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On the optimization of vectors of Katz-Bonacich's centralities related to Nash equilibria for a class of non-cooperative network games

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ABSTRACT

Evaluating the importance or centrality of nodes in a graph is a fundamental issue in network analysis and design. One possible measure of centrality is Katz-Bonacich's centrality, which arises both at the steady state of a dynamical system involving local communication within an undirected graph, and at the pure Nash equilibrium of a non-cooperative game on that graph, whose players are its nodes. Given this context, the present article explores the connection between vectors of Katz-Bonacich's centralities arising, respectively, in the deterministic and stochastic cases, and its consequences for network design and for finding the so-called key player, i.e. a player/node that mostly contributes to raising the equilibrium aggregate level in the network. The theoretical analysis is complemented by numerical results and by a discussion of its possible extensions to other measures of node centrality.

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1. Introduction

A relevant issue in network analysis and design consists in quantifying the centrality of its nodes, thus providing a measure of their importance. This can help designers, for example, in detecting weak components of the network, identifying and preventing node failures, and, more generally, in improving network connectivity. Various measures of node centrality were proposed in the literature, such as degree centrality, closeness centrality, betweenness centrality, eigenvector (or eigenvalue) centrality [1], and measures of centrality inspired by cooperative [2] or non-cooperative game theory [3]. For a very recent survey on centrality measures, we mention the interesting paper [4], while for an application of centrality measures to the problem of controlling the spread of epidemics in networks, we refer to [5]. In the present work, focus is given to one non-cooperative measure of node centrality, namely, Katz-Bonacich's centrality, which has the advantage, with respect to other measures of node centrality, of lending itself to a variety of interpretations, since it arises both at the steady state of a dynamical system involving local communication within an undirected graph, and at the pure Nash equilibrium of a non-cooperative game on that

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graph, whose players are its nodes. Given this context, the work explores the relationship between vectors of Katz-Bonacich's centralities (one component for each node) that arise, respectively, in the deterministic and stochastic cases, and its consequences for network optimization problems.

In more detail, the following is a list of the main contributions of the present work:

- The extension to the stochastic case of the connection between the vector of Katz-Bonacich's centralities and the vector of pure Nash equilibrium strategies for a linear-quadratic non-cooperative game model on an undirected graph (see Section 3).
- An investigation of the relationship between the set of admissible vectors of stochastic Katz-Bonacich's centralities (feasible region), which is a bounded convex polyhedral set, and the set of admissible vectors of deterministic Katz-Bonacich's centralities (see Proposition 4.1 in Section 4).
- An analysis of the connection between the linear optimization of vectors of deterministic (respectively, stochastic) Katz-Bonacich's centralities (see Remark 4.2 in Section 4).
- An explicit expression for the cardinality of the set of vertices of the feasible region for the stochastic case (see Proposition 4.2 in Section 4).
- A study of the relationship between two conditions used in the analysis (see Proposition 4.3 in Section 4), where one of these conditions was introduced in a previous work [6] to define the vector of stochastic Katz-Bonacich's centralities, whereas the other is a simpler but stronger condition.
- A Mixed Integer Linear Programming (MILP) formulation (see Proposition 5.1 in Section 5) and a solution algorithm for the key player problem (see Section 5), which is the problem of finding a player/node that mostly contributes to raising the equilibrium aggregate level in the network [3].

The work is organized as follows. Section 2 provides a background on Katz-Bonacich's centrality, which arises in the context of either deterministic or stochastic dynamical systems. Section 3 reports known results about the relationship between Katz-Bonacich's centrality and Nash equilibrium for a non-cooperative game-theoretical model on a graph, whose players are its nodes, and then it extends such a relationship from the deterministic to the stochastic case. Section 4 reports the main theoretical results of the work on the relationship between the vectors of Katz-Bonacich's centralities arising in deterministic and stochastic cases and its consequences for related optimization problems; then it complements such a theoretical analysis with numerical results. Section 5 presents a MILP formulation for the key player problem, and proposes an algorithm to solve it, inspired by the results of Section 4. Finally, Section 6 discusses possible extensions of the analysis made in the present work.

2. Background on Katz-Bonacich's centrality

In the following, we summarize some results from Reiffers-Masson et al. [6] about vectors of deterministic and stochastic Katz-Bonacich's centralities.

- Given an undirected¹ graph (or network) $G := (V, \mathbf{A})$ described by its set of vertices (or nodes) $V := \{1, \dots, |V|\}$ and its adjacency matrix $\mathbf{A} \in \{0, 1\}^{|V| \times |V|}$ with no self-loops

(i.e. with $A_{ii} = 0$ for every $i \in V$), and a scalar $\rho > 0$ such that $\lambda_{\max}(\rho \mathbf{A}) < 1$, the vector of classical (deterministic) Katz-Bonacich's centralities is defined as the steady state $\mathbf{k} \in \mathbb{R}^{|V|}$ of the deterministic dynamical system with state vector $\mathbf{x}(t) \in \mathbb{R}^{|V|}$, whose components evolve as follows:

$$x_i(t+1) = 1 + \sum_{j \in V} \rho A_{ij} x_j(t), \quad \forall i \in V. \quad (1)$$

Such a steady state surely exists when $\lambda_{\max}(\rho \mathbf{A}) < 1$, and is expressed as

$$\mathbf{k} = (\mathbf{I}_{|V|} - \rho \mathbf{A})^{-1} \mathbf{1}_{|V|}, \quad (2)$$

where $\mathbf{I}_{|V|}$ is the identity matrix of dimension $|V|$, and $\mathbf{1}_{|V|}$ is a $|V|$ -dimensional column vector made of all 1's. Moreover, all the components of $\mathbf{k}$ in Equation (2) are larger than or equal to 0, since $(\mathbf{I}_{|V|} - \rho \mathbf{A})^{-1} = \sum_{h=0}^{+\infty} (\rho \mathbf{A})^h$ (i.e. it is the limit of a geometric series of matrices), $\rho \mathbf{A}$ is a non-negative matrix, and $\mathbf{1}_{|V|}$ is a positive vector. Finally, convergence for $t \rightarrow +\infty$ to the steady state $\mathbf{k}$ in Equation (2) of the state vector $\mathbf{x}(t)$ holds regardless of the initial condition $\mathbf{x}(0)$, because the residual error vector $\mathbf{r}(t) := \mathbf{x}(t) - \mathbf{k}$ satisfies

$$\mathbf{r}(t+1) = \rho \mathbf{A} \mathbf{r}(t),$$

and, as already stated, one has $\lambda_{\max}(\rho \mathbf{A}) < 1$.

- Given an undirected graph $G = (V, \mathbf{A})$ described by its set of vertices V and its adjacency matrix $\mathbf{A} \in \{0, 1\}^{|V| \times |V|}$ with no self-loops, a scalar $\rho > 0$ such that $\lambda_{\max}(\rho \mathbf{A}) < 1$, and a probability vector $\mathbf{p} \in [0, 1]^{|V|}$, the vector of stochastic Katz-Bonacich's centralities is defined as the (expected) steady state $\mathbf{k} \in \mathbb{R}^{|V|}$ of the stochastic dynamical system with state vector $\mathbf{x}(t) \in \mathbb{R}^{|V|}$, whose expected components evolve as follows:

$$\mathbb{E}[x_i(t+1)] = p_i \left(1 + \sum_{j \in V} \rho A_{ij} \mathbb{E}[x_j(t)] \right), \quad \forall i \in V. \quad (3)$$

This (expected) steady state surely exists when $\lambda_{\max}(\rho \mathbf{A}) < 1$, since it implies² $\lambda_{\max}(\rho \text{diag}(\mathbf{p}) \mathbf{A}) < 1$, where $\text{diag}(\mathbf{p})$ is a diagonal matrix of dimension $|V|$ with its elements taken from $\mathbf{p}$, and is expressed as

$$\mathbf{k} = (\mathbf{I}_{|V|} - \rho \text{diag}(\mathbf{p}) \mathbf{A})^{-1} \mathbf{p}. \quad (4)$$

Also in this case, all the components of $\mathbf{k}$ in Equation (4) are larger than or equal to 0, since $(\mathbf{I}_{|V|} - \rho \text{diag}(\mathbf{p}) \mathbf{A})^{-1} = \sum_{h=0}^{+\infty} (\rho \text{diag}(\mathbf{p}) \mathbf{A})^h$, $\rho \text{diag}(\mathbf{p}) \mathbf{A}$ is a non-negative matrix, and $\mathbf{p}$ is a non-negative vector. In more detail, the dynamical system (3) refers to the case in which, for any $t = 0, 1, \dots$, each vertex $i \in V$ is present with probability p_i , and absent with probability $1 - p_i$ (in the latter case, $x_i(t) = 0$). Moreover, for any finite set of times, the presences/absences of the same vertex at different times and/or of different vertices at the same or different times form a set of mutually independent events. Finally, convergence for $t \rightarrow +\infty$ to the (expected) steady state $\mathbf{k}$ in Equation (4)

of the expected state vector $\mathbb{E}[\mathbf{x}(t)]$ holds regardless of the initial condition $\mathbb{E}[\mathbf{x}(0)]$, because the expected residual error vector $\mathbb{E}[\mathbf{r}(t)] := \mathbb{E}[\mathbf{x}(t)] - \mathbf{k}$ satisfies

$$\mathbb{E}[\mathbf{r}(t+1)] = \rho \operatorname{diag}(\mathbf{p})\mathbf{A}\mathbb{E}[\mathbf{r}(t)],$$

and, as already stated, one has $\lambda_{\max}(\rho \operatorname{diag}(\mathbf{p})\mathbf{A}) < 1$.

It is worth observing that, when $\mathbf{p} \in \{0, 1\}^{|V|}$, the vector of stochastic Katz-Bonacich's centralities expressed by Equation (4) can be interpreted as a vector of deterministic Katz-Bonacich's centralities, in which the nodes i associated with $p_i = 0$ have been removed from the graph and assigned by convention a deterministic Katz-Bonacich's centrality equal to 0, and Equation (2) is used to express the deterministic Katz-Bonacich's centralities of the remaining vertices (by reducing appropriately the dimensions of all matrices and vectors).

3. Relationship between Katz-Bonacich's centrality and Nash equilibrium: extension from the deterministic to the stochastic case

It is well-known [3] that, when $\lambda_{\max}(\rho\mathbf{A}) < 1$, the vector $\mathbf{k}$ of deterministic Katz-Bonacich's centralities, reported in Equation (2), can be interpreted as the vector of pure strategies at the unique pure Nash equilibrium [8] of a non-cooperative game model with $|V|$ players (i.e. the vertices of the undirected graph G), where each player $i \in V$ chooses its pure strategy $a_i \in \mathbb{R}$ with the aim to maximize its individual utility function u_i , which depends on its pure strategy and on the ones of its neighbours in the graph, and is expressed in the following linear-quadratic form (which is strictly concave with respect to a_i , for fixed choices of the pure strategies of the other players):

$$u_i(a_1, \dots, a_{|V|}) = -\frac{1}{2}a_i^2 + a_i + a_i \sum_{j \in V} \rho A_{ij}a_j. \quad (5)$$

Indeed, setting $\frac{\partial u_i}{\partial a_i}(a_1, \dots, a_{|V|})$ to 0 for each $i \in V$, at a pure Nash equilibrium $\mathbf{a}^* \in \mathbb{R}^{|V|}$ one has

$$-a_i^* + 1 + \sum_{j \in V} \rho A_{ij}a_j^* = 0, \quad \forall i \in V, \quad (6)$$

hence

$$\mathbf{a}^* = (\mathbf{I}_{|V|} - \rho\mathbf{A})^{-1} \mathbf{1}_{|V|}, \quad (7)$$

which coincides with the expression of $\mathbf{k}$ reported in Equation (2). The expression (7) for the unique pure Nash equilibrium is valid also if one reformulates the non-cooperative game model in such a way that the constraint $a_i \geq 0$ holds, too, for every player $i \in V$ because, as already reported in Section 2, all the components of $\mathbf{k}$ in Equation (2) are larger than or equal to 0. Moreover, for this modification of the model, one can easily check that corner solutions are not part of a pure Nash equilibrium [3] (to do this, it is enough to check that $\frac{\partial u_i}{\partial a_i}(a_1, \dots, a_{|V|}) > 0$ when $a_i = 0$)³.

In the following, we prove that, for $\lambda_{\max}(\rho\mathbf{A}) < 1$, a similar interpretation holds for the vector of stochastic Katz-Bonacich's centralities reported in Equation (4), under a suitable modification of the non-cooperative game model detailed above. Specifically, for any given

(exogenous) vector of probabilities $\mathbf{p} \in [0, 1]^{|V|}$, we still consider a framework where V is the set of players, but their participation is now governed by mutually independent events, such that each player i is present with probability p_i and absent with probability $1 - p_i$. The action available to a player depends strictly on its presence. When player i is absent, its action is forced to be $a_i = 0$. Conversely, when present, the player chooses an action denoted by $\tilde{a}_i \in \mathbb{R}$ (without being further informed about the actual presence or absence of any of the other players). Because the action a_j of any given player j is $\tilde{a}_j$ if it is present and 0 otherwise, the expected value of its action is $\mathbb{E}[a_j] = p_j \tilde{a}_j + 0(1 - p_j) = p_j \tilde{a}_j$. When player i is present, its aim is to maximize the conditional expectation of its individual utility function expressed by Equation (5). This optimization takes into account that the other players j are also independently present or absent according to their respective probabilities p_j . Consequently, the expected value of the utility function of player i , conditional on its specific presence, is expressed as:

$$\begin{aligned} & \mathbb{E}[u_i(a_1, \dots, a_{|V|}) | \text{presence of player } i] \\ &= -\frac{1}{2} \tilde{a}_i^2 + \tilde{a}_i + \tilde{a}_i \sum_{j \in V} \rho A_{ij} \mathbb{E}[a_j] \\ &= -\frac{1}{2} \tilde{a}_i^2 + \tilde{a}_i + \tilde{a}_i \sum_{j \in V} \rho A_{ij} p_j \tilde{a}_j. \end{aligned} \quad (8)$$

Building upon this, we search for a pure Nash equilibrium $\tilde{\mathbf{a}}^* \in \mathbb{R}^{|V|}$ of the modified non-cooperative game in which the set of players is V , the components $\tilde{a}_i$ are the pure strategies of the players $i \in V$ and, based on Equation (8), their individual utilities $\tilde{u}_i(\tilde{a}_1, \dots, \tilde{a}_{|V|})$ are defined as follows:

$$\tilde{u}_i(\tilde{a}_1, \dots, \tilde{a}_{|V|}) = -\frac{1}{2} \tilde{a}_i^2 + \tilde{a}_i + \tilde{a}_i \sum_{j \in V} \rho A_{ij} p_j \tilde{a}_j. \quad (9)$$

It is worth remarking that the Nash equilibrium $\tilde{\mathbf{a}}^*$ exists and is unique because each utility function $\tilde{u}_i$ defined in Equation (9) is strictly concave with respect to $\tilde{a}_i$, and in this case imposing the first-order necessary and sufficient conditions for a pure Nash equilibrium generate the linear system⁴

$$\tilde{\mathbf{a}}^* = \mathbf{1}_{|V|} + \rho \mathbf{A} \text{diag}(\mathbf{p}) \tilde{\mathbf{a}}^*,$$

i.e.

$$(\mathbf{I}_{|V|} - \rho \mathbf{A} \text{diag}(\mathbf{p})) \tilde{\mathbf{a}}^* = \mathbf{1}_{|V|},$$

which has a unique solution since, for $\lambda_{\max}(\rho \mathbf{A}) < 1$, the matrix $(\mathbf{I}_{|V|} - \rho \mathbf{A} \text{diag}(\mathbf{p}))$ is invertible⁵.

At this point, let us define $\mathbf{a}^* \in \mathbb{R}^{|V|}$ as the random vector of realized actions at the equilibrium, where each component a_i^* coincides with the component $\tilde{a}_i^*$ of the Nash equilibrium described above, assuming player i is present. Should player i be absent, a_i^* is equal

to 0. Therefore, for any $i \in V$, one gets the following:

$$a_i^* = \begin{cases} \tilde{a}_i^* = 1 + \sum_{j \in V} \rho A_{ij} p_j \tilde{a}_j^* = 1 + \sum_{j \in V} \rho A_{ij} \mathbb{E}[a_j^*], & \text{with probability } p_i, \\ 0, & \text{with probability } 1 - p_i, \end{cases}$$

hence

$$\mathbb{E}[a_i^*] = p_i \left(1 + \sum_{j \in V} \rho A_{ij} \mathbb{E}[a_j^*] \right), \quad \forall i \in V.$$

Concluding, the expected value vector $\mathbb{E}[\mathbf{a}^*]$ satisfies the equation

$$\mathbb{E}[\mathbf{a}^*] = (\mathbf{I}_{|V|} - \rho \operatorname{diag}(\mathbf{p})\mathbf{A})^{-1} \mathbf{p}, \quad (10)$$

which coincides with the expression of $\mathbf{k}$ reported in Equation (4).

Remark 3.1: It is worth stressing that while the stochastic model introduced above can be formally cast as a game with incomplete information in the sense of Harsanyi [11–13], it operates without hidden payoff parameters or preferences. Indeed, the randomness concerns strictly the exogenous presence/absence of players, modelled as outcomes of independent Bernoulli trials governed by the probability vector $\mathbf{p} \in [0, 1]^{|V|}$, which is assumed to be common knowledge. In this sense, it can be viewed as a “degenerate” Bayesian game of incomplete information, where each player’s type simply models the player’s presence or absence, and the absent player takes no decision. Consequently, the framework can be analyzed directly through the expected actions of the active players, and solving for pure strategies in this expected-value formulation is sufficient to obtain the Katz-Bonacich vector in the stochastic case without necessarily relying on a Bayesian-Nash formalism.

Remark 3.2: With reference to the game-theoretical interpretation presented above, we observe that, when $p_i > 0$, the action exerted by player $i \in V$ at the pure Nash equilibrium is expressed either as $1 + \sum_{j \in V} \rho A_{ij} \mathbb{E}[a_j^*]$ or as $\mathbb{E}[a_i^*]/p_i$. Otherwise, when $p_i = 0$, it is expressed only as $1 + \sum_{j \in V} \rho A_{ij} \mathbb{E}[a_j^*]$ (although in this second pathological case, that action is played with probability 0). Finally, the expression (10) is valid also if one reformulates the modified non-cooperative game model in such a way that the constraint $a_i \geq 0$ holds, too, for every player $i \in V$ because, as already reported in Section 2, all the components of $\mathbf{k}$ in Equation (4) are larger than or equal to 0. Again, for this new variation of the model, corner solutions are not part of a pure Nash equilibrium, because $\frac{\partial u_i}{\partial a_i}(a_1, \dots, a_{|V|}) > 0$ when $a_i = 0$.

4. Main theoretical results of the work

The next result provides a characterization of the feasible set $\mathcal{F}$ of admissible vectors of stochastic Katz-Bonacich’s centralities for an undirected graph and relates it with the corresponding set of admissible vectors of deterministic Katz-Bonacich’s centralities.

Proposition 4.1: *Given an undirected graph $G = (V, \mathbf{A})$ described by its set of vertices V and its adjacency matrix $\mathbf{A} \in \{0, 1\}^{|V| \times |V|}$ with no self-loops, and a scalar $\rho > 0$ such that $\lambda_{\max}(\rho \mathbf{A}) < 1$, the following holds:*

- (i) *The set of admissible vectors of stochastic Katz-Bonacich's centralities $\mathbf{k} \in [0, +\infty)^{|V|}$ obtained by varying the probability vector $\mathbf{p} \in [0, 1]^{|V|}$ is a bounded convex polyhedral set $\mathcal{F}$.*
- (ii) *The vertices of $\mathcal{F}$ correspond to classical Katz-Bonacich's centralities, i.e. the components of their associated probability vectors are equal to either 0 or 1.*

Proof: (i) A direct application of [6, Proposition 2]⁶ shows that the set of admissible vectors of stochastic Katz-Bonacich's centralities $\mathbf{k} \in [0, +\infty)^{|V|}$ obtained by varying the probability vector $\mathbf{p} \in [0, 1]^{|V|}$ is the convex polyhedral set

$$\mathcal{F} := \left\{ \mathbf{k} \in [0, +\infty)^{|V|} : k_i \leq 1 + \sum_{j \in V} \rho A_{ij} k_j, \quad \forall i \in V \right\}, \quad (11)$$

where $\mathbf{p}$ and $\mathbf{k}$ are related as follows:

$$p_i = \frac{k_i}{1 + \sum_{j \in V} \rho A_{ij} k_j}, \quad \forall i \in V. \quad (12)$$

Moreover, it follows by the proof of Reiffers-Masson et al. [6, Proposition 1] that the relationship between $\mathbf{k} \in \mathcal{F}$ and $\mathbf{p} \in [0, 1]^{|V|}$ is invertible (so, both $\mathbf{p} = \mathbf{p}(\mathbf{k})$ and $\mathbf{k} = \mathbf{k}(\mathbf{p})$ hold).

The feasibility region $\mathcal{F}$ in Equation (11) can also be written as

$$\begin{aligned} \mathcal{F} &= \left\{ \mathbf{k} \in \mathbb{R}^{|V|} : 0 \leq k_i \leq 1 + \sum_{j \in V} \rho A_{ij} k_j, \quad \forall i \in V \right\} \\ &= \left\{ \mathbf{k} \in \mathbb{R}^{|V|} : \mathbf{0}_{|V|} \leq \mathbf{k} \leq \mathbf{1}_{|V|} + \rho \mathbf{A} \mathbf{k} \right\} \\ &= \left\{ \mathbf{k} \in \mathbb{R}^{|V|} : \mathbf{k} \geq \mathbf{0}_{|V|} \text{ and } (\mathbf{I}_{|V|} - \rho \mathbf{A}) \mathbf{k} \leq \mathbf{1}_{|V|} \right\}, \end{aligned} \quad (13)$$

where $\mathbf{0}_{|V|}$ is a $|V|$ -dimensional column vector made of all 0's, and the inequalities in Equation (13) have to be interpreted element-wise. Finally, since $(\mathbf{I}_{|V|} - \rho \mathbf{A})^{-1} = \sum_{h=0}^{+\infty} (\rho \mathbf{A})^h$ is a non-negative matrix, one gets

$$\mathcal{F} \subseteq \left\{ \mathbf{k} \in \mathbb{R}^{|V|} : \mathbf{k} \geq \mathbf{0}_{|V|} \text{ and } \mathbf{k} \leq (\mathbf{I}_{|V|} - \rho \mathbf{A})^{-1} \mathbf{1}_{|V|} \right\}, \quad (14)$$

which proves the boundedness of the feasibility region $\mathcal{F}$.

- (ii) We prove that the probability vectors associated with the vertices of $\mathcal{F}$ have components in the set $\{0, 1\}$. Indeed, by introducing a vector of slack variables $\mathbf{s} \in$

$[0, +\infty)^{|V|}$, the set $\mathcal{F}$ in Equation (11) can be equivalently represented in the form

$$\mathcal{F} = \left\{ \mathbf{k} \in \mathbb{R}^{|V|} : k_i \geq 0 \text{ and } \exists s_i \geq 0 \text{ such that } 1 + \sum_{j \in V} \rho A_{ij} k_j = k_i + s_i, \forall i \in V \right\}. \quad (15)$$

Now, given any vertex $\mathbf{k} \in \mathcal{F}$ and any index $i \in V$, k_i and s_i cannot be simultaneously part of a basis associated with that vertex. Indeed, if this were not the case (i.e. if, for some $i \in V$, k_i and s_i were simultaneously part of a basis), then it would follow by a simple counting argument – e.g. by an application of the pigeonhole principle [14, Theorem 2.5] – that there would exist $\hat{i} \in V$ for which both $k_{\hat{i}}$ and $s_{\hat{i}}$ were simultaneously not part of that basis, hence they would be both equal to 0. However, from Equation (15), one would get

$$1 \leq 1 + \sum_{j \in V} \rho A_{\hat{i}j} k_j = k_{\hat{i}} + s_{\hat{i}} = 0,$$

obtaining a contradiction. Concluding, since any basis has cardinality $|V|$ (which is equal to the cardinality of each set of variables k_i and s_i), it follows that, for any $i \in V$, one (and only one) between k_i and s_i is part of that basis. Now, there are two possible cases:

- If s_i is not part of the basis, then $s_i = 0$, $k_i = 1 + \sum_{j \in V} \rho A_{ij} k_j$, hence $p_i = \frac{k_i}{1 + \sum_{j \in V} \rho A_{ij} k_j} = 1$.
- If k_i is not part of the basis, then $k_i = 0$, hence $p_i = \frac{k_i}{1 + \sum_{j \in V} \rho A_{ij} k_j} = 0$.

This proves that the probability vectors associated with the vertices of the set $\mathcal{F}$ have components in the set $\{0, 1\}$. ■

Example 4.1: The following example verifies the validity of Proposition 4.1, by reporting all the vertices of the feasible region $\mathcal{F}$, for a case (illustrated in Figure 1) in which the number of vertices is $|V| = 4$, the adjacency matrix is

$$\mathbf{A} := \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

and $\rho = 0.3$. From these data, one gets $\lambda_{\max}(\rho \mathbf{A}) \simeq 0.6510 < 1$. The enumeration of all vertices of the feasible region $\mathcal{F}$ (obtained using the MATLAB function `con2vert.m`)⁷ and of the associated probability vectors provides the results shown in Table 1, which are in accordance with Proposition 4.1.

Remark 4.1: It is worth noting that part (ii) of Proposition 4.1 implies that any feasible vector $\mathbf{k}$ of stochastic Katz-Bonacich's centralities can be written as a convex combination

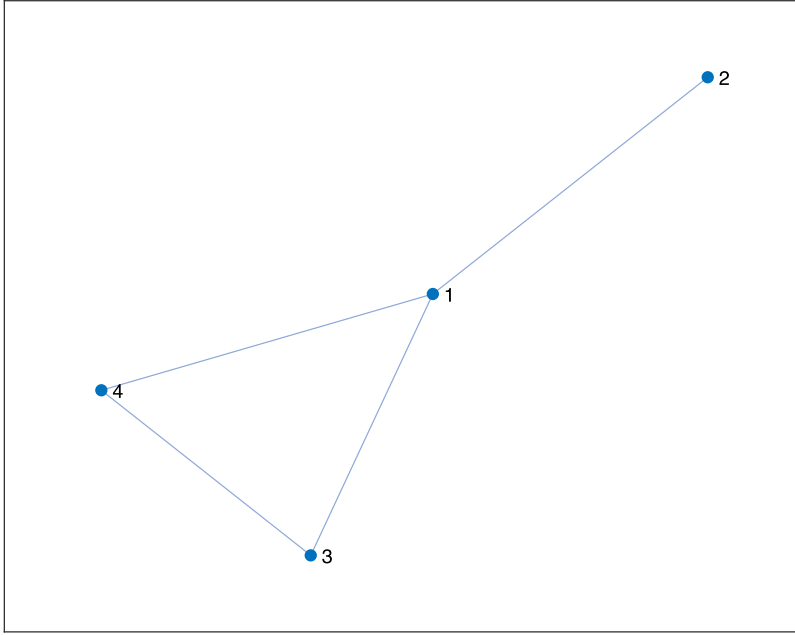


Figure 1. Graph associated with the adjacency matrix of Example 4.1 (the numbers are used to label the vertices of the graph).

Table 1. List of all the vertices of the feasible region $\mathcal{F}$ of stochastic Katz-Bonacich's centralities in Example 4.1, and associated probability vectors.

Vertex no.	Vector of Katz-Bonacich's centralities ($\mathbf{k}$)	Associated probability vector ($\mathbf{p}$)
1	$[0\ 0\ 0\ 0]^T$	$[0\ 0\ 0\ 0]^T$
2	$[0\ 0\ 0\ 1]^T$	$[0\ 0\ 0\ 1]^T$
3	$[0\ 0\ 1\ 0]^T$	$[0\ 0\ 1\ 0]^T$
4	$[0\ 0\ 1.4286\ 1.4286]^T$	$[0\ 0\ 1\ 1]^T$
5	$[0\ 1\ 0\ 0]^T$	$[0\ 1\ 0\ 0]^T$
6	$[0\ 1\ 0\ 1]^T$	$[0\ 1\ 0\ 1]^T$
7	$[0\ 1\ 1\ 0]^T$	$[0\ 1\ 1\ 0]^T$
8	$[0\ 1\ 1.4286\ 1.4286]^T$	$[0\ 1\ 1\ 1]^T$
9	$[1\ 0\ 0\ 0]^T$	$[1\ 0\ 0\ 0]^T$
10	$[1.4286\ 0\ 0\ 1.4286]^T$	$[1\ 0\ 0\ 1]^T$
11	$[1.4286\ 0\ 1.4286\ 0]^T$	$[1\ 0\ 1\ 0]^T$
12	$[2.5000\ 0\ 2.5000\ 2.5000]^T$	$[1\ 0\ 1\ 1]^T$
13	$[1.4286\ 1.4286\ 0\ 0]^T$	$[1\ 1\ 0\ 0]^T$
14	$[1.9512\ 1.5854\ 0\ 1.5854]^T$	$[1\ 1\ 0\ 1]^T$
15	$[1.9512\ 1.5854\ 1.5854\ 0]^T$	$[1\ 1\ 1\ 0]^T$
16	$[3.3042\ 1.9912\ 2.8446\ 2.8446]^T$	$[1\ 1\ 1\ 1]^T$

Note: All the vertices correspond with deterministic Katz-Bonacich's centralities.

of feasible vectors of deterministic Katz-Bonacich's centralities. Although the corresponding probability vector $\mathbf{p}(\mathbf{k})$ can be written, too, as a linear combination of vectors belonging to $\{0, 1\}^{|V|}$, the coefficients of these two combinations are generally different since the vectors $\mathbf{k}$ and $\mathbf{p}$ are non-linearly related through Equation (12)⁸. This shows that part (ii) of Proposition 4.1 is a non-trivial result. For instance, in the case of Example 4.1, one has

$$\mathbf{k} = [\frac{1}{2} \ \frac{1}{2} \ \frac{1}{2} \ \frac{1}{2}]^\top = \frac{1}{2} [1 \ 0 \ 0 \ 0]^\top + \frac{1}{2} [0 \ 1 \ 0 \ 0]^\top + \frac{1}{2} [0 \ 0 \ 1 \ 0]^\top + \frac{1}{2} [0 \ 0 \ 0 \ 1]^\top, \text{ whereas } \mathbf{p}(\mathbf{k}) = [\frac{10}{29} \ \frac{10}{23} \ \frac{5}{13} \ \frac{5}{13}]^\top = \frac{10}{29} [1 \ 0 \ 0 \ 0]^\top + \frac{10}{23} [0 \ 1 \ 0 \ 0]^\top + \frac{5}{13} [0 \ 0 \ 1 \ 0]^\top + \frac{5}{13} [0 \ 0 \ 0 \ 1]^\top.$$

Remark 4.2: For $\mathbf{p} \in [0, 1]^{|V|}$, let $\mathbf{k}(\mathbf{p})$ be the associated vector of stochastic Katz-Bonacich's centralities, which corresponds to a vector of classical Katz-Bonacich's centralities when $\mathbf{p} \in \{0, 1\}^{|V|}$. Due to Proposition 4.1, the optimal solutions of the following optimization problem (where $\mathbf{c} \in \mathbb{R}^{|V|}$, and $\cdot$ denotes the inner product):

$$\max_{\mathbf{p} \in [0, 1]^{|V|}} \mathbf{c} \cdot \mathbf{k}(\mathbf{p}), \quad (16)$$

are the optimal vertices of the following continuous Linear Programming (LP) problem:

$$\begin{aligned} & \max_{\mathbf{k} \in \mathbb{R}^{|V|}} \mathbf{c} \cdot \mathbf{k}, \\ & \text{subject to} \quad 0 \leq k_i \leq 1 + \sum_{j \in V} \rho A_{ij} k_j, \quad \forall i \in V, \end{aligned} \quad (17)$$

(i.e. $\max_{\mathbf{k} \in \mathcal{F}} \mathbf{c} \cdot \mathbf{k}$), which is associated with the following continuous relaxation of the optimization problem (16):

$$\max_{\mathbf{p} \in [0, 1]^{|V|}} \mathbf{c} \cdot \mathbf{k}(\mathbf{p}). \quad (18)$$

In other words, it follows by an application of Proposition 4.1 that the two optimization problems (16) and (17) are equivalent. The two optimization problems (17) and (18) are also equivalent since, given any admissible vector $\mathbf{k}$ for problem (17), one can generate an admissible vector $\mathbf{p}$ for problem (18) having the same objective value, and vice versa. Concluding, the three optimization problems (16), (17), and (18) are equivalent.

Remark 4.3: The equivalence between the two optimization problems (16) and (18) discussed in Remark 4.2 is a non-trivial result, since the coefficient matrix of problem (17) in general is not totally unimodular [15, Section 13.2]. It is recalled here that the total unimodularity of the coefficient matrix (i.e. the fact that the determinant of each of its square submatrices is either -1 , 0 , or 1), together with the integrality of the vector of constant terms, provides a sufficient condition for the equivalence between an integer linear programming problem and its continuous relaxation. Concluding, the two optimization problems (16) and (18) – which are related to the continuous linear programming problem (17) – are still equivalent, despite the fact that the coefficient matrix of the problem (17) in general is not totally unimodular. For instance, the matrix $\mathbf{A}$ in Example 4.1 is not totally unimodular, since its square submatrix obtained by removing its second row and its second column is

$$\hat{\mathbf{A}} := \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

whose determinant is equal to 2.

Remark 4.4: When $\mathbf{c}$ is a non-negative vector, the optimization problem (16) has the optimal solution $\mathbf{p}^\circ = \mathbf{1}_{|V|}$. This follows directly from the fact that, as already reported in Section 2, $\mathbf{k}(\mathbf{p}) = (\mathbf{I}_{|V|} - \rho \operatorname{diag}(\mathbf{p})\mathbf{A})^{-1}\mathbf{p} = \sum_{h=0}^{+\infty} (\rho \operatorname{diag}(\mathbf{p})\mathbf{A})^h \mathbf{p}$. For instance, in the case of Example 4.1, the optimal solution for either $\mathbf{c} = [1 \ 1 \ 1 \ 1]^\top$ or $\mathbf{c} = [1 \ 2 \ 3 \ 4]^\top$ is $\mathbf{p}^\circ = [1 \ 1 \ 1 \ 1]^\top$ (with corresponding optimal objective value $\mathbf{c} \cdot \mathbf{k}(\mathbf{p}^\circ)$ equal, respectively, to 10.9846 and 27.1988). However, for $\mathbf{c} = [-1 \ -2 \ 2 \ 2]^\top$, it is $\mathbf{p}^\circ = [1 \ 0 \ 1 \ 1]^\top$ (with corresponding optimal objective value $\mathbf{c} \cdot \mathbf{k}(\mathbf{p}^\circ)$ equal to 7.5).

Remark 4.5: The optimization problem (16) refers to the situation in which, for each $i \in V$, one can decide (by the choice of p_i) if each vertex i is either always present in the graph, or always absent in it (or, more precisely, either present with probability 1, or absent with probability 1). A result similar to Proposition 4.1 can be obtained by imposing that certain vertices are either always present in the graph, or always absent in it, which is actually equivalent to associating, to each such vertex i , either the constraint $p_i = 1$, or the constraint $p_i = 0$. Indeed, according to [6, Proposition 2], such constraints can be also written, respectively, as $1 + \sum_{j \in V} \rho A_{ij} k_j = k_i$, and $k_i = 0$. The only change in the proof of part (ii) of Proposition 4.1 would be that, in the first case, k_i would be constrained to be part of any basis, whereas, in the second case s_i would be constrained to be part of any basis.

Remark 4.6: A second variation of the optimization problem (16) would be achieved by replacing the graph $G = (V, \mathbf{A})$ with a second graph $G' = (V', E')$, obtained by replacing each edge e_{ij} of the original graph with an auxiliary vertex v and two edges linking v with i and j , respectively. Then, one could formulate a similar optimization problem for the vector of Katz-Bonacich's centralities on the new graph, imposing, e.g. that the vertices of the original graph are always present. In this way, keeping/removing the auxiliary vertices would correspond with keeping/removing the edges of the original graph.

The next result provides an explicit expression for the cardinality of the set of vertices of the feasible region $\mathcal{F}$, which coincide with the admissible vectors of deterministic Katz-Bonacich's centralities.

Proposition 4.2: *Under the assumptions of Proposition 4.1, the feasible region $\mathcal{F}$, expressed by Equation (11), has exactly $2^{|V|}$ vertices.*

Proof: For each vertex $i \in V$ of the graph $G = (V, \mathbf{A})$, the associated variable k_i can be either part of a basis (associated with a vertex of $\mathcal{F}$) or not (in the latter case, the slack variable s_i is part of a basis). When k_i is part of a basis, one has

$$1 \leq 1 + \sum_{j \in V} \rho A_{ij} k_j = k_i.$$

Hence, $k_i > 0$ and there are no degenerate bases. This, combined with an application of Equation (12), implies that each basis is associated in a one-to-one-way to a vector $\mathbf{p} \in \{0, 1\}^{|V|}$. The proof is concluded by observing that there are $2^{|V|}$ such vectors (each associated with a distinct vertex $\mathbf{v} \in \mathcal{F}$). ■

Remark 4.7: The number of vertices of the feasibility region $\mathcal{F}$ of Example 4.1 is $2^{|V|} = 2^4 = 16$, in accordance with Proposition 4.2.

The next result (which is a result of independent interest) investigates the relationship between the condition $\lambda_{\max}(\rho \mathbf{A}) < 1$, assumed in Proposition 4.1, and a simpler but stronger condition, i.e. $\rho \max_{i \in V} \deg(i) < 1$, where $\deg(i) := \sum_{j \in V} A_{ij}$. This latter condition becomes $\rho(|V| - 1) < 1$ in the case of a complete undirected graph.

Proposition 4.3: *The following statements hold:*

- (i) *The condition $\rho \max_{i \in V} \deg(i) < 1$ implies the condition $\lambda_{\max}(\rho \mathbf{A}) < 1$.*
- (ii) *The two conditions $\rho \max_{i \in V} \deg(i) < 1$ and $\lambda_{\max}(\rho \mathbf{A}) < 1$ are equivalent, meaning that $\lambda_{\max}(\mathbf{A}) = \max_{i \in V} \deg(i)$, if and only if the undirected graph $G = (V, \mathbf{A})$ contains a connected component which is a k -regular subgraph with $k = \max_{i \in V} \deg(i)$.*

Proof: (i) The Gershgorin's circle theorem [7, Theorem 6.1.1] guarantees that all the eigenvalues of the matrix $\mathbf{A}$ are within the circle (in the complex plane) with centre 0 and radius $\max_{i \in V} \sum_{j \in V, j \neq i} |A_{ij}| = \max_{i \in V} \deg(i)$. Since all the eigenvalues of the symmetric matrix $\mathbf{A}$ are real, we get

$$\lambda_{\max}(\mathbf{A}) \leq \max_{i \in V} \deg(i), \quad (19)$$

hence $\rho \max_{i \in V} \deg(i) < 1$ implies $\lambda_{\max}(\rho \mathbf{A}) < 1$.

- (ii) Suppose that the graph G contains a connected component (whose set of nodes is indicated as $U \subseteq V$) which is a k -regular subgraph with $k = \max_{i \in V} \deg(i)$. If we define the vector $\mathbf{v}$ as follows: $v_i = 1$ if $i \in U$ and $v_i = 0$ if $i \in V \setminus U$, then we get $\mathbf{A}\mathbf{v} = k\mathbf{v}$, i.e. k is an eigenvalue of $\mathbf{A}$ and thus $\lambda_{\max}(\mathbf{A}) = \max_{i \in V} \deg(i)$ follows from (19).

Suppose that $\lambda_{\max}(\mathbf{A}) = \max_{i \in V} \deg(i)$ and let $\mathbf{v} \in \mathbb{R}^{|V|}$ be an associated eigenvector. Moreover, let $i^* \in \arg \max_{i \in V} |v_i|$ and U the subset of nodes of the connected component of G containing i^* . Since $\mathbf{v} \neq \mathbf{0}_{|V|}$, one gets $|v_{i^*}| > 0$. Then, since the elements of the adjacency matrix $\mathbf{A}$ are either 0 or 1 and there are no self-loops, one gets

$$\deg(i^*)|v_{i^*}| \leq |v_{i^*}| \max_{i \in V} \deg(i) = |(\mathbf{A}\mathbf{v})_{i^*}| = \left| \sum_{j \in V} A_{i^*j} v_j \right| \leq \deg(i^*)|v_{i^*}|. \quad (20)$$

Hence, we get $\deg(i^*) = \max_{i \in V} \deg(i)$ and the second inequality in (20) holds as an equality, thus $|v_j| = |v_{i^*}|$ for any $j \in V$ such that $A_{i^*j} = 1$. Thus, $\deg(j) = \max_{i \in V} \deg(i)$ holds for any neighbour j of i^* . Repeating the same reasoning for the neighbours of i^* , we get that all the nodes of U have degree equal to $k = \max_{i \in V} \deg(i)$, i.e. this connected component is a k -regular subgraph. ■

Remark 4.8: In Example 4.1, the graph is connected but not regular, and $\lambda_{\max}(\mathbf{A}) \simeq 2.17 < \max_{i \in V} \deg(i) = 3$. Thus, for $\rho = 0.3$, one has $\lambda_{\max}(\rho \mathbf{A}) \simeq 0.65 < \rho \max_{i \in V} \deg(i) = 0.9$, in accordance with parts (i) and (ii) of Proposition 4.3.

5. A MILP approach to find a key player

A well-known notion related to the Katz-Bonacich's centrality is that of key player [3], i.e. a player of the first game described in Section 3 that mostly contributes (when included as a player) to raising the equilibrium aggregate level in the network. Equivalently, it is the player that mostly decreases the equilibrium aggregate of the network, when it is removed.

Definition 5.1: If $\mathbf{a}^*(V \setminus \{i\}) \in \mathbb{R}^{|V|-1}$ denotes the pure Nash equilibrium (7) of the game where player i is removed from the network G and $S_i = \sum_{j \in V \setminus \{i\}} a_j^*(V \setminus \{i\})$ is the corresponding equilibrium aggregate, then a key player of the game is a player $\hat{i} \in V$ such that $S_{\hat{i}} = \min_{i \in V} S_i$.

A straightforward method to find a key player numerically is solving $|V|$ systems of linear equations of the form (6). On the other hand, the discussion in Section 4 allows us to formulate the key player problem as the following single nonlinear integer program:

$$\begin{aligned} & \min \quad \mathbf{1}_{|V|} \cdot \mathbf{k}(\mathbf{p}), \\ & \text{subject to} \quad \sum_{i \in V} p_i = |V| - 1, \\ & \quad \mathbf{p} \in \{0, 1\}^{|V|}, \end{aligned}$$

which is a variation of the optimization problem (16) obtained by imposing the constraint that exactly one of the components of $\mathbf{p}$ equals 0. This latter problem can be reduced to an equivalent continuous linear programming problem having a form similar to that of problem (17), by replacing the feasible region $\mathcal{F}$ of the latter problem with the convex hull of the set of points $\{\mathbf{k}(\mathbf{p}^{(i)})\}$, for $i \in V$ (where each $\mathbf{p}^{(i)} \in \{0, 1\}^{|V|}$ satisfies $p_i^{(i)} = 0$ and $p_j^{(i)} = 1$ for any $j \neq i$), which is a subset of $\mathcal{F}$. For instance, in the case of the undirected graph $G = (V, \mathbf{A})$ of Example 4.1, the optimal solution of the key player problem is $[0 \ 1 \ 1 \ 1]^\top$ (with corresponding optimal objective value equal to 3.8572).

In order to avoid the explicit computation of such points $\mathbf{k}(\mathbf{p}^{(i)})$ and speed up the search for a key player, we propose a Mixed Integer Linear Programming (MILP) reformulation of the key player problem.

Proposition 5.1: Let $\rho > 0$ be such that $\lambda_{\max}(\rho \mathbf{A}) < 1$ and $M > 0$ a sufficiently large constant. Consider the following MILP problem:

$$\min_{(\mathbf{k}, \mathbf{p}) \in \mathbb{R}^{|V|} \times \mathbb{R}^{|V|}} \sum_{i \in V} k_i, \tag{21a}$$

$$\text{subject to} \quad k_i \leq 1 + \sum_{j \in V} \rho A_{ij} k_j, \quad \forall i \in V, \tag{21b}$$

$$k_i \leq M p_i, \quad \forall i \in V, \tag{21c}$$

$$1 + \sum_{j \in V} \rho A_{ij} k_j - M(1 - p_i) \leq k_i, \quad \forall i \in V, \quad (21d)$$

$$\sum_{i \in V} p_i = |V| - 1, \quad (21e)$$

$$k_i \geq 0, \quad \forall i \in V, \quad (21f)$$

$$p_i \in \{0, 1\}, \quad \forall i \in V. \quad (21g)$$

Then, $\hat{i} \in V$ is a key player of the game if and only if there exists an optimal solution $(\bar{\mathbf{k}}, \bar{\mathbf{p}})$ of (21) such that $\bar{p}_{\hat{i}} = 0$.

Proof: Constraint (21e) imposes that all binary variables p_i are equal to 1 except one, representing the node being eliminated from the network. Constraints (21c)–(21d) are big-M constraints linking variables $\mathbf{k}$ and $\mathbf{p}$. In particular, if $p_i = 0$, then (21c) and (21f) imply $k_i = 0$, while if $p_i = 1$, then (21b) and (21d) imply $k_i = 1 + \sum_{j \in V} \rho A_{ij} k_j$. Hence, Equation (12) holds for any $i \in V$, i.e. $\mathbf{k}$ is the vector of (deterministic) Katz-Bonacich's centralities associated with $\mathbf{p}$. Since Proposition 4.1 guarantees that the set $\mathcal{F}$ of all stochastic Katz-Bonacich's centrality vectors is bounded, then there exists a positive constant M sufficiently large such that constraints (21c) hold for any $i \in V$ with $p_i = 1$ and constraints (21d) hold for any $i \in V$ with $p_i = 0$. Therefore, the feasible solutions of the optimization problem (21) are the pairs $(\mathbf{k}, \mathbf{p})$, where $\mathbf{p} \in \{0, 1\}^{|V|}$ selects all the nodes of the network except one and $\mathbf{k} \in \mathbb{R}^{|V|}$ is the vector of (deterministic) Katz-Bonacich's centralities associated with $\mathbf{p}$. Since the objective function (21a) represents the equilibrium aggregate, it is clear that solving (21) is equivalent to finding a key player of the game. ■

Remark 5.1: In general, the constant M in problem (21) can be chosen equal to $\max_{j \in V} \hat{k}_j$, where $\hat{\mathbf{k}} = (\mathbf{I}_{|V|} - \rho \mathbf{A})^{-1} \mathbf{1}_{|V|}$ is the vector of (deterministic) Katz-Bonacich's centralities associated with $\mathbf{1}_{|V|}$. In fact, given any $\mathbf{p} \in \{0, 1\}^{|V|}$, the inclusion (14) guarantees that the vector $\mathbf{k}(\mathbf{p})$ of Katz-Bonacich's centralities associated with $\mathbf{p}$ satisfies the inequality $\mathbf{k}(\mathbf{p}) \leq \hat{\mathbf{k}}$. Hence, for any $i \in V$ we have:

$$k_i(\mathbf{p}) \leq \hat{k}_i \leq \max_{j \in V} \hat{k}_j,$$

$$1 + \sum_{j \in V} \rho A_{ij} k_j(\mathbf{p}) \leq 1 + \sum_{j \in V} \rho A_{ij} \hat{k}_j = \hat{k}_i \leq \max_{j \in V} \hat{k}_j.$$

In the special case where condition $\rho \max_{i \in V} \deg(i) < 1$ holds, the constant M can be chosen equal to $1/(1 - \rho \max_{i \in V} \deg(i))$, thus avoiding solving the linear system to compute the vector $\hat{\mathbf{k}}$. In fact, if for any $\mathbf{p} \in \{0, 1\}^{|V|}$ we write $\ell = \ell(\mathbf{p})$ to denote an arbitrarily chosen element of $\arg\max_{i \in V} k_i(\mathbf{p})$, then, we have

$$k_\ell(\mathbf{p}) = p_\ell \left(1 + \rho \sum_{j \in V} A_{\ell j} k_j(\mathbf{p}) \right) \leq 1 + \rho \sum_{j \in V} A_{\ell j} k_j(\mathbf{p})$$

$$\leq 1 + \rho k_\ell(\mathbf{p}) \deg(\ell) \leq 1 + k_\ell(\mathbf{p}) \rho \max_{i \in V} \deg(i).$$

Thus, for any $i \in V$ we have:

$$\begin{aligned}
 k_i(\mathbf{p}) &\leq k_\ell(\mathbf{p}) \leq \frac{1}{1 - \rho \max_{i \in V} \deg(i)}, \\
 1 + \sum_{j \in V} \rho A_{ij} k_j(\mathbf{p}) &\leq 1 + \frac{\rho \deg(i)}{1 - \rho \max_{i \in V} \deg(i)} \leq 1 + \frac{\rho \max_{i \in V} \deg(i)}{1 - \rho \max_{i \in V} \deg(i)} \\
 &= \frac{1}{1 - \rho \max_{i \in V} \deg(i)}.
 \end{aligned}$$

Some preliminary numerical tests showed that the direct solution of (21) can be computationally demanding, especially in the case of large-scale networks. In order to speed up problem resolution, we propose the simple approach described in Algorithm 1. First, the continuous relaxation of (21) is solved (line 1). Then, we find the set P of variables p_i equal to 1 in the latter optimal solution (line 2). If P is nonempty (line 3), then a restricted version of the MILP problem (21) is solved to find a good feasible solution and this solution is used as a warm start to solve (21) (lines 4-5). Otherwise, if P is empty (line 6), problem (18) is solved without any warm start (line 7).

In the following, we report some numerical experiments to compare the performance of Algorithm 1 and the standard approach based on the solution of $|V|$ linear systems to find a key player. Algorithm 1 was implemented in MATLAB R2024b and the Gurobi solver (with default options) was used to solve the LP and MILP programs. The linear systems of the standard approach were solved by exploiting the MATLAB function $\backslash$ (i.e. the left division). Instances were produced exploiting the generator of uniformly distributed pseudo-random numbers of MATLAB. The adjacency matrix $\mathbf{A}$ of any random network was generated according to the following rule: for any pair of nodes (i, j) with $i < j$, we set $A_{ij} = A_{ji} = 1$

Algorithm 1: MILP approach

1 Find an optimal solution $(\mathbf{k}^*, \mathbf{p}^*)$ of the continuous relaxation of (21)
 2 $P \leftarrow \{i \in V : p_i^* = 1\}$ /* Find the subset of variables $\mathbf{p}^*$ equal to 1 */

3 **if** $P \neq \emptyset$ **then**

/* Solve problem (21) with additional constraints to get a warm start */

4 Find an optimal solution $(\tilde{\mathbf{k}}, \tilde{\mathbf{p}})$ of the MILP problem

$$\begin{cases} \min_{(\mathbf{k}, \mathbf{p})} & \sum_{i \in V} k_i \\ \text{s. t.} & (21b) - (21g) \\ & p_i = 1 \quad \forall i \in P \end{cases} \quad (22)$$

5 Use $(\tilde{\mathbf{k}}, \tilde{\mathbf{p}})$ as a warm start to solve (21)

6 **else**

7 | Solve (21) without warm start

8 **end**

Table 2. Finding a key player: comparison between the MILP approach of Algorithm 1 and the standard approach based on the solution of $|V|$ linear systems (CPU times in seconds).

$ V $	$\delta = 0.2$		$\delta = 0.5$		$\delta = 0.8$	
	Algorithm 1	Standard approach	Algorithm 1	Standard approach	Algorithm 1	Standard approach
100	0.14	0.02	0.30	0.02	0.62	0.02
200	0.36	0.16	0.59	0.19	3.05	0.17
300	0.86	0.83	1.15	0.87	1.18	0.80
400	0.84	2.06	1.52	2.33	2.13	2.10
500	1.34	4.22	2.41	4.56	3.64	4.25
600	2.00	7.11	4.36	8.02	4.93	8.14
700	3.00	12.61	5.72	13.66	7.10	12.46
800	4.27	15.67	7.48	18.25	9.63	18.08
900	6.18	25.97	11.92	28.05	13.08	25.89
1000	9.40	33.36	15.28	36.61	16.55	33.65
2000	42.07	270.84	61.33	293.62	104.77	277.83
3000	127.97	1079.49	258.37	1147.50	308.56	1097.72
4000	282.14	2947.34	561.93	3046.48	773.94	2916.73
5000	679.27	6457.70	1044.05	6756.47	1270.57	6443.85

if $\text{rand}(1) < \delta$ and $A_{ij} = A_{ji} = 0$ otherwise, where the parameter $\delta \in (0, 1)$ represents the density of the network, e.g. $\delta = 0$ corresponds to an empty network while $\delta = 1$ to a complete network. We considered a set of random instances, generated as described above, where the number of nodes $|V|$ varies from 100 to 5,000 and the density δ equals 0.2, 0.5, or 0.8. In all instances we set $\rho = 0.8/(|V| - 1)$ and $M = 5$ according to the second part of Remark 5.1. The numerical tests were performed on an Apple M1 Max system with 64 GB of RAM running under macOS 15.2.

Table 2 shows the CPU times (in seconds) of the two solution approaches for any value considered of $|V|$ and δ . The figures reported are the average values obtained on a set of ten random instances. We note that in all numerical tests performed, the set P defined in Algorithm 1 was never empty, allowing problem (21) to be efficiently solved with the warm start solution obtained by solving the restricted problem (22). Indeed, the results show that, for large-scale problems, our approach outperforms the standard approach by 5 to 10 times in terms of CPU time.

6. Possible developments

One relevant research direction concerns the possibility of extending the analysis made in this present work to other measures of node centrality, such as the Shapley value, which is associated with suitable cooperative game models whose players are the nodes of an undirected graph (see, e.g. [16]). For instance, one could develop a suitable stochastic cooperative game model on an undirected graph and determine the feasible region of the resulting vectors of Shapley-value centralities obtained by varying a vector of probabilities of node presences, then examine possible relationships between the vectors of Shapley-value centralities arising, respectively, in the stochastic model and in its deterministic counterpart. Another promising research avenue is represented by the vulnerability analysis of congested networks, see, e.g. [17].

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Notes

1. We recall that a graph (also called network) is a collection of nodes (also called vertices) and edges (also called links) that connect pairs of these nodes. The presence/absence of an edge between two nodes i and j is represented by the element A_{ij} of the adjacency matrix $\mathbf{A}$ of the graph. Such an element equals 0 if the edge is absent, otherwise it equals a nonzero real number. For simplicity and to make the presentation uniform, only undirected (unweighted) graphs are considered in this work, i.e. graphs whose edges have no associated direction (nor weights). Such graphs are represented by symmetric adjacency matrices with binary (i.e. either 0, or 1) entries. However, at least part of the analysis could be extended to undirected weighted graphs and directed (unweighted or weighted) graphs.
2. Indeed, bounding from above the spectral radius $\lambda_{\max}(\rho \operatorname{diag}(\mathbf{p})\mathbf{A})$ of the matrix $\rho \operatorname{diag}(\mathbf{p})\mathbf{A}$ with its spectral norm $\|\rho \operatorname{diag}(\mathbf{p})\mathbf{A}\|_2$, and using properties of the spectral norm and the symmetry of the matrices $\operatorname{diag}(\mathbf{p})$ and $\mathbf{A}$ [7, Chapter 5], one gets

$$\begin{aligned}\lambda_{\max}(\rho \operatorname{diag}(\mathbf{p})\mathbf{A}) &\leq \|\rho \operatorname{diag}(\mathbf{p})\mathbf{A}\|_2 \leq \rho \|\operatorname{diag}(\mathbf{p})\|_2 \|\mathbf{A}\|_2 = \rho \lambda_{\max}(\operatorname{diag}(\mathbf{p})) \lambda_{\max}(\mathbf{A}) \\ &\leq \rho \lambda_{\max}(\mathbf{A}) = \lambda_{\max}(\rho \mathbf{A}).\end{aligned}$$

3. Alternatively, finding the unique pure Nash equilibrium for this non-cooperative game can be restated as a variational inequality problem (see, e.g. Passacantando and Raciti [9] and Raciti [10]).
4. We observe that, differently from other equations reported in the article, in this case the expression $\mathbf{A} \operatorname{diag}(\mathbf{p})$ arises, instead of $\operatorname{diag}(\mathbf{p})\mathbf{A}$.
5. Indeed, proceeding likewise in Footnote 2, one gets

$$\begin{aligned}\lambda_{\max}(\rho \mathbf{A} \operatorname{diag}(\mathbf{p})) &\leq \|\rho \mathbf{A} \operatorname{diag}(\mathbf{p})\|_2 \leq \rho \|\mathbf{A}\|_2 \|\operatorname{diag}(\mathbf{p})\|_2 \\ &= \rho \lambda_{\max}(\mathbf{A}) \lambda_{\max}(\operatorname{diag}(\mathbf{p})) \leq \rho \lambda_{\max}(\mathbf{A}) = \lambda_{\max}(\rho \mathbf{A}).\end{aligned}$$

Hence, when $\lambda_{\max}(\rho \mathbf{A}) < 1$, the matrix $(\mathbf{I}_{|V|} - \rho \mathbf{A} \operatorname{diag}(\mathbf{p}))$ cannot be singular.

6. We remark that the problem formulation considered in [6] – hence, the statement of [6, Proposition 2] – includes some additional constraints expressed in terms of various exogenous parameters. Namely: for each $i \in V$, there are a lower bound $\underline{p}_i \geq 0$ on p_i , and an upper bound $\bar{p}_i \leq 1$ on it; the graph G contains C non-overlapping subsets of vertices (regions), and the sum of the stochastic Katz-Bonacich’s centralities of the vertices in each region c is larger than or equal to a certain level $\phi_c \geq 0$. In general, the presence of such restrictions further constrains the set of feasible stochastic Katz-Bonacich’s centralities. Here, in our application of [6, Proposition 2], we make, for all these exogenous parameters, the simplifying choices $\underline{p}_i = 0$, $\bar{p}_i = 1$, and $\phi_c = 0$ (effectively removing such additional constraints) with the aim to relate the sets of feasible stochastic and classical Katz-Bonacich’s centralities, in such a way that the former includes the latter. The same remark holds for our successive application of [6, Proposition 1].
7. Michael Kleder (2024). CON2VERT – constraints to vertices (<https://www.mathworks.com/matlabcentral/fileexchange/7894-con2vert-constraints-to-vertices>), MATLAB Central File Exchange. Retrieved on December 8th, 2024.

8. We remark that, for each $i \in V$, the relationship between k_i and p_i provided by Equation (12) is actually linear, if the other components k_j of the vector $\mathbf{k}$ are taken as fixed. This is due to the absence of self-loops in the graph G . However, the overall map $\mathbf{k} \mapsto \mathbf{p}(\mathbf{k})$ is non-linear in the full vector $\mathbf{k}$.

Author contributions

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Data availability statement

The numerical examples of the article are based on artificial data, reported in the article itself, thus guaranteeing the reproducibility of its results.

References

- [1] Newman M. Networks: an introduction. Oxford, England: Oxford University Press; 2010.
- [2] Michalak TP, Aadithya K, Szczepanski PL, et al. Efficient computation of the Shapley value for game-theoretic network centrality. *J Artif Intel Res*. 2013;46:607–650.
- [3] Ballester C, Calvó-Armengol A, Zenou Y. Who's who in networks. Wanted: the key player. *Econometrica*. 2006;74:1403–1417. doi: [10.1111/j.1468-0262.2006.00709.x](https://doi.org/10.1111/j.1468-0262.2006.00709.x)
- [4] Bloch F, Jackson MO, Tebaldi P. Centrality measures in networks. *Soc Choice Welfare*. 2023;61:413–453. doi: [10.1007/s00355-023-01456-4](https://doi.org/10.1007/s00355-023-01456-4)
- [5] Dudkina E, Bin M, Breen J, et al. A comparison of centrality measures and their role in controlling the spread in epidemic networks. *Int J Control*. 2024;97:1325–1340. doi: [10.1080/00207179.2023.2204969](https://doi.org/10.1080/00207179.2023.2204969)
- [6] Reiffers-Masson A, Altman E, Hayel Y. Controlling the Katz-Bonacich centrality in social network: application to gossip in online social networks. In *Proceedings of the 2015 IEEE/ACM 8th International Conference on Utility and Cloud Computing*, Limassol, Cyprus. 2015, p. 442–447.
- [7] Horn RA, Johnson CR. Matrix analysis. Cambridge, England: Cambridge University Press; 2013.
- [8] Nash JFJ. Equilibrium points in N -Person games. *Proc Natl Acad Sci USA*. 1950;36:48–49. doi: [10.1073/pnas.36.1.48](https://doi.org/10.1073/pnas.36.1.48)
- [9] Passacantando M, Raciti F. A note on generalized nash games played on networks. In T.M. Rassias, editor. *Nonlinear analysis, differential equations, and applications*. Cham: Springer International Publishing; 2021, p. 365–380.
- [10] Raciti F. Some remarks on network games with local average. In M. Bruglieri, P. Festa, G. Macrina, and O. Pisacane, editors. *Optimization in green sustainability and ecological transition (ODS 2023)*. Cham: Springer, AIRO Springer; 2024, p. 25–35.
- [11] Harsanyi J. Games with incomplete information played by “Bayesian” players, I–III. part I. the basic model. *Manage Sci*. 1967;14:159–182. doi: [10.1287/mnsc.14.3.159](https://doi.org/10.1287/mnsc.14.3.159)
- [12] Harsanyi J. Games with incomplete information played by “Bayesian” players, I–III. part II. Bayesian equilibrium points. *Manage Sci*. 1968;14:320–334. doi: [10.1287/mnsc.14.5.320](https://doi.org/10.1287/mnsc.14.5.320)
- [13] Harsanyi J. Games with incomplete information played by “Bayesian” players, I–III. part III. the basic probability distribution of the game. *Manage Sci*. 1968;14:486–502. doi: [10.1287/mnsc.14.7.486](https://doi.org/10.1287/mnsc.14.7.486)
- [14] Bender EA, Williamson SG. Foundations of combinatorics with applications. Mineola, New York: Dover Publications; 2006.
- [15] Papadimitriou CH, Steiglitz K. Combinatorial optimization: algorithms and complexity. Mineola, New York: Dover Publications; 1998.

- [16] Hadas Y, Gnecco G, Sanguineti M. An approach to transportation network analysis via transferable utility games. *Trans Res Part B – Methodol.* 2017;105:120–143. doi: [10.1016/j.trb.2017.08.029](https://doi.org/10.1016/j.trb.2017.08.029)
- [17] Jadamba B, Pappalardo M, Raciti F. Efficiency and vulnerability analysis for congested networks with random data. *J Optim Theory Appl.* 2018;177:563–583. doi: [10.1007/s10957-018-1264-y](https://doi.org/10.1007/s10957-018-1264-y)