
A small and non-simple geometric transition

Michele Rossi

Abstract Following notation introduced in the recent paper [41], this paper is aimed to present in detail an example of a *small* geometric transition which is not a *simple* one i.e. a *deformation* of a conifold transition. This is realized by means of a detailed analysis of the Kuranishi space of a Namikawa cuspidal fiber product, which in particular improves the conclusion of Y. Namikawa in Remark 2.8 and Example 1.11 of [29]. The physical interest of this example is presenting a geometric transition which can't be immediately explained as a massive black hole condensation to a massless one, as described by A. Strominger [44].

Keywords Fiber products of rational elliptic surfaces; smoothing of singularities; resolution of singularities; Calabi–Yau threefolds; Calabi–Yau web; geometric transition; conifold transition; deformation of a Calabi–Yau threefold; deformation of a geometric transition

Mathematics Subject Classification (2000) 14B07; 32G10; 14E15; 14J32; 14J27

Contents

1	Introduction	1
2	Preliminaries and notation	3
3	The Namikawa fiber product	9
4	A small and non-simple geometric transition	19

1 Introduction

Let X be a complex projective threefold with terminal singularities and admitting a *small* resolution $\widehat{X} \xrightarrow{\phi} X$ such that \widehat{X} is a *Calabi–Yau threefold* (in the sense of Definition 1), where “small” means that the exceptional locus $\text{Exc}(\phi)$ has codimension greater than or equal to two. Then it is well known that $\text{Exc}(\phi)$ consists of a finite

The author was partially supported by MIUR-PRIN 2010-11 Research Funds “Geometria delle Varietà Algebriche” and by the I.N.D.A.M. as a member of the G.N.S.A.G.A.

M. Rossi
Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino
Tel.: +39-011-6702813
Fax: +39-011-6702878
E-mail: michele.rossi@unito.it

disjoint union of trees of rational curves of A–D–E type [37], [21], [33], [27], [7]. In his paper [29], Remark 2.8, Y. Namikawa considered the following

Problem *When does \widehat{X} have a flat deformation such that each tree of rational curves splits up into mutually disjoint $(-1, -1)$ -curves?*

Let us recall that a $(-1, -1)$ -curve is a rational curve in X whose normal bundle is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It arises precisely as exceptional locus of the resolution of an ordinary double point (a *node*) also called a *conifold* point since it is an isolated hypersurface singularity whose tangent cone is a non-degenerate quadratic cone.

Namikawa’s problem is interesting in the context of H. Clemens type problems of cycle deformations (see e.g. [7], Corollary (4.11)). Moreover, it is of significant interest in the context of (deformations of) geometric transitions and therefore in the study of the moduli space for Calabi–Yau threefolds. Let us recall that a *geometric transition* (gt) between two Calabi–Yau threefolds is the process obtained by “composing” a birational contraction to a normal threefold with a complex smoothing (see Definition 4). If the normal intermediate threefold has only nodal singularities then the considered gt is called a *conifold transition*. The interest in geometric transitions goes back to the ideas of H. Clemens [5] and M. Reid [38] which gave rise to the so called *Calabi–Yau Web Conjecture* (see also [12] for a revised and more recent version) stating that (more or less) all Calabi–Yau threefolds can be connected to each other by means of a chain of geometric transitions, giving a sort of (unexpected) “connectedness” of the moduli space for Calabi–Yau threefolds. There is also a considerable physical interest in geometric transitions owing to the fact that they connect topologically distinct models of Calabi–Yau vacua: the physical version of the Calabi–Yau Web Conjecture is a sort of (in this case expected) “uniqueness” of a space–time model for supersymmetric string theories (see e.g. [4] and references therein).

In this context, Namikawa’s problem can then be rephrased as follows

Problem(for small geometric transitions) *When does a small gt have the same “deformation type” (see Definition 5) of a conifold transition?*

Since the geometry of a general gt can be very intricate, while the geometry of a conifold transition is relatively easy and well understood as a topological surgery [5], the mathematical interest of such a problem is evident.

On the other hand, conifold transitions were the first (and among the few) geometric transitions to be physically understood as massive black holes condensation to massless ones, by A. Strominger [44]. Answering the given problem would then give a significant improvement in the physical interpretation of (at least the small) geometric transitions bridging topologically distinct Calabi–Yau vacua.

Unfortunately in [29], Remark 2.8, Namikawa observed that a flat deformation positively resolving the given problem “*does not hold in general*” and produced an example of a *cuspidal fiber self-product of an elliptic rational surface with sections* whose resolution admits exceptional trees, composed of couples of rational curves intersecting at one point, which should not deform to a disjoint union of $(-1, -1)$ -curves. Nevertheless, Example 1.11 in [29], supporting such a conclusion, did not give a correct argument since the proposed deformation is actually a trivial one.

In the present paper we will overcome this argument by giving a detailed analysis of the Kuranishi space of the Namikawa cuspidal fiber product, allowing us to go far beyond his conclusion in [29], Remark 2.8. Moreover this will give rise to an explicit example of a small gt which is not a simple one i.e. it has not the same deformation type of a conifold transition: such an example has already been sketched in §9.2 of [41], without any proof. Here all the needed details will be given.

The paper is organized as follows.

In Section 2 we introduce notation, preliminaries and main facts needed throughout the paper. Section 3 is then devoted to present the Namikawa construction of a fiber self-product of a particular elliptic rational surface with sections and singular “cuspidal” fibers (which will be called *cuspidal fiber product*). These are threefolds admitting six isolated singularities of Kodaira type $II \times II$ which have been rarely studied in either the pioneering work of C. Schoen [42] or the recent [16]. For this reason, their properties, small resolutions and local deformations are studied in detail. In particular, all the local deformations induced by global versal deformations are studied in Proposition 6, while all the local deformations of a cuspidal singularity to three distinct nodes are studied in Proposition 7. They actually do not lift globally to the given small resolution, as stated by Theorem 4, revising the Namikawa considerations of [29], Remark 2.8 and Example 1.11: see Remark 3 and Theorem 5. The last section 4 is dedicated to apply Theorem 4 to deformations of geometric transitions: for further details the interested reader is referred to [41] §7.1.

2 Preliminaries and notation

Definition 1 (Calabi–Yau 3-folds) A smooth, complex, projective 3-fold X is called *Calabi–Yau* if

1. $\mathcal{K}_X \cong \mathcal{O}_X$,
2. $h^{1,0}(X) = h^{2,0}(X) = 0$.

The standard example is the smooth quintic threefold in \mathbb{P}^4 . The given definition is equivalent to require that X has holonomy group a subgroup of $SU(3)$ (see [15] for a complete description of equivalences and implications).

2.1 Deformations of complex spaces

Let us start by recalling that a *complex space* is a ringed space (X, \mathcal{O}_X) where X is a Hausdorff topological space locally isomorphic to a locally closed analytic subset of some \mathbb{C}^N and \mathcal{O}_X is the induced sheaf of holomorphic functions. A *pointed complex space* is a pair (X, x) consisting of a complex space X and a distinguished point $x \in X$. A *morphism* $f : (X, x) \rightarrow (Y, y)$ of *pointed complex spaces* is a morphism $f : X \rightarrow Y$ of complex spaces such that $f(x) = y$.

Complex space germs are pointed complex spaces whose morphisms are given by equivalence classes of morphisms of pointed complex spaces defined in some open neighborhood of the distinguished point. Let (X, x) be a complex space germ and $U \subset X$ an open neighborhood of x : then the inclusion map $U \hookrightarrow X$ gives an isomorphism of complex space germs $(U, x) \cong (X, x)$. Then U is called a *representative of the germ* (X, x) .

A complex space germ is also called a *singularity*.

Definition 2 (Deformation of a complex space, [31] § 5, [1] §XI.2) Let X be a complex space. A *deformation* of X is a flat, holomorphic map $f : \mathcal{X} \rightarrow (B, o)$ from the complex space \mathcal{X} over a complex space germ (B, o) , endowed with an isomorphism $X \cong \mathcal{X}_o := f^{-1}(o)$ on the central fibre, for short denoted by $f : (\mathcal{X}, X) \rightarrow (B, o)$. If X is compact then we will also require that f is a proper map. If X is singular and the fibre $\mathcal{X}_b = f^{-1}(b)$ is smooth, for some $b \in B$, then \mathcal{X} is called a *smoothing family* of X . With a slight abuse of notation, if $b \neq o$ then the fiber \mathcal{X}_b is called either a *deformation* or a *smoothing* of X when \mathcal{X} is either a deformation or a smoothing family of X , respectively.

Let Ω_X be the sheaf of holomorphic differential forms on X and consider the *Lichtenbaum–Schlessinger cotangent sheaves* [22] of X , $\Theta_X^i = \mathcal{E}xt^i(\Omega_X, \mathcal{O}_X)$. Then $\Theta_X^0 = \mathcal{H}om(\Omega_X, \mathcal{O}_X) =: \Theta_X$ is the “tangent” sheaf of X and Θ_X^i is supported over $\text{Sing}(X)$, for any $i > 0$. Consider the associated *local* and *global deformation objects*

$$T_X^i := H^0(X, \Theta_X^i) \quad , \quad \mathbb{T}_X^i := \text{Ext}^i(\Omega_X^1, \mathcal{O}_X) \quad , \quad i = 0, 1, 2.$$

Then by the *local to global spectral sequence* relating the global Ext and sheaf $\mathcal{E}xt$ (see [13] and [8] II, 7.3.3) we get

$$E_2^{p,q} = H^p(X, \Theta_X^q) \implies \mathbb{T}_X^{p+q}$$

giving that

$$\mathbb{T}_X^0 \cong T_X^0 \cong H^0(X, \Theta_X) \quad , \quad (1)$$

$$\text{if } X \text{ is smooth then } \mathbb{T}_X^i \cong H^i(X, \Theta_X) \quad , \quad (2)$$

$$\text{if } X \text{ is Stein then } T_X^i \cong \mathbb{T}_X^i \quad . \quad (3)$$

Given a deformation family $\mathcal{X} \xrightarrow{f} B$ of X for each point $b \in B$ there is a well defined *linear (and functorial) map*

$$D_b f : T_b B \longrightarrow \mathbb{T}_{X_b}^1 \quad (\text{Generalized Kodaira–Spencer map})$$

where $T_b B$ denotes the Zariski tangent space to B at b ([11] Lemma II.1.20, [32] § 2.4).

For the following terminology the reader is referred to [23] § 6.C, [11] § 1.3, [31] Def. 5.1, [32] § 2.6, among many others. A deformation $f : (\mathcal{X}, X) \rightarrow (B, o)$ of a complex space X is called *versal* (some authors say *complete*) if, for any deformation $g : (\mathcal{Y}, X) \rightarrow (C, p)$ of X , there exists a holomorphic map of germs of complex spaces $h : (C, p) \rightarrow (B, o)$ such that $g = h^*(f)$ i.e. the following diagram commutes

$$\begin{array}{ccc} \mathcal{Y} = U \times_B \mathcal{X} & \longrightarrow & \mathcal{X} \\ \downarrow g = h^*(f) & & \downarrow f \\ C & \xrightarrow{h} & B \end{array}$$

In particular the generalized Kodaira–Spencer map $\kappa(f)$ turns out to be surjective ([32], § 2.6). Moreover the deformation f is said to be an *effective versal* (or *miniversal*) deformation of X if it is versal and $\kappa(f)$ is injective, hence an isomorphism. Finally the deformation f is called *universal* if it is versal and h is uniquely defined. This suffices to imply that f is an effective versal deformation of X ([32], § 2.7.1).

The following result is a central one in the theory of deformation of complex spaces: it is due to many authors as A. Douady [6], H. Grauert [9] and [10], V.P. Palamodov [30], among many others.

Theorem 1 (Existence of a versal deformation)

1. Any representative U of an isolated singularity (X, x) has a miniversal deformation

$$f : (U, U) \rightarrow (B, o)$$

(see [9], [11]).

2. Any (compact) complex space X has an effective versal deformation

$$f : (\mathcal{X}, X) \rightarrow (B, o)$$

(see [30] for the non compact case and [6], [10], [31] for the compact case).

3. The germ of complex space (B, o) obtained by the previous parts (1) and (2) is isomorphic to the germ $(q_X^{-1}(0), 0)$, where $q_X : \mathbb{T}_X^1 \rightarrow \mathbb{T}_X^2$ is a suitable holomorphic map (the obstruction map) such that $q_X(0) = 0$ (see [11] II.1.5, [32] § 2.5)

In particular if $q_X \equiv 0$ (e.g. when $\mathbb{T}_X^2 = 0$) then (B, o) turns out to be isomorphic to the complex space germ $(\mathbb{T}_X^1, 0)$.

Definition 3 (Kuranishi space and number) The germ of complex space (B, o) of parts (1) and (2) of the previous Theorem 1 is called the *Kuranishi space of either (X, x) or X , respectively*: in the first case one can easily check that it does not depend on the choice of the representative U . The Kuranishi space is said to give an *analytic representative* of the deformation functors $\text{Def}(U)$ and $\text{Def}(X)$, respectively, allowing one to set the identifications

either $(\text{Def}(X, x) :=) \text{Def}(U) = (B, o)$ or $\text{Def}(X) = (B, o)$, respectively.

The *Kuranishi numbers* $\text{def}(X, x)$ of (X, x) and $\text{def}(X)$ of X , are then the maximum dimensions of irreducible components of $\text{Def}(X, x)$ and $\text{Def}(X)$, respectively. $\text{Def}(X, x)$ and $\text{Def}(X)$ are said to be *smooth*, or *unobstructed*, if the obstruction map q_X is the constant map $q_X \equiv 0$. This means that any first order deformation arises to give a deformation of either U or X , respectively. In this case either $\text{Def}(X, x) \cong (T_U^1, 0)$ or $\text{Def}(X) \cong (T_X^1, 0)$, respectively, giving either $\text{def}(X, x) = \dim_{\mathbb{C}} \mathbb{T}_U^1$ or $\text{def}(X) = \dim_{\mathbb{C}} \mathbb{T}_X^1$, respectively. By a slight abuse of notation, we will write

either $\text{Def}(X, x) \cong \mathbb{T}_U^1$ or $\text{Def}(X) \cong \mathbb{T}_X^1$, respectively.

Theorem 2 ([31] Thm. 5.5) Let X be a compact complex space such that $\mathbb{T}_X^0 = 0$. Then the versal effective deformation of X , given in part (2) of Theorem 1, is actually a universal deformation of X .

Example 1 (The germ of complex space of an isolated hypersurface singularity) Let $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$, $n \geq 0$, be a holomorphic map admitting an isolated critical point in the origin $0 \in \mathbb{C}^{n+1}$ and consider the local ring \mathcal{O}_0 of germs of holomorphic function of \mathbb{C}^{n+1} in the origin. It is a well known fact that \mathcal{O}_0 is isomorphic to the ring of convergent power series $\mathbb{C}\{x_1, \dots, x_{n+1}\}$. A *germ of hypersurface singularity* $(U_0, 0)$ is defined by means of the Stein complex space

$$U_0 := \text{Spec}(\mathcal{O}_{f,0}) \quad (4)$$

where $\mathcal{O}_{f,0} := \mathcal{O}_0/(\bar{f})$ and \bar{f} is the germ represented by the series expansion of the above given holomorphic function f . Let $J_f \subset \mathbb{C}\{x_1, \dots, x_{n+1}\}$ be the *Jacobian ideal* of f i.e. the ideal generated by the partial derivatives of f . Then the following facts hold:

- since U_0 is Stein, (3) gives that $\mathbb{T}_{U_0}^i \cong T_{U_0}^i$,
- since we are dealing with an isolated hypersurface singularity,

$$\mathbb{T}_{U_0}^i \cong T_{U_0}^i = 0, \quad \text{for } i \geq 2 \quad (5)$$

(for the case $i = 2$, which is all what is useful in the following, see e.g. [43] § 3 Example on pg. 26; for $i \geq 2$ see e.g. [11] Prop. C.4.6(3)),

- the Kuranishi space of an isolated hypersurface singularity is then completely described as follows

$$\text{Def}(U_0, 0) \cong \mathbb{T}_{U_0}^1 \cong \mathcal{O}_{f,0}/J_f \cong \mathbb{C}\{x_1, \dots, x_{n+1}\}/((\bar{f}) + J_f) \quad (6)$$

(the first isomorphism follows by Theorem 1 and the previous (5); for the second isomorphism see e.g. [43] § 3 Example on pg. 24, [11] Corollary II.1.17).

Example 2 (The deformation theory of a Calabi–Yau threefold) Let us now consider the case of a Calabi–Yau threefold X . A central result in the deformation theory of Calabi–Yau manifolds is the well-known Bogomolov–Tian–Todorov–Ran Theorem [3],[45],[46],[35] asserting that the Kuranishi space $\text{Def}(X)$ is smooth, hence $\text{Def}(X) \cong \mathbb{T}_X^1$ and (2) gives that

$$\text{def}(X) = \dim_{\mathbb{C}} \mathbb{T}_X^1 = h^1(X, \Theta_X) = h^{2,1}(X) \quad (7)$$

where the last equality on the right is obtained by the Calabi–Yau condition $\mathcal{K}_X \cong \mathcal{O}_X$. Applying the Calabi–Yau condition once again gives $h^0(\Theta_X) = h^{2,0}(X) = 0$. Therefore (1) and Theorem 2 give the existence of a *universal effective family of Calabi–Yau deformations of X* . In particular $h^{2,1}(X)$ turns out to be the dimension of the complex moduli space of X .

2.2 Deformation of a morphism

Let us quickly recall the concept of *deformation of a morphism* as defined by Z. Ran in [34].

Consider a morphism $\phi : Y \rightarrow X$ of complex spaces and let B be a connected complex space with a special point $o \in B$ such that $g : (\mathcal{Y}, Y) \rightarrow (B, o)$ and $f : (\mathcal{X}, X) \rightarrow (B, o)$ are deformation families of Y and X , respectively. Then a *deformation family of the morphism ϕ* is a morphism $\Phi : \mathcal{Y} \rightarrow \mathcal{X}$ such that the following diagram commutes

$$\begin{array}{ccc} Y \subset & \xrightarrow{\quad} & \mathcal{Y} \\ \phi \searrow & & \swarrow \Phi \\ & X \subset & \mathcal{X} \\ \downarrow & \xrightarrow{\quad} & \downarrow f \\ o & \in & B \end{array} \quad (8)$$

with $Y = g^{-1}(o)$, $X = f^{-1}(o)$ and $\phi = \Phi|_{g^{-1}(o)}$. Given two distinct points $b_1, b_2 \in B$ the morphism $\phi_2 := \Phi|_{g^{-1}(b_2)}$ is called a *deformation* of the morphism $\phi_1 := \Phi|_{g^{-1}(b_1)}$ and viceversa.

2.3 The Friedman diagram

Let $\phi : Y \rightarrow X$ be a birational contraction of a Calabi–Yau manifold Y , of dimension $n \geq 3$, to a normal variety X with isolated rational singularities and assume that the codimension of the exceptional locus $E := \text{Exc}(\phi)$ is greater than or equal to 2: hence ϕ is what is usually called a *small resolution* of X . This latter assumption ensures that the Friedman argument of Lemma (3.1) in [7] still applies to give that $R^0\phi_*\Theta_Y \cong \Theta_X$. Then we get the following commutative diagram, to which we refer as the *Friedman diagram* (see [7] (3.4)),

$$\begin{array}{ccccccc} H^1(R^0\phi_*\Theta_Y) \subset & \xrightarrow{\quad} & \mathbb{T}_Y^1 & \xrightarrow{\lambda_E} & H^0(R^1\phi_*\Theta_Y) & \longrightarrow & H^2(R^0\phi_*\Theta_Y) & \longrightarrow & \mathbb{T}_Y^2 \\ \parallel & & \downarrow \delta_1 & & \downarrow \delta_{loc} & & \parallel & & \downarrow \delta_2 \\ H^1(\Theta_X) \subset & \xrightarrow{\quad} & \mathbb{T}_X^1 & \xrightarrow{\lambda_P} & T_X^1 & \longrightarrow & H^2(\Theta_X) & \longrightarrow & \mathbb{T}_X^2 \end{array} \quad (9)$$

where:

- the first row is the lower terms exact sequence of the Leray spectral sequence of $\phi_*\Theta_Y$, where we can use (2) as Y is smooth;
- the second row is the lower terms exact sequence of the Local to Global spectral sequence converging to $\mathbb{T}_X^n = \text{Ext}^n(\Omega_X^1, \mathcal{O}_X)$;
- the vertical maps

$$\mathbb{T}_Y^i = \text{Ext}^i(\Omega_Y, \mathcal{O}_Y) \xrightarrow{\delta_i} \mathbb{T}_X^i = \text{Ext}^i(\Omega_X, \mathcal{O}_X)$$

arise from the natural map $\phi^*\Omega_X \rightarrow \Omega_Y$ just recalling that X has rational singularities, hence giving

$$\text{Ext}^i(\Omega_X, \mathcal{O}_X) = \text{Ext}^i(\Omega_X, \phi_*\mathcal{O}_Y) = \text{Ext}^i(\phi^*\Omega_X, \mathcal{O}_X) ;$$

Proposition 1 ([7] Prop. (2.1)) *Let U_p be a Stein neighborhood of the singular point $p \in P = \text{Sing}(X)$ and set $V_p := \phi^{-1}(U_p)$. Let $\text{Def}(U_p)$ and $\text{Def}(V_p)$ be the Kuranishi spaces defined in parts 1 and 2 (the non-compact case) of Thm. 1, respectively. Then, under hypothesis given above, one gets:*

1. $\mathbb{T}_{V_p}^1 \cong H^1(V_p, \Theta_{V_p}) \cong H^0(R^1\phi_*\Theta_{V_p})$ and $H^0(R^1\phi_*\Theta_Y) \cong \bigoplus_{p \in P} \mathbb{T}_{V_p}^1$ is the tangent space to $\prod_{p \in P} \text{Def}(V_p)$,
2. $\mathbb{T}_{U_p}^1 \cong T_{U_p}^1$ and $\mathbb{T}_X^1 \cong \bigoplus_{p \in P} T_{U_p}^1$ is the tangent space to $\prod_{p \in P} \text{Def}(U_p)$,
3. morphisms δ_{loc} and δ_1 in the Friedman diagram (9) are injective.

Proof Parts 1 and 2 follow immediately by Prop. (2.1), part 1, in [7], recalling the previously displayed formulas (2) and (3). To prove part 3 notice, on the one hand, that the injectivity of δ_1 follows from the injectivity δ_{loc} , by an easy diagram chase. On the other hand, δ_{loc} is injective by Prop. (2.1), part 2, in [7].

Remark 1 Since Y is a Calabi–Yau 3-fold, the Bogomolov–Tian–Todorov–Ran Theorem gives that $\text{Def}(Y) \cong H^1(\Theta_Y)$. Moreover ϕ is a small resolution giving that $\text{Sing}(X)$ is composed at most by terminal singularities of index 1: in [28], Theorem A, Y. Namikawa proved an extension of the Bogomolov–Tian–Todorov–Ran Theorem allowing to conclude that $\text{Def}(X)$ is smooth also in the present situation, hence giving $\text{Def}(X) \cong \mathbb{T}_X^1$. By the previous Prop. 1, the localization near to a singular point $p \in \text{Sing}(X)$ of the second square on the left of the Friedman diagram (9) can then be rewritten as follows

$$\begin{array}{ccc} \text{Def}(Y) \cong H^1(\Theta_Y) & \xrightarrow{\lambda_{E_p}} & \text{Def}(V_p) \cong H^0(R^1\phi_*\Theta_{V_p}) \\ \downarrow \delta_1 & & \downarrow \delta_{loc,p} \\ \text{Def}(X) \cong \mathbb{T}_X^1 & \xrightarrow{\lambda_p} & \text{Def}(U_p) \cong T_{U_p}^1 \end{array} \quad (10)$$

where $E_p = \phi^{-1}(p)$ is the exceptional locus over p . In the following we will refer to this diagram as the *local Friedman diagram*. Notice that all the maps involved in this diagram are compatible with corresponding natural transformations between the associated deformation functors. For a deeper understanding of this fact the interested reader is referred to [47, § 1], [7, (3.4)], [20, (11.3-4)] and references therein.

2.4 Fiber products of rational elliptic surfaces with sections

In the present subsection we will review some well known facts about rational elliptic surfaces with section and their fiber products. For further details the reader is referred to [25], [26] and [42].

Let Y and Y' be *rational elliptic surfaces with sections* i.e. rational surfaces admitting elliptic fibrations over \mathbb{P}^1

$$r : Y \longrightarrow \mathbb{P}^1 \quad , \quad r' : Y' \longrightarrow \mathbb{P}^1$$

with distinguished sections σ_0 and σ'_0 , respectively (notation as in [42] and [26]). Define

$$X := Y \times_{\mathbb{P}^1} Y' . \quad (11)$$

Write S (resp. S') for the images of the singular fibers of Y (resp. Y') in \mathbb{P}^1 .

Proposition 2

1. *The fiber product X is smooth if and only if $S \cap S' = \emptyset$. In particular, if smooth, X is a Calabi–Yau threefold ([42] §2) and $\chi(X) = 0$.*
2. *Y (resp. Y') is the blow-up of \mathbb{P}^2 at the base locus of a rational map $\varrho : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ (resp. $\varrho' : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$) ([26] Prop. 6.1).*
3. *If $Y = Y'$ is sufficiently general and $r(=r')$ admits at most nodal fibers, then there always exists a small projective resolution \widehat{X} of X ([42] Lemma (3.1)). Moreover if Y has exactly $\nu \geq 0$ nodal fibers then $\chi(X) = \nu$ and $\chi(\widehat{X}) = 2\nu$.*

Theorem 3 (Weierstrass representation of an elliptic surface with section, [17] Thm. 1, [25] Thm. 2.1, [14] §2.1 and proof of Prop. 2.1) *Let $r : \widetilde{Y} \rightarrow C$ be a relatively minimal elliptic surface over a smooth base curve C , whose generic fibre is smooth and admitting a section $\sigma : C \rightarrow \widetilde{Y}$ (then \widetilde{Y} is algebraic [18]). Let \mathcal{L} be the co-normal sheaf of $\sigma(C) \subset \widetilde{Y}$.*

Then \mathcal{L} is invertible and there exists

$$A \in H^0(C, \mathcal{L}^{\otimes 4}) \quad , \quad B \in H^0(C, \mathcal{L}^{\otimes 6})$$

such that \widetilde{Y} is the minimal resolution of the closed subscheme Y of the projectivized bundle $\mathbb{P}(\mathcal{E}) := \mathbb{P}(\mathcal{L}^{\otimes 3} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{O}_C)$ defined by the zero locus of the homomorphism

$$(A, B) : \mathcal{E} = \mathcal{L}^{\otimes 3} \oplus \mathcal{L}^{\otimes 2} \oplus \mathcal{O}_C \longrightarrow \mathcal{L}^{\otimes 6} \quad (12)$$

$$(x, y, z) \longmapsto -x^2z + y^3 + A yz^2 + B z^3 .$$

The pair (A, B) (hence the homomorphism (12)) is uniquely determined up to the transformation $(A, B) \mapsto (c^4A, c^6B)$, $c \in \mathbb{C}^$ and the discriminant form*

$$\delta := 4 A^3 + 27 B^2 \in H^0(C, \mathcal{L}^{\otimes 12})$$

vanishes at a point $\lambda \in C$ if and only if the fiber $Y_\lambda := r^{-1}(\lambda)$ is singular.

Remark 2 *Assume that the elliptic surface $r : Y \rightarrow C$ is rational. Then $C \cong \mathbb{P}^1$ is a rational curve and the section $\sigma(C)$ is a (-1) -curve in Y (see [25] Proposition (2.3) and Corollary (2.4)). In particular $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ and (12) is a homomorphism*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \xrightarrow{(A, B)} \mathcal{O}_{\mathbb{P}^1}(6) . \quad (13)$$

Consider the fiber product

$$X := Y \times_{\mathbb{P}^1} Y$$

of the Weierstrass fibration defined as the zero locus $Y \subset \mathbb{P}(\mathcal{E})$ of the bundle homomorphism (13). Hence, for generic A, B , the rational elliptic surface Y has smooth generic fiber and a finite number of distinct singular fibers associated with the zeros of the discriminant form $\delta = 4A^3 + 27B^2$. In general the singular fibers are nodal and $\text{Sing}(X)$ is composed by a finite number $\nu = 12$ of distinct nodes. We can then apply Proposition 2(3) to guarantee the existence of a small resolution $\widehat{X} \rightarrow X$ whose exceptional locus is the union of disjoint $(-1, -1)$ -curves, i.e. rational curves $C \cong \mathbb{P}^1$ in X whose normal bundle is $\mathcal{N}_{C|X} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

Anyway, if either A and B have a common root or $A \equiv 0$, the Weierstrass fibration Y may admit *cuspidal fibers*: in this case the existence of a small resolution for X is no more guaranteed by Proposition 2(3).

3 The Namikawa fiber product

In [29], §0.1, Y. Namikawa considered the Weierstrass fibration associated with the bundle homomorphism

$$(0, B) : \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(3) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(6) \quad (14)$$

$$(x, y, z) \longmapsto -x^2z + y^3 + B(\lambda)z^3$$

i.e. its zero locus $Y \subset \mathbb{P}(\mathcal{E})$. The associated discriminant form is $\delta(\lambda) = 27B(\lambda)^2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(12))$ whose roots are precisely those of $B \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$. Hence, for a generic B , the rational elliptic surface $Y \rightarrow \mathbb{P}^1$ has smooth generic fiber and six distinct *cuspidal fibers*.

Proposition 3 *The fiber product $X := Y \times_{\mathbb{P}^1} Y$ is a threefold admitting 6 threefold cups whose local type is described by the singularity*

$$X^2 - U^2 - Y^3 + V^3 \in \mathbb{C}[X, Y, U, V] . \quad (15)$$

In the standard Kodaira notation these are singularities of type $II \times II$ ([18], Theorem 6.2).

Proof Our hypothesis give

$$\mathbb{P}(\mathcal{E}) \times \mathbb{P}^1[\lambda] \supset Y : x^2z = y^3 + B(\lambda)z^3 . \quad (16)$$

Then its fiber self-product can be represented as follows

$$\mathbb{P} := \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1[\lambda] \supset X : \begin{cases} x^2z = y^3 + B(\lambda)z^3 \\ u^2w = v^3 + B(\lambda)w^3 \end{cases} . \quad (17)$$

Since the problem is a local one, let us consider the open subset $\mathcal{A} \subset \mathbb{P}$ defined by setting

$$\mathcal{A} := \{(x : y : z) \times (u : v : w) \times (\lambda_0 : \lambda_1) \mid z \cdot w \cdot \lambda_1 \neq 0\} \cong \mathbb{C}^5(X, Y, U, V, t) \quad (18)$$

where $X = x/z, Y = y/z, U = u/w, V = v/w$ and $t = \lambda_0/\lambda_1$. Then $\mathcal{A} \cap X$ can be locally described by equations

$$\begin{cases} X^2 = Y^3 + B(t) \\ U^2 = V^3 + B(t) \end{cases} , \quad (19)$$

where $B(\lambda) = \lambda_1^6 B(t)$. If t_0 is a zero of the discriminant $\delta(t) = 27B(t)^2$ then $p_{t_0} = ((0 : 0 : 1), (0 : 0 : 1), (t_0 : 1)) \in X$ is a singular point, whose local equations are obtained from (19) by replacing $B(t)$ with its Taylor expansion in a neighborhood of t_0 , giving

$$\begin{cases} X^2 = Y^3 + t^i B^{(i)}(t_0)/i! + o(t^i) \\ U^2 = V^3 + t^i B^{(i)}(t_0)/i! + o(t^i) \end{cases}, \quad (20)$$

where i is the minimum order of derivatives such that $B^{(i)}(t_0) = dB^{(i)}(t_0)/dt^i$ does not vanish. Then the germ described by equations (20) turns out to be the same described by the local equation $X^2 - U^2 - Y^3 + V^3 = 0$.

As already observed above, Proposition 2(3) can then no more be applied to guarantee the existence of a small resolutions $\widehat{X} \rightarrow X$. In this case Y. Namikawa proved the following

Proposition 4 ([29] Example in §0.1) *The cuspidal fiber product $X = Y \times_{\mathbb{P}^1} Y$ associated with the Weierstrass fibration Y , defined as the zero locus in $\mathbb{P}(\mathcal{E})$ of the bundle homomorphism (14), admits six small resolutions which are connected to each other by flops of $(-1, -1)$ -curves. The exceptional locus of any such resolution is given by six disjoint couples of $(-1, -1)$ -curves intersecting in one point.*

Proof Let us first of all sketch the idea of the proof. The resolutions of X are constructed by considering the strict transform of X into a successive double blow up of the projectivized bundle $\mathbb{P} := \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1$ in which X is embedded. Then the cusps of X are firstly resolved to nodes and then finally resolved to smooth points. The key point of the construction is that $\text{Sing}(X)$ is contained in the diagonal locus Δ of \mathbb{P} ; moreover X turns out to be invariant under the action of a cyclic group of order 6 acting on \mathbb{P} . Then the six resolutions of X are constructed by a successive blow up of suitable couples of images of Δ under the action of this cyclic group.

Give X by equations (17) and consider the following cyclic map on \mathbb{P}

$$\begin{aligned} \tau : \quad \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 &\longrightarrow \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 \\ ((x : y : z), (u : v : w), \lambda) &\longmapsto ((x : y : z), (-u : \epsilon v : w), \lambda) \end{aligned}, \quad (21)$$

where ϵ is a primitive cubic root of unity. The second equation in (17) ensures that $\tau X = X$. Since τ generates a cyclic group of order 6, the orbit of the codimension 2 diagonal locus

$$\Delta := \{((x : y : z), (u : v : w), \lambda) \in \mathbb{P}(\mathcal{E}) \times \mathbb{P}(\mathcal{E}) \times \mathbb{P}^1 \mid (x : y : z) = (u : v : w)\}$$

is given by six distinct codimension 2 cycles $\{\tau^i \Delta \mid 0 \leq i \leq 5\}$. For any i , $\tau^i \Delta$ cuts on X a Weil divisor $D_i := \tau^i \Delta \cap X$ containing $\text{Sing}(X)$: in fact

$$\begin{aligned} \text{Sing}(X) &= \{((0 : 0 : 1), (0 : 0 : 1), \lambda) \in \mathbb{P} \mid B(\lambda) = 0\} \subset \\ D_i &= \{((x : y : z), ((-1)^i x : \epsilon^i y : z), \lambda) \in \mathbb{P} \mid x^2 z - y^3 - B(\lambda) z^3 = 0\}. \end{aligned}$$

Then

$$\text{Sing}(X) = \bigcap_{i=0}^5 D_i.$$

Let \mathbb{P}_i be the blow-up of \mathbb{P} along $\tau^i \Delta$: the exceptional divisor is a \mathbb{P}^1 -bundle over $\tau^i \Delta$. Let X_i be the strict transform of X in the blow-up $\mathbb{P}_i \rightarrow \mathbb{P}$. Since $\text{Sing}(X)$ is entirely composed by singularities of type (15), X_i is singular and $\text{Sing}(X_i)$ contains only nodes. Moreover $X_i \rightarrow X$ turns out to be a small partial resolution whose

exceptional locus is a union of disjoint $(-1, -1)$ -curves, one over each singular point of X .

Consider the strict transform $(\tau^{i+1}\Delta)_i$ of $\tau^{i+1}\Delta$ in the blow-up $\mathbb{P}_i \rightarrow \mathbb{P}$. Let $\widehat{\mathbb{P}}_i$ be the blow-up of \mathbb{P}_i along $(\tau^{i+1}\Delta)_i$ and \widehat{X}_i be the strict transform of X_i in $\widehat{\mathbb{P}}_i$. Then

– $\widehat{X}_i \rightarrow X$ is a smooth small resolution satisfying the statement, for any $0 \leq i \leq 5$.

To prove this fact we have to check that:

- i.* the exceptional locus of the resolution $\widehat{X}_i \rightarrow X$ is actually composed by disjoint couples of $(-1, -1)$ -curves intersecting in one point,
- ii.* the resolutions $\widehat{X}_i \rightarrow X$ are to each other connected by flops of $(-1, -1)$ -curves.

Let us prove *i* locally, by explicitly computing the induced resolution of a singular point of type (15) over the open subset $\mathcal{A} \subset \mathbb{P}$ defined in (18). Up to an isomorphism we may always assume that $B(1 : 0) \neq 0$, implying that $\text{Sing}(X) \subset \mathcal{A} \cap X$. Let us assume that $B(t_0 : 1) = 0$, then

$$p_{t_0} := ((0 : 0 : 1), (0 : 0 : 1), (t_0 : 1)) \in \text{Sing}(X)$$

is a threefold cusp whose local equation (15) can be factored as follows

$$(X - U)(X + U) = (Y - V)(Y - \epsilon V)(Y - \epsilon^2 V). \quad (22)$$

Let \mathcal{A}' be the section of \mathcal{A} with the hyperplane $t = t_0$. Then

$$\mathcal{A}' \cap \tau^i \Delta = \{(X, Y, U, V, t_0) \in \mathbb{C}^5 \mid X - (-1)^i U = Y - \epsilon^i V = 0\} \cong \mathbb{C}^2.$$

Rewrite (22) as $xy = uv[(1 + \epsilon)v - \epsilon u]$, where

$$\begin{aligned} x &= X - (-1)^i U, & y &= X - (-1)^{i+1} U, \\ u &= Y - \epsilon^i V, & v &= Y - \epsilon^{i+1} V, & w &= Y - \epsilon^{i+2} V. \end{aligned}$$

The blow up $\mathbb{P}_i \rightarrow \mathbb{P}$ of $\tau^i \Delta$ induces over \mathcal{A}' the blow up $\mathcal{A}'_i \rightarrow \mathcal{A}'$ of the plane $x = u = 0$. The strict transform X_i is then locally given by

$$\mathcal{A}'_i \cap X_i : \begin{cases} \mu_1 x = \mu_0 u \\ \mu_0 y = \mu_1 v[(1 + \epsilon)v - \epsilon u] \end{cases}$$

where $\mathbb{P}^1[\mu_0, \mu_1]$ is the small exceptional locus of $\mathcal{A}'_i \cap X_i \rightarrow \mathcal{A}' \cap X$. Notice that $\mathcal{A}'_i \cap X_i$ is still singular admitting a node in the point $((\mathbf{0}, t_0), (0 : 1)) \in \mathcal{A}'_i \times \mathbb{P}^1$. On the other hand, the strict transform $(\tau^{i+1}\Delta)_i$ of $\tau^{i+1}\Delta$ is locally given by

$$\begin{cases} \mu_1 x = \mu_0 u \\ y = v = 0 \end{cases} \quad (23)$$

The blow up $\widehat{\mathbb{P}}_i \rightarrow \mathbb{P}_i$ of \mathbb{P}_i along $(\tau^{i+1}\Delta)_i$ induces over \mathcal{A}'_i the blow up $\widehat{\mathcal{A}}'_i \rightarrow \mathcal{A}'_i$, along (23). Then the strict transform \widehat{X}_i of X is locally described as the following codimension three closed subset of $\mathcal{A}' \times \mathbb{P}^1[\mu] \times \mathbb{P}^1[\nu]$

$$\widehat{\mathcal{A}}'_i \cap \widehat{X}_i : \begin{cases} \mu_1 x = \mu_0 u \\ \nu_1 y = \nu_0 v \\ \mu_0 \nu_0 = \mu_1 \nu_1 [(1 + \epsilon)v - \epsilon u] \end{cases}.$$

Observe that:

- $\widehat{\mathcal{A}}'_i \cap \widehat{X}_i$ is smooth,
- $\widehat{\mathcal{A}}'_i \cap \widehat{X}_i \rightarrow \mathcal{A}' \cap X$ is an isomorphism outside of $(\mathbf{0}, t_0) \in \mathcal{A}' \cap X$, which locally represents $p_{t_0} \in \text{Sing}(X)$,

- the exceptional fiber over $(\mathbf{0}, t_0) \in \mathcal{A}' \cap X$ is described by the closed subset $\{\mu_0\nu_0 = 0\} \subset \mathbb{P}^1[\mu] \times \mathbb{P}^1[\nu]$, which is precisely a couple of \mathbb{P}^1 's meeting in the point $((\mathbf{0}, t_0), (0 : 1), (0 : 1)) \in \widehat{\mathcal{A}}'_i \cap \widehat{X}_i \subset \mathcal{A}' \times \mathbb{P}^1 \times \mathbb{P}^1$,
- by construction any exceptional \mathbb{P}^1 is a $(-1, -1)$ -curve.

To prove *ii*, it suffices to show that:

- for any $0 \leq i \leq 5$ the following flops of $(-1, -1)$ -curves exist:

$$\begin{array}{ccc} X_i & \leftarrow \text{-----} \rightarrow & X_{i+2} \\ & \searrow & \swarrow \\ & X & \end{array} \quad , \quad \begin{array}{ccc} X_i & \leftarrow \text{-----} \rightarrow & X_{i+3} \\ & \searrow & \swarrow \\ & X & \end{array} .$$

As before rewrite the local equation (22) as

$$xy = uv[(1 + \epsilon)v - \epsilon u] \quad \text{in } \mathbb{C}[x, y, u, v] .$$

Then locally X_i corresponds to blow up the plane $x = u = 0$ of \mathbb{C}^4 while X_{i+2} corresponds to blow up the plane $x = v = 0$. Ignore the term $[(1 + \epsilon)v - \epsilon u]$: then our situation turns out to be similar to the well known *Kollár quadric* ([19] Example 3.2) giving a flop

$$\begin{array}{ccc} X_i & \leftarrow \text{-----} \rightarrow & X_{i+2} \\ & \searrow & \swarrow \\ & X & \end{array} .$$

Analogously X_{i+3} corresponds to blow up $y = u = 0$ still getting a flop

$$\begin{array}{ccc} X_i & \leftarrow \text{-----} \rightarrow & X_{i+3} \\ & \searrow & \swarrow \\ & X & \end{array} .$$

3.1 Deformations and resolutions

Let $X = Y \times_{\mathbb{P}^1} Y$ be the Namikawa fiber product defined above, starting from the bundle's homomorphism (14). For a general $b \in H^0(\mathcal{O}_{\mathbb{P}^1}(6))$, the singular locus $\text{Sing}(X)$ is composed by six cusps of type (15). Let us rewrite the local equation of this singularity as follows

$$x^2 - y^3 = z^2 - w^3 . \quad (24)$$

It is a singular point of Kodaira type $II \times II$. Moreover it is a *compound Du Val singularity* of cA_2 type i.e. a threefold point p such that, for a hyperplane section H through p (in the present case assigned e.g. by $w = 0$), $p \in H$ is a Du Val surface singularity of type A_2 (see [36], §0 and §2, and [2], chapter III).

Recalling (6), the Kuranishi space of the cusp (24) is the \mathbb{C} -vector space

$$\mathbb{T}^1 \cong T^1 \cong \mathcal{O}_{F,0}/J_F \cong \mathbb{C}\{x, y, z, w\}/((F) + J_F) \cong \langle 1, y, w, yw \rangle_{\mathbb{C}} \quad (25)$$

where $F = x^2 - y^3 - z^2 + w^3$ and J_F is the associated Jacobian ideal. Then a miniversal deformation of (24) is given by the zero locus of

$$F_A : x^2 - y^3 - z^2 + w^3 + \lambda + \mu y - \nu w + \sigma yw \in \mathbb{C}[x, y, z, w] , \quad A = (\lambda, \mu, -\nu, \sigma) \in \mathbb{T}^1 . \quad (26)$$

The fibre $\mathcal{X}_\Lambda = \{F_\Lambda = 0\}$, of the miniversal deformation family $\mathcal{X} \rightarrow \mathbb{T}^1$, is singular if and only if the Jacobian rank of the polynomial function F_Λ is not maximum at some zero point of F_Λ . Singularities are then given by $(0, y, 0, w) \in \mathbb{C}^4$ such that

$$\begin{aligned} 3y^2 - \sigma w - \mu &= 0 \\ 3w^2 + \sigma y - \nu &= 0 \\ \sigma y w + 2\mu y - 2\nu w + 3\lambda &= 0 \end{aligned} \tag{27}$$

where the first two conditions come from partial derivatives of F_Λ and the latter is obtained by applying the first two conditions to the vanishing condition $F_\Lambda(0, y, 0, w) = 0$.

Proposition 5 *A fibre of the miniversal deformation family $\mathcal{X} \rightarrow \mathbb{T}^1$ of the cusp (24) admits at most three singular points.*

Proof It is a direct consequence of conditions (27). Fix a point

$$\Lambda = (l, \mu, -\nu, \sigma) \in \mathbb{T}^1 .$$

If $\sigma = 0$ then conditions (27) become conditions (31) below. The argument given in the proof of Proposition 6 shows that common solutions of (31) cannot be more than two.

Let us then assume that $\sigma \neq 0$. The first equation in (27) gives

$$\sigma w = 3y^2 - \mu \tag{28}$$

and the third equation multiplied by σ gives

$$\sigma^2 y w + 2\mu \sigma y - 2\nu \sigma w + 3\lambda \sigma = 0 . \tag{29}$$

Put (28) in (29) to get

$$3\sigma y^3 - 6\nu y^2 + \mu \sigma y + 2\mu \nu + 3\lambda \sigma = 0 .$$

Fixing $\Lambda = (l, \mu, -\nu, \sigma) \in \mathbb{T}^1$, the latter is a cubic polynomial in the unique unknown variable y . Then the common solutions of equations (27) cannot be more than 3.

For any $p \in \text{Sing}(X)$ let $U_p = \text{Spec } \mathcal{O}_{F,p}$ be a representative of the complex space germ locally describing the singularity $p \in X$. By Theorem 1, Definition 3, (25) and (26)

$$\text{Def}(U_p) \cong \mathbb{T}^1 \cong \langle 1, y, w, yw \rangle_{\mathbb{C}} \tag{30}$$

Recalling the local Friedman diagram (10), consider the localization map

$$\lambda_p : \text{Def}(X) \cong \mathbb{T}_X^1 \longrightarrow \text{Def}(U_p) \cong \mathbb{T}^1 .$$

Proposition 6 *The deformations of the cusp (24) induced by localizing a deformation of the fiber product X are based on a hyperplane of the Kuranishi space \mathbb{T}^1 in (25). More precisely, using coordinates introduced in (26),*

$$\forall p \in \text{Sing}(X) \quad \text{Im}(\lambda_p) = S := \{\sigma = 0\} \subset \mathbb{T}^1 .$$

In particular every deformation parameterized by S may admit at most 2 singular points which are

- ordinary double points if $\mu \cdot \nu \neq 0$,

– compound Du Val of type cA_2 if $\mu \cdot \nu = 0$ and precisely of Kodaira type

$$\begin{aligned} & II \times II \text{ if } \mu = \nu = 0, \\ & I_1 \times II \text{ otherwise.} \end{aligned}$$

Proof Let $X = Y \times_{\mathbb{P}^1} Y$ be the Namikawa fiber product given by equations (17). A deformation family \mathcal{Y} of the rational elliptic surface $r : Y \rightarrow \mathbb{P}^1$ comes equipped with a natural morphism $\mathcal{R} : \mathcal{Y} \rightarrow \mathbb{P}^1$ such that $\mathcal{R}|_Y = r$. Then a deformation family of X is obtained as the fiber product $\mathcal{X} = \mathcal{Y} \times_{\mathbb{P}^1} \mathcal{Y}'$ of two deformation families of Y (this fact is explained in several points of the Namikawa's paper [29], e.g. in the Introduction and in Remark 2.8). The Kuranishi space of the cusp $\{x^2 - y^3 = 0\}$ is given by

$$\mathbb{T}_{\text{cusp}}^1 \cong \mathbb{C}\{x, y\}/(x^2 - y^3, x, y^2) \cong \langle 1, y \rangle_{\mathbb{C}}$$

meaning that a deformation of Y can be locally given by the germ

$$x^2 - y^3 + \lambda + \mu y - t^i B^i(t_0) = 0 \quad , \quad (\lambda, \mu) \in \mathbb{T}_{\text{cusp}}^1 ,$$

where t_0 is a zero of $B(t)$ and $B^i(t_0)$ has the same meaning as in (20). Therefore, by applying an analogous local analysis as that given in the proof of Proposition 3 and recalling (26), it turns out that a deformation of the threefold cusp (24) induced by localizing a deformation of X , is given by the germ of complex space

$$\begin{cases} x^2 - y^3 + \lambda_1 + \mu y - t^i B^i(t_0) = 0 \\ z^2 - w^3 + \lambda_2 + \nu w - t^i B^i(t_0) = 0 \end{cases}$$

giving the following local equation

$$x^2 - y^3 + \lambda_1 + \mu y = z^2 - w^3 + \lambda_2 + \nu w .$$

Then all such deformations span the subspace

$$\{(\lambda_1 - \lambda_2, \mu, -\nu, 0)\} = \{\sigma = 0\} \subset \mathbb{T}^1 .$$

Setting $\sigma = 0$ in conditions (27) gives the following equations

$$\begin{aligned} 3y^2 - \mu &= 0 \\ 3w^2 - \nu &= 0 \\ 2\mu y - 2\nu w + 3\lambda &= 0 \end{aligned} \tag{31}$$

which can be visualized in the y, w -plane as follows:

- the first condition as two parallel and symmetric lines with respect to the y -axis; they may coincide with the y -axis when $\mu = 0$;
- the second condition as two parallel and symmetric lines with respect to the w -axis; they may coincide with the w -axis when $\nu = 0$;
- the last condition as a line in general position in the y, w -plane.

Clearly, fixing the point $A = (l, \mu, -\nu, \sigma) \in \mathbb{T}^1$, it is not possible to have more than two distinct common solutions of (31) with respect to the variables y, w .

To analyze the singularity type, let $p_A = (0, y_\mu, 0, w_\nu)$ be a singular point of

$$F_A : x^2 - y^3 - z^2 + w^3 + \lambda + \mu y - \nu w = 0$$

and translate p_A to the origin by replacing

$$y \mapsto y + y_\mu \quad , \quad w \mapsto w + w_\nu .$$

Then conditions (31) impose that the translated F_A is

$$\tilde{F}_A = x^2 - y^3 - z^2 + w^3 - 3y_\mu y^2 + 3w_\nu w^2$$

giving the classification above.

Corollary 1 *If the deformation \mathcal{X}_Λ of the cusp (24), associated with $\Lambda \in \mathbb{T}^1$, admits three distinct singular points then $\Lambda \in \mathbb{T}^1 \setminus S$, which is*

$$\Lambda = (\lambda, \mu, -\nu, \sigma) \quad \text{with } \sigma \neq 0 .$$

Proposition 7 *The locus of the Kuranishi space \mathbb{T}^1 in (25), parameterizing deformations of the cusp (24) to 3 distinct nodes, is described by the plane smooth curve*

$$C = \{ \sigma^3 - 27\lambda = \mu = \nu = 0 \} \subset \mathbb{T}^1$$

transversally meeting the hyperplane S in the origin of \mathbb{T}^1 . In particular, $(0, 0, 0, 1) \in \mathbb{T}^1$ generates the tangent space in the origin to the base of a 1st-order deformation of the cusp (24) to three distinct nodes.

Proof Consider conditions (27): since, by Corollary 1, we can assume $\sigma \neq 0$, the first equation gives

$$w = \frac{3y^2 - \mu}{\sigma} .$$

Putting this in the second equation gives that

$$R_1 := 27y^4 - 18\mu y^2 + \sigma^3 y + 3\mu^2 - \nu\sigma^2 = 0 ,$$

and in the third equation gives that

$$R_2 := 3\sigma y^3 - 6\nu y^2 + \mu\sigma y + 2\mu\nu + 3\lambda\sigma = 0 .$$

Therefore, fixing $\Lambda = (l, \mu, -\nu, \sigma) \in \mathbb{T}^1$, consider R_1 and R_2 as polynomials in $\mathbb{C}[l, \mu, \nu, \sigma][y]$, i.e. in the unique unknown variable y . Then (27) admit three common solutions if and only if R_2 divides R_1 . Since the remainder of the division of R_1 by R_2 in $\mathbb{C}[l, \mu, \nu, \sigma][y]$ is

$$\frac{27}{\sigma^2}(4\nu^2 - \mu\sigma^2)y^2 + \frac{1}{\sigma}(\sigma^4 - 36\mu\nu - 27\lambda\sigma)y + \frac{1}{\sigma^2}(3\mu^2\sigma^2 - \nu\sigma^4 - 36\mu\nu^2 - 54\lambda\nu\sigma)$$

it turns out to be 0 if and only if Λ satisfies the following conditions

$$\begin{aligned} 4\nu^2 - \mu\sigma^2 &= 0 & (32) \\ \sigma^4 - 36\mu\nu - 27\lambda\sigma &= 0 \\ 3\mu^2\sigma^2 - \nu\sigma^4 - 36\mu\nu^2 - 54\lambda\nu\sigma &= 0 \end{aligned}$$

Then the first equation gives

$$\mu = 4\frac{\nu^2}{\sigma^2} .$$

Putting this in the second equations we get

$$\lambda = \frac{\sigma^3}{27} - \frac{16\nu^3}{3\sigma^3} .$$

Then, from the third equation in (32), we get the following factorization

$$\nu(4\nu - \sigma^2)(4\nu - \epsilon\sigma^2)(4\nu - \epsilon^2\sigma^2) = 0$$

where ϵ is a primitive cubic root of unity. All the solutions of (32) are then the following

$$\begin{aligned} A_0 &= \left(\frac{1}{27}\sigma^3, 0, 0, \sigma \right) , & A_1 &= \left(-\frac{5}{108}\sigma^3, \frac{1}{4}\sigma^2, -\frac{1}{4}\sigma^2, \sigma \right) , & (33) \\ A_2 &= \left(-\frac{5}{108}\sigma^3, \frac{\epsilon^2}{4}\sigma^2, -\frac{\epsilon}{4}\sigma^2, \sigma \right) , & A_3 &= \left(-\frac{5}{108}\sigma^3, \frac{\epsilon}{4}\sigma^2, -\frac{\epsilon^2}{4}\sigma^2, \sigma \right) . \end{aligned}$$

Let us first consider the second solution Λ_1 . In this particular case R_2 becomes

$$R_2 = 3\sigma y^3 - \frac{3}{2}\sigma^2 y^2 + \frac{1}{4}\sigma^3 y - \frac{1}{72}\sigma^4 = 3\sigma \left(y - \frac{\sigma}{6} \right)^3 ,$$

meaning that Λ_1 is actually the base of a trivial deformation of (24) since the fiber associated with σ admits the unique singular point $(0, \sigma/6, 0, -\sigma/6)$ which is still a cusp of type (24). Moreover solutions Λ_2 and Λ_3 give trivial deformations too, since they can be obtained from Λ_1 by replacing

$$\begin{aligned} \text{either } y &\longmapsto \epsilon y , \quad w \longmapsto \epsilon^2 w \quad (\text{giving } \Lambda_2) \\ \text{or } y &\longmapsto \epsilon^2 y , \quad w \longmapsto \epsilon w \quad (\text{giving } \Lambda_3) . \end{aligned}$$

It remains to consider the first solution Λ_0 . In this case R_2 becomes

$$R_2 = y^3 + \frac{\sigma^3}{27} = \left(y + \frac{\sigma}{3} \right) \left(y + \frac{\epsilon\sigma}{3} \right) \left(y + \frac{\epsilon^2\sigma}{3} \right) ,$$

then the deformation X_σ , $\sigma \neq 0$, turns out to admit three distinct nodes given by

$$\left(0, -\frac{\sigma}{3}, 0, \frac{\sigma}{3} \right) , \quad \left(0, -\frac{\epsilon\sigma}{3}, 0, \frac{\epsilon^2\sigma}{3} \right) , \quad \left(0, -\frac{\epsilon^2\sigma}{3}, 0, \frac{\epsilon\sigma}{3} \right) . \quad (34)$$

Notice that the base curve $\Lambda_0 \subset \mathbb{T}^1 \cong \mathbb{C}^4$ is actually the plane smooth curve $C = \{\sigma^3 - 27\lambda = \mu = \nu = 0\}$ meeting the hyperplane $S = \{\sigma = 0\}$ only in the origin, where they are transversal since a tangent vector to C in the origin is a multiple of $(0, 0, 0, 1)$. The statement is then proved by thinking \mathbb{T}^1 as the tangent space in the origin to the germ of complex space $\text{Def}(U_0)$ and representing the functor of 1st-order deformation of the cusp (24).

Let $\widehat{X} \xrightarrow{\phi} X$ be one of the six small resolutions constructed in Proposition 4 and consider the *localization near to* $p \in \text{Sing}(X)$

$$\begin{array}{ccc} \widehat{U}_p := \phi^{-1}(U_p) & \hookrightarrow & \widehat{X} \\ \downarrow \varphi & & \downarrow \phi \\ U_p := \text{Spec } \mathcal{O}_{F,p} & \hookrightarrow & X \end{array} \quad (35)$$

and consider the associated local Friedman diagram (10), which is the following commutative diagram between Kuranishi spaces

$$\begin{array}{ccc} \text{Def}(\widehat{X}) \cong H^1(\Theta_{\widehat{X}}) & \xrightarrow{\lambda_{E_p}} & \text{Def}(\widehat{U}_p) \cong H^0(R^1\phi_*\Theta_{\widehat{U}_p}) \\ \downarrow \delta_1 & & \downarrow \delta_{loc,p} \\ \text{Def}(X) \cong \mathbb{T}_X^1 & \xrightarrow{\lambda_p} & \text{Def}(U_p) \cong \mathbb{T}^1 \end{array} \quad (36)$$

Theorem 4 *The image of the map $\delta_{loc,p}$ in diagram (36) is the plane smooth curve $C \subset \mathbb{T}^1$ defined in Proposition 7. In particular this means that*

- (a) $\text{def}(\widehat{U}_p) = 1$,
- (b) $\text{Im}(\lambda_p) \cap \text{Im}(\delta_{loc,p}) = 0$,
- (c) $\text{Im}(\lambda_{E_p}) = 0$.

Proof By the construction of the resolution $\widehat{X} \xrightarrow{\phi} X$, $\text{Im}(\delta_{loc,p}) \subset \mathbb{T}^1$ parameterizes all the deformations of U_p induced by deformations of \widehat{U}_p . Then a fiber of the versal deformation family of U_p over a point in $\text{Im}(\delta_{loc,p}) \subset \mathbb{T}^1$ has local equation respecting the factorization (22) of the local equation (15). A general deformation respecting such a factorization can be written as follows

$$(X - U + \xi)(X + U + v) = (Y - V + \alpha)(Y - \epsilon V + \beta)(Y - \epsilon^2 V + \gamma) \quad (37)$$

for $(\xi, v, \alpha, \beta, \gamma) \in \mathbb{C}^5$. By means of the translation

$$X \mapsto X - \frac{\xi + v}{2}, \quad U \mapsto U + \frac{\xi - v}{2}$$

the left part of (37) becomes $X^2 - U^2$. After some calculation on the right part, (37) can then be rewritten as follows

$$\begin{aligned} F_{\mathbf{a}} := & F - (\alpha + \beta + \gamma)Y^2 - (\alpha + \epsilon^2\beta + \epsilon\gamma)V^2 - (\alpha + \epsilon\beta + \epsilon^2\gamma)YV \\ & - (\alpha\gamma + \alpha\beta + \beta\gamma)Y + (\beta\gamma + \epsilon\alpha\gamma + \epsilon^2\alpha\beta)V - \alpha\beta\gamma = 0 \end{aligned} \quad (38)$$

where $F := X^2 - U^2 - Y^3 + V^3$ and $\mathbf{a} := (\alpha, \beta, \gamma) \in A \cong \mathbb{C}^3$. Consider the deformation $f : \mathcal{U} \rightarrow (A, 0)$ defined by setting

$$\forall \mathbf{a} \in A \quad \mathcal{U}_{\mathbf{a}} := f^{-1}(\mathbf{a}) = \{F_{\mathbf{a}} = 0\}.$$

The following facts then occur:

1. the fibre $\mathcal{U}_{\mathbf{a}}$ is isomorphic to the central fibre \mathcal{U}_0 if and only if \mathbf{a} is a point of the plane $\pi = \{\alpha + \epsilon\beta + \epsilon^2\gamma = 0\} \subset A$; in particular $\mathcal{U}|_{\pi} \rightarrow \pi$ is a trivial deformation;
2. the open subset $V := A \setminus \pi$ is the base of a deformation of the cusp $\mathcal{U}_0 = \{F = 0\}$ to 3 distinct nodes;
3. there exists a morphism of germs of complex spaces $g : (A, 0) \rightarrow (\mathbb{T}^1, 0)$ to the Kuranishi space \mathbb{T}^1 described in (25), such that $\text{Im } g$ turns out to be precisely the plane smooth curve C defined in Proposition 7 and parameterizing the deformation of the cusp (24) to three distinct nodes.

Let us postpone the proof of these facts to observe that fact (3) means that the deformation $\mathcal{U} \rightarrow A$ is the *pull-back* by g of the versal deformation $\mathcal{V} \rightarrow \mathbb{T}^1$ i.e. $\mathcal{U} = A \times_{\mathbb{T}^1} \mathcal{V}$. Then $C = \text{Im}(\delta_{loc,p})$ proving the first part of the statement. Part (a) of the statement then follows by recalling that $\delta_{loc,p}$ is injective. Moreover Propositions 6 and 7 allow one to prove part (b). Finally part (c) follows by (b), the injectivity of $\delta_{loc,p}$ and the commutativity of diagram (36).

Let us then prove facts (1), (2) and (3) stated above.

(1), (2) : this facts are obtained analyzing the common solutions of

$$F_{\mathbf{a}} = \partial_X F_{\mathbf{a}} = \partial_Y F_{\mathbf{a}} = \partial_U F_{\mathbf{a}} = \partial_V F_{\mathbf{a}} = 0.$$

Since $\partial_X F_{\mathbf{a}} = 2X$ and $\partial_U F_{\mathbf{a}} = 2U$, we can immediately reduce to look for the common solutions $(0, Y, 0, V)$ of

$$F_{\mathbf{a}}(0, Y, 0, V) = \partial_Y F_{\mathbf{a}} = \partial_V F_{\mathbf{a}} = 0. \quad (39)$$

After some calculations one finds that these common solutions are given by

$$\begin{aligned} p_1 &= \left(0, \frac{\epsilon\beta - \gamma}{1 - \epsilon}, 0, \frac{\beta - \gamma}{\epsilon(1 - \epsilon)}\right) \\ p_2 &= \left(0, \frac{\epsilon^2\alpha - \gamma}{1 - \epsilon^2}, 0, \frac{\alpha - \gamma}{1 - \epsilon^2}\right) \\ p_3 &= \left(0, \frac{\epsilon\alpha - \beta}{1 - \epsilon}, 0, \frac{\alpha - \beta}{1 - \epsilon}\right) \end{aligned} \quad (40)$$

which have to be necessarily distinct since

$$\begin{aligned} \frac{\beta - \gamma}{\varepsilon(1 - \varepsilon)} = \frac{\alpha - \gamma}{1 - \varepsilon^2} &\iff \frac{\alpha - \gamma}{1 - \varepsilon^2} = \frac{\alpha - \beta}{1 - \varepsilon} \iff \frac{\alpha - \beta}{1 - \varepsilon} = \frac{\beta - \gamma}{\varepsilon(1 - \varepsilon)} \\ &\iff \alpha + \varepsilon\beta + \varepsilon^2\gamma = 0 . \end{aligned}$$

On the other hand, if $\alpha + \varepsilon\beta + \varepsilon^2\gamma = 0$ then we get the unique singular point $p_1 = p_2 = p_3$ which is still a threefold cusp.

(3) : Look at the definition (38) of $F_{\mathbf{a}}$ and construct g as a composition $g = i \circ p$ where

- $p : (A, 0) \cong (\mathbb{C}^3, 0) \longrightarrow (A, 0) \cong (\mathbb{C}^3, 0)$ is a linear map of rank 1 whose kernel is the plane $\pi \subset A$ defined in (1),
- $i : (A, 0) \cong (\mathbb{C}^3, 0) \rightarrow (\mathbb{T}^1, 0) \cong (\mathbb{C}^4, 0)$ is the map $(\alpha, \beta, \gamma) \mapsto (\lambda, \mu, -\nu, \sigma)$ given by

$$\begin{aligned} \lambda &= -\alpha\beta\gamma \quad , \quad \mu = -\alpha\gamma - \alpha\beta - \beta\gamma \quad , \\ \nu &= -\beta\gamma - \varepsilon\alpha\gamma - \varepsilon^2\alpha\beta \quad , \quad \sigma = -\alpha - \varepsilon\beta - \varepsilon^2\gamma \quad ; \end{aligned}$$

then, by (2) and Proposition 7, necessarily $\text{Im } i = C$ and $i|_{\text{Im } p}$ is the rational parameterization Λ_0 given in (33).

The linear map p has to annihilate the coefficients of Y^2 and V^2 in (38) i.e.

$$\alpha + \beta + \gamma = \alpha + \varepsilon^2\beta + \varepsilon\gamma = 0 .$$

Then we get the following conditions

$$\text{Im } p = \langle (\varepsilon, 1, \varepsilon^2) \rangle_{\mathbb{C}} \subset A \quad , \quad \ker p = \pi = \langle (-\varepsilon, 1, 0), (-\varepsilon^2, 0, 1) \rangle_{\mathbb{C}} \subset A$$

which determine p , up to a multiplicative constant $k \in \mathbb{C}$, as the linear map represented by the rank 1 matrix $k \cdot \begin{pmatrix} 1 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & 1 & \varepsilon \\ \varepsilon & \varepsilon^2 & 1 \end{pmatrix}$. Then

$$p(\mathbf{a}) = k(\varepsilon^2\alpha + \beta + \varepsilon\gamma) \cdot (\varepsilon, 1, \varepsilon^2)$$

and

$$g(\mathbf{a}) = i \circ p(\mathbf{a}) = (-k^3(\alpha + \varepsilon\beta + \varepsilon^2\gamma)^3, 0, 0, -3k(\alpha + \varepsilon\beta + \varepsilon^2\gamma))$$

which satisfies equations $\sigma^3 - 27\lambda = \mu = \nu = 0$ of $C \subset \mathbb{T}^1$.

Remark 3 Propositions 6 and 7 and Theorem 4 give a detailed and revised version of what observed by Y.Namikawa in [29] Examples 1.10 and 1.11 and Remark 2.8. In fact point (c) of Theorem 4 proves the following

Theorem 5 *In the notation introduced above, every global deformation of the small resolution \widehat{X} induces only trivial local deformations of a neighborhood of the exceptional fibre $\phi^{-1}(p)$ over a cusp $p \in \text{Sing}(X)$.*

4 A small and non-simple geometric transition

Finally we propose an example of a small geometric transition which is not a simple gt i.e. it is not a *deformation of a conifold transition*, as defined in the following. This example has been already sketched in §9.2 of [41]. Thanks to the detailed analysis of the Kuranishi space of a Namikawa fiber product presented above, all the following statements are now completely proved.

Let us first of all recall the main definitions.

Definition 4 (see [39] and references therein) Let \widehat{X} be a Calabi–Yau threefold and $\phi : \widehat{X} \rightarrow X$ be a *birational contraction* onto a *normal* variety. Assume that there exists a *Calabi–Yau smoothing* \widetilde{X} of X . Then the process of going from \widehat{X} to \widetilde{X} is called a *geometric transition* (for short *transition* or *gt*) and denoted either by $T(\widehat{X}, X, \widetilde{X})$ or by the diagram

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\phi} & X \rightleftarrows \widetilde{X} \\ & \searrow & \nearrow \\ & T & \end{array} \quad (41)$$

A gt $T(\widehat{X}, X, \widetilde{X})$ is called *conifold* if X admits only *ordinary double points* (nodes or o.d.p.’s) as singularities. Moreover a gt $T(\widehat{X}, X, \widetilde{X})$ will be called *small* if $\text{codim}_{\widehat{X}} \text{Exc}(\phi) > 1$, where $\text{Exc}(\phi)$ denotes the exceptional locus of ϕ .

The most well known example of a gt is given by a generic quintic threefold $X \subset \mathbb{P}^4$ containing a plane. One can check that $\text{Sing}(X)$ is composed by 16 nodes. Look at the strict transform of X , in the blow-up of \mathbb{P}^4 along the contained plane, to get the resolution \widehat{X} , while a generic quintic threefold in \mathbb{P}^4 gives the smoothing \widetilde{X} . Due to the particular nature of $\text{Sing}(X)$ the gt $T(\widehat{X}, X, \widetilde{X})$ is actually an example of a conifold transition and hence of a small gt, too.

Remark 4 Let $T(\widehat{X}, X, \widetilde{X})$ be a small gt. Then $\text{Sing}(X)$ is composed at most by terminal singularities of index 1 which turns out to be isolated hypersurface singularities (actually of *compound Du Val type* [36], [37]). The exceptional locus $\text{Exc}(\phi : \widehat{X} \rightarrow X)$ is then composed by a finite number of trees of transversally intersecting rational curves, dually represented by ADE Dynkin diagrams [21], [33], [27], [7]. Moreover [28, Thm. A] allows us to conclude that $\text{Def}(X)$ is smooth. Therefore

$$\text{def}(X) = \dim_{\mathbb{C}} \mathbb{T}_X^1. \quad (42)$$

Moreover the Leray spectral sequence of the sheaf $\phi_* \Theta_{\widehat{X}}$ gives

$$h^0(\Theta_X) = h^0(\phi_* \Theta_{\widehat{X}}) = h^0(\Theta_{\widehat{X}}) = h^{2,0}(\widehat{X}) = 0$$

where the first equality on the left is a consequence of the isomorphism $\Theta_X \cong \phi_* \Theta_{\widehat{X}}$ (see [7] Lemma (3.1)) and the last equality on the right is due to the Calabi–Yau condition for \widehat{X} . Then Theorem 2 and (1) allow to conclude that X admits a *unique smoothing component giving rise to a universal effective family of Calabi–Yau deformations*.

In the case of small geometric transitions we may then set the following

Definition 5 Two small geometric transitions

$$T_1(\widehat{X}_1, X_1, \widetilde{X}_1), T_2(\widehat{X}_2, X_2, \widetilde{X}_2)$$

are *direct deformation equivalent* (i.e. *direct def-equivalent*) if the associated birational morphisms $\phi_i : \widehat{X}_i \rightarrow X_i$, $i = 1, 2$, are deformations of each other as defined in § 2.2. The equivalence relation of small geometric transitions generated by direct def-equivalence is called *def-equivalence* (or *deformation type*) of small geometric transitions.

In particular a small gt $T(\widehat{X}, X, \widetilde{X})$ is called *simple* if it is def-equivalent to a conifold transition.

Actually def-equivalence of geometric transitions is a more complicated concept which reduces to the given Definition 5 in the case of small geometric transitions. The interested reader is referred to [41] §8.1 for any further detail.

A direct consequence of Theorem 4 is then the following

Corollary 2 *There exist small geometric transitions which are not def-equivalent to a conifold transition.*

Proof Consider the Namikawa fiber product $X := Y \times_{\mathbb{P}^1} Y$, a smooth resolution $\phi : \widehat{X} \rightarrow X$, as defined in Proposition 4 and a fiber product of generic rational elliptic surfaces $\widetilde{X} := Y' \times_{\mathbb{P}^1} Y''$. Then $T(\widehat{X}, X, \widetilde{X})$ is a small gt which can't be direct def-equivalent to a conifold transition for what observed in Remark 3.

Assume now that $T(\widehat{X}, X, \widetilde{X})$ is def-equivalent to a conifold transition. This means that there exists a finite chain of morphism deformations connecting the resolution $\phi : \widehat{X} \rightarrow X$ with a conifold resolution. Hence there exists at least one of those morphism deformations locally inducing a non-trivial deformation of the exceptional locus, so violating condition (c) of Theorem 4.

Remark 5 (The Friedman diagram of the Namikawa fiber product) As a final result let us give a complete account of the Friedman diagram (9) in the case of the small resolution $\phi : \widehat{X} \rightarrow X$ of a cuspidal Namikawa fiber product $X = Y \times_{\mathbb{P}^1} Y$.

Let us start by computing the Kuranishi number $\text{def}(X)$, by an easy moduli computation. In fact the moduli of the elliptic surface Y are 8 since they are given by the moduli of an elliptic pencil in \mathbb{P}^2 . Moreover the 6 cuspidal fibers are parameterized by the roots in \mathbb{P}^1 of a general element in $H^0(\mathcal{O}_{\mathbb{P}^1}(6))$ up to the action of the projective group $\mathbb{P}\text{GL}(2)$. Then

$$\text{def}(X) = 2 \cdot \text{def}(Y) + h^0(\mathcal{O}_{\mathbb{P}^1}(6)) - \dim \text{GL}(2, \mathbb{C}) = 2 \cdot 8 + 7 - 4 = 19 .$$

Recalling the existence of the small geometric transition $T(\widehat{X}, X, \widetilde{X})$, described in the proof of Corollary 2, it turns out that

$$\text{def}(X) = \text{def}(\widetilde{X}) = h^{1,2}(\widetilde{X}) = 19 = h^{1,1}(\widetilde{X})$$

since \widetilde{X} is a Calabi–Yau 3-fold with $\chi(X) = 0$ ([42] §2). Recalling that

- the small resolution $\phi : \widehat{X} \rightarrow X$ is obtained as a composition of 2 blow-ups,
- the exceptional locus $\text{Exc}(\phi)$ has 12 irreducible components,
- every cusp of X has Milnor (and Tyurina) number 4,

then Calabi–Yau conditions on \widehat{X} and Theorem 7 in [40] gives that

$$\begin{aligned} \text{def}(\widehat{X}) &= h^1(\Theta_{\widehat{X}}) = h^{1,2}(\widehat{X}) = h^{1,2}(\widetilde{X}) - 16 = 3 \\ \dim \mathbb{T}_{\widehat{X}}^2 &= h^2(\Theta_{\widehat{X}}) = h^{1,1}(\widehat{X}) = h^{1,1}(\widetilde{X}) + 2 = 21 \end{aligned}$$

Moreover recall that:

- by Proposition 1(1) and Theorem 4(a), $h^0(R^1\phi_*\Theta_{\widehat{X}}) = 6$,
- by Theorem 4(c) the localization map l_E in (9) is trivial,

- by Proposition 1(2) and relation (25), $\dim T_X^1 = 24$,
- the last horizontal maps on the right in the Friedman diagram (9) are, in this case, surjective since both related with spectral sequences having $E_\infty^{1,1} = E_\infty^{0,2} = 0$.

Putting all together, the Friedman diagram (9) becomes the following one

$$\begin{array}{ccccccccccccccc}
0 & \longrightarrow & \mathbb{C}^3 & \xrightarrow{\cong} & \mathbb{C}^3 & \xrightarrow{\lambda_E=0} & \mathbb{C}^6 & \xrightarrow{d_2} & \mathbb{C}^{27} & \twoheadrightarrow & \mathbb{C}^{21} & \longrightarrow & 0 \\
& & \parallel & & \downarrow \delta_1 & \searrow & \downarrow \delta_{loc} & & \parallel & & \downarrow \delta_2 & & \\
0 & \longrightarrow & \mathbb{C}^3 & \hookrightarrow & \mathbb{C}^{19} & \xrightarrow{\lambda_P} & \mathbb{C}^{24} & \longrightarrow & \mathbb{C}^{27} & \twoheadrightarrow & \mathbb{C}^{19} & \longrightarrow & 0
\end{array}$$

In particular notice that

$$h^1(X, \Theta_X) = h^1(\widehat{X}, \Theta_{\widehat{X}}) = 3$$

contradicting the sufficient condition (13) in Proposition 32 of [41] which has to be satisfied by a simple gt. This is consistent with the previous Corollary 2.

Acknowledgements A first draft of the present paper was written on a visit to the Dipartimento di Matematica e Applicazioni of the Università di Milano Bicocca and the Department of Mathematics of the University of Pennsylvania. I would like to thank the Faculties of both Departments for warm hospitality and in particular F. Magri, R. Paoletti and S. Terracini from the first Department and R. Donagi, A. Grassi, T. Pantev and J. Shaneson from the second Department. Special thanks are due to A. Grassi for beautiful and stimulating conversations. I am also greatly indebted to B. van Geemen for useful suggestions.

References

1. Arbarello E., Cornalba M. and Griffiths P.A. *Geometry of algebraic curves, II* Grundlehren der Mathematischen Wissenschaften, **268**, Springer, Heidelberg (2011).
2. Barth W., Peters C. and Van de Ven A. *Compact complex surfaces* vol. **4** E.M.G, Springer-Verlag (1984)
3. Bogomolov F. “Hamiltonian Kähler manifolds” *Dokl. Akad. Nauk. SSSR* **243/5**, 1101–1104 (1978).
4. Candelas P., Green P.S. and Hübsch T. “Rolling among Calabi–Yau vacua” *Nucl. Phys.* **B330**, 49–102 (1990).
5. Clemens C.H. “Double Solids” *Adv. in Math.* **47**, 107–230 (1983).
6. Douady A. “Le problème des modules locaux pour les espaces C-analytiques compacts” *Ann. scient. Éc. Norm. Sup.* 4e série, **7** 569–602 (1974).
7. Friedman R. “Simultaneous resolution of threefold double points” *Math. Ann.* **247**, 671–689 (1986).
8. Godement R. *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris (1958).
9. Grauert H. “Über die Deformation isolierter Singularitäten analytischer Mengen” *Invent. Math.* **15** (1972), 171–198.
10. Grauert H. “Der Satz von Kuranishi für Kompakte Komplexe Räume” *Invent. Math.* **25**, 107–142 (1974).
11. Greuel G.-M., Lossen C. and Shustin E. *Introduction to singularities and deformations* Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2007.
12. Gross M. “Primitive Calabi–Yau threefolds” *J. Diff. Geom.* **45**, 288–318 (1997); [math.AG/9512002](#).
13. Grothendieck A. “Sur quelques points d’algèbre homologique” *Tôhoku Math. J.* **9**, 199–221 (1957).
14. Heckman G. and Looijenga E. *The moduli space of rational elliptic surfaces*, Algebraic geometry 2000, Azumino (Hotaka), Adv. Stud. Pure Math., **36**, 185–248 (2002).

15. Joyce D. *Compact manifolds with Special Holonomy*, Oxford Science Publications, Oxford–New York (2000).
16. Kapustka G. and M. “Fiber products of elliptic surfaces with sections and associated Kummer fibrations” *Manuscripta Math.* **130**, 121–135 (2009); [arXiv:0802.3760](https://arxiv.org/abs/0802.3760) [math.AG].
17. Kas A. “Weierstrass normal forms and invariants of elliptic surfaces” *Trans. Amer. Math. Soc.* **225**, 259–266 (1977).
18. Kodaira K. “On compact analytic surface. II” *Ann. of Math.* **77**(2), 563–626 (1963).
19. Kollár J. *Minimal models of algebraic threefolds: Mori’s program* Séminaire Bourbaki, Astérisque **177–178**, Exp. No. 712, 303–326 (1989).
20. Kollár J. and Mori S. “Classification of three–dimensional flips” *J. Amer. Math. Soc.* **5**, 533–703 (1992).
21. Laufer H. “On $\mathbb{C}P^1$ as an exceptional set” in *Recent developments in several complex variables* Ann. Math. Stud. **100**, 261–276 (1981).
22. Lichtenbaum S. and Schlessinger M. “On the cotangent complex of a morphism” *Trans. A.M.S.* **128**, 41–70 (1967).
23. E. J. N. Looijenga *Isolated singular points on complete intersections* London Mathematical Society Lecture Note Series, **77**, Cambridge University Press, Cambridge, 1984.
24. Miranda R. “On the stability of pencils of cubic curves” *Amer. J. Math.* **102**(6), 1177–1202 (1980).
25. Miranda R. “The moduli of Weierstrass fibrations over \mathbb{P}^1 ” *Math. Ann.* **255**(3), 379–394 (1981).
26. Miranda R. and Persson U. “On extremal rational elliptic surfaces” *Math. Z.* **193**(4), 537–558 (1986).
27. Morrison D.R. “The birational geometry of surfaces with rational double points” *Math. Ann.* **271**, 415–438 (1985).
28. Namikawa Y. “On deformations of Calabi–Yau 3-folds with terminal singularities” *Topology* **33**(3), 429–446 (1994).
29. Namikawa Y. “Stratified local moduli of Calabi–Yau threefolds” *Topology* **41**(6), 1219–1237 (2002).
30. Palamodov V. P. “The existence of versal deformations of complex spaces” *Dokl. Akad. Nauk SSSR* **206**, 538–541 (1972).
31. Palamodov V. P. “Deformations of complex spaces” *Russian Math. Surveys* **31**(3) (1976), 129–197; from russian *Uspekhi Mat. Nauk* **31**(3), 129–194 (1976).
32. Palamodov V. P. “Deformations of complex spaces” in *Several complex variables IV*, Encyclopedia of math.sciences **10**, Springer (1990).
33. Pinkham H. “Factorization of birational maps in dimension three” in *Singularities Proc. Symp.* Pure Math. **40**, 343–372 (1981).
34. Ran Z. “Deformations of maps” in *Algebraic curves and projective geometry*, Ballico E. and Ciliberto C., Eds. LNM **1389**, Springer–Verlag (1989).
35. Ran Z. “Deformations of manifolds with torsion or negative canonical bundle” *J. Alg. Geom.* **1**, 279–291 (1992).
36. Reid M. “Canonical 3–folds” in *Journées de géométrie algébrique d’Angers*, 671–689, Sijthoff & Norddhoff (1980).
37. Reid M. “Minimal model of canonical 3–folds” in *Algebraic varieties and analytic varieties*, 131–180, Adv. Stud. Pure Math. **1**, North–Holland (1983).
38. Reid M. “The moduli space of 3–folds with $K = 0$ may nevertheless be irreducible” *Math. Ann.* **287**, 329–334 (1987).
39. Rossi M. “Geometric transitions” *J. Geom. Phys.* **56**(9), 1940–1983 (2006).
40. Rossi M. “Homological type of geometric transitions” *Geom. Dedicata* **151** (2011), 323–359.
41. Rossi M. “Deforming geometric transition” *Int. J. Geom. Methods Mod. Phys.* **12**(5), (2015).
42. Schoen C. “On fiber products of rational elliptic surfaces with section” *Math. Z.* **197**(2), 177–199 (1988).
43. J. Stevens *Deformations of singularities* Lecture Notes in Mathematics **1811**, Springer-Verlag, Berlin, 2003.
44. Strominger A. “Massless black holes and conifolds in string theory” *Nucl. Phys.* **B451**, 97–109 (1995); [hep-th/9504145](https://arxiv.org/abs/hep-th/9504145).
45. Tian G. “Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Weil–Petersson metric” in *Mathematical aspects of string theory* (S.-T. Yau, ed.), 629–646. World Scientific, Singapore (1987).
46. Todorov A. “The Weil-Petersson geometry of the moduli space of $SU(n \geq 3)$ (Calabi–Yau) manifolds” *Comm.Math.Phys.* **126**, 325–346 (1989).
47. Wahl, J.M. “Equisingular deformations of normal surface singularities, I” *Ann. of Math.* **104**, 325–356 (1976).