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Variance of Lattice Point Counting in Thin Annuli

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Abstract
We give asymptotic estimates of the variance of the number of integer points in translated thin annuli in any dimension.

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1 Introduction

Sinai proved in [17] that the number of integer points in the plane inside a thin annulus of fixed area λ, of random shape and large random radius, with a suitable definition of randomness, converges in distribution to a Poisson random variable with parameter λ. The probabilistic proof does not exhibit a specific annulus. See also [13,14]. Indeed in [13] it is shown that the number of integer points in the circular annulus $\{r - 1/4r < |x| \leq r + 1/4r\}$ in the plane does not converge to a Poisson distribution when r varies randomly and uniformly in $[a_1 L, a_2 L]$ and L goes to $+\infty$. The reason is that, under the condition that the annulus contains some integer points, then with probability almost one the number of integer points in the annulus tends to infinity. On the other hand,

Dedicated to Guido Weiss, our teacher and friend. Grazie Guido.

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a translation of the annulus breaks the symmetry, and the situation changes. Indeed Cheng et al. proved in [3] that if Ω is a convex set in the plane with a smooth boundary with positive curvature, then the expectation and variance for the number of integer points in a shifted annular region of radius r and thickness c/r

$$[(r+c/(2r))\,\Omega - x] \setminus [(r-c/(2r))\,\Omega - x],$$

where x is uniformly distributed in the unit square, are both asymptotic to the area of the annulus $2c\,|\Omega|$ as c is fixed and $r \to +\infty$. Since the mean and the variance of a Poisson distribution coincide, this is consistent with the conjecture that this random variable converges in distribution to a Poisson random variable. Indeed these authors briefly mention higher dimensional analogues. The following is a proof of these higher dimensional analogues via Fourier analysis.

Theorem 1.1 *Assume that Ω is a convex body in $\mathbb{R}^d$ with smooth boundary with everywhere positive Gaussian curvature, which contains in its interior the origin. Denote by $\Omega(r,t)$ the annulus $(r+(t/2))\,\Omega \setminus (r-(t/2))\,\Omega$, and by $|\Omega(r,t)|$ its volume. Then for every $\alpha > (d-1)/(d+1)$ there exists $0 < \beta < 1$ and a positive constant C such that for every $1 \le r < +\infty$ and every $0 < t \le r^{-\alpha}$ one has*

$$\left| \int_{\mathbb{T}^d} \left| \sum_{k\in\mathbb{Z}^d} \chi_{\Omega(r,t)-x}(k) - |\Omega(r,t)| \right|^2 \mathrm{d}x - |\Omega(r,t)| \right| \le C\,|\Omega(r,t)|\,t^{\beta}.$$

The mean of the random variable that counts the number of integer points in the annulus is the volume of the annulus, hence the above is an estimate of the variance of this random variable. In particular, the theorem can be rephrased by saying that the expectation and variance of the number of integer points in translated annuli are asymptotic when $r \to +\infty$ and $t \to 0+$, with $t \le r^{-\alpha}$ for some $\alpha > (d-1)/(d+1)$. Observe that when $t = o(r)$, then

$$\begin{aligned} |\Omega(r,t)| &= |(r+t/2)\,\Omega| - |(r-t/2)\,\Omega| \\ &= \left((r+t/2)^d - (r-t/2)^d\right)|\Omega| \sim d r^{d-1} t\,|\Omega|. \end{aligned}$$

In particular, under the assumption that $0 < t \le r^{-\alpha}$ with $\alpha > (d-1)/(d+1)$, the measure of the annulus $|\Omega(r,t)| \sim d r^{d-1} t\,|\Omega|$ may diverge. Also observe that with the above theorem in dimension $d=2$ and with $r = c/t$ one recovers the results in [3], and indeed the assumption $t = c/r$ can be replaced by the weaker assumption $t \le r^{-\alpha}$ for some $\alpha > 1/3$. We do not know if this assumption $0 < t \le r^{-\alpha}$ with $\alpha > (d-1)/(d+1)$ can be weakened, but it follows from some results in [15] that the only assumption that the widths of the annuli converge to zero does not suffice, and one has to require a suitable speed. Finally, also the curvature assumption is necessary. The variance of annuli with boundary points of zero curvature may be much larger than the mean, and an asymptotic estimate of the variance may fail. An example are the annuli generated by polyhedra with faces with rational orientation. See Remark 3.2 below.

2 Proof of the Main Result

The main tool in our proof is the Fourier expansion of the random variable that counts the integer points. As shown by Kendall in [9], an estimate from above of the variance of the number of integer points in shifted ovals follows easily from estimates of the order of decay of the Fourier transform of an oval. Here, in order to obtain an asymptotic for the variance, we shall need to extract from the Fourier transform more precise geometric informations. The proof is split in a number of lemmas. The first two lemmas, Lemma 2.1 on the Fourier expansion of the discrepancy function, and Lemma 2.2 on the asymptotic expansion of the Fourier transform of a convex set, are well known and the proofs are included only for the sake of completeness. Lemma 2.3 on the Fourier transform of an annulus, and Lemma 2.4 on the asymptotic expansion of the variance, are easy consequences of the first two lemmas and allow us to split the variance into a sum of different terms. The actual estimate of these terms is the core of the proof and is contained in the remaining lemmas.

Lemma 2.1 *If Ω is a bounded domain in $\mathbb{R}^d$, then the number of integer points in $\Omega - x$*

$$\sum_{k\in\mathbb{Z}^d} \chi_{\Omega-x}(k)$$

is a periodic function of the translation x, and it has the Fourier expansion

$$\sum_{n\in\mathbb{Z}^d} \widehat{\chi}_{\Omega}(n) \exp(2\pi i n x) .$$

In particular, this Fourier expansion converges in the square metric, and

$$\int_{\mathbb{T}^d} \left| \sum_{k\in\mathbb{Z}^d} \chi_{\Omega-x}(k) - |\Omega| \right|^2 \mathrm{d}x = \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |\widehat{\chi}_{\Omega}(n)|^2 .$$

Proof The first part is just the Poisson summation formula. See [18, Theorem 2.4 in Chapter VII]. Here is a quick proof. Letting $Q = \left\{-1/2 \le x_j < 1/2\right\}$ be the unit cube, then

$$\begin{aligned}\int_{\mathbb{T}^d} \sum_{k\in\mathbb{Z}^d} \chi_{\Omega-x}(k) \exp(-2\pi i n x)\,\mathrm{d}x &= \int_Q \sum_{k\in\mathbb{Z}^d} \chi_{\Omega}(x+k) \exp(-2\pi i n x)\,\mathrm{d}x \\ = \sum_{k\in\mathbb{Z}^d} \int_{Q+k} \chi_{\Omega}(x) \exp(-2\pi i n x)\,\mathrm{d}x &= \int_{\mathbb{R}^d} \chi_{\Omega}(x) \exp(-2\pi i n x)\,\mathrm{d}x = \widehat{\chi}_{\Omega}(n) .\end{aligned}$$

Since Ω is bounded, the number of non zero terms in the above sums is finite, and the above identities are fully justified. The final part of the lemma is Parseval's identity, just observe that $\widehat{\chi}_{\Omega}(0) = |\Omega|$. $\square$

We emphasize that the above lemma does not claim that the Fourier expansions of the random variables converge pointwise. Anyhow, these series are summable pointwise with suitably strong summability methods at every point x with $\mathbb{Z}^d \cap \partial\{\Omega - x\} = \varnothing$, where the function $\sum_{k\in\mathbb{Z}^d} \chi_{\Omega-x}(k)$ is smooth. If $\varphi(\xi)$ is a smooth even radial function with compact support and with $\varphi(0) = 1$, then at every point of continuity of the function $\sum_{k\in\mathbb{Z}^d} \chi_{\Omega-x}(k)$ one has

$$\lim_{\varepsilon\to 0+}\left\{\sum_{k\in\mathbb{Z}^d} \varphi(\varepsilon n)\widehat{\chi}_{\Omega}(n)\exp(2\pi i n x)\right\} = \sum_{k\in\mathbb{Z}^d} \chi_{\Omega-x}(k).$$

Indeed it can be shown that in dimensions $d = 1$ and $d = 2$ and for domains with smooth boundaries the above Fourier expansions are pointwise spherically convergent, that is the above equality holds also when $\varphi(\xi)$ is the characteristic function of the unit ball $\{|\xi| \leq 1\}$, but this is not necessarily the case if $d \geq 3$.

The above lemma suggests the search of precise estimates of the Fourier transform of an annulus. In order to guess the correct result, it may be helpful to have an explicit example. The Fourier transform of the sphere $\{|x| \leq r\}$ is a Bessel function,

$$\widehat{\chi}_{\{|x|\leq r\}}(\xi) = r^d \widehat{\chi}_{\{|x|\leq 1\}}(r\xi) = r^d |r\xi|^{-d/2} J_{d/2}(2\pi r|\xi|).$$

See [18, Theorem 4.15, Chapter IV]. Hence, the Fourier transform of the annulus $\{r - t/2 < |x| \leq r + t/2\}$ is

$$\begin{aligned}
&\widehat{\chi}_{\{r-t/2<|x|\leq r+t/2\}}(\xi) \\
&\quad = \widehat{\chi}_{\{|x|\leq r+t/2\}}(\xi) - \widehat{\chi}_{\{|x|\leq r-t/2\}}(\xi) \\
&\quad = r^{d/2}|\xi|^{-d/2}\left(J_{d/2}(2\pi(r+t/2)|\xi|) - J_{d/2}(2\pi(r-t/2)|\xi|)\right) \\
&\qquad + \left((r+t/2)^{d/2} - r^{d/2}\right)|\xi|^{-d/2} J_{d/2}(2\pi(r+t/2)|\xi|) \\
&\qquad - \left((r-t/2)^{d/2} - r^{d/2}\right)|\xi|^{-d/2} J_{d/2}(2\pi(r-t/2)|\xi|).
\end{aligned}$$

Recall the asymptotic expansions of Bessel functions, for z real and positive,

$$\begin{aligned}
J_\nu(z) &= 2^{1/2}\pi^{-1/2}z^{-1/2}\cos(z - \pi(2\nu+1)/4) + \mathcal{O}\left(z^{-3/2}\right), \\
\frac{d}{\mathrm{d}z}J_\nu(z) &= 2^{-1}\left(J_{\nu-1}(z) - J_{\nu+1}(z)\right) \\
&= -2^{1/2}\pi^{-1/2}z^{-1/2}\sin(z - \pi(2\nu+1)/4) + \mathcal{O}\left(z^{-3/2}\right).
\end{aligned}$$

Then, from these formulas and with some trigonometry, one obtains the asymptotic expansion of the Fourier transform of a spherical shell,

$$\begin{aligned}&\widehat{\chi}_{\{r-t/2<|x|\leq r+t/2\}}\left(\xi\right)\\&\quad=2\pi^{-1}r^{(d-1)/2}\left|\xi\right|^{-(d+1)/2}\cos\left(2\pi r\left|\xi\right|-\pi\left(d-1\right)/4\right)\sin\left(\pi t\left|\xi\right|\right)\\&\qquad+\mathcal{O}\left(r^{(d-3)/2}t\left|\xi\right|^{-(d+1)/2}\right).\end{aligned}$$

When the dimension of the space is odd, the Bessel functions can be written explicitly in terms of trigonometric functions, and one can also obtain an exact formula for this Fourier transform in terms of elementary functions. The behavior of the Fourier transforms of convex bodies and annuli is similar, although a bit more complicated.

Lemma 2.2 *The Fourier transform of a characteristic function of a convex body Ω in $\mathbb{R}^d$ with smooth boundary with everywhere positive Gaussian curvature for $|\xi|\to+\infty$ has the asymptotic expansion*

$$\widehat{\chi}_{\Omega}\left(\xi\right)=a\left(\xi\right)\left|\xi\right|^{-(d+1)/2}+E\left(\xi\right).$$

If $\sigma\left(\pm\xi\right)$ are the points of the boundary of Ω with outward unit normals $\pm\xi/\left|\xi\right|$, and if $K\left(\sigma\left(\pm\xi\right)\right)$ are the Gaussian curvatures at the points $\sigma\left(\pm\xi\right)$, then

$$\begin{aligned}a\left(\xi\right)=&\left(2\pi i\right)^{-1}\exp\left(-2\pi i\sigma\left(-\xi\right)\cdot\xi-\pi i\left(d-1\right)/4\right)K\left(\sigma\left(-\xi\right)\right)^{-1/2}\\&-\left(2\pi i\right)^{-1}\exp\left(-2\pi i\sigma\left(\xi\right)\cdot\xi+\pi i\left(d-1\right)/4\right)K\left(\sigma\left(\xi\right)\right)^{-1/2}.\end{aligned}$$

The remainder $E\left(\xi\right)$ satisfies the estimates

$$\left|E\left(\xi\right)\right|+\left|\nabla E\left(\xi\right)\right|\leq C\left|\xi\right|^{-(d+3)/2}.$$

Proof This is a classical result. See [4–6], or [7, Corollary 7.7.15], or [19, Chapter VIII]. In particular, as shown before, the lemma for a ball follows straightly from the asymptotic expansion of Bessel functions. Anyhow, since in most references the exact constants in this asymptotic expansion are not explicit and a control on the derivative of the remainder is omitted, it may be helpful to recall a proof. Write $\xi=\rho\vartheta$, with $\rho>0$ and $|\vartheta|=1$, and denote by $n\left(x\right)$ the outward unit normal to the boundary at the point x. By the divergence theorem,

$$\int_{\Omega}\exp\left(-2\pi i\rho\vartheta\cdot x\right)\mathrm{d}x=-\left(2\pi i\rho\right)^{-1}\int_{\partial\Omega}\vartheta\cdot n\left(x\right)\exp\left(-2\pi i\rho\vartheta\cdot x\right)\mathrm{d}x.$$

In the surface integral the phase $\vartheta\cdot x$ is stationary at the points $\sigma\left(\pm\vartheta\right)$ with normals $\pm\vartheta$, and one can isolate these points with a smooth cutoff $\varphi\left(s\right)$, with $\varphi\left(s\right)=0$ if

$s \leq 1-2\varepsilon$ and $\varphi(s)=1$ if $s \geq 1-\varepsilon$ for some small positive ε,

$$\begin{aligned}
&\int_{\partial\Omega} \vartheta \cdot n(x) \exp(-2\pi i \rho \vartheta \cdot x)\,\mathrm{d}x \\
&\quad = \int_{\partial\Omega} \varphi(\vartheta \cdot n(x))\, \vartheta \cdot n(x) \exp(-2\pi i \rho \vartheta \cdot x)\,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} \varphi(-\vartheta \cdot n(x))\, \vartheta \cdot n(x) \exp(-2\pi i \rho \vartheta \cdot x)\,\mathrm{d}x \\
&\qquad + \int_{\partial\Omega} (1-\varphi(\vartheta \cdot n(x)) - \varphi(-\vartheta \cdot n(x)))\, \vartheta \cdot n(x) \exp(-2\pi i \rho \vartheta \cdot x)\,\mathrm{d}x.
\end{aligned}$$

Since in the domain of integration of the third integral there are no critical points, this integral decays faster than any power ρ^{-N} when $\rho \to +\infty$, and the same is true for the derivatives of this integral. The first and second integrals are similar to each other. Let us consider the first one. By a suitable choice of the coordinates $x = \sigma(\vartheta) + (y, z)$, with $y \in \mathbb{R}^{d-1}$ and $z \in \mathbb{R}$, one can move the singular point of the phase to the origin, and one can assume that in a neighborhood of the origin the boundary $\partial\Omega$ is the graph $z = \Phi(y)$ and the unit normal at the origin is $(0, -1)$. In particular, $\nabla\Phi(0) = 0$. Then, setting $(0, -1) \cdot n(x) = n(y)$, one obtains

$$\begin{aligned}
&\int_{\partial\Omega} \varphi(\vartheta \cdot n(x))\, \vartheta \cdot n(x) \exp(-2\pi i \rho \vartheta \cdot x)\,\mathrm{d}x \\
&\quad = \exp(-2\pi i \rho \sigma(\vartheta) \cdot \vartheta) \int_{\mathbb{R}^{d-1}} \varphi(n(y))\, n(y) \exp(2\pi i \rho \Phi(y)) \sqrt{1+|\nabla\Phi(y)|^2}\,\mathrm{d}y.
\end{aligned}$$

By [19, Proposition 6, Chapter VIII, §2], if $\{\mu_k\}_{k=1}^{d-1}$ are the eigenvalues of the Hessian matrix $\left[\partial\Phi(y)/\partial y_i \partial y_j\right]$ at the point $y=0$, then

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \varphi(n(y))\, n(y) \exp(2\pi i \rho \Phi(y)) \sqrt{1+|\nabla\Phi(y)|^2}\,\mathrm{d}y \\
&\quad = \rho^{-(d-1)/2} \prod_{k=1}^{d-1} (-i\mu_k)^{-1/2} + \mathcal{O}\left(\rho^{-(d+1)/2}\right).
\end{aligned}$$

The eigenvalues of the Hessian matrix are the principal curvatures of $\partial\Omega$ at $\sigma(\vartheta)$, and the product of these eigenvalues is the Gaussian curvature,

$$\prod_{k=1}^{d-1} (-i\mu_k)^{-1/2} = \exp((d-1)\pi i/4)\, K(\sigma(\vartheta))^{-1/2}.$$

Hence,

$$
\begin{aligned}
&-(2\pi i\rho)^{-1}\int_{\partial\Omega}\varphi\left(\vartheta\cdot n\left(x\right)\right)\vartheta\cdot n\left(x\right)\exp\left(-2\pi i\rho\vartheta\cdot x\right)\mathrm{d}x\\
&\quad=-\left(2\pi i\right)^{-1}\exp\left(-2\pi i\sigma\left(\xi\right)\cdot\xi+\left(d-1\right)\pi i/4\right)K\left(\sigma\left(\xi\right)\right)^{-1/2}\left|\xi\right|^{-(d+1)/2}\\
&\qquad+\mathcal{O}\left(\left|\xi\right|^{-(d+3)/2}\right).
\end{aligned}
$$

In order to obtain the main term in the asymptotic expansion one has to sum the contribution of the point $\sigma\left(\vartheta\right)$ with the one of the antipodal point $\sigma\left(-\vartheta\right)$. In this way one obtains the decomposition

$$\widehat{\chi}_{\Omega}\left(\xi\right)=a\left(\xi\right)\left|\xi\right|^{-(d+1)/2}+E\left(\xi\right).$$

The remainder has the property $\left|E\left(\xi\right)\right|\leq C\left|\xi\right|^{-(d+3)/2}$ as $\left|\xi\right|\to\infty$. Since $\widehat{\chi}_{\Omega}\left(\xi\right)$ is an entire function of finite exponential type, the above equality can be differentiated and one obtains

$$\nabla\widehat{\chi}_{\Omega}\left(\xi\right)=\left|\xi\right|^{-(d+1)/2}\nabla a\left(\xi\right)-((d+1)/2)a\left(\xi\right)\left|\xi\right|^{-(d+5)/2}\xi+\nabla E\left(\xi\right).$$

This is the same as

$$\nabla E\left(\xi\right)=\nabla\widehat{\chi}_{\Omega}\left(\xi\right)-\left|\xi\right|^{-(d+1)/2}\nabla a\left(\xi\right)+((d+1)/2)a\left(\xi\right)\left|\xi\right|^{-(d+5)/2}\xi.$$

The term $((d+1)/2)a\left(\xi\right)\left|\xi\right|^{-(d+5)/2}\xi$ is $\mathcal{O}\left(\left|\xi\right|^{-(d+3)/2}\right)$, and both terms $\nabla\widehat{\chi}_{\Omega}\left(\xi\right)$ and $\left|\xi\right|^{-(d+1)/2}\nabla a\left(\xi\right)$ are $\mathcal{O}\left(\left|\xi\right|^{-(d+1)/2}\right)$, but the main parts of these last terms are the same and they cancel, and what is left is $\mathcal{O}\left(\left|\xi\right|^{-(d+3)/2}\right)$. Let us first identify the main part of $\left|\xi\right|^{-(d+1)/2}\nabla a\left(\xi\right)$ that comes from the point $\sigma\left(\vartheta\right)$. Recall that $\sigma\left(\xi\right)\cdot\xi=\sup_{x\in\Omega}\left\{x\cdot\xi\right\}$, the support function of the convex body, has gradient $\nabla\left(\sigma\left(\xi\right)\cdot\xi\right)=\sigma\left(\xi\right)$. See [1], or [16, Corollary 1.7.3]. Hence,

$$
\begin{aligned}
&\nabla\left(-\left(2\pi i\right)^{-1}\exp\left(-2\pi i\sigma\left(\xi\right)\cdot\xi+\left(d-1\right)\pi i/4\right)K\left(\sigma\left(\xi\right)\right)^{-1/2}\right)\\
&\quad=-\left(2\pi i\right)^{-1}\exp\left(-2\pi i\sigma\left(\xi\right)\cdot\xi+\left(d-1\right)\pi i/4\right)\nabla\left(K\left(\sigma\left(\xi\right)\right)^{-1/2}\right)\\
&\qquad+\exp\left(-2\pi i\sigma\left(\xi\right)\cdot\xi+\left(d-1\right)\pi i/4\right)K\left(\sigma\left(\xi\right)\right)^{-1/2}\sigma\left(\xi\right).
\end{aligned}
$$

Since $\sigma\left(\xi\right)$ is homogeneous of degree 0, $\nabla\left(K\left(\sigma\left(\xi\right)\right)^{-1/2}\right)$ is homogeneous of degree -1, so that the main contribution to $\left|\xi\right|^{-(d+1)/2}\nabla a\left(\xi\right)$ that comes from the point $\sigma\left(\vartheta\right)$ is

$$\exp\left(-2\pi i\sigma\left(\xi\right)\cdot\xi+\left(d-1\right)\pi i/4\right)K\left(\sigma\left(\xi\right)\right)^{-1/2}\left|\xi\right|^{-(d+1)/2}\sigma\left(\xi\right).$$

Let us now identify the main part of $\nabla\widehat{\chi}_{\Omega}\left(\xi\right)$ that comes from the point $\sigma\left(\vartheta\right)$. The gradient $\nabla\widehat{\chi}_{\Omega}\left(\xi\right)$ is defined by an integral similar to the one that defines $\widehat{\chi}_{\Omega}\left(\xi\right)$, and it has a similar asymptotic expansion,

$$\nabla\left(\int_{\Omega}\exp\left(-2\pi i\xi\cdot x\right)\mathrm{d}x\right)=-2\pi i\int_{\Omega}x\exp\left(-2\pi i\xi\cdot x\right)\mathrm{d}x$$
$$=-\left|\xi\right|^{-2}\xi\int_{\Omega}\exp\left(-2\pi i\xi\cdot x\right)\mathrm{d}x+\left|\xi\right|^{-2}\int_{\partial\Omega}x\exp\left(-2\pi i\xi\cdot x\right)\xi\cdot n\left(x\right)\mathrm{d}x.$$

The first integral is similar to the previous one, but the factor $|\xi|^{-2}\,\xi$ gives an extra decay,

$$\left|\left|\xi\right|^{-2}\xi\int_{\Omega}x\exp\left(-2\pi i\xi\cdot x\right)\mathrm{d}x\right|\leq C\left|\xi\right|^{-(d+3)/2}.$$

Arguing as before and isolating the critical point $\sigma\left(\vartheta\right)$, with the change of variables $x=\sigma\left(\vartheta\right)+\left(y,z\right)$ one obtains

$$\begin{aligned}
&\rho^{-1}\int_{\partial\Omega}x\varphi\left(\vartheta\cdot n\left(x\right)\right)\vartheta\cdot n\left(x\right)\exp\left(-2\pi i\rho\vartheta\cdot x\right)\mathrm{d}x\\
&\quad=\rho^{-1}\exp\left(-2\pi i\rho\sigma\left(\vartheta\right)\cdot\vartheta\right)\\
&\qquad\times\int_{\mathbb{R}^{d-1}}\left(y,\Phi\left(y\right)\right)\varphi\left(n\left(y\right)\right)n\left(y\right)\exp\left(2\pi i\rho\Phi\left(y\right)\right)\sqrt{1+\left|\nabla\Phi\left(y\right)\right|^{2}}\mathrm{d}y\\
&\qquad+\rho^{-1}\exp\left(-2\pi i\rho\sigma\left(\vartheta\right)\cdot\vartheta\right)\\
&\qquad\times\sigma\left(\vartheta\right)\int_{\mathbb{R}^{d-1}}\varphi\left(n\left(y\right)\right)n\left(y\right)\exp\left(2\pi i\rho\Phi\left(y\right)\right)\sqrt{1+\left|\nabla\Phi\left(y\right)\right|^{2}}\mathrm{d}y.
\end{aligned}$$

In the first integral the factor $\left(y,\Phi\left(y\right)\right)$ vanishes at the singular point $y=0$ of the phase, and again by [19, Proposition 6, Chapter VIII, §2] and the note that follows it, this implies that

$$\begin{aligned}
&\left|\rho^{-1}\exp\left(-2\pi i\rho\sigma\left(\vartheta\right)\cdot\vartheta\right)\right|\\
&\qquad\times\left|\int_{\mathbb{R}^{d-1}}\left(y,\Phi\left(y\right)\right)\varphi\left(n\left(y\right)\right)n\left(y\right)\exp\left(2\pi i\rho\Phi\left(y\right)\right)\sqrt{1+\left|\nabla\Phi\left(y\right)\right|^{2}}\mathrm{d}y\right|\\
&\quad\leq C\rho^{-(d+3)/2}.
\end{aligned}$$

In other words, one obtains the exponent $(d+3)/2$ rather than the usual exponent $(d+1)/2$ because the amplitude of the oscillatory integral vanishes at the critical point.

The second integral is exactly the same that appears in the computation of $\widehat{\chi}_\Omega(\xi)$,

$$\begin{aligned}
&\rho^{-1}\exp\left(-2\pi i\rho\sigma(\vartheta)\cdot\vartheta\right)\\
&\quad\times\sigma(\vartheta)\int_{\mathbb{R}^{d-1}}\varphi(n(y))\,n(y)\exp(2\pi i\rho\Phi(y))\sqrt{1+|\nabla\Phi(y)|^2}\mathrm{d}y\\
&\;=\exp\left(-2\pi i\rho\sigma(\vartheta)\cdot\vartheta+(d-1)\pi i/4\right)K(\sigma(\vartheta))^{-1/2}\rho^{-(d+1)/2}\sigma(\vartheta)\\
&\quad+\mathcal{O}\left(\rho^{-(d+3)/2}\right).
\end{aligned}$$

In conclusion, the main parts of $\nabla\widehat{\chi}_\Omega(\xi)$ and $|\xi|^{-(d+1)/2}\nabla a(\xi)$ cancel, and all that is left is $\mathcal{O}\left(|\xi|^{-(d+3)/2}\right)$. □

Lemma 2.3 *The Fourier transform of the annulus $\Omega(r,t)=(r+t/2)\Omega\setminus(r-t/2)\Omega$ can be decomposed into*

$$\widehat{\chi}_{\Omega(r,t)}(\xi)=A(r,t,\xi)+B(r,t,\xi).$$

The main term is

$$\begin{aligned}
A(r,t,\xi)=&-\pi^{-1}r^{(d-1)/2}|\xi|^{-(d+1)/2}\exp\left(-2\pi ir\sigma(-\xi)\cdot\xi-\pi i(d-1)/4\right)\\
&\times K(\sigma(-\xi))^{-1/2}\sin(\pi t\sigma(-\xi)\cdot\xi)\\
&+\pi^{-1}r^{(d-1)/2}|\xi|^{-(d+1)/2}\exp\left(-2\pi ir\sigma(\xi)\cdot\xi+\pi i(d-1)/4\right)\\
&\times K(\sigma(\xi))^{-1/2}\sin(\pi t\sigma(\xi)\cdot\xi).
\end{aligned}$$

The remainder has the property that there exists $C>0$ such that for every $r|\xi|\geq 1$ and for every $0<t\leq r$,

$$|B(r,t,\xi)|\leq Cr^{(d-3)/2}t\,|\xi|^{-(d+1)/2}.$$

Proof With the notation of the previous lemma $\widehat{\chi}_\Omega(\xi)=a(\xi)|\xi|^{-(d+1)/2}+E(\xi)$,

$$\begin{aligned}
\widehat{\chi}_{\Omega(r,t)}(\xi)=&(r+t/2)^d\,\widehat{\chi}_\Omega((r+t/2)\xi)-(r-t/2)^d\,\widehat{\chi}_\Omega((r-t/2)\xi)\\
=&\,r^{(d-1)/2}\left(a((r+t/2)\xi)-a((r-t/2)\xi)\right)|\xi|^{-(d+1)/2}\\
&+\left((r+t/2)^{(d-1)/2}-r^{(d-1)/2}\right)a((r+t/2)\xi)\,|\xi|^{-(d+1)/2}\\
&-\left((r-t/2)^{(d-1)/2}-r^{(d-1)/2}\right)a((r-t/2)\xi)\,|\xi|^{-(d+1)/2}\\
&+(r+t/2)^d\left(E((r+t/2)\xi)-E((r-t/2)\xi)\right)\\
&+\left((r+t/2)^d-(r-t/2)^d\right)E((r-t/2)\xi).
\end{aligned}$$

The estimates on $E(\xi)$ and on $\nabla E(\xi)$ give

$$\left|\left((r+t/2)^d-(r-t/2)^d\right)E\left((r-t/2)\,\xi\right)\right| \le Cr^{(d-5)/2}t\,|\xi|^{-(d+3)/2},$$
$$\left|(r+t/2)^d\left(E\left((r+t/2)\,\xi\right)-E\left((r-t/2)\,\xi\right)\right)\right| \le Cr^{(d-3)/2}t\,|\xi|^{-(d+1)/2}.$$

Similarly, one also has

$$\left|\left((r\pm t/2)^{(d-1)/2}-r^{(d-1)/2}\right)a\left((r\pm t/2)\,\xi\right)|\xi|^{-(d+1)/2}\right|$$
$$\le Cr^{(d-3)/2}t\,|\xi|^{-(d+1)/2}.$$

The main term comes from $a((r+t/2)\,\xi)-a((r-t/2)\,\xi)$, and it needs a slightly more precise analysis. Since $\sigma(\pm\xi)$ is homogeneous of degree zero, one has $\sigma(\pm(r\pm t/2)\,\xi)=\sigma(\pm\xi)$, and a little computation gives

$$\begin{aligned}
&a\left((r+t/2)\,\xi\right)-a\left((r-t/2)\,\xi\right)\\
&\quad=-\pi^{-1}\exp\left(-2\pi ir\sigma(-\xi)\cdot\xi-\pi i(d-1)/4\right)K\left(\sigma(-\xi)\right)^{-1/2}\sin\left(\pi t\sigma(-\xi)\cdot\xi\right)\\
&\qquad+\pi^{-1}\exp\left(-2\pi ir\sigma(\xi)\cdot\xi+\pi i(d-1)/4\right)K\left(\sigma(\xi)\right)^{-1/2}\sin\left(\pi t\sigma(\xi)\cdot\xi\right).
\end{aligned}$$

$\square$

At this point one can already show that the variance is bounded up to a constant by the mean. Indeed, it follows from the above lemma that if $t\le r$ and $r|\xi|\ge 1$, then

$$\left|\widehat{\chi}_{\Omega(r,t)}(\xi)\right| \le Cr^{(d-1)/2}|\xi|^{-(d+1)/2}\min\{1,t\,|\xi|\}.$$

Hence, by Parseval's equality,

$$\begin{aligned}
&\int_{\mathbb{T}^d}\left|\sum_{k\in\mathbb{Z}^d}\chi_{\Omega(r,t)-x}(k)-|\Omega(r,t)|\right|^2 \mathrm{d}x=\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\left|\widehat{\chi}_{\Omega(r,t)}(n)\right|^2\\
&\quad\le Cr^{d-1}t^2\sum_{0<|n|\le 1/t}|n|^{1-d}+Cr^{d-1}\sum_{1/t<|n|<+\infty}|n|^{-1-d}\le Cr^{d-1}t.
\end{aligned}$$

Proving an asymptotic estimate of the variance is a more difficult task. One has to take into account not only the size of the Fourier transform, but also the oscillations. In particular, the curvature $K(x)$ and the support function $\sup_{y\in\Omega}\{x\cdot y\}$, which determine the geometry of the convex body, will play a crucial role.

Lemma 2.4 *The variance of the number of integer points in the shifted annulus can be decomposed into*

$$\int_{\mathbb{T}^d}\left|\sum_{k\in\mathbb{Z}^d}\chi_{\Omega(r,t)-x}(k)-|\Omega(r,t)|\right|^2 \mathrm{d}x=X(r,t)+Y(r,t)+Z(r,t),$$

where

$$X(r,t) = 2\pi^{-2} r^{d-1} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} K(\sigma(n))^{-1} \sin^2(\pi t \sigma(n)\cdot n) |n|^{-d-1},$$

$$Y(r,t) = -2\pi^{-2} r^{d-1} \sum_{n\in\mathbb{Z}^d\setminus\{0\}} \cos(2\pi r(\sigma(n)-\sigma(-n))\cdot n - \pi(d-1)/2)$$
$$\times K(\sigma(n))^{-1/2} K(\sigma(-n))^{-1/2} \sin(\pi t\sigma(n)\cdot n)\sin(\pi t\sigma(-n)\cdot n)|n|^{-d-1}.$$

The remainder $Z(r,t)$ has the property that there exists a constant C such if $r \geq 1$ and $t \leq r$ then

$$|Z(r,t)| \leq C|\Omega(r,t)| r^{-1} t \log(2+1/t).$$

Proof By Lemmas 2.1 and 2.3, the variance equals

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d\setminus\{0\}} \left|\widehat{\chi}_{\Omega(r,t)}(n)\right|^2 \\
&= \sum_{n\in\mathbb{Z}^d\setminus\{0\}} A(r,t,n)\overline{A(r,t,n)} + \sum_{n\in\mathbb{Z}^d\setminus\{0\}} A(r,t,n)\overline{B(r,t,n)} \\
&\quad + \sum_{n\in\mathbb{Z}^d\setminus\{0\}} B(r,t,n)\overline{A(r,t,n)} + \sum_{n\in\mathbb{Z}^d\setminus\{0\}} B(r,t,n)\overline{B(r,t,n)}.
\end{aligned}$$

Since $c|n| \leq \sigma(n)\cdot n \leq C|n|$ for some $C \geq c > 0$, Lemma 2.3 implies that

$$\begin{aligned}
&|A(r,t,n)| \leq Cr^{(d-1)/2}|n|^{-(d+1)/2}\min\{1, t|n|\},\\
&|B(r,t,n)| \leq Cr^{(d-3)/2} t |n|^{-(d+1)/2}.
\end{aligned}$$

These estimates give

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |A(r,t,n)||B(r,t,n)| \\
&\leq Cr^{d-2}t^2 \sum_{0<|n|\leq 1/t} |n|^{-d} + Cr^{d-2} t \sum_{1/t<|n|<+\infty} |n|^{-d-1} \\
&\leq Cr^{d-2}t^2 \log(2+1/t),
\end{aligned}$$

and

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |B(r,t,n)|^2 \leq Cr^{d-3}t^2 \sum_{n\in\mathbb{Z}^d\setminus\{0\}} |n|^{-d-1} \leq Cr^{d-3}t^2.$$

The main term is $\sum_{n\in\mathbb{Z}^d\setminus\{0\}} |A(r,t,n)|^2$, and one can check that it is equal to $X(r,t)+Y(r,t)$. □

It follows from the Cauchy–Schwarz inequality that in the statement of the above lemma the series $Y(r,t)$ with cosines is smaller than the series $X(r,t)$. Moreover, the cancellations due to the change of sign of the cosine lead to conjecture that $Y(r,t)$ is indeed much smaller than $X(r,t)$, and it gives a negligible contribution to the variance. Also observe that the single terms in the expansions $X(r,t)$ and $Y(r,t)$ give negligible contributions to the series. This suggests that these series are asymptotic to integrals, and at least for $X(r,t)$ this is the case.

Lemma 2.5 *If $|\Omega|$ is the volume of the convex body, and with the definition of $X(r,t)$ in Lemma 2.4, we have*

$$X(r,t) = d\,|\Omega|\,r^{d-1}t + W(r,t).$$

The remainder $W(r,t)$ has the property that for some C and every $r \geq 1$ and $t \leq r$,

$$|W(r,t)| \leq C\,|\Omega(r,t)|\,t\log(2+1/t).$$

Proof Identifying the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with the unit cube $\{-1/2 \leq x_j < 1/2\}$ and decomposing $\mathbb{R}^d$ into $\bigcup_{n\in\mathbb{Z}^d}\left\{\mathbb{T}^d + n\right\}$, one gets

$$\begin{aligned}
X(r,t) &= 2\pi^{-2}r^{d-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} K(\sigma(n))^{-1}\sin^2(\pi t\sigma(n)\cdot n)\,|n|^{-d-1} \\
&= 2\pi^{-2}r^{d-1}\int_{\mathbb{R}^d} K(\sigma(x))^{-1}\sin^2(\pi t\sigma(x)\cdot x)\,|x|^{-d-1}\,\mathrm{d}x \\
&\quad - 2\pi^{-2}r^{d-1}\int_{\mathbb{T}^d} K(\sigma(x))^{-1}\sin^2(\pi t\sigma(x)\cdot x)\,|x|^{-d-1}\,\mathrm{d}x \\
&\quad - 2\pi^{-2}r^{d-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}}\int_{\mathbb{T}^d}\left(K(\sigma(n+x))^{-1} - K(\sigma(n))^{-1}\right) \\
&\quad \times \sin^2(\pi t\sigma(n+x)\cdot(n+x))\,|n+x|^{-d-1}\,\mathrm{d}x \\
&\quad - 2\pi^{-2}r^{d-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} K(\sigma(n))^{-1}\int_{\mathbb{T}^d}|n+x|^{-d-1} \\
&\quad \times \left(\sin^2(\pi t\sigma(n+x)\cdot(n+x)) - \sin^2(\pi t\sigma(n)\cdot n)\right)\mathrm{d}x \\
&\quad - 2\pi^{-2}r^{d-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} K(\sigma(n))^{-1}\sin^2(\pi t\sigma(n)\cdot n) \\
&\quad \times \int_{\mathbb{T}^d}\left(|n+x|^{-d-1} - |n|^{-d-1}\right)\mathrm{d}x.
\end{aligned}$$

First of all, one has

$$
\begin{aligned}
&2\pi^{-2} r^{d-1} \int_{\mathbb{T}^d} K(\sigma(x))^{-1} \sin^2(\pi t \sigma(x) \cdot x) |x|^{-d-1} \,\mathrm{d}x \\
&\quad \leq 2 r^{d-1} t^2 \int_{\mathbb{T}^d} K(\sigma(x))^{-1} (\sigma(x) \cdot x)^2 |x|^{-d-1} \,\mathrm{d}x \leq C r^{d-1} t^2.
\end{aligned}
$$

Then observe that $\sigma(x)$ is smooth in $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree zero. Moreover, as mentioned before, $c|x| \leq \sigma(x) \cdot x \leq C|x|$ for some $C \geq c > 0$ and every $x \in \mathbb{R}^d$. Hence also $K(\sigma(x))^{-1}$ is smooth in $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree zero, and for every $x \in \mathbb{T}^d$ and $n \in \mathbb{Z}^d \setminus \{0\}$ one has

$$
\left| K(\sigma(n+x))^{-1} - K(\sigma(n))^{-1} \right| \leq C |n|^{-1}.
$$

Hence,

$$
\begin{aligned}
&2\pi^{-2} r^{d-1} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbb{T}^d} \left| K(\sigma(n+x))^{-1} - K(\sigma(n))^{-1} \right| \\
&\quad \times \sin^2(\pi t \sigma(n+x) \cdot (n+x)) |n+x|^{-d-1} \,\mathrm{d}x \\
&\quad \leq C r^{d-1} t^2 \sum_{0<|n|\leq 1/t} |n|^{-d} + C r^{d-1} \sum_{1/t<|n|<+\infty} |n|^{-d-2} \\
&\quad \leq C r^{d-1} t^2 \log(2 + 1/t).
\end{aligned}
$$

Similarly, by the trigonometric identity $\sin^2(x) - \sin^2(y) = \sin(x+y)\sin(x-y)$, and since $|\sigma(x) \cdot x - \sigma(y) \cdot y| \leq C|x-y|$,

$$
\begin{aligned}
&2\pi^{-2} r^{d-1} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(\sigma(n))^{-1} \int_{\mathbb{T}^d} |n+x|^{-d-1} \\
&\quad \times \left| \sin^2(\pi t \sigma(n+x) \cdot (n+x)) - \sin^2(\pi t \sigma(n) \cdot n) \right| \mathrm{d}x \\
&\leq 2\pi^{-2} r^{d-1} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} K(\sigma(n))^{-1} \int_{\mathbb{T}^d} |\sin(\pi t (\sigma(n+x) \cdot (n+x) + \sigma(n) \cdot n))| \\
&\quad \times |\sin(\pi t (\sigma(n+x) \cdot (n+x) - \sigma(n) \cdot n))| \, |n+x|^{-d-1} \,\mathrm{d}x \\
&\leq C r^{d-1} t^2 \sum_{0<|n|\leq 1/t} |n|^{-d} + C r^{d-1} t \sum_{1/t<|n|<+\infty} |n|^{-d-1} \\
&\leq C r^{d-1} t^2 \log(2 + 1/t).
\end{aligned}
$$

And the last term is

$$
\begin{aligned}
&2\pi^{-2}r^{d-1}\sum_{n\in\mathbb{Z}^d\setminus\{0\}} K\left(\sigma\left(n\right)\right)^{-1}\sin^2\left(\pi t\sigma\left(n\right)\cdot n\right)\int_{\mathbb{T}^d}\left|\left|n+x\right|^{-d-1}-\left|n\right|^{-d-1}\right|\mathrm{d}x\\
&\quad\le Cr^{d-1}t^2\sum_{0<|n|\le 1/t}|n|^{-d}+Cr^{d-1}\sum_{1/t<|n|<+\infty}|n|^{-d-2}\,\mathrm{d}x\\
&\quad\le Cr^{d-1}t^2\log\left(2+1/t\right).
\end{aligned}
$$

Finally, an integration in polar coordinates $x=\rho\vartheta$ with a change of variables gives

$$
\begin{aligned}
&2\pi^{-2}r^{d-1}\int_{\mathbb{R}^d} K\left(\sigma\left(x\right)\right)^{-1}\sin^2\left(\pi t\sigma\left(x\right)\cdot x\right)|x|^{-d-1}\,\mathrm{d}x\\
&\quad=2\pi^{-2}r^{d-1}\int_0^{+\infty}\int_{\{|\vartheta|=1\}} K\left(\sigma\left(\vartheta\right)\right)^{-1}\sin^2\left(\pi t\rho\sigma\left(\vartheta\right)\cdot\vartheta\right)\rho^{-2}\mathrm{d}\rho\mathrm{d}\vartheta\\
&\quad=2\pi^{-1}r^{d-1}t\left(\int_0^{+\infty}\sin^2\left(s\right)s^{-2}\mathrm{d}s\right)\left(\int_{\{|\vartheta|=1\}} K\left(\sigma\left(\vartheta\right)\right)^{-1}\sigma\left(\vartheta\right)\cdot\vartheta\,\mathrm{d}\vartheta\right).
\end{aligned}
$$

The first integral can be evaluated using residues,

$$
\int_0^{+\infty}\frac{\sin^2\left(s\right)}{s^2}\mathrm{d}s=\int_{-\infty}^{+\infty}\frac{1-\cos\left(2s\right)}{4s^2}\mathrm{d}s=\mathrm{Re}\left(\int_{-\infty}^{+\infty}\frac{1-\exp\left(2iz\right)}{4z^2}\mathrm{d}z\right)=\frac{\pi}{2}.
$$

The integral with the curvature is d times the volume of the convex body Ω,

$$
\int_{\{|\vartheta|=1\}} K\left(\sigma\left(\vartheta\right)\right)^{-1}\sigma\left(\vartheta\right)\cdot\vartheta\,\mathrm{d}\vartheta=d\ \left|\Omega\right|.
$$

This comes from the definition of the curvature as the Jacobian determinant of the Gauss map. $K\left(\sigma\left(\vartheta\right)\right)^{-1}\mathrm{d}\vartheta=\mathrm{d}A$ is an infinitesimal element of surface area of $\partial\Omega$, and $\sigma\left(\vartheta\right)\cdot\vartheta$ is the height of the cone with vertex 0 and base $\mathrm{d}A$. Hence,

$$
2\pi^{-2}r^{d-1}\int_{\mathbb{R}^d} K\left(\sigma\left(x\right)\right)^{-1}\sin^2\left(\pi t\sigma\left(x\right)\cdot x\right)|x|^{-d-1}\,\mathrm{d}x=d\left|\Omega\right|r^{d-1}t.
$$

□

Observe that the only restriction on the indexes in the above lemmas is $r\ge 1$ and $t\le r$, and the assumption $t\le r^{-\alpha}$ with $\alpha>\left(d-1\right)/\left(d+1\right)$ in the statement of the theorem has not been used. It remains to estimate $Y\left(r,t\right)$, and this is the most delicate part of the proof. If one assumes that the series that defines $Y\left(r,t\right)$ is asymptotic to an integral, then one can easily check that this integral is negligible with respect to $X\left(r,t\right)$. We do not know under which assumptions the series that defines $Y\left(r,t\right)$ is asymptotic to an integral, as it is the case for $X\left(r,t\right)$. But, by Remark 3.1, some assumptions are necessary. For this reason we need to follow a more circuitous path. By the Cauchy–Schwarz inequality, $|Y\left(r,t\right)|\le X\left(r,t\right)\le Cr^{d-1}t$. In order to obtain some better estimates one has to take into account the cancellations in the series that defines $Y(r,t)$. We need a couple of preliminary lemmas.

Lemma 2.6 *If X and Y are two convex bodies with smooth boundaries with everywhere positive Gaussian curvature, then also the Minkowski sum $X+Y$, that is the set obtained by adding each vector in X to each vector in Y, is a convex body with smooth boundary with everywhere positive curvature.*

Proof The fact that $X+Y$ has smooth boundary is proved in [11]. The fact that the boundary has positive Gaussian curvature can be seen as follows. The strict convexity of X and Y implies that for every z on the boundary $\partial(X+Y)$ there exist only one $x\in\partial X$ and one $y\in\partial Y$ with $z=x+y$. The curvature assumption implies that there exist balls B_x and B_y with $x\in\partial B_x$, $X\subseteq B_x$, $y\in\partial B_y$, $Y\subseteq B_y$. It follows that $x+y\in\partial\left(B_x+B_y\right)$ and $X+Y\subseteq B_x+B_y$. Hence the curvature of $\partial(X+Y)$ at the point $x+y$ is at least as large as the curvature of B_x+B_y, which is a ball with radius the sum of the radii of B_x and B_y. By the way, without the curvature assumption the smoothness of the Minkowsky sum may fail. Indeed it has been proved in [10] that there exist convex sets in the plane with real analytic boundaries, but with the smoothness of the sum not exceeding $C^{20/3}$. And if the boundaries are only C^∞ then the smoothness of the sum may break out at the level C^5. □

Lemma 2.7 *Denote by $\sigma(\pm x)$ the points of the boundary $\partial\Omega$ with outward unit normals $\pm x/|x|$, and define*

$$\zeta(x)=(\sigma(x)-\sigma(-x))\cdot x.$$

Also denote by $A=\Omega+(-\Omega)$ the Minkowski sum of Ω and $-\Omega$. Finally, assume that $\psi(x)$ is a smooth function in $\mathbb{R}^d$ with support in $\varepsilon\le|x|\le 1/\varepsilon$, and such that for some η and for every multi index k,

$$\left|\frac{\partial^k}{\partial x^k}\psi(s)\right|\le C(k)\,\varepsilon^{-\eta-|k|}.$$

Then for every $j>0$ there exist positive constants C and γ, such that for every ξ in $\mathbb{R}^d$, every $\lambda>0$, and every $0<\varepsilon<1$, one has

$$\begin{aligned}&\left|\int_{\mathbb{R}^d}\psi(x)\exp\left(2\pi i\lambda\left(\zeta(x)-\xi\cdot x\right)\right)\mathrm{d}x\right|\\&\quad\le C\varepsilon^{-\gamma}\min\left\{\lambda^{-(d-1)/2},\ (\lambda\,\text{distance}\,\{\xi,\ \partial A\})^{-j}\right\}.\end{aligned}$$

Proof Recall that $\sigma(x)\cdot x=\sup_{y\in\Omega}\{y\cdot x\}$, the support function of the convex body, has gradient $\nabla(\sigma(x)\cdot x)=\sigma(x)$. See [1], or [16, Corollary 1.7.3]. Also observe that when x varies in $\mathbb{R}^d\setminus\{0\}$, then $\sigma(x)-\sigma(-x)$ describes the boundary of $A=\Omega+(-\Omega)$. Hence,

$$|\nabla((\sigma(x)-\sigma(-x))\cdot x-\xi\cdot x)|=|(\sigma(x)-\sigma(-x))-\xi|\ge\text{distance}\,\{\xi,\ \partial A\}.$$

Then a repeated integration by parts gives

$$\left|\int_{\mathbb{R}^d} \psi(x) \exp(2\pi i\lambda(\zeta(x) - \xi \cdot x))\,\mathrm{d}x\right| \le C\varepsilon^{-\gamma}(\lambda\,\text{distance}\{\xi,\ \partial A\})^{-j}.$$

See e.g., [19, Chapter VIII, §2.1]. This proves half of the lemma. In order to complete the proof, observe that the function $(\sigma(x) - \sigma(-x)) \cdot x$ is the support function of $A = \Omega + (-\Omega)$, and recall that, by the previous lemma, the boundary of this body is smooth with everywhere positive Gaussian curvature. It follows that this support function is homogeneous of degree one, and that one eigenvalue of the Hessian matrix is zero, but all other eigenvalues are positive. See [16, Corollary 2.5.2]. Hence, the Hessian of the phase $\zeta(x) - \xi \cdot x$, which is the Hessian of $(\sigma(x) - \sigma(-x)) \cdot x$, has rank $d-1$, and it follows that

$$\left|\int_{\mathbb{R}^d} \psi(x) \exp(2\pi i\lambda(\zeta(x) - \xi \cdot x))\,\mathrm{d}x\right| \le C\varepsilon^{-\gamma}\lambda^{-(d-1)/2}.$$

In order to see this, it suffices to apply the coarea formula to the level set of the function $\zeta(x)$. Then one ends up to estimate the Fourier transform of a smooth measure carried by a smooth surface with everywhere positive Gaussian curvature. See e.g. [12], or [19, Chapter VIII,§2.3 and §3.1]. □

Lemma 2.8 *With the definition of $Y(r,t)$ in Lemma 2.4, if $\alpha > (d-1)/(d+1)$ there exist positive constants C and β such that for every $1 \le r < +\infty$ and every $0 < t \le r^{-\alpha}$ one has,*

$$|Y(r,t)| \le C\,|\Omega(r,t)|\,t^{\beta}.$$

Proof In order to simplify the notation, set

$$\begin{aligned}
\vartheta &= \pi(d-1)/2,\\
\zeta(x) &= (\sigma(x) - \sigma(-x)) \cdot x,\\
\varphi(x) &= K(\sigma(x))^{-1/2} K(\sigma(-x))^{-1/2} \sin(\pi\sigma(x)\cdot x)\sin(\pi\sigma(-x)\cdot x)\,|x|^{-d-1}.
\end{aligned}$$

Then one can rewrite the series that defines $Y(r,t)$ as

$$Y(r,t) = -2\pi^{-2}r^{d-1}t\sum_{n\in\mathbb{Z}^d\setminus\{0\}} t^d\varphi(tn)\cos\left(2\pi r t^{-1}\zeta(tn) - \vartheta\right).$$

Observe that the factor $r^{d-1}t$ in front of the series is of the order of $|\Omega(r,t)|$. Hence, in order to prove the lemma it suffices to show that the series is bounded by Ct^{β} when $t \le r^{-\alpha}$. Let $0 < \varepsilon < 1/2$ and let $\chi(s)$ be a smooth function with support in $\varepsilon \le s \le 1/\varepsilon$, with $0 \le \chi(s) \le 1$ and equal to 1 in $2\varepsilon \le s \le 1/2\varepsilon$, and with

$$\left|\frac{\mathrm{d}^j}{\mathrm{d}s^j}\chi(s)\right| \le C\varepsilon^{-j}.$$

With this cut off function, one can decompose

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}^d\setminus\{0\}} t^d\varphi(tn)\cos\left(2\pi r t^{-1}\zeta(tn)-\vartheta\right)\\
&\quad=\sum_{n\in\mathbb{Z}^d\setminus\{0\}} t^d(1-\chi(|tn|))\varphi(tn)\cos\left(2\pi r t^{-1}\zeta(tn)-\vartheta\right)\\
&\qquad+\sum_{n\in\mathbb{Z}^d\setminus\{0\}} t^d\chi(|tn|)\varphi(tn)\cos\left(2\pi r t^{-1}\zeta(tn)-\vartheta\right).
\end{aligned}
$$

One has

$$
\begin{aligned}
&\left|\sum_{n\in\mathbb{Z}^d\setminus\{0\}} t^d(1-\chi(|tn|))\varphi(tn)\cos\left(2\pi r t^{-1}\zeta(tn)-\vartheta\right)\right|\\
&\quad\le\pi^2\sup\left\{|\sigma(n)|^2K(\sigma(n))^{-1}\right\}t\sum_{0<|n|<2\varepsilon/t}|n|^{1-d}\\
&\qquad+\sup\left\{K(\sigma(n))^{-1}\right\}t^{-1}\sum_{1/(2\varepsilon t)<|n|<+\infty}|n|^{-d-1}\\
&\quad\le C\varepsilon.
\end{aligned}
$$

Again, in order to simplify a bit the notation, set

$$
f(x)=\chi(|x|)\varphi(x)\cos\left(2\pi r t^{-1}\zeta(x)-\vartheta\right).
$$

Then, if $\widehat{f}(\xi)=\int_{\mathbb{R}^d}f(x)\exp(-2\pi i\xi\cdot x)\,\mathrm{d}x$ is the Fourier transform of $f(x)$, the Poisson summation formula with a change of variables gives

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}^d\setminus\{0\}} t^d\chi(|tn|)\varphi(tn)\cos\left(2\pi r t^{-1}\zeta(tn)-\vartheta\right)\\
&\quad=\sum_{n\in\mathbb{Z}^d} t^d f(tn)=\sum_{n\in\mathbb{Z}^d}\widehat{f}\left(t^{-1}n\right).
\end{aligned}
$$

Observe that the function $f(x)$ is smooth with compact support, and that $\widehat{f}(\xi)$ has fast decay at infinity. In particular, in the above series there are no problems of convergence. Writing a cosine as a sum of exponentials, one has

$$
\begin{aligned}
\widehat{f}\left(t^{-1}n\right)&=\int_{\mathbb{R}^d}\chi(|x|)\varphi(x)\cos\left(2\pi r t^{-1}\zeta(x)-\vartheta\right)\exp\left(-2\pi i t^{-1}n\cdot x\right)\mathrm{d}x\\
&=2^{-1}\exp(-i\vartheta)\int_{\mathbb{R}^d}\chi(|x|)\varphi(x)\exp\left(2\pi i r t^{-1}\left(\zeta(x)-r^{-1}n\cdot x\right)\right)\mathrm{d}x\\
&\quad+2^{-1}\exp(i\vartheta)\int_{\mathbb{R}^d}\chi(|x|)\varphi(x)\exp\left(2\pi i r t^{-1}\left(-\zeta(x)-r^{-1}n\cdot x\right)\right)\mathrm{d}x.
\end{aligned}
$$

Then the previous lemma with $\lambda = rt^{-1}$ and $\xi = \pm r^{-1}n$ gives for every j,

$$\left|\widehat{f}\left(t^{-1}n\right)\right| \leq C\varepsilon^{-\gamma} \min\left\{\left(rt^{-1}\right)^{-(d-1)/2}, \left(t^{-1}\text{distance}\left\{n, \partial\left(rA\right)\right\}\right)^{-j}\right\},$$

where the term $\pm n$ in the right-hand side has been replaced by n because A is symmetric.

At this point, without pretense of rigor one could conclude the proof as follows. The above Fourier transform is concentrated within the annulus $\{\text{distance}\{n, \partial(rA)\} \leq t\}$ which has a measure dominated by $Cr^{d-1}t$, and in this annulus $\left|\widehat{f}\left(t^{-1}n\right)\right| \leq C\varepsilon^{-\gamma}\left(rt^{-1}\right)^{-(d-1)/2}$. This should imply that

$$\sum_{n\in\mathbb{Z}^d} \left|\widehat{f}\left(t^{-1}n\right)\right| \leq C\varepsilon^{-\gamma}\left(rt^{-1}\right)^{-(d-1)/2} r^{d-1}t = C\varepsilon^{-\gamma} r^{(d-1)/2} t^{(d+1)/2}.$$

If $t \leq r^{-\alpha}$ with $\alpha > (d-1)/(d+1)$, then one can choose $\varepsilon \to 0+$ such that $\varepsilon^{-\gamma} r^{(d-1)/2} t^{(d+1)/2} \to 0+$ as $r \to +\infty$, and this would conclude this pseudo proof. The proof with full details is a bit more involved. For every $0 < s < 1$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d} \left|\widehat{f}\left(t^{-1}n\right)\right| \leq{}& C\varepsilon^{-\gamma} r^{-(d-1)/2} t^{(d-1)/2} \sum_{\text{distance}\{n, \partial(rA)\}\leq s} 1 \\ &+ C\varepsilon^{-\gamma} t^{j} s^{-j} \sum_{\text{distance}\{n, \partial(rA)\}\leq 1} 1 \\ &+ C\varepsilon^{-\gamma} t^{j} \sum_{k=1}^{+\infty} 2^{-jk} \left(\sum_{\text{distance}\{n, \partial(rA)\}\leq 2^k} 1\right).\end{aligned}$$

In order to estimate the sum over $\{\text{distance}\{n, \partial(rA)\} \leq s\}$, observe that for some positive constant c and for every $s < 1 \leq r$ one has

$$\{\text{distance}\{n, \partial(rA)\} \leq s\} \subseteq (r+cs)A \setminus (r-cs)A.$$

By Lemma 2.6 the convex body $A = \Omega + (-\Omega)$ has a smooth boundary with everywhere positive Gaussian curvature, and it has been proved in [5,6] that there exists a positive constant C such that for every $r \geq 1$,

$$\left|\sum_{n\in rA} 1 - r^d |A|\right| \leq C r^{d(d-1)/(d+1)}.$$

See also [7, Theorem 7.7.16]. This implies that

$$\begin{aligned}
&\sum_{\text{distance}\{n,\ \partial(rA)\}\le s} 1 \\
&\le \left|\sum_{n\in(r+cs)A} 1-(r+cs)^d\,|A|\right| + \left|\sum_{n\in(r-cs)A} 1-(r-cs)^d\,|A|\right| \\
&\quad + \left|(r+cs)^d-(r-cs)^d\right|\,|A| \\
&\le C\left(r^{d(d-1)/(d+1)}+r^{d-1}s\right).
\end{aligned}$$

The choice $s=r^{-(d-1)/(d+1)}$, so that $r^{d(d-1)/(d+1)}=r^{d-1}s$, then gives

$$\varepsilon^{-\gamma}r^{-(d-1)/2}t^{(d-1)/2}\sum_{\text{distance}\{n,\ \partial(rA)\}\le s} 1\le C\varepsilon^{-\gamma}r^{(d-1)^2/(2d+2)}t^{(d-1)/2}.$$

In order to estimate the sum over $\left\{\text{distance}\,\{n,\ \partial\,(rA)\}\le 2^k\right\}$, observe that

$$\sum_{\text{distance}\{n,\ \partial(rA)\}\le 2^k} 1\le\begin{cases}Cr^{d-1}2^k & \text{if } 2^k\le r,\\ C2^{dk} & \text{if } 2^k\ge r.\end{cases}$$

It follows that, with the choice $s=r^{-(d-1)/(d+1)}$,

$$\varepsilon^{-\gamma}t^js^{-j}\sum_{\text{distance}\{n,\ \partial(rA)\}\le 1} 1\le C\varepsilon^{-\gamma}r^{d-1+j(d-1)/(d+1)}t^j.$$

And if j is suitably large it also follows that

$$\varepsilon^{-\gamma}t^j\sum_{k=1}^{+\infty}2^{-jk}\left(\sum_{\text{distance}\{n,\ \partial(rA)\}\le 2^k} 1\right)\le C\varepsilon^{-\gamma}r^{d-1}t^j.$$

Collecting all these estimates, and assuming that $t\le r^{-\alpha}$ for some $\alpha>(d-1)/(d+1)$ and that j is sufficiently large, one obtains that

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}^d}\left|\widehat{f}\left(t^{-1}n\right)\right| \\
&\le C\varepsilon^{-\gamma}\left(r^{(d-1)^2/(2d+2)}t^{(d-1)/2}+r^{d-1+j(d-1)/(d+1)}t^j+r^{d-1}t^j\right) \\
&\le C\varepsilon^{-\gamma}\left(r^{(d-1)^2/(2d+2)}t^{(d-1)/2}+r^{d-1+j(d-1)/(d+1)}t^j\right)
\end{aligned}$$

$$
\leq C\varepsilon^{-\gamma} r^{(d-1)^2/(2d+2)} t^{(d-1)/2} \left(1 + r^{d-1} \left(r^{(d-1)/(d+1)} t\right)^{j-(d-1)/2}\right)
$$
$$
\leq C\varepsilon^{-\gamma} \left(r^{(d-1)/(d+1)} t\right)^{(d-1)/2}.
$$

Assuming again that $t \leq r^{-\alpha}$ for some $\alpha > (d-1)/(d+1)$, and with the choice $\varepsilon = \left(r^{(d-1)/(d+1)} t\right)^{(d-1)/(2\gamma+2)}$, one obtains

$$
\begin{aligned}
|Y(r,t)| &\leq C\,|\Omega(r,t)| \left(\varepsilon + \varepsilon^{-\gamma} \left(r^{(d-1)/(d+1)} t\right)^{(d-1)/2}\right) \\
&\leq C\,|\Omega(r,t)| \left(r^{(d-1)/(d+1)} t\right)^{(d-1)/(2\gamma+2)} \\
&\leq C\,|\Omega(r,t)| \left(t^{1-(d-1)/((d+1)\alpha)}\right)^{(d-1)/(2\gamma+2)}.
\end{aligned}
$$

Finally, in order to prove the lemma it suffices to choose

$$
\beta \leq \frac{\left(\alpha - \dfrac{d-1}{d+1}\right)(d-1)}{\alpha\,(2\gamma+2)}.
$$

□

Proof of Theorem 1.1 By the previous lemmas, choosing $\beta < 1$ in Lemma 2.8, one has

$$
\begin{aligned}
&|W(r,t)| + |Z(r,t)| + |Y(r,t)| \\
&\quad \leq C|\Omega(r,t)| \left(t \log(2+1/t) + tr^{-1} \log(2+1/t) + t^{\beta}\right) \\
&\quad \leq C|\Omega(r,t)| t^{\beta}
\end{aligned}
$$

□

3 Final Remarks

We conclude with some remarks.

Remark 3.1 As said in the introduction, for the validity of the theorem the assumption that the widths of the annuli converge to zero does not suffice, and one has to require a suitable speed. Indeed in [15] a somehow stronger failure of an asymptotic estimate is proved. In any dimension d the variance of spherical annuli $\{r - t/2 < |x| \leq r + t/2\}$ is always smaller than $Cr^{d-1}t$, and for some sequences $r \to +\infty$ it is larger than $cr^{d-1}t$. Moreover, there exist sequences $r \to +\infty$ and $t \to +\infty$ with associated variance much smaller than $cr^{d-1}t$ for every $c > 0$. In dimension $d \equiv 3$ modulo 4 this also holds for some sequences of widths that stay bounded or that tend to zero slower than any negative power of the radii. This is related to the location of the zeroes of the Fourier transform of an annulus. See also [2] for related results on higher order moments.

Remark 3.2 As said in the introduction, the variance of annuli with boundary points of zero curvature may be much larger than the mean, and an asymptotic estimate of the variance may fail. A simple example are the flat annuli in the plane generated by squares with sides parallel to the axes,

$$\begin{aligned} A &= \{x = (x_1, x_2) : \ n - t/2 < \max\{|x_1|, |x_2|\} \le n + t/2\}, \\ B &= \{x = (x_1, x_2) : \ n < \max\{|x_1|, |x_2|\} \le n + t\}. \end{aligned}$$

The diameters and thicknesses of these two annuli are approximately the same, but the random variables that count the lattice points are quite different when n is a large integer and t is a small positive number. The random variable $N(A, x)$ that counts the number of integer points in $A - x$ takes the value $8n$ on a set with measure t^2, the value $4n$ on a set with measure $2t - 2t^2$, and 0 otherwise, and the mean and variance are

$$\begin{aligned} \int_{\mathbb{T}^2} N(A, x)\mathrm{d}x &= 8nt, \\ \int_{\mathbb{T}^2} |N(A, x) - 8nt|^2 \,\mathrm{d}x &= 32n^2t - 32n^2t^2 \sim 32n^2t. \end{aligned}$$

Similarly, the random variable $N(B, x)$ that counts the number of integer points in $B - x$ takes the value $4n + 1$ on a set with measure $4t^2$, the value $2n$ on a set with measure $4t - 8t^2$, and 0 otherwise, and the mean and variance are

$$\begin{aligned} \int_{\mathbb{T}^2} N(B, x)\mathrm{d}x &= 8nt + 4t^2 \sim 8nt, \\ \int_{\mathbb{T}^2} |N(B, x) - (8nt + 4t^2)|^2\mathrm{d}x &= 16n^2t - 32n^2t^2 + 32nt^2 - 64nt^3 + 4t^2 - 16t^4 \\ &\sim 16n^2t. \end{aligned}$$

Observe that the means of $N(A, x)$ and $N(B, x)$ are approximately the same and they are much smaller than the variances, and that the variance of $N(A, x)$ is about twice the variance of $N(B, x)$. In particular, the variances of these flat annuli have a sort of oscillating behavior.

Remark 3.3 The above are estimates of the discrepancy between volume and integer points in translated annuli. As in [8,13,14,17], one may ask about similar estimates when the annuli are not translated and the averages are with respect to dilations. We suspect that the discrepancy with respect to dilations of spherical annuli may be much larger than the discrepancy with respect to translations, and indeed in [8] it is proved that this is the case for annuli in the plane, that is in dimension $d = 2$.

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