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Risk bounds under right-tail uncertainty

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Abstract
We investigate upper and lower bounds for spectral risk measures, when there exists uncertainty regarding the probability distribution of large losses. Initially, we focus on scenarios in which information is only available on the left-tail of the relevant random variable. Subsequently, we progressively incorporate knowledge of the first two moments of the distribution, culminating in uncertainty sets for both the mean and the variance. Throughout our analysis, we provide closed-form bounds and discuss their sharpness. A pivotal aspect of our study is to show that while the sole knowledge of the left-tail leaves a spectral risk measure unbounded, such partial information combined with additional assumptions on the moments of the distribution can notably improve the worst-case scenario, with respect to the conventional case explored in Li (2018), in which only the mean and variance are fixed. Furthermore, we offer a numerical analysis of our findings.

Keywords Model uncertainty · Spectral risk measures · Tail-Value-at-Risk · Tail uncertainty · Coherent risk measures

JEL codes D81 Criteria for Decision-Making under Risk and Uncertainty, G22 Insurance · Insurance Companies · Actuarial Studies

Introduction

Financial institutions often face a concrete risk of model misspecification, i.e., the risk of adopting the wrong model for risk assessment. Extrapolating a probabilistic description of single or aggregate losses requires the adoption of statistical procedures that often involves subjective judgment, and their output can be affected by data issues of various nature. These observations have fueled the discussion regarding the need

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to develop risk management procedures that take model risk into account, as reported in Blanchet et al. (2019), Kerkhof et al. (2010), Alexander and Sarabia (2012) and Embrechts et al. (2014). The literature on risk bounds has proposed, explicitly or implicitly, to tackle model risk using a two step-approach. First, it is assumed that only partial information on the loss distribution is available. This represents the fully trusted features of the model, i.e, those features that are considered reliable and not under discussion. The second step consists in computing the best- and worst-case scenario attainable by the risk measure of interest when considering all random variables that respect the fully trusted features. The interested reader can find a formal description of this approach in Barrieu and Scandolo (2015) and Bernard and Vanduffel (2015). The output of this procedure can be used to obtain, e.g., a conservative capital requirement calculation, and an assessment of the full impact of model uncertainty on the risk measure of interest. The definition of the fully trusted features is crucial in the study of model risk. Such set should be designed in a way that allows to exploit as much as possible the information that one can confidently extract from the data at hand, but taking into account that the more features are deemed reliable, the higher the chance of underestimating the riskiness of the loss distribution. As for important recent contributions in this research field, Li et al. (2018) study the worst-case Range Value-at-Risk (RVaR) for unimodal distributions, assuming that mean and variance are given, and Bernard et al. (2020) improve the Value-at-Risk (VaR) and RVaR bounds proposed in Li et al. (2018), including the case of random variables that are non-negative. Considering a more general class of risk measures, Li (2018) obtain closed formulas for the worst-case value of a given spectral risk measure under the knowledge of the mean and the variance, while Cornilly et al. (2018) shows how to compute sharp upper-bounds for spectral risk measures under the knowledge of the mean and of any other moment. Furthermore, Bernard et al. (2024) propose upper and lower bounds for distortion risk measures, given a distribution with fixed mean, variance and satisfying a constraint on the Wasserstein distance from a benchmark distribution function. It is interesting to notice that the assumption of the knowledge of at least two moments on the risk distribution is almost omnipresent in this stream of literature. Other relevant results in the model risk literature concern the study of the risk aggregation problem under dependence uncertainty. In this setting, the random loss of interest is seen as the sum of random variables for which the marginal distributions are known, but their dependence structure is only partially specified. An overview of this class of problems can be found in Puccetti and Rüschendorf (2012), Puccetti and Rüschendorf (2013), Embrechts et al. (2013), Wang et al. (2015), Bernard et al. (2017), Fontana et al. (2021) and Chen et al. (2022) and the references therein.

The risk assessment one can obtain from the literature on risk bounds essentially depends on two ingredients: the risk measures considered and the information available on the loss distribution. We study the class of spectral risk measures, introduced in Acerbi (2002), which include the Tail-Value-at-Risk (TVaR), the Wang, and the dual-power risk measures as special cases; spectral risk measures are (convex) distortion risk measures, as discussed in Dhaene et al. (2012) and Hürlimann (2004).

As for the information available, the present contribution studies various settings, having as common denominator a not-completely specified right-tail distribution for the random loss. In many cases of practical interest, the random losses are modeled using non-negative distributions, which are right-skewed and have a unimodal density function. Such features have been empirically observed in the loss data in various contexts, e.g. in the modeling of insurance contracts' claims severity, or in the analysis of losses deriving from operational risk in the banking sector. The list of probabilistic models that are able to take into account such features includes Gamma, Lognormal, and Pareto distribution. For probability distributions with such characteristics, one can reasonably expect to have most data concentrated in the central and left part of the distribution, while data on the right-tail are more sparse. Hence, this kind of data facilitate the estimation of the central and left-tail of the loss distribution, but do not provide much information on the right-tail. This is consistent with the statistical literature showing that, for the probability distributions usually considered in risk management, the values that the distribution function assumes (or equivalently, its quantiles) corresponding to tail events are more difficult to estimate than in the main body of the distributions, i.e., in those areas in which most of the probability mass of the losses is concentrated. A clear statement regarding this statistical issue was made in Moscadelli (2004) in the context of operational losses modelling: *The main lesson learnt from modelling the severity of operational risk losses by conventional inference methods is that, even though some selected distributions fit the body of the data well, these distributions would underestimate the severity of the data in the tail area.* Moreover, since the values of risk measures are usually more sensitive to changes in the right-tail rather than in the left-tail of the loss distribution (see the case of TVaR, for a clear example), we concentrate our analysis to the case in which the left-tail is given, but the right-tail distribution is not completely specified. One may argue that the distribution function values corresponding to the not-frequent but high-impact losses can be estimated using tools developed in the Extreme Value Theory (EVT). While we acknowledge the fundamental contributions of EVT to the risk management literature and practice, this field of research can not be seen as a panacea of the tails estimation problem, if we may call it such. The reason is that an efficient application of EVT tools requires a certain level of data quality and quantity which is not always available in practice, and it requires embracing certain assumptions that in some situations may be hard to accept. Therefore, there are at least some situations in which a risk manager may not be confident in specifying one particular model for extreme losses. Illustration of the properties and the limitations of EVT can be found, e.g., in Embrechts et al. (1997), Diebold et al. (1998) and Embrechts (2000). The tools that we have developed in the paper are compatible with data-driven and model-independent approaches: our analysis does not require the adoption of a specific statistical methodology, and thus the reference model for the left-tail can also be set equal to the empirical quantile function of the realized data.

In Sect. 2 we delineate the optimization problems and conduct a constraint analysis; we show how the knowledge of the left-tail of the distribution limits the feasible values of the mean and the variance. Section 3 presents upper and lower bounds for a spectral risk measure across increasing levels of information. We consider the case where only the left-tail of the distribution is known (in this case the upper bound is not finite); subsequently we add information on the mean, this guarantees finite bounds for suitable spectral risk measures; Section 3.3 introduce information also on the variance and then we conclude with the most realistic case where one has information on the left-tail of the distribution and confidence intervals for the mean and the variance. Section 3.5 further illustrates the applicability of some of these results to VaR and RVaR. In Sect. 4, through numerical analysis, we establish that in several relevant cases, the bounds that we provide are indeed best-possible, and that the additional knowledge of the left-tail, combined with the knowledge of the first two moments, can effectively improve the risk bounds with respect to the case in which only the mean and the variance are known. Finally, Section 5 provides a concise summary of our findings.

1 Problem statement

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be an atomless probability space and $L^2 := L^2(\Omega, \mathcal{F}, \mathbb{P})$ be the corresponding set of square integrable random variables. For any a real-valued random variable $X \in L^2$, we denote $G_X(x) = \mathbb{P}(X \leqslant x),\ x \in \mathbb{R}$, its cumulative distribution function (cdf), and G_X^{-1} its left-continuous inverse:

$$G_X^{-1}(u) := \inf\{x \in \mathbb{R} \mid G_X(x) \geqslant u\},\ \text{for } u \in (0, 1).$$

With a slight abuse of notation we often refer to G_X^{-1} as the quantile function of X. We define μ_X and σ_X the mean and standard deviation of a random variable $X \in L^2$. We denoted $\mathcal{U}[a, b]$, the uniform distribution on the interval $[a, b]$, with $a, b \in \mathbb{R}$.

Following Li (2018), we define a spectral risk measure as a functional:

$$\varrho_\gamma(X) = \int_0^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u, \quad \forall X \in L^2 \tag{1}$$

where the spectrum (or weight function) $\gamma : [0, 1) \to [0, +\infty)$ is non-decreasing, right-continuous and satisfies $\int_0^1 \gamma(u)\mathrm{d}u = 1$.

Furthermore, for the rest of the paper we have the following:

Assumption 1.1 X is a random variable in L^2 and $\int_0^1 |\gamma(u)|^2\mathrm{d}u < +\infty$.

From the Cauchy-Schwarz inequality, one sees that such an assumption guarantees that the integral is well defined and $\varrho_\gamma(X) < \infty$.

Spectral risk measures are coherent (i.e. satisfy the usual properties of monotonicity, translation invariance, positive homogeneity and convexity, see Artzner et al. 1999) and are widely used in different contexts. The standard example is the Tail Value-at-Risk (TVaR), introduced in Acerbi (2002). This risk measure and slightly different versions of it are available in the literature under different names, e.g. Expected Shortfall, Conditional Value at Risk, Average Value at Risk. This is the benchmark risk measure for regulatory capital requirements in the banking sector (Embrechts et al. 2014 and Chang et al. 2019) and it is defined as

$$\mathrm{TVaR}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 G_X^{-1}(u)\mathrm{d}u, \qquad \gamma(u) = \frac{\mathbb{I}_{(\alpha \leqslant u < 1)}}{1-\alpha},$$
$$\text{for } \alpha \in (0, 1) \text{ and } u \in (0, 1),$$

where $\mathbb{I}_{(A)}$ denotes the indicator function that takes value 1 if A holds and 0 otherwise. Further notable examples of spectral risk measures include the Dual power risk measure (Brandtner 2016 and Balbás et al. 2009), identified by the weight function

$$\gamma(u) = ku^{k-1}, \text{ for } k \geqslant 1, \; u \in (0, 1);$$

and the Wang transform, which plays an important role in the pricing of insurance contracts (Wang 1996) and financial options (Wang 2000), having spectrum of the form

$$\gamma(u) = \frac{\phi(\Phi^{-1}(1-u) + \Phi^{-1}(\beta))}{\phi(\Phi^{-1}(1-u))}, \text{ for } 0.5 < \beta < 1, \; u \in (0, 1),$$

in which ϕ and Φ are the density and the cdf of a standard Gaussian random variable.

The goal of this paper is to investigate the following problems

$$\underline{\varrho}_{\mathcal{M}} := \inf_{X \in \mathcal{M}} \varrho_\gamma(X), \qquad \text{and} \qquad \bar{\varrho}_{\mathcal{M}} := \sup_{X \in \mathcal{M}} \varrho_\gamma(X), \tag{2}$$

where $\mathcal{M}$ is a set of random variables with trusted information and ρ_γ is a spectral risk measure with weight function γ. The sets considered are

$$\mathcal{M}(F_q) := \left\{ X \in L^2 \;\middle|\; G_X^{-1}(u) \equiv F^{-1}(u), \text{ for all } u \in (0, q] \right\};$$
$$\mathcal{M}(F_q, \mu) := \left\{ X \in \mathcal{M}(F_q) \;\middle|\; \mathbb{E}[X] = \mu \right\};$$
$$\mathcal{M}(F_q, \mu, \sigma^2) := \left\{ X \in \mathcal{M}(F_q, \mu) \;\middle|\; \text{var}(X) = \sigma^2 \right\},$$

where $\mu \in \mathbb{R}$, $\sigma > 0$, $q \in (0, 1)$ and F represents a distribution function with finite second moment. Note that μ and σ^2 are not necessarily the mean and variance of the distribution F. In Sect. 2.4, we also consider the case in which the mean and the variance belong to a given uncertainty set, i.e. we study the case:

$$\mathcal{M}(F_q, [\underline{m}, \overline{m}], [\underline{s}, \overline{s}]) := \left\{ X \in L^2 \cap \mathcal{M}(F_q) \;\middle|\; \mathbb{E}[X] \in [\underline{m}, \overline{m}], \; \text{var}(X) \in [\underline{s}, \overline{s}] \right\},$$

where $\underline{m}, \overline{m} \in \mathbb{R}$ and $\underline{s}, \overline{s} > 0$ represent possible bounds for the mean and the variance, respectively.

Remark 1 The knowledge of the quantile function on $(0, q]$ corresponds to the knowledge of the cdf itself on the interval $(-\infty, F^{-1}(q))$. Therefore the set $\mathcal{M}(F_q)$ could be equivalently written as

$$\mathcal{M}(F_q) := \left\{ X \in L^2 \;\middle|\; G_X(x) \equiv F(x), \text{ for all } x \in (-\infty, F^{-1}(q)) \right\}.$$

The proof of the equality above can be found in Appendix A.1. Thus, the bounds derived for spectral risk measures considering the distributions in the sets $\mathcal{M}(F_q)$, $\mathcal{M}(F_q, \mu)$, $\mathcal{M}(F_q, \mu, \sigma^2)$ and $\mathcal{M}(F_q, [\underline{m}, \overline{m}], [\underline{s}, \overline{s}])$ can be seen as bounds obtained under partial knowledge on the cdf of the random loss of interest. Specifically, we assume the cdf is known in the interval $(-\infty, F^{-1}(q))$, but not in $[F^{-1}(q), \infty)$.

1.1 Constraints analysis

Spectral risk measures are law-invariant, therefore random variables with the same distribution can be considered equivalent for the risk measure evaluation. In what follows, with a slight abuse of notation, we say that $\mathcal{M}(F_q)$ and its subsets, contain infinitely many elements if they contain infinitely many random variables with different distribution functions. This section provides conditions under which the information sets are not empty and contain infinitely many elements. In particular, Proposition 1.1 below shows that information on the left-tail and information on the left-tail and the mean implies the following lower bounds for the mean and the variance of random variables in $\mathcal{M}(F_q)$ and $\mathcal{M}(F_q, \mu)$ respectively:

$$\underline{\mu}(F_q) := \int_0^q F^{-1}(u)\mathrm{d}u + (1-q)F^{-1}(q); \tag{3}$$

$$\begin{aligned}\underline{\sigma}^2(F_q, \mu) &:= \int_0^q F^{-1}(u)^2\mathrm{d}u + \frac{\left(\mu - \int_0^q F^{-1}(u)\,\mathrm{d}u\right)^2}{1-q} - \mu^2 \\ &= \int_0^q F^{-1}(u)^2\mathrm{d}u + (1-q)\mu_q^2 - \mu^2,\end{aligned} \tag{4}$$

where μ_q is given by:

$$\mu_q = \frac{\mu - \int_0^q F^{-1}(u)\mathrm{d}u}{1-q}. \tag{5}$$

To keep the notation compact, we also use:

$$\mu_{2,q} = \frac{\sigma^2 + \mu^2 - \int_0^q F^{-1}(u)^2\mathrm{d}u}{1-q}. \tag{6}$$

Remark 2 (Interpretation of q-tail moments) Given $U_q \sim \mathcal{U}[q, 1]$ and $X \sim G_X$, the random variable $G_X^{-1}(U_q)$ is called tail-risk of X beyond its q-quantile in Liu and Wang (2021), its distribution is called q-tail distribution of X in Rockafellar and Uryasev (2002). Quantities μ_q and $\mu_{2,q} - \mu_q^2$ describe the expected value and variance of the q-tail distribution of each random variable in $\mathcal{M}(F_q, \mu)$ and $\mathcal{M}(F_q, \mu, \sigma^2)$, respectively. Namely, $\mu_q = \mathbb{E}[G_X^{-1}(U_q)]$ and $var(G_X^{-1}(U_q)) = \mu_{2,q} - \mu_q^2$ hold true for any $X \in \mathcal{M}(F_q, \mu, \sigma^2)$. Hence, the constraints used in the definition of $\mathcal{M}(F_q, \mu, \sigma^2)$ provide partial information (i.e. expected value and variance) on the right-tail of the random loss X. Note that μ_q coincides also with $\mathrm{TVaR}_q(X)$. Analogously, in what follows $\mu_{\gamma(U_q)} = \frac{1}{1-q}\int_q^1 \gamma(u)\mathrm{d}u$ and $\sigma_{\gamma(U_q)} = \sqrt{\frac{\int_q^1 \gamma(u)^2\mathrm{d}u}{1-q} - \left(\frac{\int_q^1 \gamma(u)\mathrm{d}u}{1-q}\right)^2}$, represent respectively the mean and standard deviation of the r.v. $\gamma(U_q),\ U_q \sim \mathcal{U}[q,1]$, where γ is the weight function of a spectral risk measure.

Proposition 1.1 *Let X be a random variable in $\mathcal{M}(F_q)$. Then,*

$$\mathbb{E}[X] \geqslant \underline{\mu}(F_q). \tag{7}$$

Let X be a random variable in $\mathcal{M}(F_q, \mu)$ with $\mu \geqslant \underline{\mu}(F_q)$. Then,

$$var(X) \geqslant \underline{\sigma}^2(F_q, \mu). \tag{8}$$

The bounds are attainable and thus best-possible.

Proof Any random variable $X \in \mathcal{M}(F_q)$ is higher in first-order stochastic dominance[1] than the random variable $X_{st} \in \mathcal{M}(F_q)$, where

$$G^{-1}_{X_{st}}(u) = F^{-1}(u)\mathbb{I}_{(0<u\leqslant q)} + F^{-1}(q)\mathbb{I}_{(q<u<1)}. \tag{9}$$

This can be easily seen using the characterization of first-order stochastic dominance based on the cdf left-inverse[2]. Thus, X_{st} is the minimal element in first-order stochastic dominance in the set $\mathcal{M}(F_q)$. Since the expected value is consistent with the first-order stochastic dominance, we obtain that for any $X \in \mathcal{M}(F_q)$

$$\mathbb{E}[X] \geqslant \mathbb{E}[X_{st}] = \underline{\mu}(F_q),$$

which completes the proof of (7).

As for the inequality in (8), consider a random variable $X_{cx} \in \mathcal{M}(F_q, \mu),\ \mu \geqslant \underline{\mu}(F_q)$ and its inverse distribution function

$$G^{-1}_{X_{cx}}(u) = F^{-1}(u)\mathbb{I}_{(0<u\leqslant q)} + \mu_q\mathbb{I}_{(q<u<1)}. \tag{10}$$

For any $X \in \mathcal{M}(F_q, \mu)$, one has

$$\begin{aligned}\mathrm{TVaR}_\alpha(X) &= \frac{1}{1-\alpha}\int_\alpha^1 G^{-1}_X(u)\mathrm{d}u = \frac{1}{1-\alpha}\left(\int_\alpha^q F^{-1}(u)\mathrm{d}u + (1-q)\mu_q\right)\\ &= \mathrm{TVaR}_\alpha(X_{cx})\end{aligned}$$

for all $\alpha \in (0, q]$. Since TVaR_α is non-decreasing with respect to α, and $\mathrm{TVaR}_q(X) = \mu_q$, we have $\mathrm{TVaR}_\alpha(X) \geqslant \mu_q = \mathrm{TVaR}_\alpha(X_{cx})$ for all $\alpha \in (q, 1)$. Therefore, $\mathrm{TVaR}_\alpha(X) \geqslant \mathrm{TVaR}_\alpha(X_{cx})$ for all $\alpha \in (0, 1)$. From Theorem 3.5.A in Shaked and

[1] Given two random variables X an Y we say that X dominates Y in first-order stochastic dominance, and write $Y \leqslant_{st} X$ if $\mathbb{E}(\phi(X)) \geqslant \mathbb{E}(\phi(Y))$ for every increasing function $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectation exists.

[2] Recall that $Y \leqslant_{st} X$ is equivalent to $G^{-1}_Y(u) \leqslant G^{-1}_X(u)$ for all $u \in (0, 1)$.

Shanthikumar (2007), it follows that X dominates X_{cx} in convex order.[3] Thus, X_{cx} is the minimal element in convex order for the random variables in $\mathcal{M}(F_q,\mu)$ and one gets

$$var(X) \geqslant var(X_{cx}) = \underline{\sigma}^2(F_q,\mu),$$

which completes the proof of inequality (8). □

Remark 3 Condition $\mu \geqslant \underline{\mu}(F_q)$ is necessary for $\mathcal{M}(F_q,\mu)$ to be not-empty. Together with condition $\sigma^2 \geqslant \underline{\sigma}^2(F_q,\mu)$ it is necessary for the set $\mathcal{M}(F_q,\mu,\sigma^2)$ to be not-empty. Furthermore, the set $\mathcal{M}(F_q,\underline{\mu}(F_q))$ contains only random variables with inverse distribution function $G^{-1}_{X_{st}}$ defined in (9), while the set $\mathcal{M}(F_q,\mu,\underline{\sigma}^2(F_q,\mu))$ contains only distributions with inverse distribution function $G^{-1}_{X_{cx}}$, defined in (10). In this case upper and lower bounds are trivial. For this reason, we restrict our attention to the sets $\mathcal{M}(F_q,\mu)$ and $\mathcal{M}(F_q,\mu,\sigma^2)$ with $\mu > \underline{\mu}(F_q)$ and $\sigma > \underline{\sigma}(F^{-1},q,\mu)$. Proposition 1.2 below, shows that these sets are not-empty and contain infinitely many elements, so that the optimization problems in (2) are well-posed and non-trivial. Part *b*) of the following proposition discusses the behaviour of the information sets when the parameter q varies.

Proposition 1.2 *a) For any $\mu > \underline{\mu}(F_q)$ and $\sigma^2 > \underline{\sigma}^2(F_q,\mu)$, the sets $\mathcal{M}(F_q)$, $\mathcal{M}(F_q,\mu)$ and $\mathcal{M}(F_q,\mu,\sigma^2)$ contain infinitely many distributions.*
b) For any $0 < q < q' < 1$, one has

$$\mathcal{M}(F_{q'}) \subseteq \mathcal{M}(F_q), \quad \mathcal{M}(F_{q'},\mu) \subseteq \mathcal{M}(F_q,\mu), \quad \mathcal{M}(F_{q'},\mu,\sigma) \subseteq \mathcal{M}(F_q,\mu,\sigma^2).$$

Proof See Appendix A.2. □

2 Risk bounds under increasing levels of information

For ease of exposition, we start considering the bounds when information limited to the left-tail is available.

2.1 Risk bounds under the sole knowledge of the left-tail

We are interested in determining the following lower and upper bounds:

$$\underline{\varrho}_{\{F_q\}} := \inf_{X\in\mathcal{M}(F_q)} \varrho_\gamma(X), \tag{11}$$

$$\bar{\varrho}_{\{F_q\}} := \sup_{X\in\mathcal{M}(F_q)} \varrho_\gamma(X). \tag{12}$$

Focusing on the lower bound, we have the following result:

[3] Given two random variables X an Y we say that X dominates Y in convex order if $\mathbb{E}(\psi(X)) \geqslant \mathbb{E}(\psi(Y))$ for every convex function $\psi:\mathbb{R}\to\mathbb{R}$, provided the expectation exists.

Proposition 2.1 *The lower bound in (11) is given by*

$$\underline{\varrho}_{\{F_q\}} = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}F^{-1}(q)$$

and it is attained by a random variable $Y \in \mathcal{M}(F_q)$ with $G_Y^{-1}(u) \equiv F^{-1}(q)$ for any $u \in (q, 1]$.

Proof The proof follows immediately by noting that for any $X \in \mathcal{M}(F_q)$, G_X^{-1} is non-decreasing and $G_X^{-1}(u) \geqslant F^{-1}(q)$ for any $u \in [q, 1)$, then

$$\begin{aligned}\underline{\varrho}_{\{F_q\}} &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \inf_{X\in\mathcal{M}(F_q)} \int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u \\ &\geqslant \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + F^{-1}(q)\int_q^1 \gamma(u)\mathrm{d}u \\ &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}F^{-1}(q).\end{aligned}$$

The lower bound is attained by the random variable $Y \in \mathcal{M}(F_q)$ with $G_Y^{-1}(u) \equiv F^{-1}(q)$ for any $u \in (q, 1)$. This result can also be obtained noting that Y is the minimal element in first-order stochastic dominance for the set $\mathcal{M}(F_q)$, see also Proposition 2.15. □

Corollary 2.2 *From Proposition 1.2 b) the lower bound $\underline{\varrho}_{\{F_q\}}$ is non-decreasing in q.*

Next proposition shows that, without further assumptions, it is not possible to guarantee the finiteness of the upper bound.

Proposition 2.3 *The upper bound in (12) is not finite:*

$$\bar{\varrho}_{\{F_q\}} = +\infty.$$

Proof Define a random variable $Y \in \mathcal{M}(F_q)$ with $G_Y^{-1}(u) \equiv b$ for $b \in [F^{-1}(q), +\infty)$ and $u \in (q, 1)$, then $\varrho_\gamma(Y) = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + b\int_q^1 \gamma(u)\mathrm{d}u \leqslant \bar{\varrho}_{\{F_q\}}$. This holds for any $b \in \left[F^{-1}(q), +\infty\right)$, therefore the upper bound cannot be finite. □

2.2 Risk bounds under the knowledge of the left-tail and the mean

In this subsection we impose the knowledge of the left-tail of the distribution as well as the mean. This guarantees a finite upper bound in case the spectrum is bounded from above. We consider the following problems:

$$\underline{\varrho}_{\{F_q,\mu\}} := \inf_{X\in\mathcal{M}(F_q,\mu)} \varrho_\gamma(X), \tag{13}$$

$$\bar{\varrho}_{\{F_q,\mu\}} := \sup_{X\in\mathcal{M}(F_q,\mu)} \varrho_\gamma(X). \tag{14}$$

First, we point out that for spectral risk measures ϱ having a spectrum γ that is constant on $(q, 1)$, the knowledge of the left-tail and of the expected value is enough to compute exactly the value of the risk measure. This is formally stated in the next proposition.

Proposition 2.4 *Assume $\gamma(u) = c \in \mathbb{R}$ for every $u \in (q, 1)$. Then:*

$$\underline{\varrho}_{\{F_q,\mu\}} = \overline{\varrho}_{\{F_q,\mu\}} = \varrho_\gamma(X) = \int_0^q F^{-1}(u)\gamma(u)du + c\left(\mu - \int_0^q F^{-1}(u)du\right),$$

for every $X \in \mathcal{M}(F_q, \mu)$.

The proof of Proposition 2.4 is straightforward and thus omitted. As for the interpretation, Proposition 2.4 implies that when one considers the distributions belonging to the set $\mathcal{M}(F_q, \mu)$ or a subset of it, they share the same values for a risk measure ϱ having a spectrum that is constant on $(q, 1)$. For instance, one does not have uncertainty on the $\mathrm{TVaR}_\alpha(X)$ for $\alpha \leqslant q$ if $X \in \mathcal{M}(F_q, \mu)$. In the rest of the paper we tacitly consider the case in which γ is not constant on the interval $(q, 1)$.

For the lower bound we get the following result:

Proposition 2.5 *The lower bound in (13) is given by*

$$\underline{\varrho}_{\{F_q,\mu\}} = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}\mu_q$$

and it is attained by the random variable $Y \in \mathcal{M}(F_q, \mu)$ with $G_Y^{-1}(u) \equiv \mu_q$ for any $u \in (q, 1)$, where μ_q has been defined in (5).

Proof Consider the r.v. $Y \in \mathcal{M}(F_q, \mu)$ with $G_Y^{-1}(u) \equiv \mu_q$ for any $u \in (q, 1)$; clearly $\underline{\varrho}_{\{F_q,\mu\}} \leqslant \varrho_\gamma(Y)$. Assume that there exists a r.v. $Y^* \in \mathcal{M}(F_q, \mu)$ such that $\varrho_\gamma(Y^*) < \varrho_\gamma(Y)$, that is

$$\int_q^1 G_{Y^*}^{-1}(u)\gamma(u)\mathrm{d}u < \mu_q \int_q^1 \gamma(u)\mathrm{d}u = (1-q)\mu_{\gamma(U_q)}\mu_q. \tag{15}$$

Consider a r.v. $U_q \sim \mathcal{U}[q, 1]$, then one has $\mathbb{E}[G_{Y^*}^{-1}(U_q)\gamma(U_q)] = \frac{1}{1-q}\int_q^1 G_{Y^*}^{-1}(u)\gamma(u)\mathrm{d}u$ and the random variables $G_{Y^*}^{-1}(U_q)$ and $\gamma(U_q)$ are comonotone since they arise as increasing functions of the same r.v. U_q, furthermore $\mathbb{E}[G_{Y^*}^{-1}(U_q)] = \mu_q$ and $\mathbb{E}[\gamma(U_q)] = \mu_{\gamma(U_q)}$. Inequality (15) leads to

$$\mathbb{E}[G_{Y^*}^{-1}(U_q)\gamma(U_q)] - \mu_q\mu_{\gamma(U_q)} = \mathrm{Cov}(G_{Y^*}^{-1}(U_q), \gamma(U_q)) < 0,$$

that is not possible since the covariance of comonotonic random variables is non-negative, see for instance Schmidt (2014). This result can also be obtained noting that

Y is the minimal element in convex order for the set $\mathcal{M}(F_q)$, see also Proposition 2.15. □

As for the upper bound, the next proposition shows that its finiteness depends on the spectrum γ. Specifically, from Proposition 2.6 we can conclude that $\overline{\varrho}_{\{F_q,\mu\}}$ is finite if and only if $\lim_{c\to 1^-}\gamma(c)$ is finite. Note that such a condition on the spectrum γ holds for the TVaR and Dual power, but not for all spectral risk measures, see the case of the Wang transform.

Proposition 2.6 *The upper bound in* (14) *is given by*

$$\overline{\varrho}_{\{F_q,\mu\}} = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}F^{-1}(q) + (1-q)(\mu_q - F^{-1}(q))\gamma(1^-), \tag{16}$$

where $\gamma(1^-) = \lim_{c\to 1^-}\gamma(c)$.

Proof We begin with the case $F^{-1}(q) = 0$. This implies $\mu_q > 0$ and equality in (16) becomes

$$\overline{\varrho}_{\{F_q,\mu\}} = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_q\gamma(1^-).$$

We prove the following inequalities:

$$\overline{\varrho}_{\{F_q,\mu\}} \leqslant \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_q\gamma(1^-), \tag{17}$$

and

$$\overline{\varrho}_{\{F_q,\mu\}} \geqslant \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_q\gamma(1^-). \tag{18}$$

Since γ is non-decreasing, $G_X^{-1}(u)\gamma(u) \leqslant G_X^{-1}(u)\gamma(1^-)$ for every $u \in (q, 1)$ and for every $X \in \mathcal{M}(F_q, \mu)$. Thus, (17) follows from

$$\begin{aligned}
\overline{\varrho}_{\{F_q,\mu\}} &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \sup_{X\in\mathcal{M}(F_q,\mu)} \int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u \\
&\leqslant \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \gamma(1^-) \sup_{X\in\mathcal{M}(F_q,\mu)} \int_q^1 G_X^{-1}(u)\mathrm{d}u \\
&= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_q\gamma(1^-).
\end{aligned}$$

Inequality (18) can be expressed as follows:

$$\sup_{X\in\mathcal{M}(F_q,\mu)} \int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u \geqslant \varepsilon, \text{ for every } \epsilon < (1-q)\mu_q\gamma(1^-).$$

Fix $\epsilon < (1-q)\mu_q\gamma(1^-)$, define

$$p_\epsilon = \inf\{p \in (q,1) \,|\, (1-q)\mu_q\gamma(p) \geqslant \epsilon\}.$$

The assumption $\epsilon < (1-q)\mu_q\gamma(1^-) = (1-q)\mu_q \sup_{u\in(0,1)} \gamma(u)$ implies that $\{p \in (q,1) \,|\, (1-q)\mu_q\gamma(p) \geqslant \epsilon\}$ is a not-empty set. Additionally, the spectrum γ is non-decreasing and $\mu_q > 0$, thus $p_\epsilon \in [q,1)$. Consider two cases.

CASE 1) Assume $p_\epsilon > q$. Fix $\tilde{X} \in \mathcal{M}(F_q,\mu)$. For every $\beta > 1$, it is possible to define

$$G^{-1}_{\beta,\epsilon}(u) = \begin{cases} F^{-1}(u) \text{ for } u \in (0,q], \\ G^{-1}_{\tilde{X}}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \text{ for } u \in (q,p_\epsilon], \\ G^{-1}_{\tilde{X}}(u)\alpha_{\beta,\epsilon} \text{ for } u \in (p_\epsilon,1), \end{cases}$$

in which

$$\alpha_{\beta,\epsilon} = \frac{(1-q)\mu_q - \int_q^{p_\epsilon} G^{-1}_{\tilde{X}}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u}{\int_{p_\epsilon}^1 G^{-1}_{\tilde{X}}(u)\mathrm{d}u}.$$

The function $\left(\frac{u}{p_\epsilon}\right)^\beta$ is increasing, continuous, positive and lower than 1 for $u \in (q,p_\epsilon]$. This implies $\alpha_{\beta,\epsilon} \geqslant 1$. Assumption $F^{-1}(q) = 0$, guarantees that $G^{-1}_{\beta,\epsilon}(q) = 0$ and $G^{-1}_{\beta,\epsilon}$ is non-decreasing, left-continuous, and satisfies $\int_0^1 G^{-1}_{\beta,\epsilon}(u) = \mu$. Thus, $G^{-1}_{\beta,\epsilon}$ is a quantile function of a random variable in $\mathcal{M}(F_q,\mu)$ for every $\beta > 1$. This last observation allows us to write

$$\sup_{X\in\mathcal{M}(F_q,\mu)} \int_q^1 G^{-1}_X(u)\gamma(u)\mathrm{d}u \geqslant \sup_{\beta>1} \int_q^1 G^{-1}_{\beta,\epsilon}(u)\gamma(u)\mathrm{d}u. \tag{19}$$

Let us focus on the right-hand side of (19). Fix $\beta > 1$, by definition

$$\int_q^1 G^{-1}_{\beta,\epsilon}(u)\gamma(u)\mathrm{d}u = \int_q^{p_\epsilon} G^{-1}_{\tilde{X}}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \gamma(u)\mathrm{d}u + \int_{p_\epsilon}^1 G^{-1}_{\tilde{X}}(u)\alpha_{\beta,\epsilon}\gamma(u)\mathrm{d}u.$$

Moreover, $\int_q^{p_\epsilon} G^{-1}_{\tilde{X}}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \gamma(u)\mathrm{d}u \geqslant 0$ implies

$$\int_q^1 G^{-1}_{\beta,\epsilon}(u)\gamma(u)\mathrm{d}u \geqslant \alpha_{\beta,\epsilon}\int_{p_\epsilon}^1 G^{-1}_{\tilde{X}}(u)\gamma(u)\mathrm{d}u.$$

Again one has $\int_{p_\epsilon}^1 G^{-1}_{\tilde{X}}(u)\gamma(u)\mathrm{d}u = (1-p_\epsilon)\mathbb{E}[G^{-1}_{\tilde{X}}(U_{p_\epsilon})\gamma(U_{p_\epsilon})]$, where $U_{p_\epsilon} \sim \mathcal{U}[p_\epsilon,1]$. From Schmidt (2014) and the comonotonicity of $G^{-1}_{\tilde{X}}(U_{p_\epsilon})$ and $\gamma(U_{p_\epsilon})$ follows

$$(1-p_\epsilon)\mathbb{E}[G_{\tilde{X}}^{-1}(U_{p_\epsilon})\gamma(U_{p_\epsilon})] \geqslant (1-p_\epsilon)\mathbb{E}[G_{\tilde{X}}^{-1}(U_{p_\epsilon})]\mathbb{E}[\gamma(U_{p_\epsilon})] \\ = \int_{p_\epsilon}^1 G_{\tilde{X}}^{-1}(u)\mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}.$$

Hence, we have obtained that

$$\alpha_{\beta,\epsilon}\int_{p_\epsilon}^1 G_{\tilde{X}}^{-1}(u)\gamma(u)\mathrm{d}u \geqslant \alpha_{\beta,\epsilon}\int_{p_\epsilon}^1 G_{\tilde{X}}^{-1}(u)\mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}.$$

By definition of $\alpha_{\beta,\epsilon}$,

$$\begin{aligned}
&\alpha_{\beta,\epsilon}\int_{p_\epsilon}^1 G_{\tilde{X}}^{-1}(u)\mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} \\
&= (1-q)\mu_q \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} - \int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} \\
&\geqslant \frac{\int_{p_\epsilon}^1 \epsilon\mathrm{d}u}{1-p_\epsilon} - \int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} \\
&= \epsilon - \int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}.
\end{aligned}$$

It follows that

$$\int_q^1 G_{\beta,\epsilon}^{-1}(u)\gamma(u)\mathrm{d}u \geqslant \epsilon - \int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} \text{ for every } \beta > 1,$$

from which we deduce

$$\sup_{\beta>1}\int_q^1 G_{\beta,\epsilon}^{-1}(u)\gamma(u)\mathrm{d}u \geqslant \sup_{\beta>1} \epsilon - \int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} \\ = \epsilon - \inf_{\beta>1}\int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}.$$

Observe now that the mapping $\beta \mapsto \int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}$ is decreasing w.r.t. β and always non-negative. Hence,

$$\inf_{\beta>1}\int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} = \lim_{\beta\to\infty}\int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}$$

$$\leqslant \lim_{\beta\to\infty}\int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(p_\epsilon)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon}$$
$$= \lim_{\beta\to\infty} G_{\tilde{X}}^{-1}(p_\epsilon)\left(\frac{1}{\beta+1}\left(1-\left(\frac{q}{p_\epsilon}\right)^{\beta+1}\right)\right)\frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} = 0.$$

This last result leads us to

$$\sup_{X\in\mathcal{M}(F_q,\mu)}\int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u \geqslant \sup_{\beta>1}\int_q^1 G_{\beta,\epsilon}^{-1}(u)\gamma(u)\mathrm{d}u \geqslant \sup_{\beta>1}\epsilon$$
$$-\int_q^{p_\epsilon} G_{\tilde{X}}^{-1}(u)\left(\frac{u}{p_\epsilon}\right)^\beta \mathrm{d}u \frac{\int_{p_\epsilon}^1 \gamma(u)\mathrm{d}u}{1-p_\epsilon} \geqslant \epsilon,$$

which completes the proof of (18) in the case $p_\epsilon > q$.

CASE 2) Assume $p_\epsilon = q$. This is true if and only if $(1-q)\mu_q\gamma(u) \geqslant \epsilon$ for every $(q, 1)$. Hence, for any $X \in \mathcal{M}(F_q, \mu)$ and r.v. U_q with uniform distribution $\mathcal{U}[q, 1]$, one can use again the results from Schmidt (2014) to obtain

$$\int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u = (1-q)\mathbb{E}[G_{\tilde{X}}^{-1}(U_q)\gamma(U_q)] \geqslant (1-q)\mathbb{E}[G_{\tilde{X}}^{-1}(U_q)]\mathbb{E}[\gamma(U_q)] =$$
$$= \int_q^1 G_{\tilde{X}}^{-1}(u)\mathrm{d}u \frac{\int_q^1 \gamma(u)\mathrm{d}u}{1-q} = \frac{\int_q^1 (1-q)\mu_q\gamma(u)\mathrm{d}u}{1-q} \geqslant \frac{\int_q^1 \epsilon\mathrm{d}u}{1-q} = \epsilon,$$

which completes the proof of (18) in the case $p_\epsilon = q$, and also the proof of this proposition in the case $F^{-1}(q) = 0$.

Let us focus on the case $F^{-1}(q) \neq 0$. Then, for any random variable $X \in \mathcal{M}(F_q, \mu)$, the random variable $Y = X - F^{-1}(q)$ belongs to $\mathcal{M}(\hat{F}_q, \hat{\mu})$, in which $\hat{F}^{-1}(u) = F^{-1}(u) - F^{-1}(q)$ and $\hat{\mu} = \mu - F^{-1}(q)$. Since $\hat{F}^{-1}(q) = 0$, we can compute $\overline{\varrho}_{\{\hat{F}_q,\hat{\mu}\}}$ using (16). Moreover, from the translation invariance of spectral risk measure and $\int_0^1 \gamma(u)du = 1$, we deduce

$$\overline{\varrho}_{\{F_q,\mu\}} = \overline{\varrho}_{\{\hat{F}_q,\hat{\mu}\}} + F^{-1}(q) = \int_0^q \hat{F}(u)\gamma(u)\mathrm{d}u + (1-q)\hat{\mu}_q\gamma(1^-) + F^{-1}(q)$$
$$= \int_0^q (F^{-1}(u) - F^{-1}(q))\gamma(u)\mathrm{d}u + (1-q)(\mu_q - F^{-1}(q))\gamma(1^-) + F^{-1}(q)$$
$$= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u - F^{-1}(q)\int_0^q \gamma(u)\mathrm{d}u$$
$$+ (1-q)(\mu_q - F^{-1}(q))\gamma(1^-) + F^{-1}(q)\int_0^1 \gamma(u)du$$
$$= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}F^{-1}(q) + (1-q)(\mu_q - F^{-1}(q))\gamma(1^-).$$

□

The bounds that we derive for any spectral risk measure on the uncertainty set $\mathcal{M}(F_q, \mu)$ are best-possible. This observation, together with Proposition 1.2 b), yields to the following corollary.

Corollary 2.7 *Let μ be equal to the mean of the reference model F. Then, the mappings*

$$q \mapsto \overline{\varrho}_{\{F_q,\mu\}} \quad \text{and} \quad q \mapsto \underline{\varrho}_{\{F_q,\mu\}}$$

are non-increasing and non-decreasing, respectively, for every $q \in (0, 1)$.

2.3 Risk bounds under the knowledge of the left-tail, the mean and the variance

In order to get more stringent bounds, in this section we impose the knowledge of the left-tail of the distribution as well as the mean and the variance. In light of the results presented in Sect. 1.1, given a distribution function F and $q \in (0, 1)$, we will tacitly assume $\mu > \underline{\mu}(F_q)$ and $\sigma^2 > \underline{\sigma}^2(F_q, \mu)$. The additional variance constraint allows us to obtain a finite worst-case scenario, also in the case γ is unbounded. Therefore, we are interested in calculating:

$$\underline{\varrho}_{\{F_q,\mu,\sigma^2\}} := \inf_{X\in\mathcal{M}(F_q,\mu,\sigma^2)} \varrho_\gamma(X), \tag{20}$$

$$\bar{\varrho}_{\{F_q,\mu,\sigma^2\}} := \sup_{X\in\mathcal{M}(F_q,\mu,\sigma^2)} \varrho_\gamma(X). \tag{21}$$

Remark 4 Since $\mathcal{M}(F_q, \mu, \sigma^2) \subseteq \mathcal{M}(F_q, \mu)$ the infimum in (20) satisfies

$$\underline{\varrho}_{\{F_q,\mu,\sigma^2\}} \geqslant \underline{\varrho}_{\{F_q,\mu\}} = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}\mu_q. \tag{22}$$

The following result discusses the upper bound in (21);

Theorem 2.8 *Assume that γ is not constant on $(q, 1)$, then*

$$\bar{\varrho}_{\{F_q,\mu,\sigma^2\}} \leqslant \overline{\varrho_q} := \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}\mu_q + (1-q)\sigma_{\gamma(U_q)}\sqrt{\mu_{2,q} - \mu_q^2}. \tag{23}$$

Moreover, if

$$\frac{\gamma(q^-) - \mu_{\gamma(U_q)}}{\sigma_{\gamma(U_q)}}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \geqslant F^{-1}(q), \tag{24}$$

then the upper bound in (23) is best-possible and attained by the quantile function $\overline{G}^{-1}$ defined as follows:

$$\overline{G}^{-1}(u) = \begin{cases} F^{-1}(u) \text{ for } u \in (0, q], \\ \frac{\gamma(u^-)-\mu_{\gamma(U_q)}}{\sigma_{\gamma(U_q)}}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \text{ for } u \in (q, 1), \end{cases} \tag{25}$$

in which $\gamma(u^-) := \lim_{c\to u^-}\gamma(c)$.

Proof For any random variable $X \in \mathcal{M}(F_q, \mu, \sigma^2)$, it holds

$$\rho_\gamma(X) = \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u.$$

Furthermore, given $U_q \sim U[q,1]$ one can define two comonotone random variables $G_X^{-1}(U_q)$ and $\gamma(U_q)$, such that $\int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u = (1-q)\mathbb{E}[G_X^{-1}(U_q)\gamma(U_q)]$, $\mathbb{E}[G_X^{-1}(U_q)] = \mu_q$, $\mathbb{E}[(G_X^{-1}(U_q)^2)] = \int_q^1 G_X^{-1}(u)^2\frac{1}{1-q}\mathrm{d}u = \mu_{2,q}$ and $\mathbb{E}[\gamma(U_q)] = \mu_{\gamma(U_q)}$. It follows that

$$\begin{aligned}\bar{\varrho}_{\{F_q,\mu,\sigma^2\}} &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \sup_{X\in\mathcal{M}(F_q,\mu,\sigma^2)}\int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u\\ &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \sup_{X\in\mathcal{M}(F_q,\mu,\sigma^2)}(1-q)\mathbb{E}[G_X^{-1}(U_q)\gamma(U_q)]\\ &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \sup_{X\in\mathcal{M}(F_q,\mu,\sigma^2)}\\ &\quad (1-q)[\mathrm{cov}(G_X^{-1}(U_q),\gamma(U_q)) + \mu_q\mu_{\gamma(U_q)}]. \end{aligned} \tag{26}$$

Let us denote with $\mathcal{Y}_q$ the set of random variables having first and second moment that coincide with μ_q and $\mu_{2,q}$ as in (5) and (6), respectively:

$$\mathcal{Y}_q = \left\{Y_q \text{ r.v. } \middle| \mathbb{E}[Y_q] = \mu_q,\ \mathbb{E}[Y_q^2] = \mu_{2,q}\right\}.$$

Note that for any $X \in \mathcal{M}(F_q, \mu, \sigma^2)$, $G_X^{-1}(U_q)$ belongs to $\mathcal{Y}_q$, and therefore one has

$$\begin{aligned}&\sup_{X\in\mathcal{M}(F_q,\mu,\sigma^2)}(1-q)[\mathrm{cov}(G_X^{-1}(U_q),\gamma(U_q)) + \mu_q\mu_{\gamma(U_q)}]\\ &\leqslant \sup_{Y_q\in\mathcal{Y}_q}(1-q)\Big[\mathrm{cov}(Y_q,\gamma(U_q)) + \mu_q\mu_{\gamma(U_q)}\Big].\end{aligned}$$

It follows from the Cauchy-Schwarz inequality that covariance between random variables is maximized by (positive) linear dependence, and $\mathcal{Y}_q$ contains only one random variable that is an increasing linear transformation of $\gamma(U_q)$.

$$Y := \frac{\gamma(U_q) - \mu_{\gamma(U_q)}}{\sigma_{\gamma(U_q)}} \sqrt{\mu_{2,q} - \mu_q^2} + \mu_q.$$

Note $Y \in \mathcal{Y}_q$ in that $\mathbb{E}[Y] = \mu_q$ and $\mathbb{E}[Y^2] = \mu_{2,q}$. An easy computation provides $\mathrm{cov}(Y, \gamma(U_q)) = \sqrt{\mu_{2,q} - \mu_q^2}\sigma_{\gamma(U_q)}$, so that:

$$\sup_{X\in\mathcal{M}(F_q,\mu,\sigma^2)} \int_q^1 G_X^{-1}(u)\gamma(u)\mathrm{d}u \leqslant (1-q)\left[\sqrt{\mu_{2,q} - \mu_q^2}\sigma_{\gamma(U_q)} + \mu_q\mu_{\gamma(U_q)}\right]. \tag{27}$$

Equations (26) together with (27), provides the upper bound in (23). Consider the function defined in (25), this corresponds to a (left-continuous) inverse distribution function if and only if condition (24) is satisfied. Furthermore, any easy calculation shows that a random variable with inverse distribution function $\overline{G}^{-1}$ belongs to the set $\mathcal{M}(F_q, \mu, \sigma^2)$. ◻

Remark 5 The case γ constant on $(q, 1)$ follows from Proposition 2.4.

Remark 6 Given two r.v.s $U_{\underline{q}} \sim \mathcal{U}(0, q)$ and $U_{\overline{q}} \sim \mathcal{U}[q, 1]$, the bound in (23) can be reinterpreted as

$$\overline{\varrho_q} = q\mathbb{E}[F^{-1}(U_{\underline{q}})\gamma(U_{\underline{q}})] + (1-q)\left[\mu_q\mu_{\gamma(U_{\overline{q}})} + \sigma_{\gamma(U_{\overline{q}})}\sqrt{\mu_{2,q} - \mu_q^2}\right],$$

where $\mu_q = \mathbb{E}[G_X^{-1}(U_q)]$ and $\sqrt{\mu_{2,q} - \mu_q^2} = \sigma_{G_X^{-1}(U_q)}$ for any $X \in \mathcal{M}(F_q, \mu, \sigma^2)$.

In Theorem 2.8 we obtain an upper bound for the values that a given spectral risk measure can assume when the mean, variance and left-tail of the loss distribution are known. Moreover, we provide a sufficient condition for the bound to be best-possible and, in this case, we also derive the quantile function attaining the bound. It is of interest to compare such results with those obtained in Li (2018), where a best-possible upper bound for a spectral risk measure is provided when only the mean and the variance are known, that is, when the information set considered is $\mathcal{M}(\mu, \sigma^2) := \{X \in L^2 \mid \mathbb{E}[X] = \mu, \mathrm{var}(X) = \sigma^2\}$.

Theorem 2.9 (Li 2018) *Let ϱ be a spectral risk measure, with corresponding weight function γ. If γ is not constant on* $(0, 1)$, *then*

$$\sup_{X\in\mathcal{M}(\mu,\sigma^2)} \varrho_\gamma(X) = \mu + \sigma\sqrt{\int_0^1 (\gamma(u) - 1)^2\mathrm{d}u}.$$

The bound is sharp and attained by a unique distribution having the following quantile function

$$h(u) = \mu + \sigma \cdot \frac{\gamma(u) - 1}{\sqrt{\int_0^1 (\gamma(u) - 1)^2\mathrm{d}u}}, \qquad u \in (0, 1).$$

Remark 7 Intuitively, when the level of knowledge of the left-tail, described by the parameter q goes to zero one would expect that the upper bound is determined solely by the moments' constraint. The upper bound $\overline{\varrho_q}$ defined in (23) from Theorem 2.8 is consistent with this observation in that if γ is not constant on $(q, 1)$:

$$\lim_{q\to 0^+} \overline{\varrho_q} = \mu + \sigma\sqrt{\int_0^1 (\gamma(u) - 1)^2 \mathrm{d}u}. \tag{28}$$

To see this point, observe that

$$\lim_{q\to 0^+} \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u = 0,$$

$$\lim_{q\to 0^+} (1-q)\mu_{\gamma(U_q)}\mu_q = \mu, \ \lim_{q\to 0^+} \sqrt{\mu_{2,q} - \mu_q^2} = \sigma,$$

$$\lim_{q\to 0^+} \sigma_{\gamma(U_q)} = \lim_{q\to 0^+} \sqrt{\frac{\int_q^1 \gamma(u)^2\mathrm{d}u}{1-q} - \left(\frac{\int_q^1 \gamma(u)\mathrm{d}u}{1-q}\right)^2} = \sqrt{\int_0^1 (\gamma(u) - 1)^2\mathrm{d}u},$$

thus we obtain the limit in (28).

Remark 8 *(Model uncertainty spread and tail standard deviations)* The model uncertainty spread is commonly interpreted in the literature as the difference between the worst- and best-scenario compatible with the available information, see e.g. Barrieu and Scandolo (2015) and Bernard et al. (2024). Our results can be used to study also such a quantity, in that

$$\overline{\varrho}_{\{F_q,\mu,\sigma\}} - \underline{\varrho}_{\{F_q,\mu,\sigma\}} \leqslant \overline{\varrho_q} - \underline{\varrho}_{\{F_q,\mu\}} = (1-q)\sigma_{\gamma(U_q)}\sqrt{\mu_{2,q} - \mu_q^2}.$$

Given U_q uniform on $[q, 1]$ and $X \in \mathcal{M}(F_q, \mu, \sigma^2)$, the model uncertainty spread for a risk measure with a weight function γ is bounded by a quantity that is proportional to the product of $\sigma_{\gamma(U_q)}$ and $\sqrt{\mu_{2,q} - \mu_q^2} = \sigma_{G_X^{-1}(U_q)}$, i.e., the q-tail standard deviations of γ and G_X^{-1}, respectively, as pointed out in Remark 2.

To conclude this subsection, under some technical conditions, we show that the upper bound that we provide in Theorem 2.8, despite not being necessarily sharp, is well-behaved in the sense that it can reflect the level of information available on the left-tail distribution, which in the present framework is described by the probability level $q \in (0, 1)$ up to which we trust the quantile function of the reference model F.

Proposition 2.10 *Let μ and σ^2 be equal to the mean and the variance of the reference model F. Given $0 < a < b < 1$, assume F^{-1} and γ are continuous on (a, b), and not constant on $(b, 1)$. Then, the mapping*

$$q \mapsto \overline{\varrho_q}$$

is non-increasing for every $q \in (a, b)$.

The proof is given in Appendix A.3. Assuming F^{-1} strictly increasing and continuous on (0, 1), it follows from Proposition 2.10 that the TVaR_α bounds that we provide in Theorem 2.8 is non-increasing w.r.t. $q \in (0, \alpha)$. The same holds for the Dual power distortion but for any $q \in (0, 1)$.

2.4 Risk bounds under the knowledge of the left-tail and moments' uncertainty set

So far, we have assumed that only point constraints are available for the loss distribution mean and variance. It can be of practical interest to consider those situations in which a risk manager faces uncertainty also on the first two moments of the loss distribution. Accounting for such circumstances requires extending the definition of the set $\mathcal{M}$ to the case in which moments' uncertainty set (MUS) are available for the loss distribution expected value and variance:

$$\mathcal{M}(F_q, [\underline{m}, \overline{m}], [\underline{s}, \overline{s}]) := \left\{ X \in L^2 \cap \mathcal{M}(F_q) \,\middle|\, \mathbb{E}[X] \in [\underline{m}, \overline{m}],\ \ \text{var}(X) \in [\underline{s}, \overline{s}] \right\}.$$

A robust risk assessment in such a setting requires to study the following bounds:

$$\begin{aligned} \underline{\varrho}_{MUS} &:= \inf_{X \in \mathcal{M}(F_q, [\underline{m}, \overline{m}], [\underline{s}, \overline{s}])} \varrho_\gamma(X), \\ \overline{\varrho}_{MUS} &:= \sup_{X \in \mathcal{M}(F_q, [\underline{m}, \overline{m}], [\underline{s}, \overline{s}])} \varrho_\gamma(X). \end{aligned} \tag{29}$$

Given the quantile function F^{-1} and $q \in (0, 1)$, we assume that the interval value for the mean $[\underline{m}, \overline{m}]$ satisfies

$$\underline{m} > \underline{\mu}(F_q), \tag{30}$$

in which the function $\underline{\mu}(F_q)$ is defined in (3) and describes the minimal value attainable by any distribution in $\overline{\mathcal{M}}(F_q)$, as showed in Proposition 7. Therefore, given the quantile function F^{-1} and $q \in (0, 1)$, the inequality in (30) is a sufficient condition such that for every $\mu \in [\underline{m}, \overline{m}]$, the set $\mathcal{M}(F_q, \mu)$ contains infinitely many elements.

Second, in the remaining part of this section we assume that the interval constraint on the variance is consistent with the left-tail and mean constraints, i.e., we consider $[\underline{s}, \overline{s}]$ such that

$$\underline{s} > \max_{\mu \in [\underline{m}, \overline{m}]} \underline{\sigma}^2(F_q, \mu). \tag{31}$$

The function $\underline{\sigma}^2(F_q, \mu)$, describing the lower bound on the variance given the mean and the left-tail, is defined in (4). An easy computation shows that $\underline{\sigma}^2(F_q, \mu)$ is increasing in μ and thus the condition in (31) can be equivalently expressed with $\underline{s} > \underline{\sigma}^2(F_q, \overline{m})$.

The inequalities in (30) and (31) ensure that for every $(\mu, \sigma^2) \in [\underline{m}, \overline{m}] \times [\underline{s}, \overline{s}]$ there exists at least one random variable with mean μ, variance σ^2 and left-tail F^{-1} up to q. The following two functions $g : [\underline{m}, \overline{m}] \times [\underline{s}, \overline{s}] \to \mathbb{R}$ and $t_{\mu,\sigma^2} : (0, 1) \to \mathbb{R}$

are auxiliary for Proposition 2.12, which is our main result regarding the computation of (29):

$$g(\mu,\sigma^2) := \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + (1-q)\mu_{\gamma(U_q)}\frac{\mu - \int_0^q F^{-1}(u)\mathrm{d}u}{1-q} + (1-q)\sigma_{\gamma(U_q)}\sqrt{\frac{\sigma^2+\mu^2-\int_0^q F^{-1}(u)^2\mathrm{d}u}{1-q} - \left(\frac{\mu - \int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)^2}, \tag{32}$$

and

$$t_{\mu,\sigma^2}(u) := \frac{\gamma(u^-)-\mu_{\gamma_q}}{\sigma_{\gamma_q}}\sqrt{\frac{\sigma^2+\mu^2-\int_0^q F^{-1}(u)^2\mathrm{d}u}{1-q} - \left(\frac{\mu - \int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)^2} + \frac{\mu - \int_0^q F^{-1}(u)\mathrm{d}u}{1-q},$$

in which $\gamma(u^-) := \lim_{c\to u^-}\gamma(c)$. Note that the function g defined in (32), corresponds to the upper bound derived in Theorem 2.8 written explicitly in terms of μ and σ^2 exploiting (5) and (6).

Lemma 2.11 *The function g defined in (32) is concave in its first argument and continuous and increasing in the second one. Furthermore, one has:*

$$\sup_{\mu\in[\underline{m},\overline{m}],\sigma^2\in[\underline{s},\overline{s}]} g(\mu,\sigma^2) = g\left(\mu^*,\overline{s}\right), \tag{33}$$

where μ^* is the unique solution to the concave problem

$$\underset{\mu\in[\underline{m},\overline{m}]}{\arg\max}\, g(\mu,\overline{s}).$$

Proof Given the consistency assumption stated in (30) and (31), we know that the function g is well defined on the compact set $\left[\underline{m},\overline{m}\right]\times\left[\underline{s},\overline{s}\right]$. Since γ is non-negative and non-constant on $(q,1)$, we have

$$(1-q)\mu_{\gamma(U_q)} > 0 \text{ and } (1-q)\sigma_{\gamma(U_q)} > 0. \tag{34}$$

Thus, for any $\mu\in\left[\underline{m},\overline{m}\right]$, the mapping $\sigma^2\mapsto g(\mu,\sigma^2)$ is continuous and increasing. We observe that the mapping $\mu\mapsto g(\mu,\overline{s})$ is continuous on $\left[\underline{m},\overline{m}\right]$. Thanks to the Weierstrass extreme value theorem, the function $g(\mu,\overline{s})$ attains at least a maximum over the interval $\left[\underline{m},\overline{m}\right]$; we study its first and second order derivative with respect to μ. The first order partial derivative of g with respect to μ writes as

$$\frac{\partial g(\mu,\overline{s})}{\partial \mu} = \mu_{\gamma(U_q)} + \frac{\sigma_{\gamma(U_q)}\left(\mu - \frac{\mu - \int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)}{\sqrt{\frac{\overline{s}+\mu^2-\int_0^q F^{-1}(u)^2\mathrm{d}u}{1-q} - \left(\frac{\mu-\int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)^2}}.$$

Note that the term $\mu - \frac{\mu-\int_0^q F^{-1}(u)\mathrm{d}u}{1-q}$ is negative, and therefore it is not possible to establish a priori the sign of the first-derivative and hence the monotonicity of the function g in its first argument.

The second-order derivative writes as

$$\frac{\partial^2 g(\mu,\overline{s})}{\partial^2 \mu} = \frac{\sigma_{\gamma(U_q)}\left(1 - \frac{1}{1-q}\right)}{\sqrt{\frac{\overline{s}+\mu^2-\int_0^q F^{-1}(u)^2\mathrm{d}u}{1-q} - \left(\frac{\mu-\int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)^2}} - \frac{\left(\mu - \frac{\mu-\int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)^2 \frac{1}{1-q}}{\left(\frac{\overline{s}+\mu^2-\int_0^q F^{-1}(u)^2\mathrm{d}u}{1-q} - \left(\frac{\mu-\int_0^q F^{-1}(u)\mathrm{d}u}{1-q}\right)^2\right)^{\frac{3}{2}}}$$

Using the inequalities in (34) and the fact that $1-\frac{1}{1-q}$ is strictly negative for $q \in (0, 1)$, we deduce that $\frac{\partial^2 g(\mu,\overline{s})}{\partial^2 \mu}$ is strictly negative for $\mu \in \left[\underline{m}, \overline{m}\right]$. Thus, g is strictly concave in μ on $\left[\underline{m}, \overline{m}\right]$ and therefore has only one maximum over this interval. It follows that

$$\sup_{\mu\in[\underline{m},\overline{m}],\sigma^2\in[\underline{s},\overline{s}]} g(\mu,\sigma^2) = \max_{\mu\in[\underline{m},\overline{m}],\sigma^2\in[\underline{s},\overline{s}]} g(\mu,\sigma^2) = g\left(\mu^*,\overline{s}\right), \tag{35}$$

where μ^* is the unique solution to the concave problem

$$\arg\max_{\mu\in[\underline{m},\overline{m}]} g(\mu,\overline{s}).$$

□

Proposition 2.12 *If γ is not constant on $(q, 1)$, then*

$$\overline{\varrho}_{MUS} \leqslant g(\mu^*,\overline{s}), \tag{36}$$

in which μ^* is the unique solution of the following concave problem

$$\arg\max_{\mu\in[\underline{m},\overline{m}]} g(\mu,\overline{s}).$$

Moreover, if the condition

$$t_{\mu^*,\overline{s}}(q) \geqslant F^{-1}(q)$$

is satisfied, the upper bound in (36) *is best-possible and attained by the quantile function* $\overline{G}^{-1}$ *defined as follows:*

$$\overline{G}^{-1}(u) = \begin{cases} F^{-1}(u) \text{ for } u \in (0, q], \\ t_{\mu^*, \overline{s}}(u) \text{ for } u \in (q, 1). \end{cases}$$

Proof Observe that for any $\tilde{X} \in \mathcal{M}(F_q, [\underline{m}, \overline{m}], [\underline{s}, \overline{s}])$ one must have $\mu_{\tilde{X}} \in [\underline{m}, \overline{m}]$ and $\sigma^2_{\tilde{X}} \in [\underline{s}, \overline{s}]$. By (31), we know that $\mathcal{M}(F_q, \mu_{\tilde{X}}, \sigma^2_{\tilde{X}})$ is not empty and thus we obtain

$$\varrho_\gamma(\tilde{X}) \leqslant \sup_{X \in \mathcal{M}(F_q, \mu_{\tilde{X}}, \sigma^2_{\tilde{X}})} \varrho_\gamma(X) \leqslant \sup_{\mu \in [\underline{m}, \overline{m}], \sigma^2 \in [\underline{s}, \overline{s}]} \left(\sup_{X \in \mathcal{M}(F_q, \mu, \sigma^2)} \varrho_\gamma(X) \right).$$

Hence,

$$\overline{\varrho}_{MUS} \leqslant \sup_{\mu \in [\underline{m}, \overline{m}], \sigma^2 \in [\underline{s}, \overline{s}]} \left(\sup_{X \in \mathcal{M}(F_q, \mu, \sigma^2)} \varrho_\gamma(X) \right). \tag{37}$$

The upper bound obtained in Theorem 2.8 and Lemma 2.11 allow us to write

$$\sup_{\mu \in [\underline{m}, \overline{m}], \sigma^2 \in [\underline{s}, \overline{s}]} \left(\sup_{X \in \mathcal{M}(F_q, \mu, \sigma^2)} \varrho_\gamma(X) \right) \leqslant \sup_{\mu \in [\underline{m}, \overline{m}], \sigma^2 \in [\underline{s}, \overline{s}]} g(\mu, \sigma^2) = g\left(\mu^*, \overline{s}\right), \tag{38}$$

where μ^* is the unique solution of the following concave problem

$$\arg\max_{\mu \in [\underline{m}, \overline{m}]} g(\mu, \overline{s}).$$

The rest of the proof follows analogously to the last part of the proof of Theorem 2.8. □

Turning our attention to the best-case scenario attainable when one faces uncertainty on the right-tail and on the first two moments, we get the following result.

Proposition 2.13 *The lower bound under right-tail and moments' uncertainty satisfies:*

$$\underline{\varrho}_{MUS} \geqslant \int_0^q F^{-1}(u)\gamma(u)du + (1-q)\mu_{\gamma(U_q)} \frac{\underline{m} - \int_0^q F^{-1}(u)du}{1-q}.$$

The proof of Proposition 2.13 follows from Remark 4 and an argument similar to the one adopted in the first part of the proof for Proposition 2.12.

2.5 Extensions beyond spectral risk measures

So far we have focused on the class of spectral risk measure, which is indeed a rich family, but does not include all risk functionals that are considered in the financial and actuarial literature. Given $0 < \alpha < \alpha' < 1$, we denote by $\mathrm{VaR}_\alpha(X)$ the left-inverse of the distribution function of a random variable $X \sim G_X$, i.e.,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} \mid G_X(x) \geqslant \alpha\},$$

while $\mathrm{VaR}^+_\alpha(X)$ is defined using the cdf right-inverse, i.e.,

$$\mathrm{VaR}^+_\alpha(X) = \sup\{x \in \mathbb{R} \mid G_X(x) \leqslant \alpha\}.$$

Finally, the Range Value-at-Risk (RVaR), introduced in Cont et al. (2010), is formally defined as

$$\mathrm{RVaR}_{\alpha,\alpha'}(X) = \frac{1}{\alpha' - \alpha} \int_\alpha^{\alpha'} \mathrm{VaR}_u(X)\mathrm{d}u, \quad 0 < \alpha < \alpha' < 1.$$

We now illustrate that Theorem 2.8 can be used to derive a closed-form upper bound for the VaR, VaR^+ and RVaR, three prominent risk measures that are not spectral in the sense of Acerbi (2002).

Corollary 2.14 *Given $0 \leqslant q < \alpha < \alpha' < 1$, let ϱ be either VaR_α, VaR^+_α, $RVaR_{\alpha,\alpha'}$ or $TVaR_\alpha$. Then,*

$$\sup_{X \in \mathcal{M}(F_q,\mu,\sigma^2)} \varrho(X) \leqslant \sqrt{\frac{\alpha - q}{(1-\alpha)(1-q)^2}} \sqrt{\mu_{2,q} - \mu_q^2} + \mu_q. \tag{39}$$

If $\frac{\sqrt{1-\alpha}}{q-\alpha}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \geqslant F^{-1}(q)$, this upper bound is attained and thus best-possible for VaR^+_α, $RVaR_{\alpha,\alpha'}$ and $TVaR_\alpha$.

The proof of Corollary 2.14 can be found in Appendix A.4. Some of our results can be easily extended to the class of risk measures that are consistent with respect to a given stochastic order. Specifically, we say that a risk measure ϱ is consistent with first-order stochastic dominance (resp. convex order) if $X \leqslant_{st}$ (resp. $\leqslant_{cx}$)Y implies $\varrho(X) \leqslant \varrho(Y)$. Such assumptions are quite mild in that essentially all law-invariant monetary risk measures are FSD consistent, and the class of convex order consistent risk measures include coherent (and thus also spectral) risk measures. In Proposition 2.1 and Proposition 2.5 we derived best-possible lower bounds on spectral risk measures on the sets $\mathcal{M}(F_q)$ and $\mathcal{M}(F_q, \mu)$. These results can be generalized assuming solely the consistency w.r.t. first-order stochastic dominance for $\mathcal{M}(F_q)$, and the consistency w.r.t. convex order for $\mathcal{M}(F_q, \mu)$. Recall that in the proof of Proposition 1.1 we showed that the random variable X_{st} defined in (9) is the minimal element in first-order stochastic dominance for the set $\mathcal{M}(F_q)$, while the random variable X_{cx} defined in (10) is the minimal element in convex order on the set $\mathcal{M}(F_q, \mu)$.

Proposition 2.15 *Let ϱ be consistent with first-order stochastic dominance. Then,*

$$\inf_{X \in \mathcal{M}(F_q)} \varrho(X) = \varrho(X_{st}).$$

Let ϱ be consistent with convex order. Then,

$$\inf_{X \in \mathcal{M}(F_q,\mu)} \varrho(X) = \varrho(X_{cx}).$$

Moreover, both bounds are attained.

Proof This result follows from the observations made in the proof of Proposition 1.1, in which we point out that the sets $\mathcal{M}(F_q)$ and $\mathcal{M}(F_q, \mu)$ admit minimal elements in first-order stochastic dominance and convex order, respectively, identified by the random variables X_{st} and X_{cx}. Thus, the lower bounds of Proposition 2.15 are a direct consequences of the consistency assumption with respect to these two stochastic dominance rule. The bounds are attained in that $X_{st} \in \mathcal{M}(F_q)$ and $X_{cx} \in \mathcal{M}(F_q, \mu)$. □

If ϱ is a spectral risk measure, one can immediately check that the bounds in Proposition 2.15 coincides with those in Proposition 2.1 and Proposition 2.5 using the definition of X_{st} and X_{cx}.

3 Numerical analysis

The goal of this section is to numerically illustrate the impact of additional left-tail constraint on the worst-case scenario of a distortion risk measure with respect to the case studied in Li (2018), in which only loss distribution's mean and variance are assumed to be known. In doing so, we also show that the upper bound in Theorem 2.8 is in fact best-possible in cases of practical interest.

Special attention will be given to the worst-case values and quantile functions' behaviour with respect to $q \in (0, 1)$. In our framework, the parameter q can be seen as the level of information available on the underlying distribution, or equivalently, q can be interpreted as a parameter that measures how much we trust the reference model identified by F^{-1}, taking into account that we focus the distributional uncertainty on the right-tail of such model. Under some technical assumptions, note that we have already established the bounds we provide in Theorem 2.8 and Remark 4 are able to reflect the amount of information available described by $q \in (0, 1)$. Here, we find a numerical confirmation of such results, and we also find empirical evidence that such results hold under weaker assumptions, e.g. for the TVaR_α, a distortion risk measure that does not have a continuous weight function on $(0, 1)$. Hereafter, we consider the possible reference distribution functions for the left-tail identified by the Lognormal, Gamma and Inverse Gaussian (IG) model, with parameters fixed according to Table 1. The numerical examples are implemented using three prominent distortion risk measures: the TVaR, the Wang transform and the Dual power risk measures, which have been introduced in Sect. 1.

Table 1 Parameters of the Lognormal, Gamma and Inverse Gaussian distribution adopted in the numerical analysis. To make the notation clear, here we denote the Lognormal and Inverse Gaussian location and scale parameters by (μ_L, σ_L) and (μ_{IG}, λ), respectively

	Location parameter	Scale parameter
Lognormal	μ_L=2.5	σ_L=0.6
Gamma	$k = 7.3$	$\theta = 1.5$
Inverse Gaussian	$\mu_{IG} = 30$	$\lambda = 675$

3.1 Worst-case values

First, we focus on the upper bound derived in Theorem 2.8 on worst-case values attainable by a distortion risk measure. Three cases are considered in Fig. 1: Wang transform with $\beta = 0.95$ (Panel 1a), TVaR with $\alpha = 0.95$ (Panel 1b), and Dual power risk measure with $k = 20$ (Panel 1c). The left-tail and moments constraints are given by the Gamma, Lognormal and Inverse Gaussian distributions, respectively.

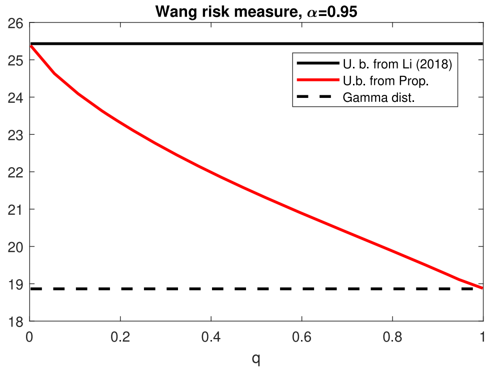
In Fig. 1 we can observe a negative relationship between the level of information available, described by q, and the worst-case scenario for the risk measures considered. This is consistent with the observation that higher levels of $q \in (0, 1)$ correspond to lower levels of uncertainty regarding the underlying loss distribution as established in Proposition 2.10, and this naturally translates into a reduction of the worst-case scenario. The worst-case scenario improvements due to the left-tail knowledge with respect to the upper bound one can obtain by exploiting solely the information on the first two moments (Theorem 2 in Li 2018) are visible in all cases considered in Fig. 1. Specifically, the impact of the left-tail constraint is already significant for relatively low levels of $q \in (0, 1)$, such as $q = 0.5$, corresponding to the cases in which one is able to specify the left-tail up to its median value.

Furthermore, we checked numerically that the sufficient condition for the bound in Theorem 2.8 to be best-possible is satisfied for every $q \in (0, 0.88] \cup (0.95, 1)$, $q \in (0, 1)$ and $q \in (0, 0.84]$, for the TVaR (Panel 1a), for the Wang transform (Panel 1b) and the for Dual power risk measure (Panel 1c), respectively, and thus for the vast majority of the probability levels considered.

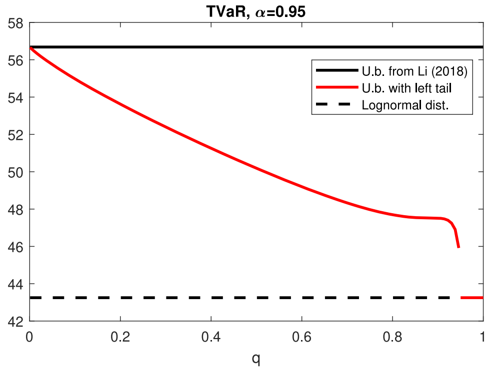
3.2 Worst-case quantile functions

We can now turn our attention to the shape of the worst-case quantile function corresponding to different values of $q \in (0, 1)$, as derived in Theorem 2.8. Panel 2a, Panel 2b and Panel 2c in Fig. 2 show the worst-case quantile functions attainable for the Wang transform, the TVaR and Dual power distortion under different scenarios for the probability value $q \in (0, 1)$ up to which we trust the left-tail of the reference model.

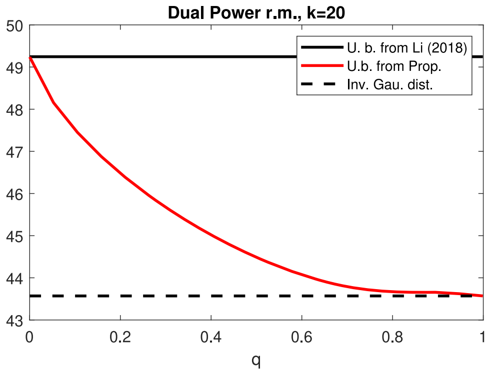
From these graphs, we can see that the value of q can have a significant impact also on the shape of the worst-case quantile function. In line with what observed in



(a) Wang risk measure, Gamma distribution

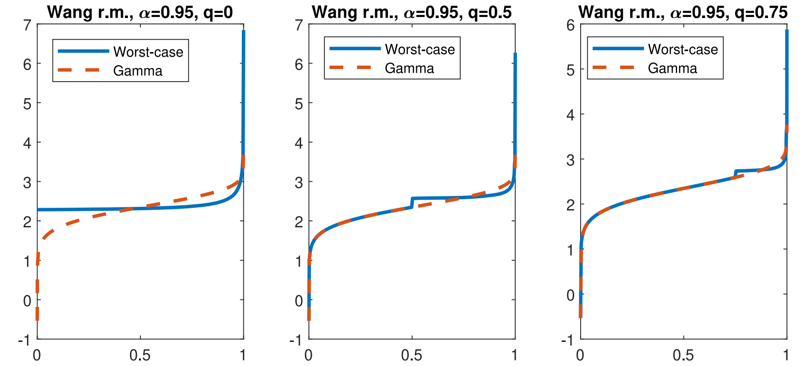


(b) TVaR, Lognormal distribution

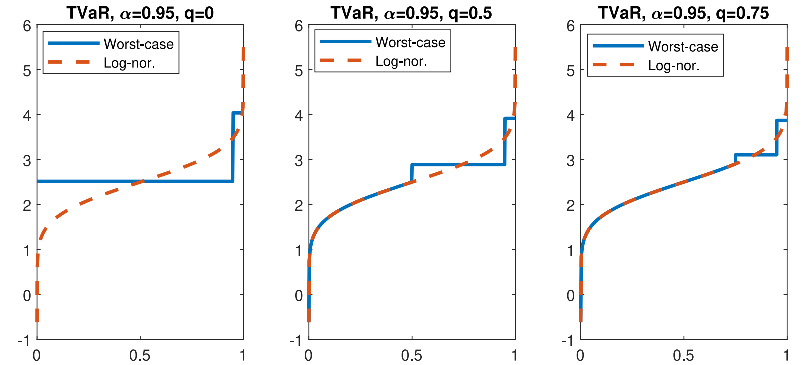


(c) Dual power risk measure, Inverse Gaussian distribution

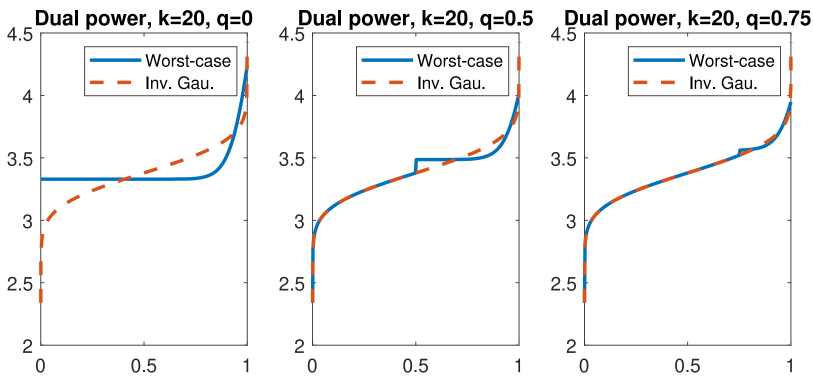
Fig. 1 Upper bounds for the Wang transform, TVaR and Dual power risk measure. In each figure, the horizontal black and dashed lines describe the risk measure upper bound derived in Li (2018) and its value computed using the reference distribution, respectively. The red line illustrates the value of the upper bound we obtained in Theorem 2.8. The reference distribution (left-tail) and the first two moments constraints are given by the Gamma (Panel 1a), Lognormal (Panel 1b), and Inverse Gaussian (Panel 1c) distributions with parameters fixed according to Table 1



(a) Wang risk measure, Gamma distribution



(b) TVaR, Lognormal distribution



(c) Dual power risk measure, Inverse Gaussian distribution

Fig. 2 Worst-case quantile functions comparison, on the logarithmic scale for Wang transform, TVaR and Dual power risk measure. The blue line in each graph shows the worst-case quantile function for the risk measure as obtained in Theorem 2.8, corresponding to $q = 0, 0.5, 0.75$. The dashed red line describes the quantile function of the reference distribution of which we trust the left-tail and the first two moments. The reference distributions are given by the Gamma (Panel 2a), the Lognormal (Panel 2b), and the Inverse Gaussian (Panel 2c) models with parameters fixed according to Table 1

Fig. 1, setting $q = 0.5$ is already enough to observe a significant difference between the worst case quantile function from Theorem 2.8 and the one obtained in Theorem 2 in Li (2018) using solely the loss distribution mean and variance. Moreover, it is interesting to point out that for higher values of $q \in (0, 1)$, the worst-case quantile function becomes closer to the reference distribution not only on the interval $(0, q)$ but also on the interval $[q, 1)$. Thus, Fig. 2 provides further numerical evidence that the left-tail knowledge is a source of partial information on the loss distribution that can be an effective tool to obtain a more realistic worst-case assessment.

4 Final remarks

In conclusion, we study upper and lower bounds for spectral risk measures under different levels of information compatible with the uncertainty on the right-tail of the loss distribution. For every set of trusted information considered, we investigate upper and lower bounds, their sharpness and properties. Through this paper, we have delved into the theoretical and practical implications of these bounds, highlighting their significance in quantifying model risk. In particular, by understanding the range of possible outcomes through these bounds, practitioners can gain valuable insights into the model risk toward which they are exposed in the context of tail uncertainty. The relevance of our results is enhanced by the numerical analysis in which it is shown that a constraint on the left-tail of the loss distribution can lead to a significant reduction for the upper bound of TVaR, Wang risk measure and Dual power risk measure.

A Appendix

A.1 Proposition A.1

Proposition A.1 *Let F and G be two distribution functions. Given $q \in (0, 1)$, the following two statements are equivalent*

1. $G^{-1}(u) = F^{-1}(u)$ *for all* $u \in (0, q]$.
2. $G(x) = F(x)$ *for all* $x \in (-\infty, F^{-1}(q))$ *and* $G(F^{-1}(q)) \geqslant q$.

Proof We recall that for any distribution function, given $u \in (0, 1)$ and $x \in \mathbb{R}$, it holds that $F(F^{-1}(u)) \geqslant u$, $F^{-1}(F(x)) \leqslant x$ and $F(x) \geqslant u \iff F^{-1}(u) \leqslant x$. See Part 1 in Denuit et al. (2005) for more details.

We start by showing 1) $\Longrightarrow$ 2). First, under assumption 1), we have $G^{-1}(q) = F^{-1}(q)$ and thus $G(F^{-1}(q)) = G(G^{-1}(q)) \geqslant q$.

Let us now fix $\overline{x} \in (-\infty, F^{-1}(q))$. Note for these values of $\overline{x}$ one has $F(\overline{x}) < q$ and $G(\overline{x}) < q$. Using the definition of left-inverse, one obtains that

$$\begin{aligned} F(\overline{x}) > 0 &\Longrightarrow \overline{x} \in \{x \in \mathbb{R} \,|\, F(x) \geqslant F(\overline{x})\} \Longrightarrow \overline{x} \geqslant F^{-1}(F(\overline{x})) = G^{-1}(F(\overline{x})) \\ &\Longrightarrow \overline{x} \in \{x \in \mathbb{R} \,|\, G(x) \geqslant F(\overline{x})\} \Longrightarrow G(\overline{x}) > 0. \end{aligned}$$

The fact that $G(\overline{x}) > 0 \implies F(\overline{x}) > 0$ follows from a similar argument. Therefore, also the equivalence $F(\overline{x}) = 0 \iff G(\overline{x}) = 0$ holds. Hence, we have proved that for $\overline{x} \in (-\infty, F^{-1}(q))$, $F(\overline{x}) = 0$ implies $G(\overline{x}) = F(\overline{x})$.

We can move to the case $F(\overline{x}) > 0$, for which already showed that $G(\overline{x}) > 0$ must hold. Thus, we can write

$$G^{-1}(F(\overline{x})) = F^{-1}(F(\overline{x})) \leqslant \overline{x} \implies G(\overline{x}) \geqslant F(\overline{x}). \tag{40}$$
$$F^{-1}(G(\overline{x})) = G^{-1}(G(\overline{x})) \leqslant \overline{x} \implies F(\overline{x}) \geqslant G(\overline{x}). \tag{41}$$

Putting together the last inequalities in (40) and (41), we end up with $G(\overline{x}) = F(\overline{x})$, which completes the first part of the proof.

We can now prove that 2) $\implies$ 1). First note that $G(F^{-1}(q)) \geqslant q$ implies $G^{-1}(q) \leqslant F^{-1}(q)$. In order to show that $G^{-1}(q) = F^{-1}(q)$ holds, we proceed by contradiction. Assume that $G^{-1}(q) < F^{-1}(q)$, and thus that $G^{-1}(q)$ belongs to the interval $(-\infty, F^{-1}(q))$ over which F and G coincide. On one side, this implies $F(G^{-1}(q)) < q$. On the other side, we have $F(G^{-1}(q)) = G(G^{-1}(q)) \geqslant q$. Hence we conclude that $G^{-1}(q) = F^{-1}(q)$.

Fix now $\overline{u} \in (0, q)$. First, we consider the case $F^{-1}(\overline{u}) = F^{-1}(q)$. Observe that for x in $\mathbb{R}$ such that $F(x) \geqslant \overline{u}$, one has $x \geqslant F^{-1}(\overline{u}) = F^{-1}(q)$. Hence, we obtain that

$$\begin{aligned} F^{-1}(\overline{u}) = F^{-1}(q) &\implies F(x) < \overline{u}, \text{ for all } x \in (-\infty, F^{-1}(q)) \\ &\implies G(x) < \overline{u}, \text{ for all } x \in (-\infty, G^{-1}(q)) \\ &\implies \{x \in \mathbb{R} \,|\, G(x) \geqslant \overline{u}\} = [G^{-1}(q), +\infty) \\ &\implies G^{-1}(\overline{u}) = G^{-1}(q). \end{aligned} \tag{42}$$

Hence, the equality $F^{-1}(\overline{u}) = G^{-1}(\overline{u})$ holds if $F^{-1}(\overline{u}) = F^{-1}(q)$.

Second, we move to the case $F^{-1}(\overline{u}) < F^{-1}(q)$. Observe that one can use a similar argument as in (42) to show that $G^{-1}(\overline{u}) = G^{-1}(\overline{q}) \implies F^{-1}(\overline{u}) = F^{-1}(q)$. Thus, we deduce that the equivalence $F^{-1}(\overline{u}) < F^{-1}(q) \iff G^{-1}(\overline{u}) < G^{-1}(q)$ holds true, and this allows us to obtain the following results

$$G(F^{-1}(\overline{u})) = F(F^{-1}(\overline{u})) \geqslant \overline{u} \implies F^{-1}(\overline{u}) \geqslant G^{-1}(\overline{u}) \tag{43}$$
$$F(G^{-1}(\overline{u})) = G(G^{-1}(\overline{u})) \geqslant \overline{u} \implies G^{-1}(\overline{u}) \geqslant F^{-1}(\overline{u}). \tag{44}$$

The inequalities in (43) and (44) together imply $G^{-1}(\overline{u}) = F^{-1}(\overline{u})$, which completes the proof. □

A.2 Proof of Proposition 1.2

For part a) it is sufficient to show that $\mathcal{M}(F_q, \mu, \sigma^2)$ contains infinitely many elements. *First*, we consider the case $F^{-1}(q) = 0$ and fix $\mu > \underline{\mu}(F_q)$, $\sigma > \underline{\sigma}(F_q^{-1}, \mu)$, and $\mu_2 = \mu^2 + \sigma^2$. Let μ_q and $\mu_{2,q}$ be defined as in (5) and (6), so that $\mu_{2,q} > \mu_q^2$ and $\mu_q > 0$.

Fix a value $\beta \in \left(q, 1 - \frac{\mu_q^2(1-q)}{\mu_{2,q}}\right)$ and define $X \sim G_X$ as a continuous non-negative random variable with $\mathbb{E}[X] = \frac{\mu_q(1-q)}{1-\beta}$ and $\mathbb{E}[X^2] = \frac{\mu_{2,q}(1-q)}{1-\beta}$.

For the values of β that we are considering one can always construct such a distribution using, e.g., a Gamma or a Log-normal model. Consider a random variable X_β with quantile function:

$$G_\beta^{-1}(u) = \begin{cases} F^{-1}(u), \text{ for } u \in (0, q], \\ 0, \text{ for } u \in (q, \beta], \\ G_X^{-1}\left(\frac{u-\beta}{1-\beta}\right) \text{ for } u \in (\beta, 1). \end{cases}$$

G_β^{-1} is a proper quantile function in that $G_X^{-1}(\frac{u-\beta}{1-\beta})$ is a non-decreasing and left-continuous function of u. Furthermore, $G_\beta^{-1}(u) = G_X^{-1}(\frac{u-\beta}{1-\beta}) > 0 = F^{-1}(q)$, for any $u \in (\beta, 1)$. It is easily calculated:

$$\int_0^1 G_\beta^{-1}(u)\mathrm{d}u = \int_0^q F^{-1}(u)\mathrm{d}u + \int_q^\beta 0\mathrm{d}u + \int_\beta^1 G_X^{-1}\left(\frac{u-\beta}{1-\beta}\right)\mathrm{d}u = \mu;$$
$$\int_0^1 G_\beta^{-1}(u)^2\mathrm{d}u = \mu^2 + \sigma^2,$$

that gives $X_\beta \in \mathcal{M}(F_q, \mu, \sigma^2)$. It is possible to define a quantile function G_β^{-1} for each $\beta \in \left(q, 1 - \frac{\mu_q^2(1-q)}{\mu_{2,q}}\right)$ therefore the set $\mathcal{M}(F_q, \mu, \sigma^2)$ contains infinitely many elements. This completes the first part of the proof. For the general case $F^{-1}(q) \neq 0$ it is sufficient to note that for every $\mu > \underline{\mu}(F_q)$ and $\sigma > \underline{\sigma}(F_q^{-1}, \mu)$ there is one-to-one relation between the sets $\mathcal{M}(F_q, \mu, \sigma^2)$ and $\mathcal{M}(\tilde{F}_q, \tilde{\mu}, \sigma)$, where $\tilde{F}^{-1}(u) := F^{-1}(u) - F^{-1}(q)$ for each $u \in (0, q]$ and $\tilde{\mu} := \mu - F^{-1}(q)$. We already proved in the first step that the latter set contains infinitely many elements and this concludes the proof of part *a*). For part *b*), for any $X \in \mathcal{M}(F_{q'})$, one has $G_X^{-1}(u) \equiv F^{-1}(u)$, for all $u \in (0, q']$ which implies that the equality holds for any $u \in (0, q]$ and therefore $X \in \mathcal{M}(F_q)$. A similar reasoning applies to the other sets. It may be that $\mathcal{M}(F_{q'}, \mu)$ and $\mathcal{M}(F_{q'}, \mu, \sigma^2)$ are empty, if $\mu < \underline{\mu}(F_{q'})$ and $\sigma < \underline{\sigma}(F_{q'}, \mu)$. A sufficient condition for these sets to be not empty is to ask that μ and σ^2 coincide with the mean and variance of the reference distribution F.

A.3 Proof of Proposition 2.10

Proof The idea of the proof is simply to compute the derivatives of $\overline{\varrho_q}$ w.r.t. $q \in (a, b)$ and show that it is non-negative.

Recall that for any real function f continuous on (a, b) with $0 < a < b < 1$ such that $\int_0^1 f(u)du < +\infty$, it follows from the fundamental theorem of calculus (see, e.g., Theorem 3.3.11 in Trench (2013)) that $\frac{\mathrm{d}}{\mathrm{d}q}\int_a^q f(u)\mathrm{d}uu = f(q)$ and $\frac{\mathrm{d}}{\mathrm{d}q}\int_q^b f(u)\mathrm{d}u = -f(q)$ for any $q \in (a,b)$. Moreover, the Riemann integral additivity yields to $\int_0^q f(u)\mathrm{d}uu = \int_0^a f(u)\mathrm{d}uu + \int_a^q f(u)\mathrm{d}uu$, and thus one has $\frac{\mathrm{d}}{\mathrm{d}q}\int_0^q f(u)\mathrm{d}uu = f(q)$ even if the function f is not continuous on $(0,a]$, provided $a < q$. Similarly, $\frac{\mathrm{d}}{\mathrm{d}q}\int_q^1 f(u)du = -f(q)$ holds for $q < b$.

We start with the lower bound $\underline{\varrho}_{\{F_q,\mu\}}$. Given $q \in (a,b)$, Corollary 4 and $\mu = \int_0^1 F^{-1}(u)\mathrm{d}u$ yield to

$$\begin{aligned}\underline{\varrho}_{\{F_q,\mu\}} &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \mu_q\int_q^1 \gamma(u)\mathrm{d}u\\ &= \int_0^q F^{-1}(u)\gamma(u)\mathrm{d}u + \frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{1-q}\int_q^1 \gamma(u)\mathrm{d}u.\end{aligned}$$

Thanks to the continuity of F^{-1} and γ on (a,b), for every $q \in (a,b)$ we can write

$$\begin{aligned}\frac{\mathrm{d}\underline{\varrho}_{\{F_q,\mu\}}}{\mathrm{d}q} &= F^{-1}(q)\gamma(q) + \left(\frac{-F^{-1}(q)(1-q) + \int_q^1 F^{-1}(u)\mathrm{d}u}{(1-q)^2}\right)\\ &\quad \int_q^1 \gamma(u)\mathrm{d}u - \gamma(q)\frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{1-q}\\ &= F^{-1}(q)\gamma(q) - \frac{F^{-1}(q)\int_q^1 \gamma(u)\mathrm{d}u}{1-q} + \frac{\int_q^1 \gamma(u)\mathrm{d}u\int_q^1 F^{-1}(u)\mathrm{d}u}{(1-q)^2}\\ &\quad -\gamma(q)\frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{1-q}.\end{aligned}$$

Hence, we obtain

$$\frac{\mathrm{d}\underline{\varrho}_{\{F_q,\mu\}}}{\mathrm{d}q} = \left(\frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{1-q} - F^{-1}(q)\right)\left(\frac{\int_q^1 \gamma(u)\mathrm{d}u}{1-q} - \gamma(q)\right) > 0, \qquad (45)$$

by $\frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{1-q} - F^{-1}(q) > 0$ and $\frac{\int_q^1 \gamma(u)\mathrm{d}u}{1-q} - \gamma(q) > 0$ (which follow from the fact that F^{-1} and γ, non-decreasing on $(0,1)$, and non-constant on $(q,1)$ for $q < b$ as direct consequence of the fact that they are not constant on $(b,1)$).

As for the upper bound given in (23), note that via a simple rearrangement, it can be written as

$$\overline{\varrho_q} = \underline{\varrho}_{\{F_q,\mu\}} + \sqrt{\int_q^1 F^{-1}(u)^2 \mathrm{d}u - \frac{\left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{1-q}}$$
$$\sqrt{\int_q^1 \gamma^2(u)du - \frac{\left(\int_q^1 \gamma(u)du\right)^2}{1-q}}du.$$

We can now compute the derivatives w.r.t. $q \in (a, b)$ of its components:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}q}\int_q^1 F^{-1}(u)^2 \mathrm{d}u - \frac{\left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{1-q} &= -F^{-1}(q)^2 \\
&- \frac{-2\int_q^1 F^{-1}(u)\mathrm{d}u F^{-1}(q)(1-q) + \left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{(1-q)^2} \\
&= -F^{-1}(q)^2 + \frac{2\int_q^1 F^{-1}(u)\mathrm{d}u F^{-1}(q)}{(1-q)} - \frac{\left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{(1-q)^2} \\
&= -\left(F^{-1}(q)^2 - \frac{2\int_q^1 F^{-1}(u)\mathrm{d}u F^{-1}(q)}{(1-q)} + \frac{\left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{(1-q)^2}\right).
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}q}\int_q^1 F^{-1}(u)^2 \mathrm{d}u - \frac{\left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{1-q} = -\left(\frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{(1-q)} - F^{-1}(q)\right)^2. \quad (46)$$

With a similar approach, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}q}\int_q^1 \gamma(u)^2 \mathrm{d}u - \frac{\left(\int_q^1 \gamma(u)\mathrm{d}u\right)^2}{1-q} = -\left(\frac{\int_q^1 \gamma(u)\mathrm{d}u}{(1-q)} - \gamma(q)\right)^2. \quad (47)$$

Let now

$$A = \frac{\left(\frac{\int_q^1 F^{-1}(u)\mathrm{d}u}{(1-q)} - F^{-1}(q)\right)\sqrt{\int_q^1 \gamma^2(u)\mathrm{d}u - \frac{\left(\int_q^1 \gamma(u)du\right)^2}{1-q}\mathrm{d}u}}{\left(\frac{\int_q^1 \gamma(u)\mathrm{d}u}{1-q} - \gamma(q)\right)\sqrt{\int_q^1 F^{-1}(u)^2\mathrm{d}u - \frac{\left(\int_q^1 F^{-1}(u)\mathrm{d}u\right)^2}{1-q}\mathrm{d}u}}.$$

Note that F^{-1} and γ not-decreasing and not constant on $(q, 1)$ for $q < b$ yield to $A > 0$. One can use the results in (45), (46), and (47) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}q}\overline{\varrho}_q = -\frac{(A-1)^2}{2A} \leqslant 0,$$

Hence, the mapping $q \mapsto \overline{\varrho}_q$ is non-increasing for $q \in (a, b)$. □

A.4 Proof of Corollary 2.14

Proof For any $X \in \mathcal{M}(F_q, \mu, \sigma^2)$ one has

$$\mathrm{VaR}_\alpha(X) \leqslant \mathrm{VaR}_\alpha^+(X) \leqslant \mathrm{RVaR}_{\alpha,\alpha'}(X) \leqslant \sup_{X \in \mathcal{M}(F_q,\mu,\sigma^2)} \mathrm{TVaR}_\alpha(X).$$

Noticing that for $0 \leqslant q < \alpha < \alpha' < 1$ the TVaR_α weight function $\gamma_\alpha(u) = \frac{\mathbb{1}_{(\alpha,1)}(u)}{1-\alpha}$ is not constant on $(q, 1)$, the upper bound in Theorem 2.8 leads to the formula in (39), which is simply the upper bound defined (23) for the specific spectrum of the $\mathrm{TVaR}_\alpha(X)$. Additionally, recall that in (24) we derive a sufficient condition for which the quantile function defined in (25) belongs to the set $\mathcal{M}(F_q, \mu, \sigma^2)$ and attains the upper bound in defined in (23). Using the TVaR_α weight function $\gamma_\alpha(u) = \frac{\mathbb{1}_{(\alpha,1)}(u)}{1-\alpha}$ one readily obtains that the sufficient condition in (24) can be written as $\frac{\sqrt{1-\alpha}}{q-\alpha}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \geqslant F^{-1}(q)$, and that the corresponding quantile function defined in (25) can be expressed as

$$\overline{G}_\alpha^{-1}(u) = \begin{cases} F^{-1}(u) \text{ for } u \in (0, q], \\ \frac{\sqrt{1-\alpha}}{q-\alpha}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \text{ for } u \in (q, \alpha], \\ \sqrt{\frac{\alpha-q}{(1-\alpha)(1-q)^2}}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \text{ for } u \in (\alpha, 1). \end{cases} \tag{48}$$

Hence, if $\frac{\sqrt{1-\alpha}}{q-\alpha}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q \geqslant F^{-1}(q)$ the random variable $X_\alpha \sim \overline{G}_\alpha$ belongs to $\mathcal{M}(F_q, \mu, \sigma^2)$. The quantile function of X_α, defined in (48), assumes the constant value $c_\alpha := \sqrt{\frac{\alpha-q}{(1-\alpha)(1-q)^2}}\sqrt{\mu_{2,q} - \mu_q^2} + \mu_q$ on the interval $(\alpha, 1)$. Hence, we obtain $\mathrm{RVaR}_{\alpha,\alpha'}(X_\alpha) = \mathrm{TVaR}_\alpha(X_\alpha) = c_\alpha$. Finally, note that condition $\mathrm{RVaR}_{\alpha,\alpha'}(X_\alpha) = c_\alpha$ for every $\alpha' \in (\alpha, 1)$ implies that $\mathrm{VaR}_{\alpha'}^+(X_\alpha) = c_\alpha$ for every $\alpha' \in (\alpha, 1)$. Since VaR^+ is right-continuous, we obtain, $\mathrm{VaR}_\alpha^+(X_\alpha) = \lim_{\alpha' \to \alpha^+} \mathrm{VaR}_{\alpha'}^+(X_\alpha) = c_\alpha$. □

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