Example 3.4.6. Consider the Lie algebra introduced in Example 1.3.13

$$\mathfrak{g} = (-2e^{23}, 3e^{13} - 3e^{34}, -3e^{12} + 3e^{24}, 2e^{23})$$

and consider on a Lie group G with Lie algebra $\mathfrak{g}$ the metric associated to the scalar product

$$g = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 - e^4 \otimes e^4.$$

Recall from Example 1.3.11 that $\mathrm{Cl}_{3,1} = M(2, \mathbb{H})$, with an orthonormal basis of $\mathbb{R}^{3,1}$ given by

$$\tilde{E}_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \tilde{E}_2 = \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix}, \quad \tilde{E}_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \tilde{E}_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and notice that in this case

$$\omega_{3,1} = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

Furthermore, recall from Example 1.3.13 that on the metric Lie algebra $(\mathfrak{g}, g)$, the spin covariant derivative assumes the form

$$\nabla^{\Sigma G} = \frac{1}{2}\Big[2e^1 \otimes E_2E_3 - e^2 \otimes (E_1E_3 + E_3E_4) + e^3 \otimes (E_1E_2 + E_2E_4) + 2e^4 \otimes E_2E_3\Big].$$

Consider the spinors $\psi = (i, 1, i, 1)$ and $\varphi = (-i, 1, i, -1)$ and the endomorphism

$$A = e^1 \otimes (2e_1 - e_4) + e^2 \otimes e_2 + e^3 \otimes e_3 + e^4 \otimes e_1.$$

Then (ψ, φ) is a harmful structure, that is, they satisfy the system (3.8) with $\lambda = i/2$ and $d\,\mathrm{tr}\,A + \delta A = 0$, and show that $\mathfrak{g}$ extends to a 5-dimensional Einstein manifold endowed with a Killing spinor.

Indeed, it suffices to show that

$$\nabla^{\Sigma G}_X \psi = \frac{1}{2}A(X)\psi + \lambda X \cdot \varphi$$

for any $X \in TG$. Notice that

$$(E_1 - E_2E_3)\psi = (E_2 - E_3E_1)\psi = (E_3 - E_1E_2)\psi = 0,$$

hence, the harmful condition is equivalent to the system

$$E_4\psi - iE_1\varphi = iE_2\varphi - E_3E_4\psi = E_2E_4\psi - iE_3\varphi = iE_4\varphi - E_2E_3\psi = 0.$$

It is easy to see that these are satisfied, hence $\nabla_X\psi = \frac{1}{2}\big(A(X)\psi + iX\varphi\big)$ for any $X \in \mathfrak{g}$. A similar computation shows that $\nabla_X\varphi = \frac{1}{2}\big(iX\psi - A(X)\varphi\big)$.

Now consider the derivation

$$D = 2e^1 \otimes (e_1 - e_5) + e^2 \otimes e_2 + e^3 \otimes e_3.$$

Its symmetric part coincides with A; it follows that the semidirect product $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \mathrm{Span}\,\{e_5\}$ satisfies the equations of Theorem 3.1.1. Explicitly, one can write

$$\tilde{\mathfrak{g}} = (2e^{15} - 2e^{23}, e^{25} + 3e^{13} - 3e^{34}, e^{35} - 3e^{12} + 3e^{24}, -2e^{15} + 2e^{23}, 0),$$

and verify that the metric

$$\tilde{g} = e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3 - e^4 \otimes e^4 + e^5 \otimes e^5$$

is Einstein with $\mathrm{Ric} = -4\,\mathrm{Id}$ and the spinor $\Psi = (i, 1, i, 1)$ is Killing with Killing number $i/2$. In fact, this is a Lorentz-Einstein-Sasaki metric; if one reverses the sign of the metric along the Reeb vector field e_4, one obtains the known η-Einstein-Sasaki metric on the Lie algebra D_{22} in the classification of [38] (see also [9, 2]).

The Lie algebra in this example is isometric (but not isomorphic) via Proposition 1.1.5 to one of the Lie algebras appearing in the list of 5-dimensional $\mathfrak{z}$-standard Einstein-Sasaki Lie algebras of Proposition 2.5.4.

Chapter 4

Diagonalization of the metric of a Lorentzian 3-manifold

In this last chapter, I will present a result concerning the diagonalization problem for Lorentzian metrics of 3-manifolds. Recall from the introduction that, in general, the diagonalization problem amounts to determine if a n-dimensional (pseudo)-Riemannian manifold (M, g) admits an atlas such that, in each chart, the metric assumes diagonal form, i.e. if $(x^1, \ldots, x^n)$ is a set of coordinates of a chart around a point $p \in M$, the metric can be written in the form

$$g = \sum_{i=1}^{n} f_i(x_1, \ldots, x_n) dx^i \otimes dx^i.$$

In the first section, I recall some classical notions and results regarding PDE's systems, in particular existence results concerning symmetric and diagonal hyperbolic systems. Next, similarly to [37], I recast the problem in terms of moving frames by applying Frobenius' Theorem in order to circumvent the appearance of a gauge invariance and fix a reference frame. The unknowns will take the form of functions $b^i_j \in \mathcal{C}^\infty(M, \mathrm{SO}(2,1))$ which will determine the appropriate coframe with respect to the reference one. There are two key aspects to the remainder of the proof. On the one hand, one needs to prove that the linearization of PDE system involving the $\{b^i_j\}$ is diagonal hyperbolic. On the other hand, it must be proved that it is possible to construct non-characteristic initial data for the associated Cauchy problem. The contents of the chapter appear in [72].

4.1 Diagonal hyperbolic systems

I will now recall some useful notions about symmetric hyperbolic and diagonal systems of PDE's, as they are crucial in the proof of the main theorem of the chapter. For the remainder of the section, M will be a compact manifold of dimension $n-1$ with coordinates $(x^1, \ldots, x^{n-1})$, while $X = M \times [0, 1]$ will have coordinates $(x^1, \ldots, x^{n-1}, t) =$

(x', t). A first order differential operator

$$P \colon \mathcal{C}^\infty(X, \mathbb{R}^m) \to \mathcal{C}^\infty(X, \mathbb{R}^m)$$

is called *symmetric hyperbolic* if it can be written in the form

$$P = \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} A^i \frac{\partial}{\partial x^i} + B, \quad A^i, B \in \mathcal{C}^\infty(X, \mathrm{End}(X, \mathbb{R}^m))$$

for some coordinates (x', t) on X, where each A^i is symmetric. These type of operators are useful as the following holds

Proposition 4.1.1 ([48, Section 6])**.** *Let P be a symmetric hyperbolic operator and consider a function $f \in \mathcal{C}^\infty(X, \mathbb{R}^m)$. Then the system*

$$\begin{cases} Pu = f \\ u(x', 0) = u_0 \end{cases}$$

admits a solution for any $u_0 \in \mathcal{C}^\infty(M, \mathbb{R}^m)$, and such solution is unique.

Interestingly, something similar can be said about non-linear operators. These operators are usually of the form

$$\frac{\partial u}{\partial t} = F\left(x', t, u, \frac{\partial u}{\partial x^1}, \dots, \frac{\partial u}{\partial x^{n-1}}\right),$$

where $F \in \mathcal{C}^\infty(V, \mathbb{R}^m)$ and $V \subset X \times \mathbb{R}^m \times \mathbb{R}^{(n-1)m}$ is open. Considering the natural splitting

$$T^*_x X = T^*_{x'}(M \times \{t\}) \oplus \mathbb{R}dt$$

the set V can be rewritten as $V = \bigcup_x V_x$, where V_x is open and not empty in the space $\mathbb{R}^m \times T^*_{x'}(M \times \{t\}) \times \mathbb{R}^m$, and, if $\frac{\partial u}{\partial x^\alpha} = u_\alpha$, in V each point is described by $(x, u, u_\alpha dx^\alpha)$, with $u, u_\alpha \in \mathbb{R}^m$. Then a first order differential operator $\frac{\partial u}{\partial t} = F(x, u, u_\alpha dx^\alpha)$ is said to be *symmetric hyperbolic* if its linearization

$$\frac{\partial u}{\partial t} - \sum_\alpha \frac{\partial F}{\partial u_\alpha} \frac{\partial}{\partial x^\alpha}$$

is symmetric hyperbolic. In this case, the matrices $\left[\frac{\partial F^i}{\partial u^j_\alpha}\right]_{ij}$ correspond to the A_α of the first definition. An existence result for symmetric hyperbolic systems, which can be found for example in [37], is the following

Theorem 4.1.2. *Let $F \colon V \to \mathbb{R}^m$ define a symmetric hyperbolic system. If $u_0 \colon M \to \mathbb{R}^m$ satisfies*

$$\left(x', 0, u_0(x'), \frac{\partial u_0}{\partial x^\alpha}(x') dx^\alpha\right) \in V \quad \forall x' \in M,$$

then there exists $\varepsilon > 0$ *and a unique smooth function* $u\colon M \times [0,\varepsilon] \to \mathbb{R}^m$ *such that for all* $x \in M \times [0,\varepsilon]$ *the equality*

$$\frac{\partial u}{\partial t} = F\left(x, u(x), \frac{\partial u}{\partial x^\alpha}(x)dx^\alpha\right)$$

holds and $u(x',0) = u_0(x')$ *on* M.

If one deforms the differential operator F in a suitable way, it is possible to obtain a solution that agrees with the solution of F on M. More precisely

Theorem 4.1.3. *Let* F *be a differential operator as in the previous theorem and let* $u \in \mathcal{C}^\infty(X,\mathbb{R}^m)$ *solve*

$$\frac{\partial u}{\partial t} = F\left(x, u, \frac{\partial u}{\partial x^\alpha}dx^\alpha\right)$$

on all X. *Then, if* G *is in a neighborhood of* F *in the* $\mathcal{C}^\infty(V,\mathbb{R}^m)$*-topology, there exists a unique solution* $v\colon X \to \mathbb{R}^m$ *of*

$$\frac{\partial v}{\partial t} = G\left(x, v(x), \frac{\partial v}{\partial x^\alpha}(x)dx^\alpha\right)$$

such that $v(x',0) = u(x',0)$ *for all* $x' \in M$.

The proof of both theorems can be found for example in Hamilton's paper [50]. A particular case of symmetric hyperbolic systems is given by the following. Let $D_1,\ldots,D_m$ be vector fields on X such that for any $1 \le \alpha \le m$ and $(x',t) \in X$ the vector $D_\alpha(x',t)$ is not tangent to $M \times \{t\}$. Furthermore, let $B \in \mathcal{C}^\infty(X,\mathrm{End}(\mathbb{R}^m))$, then the differential operator

$$P = \begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_m \end{pmatrix} + B$$

is called *diagonal hyperbolic*.

Proposition 4.1.4. *A diagonal hyperbolic operator as above is equivalent to a symmetric hyperbolic operator.*

Proof. In the coordinates $(x^1,\ldots,x^{n-1},t)$ one can write

$$D_\alpha = D_\alpha^n \frac{\partial}{\partial t} + \sum_{i=1}^{n-1} D_\alpha^i \frac{\partial}{\partial x^i}$$

for $1 \le \alpha \le m$. As by hypothesis the D_α's are not tangent to $M \times \{t\}$ and are globally defined, D_α^n never vanish. Hence,

$$\begin{pmatrix} D_1^n & & 0 \\ & \ddots & \\ 0 & & D_m^n \end{pmatrix}^{-1} P$$

is symmetric hyperbolic. □

Clearly the previous result means that Theorems 4.1.2 and 4.1.3 hold for diagonal hyperbolic systems too. Now, recall that the *symbol* of a linear operator

$$P = \sum_{i=1}^{n} A^i(x)\frac{\partial}{\partial x^i} + B(x)$$

is the linear map defined for $\xi = (\xi_1, \dots, \xi_n) \in T_x^*X$ as

$$\sigma_P(\xi) = \sum_{i=1}^{n} A^i(x)\xi_i, \quad \sigma_P\colon (x,\xi) \to \mathrm{End}(\mathbb{R}^m).$$

The map σ_P is well defined on TX and does not depend on the coordinates. The *characteristic variety* of the operator P is the set of $(x, \xi) \in T^*X$ such that $\sigma_P(\xi)$ is not invertible.

Example 4.1.5. Let P be a diagonal hyperbolic operator as before. Then the fibers of the characteristic variety of P are m hyperplanes counted with multiplicity.

4.2 Orthogonal coordinates on Lorentzian manifolds

The theorem that I will have proved by the end of the chapter is the following

Theorem 4.2.1. *Let (M, g) be a smooth Lorentzian 3-manifold. Then M admits an orthogonal atlas.*

Let $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ be an orthonormal frame for (M, g) and $(\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3)$ be the corresponding coframe. These will be the (co)frames with respect to which the orthogonal atlas will be described. The goal is to find a triplet of coordinated functions (x_1, x_2, x_3) such that, if $e_i = \partial_i$ is the coordinated frame of (x_1, x_2, x_3), then $g(e_i, e_j) = 0$ every time $i \neq j$. There are two main difficulties that one faces both in the Riemannian and in the Lorentzian setting. Assume (y^1, y^2, y^3) are fixed coordinates and $g(y)$ is the metric tensor with respect to this chart; then the coordinated frame $\{e_i\}$ can be written as

$$e_i = \frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha}$$

and hence the PDE system to be solved is

$$0 = g(\partial_i, \partial_j) = \sum_{\alpha,\beta=1}^{3} \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} g_{\alpha\beta}(y) \quad \text{for } i \neq j. \tag{4.1}$$

This system is nonlinear, and its linearization is not symmetric hyperbolic, which means that the standard results of existence of the solution do not apply. Furthermore, there is an invariance in the solution if the unknowns are the coordinates: assume $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$

are other coordinates, such that $\tilde{x}^i = f^i(x^i)$ and each f^i is a strictly monotone function. Then

$$0 = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial f^i}{\partial x^i}\frac{\partial f^j}{\partial x^j} g\left(\frac{\partial}{\partial \tilde{x}^i}, \frac{\partial}{\partial \tilde{x}^j}\right)$$

and hence also $(\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ are orthogonal coordinates.

For this reason it works best if one does not set the unknowns to be the coordinated functions (x_1, x_2, x_3), but the normalized coframe $(\omega^1, \omega^2, \omega^3)$, where $\omega^i = f^i dx^i$, no sum intended, and $f^i = 1/|dx^i|$. Applying Frobenius Integration Theorem (see [44, Section 7.3]) it is easy to get an equivalent condition to the existence of the coordinated charts given the coframe, that is there exist 1-forms ϑ^i_j such that

$$d\omega^i = \sum_j \omega^j \wedge \vartheta^i_j;$$

condition satisfied by the connection 1-forms $\vartheta^i_j = \omega^i_j$. Furthermore, if indeed the coframe exists, then it holds that $\omega^i = f^i dx^i$, hence

$$\omega^i \wedge d\omega^i = \omega^i \wedge df^i \wedge dx^i = \frac{1}{f^i}\omega^i \wedge df^i \wedge \omega^i = 0 \tag{4.2}$$

must hold, when $i = 1, 2, 3$. Due to the signature of the metric, in the Riemannian case the connection is skew-symmetric, hence $\omega^j_i = -\omega^i_j$ for any i, j, while in the Lorentzian case the symmetry is $\mathfrak{so}(2,1)$, hence

$$\omega^2_1 = -\omega^1_2, \quad \omega^3_1 = \omega^1_3, \quad \omega^3_2 = \omega^2_3. \tag{4.3}$$

Thus (4.2) becomes

$$\begin{aligned}
\omega^1 \wedge \omega^2 \wedge \omega^1_2 + \omega^1 \wedge \omega^3 \wedge \omega^1_3 = 0\\
\omega^1 \wedge \omega^2 \wedge \omega^1_2 + \omega^2 \wedge \omega^3 \wedge \omega^2_3 = 0\\
\omega^1 \wedge \omega^3 \wedge \omega^1_3 + \omega^2 \wedge \omega^3 \wedge \omega^2_3 = 0,
\end{aligned}$$

and, by alternatively subtracting one and adding the other, the system becomes

$$\omega^1 \wedge \omega^2 \wedge \omega^1_2 = 0, \quad \omega^2 \wedge \omega^3 \wedge \omega^2_3 = 0, \quad \omega^1 \wedge \omega^3 \wedge \omega^1_3 = 0. \tag{4.4}$$

The next step is to write ω^i with respect to $\bar{\omega}^j$ and vice-versa as

$$\omega^i = b^i_j \bar{\omega}^j, \quad \bar{\omega}^j = \bar{b}^i_j \omega^i$$

and solve for the b^j_i in (4.4). I first note that

$$\begin{aligned}
\omega^l \wedge \omega^i_l = d\omega^i = d\sum_j b^i_j \bar{\omega}^j &= \sum_j \left(\sum_k \bar{e}_k(b^i_j)\bar{\omega}^k \wedge \bar{\omega}^j + b^i_j \bar{\omega}^k \wedge \bar{\omega}^i_k\right)\\
&= \sum_{j,k} \bar{\omega}^k \wedge \left(\bar{e}_k(b^i_j)\bar{\omega}^j + b^i_j \bar{\omega}^i_k\right)\\
&= \sum_{j,k,l} \omega^l \wedge \left(b^l_k \bar{e}_k(b^i_j)\bar{\omega}^j + b^l_k b^i_j \bar{\omega}^i_k\right).
\end{aligned}$$

As a consequence of (4.3), the previous equality becomes

$$\omega_2^1 = \sum_{j,k} \frac{1}{2}\big\{b_k^2 \bar{e}_k(b_j^1) - b_k^1 \bar{e}_k(b_j^2)\big\}\bar{\omega}^j + b_k^2 b_j^1 \bar{\omega}_k^1,$$

$$\omega_3^1 = \sum_{j,k} \frac{1}{2}\big\{b_k^3 \bar{e}_k(b_j^1) + b_k^1 \bar{e}_k(b_j^3)\big\}\bar{\omega}^j + b_k^3 b_j^1 \bar{\omega}_k^1$$

and

$$\omega_3^2 = \sum_{j,k} \frac{1}{2}\big\{b_k^3 \bar{e}_k(b_j^2) + b_k^2 \bar{e}_k(b_j^3)\big\}\bar{\omega}^j + b_k^3 b_j^2 \bar{\omega}_k^2.$$

Notice that in the paper by DeTurck and Yang, all the terms in the braces had a minus sign, again by the $\mathfrak{so}(3)$ symmetry. Making the substitution in (4.4), the equations become

$$0 = \sum_{i,l,j,k} b_i^1 b_l^2 \bar{\omega}^i \wedge \bar{\omega}^l \wedge \left[\frac{1}{2}\big\{b_k^2 \bar{e}_k(b_j^1) - b_k^1 \bar{e}_k(b_j^2)\big\}\bar{\omega}^j + b_k^2 b_j^1 \bar{\omega}_k^1\right]$$
$$0 = \sum_{i,l,j,k} b_i^1 b_l^3 \bar{\omega}^i \wedge \bar{\omega}^l \wedge \left[\frac{1}{2}\big\{b_k^3 \bar{e}_k(b_j^1) + b_k^1 \bar{e}_k(b_j^3)\big\}\bar{\omega}^j + b_k^3 b_j^1 \bar{\omega}_k^1\right]$$
$$0 = \sum_{i,l,j,k} b_i^2 b_l^3 \bar{\omega}^i \wedge \bar{\omega}^l \wedge \left[\frac{1}{2}\big\{b_k^3 \bar{e}_k(b_j^2) + b_k^2 \bar{e}_k(b_j^3)\big\}\bar{\omega}^j + b_k^3 b_j^2 \bar{\omega}_k^2\right]$$

where the unknowns of the system are $(b_i^j) \in C^\infty(M, \mathrm{SO}(2,1))$.

The second to last step is to prove that the linearization of this system is diagonal hyperbolic. Consider the linearization $\beta_j^i = (\delta b)_j^i$ and notice that one can assume that $\{\bar{\omega}^i\} = \{\omega^i\}$ when linearizing around $\{\omega^i\}$, thus having $b_j^i(x) = \delta_j^i$. Hence, the linearized system is

$$0 = \delta_i^1 \delta_l^2 \frac{1}{2}\big(\delta_k^2 \bar{e}_k(\beta_j^1) - \delta_k^1 \bar{e}_k(\beta_j^2)\big)\bar{\omega}^i \wedge \bar{\omega}^l \wedge \bar{\omega}^j + \text{ lower order terms in } \beta$$
$$0 = \delta_i^1 \delta_l^3 \frac{1}{2}\big(\delta_k^3 \bar{e}_k(\beta_j^1) + \delta_k^1 \bar{e}_k(\beta_j^3)\big)\bar{\omega}^i \wedge \bar{\omega}^l \wedge \bar{\omega}^j + \text{ lower order terms in } \beta$$
$$0 = \delta_i^2 \delta_l^3 \frac{1}{2}\big(\delta_k^3 \bar{e}_k(\beta_j^2) + \delta_k^2 \bar{e}_k(\beta_j^3)\big)\bar{\omega}^i \wedge \bar{\omega}^l \wedge \bar{\omega}^j + \text{ lower order terms in } \beta$$

in which the only non-zero elements are

$$\frac{1}{2}\big(\bar{e}_2(\beta_3^1) - \bar{e}_1(\beta_3^2)\big) = \text{ terms of order 0 in } \beta,$$
$$\frac{1}{2}\big(\bar{e}_3(\beta_2^1) + \bar{e}_1(\beta_2^3)\big) = \text{ terms of order 0 in } \beta,$$
$$\frac{1}{2}\big(\bar{e}_3(\beta_1^2) + \bar{e}_2(\beta_1^3)\big) = \text{ terms of order 0 in } \beta.$$

As $(b^i_j(x)) \in \mathrm{SO}(2,1)$ their linearization satisfy $(\beta^i_j) \in \mathfrak{so}(2,1)$, hence the system takes the form

$$\begin{pmatrix} \bar{e}_1 & & \\ & \bar{e}_2 & \\ & & \bar{e}_3 \end{pmatrix} \begin{pmatrix} \beta^2_3 \\ \beta^1_3 \\ \beta^1_2 \end{pmatrix} = \begin{pmatrix} \text{terms of order 0 in } \beta \\ \text{terms of order 0 in } \beta \\ \text{terms of order 0 in } \beta \end{pmatrix}.$$

Hence, the linearized differential operator for the system (4.4) is

$$A = \bar{e}_1 + \bar{e}_2 + \bar{e}_3$$

that is diagonal hyperbolic as $\{\bar{e}_i\}$ is a frame, and its symbol, for $\bar{e}_i = a^j_i(x)\partial_j$, is

$$\sigma_A(\xi) = \begin{pmatrix} a^j_1(x)\xi^j & 0 & 0 \\ 0 & a^j_2(x)\xi^j & 0 \\ 0 & 0 & a^j_3(x)\xi^j \end{pmatrix} \in \mathrm{End}(\mathbb{R}^3).$$

To finally prove that the metric is diagonalizable it remains to find a solution to the Cauchy problem given by the differential operator A and a set of initial data to be chosen. To do so, the data needs to be not characteristic for the operator. It is easy to see that the symbol σ_A is not invertible at a point $x \in M$ for a covector $\xi \in T^*_x M$ if and only if $\xi(\bar{e}_i) = 0$ for some $i = 1, 2, 3$. Thus, the fiber of the characteristic variety of A at a point x is composed of three planes $\pi_1, \pi_2, \pi_3 \subset T^*_x M$ through 0 in general position. Hence, the initial data for the Cauchy problem associated to the system can be given as the coframe $\{\bar{\omega}^i\}$ on a surface $\imath\colon \Sigma \hookrightarrow M$ such that for each $x \in \Sigma$ it holds

$$\dim(\imath_* T^*_x \Sigma \cap \pi_k) = 1 \text{ for each } k = 1, 2, 3.$$

This is equivalent to requiring that no element of the frame $\{\bar{e}_i\}$ is tangent to Σ. Since the linearization of (4.4) is diagonal hyperbolic and one can construct non-characteristic initial data, Theorem 4.1.2 applies and Theorem 4.2.1 is proven.

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