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Derived Invariance of Higher Direct Image Sheaves of the Canonical Bundle

PhD Thesis

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Contents

Introduction

In the following, we consider a smooth projective complex variety X. We denote by $\mathrm{Alb}(X)$ the Albanese variety of X, which is a $q(X)$-dimensional abelian variety, where $q(X) = h^1(X, \mathcal{O}_X)$ is the *irregularity* of X. A variety X is said to be irregular if $q(X) > 0$. The Albanese map of X is the morphism $a_X \colon X \to \mathrm{Alb}(X)$. Denote by $\mathbf{D}^{\mathrm{b}}(X) = \mathbf{D}^{\mathrm{b}}(Coh(X))$ the bounded derived category of coherent sheaves on X. We call *derived equivalence* an exact equivalence of triangulated categories $\Phi \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$. Our main interest concerns the study of the invariance of the cohomology ranks $h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X)$ under derived equivalence. This problem arises as a generalization of the well-known conjecture, formulated by Orlov [Orlov, 2005] and by Kontsevich [Kontsevich, 1995], about the invariance of the Hodge numbers $h^{p,q}(X)$, which can be stated as follows:

Conjecture A. *Let X and Y be smooth projective complex varieties such that there exists a derived equivalence $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$. Then*

$$h^{p,q}(X) = h^{p,q}(Y)$$

for every p, q.

This conjecture has already been proved in several cases: in dimension one it is trivial because the only non trivial Hodge number for smooth complex projective curves is the genus, for surfaces it was proved by T. Bridgeland and A. Maciocia in [Bridgeland and Maciocia, 2001], for threefolds by M. Popa and C. Schnell in [Popa and Schnell, 2011], for varieties of dimension 4 by R. Abuaf who proved that if X and Y have the same $h^{1,1}$ then all their Hodge numbers are equals [Abuaf, 2017], for varieties of general type by Y. Kawamata in [Kawamata, 2002], for hyperkähler varieties by L. Taelmann in [Taelman, 2019].

Introduction

Rouquier in [Rouquier, 2011] proved that a derived equivalence $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ induces an isomorphism of algebraic groups

$$\varphi\colon \operatorname{Aut}^0(X) \times \operatorname{Pic}^0(X) \xrightarrow{\sim} \operatorname{Aut}^0(Y) \times \operatorname{Pic}^0(Y)$$

called *Rouquier isomorphism*. Using this result Popa and Schnell show that $\operatorname{Pic}^0(X)$ and $\operatorname{Pic}^0(Y)$ are isogenous and that if $\operatorname{Aut}^0(X)$ is not affine, then X and Y have the structures of étale locally trivial fibrations over isogenous positive dimensional abelian varieties (see [Popa and Schnell, 2011]). We mainly consider the case when $\operatorname{Aut}^0(X)$ is affine. In particular, we are interested in the case where the Rouquier isomorphism respects the Picard factors, i.e., when it induces an isomorphism which, by a slight abuse of notation, will also be denoted by $\varphi\colon \operatorname{Pic}^0(X) \xrightarrow{\sim} \operatorname{Pic}^0(Y)$. If this is the case, we say that the derived equivalence Φ is *Rouquier-stable*. In this setting, we establish the following result

Theorem B. *Let $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence. If $\dim X \leq 3$ then*

$$h^p(\operatorname{Alb}(X), R^q a_{X*}\omega_X) = h^p(\operatorname{Alb}(Y), R^q a_{Y*}\omega_Y)$$

for every $p, q \geq 0$.

Moreover, the same result holds true if $\dim X = 4$ and $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ is a Rouquier-stable derived equivalence.

The proof of this theorem involves another important result that we prove about the invariance of a certain higher direct image of the canonical bundle. Caucci, Lombardi, and Pareschi in [Caucci et al., 2022] showed the derived invariance of the relative canonical ring $\mathcal{R}(b_X) = \bigoplus_{m\geq 0} b_{X*}\omega_X^{\otimes m}$ under a morphism b_X from X to the dual of a Rouquier-stable abelian subvariety of $\operatorname{Pic}^0(X)$. By pushing forward these techniques, we establish the invariance of the top non-trivial higher direct image sheaf of the canonical bundle. We denote by

$$c(X) = \dim X - \dim a_X(X)$$

the Albanese fiber dimension of X.

Theorem C. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a Rouquier-stable equivalence. Then $c(X) =$*

$c(Y)$ and the Rouquier isomorphism induces the following isomorphism of sheaves

$$\widehat{\varphi}^* R^{c(X)} a_{X*}\omega_X \simeq R^{c(Y)} a_{Y*}\omega_Y$$

where $\widehat{\varphi}$ indicates the dual morphism of φ.

Furthermore, the works of Lombardi and Popa [Popa, 2013, Lombardi, 2014, Lombardi and Popa, 2015] explore the connection between the derived invariance of the Hodge numbers of type $h^{0,i}(X)$ and the derived invariance of the so-called *cohomology support loci* of the canonical bundle

$$V^i_m(\omega_X) = \{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \omega_X \otimes \alpha) \geq m\}$$

for $m \geq 1$ and $i \in \{0, \dots, \dim X\}$. These loci are algebraic subsets of $\mathrm{Pic}^0(X)$. Denote by $V^i_m(\omega_X)_0$ and by $V^i_m(\omega_Y)_0$ the union of the connected components containing the origin of $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$, respectively. Lombardi and Popa proved that $h^{0,i}(X)$ is a derived invariant if and only if the cohomology support loci $V^i_m(\omega_X)_0$ are derived invariants for every $m \geq 1$, via the Rouquier isomorphism.

In an analogous way we study the following loci attached to higher direct images of the canonical bundle under the Albanese morphism

$$V^q_m(R^p a_{X*}\omega_X) = \left\{\alpha \in \mathrm{Pic}^0(X) \mid h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X \otimes \alpha) \geq m\right\}$$

with $p, q \geq 0$ and $m \geq 1$. We will establish a relation between the loci $V^q_m(R^p a_{X*}\omega_X)_0$ and the cohomology ranks $h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X)$. In fact, we prove the following theorem.

Theorem D. *The loci $V^q_m(R^p a_{X*}\omega_X)_0$ are derived invariants for every $m \geq 1$ in dimension n if and only if the cohomology ranks $h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X)$ are derived invariants in dimension n.*

As an application, we study the case when $c(X) = 2$. We prove, using the above results, a generic version of the invariance of the Hodge numbers attached to the canonical bundle, for Rouquier-stable derived equivalences.

Theorem E. *Let $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ be a Rouquier-stable derived equivalence and sup-*

pose $c(X) = 2$. *Then for every* $i \geq 0$ *we have*

$$h^i(X, \omega_X \otimes \alpha) = h^i(Y, \omega_Y \otimes \varphi(\alpha))$$

for a generic $\alpha \in \mathrm{Pic}^0(X)$.

Finally, we extend to higher values of $c(X)$ a result of Caucci and Pareschi in [Caucci and Pareschi, 2019] stating that for a variety of maximal Albanese dimension, i.e. when $c(X) = 0$, the ranks $h^i(X, \omega_X)$ are derived invariants for all $i \geq 0$.

In the first chapter of this thesis we introduce the background theory and set the context of our work. We briefly review the generic vanishing theorem and its connection with the Fourier-Mukai theory. We also recall some of Kollár's theorems about higher direct images of the Albanese map.

In the second chapter we build a solid background for our work. In particular we focus on the known derived invariants for irregular varieties. We explore the relations between the different invariants and present some key ideas and techniques that are crucial for our work.

The last Chapter is dedicated to proving the main theorems of this thesis.

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Chapter 1

Generic Vanishing and Fourier-Mukai Transform

Let X be a smooth projective complex variety of dimension n. One of the most crucial tools for comprehending the geometric properties of X is the *Albanese variety*, which is denoted by Alb(X). The Albanese variety is given by the following definition

$$\text{Alb}(X) = \frac{H^0(X, \Omega_X^1)^*}{H_1(X, \mathbb{Z})}$$

and it is an abelian variety with dimension equal to $q(X) = h^0(X, \Omega_X^1) = h^1(X, \mathcal{O}_X)$. If the condition $q(X) > 0$ is satisfied, then X is said to be *irregular*. The *Albanese map* of X, denoted as $a_X : X \to \text{Alb}(X)$ can be described as follows

$$x \mapsto \left(\omega \mapsto \int_{x_0}^{x} \omega\right)$$

where x_0 is a fixed point on X. The Albanese map is characterized by a universal property, i.e. any morphism from X to an abelian variety factorizes uniquely through Alb(X). The *Albanese dimension* of X is denoted by

$$a(X) := \dim a_X(X)$$

which is the dimension of the image of the Albanese map.

In addition to the Albanese variety, its dual abelian variety, known as the *Picard variety*, plays a significant role in the study of complex varieties. The

Picard variety is denoted by $\mathrm{Pic}^0(X)$ and can be defined as follows

$$\mathrm{Pic}^0(X) = \frac{H^1(X, \mathcal{O}_X)}{H^1(X, \mathbb{Z})}.$$

The Picard variety parametrizes line bundles on X that have a trivial first Chern class. Moreover, the pullback of the Albanese map $a_X^* \colon \mathrm{Pic}^0(\mathrm{Alb}(X)) \to \mathrm{Pic}^0(X)$ is an isomorphism.

1.1 Generic Vanishing

We will now introduce the definition of cohomological support loci, and some of its geometric properties.

Definition 1.1. For a coherent sheaf $\mathcal{F}$ on X we define its *i-th cohomological support loci* as

$$V_m^i(\mathcal{F}) = \left\{\alpha \in \mathrm{Pic}^0(X) \mid h^i(X, \mathcal{F} \otimes \alpha) \geq m\right\} \subseteq \mathrm{Pic}^0(X)$$

with $i \geq 0$ and $m \geq 1$. They are algebraic subvarieties of $\mathrm{Pic}^0(X)$. For simplicity, when $m = 1$, we will denote by $V^i(\mathcal{F}) = V_1^i(\mathcal{F})$. The union of the components containing the origin is denoted by $V_m^i(\mathcal{F})_0$.

Using the generic vanishing theory we can deduce the following codimension bound for the cohomology support loci associated to the canonical bundle

$$\operatorname{codim} V^i(\omega_X) \geq i - \dim X + a(X)$$

for every $i \geq 0$.

By combining the work of Green, Lazarsfeld, Simpson and Pareschi we can get a better understanding of the geometry of the loci $V^i(\omega_X)$ with the following linearity theorem

Theorem 1.2. ([Green and Lazarsfeld, 1991, Simpson, 1993, Pareschi, 2017]). *Let W be an irreducible component of $V^i(\omega_X)$, then W is a linear subvariety, i.e. $W = T + \alpha$ where $T \subseteq \mathrm{Pic}^0(X)$ is an abelian subvariety and $\alpha \in \mathrm{Pic}^0(X)$ is a torsion point.*

Moreover, consider the composition $h = \pi \circ a_X$

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \operatorname{Alb} X \\ & \searrow^{h} & \downarrow \pi \\ & & \widehat{T} \end{array}$$

a. Let N *be the base of the fibration part of the Stein factorization, then* $\dim N \leq \dim X - i$, *with* $i > 0$.

b. Any resolution of singularities of N *has maximal Albanese dimension.*

c. $W \subseteq h^*(\operatorname{Pic}^0(N)) + \alpha$.

Remark 1.3. If we consider higher direct images of the canonical sheaf $R^j a_{X*}\omega_X$, instead of ω_X, then any irreducible components of $V^i(R^j a_{X*}\omega_X)$ is a linear subvariety of $\operatorname{Pic}^0(X)$ by [Hacon and Pardini, 2004, Theorem 2.2 (b)].

Now, assume that X has Kodaira dimension $\operatorname{kod}(X) \geq 0$, we can describe the loci $V^0(\omega_X^{\otimes m})$ using the Iitaka fibration of X. In fact, after a birational modification of X, the *Iitaka fibration* of X has the form of a morphism $f\colon X \to Z_X$ between smooth projective varieties, with $\dim Z_X = \operatorname{kod}(X)$. The universal property of the Albanese map assures the commutativity of the following diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \operatorname{Alb}(X) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle a_f} \\ Z_X & \xrightarrow[a_{Z_X}]{} & \operatorname{Alb}(Z_X) \end{array}$$

where, by [Hacon et al., 2018, Lemma 11.1 (a)], the morphism f is surjective with connected fibers.

Combining [Hacon et al., 2018, Theorem 11.2(b), comment (2) after Lemma 11.1] with [Chen and Hacon, 2004, Lemma 2.2] we get a description of the loci $V^0(\omega_X^{\otimes m})$.

Theorem 1.4. *The irreducible components of the locus* $V^0(\omega_X)$ *are translates of abelian subvarieties of* $\widehat{a_f}(\operatorname{Pic}^0(Z_X))$ *by torsion points* $\alpha_i \in \operatorname{Pic}^0(X)$

$$V^0(\omega_X) \subseteq \bigcup_i (\alpha_i + \operatorname{Pic}^0(Z_X)).$$

Moreover, for the loci $V^0(\omega_X^{\otimes m})$ with $m \geq 2$ the above relation is an equality

$$V^0(\omega_X^{\otimes m}) = \bigcup_i (\alpha_i + \mathrm{Pic}^0(Z_X)).$$

1.2 Symmetric Fourier-Mukai Transform

We introduce some preliminary definitions and a result that will be used later, for more details refer to [Schnell, 2019].

For an abelian variety A of dimension g we denote by

$$\mathbf{R}\Delta_A(-) := \mathbf{R}\mathcal{H}om_A(-, \mathcal{O}_A[g])$$

the Groethendieck duality functor. Let $\alpha \in \mathrm{Pic}^0(A)$, we denote by $\mathcal{P}$ be a normalized Poincaré bundle on $A \times \mathrm{Pic}^0(A)$, so that $\mathcal{P}|_{A\times\{\alpha\}} \simeq \alpha$. We consider the Fourier-Mukai transform

$$\mathbf{R}\Phi_A(-) := \mathbf{R}p_{2*}(p_1^*(-) \otimes \mathcal{P})$$

where p_1 and p_2 are the projections from $A \times \mathrm{Pic}^0(A)$ onto the first and second factor, respectively. Define the *symmetric Fourier-Mukai transform* as

$$\mathbf{FM}_A = \mathbf{R}\Phi_A \circ \mathbf{R}\Delta_A.$$

Definition 1.5. We say that a coherent sheaf $\mathcal{F}$ on A satisfies *the generic vanishing condition* (GV condition) or simply is a *GV-sheaf* if it satisfies one of the conditions stated in the next Theorem [Pareschi and Popa, 2009, Theorem 2.2], [Pareschi and Popa, 2011, Theorem A].

Theorem 1.6. *In the previous setting, the following conditions are equivalent*

(a.) $\mathbf{FM}_A(\mathcal{F})$ *is a complex concentrated in degree* 0*,*

(b.) $\mathrm{codim}_{\mathrm{Pic}^0(A)} V^i(\mathcal{F}) \geq i$ *for every* $i \geq 1$*.*

GV sheaves satisfy the following properties (see [Hacon, 2004, Corollary 3.2] and [Pareschi, 2012, Lemma 1.12])

Proposition 1.7. *Consider a GV-sheaf $\mathcal{F} \neq 0$ on A, then $V^0(\mathcal{F}) \neq \emptyset$, and*

$$V^0(\mathcal{F}) \supseteq V^1(\mathcal{F}) \supseteq \cdots \supseteq V^g(\mathcal{F}).$$

Moreover, if $\operatorname{codim} V^i(\mathcal{F}) > i$ *for all $i > 0$ then*

$$V^0(\mathcal{F}) = \operatorname{Pic}^0(A).$$

We denote by $\mathcal{H}^i K$ the i-th cohomology of the complex K, then we have the following relation.

Lemma 1.8. *For any $\alpha \in \widehat{A}$ denote by $j_\alpha \colon \{\alpha\} \hookrightarrow \widehat{A}$ the closed embedding. Let $\mathcal{F}$ be a coherent sheaf on A. Then for any $\alpha \in \widehat{A}$*

$$\mathcal{H}^{-j}\mathbf{L}j_\alpha^*\mathbf{FM}_A(\mathcal{F}) \simeq H^j(A, \mathcal{F} \otimes \alpha^{-1})^\vee.$$

Moreover, if $\psi \colon \mathcal{F} \to \mathcal{G}$ is a morphism of coherent sheaves on A, then the morphism

$$\mathcal{H}^{-j}\mathbf{L}j_\alpha^*\mathbf{FM}_A(\psi) \colon \mathcal{H}^{-j}\mathbf{L}j_\alpha^*\mathbf{FM}_A(\mathcal{G}) \to \mathcal{H}^{-j}\mathbf{L}j_\alpha^*\mathbf{FM}_A(\mathcal{F})$$

is identified with the linear map

$$H^j(A, \mathcal{G} \otimes \alpha^{-1})^\vee \to H^j(A, \mathcal{F} \otimes \alpha^{-1})^\vee.$$

Proof. We have the following isomorphisms

$$\mathcal{H}^{-j}\mathbf{L}j_\alpha^*\mathbf{FM}_A(\mathcal{F}) \simeq \mathcal{H}^{-j}\mathbf{L}j_\alpha^*\left(\mathbf{R}p_{2*}\left(p_1^*\mathbf{R}\mathcal{H}om(\mathcal{F}, \mathcal{O}_A[g]) \otimes \mathcal{P}\right)\right) \simeq$$
$$\mathcal{H}^{-j}\mathbf{R}\Gamma(A, \mathbf{R}\mathcal{H}om(\mathcal{F}, \alpha[g])) \simeq \operatorname{Hom}^{-j}_{\mathbf{D}^b(A)}(\mathcal{F}, \alpha[g]) \simeq H^j(A, \mathcal{F} \otimes \alpha^{-1})^\vee.$$

where we use base change and Serre duality. □

We now recall the following useful formulas (see [Schnell, 2019, Proposition 4.1]).

Proposition 1.9. *For a homomorphism of abelian varieties $f \colon A \to B$, we have the following isomorphisms of functors*

$$\mathbf{FM}_B \circ \mathbf{R}f_* \simeq \mathbf{L}\hat{f}^* \circ \mathbf{FM}_A \qquad \mathbf{FM}_A \circ \mathbf{L}f^* \simeq \mathbf{R}\hat{f}_* \circ \mathbf{FM}_B$$

Remark 1.10. In our setting, we have that

$$\mathbf{FM}_{\mathrm{Alb}(Y)} \circ \widehat{\varphi}^* \simeq \varphi_* \circ \mathbf{FM}_{\mathrm{Alb}(X)}$$

where $\varphi\colon \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Y)$ is the Rouquier isomorphism induced by a Rouquier-stable equivalence.

Remark 1.11. We want to note that if X has maximal Albanese dimension, then ω_X is a GV-sheaf [Ein and Lazarsfeld, 1997, Remark 1.6].

The same result holds true, unconditionally on the Albanese dimension of X, for the higher direct images of the canonical bundle.

Theorem 1.12 ([Hacon, 2004], Corollary 4.2)**.** *Let X be a smooth projective variety and let a_X be the Albanese morphism. Then $R^i a_{X*}\omega_X$ is a GV-sheaf on $\mathrm{Alb}(X)$ for every $i \geq 0$.*

Remark 1.13. [Hacon and Pardini, 2004, Theorem 2.2] Theorem 1.12 still holds using $\omega_X \otimes \alpha$ for a torsion point $\alpha \in \mathrm{Pic}^0(X)$.

1.3 Higher Direct Images of the Canonical Sheaf

We introduce some theorems proved by Kollár on higher direct images of canonical sheaves, because they will often be used in the next chapters.

Theorem 1.14. *[Kollár, 1986a, Theorem 2.1], [Kollár, 1986b, Theorem 3.1]. Let X and Y be projective complex varieties of dimension d and $d-k$, with X smooth, and let $f\colon X \to Y$ be a surjective morphism. Then*

(i.) *$R^i f_*\omega_X$ is torsion-free for every $i \geq 0$;*

(ii.) *$R^i f_*\omega_X = 0$ if $i > k$;*

(iii.) *Let L be an ample line bundle on Y, then*

$$H^j(Y, L \otimes R^i f_*\omega_X) = 0$$

for every $i \geq 0$ and $j > 0$;

(iv.) There is the following decomposition, in the derived category of Y

$$\mathbf{R}f_*\omega_X \simeq \bigoplus_{i=0}^{k} R^i f_*\omega_X[-i].$$

Remark 1.15. Following [Kollár, 1986b, Section §3], in the Theorem above ω_X can be replaced with $\omega_X \otimes \beta$, with β a torsion point of $\mathrm{Pic}^0(X)$.

Combining Theorem 1.14(ii.) and (iv.) and using projection formula one can obtain the following useful relation

$$H^i(X, \omega_X \otimes a_X^*\alpha) = \bigoplus_{h=0}^{\min\{i,k\}} H^{i-h}(A, R^h a_{X*}\omega_X \otimes \alpha)$$

where $\alpha \in \mathrm{Pic}^0(X)$.

Theorem 1.16. *[Kollár, 1986a, Proposition 7.6] Let X and Y be smooth projective varieties and let $f\colon X \to Y$ be a surjective morphism with connected fibres. If* $\dim X = n$ *and* $\dim Y = k$*. Then*

$$R^{n-k} f_*\omega_X \simeq \omega_Y.$$

Theorem 1.17. *[Kollár, 1986b, Theorem 3.4] Let X be a smooth projective variety and let $f\colon X \to Y$ and $g\colon Y \to Z$ be surjective morphisms between projective varieties. Then*

$$R^p(g \circ f)_*\omega_X \simeq \sum_j R^j g_* R^{p-j} f_*\omega_X$$

Chapter 2

Derived Invariants of Irregular Varieties

2.1 The irregularity

Let X and Y be two smooth complex projective irregular varieties of dimension n, such that there exists a derived equivalence $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$, where $\mathbf{D}^{\mathrm{b}}(X)$ and $\mathbf{D}^{\mathrm{b}}(Y)$ denote the bounded derived categories of coherent sheaves on X and Y, respectively. By an Orlov's result, every such equivalence can be represented by a unique, up to isomorphism, Fourier-Mukai functor $\Phi_{\mathcal{E}}$, with $\mathcal{E} \in \mathbf{D}^{\mathrm{b}}(X \times Y)$ which is called *kernel*, such that

$$\Phi_{\mathcal{E}}(\cdot) = \mathbf{R}p_{Y*}(p_X^*(\cdot) \otimes \mathcal{E})$$

where p_X and p_Y are the natural projections from $X \times Y$ onto X and Y, respectively.

Rouquier in [Rouquier, 2011, Théoréme 4.18] proved that Φ induces an isomorphism of algebraic groups

$$\varphi\colon \ \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \xrightarrow{\sim} \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y) \tag{2.1}$$

called *Rouquier isomorphism* (for more details see [Popa and Schnell, 2011, footnote p. 531]).

Rouquier's result show that if $(f, \alpha) \in \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X)$ then $\varphi(f, \alpha)$ is again of the form (g, β) for a unique pair $(g, \beta) \in \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$. In fact, we can associate (f, α) with the autoequivalence of $\mathbf{D}^{\mathrm{b}}(X)$ defined by $f_*(\alpha \otimes (\cdot))$, with kernel of the form $(\mathrm{id}, f)_* \alpha \in \mathbf{D}^{\mathrm{b}}(X \times Y)$, where $(\mathrm{id}, f)\colon X \to X \times X$ is the graph immersion $x \mapsto (x, f(x))$. Rouquier proved that if $(f, \alpha) \in \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X)$ then the following composition is again an autoequivalence with the same form

$$\Phi_{\mathcal{E}} \circ \Phi_{(\mathrm{id},f)_*\alpha} \circ \Phi_{\mathcal{E}}^{-1} \simeq \Phi_{(\mathrm{id},g)_*\beta} \tag{2.2}$$

for a unique pair $(g, \beta) \in \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$.

We can provide a direct description of the Rouquier isomorphism in terms of the kernel $\mathcal{E}$: with the previous notation, $\varphi(f, \alpha) = (g, \beta)$ if and only if

$$p_X^*\alpha \otimes (f \times \mathrm{id}_Y)^*\mathcal{E} \simeq p_Y^*\,\beta \otimes (\mathrm{id}_X \times g)_*\mathcal{E}$$

(refer to [Orlov, 2003, Corollary 5.1.10] and [Popa and Schnell, 2011, Lemma 3.1]).

The map φ has the property that its differential at the origin corresponds to the linear map

$$\mathrm{d}\varphi_0\colon H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X) \to H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O}_Y) \tag{2.3}$$

which is the map between the first Hochschild cohomology groups $HH^1(X)$ and $HH^1(Y)$. These groups can be described in the following way.

Definition 2.1. Let $\delta_X\colon X \to X \times X$ be the diagonal embedding of X, then the *Hochschild homology* of X is

$$HH_*(X) := \bigoplus_i \mathrm{Ext}^i_{X\times X}(\delta_{X*}\mathcal{O}_X, \delta_{X*}\omega_X)$$

which is a graded module over the *Hochschild cohomology* of X

$$HH^*(X) := \bigoplus_i \mathrm{Ext}^i_{X\times X}(\delta_{X*}\mathcal{O}_X, \delta_{X*}\mathcal{O}_X).$$

Remark 2.2. Any derived equivalence $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$, induces an isomorphism so that the Hochschild homology and cohomology are derived invariants (see [Orlov, 2003, Theorem 2.1.8] and [Căldăraru, 2003a, Theorem 8.1]).

Theorem 2.3. *Let* $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ *be an exact equivalence. Then* Φ *induces the following isomorphisms*

a. $\Phi^{HH^*}\colon HH^*(X) \to HH^*(Y)$ *as graded rings*

b. $\Phi_{HH_*}\colon HH_*(X) \to HH_*(Y)$ *as graded modules.*

In order to better understand these isomorphisms we will follow [Swan, 1996, Kontsevich, 2003] and [Hochschild et al., 1962] . We use the fact that the local-to-global spectral sequence for Ext

$$E_2^{p,q} = H^p(X \times X, \mathcal{E}xt^q(\delta_{X*}\mathcal{O}_X, \delta_{X*}\mathcal{O}_X)) \Rightarrow \mathrm{Ext}^{p+q}(\delta_{X*}\mathcal{O}_X, \delta_{X*}\mathcal{O}_X)$$

degenerates al the page E_2. We recall that $\mathcal{E}xt^q(\delta_{X*}\mathcal{O}_X, \delta_{X*}\mathcal{O}_X) \simeq \bigwedge^q T_X$, then we have

$$HH^i(X) = \mathrm{Ext}^i(\delta_{X*}\mathcal{O}_X, \delta_{X*}\mathcal{O}_X) \simeq \bigoplus_{p+q=i} H^p(X, \bigwedge^q T_X)$$

and

$$HH_i(X) = \mathrm{Ext}^i(\delta_{X*}\mathcal{O}_X, \delta_{X*}\omega_X) \simeq \bigoplus_{p-q=i-n} H^p(X, \Omega_X^q).$$

In particular, combining the previous results we obtain the derived invariance of the $(i-n)$-th column of the Hodge diamond

Corollary 2.4. *A derived equivalence induces the following vector spaces isomorphisms*

$$\bigoplus_{p-q=i-n} H^p(X, \Omega_X^q) \simeq \bigoplus_{p-q=i-n} H^p(Y, \Omega_Y^q).$$

Remark 2.5. From the isomorphism $\Phi^{HH^1}\colon HH^1(X) \xrightarrow{\sim} HH^1(Y)$ we obtain the following decomposition

$$\Phi^{HH^1}\colon H^0(X, T_X) \oplus H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^0(Y, T_Y) \oplus H^1(Y, \mathcal{O}_Y).$$

which is the same isomorphism described in (2.3).

Focusing on the dimensions, the invariance of the first Hochschild cohomology implies that the following sum is also invariant

$$h^0(X, T_X) + h^1(X, \mathcal{O}_X) = h^0(Y, T_Y) + h^1(Y, \mathcal{O}_Y).$$

Popa and Schnell further developed this result proving that, actually, each term of the previous sum is invariant.

Theorem 2.6. [Popa and Schnell, 2011, Theorem A] *Let* $\mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ *be a derived equivalence. Then*

a. $\mathrm{Pic}^0(X)$ *and* $\mathrm{Pic}^0(Y)$ *are isogenous.*

b. $\mathrm{Pic}^0(X)$ *and* $\mathrm{Pic}^0(Y)$ *are isomorphic unless* X *and* Y *are étale locally trivial fibrations over isogenous positive dimensional abelian varieties, and* $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y) = 0$.

The proof of Theorem 2.6 relies on a careful analysis of the Rouquier isomorphism.

As consequences of the last theorem we have the following results.

Theorem 2.7. *Let* $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \xrightarrow{\sim} \mathbf{D}^{\mathrm{b}}(Y)$ *an exact equivalence. Then*

$$h^0(X, T_X) = h^0(Y, T_Y) \text{ and } h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y).$$

This result, in particular, establishes the derived invariance of the so-called *irregularity* of X, denoted by

$$q(X) := h^1(X, \mathcal{O}_X) = h^0(X, \Omega^1_X),$$

which is the dimension of the varieties $\mathrm{Pic}^0(X)$ and $\mathrm{Alb}(X)$.

In particular, this implies that if $\dim X \leq 3$, then all the Hodge numbers are derived invariants, refer to [Popa and Schnell, 2011, Corollary C].

Theorem 2.8. *Suppose* X *and* Y *are two derived equivalent smooth projective varieties, with* $\dim X \leq 3$. *Then*

$$h^{p,q}(X) = h^{p,q}(Y)$$

for every p *and* q.

2.2 Non-vanishing Loci

Our primary interest lies in the case when the Rouquier isomorphism (2.1) respects the factors, in other words, when there exists an isomorphism, which we will also denote with φ, between $\mathrm{Pic}^0(X)$ and $\mathrm{Pic}^0(Y)$.

Definition 2.9. A closed point $\alpha \in \mathrm{Pic}^0(X)$ is said to be *Rouquier-stable* if

$$\varphi(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$$

for some $\beta \in \mathrm{Pic}^0(Y)$. In this case, we denote $\beta = \varphi(\alpha)$. We say that an abelian subvariety $B \subseteq \mathrm{Pic}^0(X)$ is *Rouquier-stable* if every point of B is Rouquier-stable. Furthermore, we say that a derived equivalence Φ is *Rouquier-stable* if $\varphi(\{\mathrm{id}_X\} \times \mathrm{Pic}^0(X)) = \{\mathrm{id}_Y\} \times \mathrm{Pic}^0(Y)$ (or equivalently, $H^1(X, \mathcal{O}_X) \simeq H^1(Y, \mathcal{O}_Y)$ via $\mathrm{d}\varphi_0$).

We study a particular case of the Conjecture A about the invariance of the Hodge numbers.

Conjecture 2.10. *If X and Y are two derived equivalent smooth projective complex varieties, then*

$$h^{0,j}(X) = h^{0,j}(Y)$$

for every $j \geq 0$.

There is another conjecture related to the derived invariance of Hodge numbers, formulated by Lombardi and Popa in [Lombardi and Popa, 2015], which concerns the cohomology support loci of the canonical sheaf of a smooth projective variety X. They conjectured the invariance of $V^i_m(\omega_X)_0$ for every $m \geq 1$.

Conjecture 2.11. *Let X and Y two smooth projective complex varieties with equivalent derived categories. Then*

$$\varphi\left(\{\mathrm{id}_X\} \times V^i(\omega_X)_0\right) = \{\mathrm{id}_Y\} \times V^i(\omega_Y)_0$$

for every $i \geq 0$. Moreover, if $\alpha \in V^i(\omega_X)_0$ is a Rouquier-stable line bundle, then there are the following equalities

$$h^i(X, \omega_X \otimes \alpha) = h^i(Y, \omega_Y \otimes \varphi(\alpha))$$

for every $i \geq 0$.

This problem was further studied in [Lombardi, 2014, Lombardi and Popa, 2015] by Lombardi and Popa, who proved that the two previous conjectures are related in the following sense.

Theorem 2.12. [Lombardi and Popa, 2015, Theorem 12]. *Conjecture 2.10 is equivalent to Conjecture 2.11. In fact, if Conjecture 2.10 is true for* $\dim X - j$ *then*

$$\varphi\left(\mathrm{id}_X, V^j_m(\omega_X)_0\right) = \left(\mathrm{id}_Y, V^j_m(\omega_Y)_0\right)$$

for every $m \geq 1$.

As a consequence they prove Conjecture 2.11 in the following cases:

Corollary 2.13. *Suppose* $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$. *Then*

a. $V^i(\omega_X)_0 \simeq V^i(\omega_Y)_0$ *for* $i = 0, 1, \dim X - 1, \dim X$,

b. the same result holds for every i *in dimension up to* 3, *and for fourfolds with maximal Albanese dimension.*

2.3 The Albanese Map

Let $a_X \colon X \to \mathrm{Alb}(X)$ be the *Albanese map* of X. Denote by $a(X) = \dim a_X(X)$, then X is said to have *maximal Albanese dimension* if

$$a(X) = \dim X.$$

Caucci and Pareschi, building on the work of Lombardi and Popa, showed that the above Conjecture 2.10 is true for every $i \geq 0$ if X has maximal Albanese dimension, as a consequence of the following theorem. We identify $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(\mathrm{Alb}(X))$.

Theorem 2.14. [Caucci and Pareschi, 2019, Theorem 1.1] *Let* X *be a smooth projective complex variety. Suppose* $\alpha \in \mathrm{Pic}^0(X)$ *is a Rouquier-stable line bundle, then*

$$h^i(\mathrm{Alb}(X), a_{X*}\omega_X \otimes \alpha) = h^i(\mathrm{Alb}(Y), a_{Y*}\omega_Y \otimes \varphi(\alpha))$$

for every $i \in \mathbb{N}$.

In particular, Theorem 2.14 implies that the cohomology support loci associated to the pushforward of the Albanese map of the canonical bundle are derived invariant, too.

Corollary 2.15. *Given two derived equivalent varieties X and Y, there is an isomorphism*

$$V^i_m(a_{X*}\omega_X) \simeq V^i_m(a_{Y*}\omega_Y)$$

for every $i \geq 0$ and $m \geq 1$.

Using Kollár's Theorem 1.14 on the degeneration of the Leray spectral sequence, one can verify that if X has maximal Albanese dimension, then $V^i(a_{X*}\omega_X) \simeq V^i(\omega_X)$, which by Corollary 2.15 implies that $V^i(\omega_X) \simeq V^i(\omega_Y)$, and this proves the Lombardi and Popa's conjecture in the maximal Albanese dimension case.

Furthermore, if X has maximal Albanese dimension, as a consequence of Theorem 2.14 one can obtain the derived invariance of the Hodge numbers. In fact, using Theorem 1.14, we deduce that $h^i(X,\omega_X) = h^i(\mathrm{Alb}(X), a_{X*}\omega_X)$ and therefore the following Corollary.

Corollary 2.16. *Let X and Y be smooth projective complex varieties such that $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$. If X has maximal Albanese dimension, then*

$$h^{0,j}(X) = h^{0,j}(Y)$$

for every $j \geq 0$.

The proof of Theorem 2.14 uses a result by Lombardi about the invariance of the twisted Hochschild homology.

Definition 2.17. Let $\delta_X \colon X \to X \times X$ be the diagonal morphism of X and let $\alpha \in \mathrm{Pic}^0(X)$ be a Rouquier-stable line bundle. Then, following [Lombardi, 2014] we can define the *twisted Hochschild homology*, for $m \in \mathbb{Z}$, as follows

$$HH^m_*(X,\alpha) = \bigoplus_k \mathrm{Ext}^k_{\mathcal{O}_{X\times X}}(\delta_{X*}\mathcal{O}_X, \delta_{X*}(\omega_X^{\otimes m} \otimes \alpha)), \tag{2.4}$$

which is a graded module over the *Hochschild cohomology*

$$HH^*(X) := \bigoplus_i \mathrm{Ext}^i_{X\times X}(\delta_*\mathcal{O}_X, \delta_*\mathcal{O}_X).$$

Lombardi proved a more general theorem than the one we are going to recall ([Lombardi, 2014, Theorem 1.1]), in our case it corresponds to the invariance of $HH_*^m(X, \alpha)$.

Theorem 2.18. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence and let $m \in \mathbb{Z}$, then Φ induces an isomorphism of graded modules*

$$HH_*^m(X, \alpha) \simeq HH_*^m(Y, \varphi(\alpha))$$

where $\alpha \in \operatorname{Pic}^0(X)$ is a Rouquier-stable line bundle and φ denotes the Rouquier isomorphism.

An important consequence of the last theorem is that the Rouquier isomorphism φ induces an isomorphism between $V^0(\omega_X^{\otimes m})$ and $V^0(\omega_Y^{\otimes m})$, in the following way.

Proposition 2.19. *Let $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence between smooth projective varieties and let $m \in \mathbb{Z}$ and $r \geq 1$. If $\alpha \in V_r^0(\omega_X^{\otimes m})$ then $\varphi(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$ with $\beta \in V_r^0(\omega_Y^{\otimes m})$. Moreover the Rouquier isomorphism acts as follows*

$$\varphi\left(\{\mathrm{id}_X\} \times V_r^0(\omega_X^{\otimes m})\right) = \{\mathrm{id}_Y\} \times V_r^0(\omega_Y^{\otimes m}).$$

2.4 The Albanese-Iitaka Fibration

We now introduce the more general definition of the Albanese-Iitaka morphism for a variety X with $\operatorname{kod} X \geq 0$.

Definition 2.20. We consider a smooth birational modification $\widetilde{X} \to X$ such that the Iitaka fibration of X can be represented as the morphism $\widetilde{f}\colon \widetilde{X} \to Z_X$ between smooth algebraic varieties. Denote with a_X and a_{Z_X} the Albanese morphisms of X and Z_X, respectively. We have the following commutative diagram

$$\begin{array}{ccccc} \widetilde{X} & \longrightarrow & X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ & \searrow^{\widetilde{f}} & \downarrow & \searrow^{c_X} & \downarrow{\scriptstyle p_X} \\ & & Z_X & \xrightarrow[a_{Z_X}]{} & \mathrm{Alb}(Z_X) \end{array} \tag{2.5}$$

where p_X is a surjective morphism with connected fibers between abelian varieties induced by $\widetilde{f}$ (see [Hacon et al., 2018, Lemma 11.1]). We define the *Albanese-Iitaka morphism* of X as the composition $c_X = p_X \circ a_X$

$$c_X \colon X \to \mathrm{Alb}(Z_X).$$

Remark 2.21. We recall that an abelian variety $B \subseteq \mathrm{Pic}^0(X)$ is called *Rouquier-stable* if $\varphi(\mathrm{id}_X, \alpha) = (\mathrm{id}_Y, \beta)$ for every point $\alpha \in B$ and for some $\beta \in \mathrm{Pic}^0(Y)$ (see Definition 2.9). As a consequence of [Caucci and Pareschi, 2019, Lemma 3.4] we have that the abelian variety $\mathrm{Pic}^0 Z_X$, seen as subvariety of $\mathrm{Pic}^0 X$ is Rouquier-stable. In fact, the Rouquier isomorphism induces an isomorphism

$$\varphi\left(\mathrm{Pic}^0(Z_X)\right) \simeq \mathrm{Pic}^0(Z_Y). \tag{2.6}$$

Recall that given a morphism $f \colon A \to B$ of abelian varieties, the dual morphism is denoted by $\widehat{f} \colon \mathrm{Pic}^0(B) \to \mathrm{Pic}^0(A)$. In our case, we consider the dual morphism of (2.6) denoted by $\widehat{\varphi}$ from $\mathrm{Alb}(Z_Y)$ to $\mathrm{Alb}(Z_X)$. Then the Stein factorization of the Albanese-Iitaka morphism is a derived invariant by the following theorem ([Caucci et al., 2022, Theorem 3.0.1]).

Theorem 2.22. *Let $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ be an exact equivalence, we consider the Stein factorizations of the Albanese-Iitaka morphisms c_X and c_Y*

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Alb}(X) \\ \downarrow & \searrow^{c_X} & \downarrow p_X \\ X' & \xrightarrow{c'_X} & \mathrm{Alb}(Z_X) \end{array} \qquad \begin{array}{ccc} Y & \longrightarrow & \mathrm{Alb}(Y) \\ \downarrow & \searrow^{c_Y} & \downarrow p_Y \\ Y' & \xrightarrow{c'_Y} & \mathrm{Alb}(Z_Y). \end{array}$$

Then there is an isomorphism $Y' \simeq X'$ such that the following diagram

$$\begin{array}{ccc} X' & \xleftarrow{\ \simeq\ } & Y' \\ {\scriptstyle c_{X'}}\downarrow & & \downarrow {\scriptstyle c_{Y'}} \\ \mathrm{Alb}(Z_X) & \xleftarrow[\widehat{\varphi}]{} & \mathrm{Alb}(Z_Y) \end{array}$$

is commutative.

2.5 The Relative twisted Hochschild structure

We start by recalling a result by Orlov [Orlov, 2003, Proposition 2.1.7]. Denote with δ_X and δ_Y the diagonal embeddings of X and Y, respectively.

Proposition 2.23. *If we denote by p_{ij} the projections from $X \times X \times Y \times Y$ to the (i, j)-th factor, then a derived equivalence $\Phi_{\mathcal{E}} \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ induces another equivalence $\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*} \colon \mathbf{D}^{\mathrm{b}}(X \times X) \to \mathbf{D}^{\mathrm{b}}(Y \times Y)$, with $\mathcal{E} \boxtimes \mathcal{E}^* = p_{13}^*\mathcal{E} \otimes p_{24}^*\mathcal{E}^*$, such that for every $m \in \mathbb{Z}$ we have*

$$\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*}(\delta_{X*}\omega_X^{\otimes m}) \simeq \delta_{Y*}\omega_Y^{\otimes m} \tag{2.7}$$

where $\mathcal{E}^* = \mathcal{E}^\vee \otimes p_X^*\,\omega_X[n] \simeq p_Y^*\,\omega_Y[n]$ and $\mathcal{E}^\vee = \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_{X\times Y})$.

The result in (2.7) was generalized by Lombardi, see [Lombardi, 2014, Lemma 2.1].

Lemma 2.24. *Consider the automorphisms $f \in \mathrm{Aut}^0(X)$ and $g \in \mathrm{Aut}^0(Y)$ and the embeddings $(\mathrm{id}_X, f) \colon X \to X \times X$ and $(\mathrm{id}_Y, g) \colon Y \to Y \times Y$ defined by $x \mapsto (x, f(x))$ and $y \mapsto (y, g(y))$, respectively. If $\varphi(f, \alpha) = (g, \beta)$, with $\alpha \in \mathrm{Pic}^0(X)$ and $\beta \in \mathrm{Pic}^0(Y)$, then*

$$\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*}\left((\mathrm{id}_X, f)_*(\omega_X^{\otimes m} \otimes \alpha)\right) \simeq (\mathrm{id}_Y, g)_*(\omega_Y^{\otimes m} \otimes \beta)$$

for all $m \in \mathbb{Z}$.

In particular, if α is a Rouquier-stable line bundle, i.e. $\varphi(\alpha) = \beta$ for some $\beta \in \mathrm{Pic}^0(Y)$ (see Definition 2.9), then the last isomorphism can be written in the following way

$$\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*}\left(\delta_{X*}(\omega_X^{\otimes m} \otimes \alpha)\right) \simeq \delta_{Y*}(\omega_Y^{\otimes m} \otimes \varphi(\alpha)). \tag{2.8}$$

Now, following [Caucci et al., 2022, Section §4.2], let e be the identity of $\mathrm{Alb}(Z_X)$, we consider a normalized Poincaré line bundle $\mathcal{P}_Z$ on $\mathrm{Alb}(Z_X) \times \mathrm{Pic}^0(Z_X)$ such that $(\mathcal{P}_Z)\,|_{\{e\}\times\mathrm{Pic}^0(Z_X)}$ is trivial. In the same way, we choose the Poincaré bundle $\mathcal{Q}_Z$ on $\mathrm{Alb}(Z_Y) \times \mathrm{Pic}^0(Z_Y)$. Recall that by Remark 2.6, there is an isomorphism $\varphi \colon \mathrm{Pic}^0(Z_X) \to \mathrm{Pic}^0(Z_Y)$. The line bundles $\mathcal{P}_Z$ and $\mathcal{Q}_Z$ are connected by the relation

$$(\widehat{\varphi}^{-1} \times \varphi)^* \mathcal{Q}_Z \simeq \mathcal{P}_Z,$$

which follows from the universal property of the Poincaré bundle. Using the notation of Theorem 2.22, we define the induced Poincaré line bundle on $X \times \mathrm{Pic}^0(Z_X)$ as follows

$$\mathcal{P}_{Z,X} := (c_X \times \mathrm{id})^* \mathcal{P}_Z,$$

in an analogous way we define

$$Q_{Z,Y} := (c_Y \times \mathrm{id})^* \mathcal{Q}_Z,$$

on $Y \times \mathrm{Pic}^0(Z_Y)$. We denote by

$$\widetilde{\delta_X} \colon X \times \mathrm{Pic}^0(Z_X) \to X \times X \times \mathrm{Pic}^0(Z_X)$$

the *relative diagonal embedding* defined by $(x, \alpha) \mapsto (x, x, \alpha)$, and similarly $\widetilde{\delta_Y}$ for Y. Through a process of globalization of the isomorphisms (2.8), Caucci, Lombardi and Pareschi proved the following theorem ([Caucci et al., 2022, Theorem 4.3.1]).

Theorem 2.25. *Let $\Phi \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence and let $\varphi \colon \mathrm{Pic}^0(Z_X) \to \mathrm{Pic}^0(Z_Y)$ be the isomorphism* (2.6) *induced by the Rouquier isomorphism. We consider $\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*} \boxtimes \varphi_*$ which is a derived equivalence between $\mathbf{D}^{\mathrm{b}}(X \times X \times \mathrm{Pic}^0(Z_X))$ and $\mathbf{D}^{\mathrm{b}}(Y \times Y \times \mathrm{Pic}^0(Z_Y))$. Then for every $m \in \mathbb{Z}$ there are isomorphisms*

$$(\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*} \boxtimes \varphi_*)(\widetilde{\delta_X}_*(p_X^* \omega_X^{\otimes m} \otimes \mathcal{P}_{Z,X})) \simeq \widetilde{\delta_Y}_*(p_Y^* \omega_Y^{\otimes m} \otimes \mathcal{Q}_{Z,Y})$$

where p_X and p_Y denote the natural projections $X \times \mathrm{Pic}^0(Z_X) \to X$ and $Y \times \mathrm{Pic}^0(Z_Y) \to Y$, respectively.

Remark 2.26. The result in Theorem 2.25 still holds if we replace $\omega_X^{\otimes m}$ with $\omega_X^{\otimes m} \otimes \alpha$, for a Rouquier-stable line bundle $\alpha \in \mathrm{Pic}^0(X)$

$$(\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*} \boxtimes \varphi_*)(\widetilde{\delta_X}_*(p_X^*(\omega_X^{\otimes m} \otimes \alpha) \otimes \mathcal{P}_{Z,X})) \simeq \widetilde{\delta_Y}_*(p_Y^*(\omega_Y^{\otimes m} \otimes \varphi(\alpha)) \otimes \mathcal{Q}_{Z,Y}).$$

The following theorem ([Caucci et al., 2022, Theorem 4.4.1]) is one of the key ingredients to prove the derived invariance of the relative twisted Hochschild structure (see below) and the relative canonical ring (section §2.6).

Theorem 2.27. *There is an isomorphism of functors*

$$\mathbf{R}c_{Y*} \circ \mathbf{L}\delta_Y^* \simeq \widehat{\varphi}^* \circ \mathbf{R}c_{X*} \circ \mathbf{L}\delta_X^* \circ \Phi_{\mathcal{E}\boxtimes\mathcal{E}^*}$$

where $\Phi_{\mathcal{E}\boxtimes\mathcal{E}^*}$ is taken in the reverse direction than before, from $\mathbf{D}^{\mathrm{b}}(Y \times Y)$ to $\mathbf{D}^{\mathrm{b}}(X \times X)$, and $\widehat{\varphi}$ denotes the dual of the Rouquier isomorphism.

We want to recall the *Hochschild-Konstant-Rosemberg* (HKR) isomorphism

$$\mathbf{L}\delta_X^*(\delta_{X*}\omega_X^{\otimes m}) \simeq \bigoplus_{i=0}^{\dim X} \Omega_X^i \otimes \omega_X^{\otimes m}[i]. \tag{2.9}$$

Definition 2.28. The *relative twisted Hochschild structure* of a smooth projective complex variety X with respect to the Albanese-Iitaka morphism c_X is defined as

$$\mathcal{HH}(c_X) := \bigoplus_{m,p} \mathbf{R}c_{X*}(\Omega_X^p \otimes \omega_X^{\otimes m})[p].$$

Theorem 2.27 implies the derived invariance of relative twisted Hochschild structure of X, the proof involves the isomorphism in (2.7) and the HKR isomorphism (see [Caucci et al., 2022, Corollary 4.5.1]).

Corollary 2.29. *There are the following isomorphisms*

$$\bigoplus_{p-q=i} R^p c_{Y*}(\Omega_Y^q \otimes \omega_Y^{\otimes m}) \simeq \bigoplus_{p-q=i} \widehat{\varphi}^* \left(R^p c_{X*}(\Omega_X^q \otimes \omega_X^{\otimes m})\right).$$

for every $m, i \in \mathbb{Z}$. As a consequence we obtain the following isomorphisms

$$c_{Y*}\omega_Y^{\otimes m} \simeq \widehat{\varphi}^*(c_{X*}\omega_X^{\otimes m})$$

for every m.

Remark 2.30. Corollary 2.29 still holds if we replace $\Omega_X^p \otimes \omega_X^m$ with $\Omega_X^p \otimes \omega_X^m \otimes \alpha$, for any Rouquier-stable line bundle $\alpha \in \mathrm{Pic}^0(X)$

$$\bigoplus_{p-q=i} R^p c_{Y*}(\Omega_Y^q \otimes \omega_Y^m \otimes \varphi(\alpha)) \simeq \bigoplus_{p-q=i} \widehat{\varphi}^* \left(R^p c_{X*}(\Omega_X^q \otimes \omega_X^m \otimes \alpha)\right).$$

In particular, we have the following isomorphisms

$$c_{Y*}(\omega_Y^{\otimes m} \otimes \varphi(\alpha)) \simeq \widehat{\varphi}^*(c_{X*}(\omega_X^{\otimes m} \otimes \alpha))$$

for every m.

2.6 The Relative Canonical Ring

We first introduce the following definition.

Definition 2.31. The *relative canonical ring* of X with respect to the Albanese-Iitaka morphism is

$$\mathcal{R}(c_X) = \bigoplus_{m \geq 0} c_{X*}\omega_X^{\otimes m}.$$

Combining the result in Corollary 2.29 with the generic vanishing theory, Caucci, Lombardi and Pareschi, in [Caucci et al., 2022, Theorem 4.6.1] proved the derived invariance of the relative canonical ring.

Theorem 2.32. *Let $\mathbf{D}^{\mathrm{b}}(X) \simeq \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence. Then there exists an isomorphism of $\mathcal{O}_{\mathrm{Alb}(Z_Y)}$-algebras*

$$\widehat{\varphi}^*(\mathcal{R}(c_X)) \simeq \mathcal{R}(c_Y).$$

In [Caucci et al., 2022] the last statement was generalized in the following way. Let $\widehat{B_X} \subseteq \mathrm{Pic}^0(X)$ be a Rouquier-stable abelian subvariety, then by definition there exists an abelian subvariety $\widehat{B_Y}$ of $\mathrm{Pic}^0(Y)$ such that $\varphi(\mathrm{id_X}, \widehat{B_X}) = (\mathrm{id_Y}, \widehat{B_Y})$. This means that the Rouquier isomorphism induces an isomorphism

$$\varphi\colon \widehat{B_X} \xrightarrow{\sim} \widehat{B_Y}.$$

In an analogous situation of Theorem 2.22, we can consider the morphisms $b_X = p_X \circ a_X$ and $b_Y = p_Y \circ a_Y$, where p_X is the dual morphism of $\widehat{B_X} \hookrightarrow \mathrm{Pic}^0(X)$ and p_Y is the dual of $\widehat{B_Y} \hookrightarrow \mathrm{Pic}^0(Y)$. Consider the Stein factorizations of b_X and

b_Y, then we get the following commutative diagrams

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \big\downarrow & \overset{b_X}{\searrow} & \big\downarrow{\scriptstyle p_X} \\ X' & \xrightarrow[b'_X]{} & B_X \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{a_Y} & \mathrm{Alb}(Y) \\ \big\downarrow & \overset{b_Y}{\searrow} & \big\downarrow{\scriptstyle p_Y} \\ Y' & \xrightarrow[b'_Y]{} & B_Y. \end{array}$$

In this way, Caucci, Lombardi and Pareschi proved the following generalization of Theorem 2.22.

Theorem 2.33. [Caucci et al., 2022, Theorem 8.1.1] *With the above notation, there exists an isomorphism $Y' \simeq X'$ such that the following diagram is commutative*

$$\begin{array}{ccc} X' & \xleftarrow{\ \simeq\ } & Y' \\ {\scriptstyle b'_X}\big\downarrow & & \big\downarrow{\scriptstyle b'_Y} \\ B_X & \xleftarrow[\widehat{\varphi}]{} & B_Y. \end{array}$$

With the same arguments they generalize Theorem 2.32. Let

$$\mathcal{R}(b_X) = \bigoplus_{m \geq 0} b_{X*}\omega_X^{\otimes m}$$

be the relative canonical ring of b_X.

Theorem 2.34. [Caucci et al., 2022, Theorem 8.1.2] *Let $b_X \colon X \to B_X$ be a morphism where B_X is the dual of a Rouquier-stable abelian subvariety of* $\mathrm{Pic}^0 X$. *Then there is an isomorphism of $\mathcal{O}_{\mathrm{Alb}(Z_Y)}$-algebras*

$$\widehat{\varphi}^*(\mathcal{R}(b_X)) \simeq \mathcal{R}(b_Y).$$

2.7 Irregular Fibrations

Let X be a smooth projective variety. We recall that a fibration of X is an algebraic fibre space $\pi \colon X \to S$, with S a positive-dimensional normal variety.

One can be define a Rouquier-stable abelian subvariety related to any fibration onto a normal projective variety in the following way. Let $\pi \colon X \to S$

be a fibration and consider a *non-singular representative of* π, which is a fibration $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{S}$ onto a smooth projective variety $\widetilde{S}$, together with a birational morphisms $g\colon \widetilde{X} \to X$ and $h\colon \widetilde{S} \to S$ such that the following

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{g} & X \\ {\scriptstyle\widetilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \widetilde{S} & \xrightarrow{h} & S \end{array}$$

is commutative. Then, independently on the choice of the non-singular representative of π, we can define the abelian subvariety $\widehat{B_S}$ as

$$\widehat{B_S} := g_*\widetilde{\pi}^*\mathrm{Pic}^0\widetilde{S} \subset \mathrm{Pic}^0(X).$$

It turns out that if S is of general type, then $\widehat{B_S}$ is Rouquier-stable (refer to [Caucci and Lombardi, 2022, Lemma 4.0.2] for more details). This result was very useful in the proof of the theorem below. First, we need to introduce the definition of irregular fibration.

Definition 2.35. In the above setting, we say that a fibration is *irregular* if a non-singular model of S, therefore any of them, has maximal Albanese dimension. Two irregular fibrations of X, $\pi_1\colon X \to S_1$ and $\pi_2\colon X \to S_2$, are said to be *equivalent* if there exists a birational morphism $f\colon S_1 \dashrightarrow S_2$ such that $\pi_2 = f \circ \pi_1$. We call *irregular k-fibration*, with $k \in \{0, \dots, \dim X\}$, an irregular fibration $X \to S$ with $\dim S = k$. We define the sets

$$G_{X,k} := \{\text{equivalence classes of irregular } k\text{-fibration } \pi\colon X \to S, \text{ with } S \text{ of general type}\}.$$

Caucci and Lombardi obtained the following generalization of [Lombardi, 2022, Theorem 1].

Theorem 2.36. [Caucci and Lombardi, 2022, Theorem 4.4.1] *Let X and Y be derived equivalent varieties. Then there exists a base-preserving bijection ν between the sets $G_{X,k}$ and $G_{Y,k}$. Moreover, suppose that $\nu(\pi_X\colon X \to S) = (\pi_Y\colon Y \to T)$, then S and T are birational and there is a derived equivalence between the generic fibres of π_X and π_Y.*

In the same article, they also proved the following theorem.

Theorem 2.37. [Caucci and Lombardi, 2022, Theorem 4.0.1] *Let X and Y be two smooth projective derived equivalent varieties. In the previous setting, suppose that $\widehat{B_S}$ is a Rouquier-stable subvariety. Then Y admits an irregular fibration $\theta\colon Y \to T$, such that T is birational to S. Moreover, the general fibers of π and θ are derived equivalent.*

Chapter 3

Rouquier-stable Equivalences

In this chapter we investigate the derived invariance of the cohomology ranks and the non-vanishing loci of higher direct images of the canonical bundle under the Albanese map of a smooth projective complex variety X. We prove the derived invariance of the top non-trivial higher direct image of the canonical bundle under the Albanese map. We first denote by $c(X)$ the general fiber dimension of the Albanese map $a_X\colon X \to \mathrm{Alb}(X)$

$$c(X) = \dim X - \dim a_X(X).$$

The main result is the following

Theorem *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a Rouquier-stable equivalence. Then $c(X) = c(Y)$ and the Rouquier isomorphism induces the following isomorphism of sheaves*

$$R^{c(Y)}a_{Y*}\omega_Y \simeq \widehat{\varphi}^* R^{c(X)}a_{X*}\omega_X.$$

Moreover, we prove the other results stated in the introduction.

3.1 The Albanese Dimension

Let X be a smooth projective complex variety of dimension n. Let $a_X\colon X \to \mathrm{Alb}(X)$ be the Albanese map of X and $a(X) = \dim a_X(X)$.

We first recall a result of Lombardi ([Lombardi, 2014, Theorem 1.6])

Theorem 3.1. *Let X and Y be two smooth projective derived equivalent varieties. If* $\dim X \leq 3$ *or if* $\dim X \geq 4$ *and* $\mathrm{kod}(X) \geq 0$ *then*

$$a(X) = a(Y).$$

In this section we extend this invariance to further cases. The following is a preliminary lemma.

Lemma 3.2. *If X has maximal Albanese dimension then* $\mathrm{kod}\, X \geq 0$.

Proof. As shown in Theorem 1.12, ω_X is a GV-sheaf when X has maximal Albanese dimension. Then $V^0(X, \omega_X) \neq \emptyset$ and so there is a non-trivial torsion point $\alpha \in \mathrm{Pic}^0(X)$ of order $r \geq 1$ such that $H^0(X, \omega_X \otimes \alpha) \neq 0$. Let $s \in H^0(X, \omega_X \otimes \alpha)$ be a non-zero section, then $s^r \neq 0$ and

$$s^r \in H^0(X, \omega_X^{\otimes r} \otimes \alpha^{\otimes r}) = H^0(X, \omega_X^{\otimes r}).$$

Clearly, $\mathrm{kod}(X) \geq 0$. □

With the following proposition we establish a correspondence between the Albanese fibre dimension of X and Y.

Proposition 3.3. *If X and Y are derived equivalent, then $a(X) = 0, 1, n$ if and only if $a(Y) = 0, 1, n$ respectively.*

Proof. We start with the case $a(X) = 0$. In this case, a_X is constant, then $q(X) = 0$ and $\mathrm{Alb}(X) = \{e_X\}$. Since the irregularity is a derived invariant by Theorem 2.6, $q(Y) = q(X) = 0$ and we have $a(Y) = 0$.

Suppose $a(X) = n$. In this case X has maximal Albanese dimension then, by Lemma 3.2, X satisfies $\mathrm{kod}(X) \geq 0$. We can conclude by Theorem 3.1 that $a(X) = a(Y) = n$.

The remaining case is $a(X) = 1$, when the image of the Albanese map is a curve C with genus $g(C) = q(X)$, and we discuss this in two sub-cases: $q(X) = 1$ and $q(X) \geq 2$. First, suppose $q(X) = 1$, so C is an elliptic curve, then $C = \mathrm{Alb}(X)$ and a_X is surjective. Since $q(Y) = 1$, as before by Theorem 2.6, the map a_Y is surjective, which implies that $a(Y) = 1$.

Finally, the case $a(X) = 1$ and $q(X) \geq 2$. The image of the Albanese map is a curve C such that $q(X) = g(C) \geq 2$. We claim that $V^0(C, \omega_C) = \mathrm{Pic}^0 C$. For $\mathcal{O}_C \neq \alpha \in \mathrm{Pic}^0 C$

$$\chi(\omega_C \otimes \alpha) = \chi(\omega_C) = h^0(C, \omega_C \otimes \alpha) - h^1(C, \omega_C \otimes \alpha). \tag{3.1}$$

We have $\chi(\omega_C \otimes \alpha) > 0$, since $g(C) \geq 2$, and $h^1(C, \omega_C \otimes \alpha) = 0$ by Serre duality. Then from equation (3.1) we get that $h^0(C, \omega_C \otimes \alpha) > 0$ for every non-trivial $\alpha \in \mathrm{Pic}^0 C$. If $\alpha = \mathcal{O}_C$, then $h^0(C, \omega_C) = g(C) \geq 2$ and the claim is proved.

Now we consider the Stein factorization of the Albanese map a_X, we have the following diagram

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \Big\downarrow f & \nearrow & \\ C & & \end{array} \tag{3.2}$$

such that f has connected fibres and

$$f^*\mathrm{Pic}^0 C = f^* V^0(C, \omega_C) \subseteq V^{n-1}(X, \omega_X)_0 \subseteq \mathrm{Pic}^0(X).$$

These are, in fact, equalities, since $\dim f^* V^0(C, \omega_C) = q(X) = \dim \mathrm{Pic}^0(X)$. There is an isomorphisms $V^{n-1}(X, \omega_X)_0 \simeq V^{n-1}(Y, \omega_Y)_0$ by Corollary 2.13. There exists a component $T \subseteq V^{n-1}(Y, \omega_Y)_0$ and a surjective morphism with connected fibres $h\colon Y \to D$ such that D is a smooth projective curve with $T = h^*\mathrm{Pic}^0 D$ ([Beauville, 1992, Corollaire 2.3]). Then we have a similar diagram as (3.2) for Y, then $a(Y) = 1$. □

Proposition 3.4. *If $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ is a Rouquier-stable equivalence, then*

$$a(X) = a(Y) \textit{ and } c(X) = c(Y).$$

Proof. By Theorem 2.22 the Stein factorization X' and Y' are isomorphic and $a(X) = \dim X' = \dim Y' = a(Y)$. □

3.2 Invariance of Higher Direct Images

Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a Rouquier-stable derived equivalence and let

$$\varphi\colon \mathrm{Aut}^0(X) \times \mathrm{Pic}^0(X) \to \mathrm{Aut}^0(Y) \times \mathrm{Pic}^0(Y)$$

be the induced Rouquier isomorphism. We denote by $\delta_X\colon X \hookrightarrow X \times X$ and $\delta_Y\colon Y \hookrightarrow Y \times Y$ the diagonal embeddings of X and Y, respectively. We recall the Hochschild-Kostant-Rosenberg isomorphism (cf. [Căldăraru, 2003b])

$$\mathbf{L}\delta_X^* \delta_{X*}\mathcal{O}_X \simeq \bigoplus_{k=0}^{n} \Omega_X^k[k].$$

Combining this result with Corollary 2.29 we get that there is an isomorphism for every $j \geq 0$

$$\bigoplus_{q-p=n-j} \widehat{\varphi}^* R^p a_{X*}\Omega_X^q \simeq \bigoplus_{q-p=n-j} R^p a_{Y*}\Omega_Y^q. \tag{3.3}$$

We consider the following isomorphisms

$$w_\alpha\colon H^0(\mathrm{Alb}(X), R^j a_{X*}\omega_X \otimes \alpha) \oplus \bigoplus_{0<k\leq j} H^0(\mathrm{Alb}(X), R^{j-k} a_{X*}\Omega_X^{n-k} \otimes \alpha) \to$$
$$H^0(\mathrm{Alb}(Y), R^j a_{Y*}\omega_Y \otimes \beta) \oplus \bigoplus_{0<k\leq j} H^0(\mathrm{Alb}(Y), R^{j-k} a_{Y*}\Omega_Y^{n-k} \otimes \beta)$$

where $\alpha \in \mathrm{Pic}^0(X)$ and $\beta = \varphi(\alpha)$.

Theorem 3.5. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a Rouquier-stable derived equivalence. If*

$$w_\alpha(H^0(\mathrm{Alb}(X), R^k a_{X*}\omega_X \otimes \alpha)) = H^0(\mathrm{Alb}(Y), R^k a_{Y*}\omega_Y \otimes \varphi(\alpha)) \quad \forall\, \alpha \in \mathrm{Pic}^0(X) \tag{3.4}$$

then there is an isomorphism of sheaves $R^k a_{Y*}\omega_Y \simeq \widehat{\varphi}^* R^k a_{X*}\omega_X$.

In particular, this theorem implies that

$$H^i(\mathrm{Alb}(X), R^k a_{X*}\omega_X) \simeq H^i(\mathrm{Alb}(Y), R^k a_{Y*}\omega_Y)$$

for every $i \geq 0$.

Proof. The sheaves $\mathbf{FM}_{\mathrm{Alb}(X)}(R^i a_{X*}\omega_X)$ and $\mathbf{FM}_{\mathrm{Alb}(Y)}(R^i a_{Y*}\omega_Y)$ are concentrated in degree zero, as $R^i a_{X*}\omega_X$ and $R^i a_{Y*}\omega_Y$ are GV-sheaves by Theorem 1.12. Their fibres are identified with

$$\mathbf{FM}_{\mathrm{Alb}(X)}(R^i a_{X*}\omega_X) \otimes \mathbb{C}(\alpha) \simeq H^0(\mathrm{Alb}(X), R^i a_{X*}\omega_X \otimes \alpha^{-1})^\vee$$

and similarly for Y

$$\mathbf{FM}_{\mathrm{Alb}(Y)}(R^i a_{Y*}\omega_Y) \otimes \mathbb{C}(\beta) \simeq H^0(\mathrm{Alb}(Y), R^i a_{Y*}\omega_Y \otimes \beta^{-1})^\vee.$$

Let $\beta = \varphi(\alpha) \in \mathrm{Pic}^0(Y)$ and let $j_\beta \colon \{\beta\} \to \mathrm{Pic}^0(Y)$ be the closed embedding. Starting from the isomorphism (3.3) and using the formula of Remark 1.10 we have the following isomorphism

$$\bigoplus_{j-i=n-k} \mathcal{H}^0 \mathbf{L} j_\beta^* \mathbf{FM}_{\mathrm{Alb}(Y)}(R^i a_{Y*}\Omega_Y^j) \to \bigoplus_{j-i=n-k} \mathcal{H}^0 \mathbf{L} j_\beta^* \varphi_* \mathbf{FM}_{\mathrm{Alb}(X)}(R^i a_{X*}\Omega_X^j) \quad (3.5)$$

for every $k \geq 0$. By Lemma 1.8, the isomorphism (3.5) is identified with the isomorphism $(w_{\alpha^{-1}})^\vee$

$$\bigoplus_{j-i=n-k} H^0(\mathrm{Alb}(Y), R^i a_{Y*}\Omega_Y^j \otimes \varphi(\alpha^{-1}))^\vee \to \bigoplus_{j-i=n-k} H^0(\mathrm{Alb}(X), R^i a_{X*}\Omega_X^j \otimes \alpha^{-1})^\vee.$$

By hypothesis (3.4) we have that the isomorphism

$$\bigoplus_{j-i=n-k} \mathbf{FM}_{\mathrm{Alb}(Y)}(R^i a_{Y*}\Omega_Y^j) \xrightarrow{\simeq} \bigoplus_{j-i=n-k} \varphi_* \mathbf{FM}_{\mathrm{Alb}(X)}(R^i a_{X*}\Omega_X^j)$$

induces a morphism

$$f_1 \colon \mathbf{FM}_{\mathrm{Alb}(Y)}(R^k a_{Y*}\omega_Y) \to \varphi_* \mathbf{FM}_{\mathrm{Alb}(X)}(R^k a_{X*}\omega_X)$$

which is surjective because it is so at the level of fibers. If f_1 is an isomorphism then the proof is complete. By repeating the same argument with a quasi-inverse of Φ, we can get a surjective morphism

$$f_2 \colon \varphi_* \mathbf{FM}_{\mathrm{Alb}(X)}(R^k a_{X*}\omega_X) \to \mathbf{FM}_{\mathrm{Alb}(Y)}(R^k a_{Y*}\omega_Y)$$

such that $f_2 \circ f_1$ is the identity at the level of the fibers. Then $f_2 \circ f_1$ is an isomorphism of sheaves. Therefore f_1 is also injective and then an isomorphism. □

We verify that the equivalence (3.4) is satisfied when $k = c(X)$. This yields the following Corollary.

Corollary 3.6. *Let* $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ *a Rouquier-stable derived equivalence, then* Φ *induces an isomorphism of sheaves*

$$R^{c(Y)} a_{Y*}\omega_Y \simeq \widehat{\varphi}^* R^{c(X)} a_{X*}\omega_X.$$

Proof. Consider the Stein factorizations of the Albanese maps a_X and a_Y

$$\begin{array}{ccc} X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \Big\downarrow f & \overset{s}{\nearrow} & \\ S & & \end{array} \qquad\qquad \begin{array}{ccc} Y & \xrightarrow{a_Y} & \mathrm{Alb}(Y) \\ \Big\downarrow g & \overset{t}{\nearrow} & \\ T & & \end{array}$$

with S and T normal varieties, f and g are surjective morphisms with connected fibers and s and t are finite morphisms onto their images. Moreover, $\dim S = a(X)$ and $\dim T = a(Y)$. Hence by Proposition 3.4, $a(X) = a(Y)$ and so

$$c(X) = c(Y).$$

We consider a non-singular representative of f, i.e. a smooth birational modification $\pi\colon \widetilde{X} \to X$ and a fibration $\widetilde{f}\colon \widetilde{X} \to \widetilde{S}$ between smooth algebraic varieties. Let $\theta\colon \widetilde{S} \to S$ be a birational morphism such that the following diagram

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\pi} & X & \xrightarrow{a_X} & \mathrm{Alb}(X) \\ \Big\downarrow \widetilde{f} & & \Big\downarrow f & \overset{s}{\nearrow} & \\ \widetilde{S} & \xrightarrow{\theta} & S & & \end{array} \tag{3.6}$$

is commutative. By Kollár's Theorem 1.17 we have the following isomorphisms

$$R^{c(X)} a_{X*}\omega_X \simeq R^{c(X)}(s \circ f)_*\omega_X \simeq s_* R^{c(X)} f_*\omega_X.$$

Then for any $\alpha \in \mathrm{Pic}^0(X)$ we have

$$\begin{aligned} H^0(\mathrm{Alb}(X), R^{c(X)} a_{X*}\omega_X \otimes \alpha) &\simeq H^0(S, R^{c(X)} f_*\omega_X \otimes s^*\alpha) \\ &\simeq H^0(S, R^{c(X)} (f \circ \pi)_*\omega_{\widetilde{X}} \otimes s^*\alpha) \\ &\simeq H^0(S, R^{c(X)} (\theta \circ \widetilde{f})_*\omega_{\widetilde{X}} \otimes s^*\alpha) \\ &\simeq H^0(S, \theta_* R^{c(X)} \widetilde{f}_*\omega_{\widetilde{X}} \otimes s^*\alpha) \\ &\simeq H^0(S, \theta_*\omega_{\widetilde{S}} \otimes s^*\alpha) \end{aligned}$$

where we use the commutative of the diagram (3.6) above, the fact that $\pi_*\omega_{\widetilde{X}} \simeq \omega_X$, because π is birational; Theorem 1.17 and in the last isomorphism that $R^{c(X)}\widetilde{f}_*\omega_{\widetilde{X}} \simeq \omega_{\widetilde{S}}$ by Theorem 1.16.

By Theorem 2.27 there is an isomorphism $\psi\colon T \xrightarrow{\sim} S$ such that

$$s \circ \psi = \widehat{\varphi} \circ t.$$

Now, we construct a specific non-singular representative of g. Set $\widetilde{T} = \widetilde{S}$ and consider the birational morphism $\theta_2\colon \widetilde{S} \to S \to T$ defined by $\theta_2 = (\psi^{-1} \circ \theta)$, and consider the fiber product $Z = \widetilde{S} \times_T Y$. Then a resolution $\widetilde{Y}$ of the normalization of the main component of Z is such that the natural projection $\widetilde{g}\colon \widetilde{Y} \to \widetilde{S}$ is a non-singular representative of g. We do the same calculations as above and we get the isomorphism

$$H^0(\mathrm{Alb}(Y), R^{c(Y)} a_{Y*}\omega_Y \otimes \varphi(\alpha)) \simeq H^0(T, \theta_{2*}\omega_{\widetilde{S}} \otimes t^*\varphi(\alpha)).$$

We have

$$\begin{aligned} H^0(S, \theta_*\omega_{\widetilde{S}} \otimes s^*\alpha) &\simeq H^0(T, \psi^*\theta_*\omega_{\widetilde{S}} \otimes \psi^* s^*\alpha) \\ &\simeq H^0(T, \theta_{2*}\omega_{\widetilde{S}} \otimes t^*\varphi(\alpha)). \end{aligned}$$

So w_α sends $H^0(\mathrm{Alb}(X), R^{c(X)} a_{X*}\omega_X \otimes \alpha)$ to $H^0(\mathrm{Alb}(Y), R^{c(Y)} a_{Y*}\omega_Y \otimes \varphi(\alpha))$. By using Theorem 3.5, we get

$$R^{c(Y)} a_{Y*}\omega_Y \simeq \widehat{\varphi}^* R^{c(X)} a_{X*}\omega_X$$

and the proof is complete. □

3.3 Cohomology of Higher Direct Images

We recall the following conjecture, which comes from a more general conjecture of Orlov on the derived invariance of motives.

Conjecture 3.7. *Let X and Y be smooth complex projective derived equivalent varieties, then $h^i(X, \omega_X) = h^i(Y, \omega_Y)$ for all $i \geq 0$.*

We focus on a slightly stronger question and study its validity in some cases.

Question 3.8. *Let X and Y be smooth complex projective derived equivalent varieties. Do we have the following equalities*

$$h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X) = h^q(\mathrm{Alb}(Y), R^p a_{Y*}\omega_Y)$$

for all $p, q \geq 0$?

Remark 3.9. Question 3.8 has a positive answer for all q when X has maximal Albanese dimension, that is $a(X) = \dim X$. In fact, if X has maximal Albanese dimension, by the Grauert-Riemenschneider vanishing theorem, we have $R^p a_{X*}\omega_X = 0$ and $R^p a_{Y*}\omega_Y = 0$ for $p > 0$. By Corollary 2.16, when $p = 0$ we have

$$h^q(\mathrm{Alb}(X), a_{X*}\omega_X) = h^q(\mathrm{Alb}(Y), a_{Y*}\omega_Y)$$

for every $q \in \mathbb{N}$. Moreover, recall that Abuaf proved that if $\dim X \leq 4\, h^i(X, \omega_X) = h^i(Y, \omega_Y)$ for all $i \geq 0$ [Abuaf, 2017].

Our result is the following.

Theorem 3.10. *Question 3.8 has a positive answer if* $\dim X \leq 3$ *or if* $\dim X = 4$ *and* $\Phi \colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ *is a Rouquier-stable derived equivalence.*

Proof. If $\dim X = 1$ there are two cases: $a(X)$ can be either 0 or 1. If $a(X) = 1$ we can conclude by Remark 3.9. Otherwise $a(X) = 0$, in this case both a_X is constant and $\mathrm{Alb}(X)$ is trivial. Then $h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X) = 0$ for every $q > 0$. By Proposition 3.3 also $a(Y) = 0$ and so $h^q(\mathrm{Alb}(Y), R^p a_{Y*}\omega_Y) = 0$ for every $q > 0$. When $q = 0$ we have $h^0(\mathrm{Alb}(X), R^p a_{X*}\omega_X) = h^p(X, \omega_X)$ and $h^0(\mathrm{Alb}(Y), R^p a_{Y*}\omega_Y) = h^p(Y, \omega_Y)$. Since Conjecture 3.7 holds true in dimension 1 we can conclude.

In the following proof, the cases $a(X) = 0$ and $a(X) = \dim X$, which is the maximal Albanese dimension case, can be proved with similar calculations.

Suppose that $\dim X = 2$ and $a(X) = 1$. By Theorem 1.14 and using the Leray spectral sequence we get that

$$h^1(X, \omega_X) = h^0(\text{Alb}(X), R^1 a_{X*}\omega_X) + h^1(\text{Alb}(X), a_{X*}\omega_X) \tag{3.7}$$

Here $h^1(X, \omega_X)$ and $h^1(\text{Alb}(X), a_{X*}\omega_X)$ are invariants by Remark 3.9 and Theorem 2.14. Then $h^0(\text{Alb}(X), R^1 a_{X*}\omega_X)$ is invariant, too.

Noitce that with a similar argument once can prove the case of $\dim X = 3$ and $a(X) = 2$.

If $\dim X = 3$ and $a(X) = 1$, then the image of a_X is a smooth curve C. Using the Leray spectral sequence as before we have

$$h^1(X, \omega_X) = h^0(\text{Alb}(X), R^1 a_{X*}\omega_X) + h^1(\text{Alb}(X), a_{X*}\omega_X)$$

and

$$h^2(X, \omega_X) = h^0(\text{Alb}(X), R^2 a_{X*}\omega_X) + h^1(\text{Alb}(X), R^1 a_{X*}\omega_X). \tag{3.8}$$

From the first equation we can use the same argument as in (3.7) to prove that $h^0(\text{Alb}(X), R^1 a_{X*}\omega_X)$ is invariant. By Theorem 1.16 we have the following isomorphism

$$R^2 a_{X*}\omega_X \simeq \omega_C.$$

Then the equation (3.8) becomes

$$h^2(X, \omega_X) = h^0(C, \omega_C) + h^1(\text{Alb}(X), R^1 a_{X*}\omega_X).$$

Since $h^0(C, \omega_C) = g(C) = q(X)$, in the previous equation both $h^2(X, \omega_X)$ and $h^0(C, \omega_C)$ are invariants. So we obtain the invariance of $h^1(\text{Alb}(X), R^1 a_{X*}\omega_X)$, which completes the proof of this case.

Now we study the last case: suppose $\dim X = 4$ and the equivalence Φ is Rouquier-stable. First note that if $a(X) \in \{0, 1, 3, 4\}$ the proof can be done in a similar way as above. We are going to discuss the case $a(X) = 2$. Using the Leray spectral sequence and Kollár's Theorem 1.14 we get the following equations

$$
\begin{aligned}
h^1(\omega_X) &= h^0(R^1 a_{X*}\omega_X) + h^1(a_{X*}\omega_X)\\
h^2(\omega_X) &= h^0(R^2 a_{X*}\omega_X) + h^1(R^1 a_{X*}\omega_X) + h^2(a_{X*}\omega_X)\\
h^3(\omega_X) &= h^1(R^2 a_{X*}\omega_X) + h^2(R^1 a_{X*}\omega_X)
\end{aligned}
$$

By the previous arguments, we only need to verify the invariance of $h^i(R^2 a_{X*}\omega_X)$ for $i = 0, 1$ to complete the proof. Since in this case $c(X) = 2$, by Corollary 3.6 we have that $h^i(R^2 a_{X*}\omega_X) = h^i(R^2 a_{Y*}\omega_Y)$ for $i = 0, 1$. This implies that $h^i(R^1 a_{X*}\omega_X) = h^i(R^1 a_{Y*}\omega_Y)$ for $i = 1, 2$ and the proof is completed. □

3.4 Comparison of Non-Vanishing Loci

Following [Lombardi and Popa, 2015], the Conjecture 3.7 is related to another problem regarding the invariance of the non-vanishing loci $V^i_m(\omega_X)_0$, as shown in Theorem 2.12

We consider a slightly more general problem.

Question 3.11. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence. Does the Rouquier isomorphism act as follow*

$$\varphi\left(\{\mathrm{id}_X\} \times V^q_m(R^p a_{X*}\omega_X)_0\right) = \{\mathrm{id}_Y\} \times V^q_m(R^p a_{Y*}\omega_Y)_0$$

for all $p, q \geq 0$ and $m \geq 1$?

By Theorem 3.10 we get the following result.

Theorem 3.12. *Question 3.8 has a positive answer in dimension n for a given pair of integer (p, q) if and only if Question 3.11 has a positive answer in dimension n for the same pair of integer (p, q).*

Proof. We recall that every irreducible component Z of $V^q(R^p a_{X*}\omega_X)$ is a torsion translate $\tau_Z + A_Z$ of an abelian variety by Remark 1.3. Moreover, using the Leray spectral sequence and the fact that any line bundle in $V^i(\omega_X)_0$ is Rouquier-stable for all $i \geq 0$, we note that also any line bundle $\alpha \in V^q(R^p a_{X*}\omega_X)_0$ is Rouquier-stable.

Let $Z \subset V^q_m(R^p a_{X*}\omega_X)_0$ be an irreducible component. Let S be the set of prime numbers that do not divide the order of any τ_Z and let $\alpha \in Z$ be an element of prime order $P \in S$. We claim that if $\beta = \varphi(\alpha)$, then $\beta \in V^q_m(R^p a_{Y*}\omega_Y)_0$. Denote by $\pi_\alpha \colon X_\alpha \to X$ and $\pi_\beta \colon Y_\beta \to Y$ the étale covers associated to α and β respectively. There are the following isomorphisms (see, e.g., [Huybrechts, 2006, Section 7.3])

$$\pi_{\alpha*}\mathcal{O}_{X_\alpha} \simeq \bigoplus_{j=0}^{P-1} \alpha^{\otimes(-j)} \quad \text{and} \quad \pi_{\beta*}\mathcal{O}_{Y_\beta} \simeq \bigoplus_{j=0}^{P-1} \beta^{\otimes(-j)}.$$

We have the following commutative diagrams

$$\begin{array}{ccc} X_\alpha & \xrightarrow{a_{X_\alpha}} & \mathrm{Alb}(X_\alpha) \\ {\scriptstyle \pi_\alpha}\downarrow & & \downarrow{\scriptstyle \rho_\alpha} \\ X & \xrightarrow{a_X} & \mathrm{Alb}(X) \end{array} \qquad \begin{array}{ccc} Y_\beta & \xrightarrow{a_{Y_\beta}} & \mathrm{Alb}(Y_\beta) \\ {\scriptstyle \pi_\beta}\downarrow & & \downarrow{\scriptstyle \rho_\beta} \\ Y & \xrightarrow{a_Y} & \mathrm{Alb}(Y). \end{array}$$

By [Lombardi and Popa, 2015, Theorem 10] the equivalence Φ can be extended to a derived equivalence $\Phi' \colon \mathbf{D}^{\mathrm{b}}(X_\alpha) \to \mathbf{D}^{\mathrm{b}}(Y_\beta)$. By hypothesis we have

$$h^q(\mathrm{Alb}(X_\alpha), R^p a_{X_\alpha *}\omega_{X_\alpha}) = h^q(\mathrm{Alb}(Y_\beta), R^p a_{Y_\beta *}\omega_{Y_\beta}).$$

Since the morphism ρ_α and π_α are finite, we have the following isomorphisms

$$\rho_{\alpha*} R^p a_{X_\alpha *}\omega_{X_\alpha} \simeq R^p a_{X*}\pi_{\alpha*}\omega_{X_\alpha} \simeq \bigoplus_{j=0}^{P-1} R^p a_{X*}\left(\omega_X \otimes \alpha^{\otimes(-j)}\right)$$

and similarly for Y

$$\rho_{\beta*} R^p a_{Y_\beta *}\omega_{Y_\beta} \simeq R^p a_{Y*}\pi_{\beta*}\omega_{Y_\beta} \simeq \bigoplus_{j=0}^{P-1} R^p a_{Y*}\left(\omega_Y \otimes \beta^{\otimes(-j)}\right).$$

These lead to the following equalities

$$h^q(\mathrm{Alb}(X_\alpha), R^p a_{X_\alpha *}\omega_{X_\alpha}) = h^q(\mathrm{Alb}(X), \rho_{\alpha *} R^p a_{X_\alpha *}\omega_{X_\alpha})$$
$$= \sum_{j=0}^{P-1} h^q(\mathrm{Alb}(X), R^p a_{X*}(\omega_X \otimes \alpha^{\otimes(-j)}))$$

and

$$h^q(\mathrm{Alb}(Y_\beta), R^p a_{Y_\beta *}\omega_{Y_\beta}) = \sum_{j=0}^{P-1} h^q(\mathrm{Alb}(Y), R^p a_{Y*}(\omega_Y \otimes \beta^{\otimes(-j)})).$$

Since $\alpha \in Z$ belongs to an abelian variety, then all its powers $\alpha^{\otimes j}$ belongs to Z. Moreover

$$h^q(\mathrm{Alb}(X), R^p a_{X*}\omega_X) = h^q(\mathrm{Alb}(Y), R^p a_{Y*}\omega_Y)$$

and there exists $0 < k \leq P-1$ such that $h^q(\mathrm{Alb}(Y), R^p a_{Y*}\omega_Y \otimes \beta^{\otimes k}) \geq m$. Hence $\beta^{\otimes k} \in V^q_m(R^p a_{Y*}\omega_Y)$. With the same argument of [Lombardi and Popa, 2015, p. 302] and the fact that torsion points of prime order form a Zariski dense subset, we can prove that $\beta^{\otimes k} \in V^q_m(R^p a_{Y*}\omega_Y)_0$ and therefore $\beta \in V^q_m(R^p a_{Y*}\omega_Y)_0$. This proves that $\varphi(Z) \subset V^q_m(R^p a_{Y*}\omega_Y)_0$ and that $\varphi(V^q_m(R^p a_{X*}\omega_X)_0) \subset V^q_m(R^p a_{Y*}\omega_Y)_0$.

By repeating the same argument with a quasi-inverse of Φ we complete the proof. □

Corollary 3.13. *Question 3.11 has a positive answer in dimension 3. Moreover it holds in dimension 4 if the equivalence Φ is Rouquier-stable.*

Finally, we study the case $(p, q) = (1, 0)$.

Proposition 3.14. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence.*

1. *Then*

$$h^0(\mathrm{Alb}(X), R^1 a_{X*}\omega_X) = h^0(\mathrm{Alb}(Y), R^1 a_{Y*}\omega_Y)$$

and

$$V^0_m(R^1 a_{X*}\omega_X)_0 \simeq V^0_m(R^1 a_{Y*}\omega_Y)_0$$

for every $m \geq 1$.

2. *If the equivalence is Rouquier-stable, then*

$$V^0_m(R^1 a_{X*}\omega_X) \simeq V^0_m(R^1 a_{Y*}\omega_Y)$$

for every $m \geq 1$ *and* $\chi(R^1 a_{X*}\omega_X) = \chi(R^1 a_{Y*}\omega_Y)$.

Proof. 1. Suppose $\alpha \in \mathrm{Pic}^0(X)$ is a Rouquier-stable line bundle, by the Hochschild-Kostant-Rosenberg isomorphism (2.9) we have the following direct sum decomposition of the first Hochschild homology

$$HH_1(X, \alpha) \simeq H^1(X, \omega_X \otimes \alpha) \oplus H^0(X, \Omega_X^{n-1} \otimes \alpha)$$

and there is a similar decomposition for Y

$$HH_1(Y, \varphi(\alpha)) \simeq H^1(Y, \omega_Y \otimes \varphi(\alpha)) \oplus H^0(Y, \Omega_Y^{n-1} \otimes \varphi(\alpha)).$$

Note that $h^1(X, \omega_X \otimes \alpha) = h^0(X, \Omega_X^{n-1} \otimes \alpha)$, and similarly for Y. Since the first Hochschild homology is a derived invariant, i.e. $HH_1(X, \alpha) \simeq HH_1(Y, \varphi(\alpha))$, then

$$h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \varphi(\alpha)).$$

By the Leray spectral sequence there are the following equalities

$$h^1(X, \omega_X \otimes \alpha) = h^0(\mathrm{Alb}(X), R^1 a_{X*}\omega_X \otimes \alpha) + h^1(\mathrm{Alb}(X), a_{X*}\omega_X \otimes \alpha)$$

and

$$h^1(Y, \omega_Y \otimes \varphi(\alpha)) = h^0(\mathrm{Alb}(Y), R^1 a_{Y*}\omega_Y \otimes \varphi(\alpha)) + h^1(\mathrm{Alb}(Y), a_{Y*}\omega_Y \otimes \varphi(\alpha)).$$

Now, since $h^1(X, \omega_X \otimes \alpha) = h^1(Y, \omega_Y \otimes \varphi(\alpha))$ and $h^1(\mathrm{Alb}(X), a_{X*}\omega_X \otimes \alpha) = h^1(\mathrm{Alb}(Y), a_{Y*}\omega_Y \otimes \varphi(\alpha))$ because they are derived invariants, then we also have

$$h^0(\mathrm{Alb}(X), R^1 a_{X*}\omega_X \otimes \alpha) = h^0(\mathrm{Alb}(Y), R^1 a_{Y*}\omega_Y \otimes \varphi(\alpha)). \tag{3.9}$$

Since the structure sheaf is Rouquier-stable, we can take $\alpha = \mathcal{O}_X$ in the equation (3.9) and then we conclude the first part of the proof by Theorem 3.12.

2. For the second point of the proposition, suppose the equivalence is Rouquier-

stable. By the equation (3.9) we have that $V^0_m(R^1a_{X*}\omega_X) \simeq V^0_m(R^1a_{Y*}\omega_Y)$ for $m \geq 1$. By Theorem 1.12 both $R^1a_{X*}\omega_X$ and $R^1a_{Y*}\omega_Y$ are GV-sheaves, then for general $\alpha \in \mathrm{Pic}^0(X)$ and $\beta \in \mathrm{Pic}^0(Y)$ the sheaves $R^1a_{X*}\omega_X \otimes \alpha$ and $R^1a_{Y*}\omega_Y \otimes \beta$ have no higher cohomology. In particular, when $\beta = \varphi(\alpha)$ there are the following equalities

$$h^j(\mathrm{Alb}(X), R^1a_{X*}\omega_X \otimes \alpha) = h^j(\mathrm{Alb}(Y), R^1a_{Y*}\omega_Y \otimes \varphi(\alpha)) = 0$$

for every $j > 0$. Then we have that

$$\chi(R^1a_{X*}\omega_X) = \chi(R^1a_{X*}\omega_X \otimes \alpha) = h^0(\mathrm{Alb}(X), R^1a_{X*}\omega_X \otimes \alpha)$$

and

$$\chi(R^1a_{Y*}\omega_Y) = \chi(R^1a_{Y*}\omega_Y \otimes \varphi(\alpha)) = h^0(\mathrm{Alb}(Y), R^1a_{Y*}\omega_Y \otimes \varphi(\alpha))$$

which concludes the proof.

□

3.5 Small Values of the Albanese Fiber Dimension

In this section we study Question 3.8 for small values of $c(X)$. When $c(X) = 1$ we have the following result.

Theorem 3.15. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence and let $c(X) = 1$.*

1. *If Φ is Rouquier-stable, then X and Y are birational and Question 3.8 holds.*
2. *If $\mathrm{kod}(X) \geq 0$, then $\chi(R^1a_{X*}\omega_X) = \chi(R^1a_{Y*}\omega_Y)$.*

Proof. 1. By Proposition 3.4 $c(Y) = 1$, too. Denote by $f\colon X \to X'$ and $g\colon Y \to Y'$ the fibration parts of the Stein factorization of a_X and a_Y, respectively. By Theorem 2.22 we have that $X' \simeq Y'$. Moreover, Φ induces, by Theorem 2.37, a derived equivalence between the general fibers of f and g. Since these fibers are curves they are isomorphic [Huybrechts, 2006, Corollary 5.46]. Then X and Y are birational and the statement follows [1].

[1] I thank Federico Caucci for sharing this proof with me.

2. By [Caucci and Lombardi, 2022, §2.1.3], if $\mathrm{Aut}^0(X)$ is affine, then $\mathrm{Pic}^0(X)$ is Rouquier-stable and the previous point applies. If $\mathrm{Aut}^0(X)$ is not affine, then there exists a positive dimensional abelian variety acting on X. By Theorem 2.6, we must have $\chi(\omega_X) = 0$. We also have $\chi(\omega_X) = \chi(a_{X*}\omega_X) - \chi(R^1 a_{X*}\omega_X)$ and therefore $\chi(a_{X*}\omega_X) = \chi(R^1 a_{X*}\omega_X)$. On Y we have an analogous situation, then there exists a positive dimensional abelian variety acting on Y and $\chi(\omega_Y) = 0$ as well. Since $\mathrm{kod}(Y) \geq 0$, the Albanese dimension of Y is $\dim Y - 1$ and $\chi(a_{Y*}\omega_Y) = \chi(R^1 a_{Y*}\omega_Y)$. Since $\chi(a_{X*}\omega_X) = \chi(a_{Y*}\omega_Y)$ by Theorem 2.14, then $\chi(R^1 a_{X*}\omega_X) = \chi(R^1 a_{Y*}\omega_Y)$. □

When $c(X) = 2$ we provide a generic version of Question 3.8.

Theorem 3.16. *Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a Rouquier-stable derived equivalence with* $\dim X \geq 3$ *and $c(X) = 2$. Then for a generic $\alpha \in \mathrm{Pic}^0(X)$ there are equalities*

$$h^q(X, \omega_X \otimes \alpha) = h^q(Y, \omega_Y \otimes \varphi(\alpha))$$

for every $q \geq 0$. Moreover, we have the following equalities $\chi(R^p a_{X*}\omega_X) = \chi(R^p a_{Y*}\omega_Y)$ for $p \geq 0$ and $\chi(\omega_X) = \chi(\omega_Y)$.

Proof. Recall that the sheaves $R^i a_{X*}\omega_X$ and $R^i a_{Y*}\omega_Y$ are GV-sheaves for every $i \geq 0$ by Theorem 1.12 and, since $c(X) = 2$, also $R^i a_{X*}\omega_X = R^i a_{Y*}\omega_Y = 0$ for $i \geq 3$ Hence for a generic $\alpha \in \mathrm{Pic}^0(X)$ and for all $j > 0$ and $i \geq 0$, we have

$$h^j(\mathrm{Alb}(X), R^i a_{X*}\omega_X \otimes \alpha) = h^j(\mathrm{Alb}(Y), R^i a_{Y*}\omega_Y \otimes \varphi(\alpha)) = 0. \tag{3.10}$$

Recall that when $q = 0, 1, 2$, respectively, the loci $V^0_m(R^q a_{X*}\omega_X)$ are derived invariants via the Rouquier isomorphism for all $m \geq 1$ (cfr. Corollary 2.15, Proposition 3.14 and Corollary 3.6, respectively). So for a generic $\alpha \in \mathrm{Pic}^0(X)$, as above, we have the equality

$$h^0(\mathrm{Alb}(X), R^q a_{X*}\omega_X \otimes \alpha) = h^0(\mathrm{Alb}(Y), R^q a_{Y*}\omega_Y \otimes \varphi(\alpha))$$

for $q = 0, 1, 2$. By the equations (3.10) and using the Leray spectral sequence we get the following

$$h^q(X, \omega_X \otimes \alpha) = h^0(\text{Alb}(X), R^q a_{X*}\omega_X \otimes \alpha)$$
$$h^q(Y, \omega_Y \otimes \varphi(\alpha)) = h^0(\text{Alb}(Y), R^q a_{Y*}\omega_Y \otimes \varphi(\alpha))$$

for $q = 0, 1, 2$. This proves the theorem. □

3.6 Hochschild Homology and Generation in Low Degrees

Let $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ be a derived equivalence and suppose X is of maximal Albanese dimension, or equivalently with Albanese fiber dimension $c(X) = 0$. Caucci and Pareschi's Theorem 2.14 shows that the map induced by Φ on the Hochschild homology induces the isomorphisms

$$H^j(X, \omega_X \otimes \alpha) \simeq H^j(Y, \omega_Y \otimes \varphi(\alpha))$$

for any Rouquier-stable line bundle $\alpha \in \text{Pic}^0(X)$ and for every $j \geq 0$. Our aim, in this section, is to extend their result to other values of $c(X)$.

Recall that, from the invariance of the twisted Hochschild homology (see (2.4)), Φ induces the following isomorphisms

$$\Phi^{HH_j,\alpha}\colon HH^j(X, \alpha) \to HH^j(Y, \varphi(\alpha))$$

$$\Phi_{HH_j,\alpha}\colon HH_j(X, \alpha) \to HH_j(Y, \varphi(\alpha))$$

for any Rouquier-stable $\alpha \in \text{Pic}^0(X)$. Using the Hochschild-Kostant-Rosenberg isomorphism (2.9) we get the following decomposition

$$HH_j(X, \alpha) \simeq \bigoplus_{p-q=j} H^q(X, \Omega_X^p \otimes \alpha).$$

Note that

$$H^j(X, \omega_X \otimes \alpha) \subset HH_{n-j}(X, \alpha)$$

for every $j \geq 0$.

Theorem 3.17. *Let* $\Phi\colon \mathbf{D}^{\mathrm{b}}(X) \to \mathbf{D}^{\mathrm{b}}(Y)$ *be a derived equivalence. Suppose that* $\mathrm{kod}(X) \geq 0$ *and that* $\alpha \in \mathrm{Pic}^0(X)$ *is a Rouquier-stable line bundle. If*

$$\Phi_{HH_{n-j},\alpha}\left(H^j(X, \omega_X \otimes \alpha)\right) = H^j(Y, \omega_Y \otimes \varphi(\alpha)) \quad \text{for} \quad 0 \leq j \leq c(X) \tag{3.11}$$

then

$$\Phi_{HH_{n-j},\alpha}\left(H^j(X, \omega_X \otimes \alpha)\right) = H^j(Y, \omega_Y \otimes \varphi(\alpha))$$

for $j > c(X)$ *.*

Proof. Consider the graded rings

$$E_X = \bigwedge^{*} H^1(X, \mathcal{O}_X) \text{ and } E_Y = \bigwedge^{*} H^1(Y, \mathcal{O}_Y).$$

Now consider the graded E_X-module

$$Q_X = H^*(X, \omega_X \otimes \alpha) = \bigoplus_i H^i(X, \omega_X \otimes \alpha).$$

By convention both the graded pieces $\wedge^i H^1(X, \mathcal{O}_X)$ and $H^i(X, \omega_X \otimes \alpha)$ of E_X and Q_X, respectively, live in degree $-i$. Moreover Q_X is $c(X)$-regular over E_X, by [Lazarsfeld and Popa, 2010, Theorem B], and therefore generated in degrees $0, -1, \ldots, -c(X)$. The group $HH_*(X, \alpha) = \bigoplus_i HH_i(X, \alpha)$ admits a natural structure of E_X-module. Let $\widetilde{W_*}(X)$ be the graded E_X-submodule of $HH_*(X, \alpha)$ generated by $H^j(X, \omega_X \otimes \alpha)$ for $j = 0, \ldots, c(X)$ so that $\widetilde{W_*}(X) = Q_X$. Now $\widetilde{W_*}(Y) := \Phi_{HH_*}\left(\widetilde{W_*}X\right)$ is generated by $H^j(Y, \omega_Y \otimes \varphi(\alpha))$ for $j = 0, \ldots, c(X)$ because Φ_{HH_*} is compatible with the isomorphism $E_X \simeq E_Y$ (as proved in [Caucci and Pareschi, 2019]). Since $c(X) = c(Y)$ then

$$Q_Y = \bigoplus_i H^i(Y, \omega_Y \otimes \varphi(\alpha))$$

is generated in degrees $0, -1, \ldots, -c(X)$ as E_Y-module. Hence $\widetilde{W_*}(Y) = Q_Y$ and the statement follows. □

Remark 3.18. In the result of Caucci and Pareschi, corresponding to the case

$c(X) = c(Y) = 0$, the condition

$$\Phi_{HH_0,\alpha}\left(H^0(X, \omega_X \otimes \alpha)\right) = H^0(Y, \omega_Y \otimes \varphi(\alpha))$$

is automatically satisfied by Theorem 2.18.

Bibliography

[Abuaf, 2017] Abuaf, R. (2017). Homological units. *International Mathematics Research Notices*, 2017(22):6943–6960. 5, 40

[Beauville, 1992] Beauville, A. (1992). Annulation du H^1 pour les fibrés en droites plats. In *Complex Algebraic Varieties*, pages 1–15, Berlin, Heidelberg. Springer. 35

[Bridgeland and Maciocia, 2001] Bridgeland, T. and Maciocia, A. (2001). Complex surfaces with equivalent derived categories. *Mathematische zeitschrift*, 236(4):677–697. 5

[Caucci and Lombardi, 2022] Caucci, F. and Lombardi, L. (2022). Irregular fibrations of derived equivalent varieties. In *arXiv:2207.14081*. 31, 32, 47

[Caucci et al., 2022] Caucci, F., Lombardi, L., and Pareschi, G. (2022). Derived invariance of the Albanese relative canonical ring. In *arXiv:2206.10739*. 6, 25, 26, 27, 28, 29, 30

[Caucci and Pareschi, 2019] Caucci, F. and Pareschi, G. (2019). Derived invariants arising from the Albanese map. *Algebraic Geometry*, 6(6):730–746. 8, 22, 25, 49

[Chen and Hacon, 2004] Chen, J. A. and Hacon, C. D. (2004). On the irregularity of the image of the Iitaka fibration. *Communications in Algebra*, 32(1):203–215. 11

[Căldăraru, 2003a] Căldăraru, A. (2003a). The Mukai pairing, I: the Hochschild structure. 18

[Căldăraru, 2003b] Căldăraru, A. (2003b). The Mukai pairing, II: the Hochschild-Kostant-Rosenberg isomorphism. *arXiv:math/0308080*. 36

[Ein and Lazarsfeld, 1997] Ein, L. and Lazarsfeld, R. (1997). Singularities of theta divisors and the birational geometry of irregular varieties. *Journal of the American Mathematical Society*, 10(1):243–258. 14

[Green and Lazarsfeld, 1991] Green, M. L. and Lazarsfeld, R. (1991). Higher obstructions to deforming cohomology groups of line bundles. *Journal of the American Mathematical Society*, 4:87–103. 10

[Hacon, 2004] Hacon, C. (2004). A derived category approach to generic vanishing. *J. Reine Angew. Math*, (575):173–187. 12, 14

[Hacon et al., 2018] Hacon, C., Popa, M., and Schnell, C. (2018). Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun. *Contemp. Math.*, Local and global methods in Algebraic Geometry: volume in honor of L. Ein's 60th birthday, 712:143–195. 11, 25

[Hacon and Pardini, 2004] Hacon, C. D. and Pardini, R. (2004). Birational characterization of products of curves of genus 2. *Mathematical Research Letters*, 12:129–140. 11, 14

[Hochschild et al., 1962] Hochschild, G., Kostant, B., and Rosenberg, A. (1962). Differential forms on regular affine algebras. *Transactions of the American Mathematical Society*, 102(3):383–408. 19

[Huybrechts, 2006] Huybrechts, D. (2006). *Fourier-Mukai Transforms in Algebraic Geometry*. Oxford Mathematical Monographs. 43, 46

[Kawamata, 2002] Kawamata, Y. (2002). D-Equivalence and K-Equivalence. *Journal of Differential Geometry*, 61:147–171. 5

[Kollár, 1986a] Kollár, J. (1986a). Higher direct images of dualizing sheaves, I. *Annals of Mathematics*, 123:11–42. 14, 15

[Kollár, 1986b] Kollár, J. (1986b). Higher direct images of dualizing sheaves, II. *Annals of Mathematics*, 124:171–202. 14, 15

[Kontsevich, 1995] Kontsevich, M. (1995). Homological algebra of mirror symmetry. In *Proceedings of the International Congress of Mathematicians: August 3–11, 1994 Zürich, Switzerland*, pages 120–139. Springer. 5

[Kontsevich, 2003] Kontsevich, M. (2003). Deformation quantization of Poisson manifolds. *Letters in Mathematical Physics*, 66:157–216. 19

[Lazarsfeld and Popa, 2010] Lazarsfeld, R. and Popa, M. (2010). BGG correspondence, cohomology of compact Kähler manifolds, and numerical inequalities. *Invent. Math.*, 182(3):605–633. 49

[Lombardi, 2014] Lombardi, L. (2014). Derived invariants of irregular varieties and Hochschild homology. *Algebra and Number Theory*, 8:513–542. 7, 22, 23, 24, 26, 34

[Lombardi, 2022] Lombardi, L. (2022). Derived equivalence and fibrations over curves and surfaces. *Kyoto Journal of Mathematics*, 62(4):683–706. 31

[Lombardi and Popa, 2015] Lombardi, L. and Popa, M. (2015). *Derived equivalence and non-vanishing loci II*, pages 291–306. London Mathematical Society Lecture Note Series. Cambridge University Press, United Kingdom. 7, 21, 22, 42, 43, 44

[Orlov, 2002] Orlov, D. O. (2002). Derived categories of coherent sheaves on abelian varieties and equivalences between them. *Izvestiya: Mathematics*, 66(3):569.

[Orlov, 2003] Orlov, D. O. (2003). Derived categories of coherent sheaves and equivalences between them. *Russian Mathematical Surveys*, 58(3):511. 18, 26

[Orlov, 2005] Orlov, D. O. (2005). Derived categories of coherent sheaves and motives. *arXiv preprint math/0512620*. 5

[Pareschi, 2012] Pareschi, G. (2012). Basic results on irregular varieties via Fourier-Mukai methods. *Current developments in algebraic geometry. Math. Sci. Res. Inst. Publ*, 59:379–403. 12

[Pareschi, 2017] Pareschi, G. (2017). Standard canonical support loci. *Rendiconti del Circolo Matematico di Palermo Series 2*, 66:137–157. 10

[Pareschi and Popa, 2009] Pareschi, G. and Popa, M. (2009). Strong generic vanishing and a higher dimensional Castelnuovo-de Franchis inequality. *Duke Mathematical Journal*, 150(2):269–285. 12

[Pareschi and Popa, 2011] Pareschi, G. and Popa, M. (2011). GV-sheaves, Fourier-Mukai transform, and generic vanishing. *American journal of mathematics*, 133(1):235–271. 12

[Popa, 2013] Popa, M. (2013). Derived equivalence and non-vanishing loci. In Hassett, B., McKernan, J., Starr, J., and Vakil, R., editors, *A Celebration of Algebraic Geometry: a conference in honor of Joe Harris's 60th birthday*, volume 18 of *Clay Mathematics Proceedings*, pages 567–575. AMS-CMI. 7

[Popa and Schnell, 2011] Popa, M. and Schnell, C. (2011). Derived invariance of the number of holomorphic 1-forms and vector fields. *Ann. Sci. Éc. Norm. Supéer.*, 44(2):201–213. 5, 6, 17, 18, 20

[Rouquier, 2011] Rouquier, R. (2011). Automorphismes, graduations et catégories triangulées. *Journal of the Institute of Mathematics of Jussieu*, 10(3):713–751. 6, 17

[Schnell, 2019] Schnell, C. (2019). The Fourier–Mukai transform made easy. *Pure and Applied Mathematics Quarterly*. 12, 13

[Simpson, 1993] Simpson, C. (1993). Subspaces of moduli spaces of rank one local systems. *Annales Scientifiques De L Ecole Normale Superieure*, 26:361–401. 10

[Swan, 1996] Swan, R. G. (1996). Hochschild cohomology of quasiprojective schemes. *Journal of Pure and Applied Algebra*, 110(1):57–80. 19

[Taelman, 2019] Taelman, L. (2019). Derived equivalences of hyperkähler varieties. *arXiv preprint arXiv:1906.08081*. 5