Endogenous interdependent preferences in a dynamical contest model

Fausto Cavalli[a], Mario Gilli[1], Ahmad Naimzada[1]

[a]*Department of Economics, Management and Statistics, University of Milano-Bicocca, Milan, Italy*

Abstract

Outcomes observed in laboratory experiments on contests are often not consistent with the results expected by theoretical models, with phenomena that frequently occur like overbidding or persisting oscillations in strategic choices. Several explanations have been suggested to understand such phenomena, dealing primarily with equilibrium analysis. We propose a dynamical model based on the coevolution of strategic choices and agent preferences. Each agent can have non self-interested preferences, which influence strategic choices and in turn evolve according to them. We show that multiple coexisting steady states characterized by non self-interested preferences can exist, and they lose stability as the prize increases, leading to endogenous oscillating dynamics. Finally, with an emphasis on two specific kinds of agents, we explain how overbidding can emerge. The numerical results show a good qualitative agreement with the experimental data.

Keywords: Contest models, Endogenous interdependent preferences, Coevolution of strategies and preferences, Multistability, Non convergent dynamics

1. Introduction

The variety of economic, social and political situations that can be described by contest models has been extensively analyzed by theoretical and applied social scientists. The first, seminal model of contests was proposed by Tullock [1], who examined a game in which players, exerting an effort, compete for a prize. The model introduced in [1] has a unique symmetric Nash equilibrium at which efforts are proportional to the prize. Predictions of this model have been extensively tested in controlled laboratory experiments[1]. In particular, in contest experiments (see [3] and references therein) it is observed that players typically select strategies that differ from those at Tullock Nash equilibrium $\mathbf{x}^*$. Many experiments show that players exert an actual average effort that can even amount or exceed the double of $\mathbf{x}^*$, a phenomenon known as "overdissipation" or "overbidding". Even if in a small set of experiments, underbidding has also been observed, in particular when pro-social behavior among contestants is promoted. Furthermore, strategic choices exhibit persistent erratic oscillations.

The classic Tullock approach cannot explain the aforementioned phenomena, thus several explanations have been proposed[2]. A first family of these grounds on behavioral theories. For example, Sheremeta [5] explained the overbidding observed in comparing different experimental results of several Tullock contest models in terms of noise in rational decision making. This can be ascribed to many behavioral and demographic factors, which may be the source of the differences in the behavior of players, as well as their proneness to make mistakes. Baharad and Nitzan [6] proposed an explanation based on probability distortions, modelled according to the prospect theory of Kahneman and Tversky [7]. Results in [6] were improved by Sheremeta and Zhang [8] and Chowdhury et al. [9] introducing an autocorrelation bias in winning probability evaluations, a sort of "hot-hand" phenomenon that is well-known in the literature about

Email addresses: fausto.cavalli@unimib.it (Fausto Cavalli), mario.gilli@unimib.it (Mario Gilli), ahmad.naimzada@unimib.it (Ahmad Naimzada)

[1]For a comprehensive survey, see Dechenaux *et al.* [2].

[2]For a deeper insight of the following contributions, we refer to Section 2 in [4].

gambling. Another research strand investigated the possibility of adjusted utilities for the agents, who did not make their decisions solely on the basis of the material payoff. For example, Sheremeta [10] suggested that agents can take into account an additional value induced by the utility of winning, which leads them to overestimate the prize. Fonseca [11] introduced a concern for other player payoff in the utility of each player, reconsidering the idea by Konrad [12] and allowing for altruistic/spiteful players. In [11] players have other regarding preferences and are inequality averse, so their utility reduces as payoffs become different. However, none of the previous approaches introduce a link between the dynamical evolution of preferences and strategies. Concerning experiments in which underbidding has been observed, we mention [13, 14, 15].

We stress that experimental studies also highlighted the occurrence of erratic fluctuations in agent choices, which, in the aforementioned literature, are explained in terms of exogenous noise. However, Wärneryd in [16] argued that the intrinsic nonlinearity encompassed in contest models may prevent the agents from learning to converge. By analogy with the well-known dynamical behavior characterizing nonlinear Cournot games, he noted that if the agents adopt simple rules to choose their strategies (e.g., best response mechanisms in which they are not able to know their opponent future choices), complex dynamics may arise and may be used to explain erratic trajectories of strategies.

The approaches adopted in the aforementioned literature have some limitations and unsatisfactory facets, and leave open several questions that deserve investigation. Among them, the following are particularly important and represent the research motivations behind the present contribution. Is it possible that strategic behaviors that are not consistent with the Tullock Nash equilibrium do endogenously emerge from the dynamical evolution, and not from exogenous, ex-ante assumptions on the characterization of the players? What elements do foster robust and persistent underbidding and, in particular, overbidding phenomena? In fact, in general, preferences are exogenously assigned to the agents, and do not evolve depending on what the agents experiment during the play of the contest, and hence their effect on the strategic behavior does not change. Besides those models in which some evolutionary robustness of the considered kinds of agents is studied, it is not even clear if the non self-interested preferences can endogenously emerge and last. Moreover, it is interesting to understand if it's possible to explain fluctuations in the choices of the agents in terms of endogenous oscillations and what is the role of out-of-equilibrium dynamics in the selection of strategic behaviors characterized by overbidding? In the modelling carried on in the mentioned contributions, out-of-equilibrium dynamics are not even taken into account, and it is consequently not possible to study non convergent trajectories and the emergence of the oscillations observed in laboratory experiments.

The theoretical approach we pursue to tackle the previous questions is based on a dynamical coevolution of strategic choices and preferences, considering agents who, in addition to the classic self-interested preferences, can have both positive or negative regard for their opponents. Agent preferences are influenced by the competitor behavior, which in turn affects strategic choices. We introduce a general setting that allows us to formulate a four-dimensional discrete dynamical model, for which we study possible steady states and their dynamical properties. Concerning static analysis, the main result is that possibly multiple coexisting steady states characterized by non self-interested preferences and either overbidding or underbidding can endogenously emerge. Regarding dynamics, we show that the prize value has an unambiguous destabilizing effect on steady states, leading to persistent oscillations in strategic choices. Finally, we introduce two particular kinds of agents, modelled in terms of the way they evaluate their competitor behaviors, for which we study, also with the aid of numerical simulations, the possible out-of-equilibrium dynamics, the path dependency of dynamical outcomes and their selection depending on initial configurations. In particular, for the class of inequality averse agents, we show that dynamics characterized by overbidding strategies endogenously emerge from the coevolution of preferences and strategic behavior even if no steady states characterized by overbidding are possible. This also occurs even if we consider agents that initially have both a positive regard for their competitors and exert very low efforts. Numerical results show a good agreement with experimental evidence.

We remark that our focus is limited to contests of the Tullock type, since they have mostly attracted experimental researchers. Moreover, we do not aim to reproduce quantitative characteristics of outcomes of contest experiments, which can differ depending on the adopted setting. The purpose is to provide a theoretical explanation and understanding of some key phenomena and to reproduce them qualitatively.

The remainder of the paper is organized as follows. In Section 2, we present the coevolutive model with endogenous interdependent preferences in a general setting. The static analysis is carried on in Section 3. In Section 4, we study the stability of possible steady states for the general model, then we introduce two particular kinds of agents, for which we specialize analytical results and which we numerically investigate. Finally, we conclude and suggest directions for future research. Proofs are collected in Appendix.

2. A dynamical model with endogenously evolving interdependent preferences

We consider an infinitely repeated contest in which two players (indexed by $i = 1$ and 2) compete at each game stage (indexed by discrete time sequence $t \geq 0$) for a prize $v > 0$, exerting efforts $x_{i,t} \geq 0$ at time t and facing homogeneous and constant marginal costs $c > 0$. Without loss of generality, we can normalize marginal cost parameter[3], setting $c = 1$. We recall that in the classic static setting proposed by Tullock, expected payoff of player i is expressed by

$$E(\pi_i(x_i, x_{-i})) = \begin{cases} v\dfrac{x_i}{x_i + x_{-i}} - x_i & (x_i, x_{-i}) \neq (0,0), \\ \dfrac{v}{2} & x_i = x_{-i} = 0, \end{cases} \tag{1}$$

where x_{-i} can be interpreted as the actual effort of player $-i$ or as player $-i$ effort as expected by player i. In the present dynamical setting, we adopt the latter meaning, and hence the expected payoff that, at the end of stage t, each player expects for next game stage at $t+1$ can be described by

$$E(\pi_i(x_{i,t+1}, x^e_{-i,t+1})|I_{i,t+1}) = \begin{cases} v\dfrac{x_{i,t+1}}{x_{i,t+1} + x^e_{-i,t+1}} - x_{i,t+1} & (x_{i,t+1}, x^e_{-i,t+1}) \neq (0,0), \\ \dfrac{v}{2} & x_{i,t+1} = x^e_{-i,t+1} = 0, \end{cases} \tag{2}$$

where

- $x^e_{-i,t+1}$ is the effort that agent i expects his competitor will exert at time $t+1$,
- $x_{i,t+1}$ is the effort of player i at time $t+1$, and
- $I_{i,t+1}$ is the information of player i at the end of stage t, before playing stage $t+1$.

We stress that if we proportionally rescale both efforts, the larger they are, the smaller are expected payoffs.

2.1. Interdependent preferences

First of all, we assume that players have *other regarding (interdependent) preferences*, i.e. their expected utility for period $t+1$ is a linear combination between his own and his competitor material payoff function

$$E\left(U_i(x_{i,t+1}, x^e_{-i,t+1})|I_{i,t+1}\right) = E\left(\pi_i\left(x_{i,t+1}, x^e_{-i,t+1}\right)|I_{i,t+1}\right) + \omega_{i,t}E\left(\pi_{-i}\left(x_{i,t+1}, x^e_{-i,t+1}\right)|I_{i,t+1}\right), \tag{3}$$

where $\omega_{i,t} \in (-1, 1)$ is the weight given at time t by agent i to the expected payoff of the competitor.

According to the terminology used by Levine [17], $U_i(x_{i,t+1}, x^e_{-i,t+1})$ is the *adjusted utility* of player i, which reflects player's own utility and his regard for the opponent, while $\omega_{i,t}$ is the *coefficient of altruism*. In particular,

- if $\omega_{i,t} > 0$, the player is referred to as altruistic, as such a player has a positive regard for the opponent;
- if $\omega_{i,t} = 0$, the player is referred to as selfish, corresponding to the usual case;
- if $\omega_{i,t} < 0$, the player is referred to as spiteful, as such a player has a negative regard for the opponent;

[3]Setting $c = 1$ actually corresponds to consider prize v as a relative prize with respect to costs.

- the assumption that $\omega_{i,t} \in (-1, 1)$ means that no player has a higher (positive or negative) regard for the opponent than for himself.

We stress that the coefficient of altruism encompasses both altruistic and spiteful inclinations. An increase in the coefficient of altruism could then involve either more altruistic or less spiteful preferences, as well as its decrease can mean either less altruistic or more spiteful preferences. As a consequence of this, saying that a player more (respectively, less) altruistically behaves encompasses both the case in which the positive regard for his opponent increases (respectively, decreases) and that in which the negative regard for his opponent decreases (respectively, increases), and this accordingly affects his strategic behavior.

Obviously, coefficient $\omega_{i,t}$ is not independent of the units in which utility is measured, and utility must be measured in the "same"interpersonally comparable units for all players. Finally, the linearity of the adjusted utility in the opponent's payoff may be taken as a convenient approximation. A similar approach in oligopoly modelling can be found in [18].

As previously summed up, standard theory applied to contests assumes that players are selfish in the sense that they only care about their own expected prize. The alternative line of analysis pursued in the present contribution is that players are not actually selfish, but they also consider the other player payoff. In alternative contest theory, it is frequently discussed that some notion of fairness plays a role in individual decision making. For example, Rabin [19] has proposed a formal model of what this might mean. However, the model we propose, even if similar in spirit to Rabin's model, more radically departs from the ordinary assumptions of game theory, because we suppose that player altruistic or spiteful preferences endogenously evolve through time because of the past play of the contest. This notwithstanding, the spirit is similar to the psychological game approach in [20, 19], even if the formal model we propose is completely different. In psychological games, the starting point is that player attitudes toward other players depend on how they feel they are being treated, and this aspect is modelled assuming that the utility of players depends not only on strategies, but also on their beliefs. The resulting game is then analyzed using equilibrium theory, for example, considering consistency among first and higher order beliefs. In the present contribution, we suppose that players do not care if their opponents play "fairly,"but rather if they are seen as nice people. Fairness usually requires defining an "exogenous" reference value (e.g., average payoff) related to which the agents establish if the competitor behavior was fair or not. Conversely, the approach we pursue is based on a mechanism of endogenous coevolution of strategic behavior and preferences, which depends on how a player evaluates his opponent choices. This does not require introducing a reference value encompassing "fairness", but it, likewise, retains the flavor that player weights on opponent materialistic payoffs depend both on his own coefficient of altruism or spitefulness (similar to the approach based on beliefs), but they will also evolve through time. The point we wish to make with this approach is that the phenomena observed during experiments do not seem to simply reflect social preferences, but their adjustment through time on the basis of observed behavior.

2.2. Endogenous dynamics

We assume that, at the end of each game stage, the efforts played by each agent are disclosed,[4] and hence the information for game stage $t+1$ is $I_{i,t+1} = \{x_{i,t}, x_{-i,t}\}$. In addition to this, we make the following assumptions, in particular on the coevolutive mechanism of strategies $x_{i,t}$ and coefficients $\omega_{i,t}$.

a) the agent strategic behaviors, which are influenced by their interdependent preferences, dynamically evolves on the basis of a *best response mechanism with respect to each agent expectations*, i.e. each agent, at the end of stage t, chooses for stage $t+1$ the strategy that maximizes his adjusted utility:

$$x_{i,t+1} \in \arg\max E\left(U_i(x_{i,t+1}, x^e_{-i,t+1})|I_{i,t+1}\right); \tag{4}$$

b) each player i has *static expectations*, i.e. each agent, at the end of stage t, assumes $x^e_{-i,t+1} = x_{-i,t}$;

[4]A common setting in contest experiments is that, at the end of each game stage, each player is informed about the aggregate exerted effort. Since we consider a two player game, this indeed implies that a player knows the last period strategy of his competitor.

c) the *agent preferences are in turn influenced by the behavior of their competitors.*

In what follows, in describing the dynamical coevolutive mechanism, we assume $x_{i,0} > 0$ and we focus on those trajectories for which $x_{1,t}$ and $x_{2,t}$ do not simultaneously vanish.

2.2.1. Dynamics for the agent's strategic behavior

Based on assumption a), each player chooses his next period strategy adopting a myopic best response to each player's conjecture on opponent's choice, in the sense that each player at each stage maximizes his stage adjusted expected utility $E\left(U_i(x_{i,t+1}, x^e_{-i,t+1})|I_{i,t+1}\right)$. This assumption can be justified as a result of impatience, so to explain why players don't maximize their inter-temporal utility function, thus considering the possibility of influencing the future play of the opponent. If we assumed perfect forecast ($x^e_{-i,t+1} = x_{-i,t+1}$) and time constant interdependent preferences ($\omega_{i,t} = \omega_i$), the dynamical model would become static, as the agent would be able to reach the steady state in just one shot. In this case, the model would be described in terms of a game $\Gamma_{\boldsymbol{\omega}} = (\{1,2\}, [0,+\infty)^2, E(\pi_i(x_i, x_{-i})))$, for which it can be shown that the unique Nash equilibrium $\boldsymbol{x}^* = (x_1^*, x_2^*)$, with

$$x_1^* = \frac{v(1-\omega_1)^2(1-\omega_2)}{(2-\omega_2-\omega_1)^2},\ x_2^* = \frac{v(1-\omega_2)^2(1-\omega_1)}{(2-\omega_2-\omega_1)^2},$$

always exists. As a particular situation, if we consider selfish preferences (setting $\omega_i = 0$), we retrieve the classic Tullock game $\Gamma_0 = (\{1,2\}, [0,+\infty)^2,\ E(\pi_i(x_i, x_{-i})))$.

We assume that players have static expectations, i.e. for any $i \in \{1,2\}$, and for any $t \in \mathbb{N}$ we have $x^e_{-i,t+1} = x_{-i,t}$.

According to assumption b), we have that adjusted utility (4) can be written as

$$E(U_i(x_{i,t+1}, x_{-i,t})|I_{i,t+1}) = \frac{x_{i,t+1}}{x_{i,t+1} + x_{-i,t},}v - x_{i,t} + \omega_{i,t}\left(\frac{x_{-i,t}}{x_{i,t+1} + x_{-i,t}}v - x_{i,t}\right). \tag{5}$$

Solving the optimization problem (assumption a)), we get $x_{i,t+1} \in \arg\max E(U_i(x_{i,t+1}, x_{-i,t})|I_{i,t+1})$, so that, for $x_{-i,t} > 0$, we obtain the standard best response mechanism with static expectations

$$x_{i,t+1} = \max\{\sqrt{vx_{-i,t}(1-\omega_{1,t})} - x_{-i,t}, 0\}.$$

We stress that if $x_{-i,t} = 0$, from (5) we have that a whatever small effort would guarantee to player i an expected payoff equal to v, when for a null best response it would reduce to $v/2$. Since we are mainly interested in trajectories for which both players exert a positive effort, we simply assume an exogenous minimum effort ε that is played as the reaction to a null expected strategy. We stress that this is in line with the setting of lab experiments, in which players usually have a discrete set of strategic choices, and hence there is a minimum non null effort level that can be exerted.

In Figure 1 (a-b) we report the graphs of some best response functions on varying the prize and the coefficients of altruism. As the prize increases, the best response to a given effort $x_{-i,t}$ increases as well. This reflects the increasingly high player involvement in the game, as the prize for which they compete is more and more relevant. Moreover, altruistic preferences (i.e. when $\omega_{i,t} > 0$) induce milder responses to the opponent behavior, while the opposite occurs with spiteful preferences. In particular, with altruistic preferences, best response strategy is always smaller than the classic Tullock Nash equilibrium strategy (denoted by asterisks in Figure 1 (a-b)), while larger replies are possible for spiteful preferences.

The best response dynamics is the oldest, most familiar and simple instance of adaptive dynamics, widely studied in economic theory (see e.g., [21]) for a general analysis of this kind of models. Although based on myopic behavior and naive expectation formation, it can be justified within a general frame. In lab experiments, best reply mechanisms often describe dynamical paths well (see [22]). However, the main point of the present contribution is to couple this simple dynamical mechanism with the adaptation of preferences, with coevolution driving the observed player behaviors and allows mimicking the lab results. From this point of view, the use of the well-known best response dynamics coupled with the coevolution of player interdependent preferences allows a deeper understanding of the interaction between the player maximizing behavior, their expectation formation and the endogenous evolution of the coefficient of altruism.

2.2.2. Dynamics for the agent's preferences

There have been a great many experimental studies of infinitely repeated games (see e.g., [23]) and one main takeaway is that in contrast to predictions based on equilibrium analysis, players tend to reciprocate the past behaviors of their opponents. We formalize this by assuming that the coefficients of altruism of each agent evolve according to how they assess whether their competitor has played well or not.

Indeed, the agents do not know altruistic, selfish or spiteful preferences of their competitors, but their strategic choices at stage t depends on such preferences, which consequently affect the probability of winning the prize. So agents can compare their own and their competitor expected payoffs[5] and evaluate how much they liked/disliked their competitor behavior, and, based on that, their preferences consequently adapt.

We stress that it is possible to show that if preferences adapted on the basis of the comparison of the exerted efforts instead of expected payoffs, all the analytical and numerical results we are going to present would not essentially change. The results we present in the remainder of the paper are discussed in terms of expected payoff comparison, which allows providing a simpler and clearer economic interpretation, but they can be rephrased in terms of exerted effort comparison and basically remain identical.

The process of preference evolution can then be outlined in three steps.

a) Agents are informed of the exerted efforts and compare the expected payoffs

In particular, we assume that agents compare the last period expected payoff difference $\Delta\pi_{i,t} = \pi_{i,t} - \pi_{-i,t}$, for which we have

$$\pi_{i,t} = v\frac{x_{i,t}}{x_{i,t} + x_{-i,t}} - x_{i,t} \Rightarrow \Delta\pi_{i,t} = v\frac{x_{i,t} - x_{-i,t}}{x_{i,t} + x_{-i,t}} + x_{-i,t} - x_{i,t} = -\Delta\pi_{-i,t} \; i = 1, 2. \tag{6}$$

The payoff comparison is modelled in additive terms instead of a ratio, a possible viable alternative, following the most common use in the literature, as in [19, 24, 17].

b) Agents evaluate how much they like/dislike the expected payoff difference

The agents positively or negatively evaluate the expected payoff difference $\Delta\pi_{i,t}$. Such evaluation can be made in terms of own convenience, fairness/unfairness evaluation, status-seeking behavior and so on. Since a given scenario can be differently evaluated by distinct agents, the way the expected payoff difference is evaluated defines the *agent kind*, which can be modelled by introducing a function $m : \mathbb{R} \to \mathbb{R}, \Delta\pi_i \mapsto m(\Delta\pi_{i,t})$ which quantifies the evaluation given by agent i to the behavior of agent $-i$, inferred from the expected payoff difference. We investigate some possible relevant kinds of players in Section 4 and the corresponding functional shapes m, but they have in common that the more $m(\Delta\pi_{i,t})$ positively increases, the more agent i liked what he experienced after stage t and, symmetrically, the more $m(\Delta\pi_{i,t})$ negatively decreases, the more agent i disliked it. Such an assessment of the economic observables is actually an evaluation on the competitor behavior, and it is the element on the basis of which the preferences of each player adapt.

We assume that $m(\Delta\pi_{i,t})$ is a strictly monotone function on $(-\infty, 0]$ and on $[0, +\infty)$, respectively. This means that, for example, depending on the kind of player, a positive expected payoff difference can be either evaluated in a positive (like) or negative (dislike) way, but given one of these alternatives, as $\Delta\pi_{i,t}$ increases, the liking/disliking degree increases as well. Finally, we assume that m is a Lipschitz continuous function, i.e. there are no sudden changes in the (marginal) evaluation of the competitor behavior if it slightly changes.

c) Agent preferences adapt accordingly to the evaluation of the expected payoff difference and on their past experience

How preferences change depends not only on the evaluation of the last game outcome, but it is also affected by the player same preferences, i.e. by his identity at stage t, which is a consequence of his history and past experiences. For example, the reaction of a player to a positive evaluation of the competitor behavior in the last stage could considerably change depending on the current player preferences. For example, let us assume that at stage t player 1 negatively evaluates the behavior of his opponent. If player 1 evaluated in a positive way the behavior of player 2 for many periods, so that he currently has a positive

[5]From now on, for the sake of notation, we drop $E(\cdot)$ and we simply denote the expected payoffs by π_i and expected payoff difference by $\Delta\pi_i$.

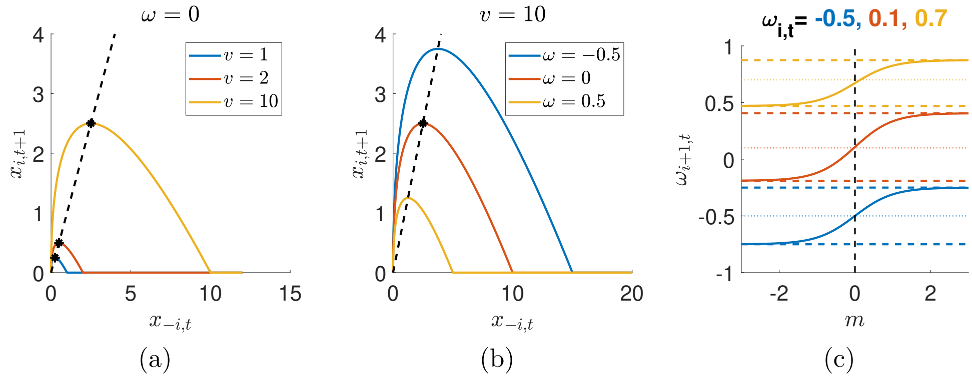


Figure 1: Best response function for different values of prize (panel (a)) and coefficients of altruism (panel (b)). Asterisks denote the classic Tullock Nash equilibrium strategies. Panel (c): graph of function $\tilde{f}$ on varying its former component x, for different fixed values of its latter component. Horizontal dotted lines represent $\omega_{i,t+1} = \omega_{i,t}$.

regard for his opponent, a negative evaluation for one period could weaken such regard, but his preferences will remain characterized in terms of altruism. We could face an opposite situation if player $1's$ preferences evolved toward a negative regard for player 2, as in this case he could become less altruistic. In addition to this, the intensity of the effect of, for instance, a negative evaluation of the expected payoff difference can change depending on how much each player current preferences are polarized in terms of positive or negative regard for his competitor. From the mathematical point of view, this can be described by the process

$$\omega_{i,t+1} = \tilde{f}(m(\Delta\pi_{i,t}), \omega_{i,t}),$$

where $\tilde{f} : \mathbb{R} \times (-1, 1) \to (-1, 1), (m, \omega) \mapsto \tilde{f}(m, \omega)$ is such that, for any given $\omega \in (-1, 1)$, $x \mapsto \tilde{f}(x, \omega)$ is a smooth, strictly increasing function, whose range is $(\sigma_1(\omega), \sigma_2(\omega)) = (\omega - k_1(\omega), \omega + k_2(\omega))$ and which is symmetric with respect to point $(0, \tilde{f}(0, \omega))$. We stress that monotonicity with respect to the former component obviously guarantees that, for each given preference, the higher the evaluation of payoff difference is, the larger the coefficient of altruism is. Moreover, since we want to avoid exogenous biases toward altruism or spitefulness, we introduced suitable symmetry assumptions on the involved modelling functions. Based on this, we can recast function $\tilde{f}$ into the form

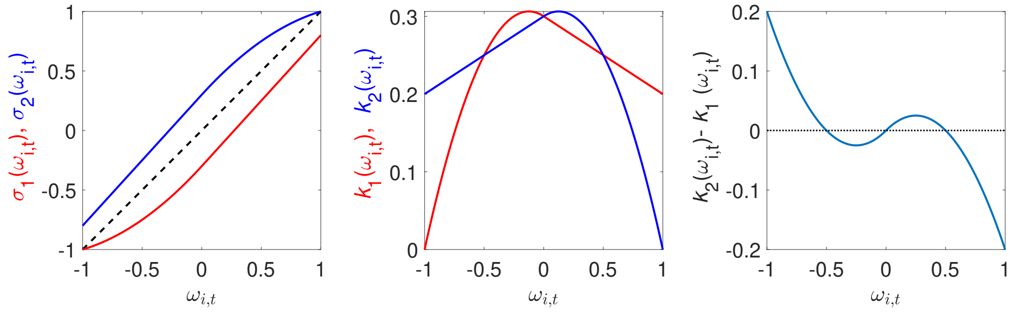
$$\tilde{f}(m, \omega) = \frac{\sigma_2(\omega) - \sigma_1(\omega)}{2} f(m) + \frac{\sigma_1(\omega) + \sigma_2(\omega)}{2} = \omega + \frac{k_1(\omega) + k_2(\omega)}{2} f(m) + \frac{k_2(\omega) - k_1(\omega)}{2}, \tag{7}$$

where $f(m)$ is an odd function[6] (so $f(0) = 0$), for which $\lim_{m \to \pm\infty} f(m) = \pm 1$.

Function $\sigma_1(\omega_{i,t}) = \omega_{i,t} - k_1(\omega_{i,t})$ (respectively, $\sigma_2(\omega_{i,t}) = \omega_{i,t} + k_2(\omega_{i,t})$) represents, given the current preferences, the lower bound (respectively, the upper bound) to the coefficient of altruism that can characterize player i at time $t+1$. In Figure 1 (c) we report three possible graphs of function $\tilde{f}(\cdot, \omega_{i,t})$, for different values of $\omega_{i,t}$. Let us refer to the upper graph in yellow, corresponding to the case of player i with altruistic preferences at time t. The yellow dotted line represents the value at time t of the coefficient of altruism.

Depending on the positive or negative evaluation of the stage outcome (abscissa m) at time t, his coefficient of altruism could increase (part of the curve above the yellow dotted line) or decrease (part of the curve below the yellow dotted line). In such latter case, altruistic preferences could also turn into spiteful ones. We note that in general $\tilde{f}(0, \omega) \neq \omega$, i.e. a neutral evaluation of opponent behavior can lead to a change in the preferences. For example, preferences at time t are represented by the yellow dotted line, but

[6]For the sake of completeness, in the most general case, function $\tilde{f}$ should be rewritten as in (7) but in terms of a function f that explicitly depends on ω. Since taking such dependence into account would not introduce new additional, economically relevant phenomena to those reported in the present contribution, we do not consider dependence of f on ω. This also allows for a more direct and clear explanation of the results.



(a) (b) (c)

Figure 2: Functions σ_1 and σ_2 (panel (a)) for which k_1 and k_2 are not monotonic (panel (b)) and disposition k_2/k_1 is unimodal on $[0,1)$ (panel (c)).

the next period coefficient of altruism $\omega_{i,t+1} = \tilde{f}(0,\omega)$, which graphically corresponds to the intersection between the yellow graph and the vertical dashed line, lies below the yellow dotted line. The opposite occurs for the red graph related to $\omega_{i,t} = 0.1$, while for the blue graph related to $\omega_{i,t} = -0.5$ we have $\tilde{f}(0,\omega) = \omega$. The occurrence of each of these situations is in principle unrelated to the sign of the coefficient of altruism, i.e. they all can hold for both altruistic, spiteful or self-interested preferences. We delve deeper into this point at the beginning of Section 3, when discussing the dynamical evolution of variables of the model. Possible values of coefficients of altruism at time $t+1$ can range between the two dashed yellow lines, respectively representing values $\sigma_1(\omega_{i,t})$ (lower yellow dashed line) and $\sigma_2(\omega_{i,t})$ (upper yellow dashed line). These comments can be adapted to discuss red and blue graphs as well.

In what follows, we refer to term $f(m(\Delta\pi_{i,t}))$ in (7) as *drift toward altruism*, and it encompasses the effect of the evaluation of the stage t outcome on the next period preferences. In line with the discussion on the meaning of coefficient of altruism, a negative value of the drift toward altruism must be intended as a drift toward spitefulness.

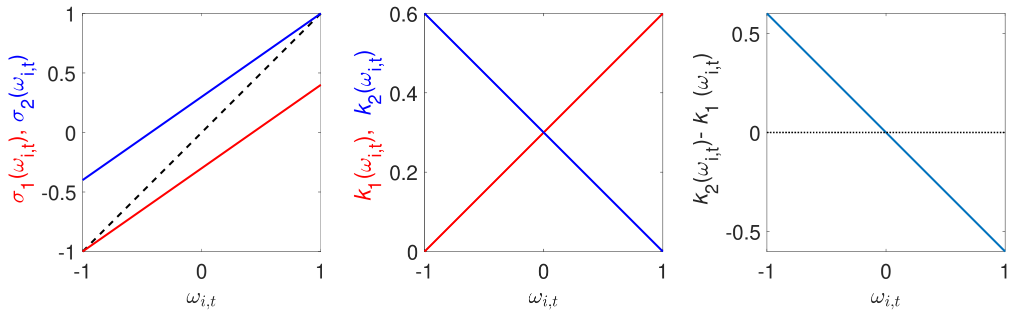
Conversely, in (7), the difference $k_2(\omega_{i,t}) = \sigma_2(\omega_{i,t}) - \omega_{i,t}$ between the maximum next period value of the coefficient of altruism and the current one represents how much the regard for player $-i$ can increase from period t to period $t+1$. Similarly, difference $k_1(\omega_{i,t}) = \omega_{i,t} - \sigma_1(\omega_{i,t})$ between the current value of the coefficient of altruism and the maximum possible at $t+1$ describes how much the regard for player $-i$ can decrease from period t to period $t+1$. They encompass an intrinsic characteristic of the players, which is independent of the behavior of their competitors, but which endogenously depends on the current preferences. In Figures 2 (a-b) and 3 (a-b) we report some possible shapes for functions σ_i and corresponding functions k_i.

Before presenting the model, we make some assumptions on functions σ_i, to take into account only economically relevant functional shapes and to rule out the occurrence of some trivial or uninteresting scenarios. We stress that even under the following assumptions, the static and dynamical results are able to depict the multiplicity of the economically relevant scenarios that are able to describe outcomes of contest experiments. Essentially, no new scenarios can arise by removing such assumptions, but the discussion of the result would simply become more elaborated, as branching, marginal situations should be taken into account.

Firstly, we assume that $\sigma_i(\omega)$, $i = 1,2$, are smooth, strictly increasing functions. This guarantees that, for example, the greater is the positive regard of an agent for his competitor, the more such regard can potentially increase and the less it can decrease. Moreover, we implicitly assume that functions σ_i (and, consequently, k_i as well) are sufficiently regular to compute the required derivatives for the theorem proofs.

Since we want to disregard those functional shapes for σ_i for which dynamical evolution of variables is not depending on a coevolution of preferences and of player strategic behavior, we assume that

$$\sigma_2(\omega) > \omega > \sigma_1(\omega), \text{ for any } \omega \in (-1,1) \text{ and } \lim_{\omega \to \pm 1^{\mp}} \sigma_2(\omega) - \sigma_1(\omega) \neq 0. \quad (8)$$



(a) (b) (c)

Figure 3: Functions σ_1 and σ_2 (panel (a)) for which k_1 and k_2 are monotonic (panel (b)) and disposition k_2/k_1 is strictly decreasing $[0, 1)$ (panel (c)).

Assumption (8) means that, independently of the current agent preferences, in principle he can always become more altruistic and less altruistic. If we allowed for example $\sigma_1(\omega) = \omega$, we would have situations in which the agent cannot become less altruistic, even when facing a previous stage stage outcome that is evaluated very negatively. In addition to this, if we allowed for $\sigma_2(\omega) = \sigma_1(\omega)$ for some $\omega_0 \in (-1, 1)$ this would mean that the evolution of preferences would be "exogenously" locked at ω_0, independently of the evaluation of how much an agent liked/disliked the previous stage outcome. In this case, it is easy to see that (ω_0, ω_0) would be a steady state for the equations governing the evolution of ω for every strategic behavior.

Moreover, the present assumption also rules out that $\sigma_2(\omega) - \sigma_1(\omega) \to 0$ as ω approaches some $\omega_0 \in \{-1, 1\}$. This would mean that the evolution of preferences could converge toward ± 1, independently of the evaluation of how much an agent likes/dislikes the outcome of the previous stage outcome.

In addition to this, in line with the symmetric framework we assumed for the whole game and the model setting, we impose a symmetric behavior for functions k_i when the coefficients of altruism have opposite signs, i.e.

$$k_i(\omega) = k_{-i}(-\omega),\ i = 1, 2,\ \omega \in (-1, 1). \tag{9}$$

This guarantees that if the current coefficient of altruism is positive, the regard for the competitor can decrease as much as it could increase under an opposite value for the coefficient of altruism. We stress that a consequence of (9) is that $k_1(0) = k_2(0)$, i.e. when an agent has selfish preferences, he has neither disposition toward altruism nor toward spitefulness.

Finally, as it will become evident from the analysis in Section 3, the model allows for a multiplicity of equilibria. To simplify the discussion, we will focus on situations in which up to three symmetric equilibria arise, as the explanation of more general situations fall within this prototypical one. To guarantee this, we can assume that either

$$\frac{k_2(\omega)}{k_1(\omega)} \text{ on } \omega \in [0, 1) \text{ is monotonic,} \tag{10a}$$

or

$$\frac{k_2(\omega)}{k_1(\omega)} \text{ on } \omega \in [0, 1) \text{ is unimodal.} \tag{10b}$$

Function $k_2(\omega)/k_1(\omega)$ endogenously determines if an agent is more prone to becoming more altruistic/less spiteful or to becoming more spiteful/less altruistic.

When $k_2(\omega)/k_1(\omega)$ is above (respectively below) 1, we can say that the player has an endogenous *disposition toward altruism* (respectively *toward spitefulness*), since he is potentially more disposed to become more altruistic/less spiteful than to become more spiteful/less altruistic (respectively he is more disposed to become more spiteful/less altruistic than to become more altruistic/less spiteful).

We emphasize that disposition is endogenous, since it can change depending on the current coefficient of altruism, which evolves over time. Moreover, the disposition toward altruism or spitefulness determines the positive or negative sign of $k_2(\omega) - k_1(\omega)$, so in what follows we can discuss the role of the disposition of a player also inspecting the sign of $k_2(\omega) - k_1(\omega)$. For example, Figure 2 (c) depicts a situation in which player i exhibits disposition toward altruism for $0 < \omega_{i,t} < 0.5$ and $\omega_{i,t} < -0.5$ and disposition toward spitefulness for $-0.5 < \omega_{i,t} < 0$ and $\omega_{i,t} > 0.5$, while Figure 3 (c) depicts a situation in which player i exhibits a uniform disposition toward altruism when he is currently spiteful and a uniform disposition toward spitefulness when he is currently altruistic. Assumptions (10) then restrict the maximum possible number of changes in the disposition of a player as his preferences range from spitefulness to altruism.

As it will become evident in the remainder of the paper, drift toward altruism and disposition toward altruism/spitefulness play a crucial role in explaining static and dynamical behaviors of the model.

The graphs of functions σ_i and k_i reported in Figure 2 (a-b) and 3 (a-b) are consistent with all the previous assumptions. In particular, in Figure 2 (b) functions k_i fulfill assumption (10b), whereas in Figure 3 (b) they fulfill assumption (10a).

Finally, we limit the possible heterogeneity between the agents to the initial strategy and/or coefficient of altruism. All the remaining elements will be the same for both players (i.e. functions f, σ_i and m).

The resulting model consists of the following four dimensional dynamical system

$$\begin{cases} x_{1,t+1} = \max\{\sqrt{v x_{2,t}(1-\omega_{1,t})} - x_{2,t}, 0\}, \\ x_{2,t+1} = \max\{\sqrt{v x_{1,t}(1-\omega_{2,t})} - x_{1,t}, 0\}, \\ \omega_{1,t+1} = \omega_{1,t} + \dfrac{k_2(\omega_{1,t}) + k_1(\omega_{1,t})}{2} f(m(\Delta\pi_{1,t})) + \dfrac{k_2(\omega_{1,t}) - k_1(\omega_{1,t})}{2}, \\ \omega_{2,t+1} = \omega_{2,t} + \dfrac{k_2(\omega_{2,t}) + k_1(\omega_{2,t})}{2} f(m(\Delta\pi_{2,t})) + \dfrac{k_2(\omega_{2,t}) - k_1(\omega_{2,t})}{2}, \end{cases} \tag{11}$$

where $\Delta\pi_{i,t}$ are defined by (6) and depend on $x_{i,t}$ and $x_{-i,t}$. Model (11) can be written in a compact way as $\mathbf{s}_{t+1} = F(\mathbf{s}_t)$ by introducing function $F : (0, +\infty)^2 \times (-1, 1)^2 \to \mathbb{R}^4$, $\mathbf{s} = (x_1, x_2, \omega_1, \omega_2) \mapsto F(\mathbf{s}) = F(x_1, x_2, \omega_1, \omega_2)$, defined by the right hand side in (11).

3. Static analysis

Before analyzing the static properties of model (11), we discuss how variables evolve.

Concerning the equations describing the dynamics of the coefficients of altruism, the preferences evolve according to an anchoring-and-adjusting mechanism. The anchor corresponds to the current preferences, encompassed in $\omega_{i,t}$, and it is adjusted according to the evaluation given by players to the outcome of the previous stage (encompassed in $f(m(\Delta\pi_{i,t}))$), tuned by the potential changes in the regard for the opponent, (respectively encompassed in $k_1(\omega_{i,t})$ and $k_2(\omega_{i,t})$).

Let us start considering the case in which no anchoring is present, i.e. the preference evolution is independent of the current preferences. This occurs for $\sigma(\omega_{1,t}) = -1$ and $\sigma(\omega_{2,t}) = 1$. In this situation, the next period coefficient of altruism would be a direct consequence of the evaluation of the expected payoff difference alone, resulting $\omega_{i,t+1} = f(m(\Delta\pi_{i,t}))$.

However, in general, the anchor to past experience acts as a dampening factor for the drift. Assume for now that, given the current preferences, a player is neither disposed toward altruism nor toward spitefulness (i.e. $k_1(\omega_{i,t}) = k_2(\omega_{i,t})$). For example, let us consider current spiteful preferences. We can make reference to the blue graph in Figure 1, for which $k_1(\omega_{i,t}) = k_2(\omega_{i,t})$ holds true. In this situation, a negative (respectively positive) evaluation of the opponent behavior drives the agent preferences toward an increase (respectively decrease) of spitefulness. Due to a conservatism bias, the previous preferences are under-revised with respect to the signal represented by the expected payoff difference, so that the result is just a reduced adaption toward altruism. This mechanism is encompassed in coefficient $0 < (k_2(\omega_{i,t}) + k_1(\omega_{i,t}))/2 \leq 1$ that rescales $f(m(\Delta\pi_{i,t}))$ in the latter couple of equations in (11).

In addition to the drift toward altruism, we can however highlight another mechanism that can lead to a change of preferences, i.e. an endogenous disposition toward altruism or spitefulness. This latter mechanism is encompassed in additive term $(k_2(\omega_{i,t}) - k_1(\omega_{i,t}))/2$, whose sign is determined by the disposition toward altruism/spitefulness $k_2(\omega_{i,t})/k_1(\omega_{i,t})$.

Let us focus for simplicity on the case of a player with spiteful preferences at stage t who neutrally evaluates (neither he likes/nor he dislikes) the stage outcome, so that no endogenous drift toward either altruism or spitefulness is present as a consequence of strategic choices. For the following discussion, we can refer to Figure 2.

Depending on both the current, negative, coefficient of altruism and his disposition toward either altruism or spitefulness, the player could both become less altruistic, less spiteful (or even become altruistic) or he could not change his preferences. If a player exhibits a disposition toward spitefulness, he could become less altruistic, even in the presence of a neutral evaluation of the stage outcome. However, as already mentioned, this can change depending on the current degree of spitefulness/altruism, i.e. on the past experiences. For example, when the actual degree of spitefulness is mild and there is room for a more pronounced spiteful behavior, the player could decide to act less altruistically if he has a disposition toward spitefulness. For example, this occurs in the situation reported in Figure 2 (c) for small, negative coefficients of altruism $\omega_{i,t} \in (-0.5, 0)$. Conversely, if the current level of spitefulness of a player is very high, it could depict a scenario in which past strong drifts toward spitefulness (due to particular strategic behaviors) induced an excess of spitefulness, so that when such drift is no more present, the disposition of the player tends to counterbalance the spitefulness excess and the negative regard toward the opponent decreases. For example, this occurs in the situation reported in Figure 2 (c) for large, negative coefficients of altruism $\omega_{i,t} \in (-1, -0.5)$. Finally, another possible scenario in the presence of an elevated degree of spitefulness could depict a player who is unwilling to change his behavior and lock into spitefulness.

To summarize, preference evolution can be described in terms of an anchoring toward past experience, a drift toward altruism induced by the evaluation of the stage outcome and a path dependent disposition toward becoming more altruistic/spiteful.

We already discussed the role of prize and preferences on the strategies in Section 2.2.1. In light of the dynamical evolution of preferences, we stress that overbidding with respect to the classic Tullock Nash equilibrium strategy can emerge due to the presence of spiteful preferences, which, however, can in turn emerge from the negative evaluation of the strategic behavior of the competitors, giving rise to the dynamical coevolution of strategies and preferences.

Now we study possible steady states $\mathbf{s}^*(x_1^*, x_2^*, \omega_1^*, \omega_2^*)$ of model (11). The first result concerns symmetric steady states, i.e. those for which $x_1^* = x_2^* = x^*$ and $\omega_1^* = \omega_2^* = \omega^*$. In what follows, we refer to a steady state at which $\omega_i \neq 0$ for at least one $i = 1, 2$ as *non self-interested* steady state, while in the opposite case we refer to it as *self-interested steady state.* In the former case, we refer to a steady state in which $\omega_i^* > 0,\ i = 1, 2$ as *altruistic steady state*, meaning that the steady state is characterized by altruistic preferences. Similarly, a steady state in which $\omega_i^* < 0,\ i = 1, 2$ is called *spiteful steady state.*

Proposition 1. *Vector* $(x^*, x^*, \omega^*, \omega^*)$ *is a symmetric steady state for dynamical system* (11) *if and only if*

$$x^* = \frac{v(1-\omega^*)}{4}, \qquad \frac{k_2(\omega^*)}{k_1(\omega^*)} = \frac{1-f(\delta)}{1+f(\delta)}, \tag{12}$$

where we set $\delta = m(0)$.

At least a symmetric steady state always exists for any δ, *and it is the unique one for all* δ *if and only if* $k_2(\omega)/k_1(\omega)$ *is strictly monotonic. In this case,* ω^* *increases as* δ *increases.*

A self-interested steady state is possible if and only if $\delta = m(0) = 0$.

The components of the steady state related to the strategies correspond to the Nash equilibrium of game $\Gamma_{\boldsymbol{\omega}}$ when $\omega_1^* = \omega_2^* = \omega^*$, i.e. are the strategic choices of players whose utility function is the modified utility (3) with exogenous coefficients of altruism ω^*. We remark that x^* is larger (respectively smaller) than the Nash equilibrium strategy of a Tullock contest if and only if it corresponds to the steady state strategies of a spiteful (respectively altruistic) steady state. This means that overbidding (respectively underbidding) at symmetric steady states is a direct consequence of spiteful (respectively altruistic) preferences.

This is in line with the literature about other regarding preferences in lab experiments (see [25]). The relevant facet is that, in the present setting, non self-interested preferences can endogenously emerge from the coevolution of player behaviors and preferences, and are determined by latter condition in (12), in which both identities (i.e. disposition toward altruism/spitefulness) and player evaluation of the stage outcome (i.e. δ) are involved. Moreover, the extent of overbidding and underbidding is magnified by the prize, as evident from the former expression in (12), which is again in agreement with experimental findings (see [3]), in which players more and more overbid as the prize increases.

Let us note that, in such a symmetric scenario, the steady expected payoff differential is null, so $\delta = m(0)$ represents the evaluation that the agents make for a scenario in which $\Delta\pi_i^* = 0$. A consequence of this is that the steady state coefficient of altruism does not depend on v.

The left-hand side in the latter condition in (12) encompasses the player disposition toward either altruism of spitefulness. Conversely, its right hand side is a decreasing function of δ, for which we have $(1 - f(\delta))/(1 + f(\delta)) = 1$ if and only if $\delta = 0$. Moreover, $f(m(\Delta\pi_i))$ represents the coefficient of altruism that a player would adopt just relying on the expected payoff difference, independently of his current preferences. Since in this case case we would have $\sigma_1(\omega) = -1$ and $\sigma_2(\omega) = 1$ and hence $k_1(\omega) = \omega + 1$ and $k_2(\omega) = 1 - \omega$ and noting that the latter condition in (12) can be rewritten as

$$f(\delta) = \frac{k_1(\omega^*) - k_2(\omega^*)}{k_2(\omega^*) + k_1(\omega^*)}, \tag{13}$$

the steady state would be $\omega^* = f(\delta)$, as immediately predictable also from equation (7). This means that $f(\delta)$ represents the steady state coefficient of altruism corresponding to the symmetric scenario in which the player preferences evolve just under the drift induced by the evaluation of the strategic behavior.

Conversely, let us now take into account non constant functions $\sigma_i(\omega)$ and start assuming $\delta = 0$, i.e. we focus on the case for which the evaluation of a symmetric scenario is neutral, neither positive nor negative. In this case, player disposition toward altruism would result in an increase of altruism, since, even if the observation of stage t outcome would not induce any change in the preferences, player disposition would drive preferences toward an increased level of altruism (or reduced spitefulness). The reverse would occur in the opposite situation, with disposition toward spitefulness driving preferences toward a reduced level of altruism (or increased spitefulness). So if a symmetric scenario is neutrally evaluated, a steady state could only realize if there is disposition toward neither altruism nor spitefulness.

In general, a symmetric configuration of coefficients of altruism is a steady state if and only if the two above described mechanisms act in opposite[7] ways and balance out. So a positive evaluation of a symmetric scenario must be counter-balanced by a disposition toward spitefulness (otherwise the two forces would add together, resulting in increasingly altruistic preferences). Similarly, to a negative evaluation of a symmetric scenario must correspond a disposition toward altruism (otherwise spitefulness would increase).

A unique symmetric steady state is possible for all δ if and only if players have disposition toward altruism when they have spiteful preferences and disposition toward spitefulness when they have altruistic preferences, as in the scenario reported in Figure 3. In this case, we can say that players globally have disposition toward self-interest, as if they are currently altruistic (Figure 3 (c) for $\omega_{i,t} > 0$), their disposition toward spitefulness would induce a decrease of coefficient of altruism (the graph in Figure 3 (c) lies below the horizontal axis), with the opposite occurring when they are currently spiteful. If symmetric behaviors are neutrally evaluated (i.e. $\delta = 0$), this would lead to a decrease of the coefficient of altruism when it is positive and an increase when it is negative, so only self-interested preferences can emerge at the steady state. Conversely, if symmetric behaviors are positively or negatively evaluated, this can respectively counterbalance disposition toward spitefulness or for altruism, allowing for the existence of spiteful or altruistic steady states, respectively. For example, if players evaluate that their competitors nicely behaved, they would be disposed, to a certain point, to go against their intrinsic inclination toward self-interested preferences, and they would settle to a positive regard for their opponent. The self-interested steady state then occurs just in a particular situation, which requires that players are also neutral with respect to the evaluation of identical expected payoffs. This actually portrays a situation in which agents are completely self-interested.

Conversely, when players do not globally have disposition toward self-interest, multiple equilibria can occur. Their number and distribution strongly rely on how endogenous disposition toward altruism/spitefulness changes. In particular, the simplest scenario occurs under assumption (10b).

[7] This is the reason for which the right hand side in the latter condition in (12) is a decreasing function of $f(\delta)$.

Corollary 1. *If $k_2(\omega)/k_1(\omega)$ is unimodal for $\omega \in [0,1)$, then there exists $\bar{\delta} > 0$ such that for $\delta \in (-\bar{\delta}, \bar{\delta})$ model* (11) *has three symmetric steady states, with at least an altruistic and a spiteful steady state.*

For $\delta < -\bar{\delta}$ or $\delta > \bar{\delta}$ model (11) *has a unique symmetric steady state, which is characterized in terms of altruism if $\delta > 0$ and of spitefulness $\delta < 0$.*

The scenario of Corollary 1 is that reported in Figure 2. In particular, steady state coefficients of altruism are identified by the intersection between a horizontal line, representing $f(\delta)$, and a cubic-like function like that in Figure 2 (c). In such case, at least a non-self interested steady state always exists, at which either underbidding or underbidding strategies are chosen.

To summarize, all the previous results show that non self-interested steady states can endogenously emerge from the coevolution of player strategic behaviors and preferences.

Let us now focus on the possible existence of non-symmetric steady states, i.e. at which $x_1^* \neq x_2^*$ and $\omega_1^* \neq \omega_2^*$.

Proposition 2. *Vector $(x_1^*, x_2^*, \omega_1^*, w_2^*)$ is an asymmetric steady state for dynamical system* (11) *if and only if*

$$x_2 = \frac{v(1-\omega_1^*)(1-\omega_2^*)^2}{(2-\omega_2^*-\omega_1^*)^2}, \quad x_1^* = \frac{v(1-\omega_1^*)^2(1-\omega_2^*)}{(2-\omega_2^*-\omega_1^*)^2}, \tag{14}$$

and

$$\begin{cases} f(m(\Delta\pi_1^*(\omega_1^*, \omega_2^*))) = \dfrac{k_1(\omega_1^*) - k_2(\omega_1^*)}{k_1(\omega_1^*) + k_2(\omega_1^*)}, \\ f(m(-\Delta\pi_1^*(\omega_1^*, \omega_2^*))) = \dfrac{k_1(\omega_2^*) - k_2(\omega_2^*)}{k_1(\omega_2^*) + k_2(\omega_2^*)}, \end{cases} \tag{15}$$

Moreover, as $v \to +\infty$ we have that asymmetric steady states either vanish or they converge toward a symmetric steady state.

We start noting that (x_1^*, x_2^*) corresponds to the Nash equilibrium of game $\Gamma_{\boldsymbol{\omega}}$.

Condition (15) is a generalization of the latter one in (12) to the case of $\omega_1^* \neq \omega_2^*$. It can be again explained in terms of the balancing between the disposition toward altruism/spitefulness and the drift toward altruism induced by stage outcome evaluation. The main difference between the symmetric and the asymmetric steady state scenarios is that, in this latter one, the two players can have different evaluations of the stage outcome, as $\Delta\pi_1^*(\omega_1^*, \omega_2^*) = -\Delta\pi_2^*(\omega_1^*, \omega_2^*)$. For simplicity, let us assume that player 1 and player 2 evaluate in opposite ways the expected profits, with a negative evaluation by player 1. The drift toward spitefulness of player 1 has the same extent of the drift toward altruism of player 2. Recalling the discussion following Proposition 1, a steady state configuration occurs if player 1 has a disposition toward altruism that counterbalances the drift toward spitefulness. The difference is that now, to be in a steady state configuration, player 2, differently from player 2, must have disposition toward spitefulness, since for player 2 the drift is toward altruism.

We stress that even if several asymmetric equilibria could arise, they are relevant only for suitably small values of the prize. In fact, as v increases, they either vanish or they are close to a symmetric steady state. Since the model is strongly characterized in terms of symmetry (in particular assuming the same kind of players, i.e. function m is the same for both players), it is quite predictable that symmetric steady states play the main role, in particular as the prize increases and asymmetric configurations become hardly sustainable without an exogenous player asymmetry[8].

Therefore, we limit the study of stability to symmetric steady states. We will give some more details about asymmetric steady states in the numerical simulations reported in Section 4.

[8]We stress that the asymptotic behavior of asymmetric equilibria is not due to assumptions (9), which is not used in the proof of Proposition 2.

4. Dynamical analysis

The analysis carried on in Section 3 showed that non self-interested steady states can emerge from the coevolution of preferences and strategies. However, to understand how dynamics select a particular steady state (for example, those spiteful, linked to overbidding phenomena) and their role in the possible emergence of endogenous oscillating dynamics, we must investigate out-of-steady state dynamical behaviors.

We begin to analyze the local asymptotic stability of symmetric steady states, in particular by focusing on the role of the v. We distinguish the general case, in which the two players have non identical initial strategies and preferences, from the very particular one in which both players are identical also in regard to this aspect.

Proposition 3. *Let $\boldsymbol{s}^* = (x^*, x^*, \omega^*, \omega^*)$ be a symmetric steady state of* (11) *and let us define*

$$\theta^*(\omega^*) = f(\delta)\left(\frac{k_1'(\omega^*)}{2} + \frac{k_2'(\omega^*)}{2}\right) + \frac{k_2'(\omega^*)}{2} - \frac{k_1'(\omega^*)}{2}.$$

For general initial conditions, we have that if $-2 < \theta^*(\omega^*) < 0$ there exists $\bar{v}(\omega^*, \theta^*(\omega^*))$ such that $\boldsymbol{s}^*$ is locally asymptotically stable for $v < \bar{v}(\omega^*, \theta^*(\omega^*))$ and unstable for $v > \bar{v}(\omega^*, \theta^*(\omega^*))$.

If $\theta^*(\omega^*) < -2$ or $\theta^*(\omega^*) > 0$, $\boldsymbol{s}^*$ is never locally asymptotically stable.

In the particular case of $\omega_{1,0} = \omega_{2,0}$ and $x_{1,0} = x_{2,0}$, model (11) *is equivalent to a two-dimensional system for which steady state $\boldsymbol{s}^*$ is either unconditionally stable or unstable.*

The previous proposition shows that, as the prize increases, non-convergent dynamics can arise, giving rise to endogenous oscillations in line with what is observed in lab experiments, in which erratic behavior and overbidding more frequently occur when the prize is large [3, 16]. The economic rationale of such phenomena can be ascribed to overreactive behaviors of the agents in the presence of large prizes, self sustained by the coevolution of preferences. If at any time players have either different strategies or preferences, since the best response to the competitor strategic behavior is larger the more the prize increases, we have that a whatever a small discrepancy between the behavior of players is magnified. Moreover, if, for example, a player exerted a small effort, the competitor can be induced to play a large effort. This leads to relevant expected payoff differences, which, depending on the kind of player and his disposition, significantly alter preferences. This in turn affects the strategic behavior of the players, leading to a strong response to the competitor strategy due to the large prize.

In the remained of this section, we introduce two analytic expressions for function m, which describe two particularly economically relevant kinds of players, namely *tit-for-tat* and *inequality averse* agents, for which we specify the previous analytical results and which we numerically investigate.

A tit-for-tat agent positively evaluates a situation in which his expected payoff is larger than that of his competitor, as he recognizes this as a situation in which the competitor nicely behaved in his regards, while he negatively evaluates a situation in which his expected payoff is smaller than that of his competitor, as he recognizes this as a situation in which the competitor badly behaved in his regards. We stress we use "tit-for-tat" for this kind of players by analogy with the homonymous game strategy, in which players replicate competitive and cooperative opponent behaviors. Indeed, in the present contribution, tit-for-tat is related to the way players evaluate their opponent behaviors, and not to a strategic behavior. Moreover, tit-for-tat assumes that a player is initially cooperative, conversely, we do not restrict initial preferences.

To model a tit-for-tat kind of player, a possible simple functional shape is

$$m(\Delta\pi_{i,t}) = \gamma\Delta\pi_{i,t}, \tag{16}$$

where $\gamma > 0$ encompasses the strength of the reaction of the player.

An inequality averse agent evaluates in an increasingly negative way any situation in which the expected payoffs of the two agents are different, since this points out a bad behavior by either himself or his opponent. According to Fehr and Schmidt [26], agents exhibit a dislike toward unfair material outcomes, both if they experience unfairness against them and if they are favored by it. Nevertheless, agents more dislike inequities potentially causing disadvantage to themselves than those causing advantage to them. Mimicking what is used in [26], we consider function

$$m(\Delta\pi_{i,t}) = \delta - \alpha \max\{-\Delta\pi_{i,t}, 0\} - \beta \max\{\Delta\pi_{i,t}, 0\} = \begin{cases} \delta + \alpha\Delta\pi_{i,t} & \text{if } \Delta\pi_{i,t} \leq 0, \\ \delta - \beta\Delta\pi_{i,t} & \text{if } \Delta\pi_{i,t} > 0, \end{cases} \tag{17}$$

to describe how inequality averse player i evaluates expected payoff differences. In (17) $\alpha > \beta > 0$ respectively weight the negative evaluation of player i of the expected material disadvantage $-\Delta\pi_{i,t} = \Delta\pi_{-i,t}$ and of expected material advantage $\Delta\pi_{i,t}$, while $\delta > 0$ represents the evaluation of a completely inequality-free scenario.

Large coefficients α and β with respect to δ points out a strong inequality aversion, and hence the more there are differences in the player behaviors the smaller or more negative the evaluation of the expected payoff difference is. Conversely, small coefficients α and β with respect to δ encompass a mild or small inequality aversion.

In the next two subsections we specialize the results of Proposition 1 and Corollary 1 to tit-for-tat and inequality averse players, and we perform numerical investigations of the possible dynamical behaviors. To better focus on the main aspects and explanation of results, in what follows we always assume that monotonicity assumptions (10) on endogenous disposition hold true. Moreover, without loss of generality, we assume that $f'(0) = 1$, as scenarios arising for different values of $f'(0)$ occur for suitable rescaling of parameters defining function m. For the numerical simulations, we use functions $f(z) = \tanh(z)$ and $\sigma_2(\omega) = \max(\min(a\omega + b, 1), -1)$. Expressions for function k_1, k_2 and σ_1 can be obtained from $k_2(\omega) = \sigma_2(\omega) - \omega, k_1(\omega) = \omega - \sigma_1(\omega)$ and, thanks to assumption (9), from $k_1(\omega) = k_2(-\omega)$.

4.1. Tit-for-Tat players

Firstly, in the next proposition, we specialize general static and dynamical results for the model with two tit-for-tat players.

Proposition 4. *Vector $(x^*, x^*, \omega^*, \omega^*)$ is a symmetric steady state for* (11) *if and only if*

$$x^* = \frac{v(1-\omega^*)}{4}, \quad k_1(\omega^*) = k_2(\omega^*). \tag{18}$$

Under assumption (10a)*, the self-interested steady state is the unique symmetric one, and if $k_2'(0) - k_1'(0) \in (-4, 0)$ it is locally asymptotically stable provided that*

$$v < \frac{4}{\gamma(k_1(0) + k_2(0))}, \tag{19}$$

otherwise it is unconditionally unstable.

Under assumption (10b)*, model* (11) *has three symmetric steady states $\boldsymbol{s}_+^* = (x_a^*, x_a^*, \omega^*, \omega^*)$, $\boldsymbol{s}_0^* = (v/4, v/4, 0, 0)$ and $\boldsymbol{s}_-^* = (x_s^*, x_s^*, -\omega^*, -\omega^*)$, where $\boldsymbol{s}_0^*$ is locally unconditionally unstable and $\boldsymbol{s}_\pm^*$ are locally asymptotically unstable if $k_2'(\omega^*) - k_1'(\omega^*) < -4$, while otherwise they are locally asymptotically stable provided that*

$$v < \frac{4(1 \mp \omega^*)}{\gamma(k_1(\omega^*) + k_2(\omega^*)(1 \pm \omega^*)}. \tag{20}$$

If $\omega_{1,0} = -\omega_{2,0}$, model (11) *reduces to a three-dimensional dynamical system for which $\boldsymbol{s}_0^*$ is locally asymptotically stable provided that*

$$\max\left\{-2 - \frac{k_2'(0)}{2} + \frac{k_1'(0)}{2}, \frac{k_2'(0)}{2} - \frac{k_1'(0)}{2}, 0\right\} < v < \frac{4}{\gamma(k_1(0) + k_2(0))}.$$

First, we start noting that since a tit-for-tat player neutrally evaluates a stage outcome in which players exert identical efforts (and hence no drift toward more/less altruism is present), at symmetric steady states a player must have disposition neither toward altruism nor spitefulness, as encompassed in the latter condition in (18). For comments on steady states existence and uniqueness, we refer to the discussion after Proposition 1 and Corollary 1.

Concerning stability, we highlight the predictable destabilizing effect of the strength of reaction of players to the expected payoff differences. Moreover, the more the coefficient of altruism can potentially increase, the more $\boldsymbol{s}^*$ becomes unstable for smaller values of the prize. Greater $k_j(\omega_i^*)$ potentially allows for larger preference variations, which in turn causes larger changes in strategic behaviors. Effort differences can be relevant on payoffs even in the presence of a small prize, leading to self sustained oscillations in the coevolution of strategies and coefficients of altruism. Conversely, the opposite occurs in the presence of a small potential variations of preferences.

When three steady states exist, noting that factor $(1-\omega^*)/(1+\omega^*)$ is decreasing with respect to ω^*, the stability interval for the altruistic steady state is always smaller than for that spiteful. The explanation for this can be found in the larger payoffs expected when players underbid with respect to when they overbid. Assume that both players have slightly different altruistic preferences, close to those at the altruistic steady state. We recall that at $\mathbf{s}_+$ both players have neither disposition toward altruism nor for spitefulness, since the latter condition in (18) holds true. Due to the difference in the coefficients of altruism, there are inequalities in exerted efforts that induce inequality in expected payoffs. Since altruistic players underbid, their expected payoffs are large even in the presence of a small prize, so reduced effort inequalities can lead to significant payoff differences, which in turn induce relevant changes in the preferences. Since agents have opposite evaluations of the expected payoff differences, one agent becomes more altruistic, the other less altruistic and this goes on until the disposition toward spitefulness of the former player and that for altruism of the latter one become dominating and start counterbalancing the drifts toward altruism induced by stage outcome evaluation, reversing the phenomenon.

Conversely, when the prize is suitably small and agents are spiteful, expected payoffs are small, which induces small payoff differences and this does not significantly affect preferences.

Moreover, the self-interested steady state is locally asymptotically unstable. However, if the initial preferences are characterized in terms of coefficients of altruism with opposite signs, i.e. the degree of altruism of a player exactly balances that of spitefulness of the competitor, $\boldsymbol{s}_0^*$ can attract trajectories for intermediate values of the prize. The reason for which for small values of v convergence to $\boldsymbol{s}_0^*$ is not possible is that in such prize range asymmetric steady states exist, and attract trajectories starting with $\omega_{1,0} = -\omega_{2,0}$. We give numerical evidence and explanation of such phenomena in Figure 4, which is obtained setting $\gamma = 4$ and $\sigma_2(\omega) = \max(\min(1.1\omega + 0.3), 1), -1)$, which provides a unimodal function $k_2(\omega)/k_1(\omega)$. In the first two rows of Figure 4, we have possible steady states as graphical solutions to System (15) and the corresponding basins of attraction, obtained on varying the initial coefficients of altruism and setting $x_{1,0} = 0.1$ and $x_{2,0} = 0.4$ (accurate numerical investigations show that the reported basins of attraction are just marginally affected by the initial strategy choices). We discuss the simulations just for $\omega_2 \geq \omega_1$, due to the symmetric behavior induced by tit-for-tat players. For very reduced prize values (Figure 4 (a)), we have three distinct asymmetric steady states (one of which is P_1, which lines on $\omega_2 = -\omega_1$), in addition to the symmetric steady states $\boldsymbol{s}_\pm^*$ and $\boldsymbol{s}_0^*$ (the self-interested steady state, denoted by an asterisk). Among them, the attractor around P_1 (together with the corresponding one around P_1') and $\boldsymbol{s}_\pm^*$ attract almost any trajectory, as it can be inferred by the corresponding basins of attraction reported in Figure 4 (e). Conversely, steady states P_2 and P_3 do not play a significant active role on dynamics. As v increases, P_2 arrives at P_1 and then moves toward P_3 (Figure 4 (b)), until they merge (Figure 4 (c)) and disappear (Figure 4 (d)). Conversely, as the prize increases, steady state P_1 becomes locally stable and it moves closer to $\boldsymbol{s}_0^*$. Its basin of attraction progressively reduces (Figure 4 (f-h)) to a very small region around line $\omega_2 = \omega_1$. When asymmetric equilibria still exist, convergence toward them is possible with suitably heterogeneous initial preferences, otherwise trajectories converge toward symmetric steady states. When P_1 and P_1' merge with $\boldsymbol{s}_0^*$, it attracts trajectories characterized by opposite coefficients of altruism. The previous considerations are also evident by looking at the bifurcation diagrams reported in Figure 4 (i-l).

We stress that in-depth, numerical investigations with different functions σ_i always showed that asymmetric steady states can exist just for very small values of v or γ. Both these settings are not particularly interesting from the economic point of view, as in the former one we have that the prize is even smaller than

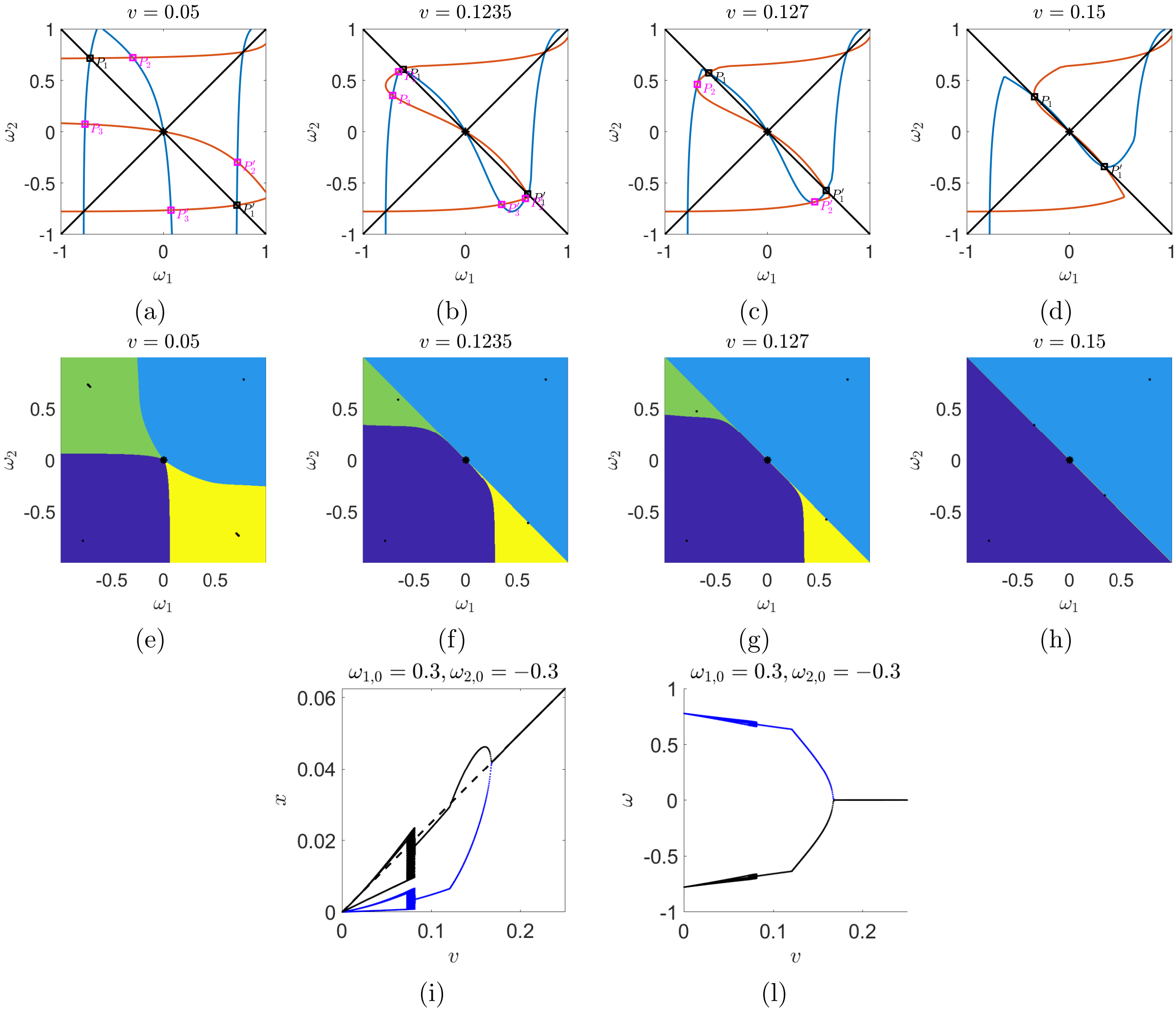
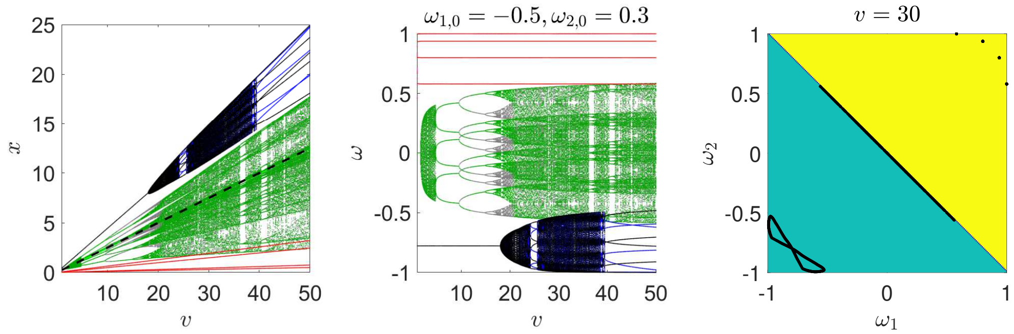


Figure 4: Tit-for-tat players: asymmetric steady states and related dynamics. First row: steady states as intersections of the curves implicitly defined by (15), for different small values of the prize v. Second row: basins of attraction corresponding to the steady state scenarios in row 1. Third row: bifurcation diagrams for variables x_1 and x_2 (panel (i)) and ω_1 and ω_2 (panel (l)), in which black and blue colors are used for player 1 and 2, respectively, while the black dashed line in panel (i) corresponds to the self-interested steady state strategy.

the marginal costs, while in the latter one the players are not particularly involved in the competition, since their reaction to the stage outcomes is very reduced. Both these situations are not realistically compatible with contest lab experiments.

Now we study the possible dynamics arising for prizes larger than marginal costs (i.e. $v > 1$), again setting $\sigma_2(\omega) = \max(\min(1.1\omega + 0.3), 1), -1)$ and $\gamma = 4$. We stress that for such functions $\sigma_i(\omega)$ we have three steady states $\boldsymbol{s}^*_+ = (v/18, v/18, 7/9, 7/9)$, $\boldsymbol{s}^*_0 = (v/4, v/4, 0, 0)$ and $\boldsymbol{s}^*_- = (4v/9, 4v/9, -7/9, -7/9)$, where $\boldsymbol{s}^*_+$ is locally asymptotically stable for $v < 18$, $\boldsymbol{s}^*_-$ is unstable (it would be stable for $v < 0.2812$), while, for $\omega_{1,0} = -\omega_{2,0}$, the self-interested steady state $\boldsymbol{s}^*_0$ is stable for $v < 1.6667$.

In Figure 5 (a-b) we report bifurcation diagrams of variables x_i and ω_i for different initial settings. In all the diagrams the initial strategies are set equal to $x_{1,0} = 3$ and $x_{2,0} = 4$. The black and blue bifurcation diagrams are obtained setting $\omega_{1,0} = -0.5$ and $\omega_{2,0} = 0.3$, i.e. the average coefficient of altruism is negative, and hence the initial preferences are, on average, characterized by spitefulness. For $v < 18$, trajectories converge toward the spiteful steady state, while at $v = 18$ it loses stability and we have numerical evidence



(a) (b) (c)

Figure 5: Tit-for-tat players: symmetric steady states and related dynamics. First row: bifurcation diagrams for variables x_1 and x_2 (panel (a)) and ω_1 and ω_2 (panel (b)), when initial preferences are characterized on average by spitefulness (black and blue colors), by altruism (red and magenta) and by self-interest (gray and green). Basins of attraction (panel (c)) when $v = 30$.

of the occurrence of a Neimark-Sacker bifurcation. We note that quasi-periodic trajectories are always characterized by overbidding (the black/blue bifurcation diagrams in Figure 5 (a) always lie above the dashed line representing the Nash equilibrium of the classic Tullock game) and spiteful preferences (the black/blue bifurcation diagrams in Figure 5 (b) always lie below horizontal line $\omega = 0$, corresponding to self-interested preferences).

Similarly, setting $\omega_{1,0} = -0.3 = \omega_{2,0}$, $\boldsymbol{s}_0^*$ loses stability through a Neimark-Sacker bifurcation at $v = 1.6667$ (green and gray bifurcation diagram). In this case, trajectories of strategies have large oscillations around the Nash equilibrium of the classic Tullock game, with alternating altruistic and spiteful preferences.

Finally, if initial preferences are, on average, characterized by altruism, as for example setting $\omega_{1,0} = 0.5$ and $\omega_{2,0} = -0.3$, we have periodic dynamics around the altruistic steady state, which, as already noted, is unstable when $v > 1$.

Convergence toward the altruistic or spiteful steady state basically depends on the characterization of the initial preferences, on average, in terms of altruism or spitefulness, as evident from the basins of attraction reported in Figure 5 (c). If $\omega_{1,0} + \omega_{2,0} < 0$, trajectories converge toward an attractor characterized by spitefulness, while if $\omega_{1,0} + \omega_{2,0} > 0$ trajectories converge toward an attractor characterized by altruism (and, as analytically proved, only for $\omega_{1,0} + \omega_{2,0} = 0$, convergence is toward an attractor characterized, at least on average, by self-interest). We stress that these phenomena are mainly driven by the initial preferences alone, while they are essentially independent of the initial strategic choices, at least when $x_{i,0}$ are not too extreme to give rise at first to overreaction phenomena in the best response mechanism.

To summarize, both Proposition 4 and numerical investigations show that tit-for-tat players, in reciprocating their competitor behaviors, allow for the emergence of non self-interested dynamics, self-sustained by the coevolution of their preferences and strategic behavior. Both overbidding and underbidding phenomena are possible. However, since contest experiments are characterized by a strong degree of competitiveness and, the initial attitude of players can be encompassed in initial spiteful preferences, a scenario with tit-for-tat kind of players coevolves giving rise to dynamics driven by spiteful preferences, with overbidding phenomena. Conversely, when pro-social attitude is promoted (e.g., [15]), reduced efforts and underbidding are observed. This can be explained in terms of stimulating altruistic preferences, which then bolster dynamics characterized in terms of positive coefficients of altruism and underbidding.

4.2. Inequality averse players

The next proposition presents specialized static and dynamical results for the model with two inequality averse players. Since the conditions defining steady states do not become simpler than those in (15), we avoid repeating them. Moreover, since the discussion on asymmetric steady states with tit-for-tat players is still applicable for inequality averse players, we just focus on symmetric steady states.

Proposition 5. *Under assumption* (10a)*, or under assumption* (10b) *and if* δ *is suitably large, model* (11) *has a unique symmetric steady state* $\boldsymbol{s}^*_+ = (x^*_a, x^*_a, \omega^*, \omega^*)$ *with* $\omega^* > 0$.

Under assumption (10b) *and if* δ *is suitably small, model* (11) *has three steady states* $\boldsymbol{s}^*_+ = (x^*_a, x^*_a, \omega^*_a, \omega^*_a)$, $\boldsymbol{s}^*_{-,1} = (x^*_{s,1}, x^*_{s,1}, \omega^*_{s,1}, \omega^*_{s,1})$, $\boldsymbol{s}^*_{-,2} = (x^*_{s,2}, x^*_{s,2}, \omega^*_{s,2}, \omega^*_{s,2})$ *with* $\omega^*_{s,1} < \omega^*_{s,2} < 0 < \omega^*_a$.

Each of the previous steady states is locally asymptotically stable provided that $k'_2(\omega^*) - k'_1(\omega^*) \in (-4, 0)$ *and*

$$v < \frac{8(1-\omega^*)}{(\alpha-\beta)f'(\delta)(k_1(\omega^*)+k_2(\omega^*))(1+\omega^*)}, \tag{21}$$

in which ω^* *is replaced by the corresponding steady state coefficient of altruism.*

If $k'_2(\omega^*) - k'_1(\omega^*) < -4$ *or* $k'_2(\omega^*) - k'_1(\omega^*) > 0$*, the steady state is unconditionally unstable, in particular* $\boldsymbol{s}^*_{-,2}$ *is always unconditionally unstable.*

Since an inequality averse player positively evaluates a stage outcome at which no inequality is present between player behaviors, it is understandable that a steady state characterized in terms of altruism always exists in a symmetric scenario. However, if the disposition toward altruism/spitefulness non-monotonically depends on the coefficient of altruism, also spiteful steady states can exist, at least for small values of δ, as shown and discussed in Corollary 1.

Concerning stability, we have that the greater is the difference between aversion toward his own material disadvantage with respect to that toward the competitor material disadvantage, the smaller is the threshold prize value after which a steady state becomes unstable. This can be explained as follows. The more α is different from β, the more the effect on the preferences of the evaluation of the stage outcome is different. Note that, differently from tit-for-tat players, inequality averse players can in principle provide the same evaluation of payoff differences (when $\alpha = \beta$), since they both dislike inequalities. However, even if, as $\Delta\pi_i$ diverts from zero, their dislike increases, the more $\alpha - \beta$ is large, the more $\Delta\pi_i$ differently increases for each player. This can amplify the difference between preferences, which can self-sustain due to the consequent inequalities in player strategies that cause differences in expected payoffs.

Once more, an altruistic steady state has, ceteris paribus, a reduced interval of prize values above which instability arises. This is again due to larger payoffs expected by altruistic players, possibly inducing larger payoff differences that, in the case of inequality averse players, self-sustain.

At a first glance, Proposition 5 seems to show that when inequality averse players are involved, dynamics characterized in terms of altruism should dominate. An altruistic steady state always exists, and it can also be the unique one. Even when it becomes unstable, one could expect that oscillating dynamics around positive values of coefficient of altruism should arise, with consequent underbidding in strategies. However, the scenarios that arise can be quite surprising.

We discuss dynamics with the help of numerical simulations. We thoroughly checked, by changing the involved parameters and the shapes of functions σ_i, that the reported simulative results are robust and the consequent comments hold true in general.

We consider $\sigma_2(\omega) = \max(\min(1.1\omega + 0.3), 1), -1)$ and we set $\alpha = 4$ and $\beta = 0.6$ (such values are among those proposed in [26]). If we set $\delta = 0.1$, we have three symmetric steady states, as shown in Figure 6 (a), corresponding to $\boldsymbol{s}^*_{-,1} = (0.43v, 0.43v, -0.72, -0.72)$, $\boldsymbol{s}^*_{-,2} = (0.32v, 0.32v, -0.30, -0.30)$, $\boldsymbol{s}^*_+ = (0.04v, 0.04v, 0.82, 0.82)$.

Moreover, from condition (21), $\boldsymbol{s}^*_+$ is locally asymptotically stable for $v < 0.58$, while $\boldsymbol{s}^*_{-,1}$ for $v < 29.05$. So, the prize values for which $\boldsymbol{s}^*_+$ is stable are very small, in particular less than marginal costs. In Figure 6 (b-c) we report the bifurcation diagrams for strategies and coefficients of altruism considering initial values close to $\boldsymbol{s}^*_+$. Notwithstanding, for $v > 3.3$ trajectories converges toward the spiteful steady state and, when $\boldsymbol{s}^*_{-,1}$ becomes unstable, toward an attractor at which preferences are always characterized by spitefulness and strategies by overbidding. As evident from Figure 6 (d-h), the spiteful steady state $\boldsymbol{s}^*_{-,1}$ attracts trajectories that also start from initial altruistic preferences. We stress that the basins reported in Figure 6 (d-h) are obtained by setting initial strategies close to $0.04v$, i.e. to the steady state strategies related to $\boldsymbol{s}^*_+$. The attractor arising from the loss of stability of $\boldsymbol{s}^*_+$ very quickly grows and as $v \to\sim 3.2$ it tends to collide with the boundary of the basin of attraction of $\boldsymbol{s}^*_{-,1}$, and hence to disappear. So, even for quite small values of the prize, $\boldsymbol{s}^*_+$ does not play any economically significant role in the dynamics (we recall that, in agreement

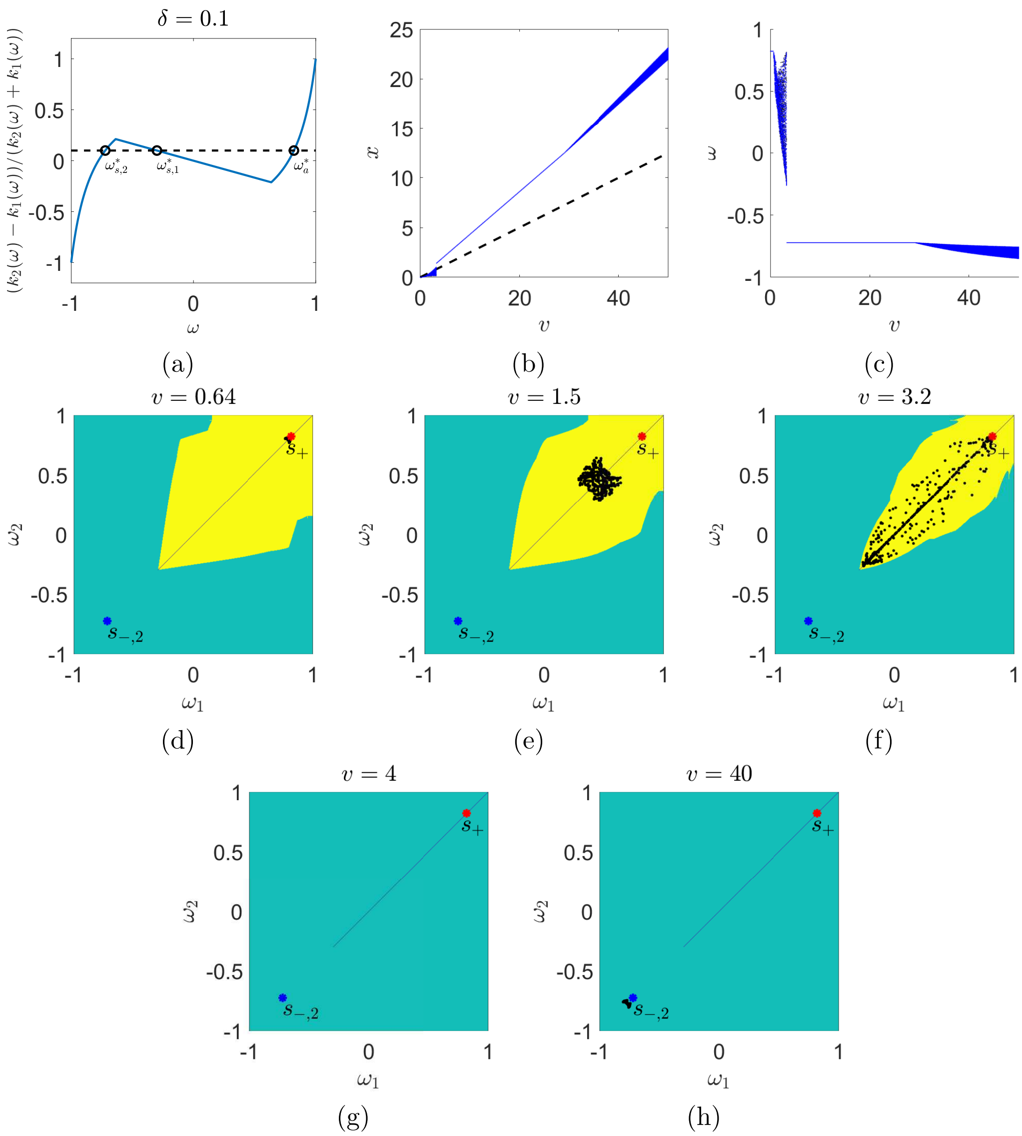


Figure 6: Inequality averse players, $\delta = 0.1$. First row: three symmetric steady states characterized in terms of either spitefulness or altruism (panel (a)). Panels (b-c): bifurcation diagrams for variables x_1 and x_2 and ω_1 and ω_2, when initial preferences are characterized by altruism (black and blue colors are respectively used for player 1 and 2). Second row: $\boldsymbol{s}^*_{-,1}$ (blue asterisk, turquoise basin of attraction) coexisting with another attractor (yellow basin of attraction) arisen from the loss of stability of $\boldsymbol{s}^*_+$ (red asterisk). Third row: $\boldsymbol{s}^*_{-,1}$ (blue asterisk) attracts almost any trajectory (panel (g)) and, as v further increases, it loses stability and a closed invariant curve emerges (panel (h)).

with Proposition (26), some trajectories starting with $\omega_{1,0} = \omega_{2,0}$ can still converge toward $\boldsymbol{s}^*_+$) and starting from almost any initial condition we have convergence toward the spiteful steady state or the attractor arising when it becomes unstable. The scenario reported in Figure 6 shows that, at least when multiple steady states are present, the spiteful steady state is that dynamically relevant, giving rise to overbidding phenomena even if the players initially have altruistic preferences, especially when the prize increases.

What is most surprising is that dynamics characterized by spiteful preferences can emerge even in the absence of spiteful steady states. To this end, we consider the same setting we used for the simulations reported in Figure 6, but we increase δ, setting it equal to 0.4. In this case, as shown in Figure 7 (a), we have a unique altruistic symmetric steady state $\boldsymbol{s}^*_+ = (0.02v, 0.02v, 0.91, 0.91)$, which is locally asymptotically stable just for $v < 0.45$. As v increases, a closed invariant curve emerges around $\boldsymbol{s}_+^*$ giving rise to quasi-periodic dynamics. However, such attractor moves away from $\boldsymbol{s}_+^*$, and it quickly reaches regions of the phase space in which coefficients of altruism are negative and hence strategic choices are larger than those corresponding to the classic Tullock Nash equilibrium. The bifurcation diagrams reported in Figure 7 (b-c) are obtained setting $\omega_{1,0} = 0.9, \omega_{2,0} = 0.905$ and $x_{1,0} = x_{2,0} = 0.31$, i.e. coefficients of altruism are very close to those at $\boldsymbol{s}_+^*$. They show that, even if for small values of the prize, dynamics are characterized in terms of altruism, strategies (respectively coefficients of altruism) shift upward (respectively downward), highlighting overbidding phenomena. As evident from Figure 7 (d-g), the attractor arising when $\boldsymbol{s}_+^*$ becomes unstable "travels" along the diagonal of phase plane (ω_1, ω_2), shifting toward the region in which both coefficients of altruism are negative. To better explain such dynamical behavior, we make reference to the initial parts of the time series of x_i and ω_i reported in Figure 7 (h-i). The initial conditions are close to $\boldsymbol{s}_+^*$, and both players have initial altruistic preferences and underbid. However, the small difference between their initial preferences causes slightly different strategic behaviors, which lead to inequalities in the expected payoffs, which is significant due to the underbidding of the agents. Inequality aversion then brings about a change in the preferences. Since they both dislike the stage outcome, this creates a drift toward spitefulness that starts decreasing the coefficient of altruism. However, the effects of aversion toward his own material disadvantage with respect to that toward the competitor material disadvantage are different, and hence the extent of the decrease of the coefficients of altruism is different for each player, and this amplifies inequality between player behaviors and hence between payoffs. The decrease of coefficients of altruism then goes on, and altruistic preferences turn into spiteful ones, due to the persistence of a behavior of the competitor that is evaluated as bad. As the players become more spiteful, they increase the exerted effort and they start overbidding ($t = 5$). This progressively reduces expected payoffs, and hence the difference between them. However, such differences persist, as they are sustained by inequality aversion toward own material disadvantages, which, due to the erratic evolution of both preferences and strategies, alternately affect different players. Spitefulness and overbidding increases until they reach a point ($t = 15$), at which, due to the large, negative coefficient of altruism, each player disposition toward altruism is dominating, leading to a slow increase of the coefficient of altruism, also because of reduced expected payoffs and consequently reduced inequality. Such an increase goes on until inequalities are bearable. As the overbidding reduces and payoff inequality tends to increase, an abrupt inversion of preference trajectories occurs, with a fast return of spitefulness sustained by inequality aversion.

We note that maximum positive drift toward altruism is bounded by the value of δ. However, increasing δ, the altruistic steady state is more altruistically polarized, and hence it's stability region further shrinks. So the increased possible drift toward altruism is opposed by the larger inequalities that steady state instability creates, so the final dynamical behavior is essentially the same of that described for reduced values of δ.

If we considered functions σ_i for which $k_2(\omega)/k_1(\omega)$ is strictly increasing and just one symmetric altruistic steady state exists, we would observe dynamical behaviors that are similar in all and for all to those reported in Figure 7.

We note that simulations show that altruistic steady states can play a more relevant dynamical role if we decrease α and β. This means that players are weakly inequality averse, i.e. they are quite indifferent to what they observe from stage outcomes. So it is not surprising that in these cases dynamics are characterized more by the initial preferences than by a coevolution of preferences and strategies. However, these scenarios are not particularly interesting from the economic point of view, as they are not consistent with the strong competitiveness degree encompassed in contest lab experiments, for which strategic behavior is expected to have greater influence on players.

We stress that, as we mentioned in the Introduction, in the literature it was already shown that inequality aversion could be an explanation of overbidding, but assuming exogenous spitefulness for players. The previous results show that overbidding due to inequality aversion is robust with respect to endogenous preference evolution and can emerge even if the players are (initially) altruistic.

To summarize, when inequality averse players are involved, the static analysis would suggest that altruism and underbidding should be dominant, but the dynamical analysis highlights that spitefulness and overbidding can endogenously emerge, giving rise to non convergent dynamics, once more self-sustained by the coevolution of strategies and preferences.

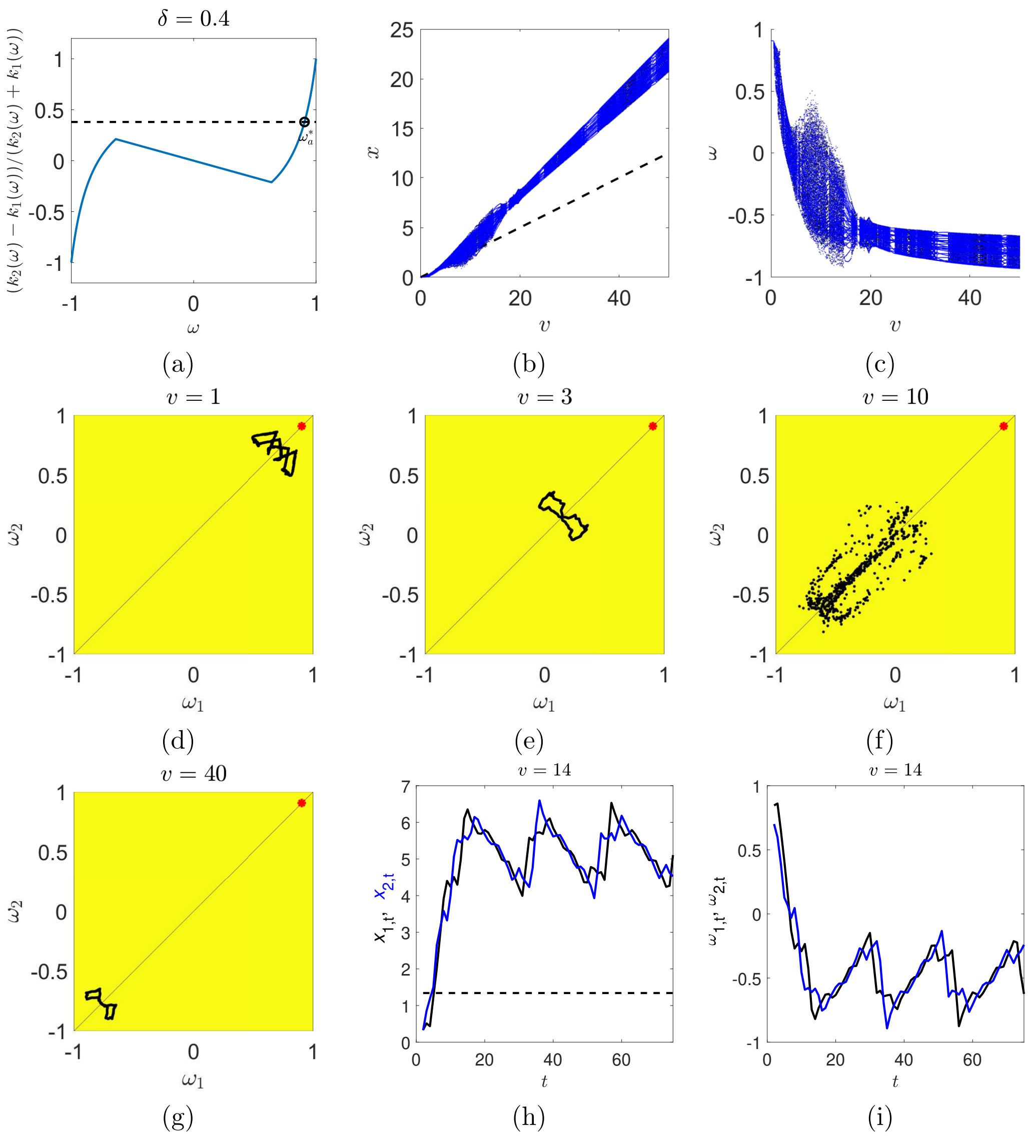


Figure 7: Inequality averse players, $\delta = 0.4$. First row: the unique symmetric steady state is characterized in terms of altruism (panel (a)). Panels (b-c): bifurcation diagrams of variables x_1, x_2 and ω_1, ω_2, when initial preferences are characterized by altruism (black and blue colors are respectively used for player 1 and 2). Second and third rows: attractor evolution as v increases (panels (d-g)) and related position with respect to locally asymptotically unstable altruistic steady state (red asterisk). Third row, panels (h-i): time series of variables x_1, x_2 and ω_1, ω_2 when $v = 14$.

5. Conclusions

Coevolution of other regarding preferences and agent behaviors proved to be an effective tool to understand the emergence of non self-interested preferences, and consequently of strategic choices that can explain overbidding or underbidding phenomena, as well as erratic behavior. The carried on analysis shows the importance played by agent preferences in determining both possible steady and dynamical outcomes. Moreover, it highlights how a static investigation of contest models can be misleading, as preference evolution can drive trajectories very far from steady states. The way player preferences evolve depending on the observation and evaluation of the contest outcomes leads to the selection of particular steady configurations, when multiple of them coexist. In addition, for example in the case of inequality averse players, it can even sustain the endogenous occurrence of overbidding when no steady states characterized in terms of overbidding do exist. Moreover, the coevolutive approach that we pursued allows a clear explanation of the rationale behind each static or dynamical result.

We aim at developing future research in several directions. First, we have so far considered a symmetric setting, with homogeneous players. Introducing heterogeneity also allows for testing the robustness of the results in an evolutionary perspective. Moreover, different preference evolution mechanisms can be taken into account, considering, for instance, an exogenous reference value on which basis the agents evaluate the behavior of their opponents. Finally, the model we proposed is related to a classic Tullock contest setting, but this approach can be extended to different situations, as for example in public good games. In addition to this, it could be applied to conflict models, to understand the ebb and flow often observed in the level of hostilities, with waves of extreme contrasts followed by an ostensible quiet, and again by extreme violence.

Appendix

Proof [Proposition 1 and Corollary 1] For a symmetric equilibrium we set $\omega_1 = \omega_2 = \omega$. From the former couple of equations we immediately obtain the former expression in (12). Since $\Delta\pi_i(x_1(\omega,\omega), x_2(\omega,\omega)) = 0$ we have

$$\begin{cases} 0 = \frac{k_1(\omega)+k_2(\omega)}{2} f(m(0)) + \frac{k_2(\omega)-k_1(\omega)}{2}, \\ 0 = \frac{k_1(\omega)+k_2(\omega)}{2} f(m(0)) + \frac{k_2(\omega)-k_1(\omega)}{2}, \end{cases}$$

which provides the latter condition in (12). Note that such condition can be rewritten as

$$f(\delta) = \frac{k_1(\omega^*) - k_2(\omega^*)}{k_2(\omega^*) + k_1(\omega^*)}. \tag{22}$$

Let $g : (-1, 1) \to \mathbb{R}$ defined by $g(\omega) = (k_1(\omega) - k_2(\omega))/(k_2(\omega) + k_1(\omega))$. From $k_1(0) = k_2(0)$ we have $g(0) = 0$, and since $k_1(\omega) = k_2(-\omega)$, function $g(\omega)$ is odd. Moreover, since $k_2(\omega) \to 0^+$ as $\omega \to 1^-$ and $k_1(\omega) \to 0^+$ as $\omega \to -1^+$, from assumption (8), since $|\sigma_2(\omega) - \sigma_1(\omega)| > \mu$ we have $k_2(\omega) + k_1(\omega) > \mu$, and hence $\lim_{\omega\to-1^-} k_2(\omega) \geq \mu$ and $\lim_{\omega\to1^-} k_1(\omega) \geq \mu$, from which $g(\omega) \to \pm1$ as $\omega \to \pm1$. Since $f(\delta) \in (-1, 1)$, thanks to the Intermediate Values Theorem, we have that equation (22) always has at least a solution. The conclusion on the self-interested steady state is straightforward from (12).

Setting $\rho(\omega) = k_2(\omega)/k_1(\omega)$, we have that the right hand side in (22) can be rewritten as $g(\omega) = (1 - \rho(\omega))/(\rho(\omega) + 1)$, for which $g'(\omega) = (-2\rho'(\omega))/((1 + \rho(\omega))^2)$, i.e. the monotonicity of g is opposite to that of $\rho(\omega)$. This means that if $\rho(\omega)$ is strictly monotonic, we then have that (22) has a unique solution. Note that, thanks to assumption (9), we have $g(0) = 0$ and, thanks to assumption (8), we have $g(\pm1^{\mp}) = \pm1$, so if $\rho(\omega)$ is strictly monotonic, function g must be strictly increasing (which means that $k_2(\omega)/k_1(\omega)$ must be strictly decreasing). So a straightforward geometric consideration shows that if δ increases, also the solution to (22) increases as well. The strict monotonicity of $\rho(\omega)$ is also necessary for uniqueness. It is clear that if $\rho(\omega)$ is monotonic but not strictly monotonic, then (22) has infinitely many solutions for some δ. If $\rho(\omega)$ is not monotonic, then for some $\bar{\omega}$ we must have $\rho'(\bar{\omega}) = 0$ and $\bar{\omega}$ is an extremum point. If it is for example a maximum point, for δ belonging to a suitably small left neighborhood of $f^{-1}(\rho(\bar{\omega}))$ we have at least a couple of solutions to (22). Moreover, since $k_1(\omega) = k_2(-\omega)$, function g is odd and unimodal on $[0, 1)$, it is a cubic-like function, so we immediately have the result about the maximum number and characterization of symmetric steady states of Corollary 1. □

Proof [Proposition 2] From the former couple of equations in (11) we immediately obtain (14). Recalling that $\Delta\pi_{2,t} = -\Delta\pi_{1,t}$ the latter couple of equations requires

$$\begin{cases} 0 = \dfrac{k_1(\omega_1) + k_2(\omega_1)}{2} f(m(\Delta\pi_1)) + \dfrac{k_2(\omega_1) - k_1(\omega_1)}{2}, \\ 0 = \dfrac{k_1(\omega_2) + k_2(\omega_2)}{2} f(m(-\Delta\pi_1)) + \dfrac{k_2(\omega_2) - k_1(\omega_2)}{2}, \end{cases}$$

which immediately provides (15).

We limit to detail the proof for those ω_1 for which $(k_1(\omega_1) - k_2(\omega_1))/(k_1(\omega_1) + k_2(\omega_1)) > 0$, the other case can be similarly handled. For simplicity, let us again denote by m the restriction of function $m(x)$ to $x \geq 0$. Since $\sigma_2(\omega) \geq \omega + \mu > \omega > \omega - \mu \geq \sigma_1(\omega)$ we have that $k_i(\omega) \neq 0$ on $(-1, 1)$, so $-1 < (k_1(\omega) - k_2(\omega))/(k_1(\omega) + k_2(\omega)) < 1$ and hence the latter condition in (15) can be then rewritten as

$$\frac{(1 - \omega_1^*\omega_2^*)(\omega_2^* - \omega_1^*)}{(2 - \omega_1^* - \omega_2^*)^2} = \frac{1}{v} m^{-1}\left(f^{-1}\left(\frac{k_1(\omega_1^*) - k_2(\omega_1^*)}{k_1(\omega_1^*) + k_2(\omega_1^*)}\right)\right). \tag{23}$$

As $v \to +\infty$, the right hand side of the previous identity pointwise converges toward the null function for any given ω_1^*. This means that, for any ω_1^*, for suitably large values of v, we have that ω_2^* solving the previous identity can be made as close as we like to ω_1^*. This concludes the proof. □

Proof [Proposition 3] We start considering general initial conditions. We note that function F defining model (11) is Lipschitz continuous but not differentiable at $x_1 = x_2$. However, the classic argument on the eigenvalues of the Jacobian matrix of F at the steady state can be adapted and still holds at a symmetric steady state $\boldsymbol{s}^*$ of (11). For the reader's sake, we detail the proof.

Let us consider a symmetric steady state $\boldsymbol{s}^*$ of (11). Note that $x_1^* = x_2^* \neq 0$, so it is possible to find a suitable neighborhood $\Omega \subset (0, +\infty)^2 \times (-1, 1)^2$ of $\boldsymbol{s}^*$ such that if $(x_1, x_2, \omega_1, \omega_2) \in \Omega$, we have $x_i \geq \mu > 0$ for some $\mu > 0$. This guarantees that on each subset

$$\Omega^+ = \Omega \cap \{(x_1, x_2, \omega_1, \omega_2) \in \mathbb{R}^4 : x_2 \geq x_1\},\ \Omega^- = \Omega \cap \{(x_1, x_2, \omega_1, \omega_2) \in \mathbb{R}^4 : x_2 \leq x_1\},$$

function F ha continuous partial derivatives, and hence it is Frechet differentiable. Let $J^\pm$ be the Jacobian matrices of F on $\Omega^\pm$, for any norm, we then have that there are two balls $B(\boldsymbol{s}^*, \delta^+)$ and $B(\boldsymbol{s}^*, \delta^-)$ such that, respectively, for any $\boldsymbol{h} \in \mathbb{R}^4 : \boldsymbol{s}^* + \boldsymbol{h} \in \Omega^\pm$

$$\lim_{\|\boldsymbol{h}\| \to 0} \frac{\|F(\boldsymbol{s}^* + \boldsymbol{h}) - F(\boldsymbol{s}^*) - J^\pm(\boldsymbol{s}^*)\boldsymbol{h}\|}{\|\boldsymbol{h}\|} = 0,$$

Let us assume that $\rho(J^\pm(\boldsymbol{s}^*)) < 1$, and let us consider a norm $\|\|$ for which $J^\pm(\boldsymbol{s}^*)\| < \rho(J^\pm(\boldsymbol{s}^*)) + \varepsilon < 1 - \varepsilon$, for some suitable $\varepsilon > 0$. Thanks to Frechet differentiability and recalling that $\boldsymbol{s}^*$ is a steady state, there is a ball $B(\boldsymbol{s}^*, \delta^+)$ such that for any $\boldsymbol{h} \in \mathbb{R}^4 : \boldsymbol{s}^* + \boldsymbol{h} \in B(\boldsymbol{s}^*, \delta^+)$ we have $\|F(\boldsymbol{s}^* + \boldsymbol{h}) - \boldsymbol{s}^* - J^+(\boldsymbol{s}^*)\boldsymbol{h}\| \leq \varepsilon\|\boldsymbol{h}\|$ and there is a ball $B(\boldsymbol{s}^*, \delta^-)$ such that for any $\boldsymbol{h} \in \mathbb{R}^4 : \boldsymbol{s}^* + \boldsymbol{h} \in B(\boldsymbol{s}^*, \delta^-)$ we have $\|F(\boldsymbol{s}^* + \boldsymbol{h}) - F(\boldsymbol{s}^*) - J^-(\boldsymbol{s}^*)\boldsymbol{h}\| \leq \varepsilon\|\boldsymbol{h}\|$. We then have, respectively on $B(\boldsymbol{s}^*, \delta^\pm)$ that

$$\|F(\boldsymbol{s}^* + \boldsymbol{h}) - \boldsymbol{s}^*\| \leq \|F(\boldsymbol{s} + \boldsymbol{h}) - \boldsymbol{s}^* - J^\pm(\boldsymbol{s}^*)\boldsymbol{h}\| + \|J^\pm(\boldsymbol{s}^*)\boldsymbol{h}\| \leq \varepsilon\|\boldsymbol{h}\| + (\rho(J^\pm) + \varepsilon)\|\boldsymbol{h}\|,$$

so that on $B = B(\boldsymbol{s}^*, \delta^+) \cup B(\boldsymbol{s}^*, \delta^-)$ we have $\|F(\boldsymbol{s}^* + \boldsymbol{h}) - \boldsymbol{s}^*\| \leq k\|\boldsymbol{h}\|$ with $k < 1$, and this guarantees convergence toward $\boldsymbol{s}^*$ of iterations that start in B.

The Jacobian matrix of System (11) is defined for any $x_1 \neq x_2$ and results

$$J = \begin{pmatrix} 0 & j_{12} & j_{13} & 0 \\ j_{21} & 0 & 0 & j_{24} \\ j_{31} & j_{32} & j_{33} & 0 \\ j_{41} & j_{42} & 0 & j_{44} \end{pmatrix},$$

where

$$j_{12} = \frac{1}{2}\sqrt{\frac{v(1 - \omega_1)}{x_2}} - 1, j_{13} = -\frac{1}{2}\sqrt{\frac{vx_2}{1 - \omega_1}}, j_{21} = \frac{1}{2}\sqrt{\frac{v(1 - \omega_2)}{x_1}} - 1, j_{24} = -\frac{1}{2}\sqrt{\frac{vx_1}{1 - \omega_1}},$$

and

$$\begin{aligned}
j_{31} &= -\frac{m'(\Delta\pi) f'(m(\Delta\pi))\left(\frac{k_1(\omega_1)}{2} + \frac{k_2(\omega_1)}{2}\right)(x_1^2 + 2x_1x_2 + x_2^2 - 2vx_2)}{(x_1 + x_2)^2} = -j_{32}, \\
j_{33} &= f(m(\Delta\pi))\left(\frac{k_1'(\omega_1)}{2} + \frac{k_2'(\omega_1)}{2}\right) + \frac{k_2'(\omega_1)}{2} - \frac{k_1'(\omega_1)}{2} + 1, \\
j_{41} &= \frac{f'(m(-\Delta\pi)) m'(-\Delta\pi)\left(\frac{k_1(\omega_2)}{2} + \frac{k_2(\omega_2)}{2}\right)(x_1^2 + 2x_1x_2 + x_2^2 - 2vx_2)}{(x_1 + x_2)^2} = -j_{42}, \\
j_{44} &= f(m(-\Delta\pi))\left(\frac{k_1'(\omega_2)}{2} + \frac{k_2'(\omega_2)}{2}\right) + \frac{k_2'(\omega_2)}{2} - \frac{k_1'(\omega_2)}{2} + 1,
\end{aligned}$$

in which we set $\Delta\pi = ((x_2 - x_1)(x_1 - v + x_2))/(x_1 + x_2)$.

We have that $J^+(\boldsymbol{s}^*)$ (i.e. the Jacobian matrix of the restriction of F to Ω^+ evaluated at $\boldsymbol{s}^*$) is

$$J^+(\boldsymbol{s}^*) = \begin{pmatrix} 0 & 0 & -\frac{v}{4} & 0 \\ 0 & 0 & 0 & -\frac{v}{4} \\ m'(0^+)z & -m'(0^+)z & 1+\theta & 0 \\ -m'(0^-)z & m'(0^-)z & 0 & 1+\theta \end{pmatrix},$$

where we set

$$z = f'(\delta)\left(\frac{k_1(\omega^*)}{2} + \frac{k_2(\omega^*)}{2}\right)\frac{1+\omega^*}{1-\omega^*},\ \theta = f(\delta)\left(\frac{k_1'(\omega^*)}{2} + \frac{k_2'(\omega^*)}{2}\right) + \frac{k_2'(\omega^*)}{2} - \frac{k_1'(\omega^*)}{2}.$$

A straightforward computation shows that the characteristic polynomial of $J^+(\boldsymbol{s}^*)$ is

$$p(\lambda) = \lambda\left(\lambda^3 - 2(\theta+1)\lambda^2 + \left(\theta^2 + 2\theta + \frac{\phi}{4} + 1\right)\lambda + \left(-\frac{\phi}{4} - \frac{\phi\theta}{4}\right)\right), \tag{24}$$

where we set $\phi = vz(m'(0^-) + m'(0^+))$.

Similarly, we have that $J^-(\boldsymbol{s}^*)$ (i.e. the Jacobian matrix of the restriction of F to Ω^- evaluated at $\boldsymbol{s}^*$) is

$$J^-(\boldsymbol{s}^*) = \begin{pmatrix} 0 & 0 & -\frac{v}{4} & 0 \\ 0 & 0 & 0 & -\frac{v}{4} \\ m'(0^-)z & -m'(0^-)z & 1+\theta & 0 \\ -m'(0^+)z & m'(0^+)z & 0 & 1+\theta \end{pmatrix}.$$

A straightforward computation shows that the characteristic polynomial of $J^-(\boldsymbol{s}^*)$ is again (24), so the conditions under which the eigenvalues of $J^+(\boldsymbol{s}^*)$ and $J^-(\boldsymbol{s}^*)$ lie inside the unit circle are the same. This also means that conditions under which their eigenvalues lie outside the unit circle are the same, and hence reverting the following inequalities we will obtain conditions under which a symmetric steady state is unstable.

Polynomial (24) has indeed a null eigenvalue, so conditions under which its eigenvalues lie inside the unit circle are those for the roots of a third degree polynomial. We recall that, as reported, for example, in [27], such conditions are

$$\begin{cases} T + D - 1 + M < 0 \\ -(T + D + 1 + M) < 0 \\ M - T\cdot D - 1 + D^2 < 0 \\ -(M - T\cdot D + 1 + D^2) < 0 \end{cases} \quad \text{where } T = 2\theta + 2, M = \theta^2 + 2\theta + \frac{\phi}{4} + 1, D = \frac{\phi(\theta+1)}{4}, \tag{25}$$

so conditions (25) become

$$\begin{cases} \frac{\theta(\phi - 4\theta)}{4} < 0, \\ -\frac{(\theta+2)(\phi+4\theta+8)}{4} < 0, \\ \left(\frac{\theta+1}{4}\right)^2\phi^2 - \left(\frac{\theta^2}{2} + \theta + \frac{1}{4}\right)\phi + \theta^2 + 2\theta < 0, \\ \frac{(\theta+1)^2}{16}\phi^2 + \left(\frac{2(\theta+1)^2 - 1}{4}\right)\phi - (\theta+1)^2 - 1 < 0. \end{cases} \tag{26}$$

Since we aim at studying stability on varying v, we solve each condition with respect to ϕ. Let

$$\phi_1(\theta) = 4\theta,\ \phi_2(\theta) = -4\theta - 8,\ \phi_3(\theta) = \frac{4\theta(\theta+2)}{(\theta+1)^2}$$

and

$$\phi_-(\theta) = \frac{2\left(-2\theta^2 - 4\theta - 1 - \sqrt{8(\theta+1)^2 + 1}\right)}{(\theta+1)^2}, \phi_4(\theta) = \frac{2\left(-2\theta^2 - 4\theta - 1 + \sqrt{8(\theta+1)^2 + 1}\right)}{(\theta+1)^2},$$

solving (26) for ϕ, depending on θ, we find

	$\theta < -2$	$-2 < \theta < 0$	$0 < \theta < 2$
Condition1	$\phi > \phi_1(\theta)$	$\phi > \phi_1(\theta)$	$\phi < \phi_1(\theta)$
Condition2	$\phi < \phi_2(\theta)$	$\phi > \phi_2(\theta)$	$\phi > \phi_2(\theta)$
Condition3	$\phi_3(\theta) < \phi < 4$	$\phi_3(\theta) < \phi < 4$ (if $\theta = -1, \phi < 4$)	$\phi_3(\theta) < \phi < 4$
Condition4	$\phi_-(\theta) < \phi < \phi_4(\theta)$	$\phi_-(\theta) < \phi < \phi_4(\theta)$ (if $\theta = -1, \phi > -4$)	$\phi_-(\theta) < \phi < \phi_4(\theta)$

We start focusing on $\theta < -2$, whose study we subdivide into two cases.

Let $-\frac{3+\sqrt{5}}{2} \leq \theta < -2$. We have

$$\phi_2(\theta) \leq \phi_3(\theta) \Leftrightarrow -4\theta - 8 \leq \frac{4\theta(\theta+2)}{(\theta+1)^2} \Leftrightarrow \frac{4(\theta+2)(\theta^2+3\theta+1)}{(\theta+1)^2} \geq 0, \tag{27}$$

in which the last expression is true for $\theta^2 + 3\theta + 1 \geq 0$, i.e. when $-\frac{3+\sqrt{5}}{2} \leq \theta < -2$, so conditions 2 and 3 are not compatible in such interval.

Let $\theta < -\frac{3+\sqrt{5}}{2}$. We have

$$\begin{aligned}\phi_4(\theta) < \phi_3(\theta) \quad &\Leftrightarrow \quad \frac{2\left(-2\theta^2-4\theta-1+\sqrt{8(\theta+1)^2+1}\right)}{(\theta+1)^2} < \frac{4\theta(\theta+2)}{(\theta+1)^2} \Leftrightarrow \frac{16\theta-2\sqrt{8\theta^4+32\theta^3+48\theta^2+32\theta+9}+8\theta^2+2}{(\theta+1)^2} > 0 \\ &\Leftrightarrow \quad 32(\theta^2+3\theta+1)(\theta^2+\theta-1) > 0,\end{aligned} \tag{28}$$

which is true since $\theta^2 + 3\theta + 1$ is positive for $\theta < -\frac{3+\sqrt{5}}{2}$ as well as $\theta^2 + \theta - 1$, since it is positive for $\theta < -\frac{1+\sqrt{5}}{2}$. So condition 3 and 4 are not compatible for $\theta < -\frac{3+\sqrt{5}}{2}$ and (26) has empty solution.
So we conclude that $\boldsymbol{s}^*$ is unconditionally unstable for $\theta < -2$.

Now we focus on $\theta < -2$, whose study we again subdivide into two cases.

Let $0 < \theta \leq \frac{\sqrt{5}-1}{2}$. We have

$$\phi_1(\theta) \leq \phi_3(\theta) \Leftrightarrow 4\theta \leq \frac{4\theta(\theta+2)}{(\theta+1)^2} \Leftrightarrow \frac{4\theta(\theta^2+\theta-1)}{(\theta+1)^2} \leq 0, \tag{29}$$

in which the last expression is true for $\theta^2 + \theta - 1 \leq 0$, i.e. for $0 < \theta \leq \frac{\sqrt{5}-1}{2}$, so condition 1 and 3 are not compatible in such interval.

Let $\theta > \frac{\sqrt{5}-1}{2}$. Recalling (28), we have that $\phi_4(\theta) < \phi_3(\theta)$ since $\theta^2 + 3\theta + 1$ is positive for $\theta > \frac{\sqrt{5}-3}{2}$ as well as $\theta^2 + \theta - 1$, since it is positive for $\theta > \frac{\sqrt{5}-1}{2}$. So condition 3 and 4 are not compatible for $\theta > \frac{\sqrt{5}-1}{2}$ and hence (26) has empty solution.

So we conclude that $\boldsymbol{s}^*$ is unconditionally unstable also for $\theta > 0$.

Let us now consider the case of $\theta \in (-2, 0)$. We note that if $m'(0^-) + m'(0^+) > 0$, we have that as v increases in $(0, +\infty)$, ϕ increases in $(0, +\infty)$, while if $m'(0^-) + m'(0^+) < 0$, we have that as v increases in $(0, +\infty)$, ϕ decreases in $(-\infty, 0)$.

So we start considering the solution to (26) in the case of $m'(0^-) + m'(0^+) > 0$, so we are interested only in solutions $\phi \in (0, +\infty)$. Since $\theta \in (-2, 0)$, both conditions 1 and 2 are always fulfilled, while condition 3 reduces to $\phi \in (0, 4)$, and we set

$$\bar{v}(\omega, \theta) = \frac{4}{(m'(0^-) + m'(0^+))f'(\delta)\left(\frac{k_1(\omega)}{2} + \frac{k_2(\omega)}{2}\right)\frac{1+\omega}{1-\omega}}. \tag{30}$$

We have

$$\frac{2\left(-2\theta^2 - 4\theta - 1 + \sqrt{8(\theta+1)^2+1}\right)}{(\theta+1)^2} < 4,$$

as it can be equivalently rewritten into

$$\frac{2\left(-2\theta^2-4\theta-1+\sqrt{8(\theta+1)^2+1}\right)}{(\theta+1)^2}<4 \Leftrightarrow$$

$$\frac{2\left(4\theta^2+3+8\theta-\sqrt{8\theta^4+32\theta^3+48\theta^2+32\theta+9}\right)}{(\theta+1)^2}>0 \Leftrightarrow 8\theta(\theta+1)^2(\theta+2)>0,$$

which is indeed true recalling that $\theta \in (-2,0)$.

Now we focus on the solution to (26) in the case of $m'(0^-)+m'(0^+)<0$, so we are interested only in solutions $\phi \in (-\infty,0)$. Conditions 1 and 2 can be summarized as

$$\phi > \begin{cases} 4\theta & -1 \le \theta < 0 \\ -4\theta-8 & -2<\theta<-1 \end{cases} \tag{31}$$

and we set

$$\bar{v}(\omega,\theta)=\begin{cases} \frac{4\theta}{(m'(0^-)+m'(0^+))f'(\delta)\left(\frac{k_1(\omega)}{2}+\frac{k_2(\omega)}{2}\right)\frac{1+\omega}{1-\omega}} & -1\le\theta<0, \\ \frac{-4\theta-8}{(m'(0^-)+m'(0^+))f'(\delta)\left(\frac{k_1(\omega)}{2}+\frac{k_2(\omega)}{2}\right)\frac{1+\omega}{1-\omega}} & -2<\theta<-1. \end{cases}$$

Moreover, recalling (29), we have $\phi_3(\theta)<\phi_1(\theta)$ on $-1\le\theta<0$ and, recalling (27), we have $\phi_3(\theta)<\phi_2(\theta)$ on $-2<\theta<-1$, so condition (31) guarantees condition 3.

Finally, also condition 4 is guaranteed by condition (31). In fact, we can note that the right-hand side in (31) is greater or equal than -4 on $(-2,0)$, $\phi_4>0$ and

$$\frac{2\left(-2\theta^2-4\theta-1-\sqrt{8(\theta+1)^2+1}\right)}{(\theta+1)^2}<-4 \Leftrightarrow \frac{2\left(\sqrt{8(\theta+1)^2+1}-1\right)}{(\theta+1)^2}>0,$$

which is true since $\sqrt{8(\theta+1)^2+1}\ge 1$.

Now let us discuss what happens if $x_{1,0}=x_{2,0}$ and $\omega_{1,0}=\omega_{2,0}$. In this case, both equations describing the dynamics of strategic behavior are the identical, as well as those describing preference adjustment mechanism. This means that four dimensional model (11) reduces to the two dimensional model

$$\begin{cases} x_{t+1}=\sqrt{vx_t(1-\omega_t)}-x_t, \\ \omega_{t+1}=\omega_t+\dfrac{k_2(\omega_t)+k_1(\omega_t)}{2}f(m(0))+\dfrac{k_2(\omega_t)-k_1(\omega_t)}{2}, \end{cases} \tag{32}$$

where x_t and ω_t represent the time invariant strategies of both players. The Jacobian matrix of the map defining the right-hand side of (32) is

$$J=\begin{pmatrix} \sqrt{\frac{v(1-\omega)}{x}}-1 & -\frac{\sqrt{xv}}{2\sqrt{1-\omega}} \\ 0 & \frac{k_2'(\omega)}{2}-\frac{k_1'(\omega)}{2}+f(\delta)\left(\frac{k_1'(\omega)}{2}+\frac{k_2'(\omega)}{2}\right)+1 \end{pmatrix},$$

which evaluated at a symmetric equilibrium becomes

$$J^*=\begin{pmatrix} 0 & -\frac{v}{4} \\ 0 & \frac{k_2'(\omega^*)}{2}-\frac{k_1'(\omega^*)}{2}+f(\delta)\left(\frac{k_1'(\omega^*)}{2}+\frac{k_2'(\omega^*)}{2}\right)+1 \end{pmatrix},$$

from which it is evident that stability does not depend on v. □

Proof [Proposition 4] Characterization of the steady state coefficient of altruism in (18) immediately follows from the latter condition in (12) and $f(\delta)=0$. Concerning stability, we start noting that at a symmetric steady state, since $k_1(\omega^*)=k_2(\omega^*)$, we have

$$\left(\frac{k_2(\omega^*)}{k_1(\omega^*)}\right)'=\frac{k_2'(\omega^*)k_1(\omega^*)-k_2(\omega^*)k_1'(\omega^*)}{k_1^2(\omega^*)}=\frac{k_2'(\omega^*)-k_1'(\omega^*)}{k_1(\omega^*)},$$

so the sign of $k_2'(\omega^*) - k_1'(\omega^*)$ is determined by the monotonicity of $k_2(\omega^*)/k_1(\omega^*)$. We already showed in the proof of Proposition 1 that if $k_2(\omega^*)/k_1(\omega^*)$ is strictly monotonic, it must be strictly decreasing, so $k_2'(\omega^*) - k_1'(\omega^*) < 0$ and from Proposition 3 $\boldsymbol{s}_0^*$ is unconditionally unstable if $\theta(0) = (k_2'(0) - k_1'(0))/2 < -2$. Conversely, if $k_2'(0) - k_1'(0) > -4$, from the proof of Proposition 3 we have that stability is guaranteed provided that (30) holds true, that in the present case reduces to (19).

Under assumption (10b), recalling that the right-hand side in (22) vanishes for $\omega^* = 0$ and approaches 1 as $\omega^* \to 1^-$, we have that $k_2(\omega)/k_1(\omega)$ is strictly increasing at $\boldsymbol{s}_0^*$, while it is strictly decreasing at $\boldsymbol{s}_\pm^*$, so $\boldsymbol{s}_0^*$ is unconditionally unstable since $k_2'(0) - k_1'(0) > 0$, while for $\boldsymbol{s}_\pm^*$ conditional stability is again guaranteed by (30) holds true, that in the present case reduces to (20).

Noting that for a tit-for-tat player m is an odd function, we have that the equations of model (11) governing dynamics of ω can be written as

$$\begin{aligned} \omega_{1,t+1} &= \omega_{1,t} + \frac{k_2(\omega_{1,t}) + k_1(\omega_{1,t})}{2} f(m(\Delta\pi_{1,t})) + \frac{k_2(\omega_{1,t}) - k_1(\omega_{1,t})}{2}, \\ \omega_{2,t+1} &= \omega_{2,t} - \frac{k_2(\omega_{2,t}) + k_1(\omega_{2,t})}{2} f(m(\Delta\pi_{1,t})) + \frac{k_2(\omega_{2,t}) - k_1(\omega_{2,t})}{2}, \end{aligned}$$

so, if $\omega_{1,0} = -\omega_{2,0}$, model (11) becomes a three dimensional system with a unique variable $\omega_t = \omega_{1,t} = -\omega_{2,t}$ for coefficients of altruism, i.e.

$$\begin{cases} x_{1,t+1} = \sqrt{v x_{2,t}(1-\omega_t)} - x_{2,t}, \\ x_{2,t+1} = \sqrt{v x_{1,t}(1+\omega_t)} - x_{1,t}, \\ \omega_{t+1} = \omega_t + \dfrac{k_2(\omega_t) + k_1(\omega_t)}{2} f(m(\Delta\pi_{1,t})) + \dfrac{k_2(\omega_t) - k_1(\omega_t)}{2}. \end{cases}$$

Evaluating at $(x_1, x_2, \omega) = (v/4, v/4, 0)$ the Jacobian matrix corresponding to the function defining the right-hand side of the last system we find

$$J^* = \begin{pmatrix} 0 & 0 & -\frac{v}{4} \\ 0 & 0 & \frac{v}{4} \\ \gamma\left(\frac{k_1(0)}{2} + \frac{k_2(0)}{2}\right) & -\gamma\left(\frac{k_1(0)}{2} + \frac{k_2(0)}{2}\right) & \frac{k_2'(0)}{2} - \frac{k_1'(0)}{2} + 1 \end{pmatrix},$$

whose characteristic polynomial is

$$p(\lambda) = -\lambda\left(\lambda^2 - \left(\frac{k_2'(0)}{2} - \frac{k_1'(0)}{2} + 1\right)\lambda + \frac{\gamma v}{4}(k_1(0) + k_2(0))\right).$$

Setting

$$T = \frac{k_2'(0)}{2} - \frac{k_1'(0)}{2} + 1, D = \frac{\gamma v}{4}(k_1(0) + k_2(0)),$$

the eigenvalues of $p(\lambda)$ lie inside the unit circle provided that

$$\begin{cases} 1 + T + D > 0 \\ 1 - T + D > 0 \\ 1 - D > 0 \end{cases} \Leftrightarrow \begin{cases} 2 + \frac{k_2'(0)}{2} - \frac{k_1'(0)}{2} + \frac{\gamma v}{4}(k_1(0) + k_2(0)) > 0 \\ -\frac{k_2'(0)}{2} + \frac{k_1'(0)}{2} + \frac{\gamma v}{4}(k_1(0) + k_2(0)) > 0 \\ 1 - \frac{\gamma v}{4}(k_1(0) + k_2(0)) > 0 \end{cases}$$

which concludes the proof. □

Proof [Proposition 5] Under assumption (10a), characterization of the steady state coefficient of altruism immediately follows from (12) and $f(\delta) > 0$. Similarly, under assumption (10b), recalling Corollary (1), for suitably large values of δ, we have a unique symmetric steady state with $\omega^* > 0$. For its stability, we start noting that for (17) we have $m'(0^-) = \alpha$ and $m'(0^-) = -\beta$, with $\alpha - \beta > 0$. From Proposition 3 any symmetric steady state is unconditionally unstable if $\theta(\omega^*) = (k_2'(\omega^*) - k_1'(\omega^*))/2 < -2$ or if $k_2'(\omega^*) - k_1'(\omega^*) > 0$, which holds true for $\boldsymbol{s}_{-,2}^*$ at which $k_2(\omega_{s,2}^*)/k_1(\omega_{s,2}^*)$ is strictly increasing, and hence $k_2(\omega^*) - k_1(\omega^*)$ as well, recalling the first part of the proof of Proposition 4. Conversely, if $k_2'(\omega^*) - k_1'(\omega^*) \in (-4, 0)$, since have $m'(0^-) + m'(0^-) > 0$, the steady state is stable under condition (30), which in the present case becomes (21). □

References

[1] G. Tullock, Toward a theory of the Rent-Seeking Societ, College Station, TX: Texas A&M University Press, 1980, Ch. Efficient rent seeking, pp. 97–112.

[2] E. Dechenaux, D. Kovenock, R. M. Sheremeta, A survey of experimental research on contests, all-pay auctions and tournaments, Experimental Econonomics 18 (2015) 609–669.

[3] R. M. Sheremeta, Overbidding and heterogeneous behavior in contest experiments, Journal of Economic Surveys 27 (2013) 491–514.

[4] F. Cavalli, M. Gilli, A. Naimzada, Endogenous interdependent preferences in a dynamical contest model, Tech. rep., University of Milan Bicocca Department of Economics, Management and Statistics Working Paper No. 492,, available at SSRN: https://ssrn.com/abstract=4060143 (2022).

[5] R. M. Sheremeta, Contest design: An experimental investigation, Economic Inquiry 49 (2011) 573–590.

[6] E. Baharad, S. Nitzan, Contest efforts in light of behavioural considerations, The Economic Journal 118 (2008) 2047–2059.

[7] D. Kahneman, A. Tversky, Prospect theory: an analysis of decision under risk, Econometrica 47 (1979) 263–291.

[8] R. M. Sheremeta, J. Zhang, Can groups solve the problem of over-bidding in contests?, Social Choice and Welfare 35 (2010) 175–197.

[9] S. Chowdhury, D. Kovenock, R. M. Sheremeta, An experimental investigation of colonel blotto games, Economic Theory 52 (2013) 833–861.

[10] R. M. Sheremeta, Experimental comparison of multi-stage and one-stage contests, Games and Economic Behavior 68 (2010) 731–747.

[11] M. A. Fonseca, An experimental investigation of asymmetric contests, International Journal of Industrial Organization 27 (2009) 582–591.

[12] K. Konrad, Altruism and envy in contests: an evolutionary stable symbiosis, Social Choice and Welfare 22 (2004) 479–490.

[13] T. Cason, R. Sheremeta, J. Zhang, Communication and efficiency in competitive coordination games, Games and Economic Behavior 76 (2012) 26–43.

[14] A. Savikhin, R. Sheremeta, Simultaneous decision-making in competitive and cooperative games, Economic Inquiry 51 (2013) 1311–1323.

[15] S. D. Mago, A. C. Samak, R. M. Sheremeta, Facing your opponents: Social identification and information feedback in contests, Journal of Conflict Resolution 60 (2016) 459–481.

[16] K. Wärneyrd, Chaotic dynamics in contests, Economic Inquiry 56 (2018) 1486–1491.

[17] D. Levine, Modeling altruism and spitefulness in experiment, Review of Economic Dynamics 1 (1998) 593–622.

[18] G. I. Bischi, F. Lamantia, Evolutionary oligopoly games with cooperative and aggressive behaviors, Journal of Economic Interaction and CoordinationOnline (2020). doi:https://doi.org/10.1007/s11403-020-00298-y.

[19] M. Rabin, Incorporating fairness into game theory and economics, The American economic review 83 (1993) 1281–1302.

[20] J. Geanakoplos, D. Pearce, E. Stacchetti, Psychological games and sequential rationality, Games and Economic Behavior 1 (1989) 60–79.

[21] S. Hart, A. Mas-Colell, Simple Adaptive Strategies: From Regret-Matching to Uncoupled Dynamics, Vol. 4, World Scientific Series in Economic Theory, 2013.

[22] D. Fudenberg, A. Peysakhovich, Recency, records and recaps: Learning and non-equilibrium behavior in a simple decision problem, in: Proceedings of the Fifteenth ACM Conference on Economics and Computation, 2014.

[23] P. Dal Bó, G. R. Fréchette, The evolution of cooperation in infinitely repeated games: Experimental evidence, American Economic Review 101 (2011) 411–429.

[24] J. Ledyard, Handbook of Experimental Economics, Princeton University Press, 1995, Ch. Public goods: a survey of experimental research, pp. 111–194.

[25] L. Balafoutas, R. Kerschbamer, M. Sutter, Distributional preferences and competitive behavior, Journal of Economic Behavior and Organization 83 (2012) 125–135.

[26] E. Fehr, K. M. Schmidt, A theory of fairness, competition, and cooperation, The Quarterly Journal of Economics 114 (1999) 817–868.

[27] L. Gardini, N. Schmitt, I. Sushko, F. Tramontana, F. Westerhoff, Necessary and sufficient conditions for the roots of a cubic polynomial and bifurcations of codimension-1, -2, -3 for 3D maps, Journal of Difference Equations and Applications 27 (2021) 557–578.