IIASA - Sopron, H

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M. chairman, ladies & gentlemen (Y. Svirezhev)

I will provide some stability results pertaining to a composite map ~~identification for control~~, with arising from a problem ~~specific reference to a model~~ of environmental interest.

(P3) the physical process I refer to is steady flow across a one-dimensional, medium (non homogeneous).

In the open interval (x_0, x_1) the potential u, conductivity a and the source term f are related by this ODE.

Nomenclature: a direct problem consists of determining u inside the interval, given a, f and boundary values of u

An inverse problem is met when a has to be found from f, u and supplementary information

(L1) Let me raise the following question:

is the determination of conductivity really the ultimate goal, given the application we have in mind?

The answer is no, as stressed by the following statement by prof T. Seidman

Let us look at this diagram, which summarizes information flow:

~~The identification of coefficients appearing in an ordinary or partial DE is just a part of a procedure: the identified coefficients, together with a new pair of boundary values and a new source term give rise to a direct (a control) problem, the purpose of which is to determine a new result, the~~

First of all, given $\{u, f\}$ and extra information, the distributed parameter $\underline{a}$ has to be determined uniquely - this involves the data-to-parameter map.

Next, the parameter $\underline{a}$ together with a new source term h and new BV is used to determine the new potential v; this involves the parameter-to-result map

What I am interested in is the well-posedness of the overall map.

(L2) And this is the plan.

I will provide some conditions under which $\underline{a}$ is uniquely determined from the available pieces of information.

Since the original flow equation is in fact a 1st order linear ODE wr to $\underline{a}$, if existence of a solution is given, then the uniqueness of $\underline{a}$ can be provided by a Cauchy datum.

Cauchy problems can be either regular or singular this remark provides a unifying view not only over identifiability conditions but also over stability estimates, as we shall see.

About the control map, uniqueness and well-posedness are trivial therefore I can assess the stability of the composite map.

(P4) Before going into details, let me examine the general context.

Conductivity characterizes the ~~physical~~ system out of which ~~first~~ we get both the data (map $\underline{\underline{P}}$) and the results (map $\underline{\underline{Q}}$). Since we have to go from the data to the results, we have to consider the composite map $\underline{\underline{Q}} \circ \underline{\underline{P}}^{-1}$

hence we want $\underline{\underline{P}}^{-1}$ to behave in the best possible way. Among the required properties is continuity of $\underline{\underline{P}}^{-1}$ (often referred to as stability)

This theorem provides a sufficient condition for the continuity of $\underline{\underline{P}}^{-1}$.

The typical situation is depicted by this abstract example, based on Feldman's paper.

Usually the parameter set A is not compact.

however there ∃ a compact imbedding $\underline{\underline{E}}$ of the normed space X into A

Define B_M the sphere of radius M in X. Assume the inverse pbm consists of finding $\bar{a}$ in X ∋ this operator equation is ~~not~~ solved

Define A_* as the closure of the image of B_M according to $\underline{\underline{E}}$.

Assume that prior knowledge leads to seeking for solutions in A_* i.e. that the restriction P_* of P to A_* needs to be considered.

Under these circumstances the Theorem applies i.e., P_*^{-1} is continuous (stable).

(23) So far for a sufficient stability condition holding for the most general situation

Let me go back to the specific problem. I will go through some details and eventually ~~compare the final stability result with the~~ make a comparison between the two procedures; the specific and the general one.

One way of achieving uniqueness for the data-to-parameter map is by providing two suitable data pairs

Everything here is set in classical function spaces. In fact, it could not be otherwise, if this is to make any sense at all.

The two potentials are of class C^1. Let E_u, E_v denote the sets of critical points of u, resp. v.

Let the source terms be in L^1 and let at least one of them not vanish identically which vanish at the left endpoint

Antiderivatives of f, g are denoted by capitals

The set of admissible parameters consists of continuous functions, which are bounded away from zero.

Let there ∃ at least one admissible conductivity s.t., the two following flow equations hold simultaneously.

Let the potentials be nowhere stationary.

Finally let the following independence condition hold. :-

there ∃ a pair of points y_1, y_2 in D, ∋ this dickence does not vanish / the pilot points

then conductivity is uniquely and is represented by this quotient, where the constant c_1 is explicitly shown here as a function of u', v', F, G.

u'_1 stands for the value of u' at y_1 & so on

(L4) Now assume u is stationary at just one point, x_u.

then conductivity can be uniquely identified from the {potential, source term} pair and is given by this quotient.

this Cauchy problem is singular, because it relies on information supplied at the point x_u, where the coefficient u' of a' vanishes.

this is the simplest example of the uniqueness conditions K + N stated in 1977

(L5) Let me come to stability for the composite map.

to begin with, I assume uniqueness is due to independence.

Information flow is summarized here.

The two top data pairs, where v is named the reference potential, yield a; this a enters the direct problem when combined with these BVs and this source term the result is r, the reference result.

If I replace v by say a "noisy" replica w, s.t. the independence condition continues to hold at the same pilot points, and leave everything else unchanged in the bottom data pair, then I get another conductivity, b. the corresponding potential is s, the "noisy" result.

The purpose of a stability estimate is to find suitable Banach spaces, V and Z ∋ an inequality of this type is established. The quantity $C_\pm$ here is independent of v, w and may depend on the listed quantities.

It shall be pointed out that the most general stability estimate may have a more complicated form

(2.6) ~~In order to achieve~~ this goal, additional hypotheses are requested. Essentially, I need a-priori bounds ...

– for L^1 norms of f & g

– for the abs values of derivatives of u & v evaluated at the pilot points

This abs value shall be bounded away from zero; this makes sense. In fact, it distinguishes regular Cauchy problems from singular ones.

Moreover these determinants shall comply with this bound and finally, the reference conductivity shall be bounded above. Note that no such inequality is needed for the second conductivity σ; if this were the case, every estimate would become almost trivial!

This is the result: a uniform $W^{1,\infty}$ estimate is established for $s-r$. The data "space" is just a set of number pairs.

The difference between derivatives is estimated as follows.

I did not choose a more synthetic way to write the inequality, because ~~Note that track has been kept for all constants appearing in the hypotheses~~ | I wanted to keep track of all constants defined in the hypotheses.

Similarly, the $W^{1,\infty}$ estimate for $s-r$ follows in a straightforward way, because the L^∞ norm of $s'-r'$ is equivalent to the $W^{1,\infty}$ norm, at least in connection with the direct two point BV problem.

(2.7) A similar, though simpler scheme applies to the data to results map when uniqueness comes from a singular Cauchy problem. Only 2 data pairs are needed.

(2.8) The regularizing hypothesis, which makes the difference from the regular Cauchy case is this:

Since u' and v' shall vanish somewhere, the only bound I can apply is of integral type. For simplicity I choose to control the L^1 norm of the reciprocals. In fact I could do the same by selecting an L^p norm, where p is finite.

As a consequence, the involved Banach spaces are $Z = W^{1,1}$ and $V = W^{1,\infty}$.

Assume u' and v' vanish at one and the same point, then the L^1 norm of $s'-r'$ is estimated as shown here.

An inequality for the $W^{1,1}$ norm of $s-r$ is obtained by recalling the already mentioned equivalence property between norms.

(L9) The stability estimates obtained so far summarize as follows.

– when the uniqueness of $\underline{a}$ is due to a regular Cauchy problem, then estimates for the ~~composite~~ (data-to-parameter $\underline{\underline{P}}^{-1}$) map are of uniform type

– when ~~the~~ uniqueness comes from a singular Cauchy problem, estimates are of integral type, L^1 of L^p, with p ~~ranging from~~ greater than 1 ~~but~~ and finite.

Since the ~~composite map~~ parameter to result (8) map is well posed, the composite map inherits the stability properties of $\underline{\underline{P}}^{-1}$.

(P5) Let me go back to the general framework – I may now wonder whether there has been any gain in going through all of the above details instead of just applying a general compactness argument

If I had stuck to the general scheme, then I ought to have looked for a compact subset of the admissible parameter set.

– By e.g. requiring f to be continuous and uniformly bounded

– u to be twice differentiable and bounded as shown here, then

the compactness of this set would have been obtained by applying Arzelà's theorem.

The hypotheses I have just cited shall be compared to those which I have actually used.
There has been a substantial gain, which is emphasized by this synopsis.
In other words, a proof based on the explicit representation of the solutions has been worthwhile.

To conclude, let me warmly thank IIASA, the meeting organizers in particular for inviting me, and
- this Italian research project for providing partial support

Thank You for Your attention.

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