J. Group Theory, Ahead of Print
DOI 10.1515/jgth-2025-0080

The maximal rank of a string group generated by involutions for alternating groups

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Communicated by Gareth Tracey

Abstract. A string group generated by involutions, or SGGI, is a pair $\Gamma = (G, S)$, where G is a group and $S = \{\rho_0, \dots, \rho_{r-1}\}$ is an ordered set of involutions generating G and satisfying the following commuting property: for all $i, j \in \{0, \dots, r-1\}$, $|i-j| \neq 1$ implies $(\rho_i \rho_j)^2 = 1$. When S is an independent set, the rank of Γ is the cardinality of S. We determine an upper bound for the rank of an SGGI over the alternating group of degree n. Our bound is tight when $n \equiv 0, 1, 4 \pmod 5$.

1 Introduction

Let G be a group and S a subset of G. We say that S is an *independent generating set* for G if

$$G = \langle S \rangle \quad \text{and} \quad s \notin \langle S \setminus \{s\} \rangle \text{ for all } s \in S.$$

A *string group generated by involutions* (SGGI) is a pair $\Gamma = (G, S)$, where G is a group and $S = \{\rho_0, \dots, \rho_{r-1}\}$ is an ordered set of involutions generating G, satisfying the following property:

$$\text{for all } i, j \in \{0, \dots, r-1\}, \quad |i-j| \neq 1 \implies (\rho_i \rho_j)^2 = 1.$$

To avoid cumbersome notation, when S is understood, we simply say that G is an SGGI. The *dual* of an SGGI is obtained by reversing the order of the string of generators. A group G is an SGGI of *rank* r if (G, S) is an SGGI for some independent generating set S of cardinality r.

SGGIs play a prominent role in the study of polytopes. In fact, there is a one-to-one correspondence between abstract regular polytopes and string C-groups, which are SGGIs whose generating set $S = \{\rho_0, \dots, \rho_{r-1}\}$ satisfies the following

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This paper is funded by the European Union – Next Generation EU, Missione 4 Componente 1 CUP B53D23009410006, PRIN 2022 – 2022PSTWLB – Group Theory and Applications. Maria Elisa Fernandes was also supported by CIDMA under the Portuguese Foundation for Science and Technology (FCT, https://ror.org/00snfqn58) Multi-Annual Financing Program for R&D Units.

condition, known as the *intersection property*:

$$\langle \rho_i \mid i \in J \rangle \cap \langle \rho_i \mid i \in K \rangle = \langle \rho_i \mid i \in J \cap K \rangle$$

for all subsets $J, K \subseteq \{0, \ldots, r-1\}$. This correspondence is described with detail in [8].

The maximal rank of the symmetric group of degree n is $n-1$ and the bound is attained, for instance, by the Coxeter generators; see [10]. In this paper, we show that the maximal rank of an SGGI for the alternating group $\mathrm{Alt}(n)$ of degree n is quite a bit smaller than $n-1$.

Theorem 1.1. *The following holds:*

- *the group* $\mathrm{Alt}(5)$ *admits SGGIs and they all have rank* 3*;*
- *the groups* $\mathrm{Alt}(3)$*,* $\mathrm{Alt}(4)$*,* $\mathrm{Alt}(6)$*,* $\mathrm{Alt}(7)$ *and* $\mathrm{Alt}(8)$ *do not admit SGGIs;*
- *for* $n \geq 9$*, the maximum rank of an SGGI for* $\mathrm{Alt}(n)$ *is at most* $\lfloor 3(n-1)/5 \rfloor$*. Moreover, this bound is attained when* $n \equiv 0, 1, 4 \pmod 5$*.*

In Table 2, we present permutation representation graphs of SGGIs for $\mathrm{Alt}(n)$ (see Section 2 for the definition of a permutation representation graph). These graphs have rank $\lfloor 3(n-1)/5 \rfloor$ when $n \equiv 0, 1, 4 \pmod 5$ and rank $\lfloor (3n-8)/5 \rfloor$ when $n \equiv 2, 3 \pmod 5$. The constructions depend on the parity of n modulo 5. Moreover, in Table 2, we require $n \geq 22$ when $n \equiv 2 \pmod 5$ and $n \geq 18$ when $n \equiv 3 \pmod 5$. When $n \in \{12, 13, 17\}$, SGGIs for $\mathrm{Alt}(n)$ of rank $\lfloor (3n-8)/5 \rfloor$ have been found with the help of a computer. We are using the computer algebra system Magma [2] in this paper. Some of our arguments require extensive computations. We do not include the code for these computations in this manuscript, but the interested reader can find it in our companion paper submitted to arXiv [1].

Computational evidence suggests that, for $n \equiv 2, 3 \pmod 5$, the maximum rank of an SGGI for $\mathrm{Alt}(n)$ is $\lfloor (3n-8)/5 \rfloor$. Consequently, Theorem 1.1 might be improved for these two congruence classes modulo 5.

An immediate application of our arguments yields the following result. We are grateful to the anonymous referee for suggesting the inclusion of this application.

Theorem 1.2. *Let* G *be a transitive subgroup of the symmetric group* $\mathrm{Sym}(n)$ *with* $\mathrm{Alt}(n) \not\leq G$ *and let* r *be the maximal rank of an SGGI for* G*. Then either*

$$r < \left\lfloor \frac{3(n-1)}{5} \right\rfloor,$$

or G *is* `TransitiveGroup`(n, g)*, where* (n, g) *are recorded in Table 1, using the library of transitive groups in the computer algebra system Magma.*

Degree n	Label g	Rank r	Comments
4	2	2	$C_2 \times C_2$
	3	2	D_4
5	2	2	D_5
6	3	3	D_6
	7	3	Sym(4)
	8	3	Sym(4)
	9	4	$\langle (1\,2\,3), (1\,2)(4\,5), (1\,4)(2\,5)(3\,6) \rangle$
	11	4	Sym(2) wr Sym(3)
	12	3	$\mathrm{PSL}_2(5)$
	13	3	Sym(3) wr Sym(2)
	14	4	$\mathrm{PGL}_2(5)$
8	22, 24, 34, 41	4	
	44	4	Sym(2) wr Sym(4)
	45	5	
	47	4	Sym(4) wr Sym(2)
9	18, 24	3	
	31	3	Sym(3) wr Sym(3)
10	22	5	
	32	5	$\mathrm{P\Sigma L}_2(9)$
	37	5	
	38	5	
	39	6	Sym(2) wr Sym(5)
	41	5	
	43	5	Sym(5) wr Sym(2)
12	117, 139, 195, 219, 239	6	
	293	6	Sym(2) wr Sym(6)
	299	6	Sym(6) wr Sym(2)
14	49, 54, 55	7	
	57	8	Sym(2) wr Sym(7)
	61	7	Sym(7) wr Sym(2)
18	968	10	Sym(2) wr Sym(9)
22	53	12	Sym(2) wr Sym(11)

Table 1. Exceptional cases in Theorem 1.2.

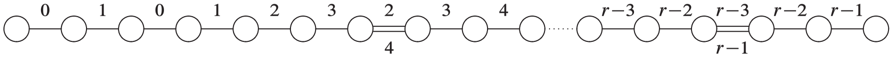
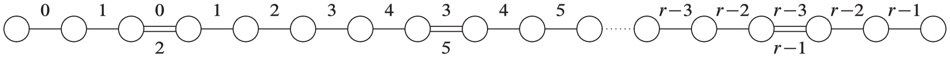
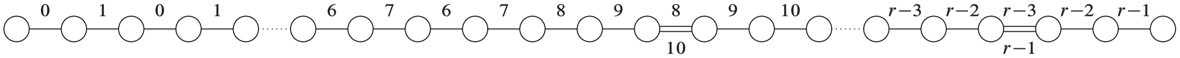
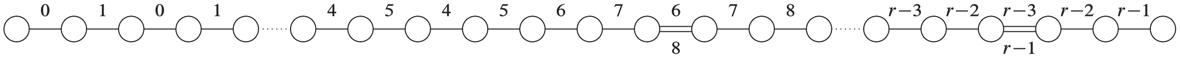
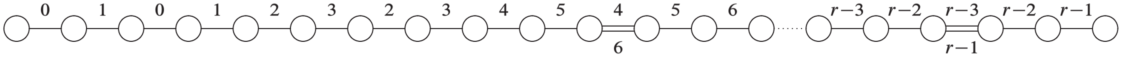
Conditions	Permutation representation graph
$n \equiv 0 \bmod 5$, $n \geq 10$	
$n \equiv 1 \bmod 5$, $n \geq 11$	
$n \equiv 2 \bmod 5$, $n \geq 22$	
$n \equiv 3 \bmod 5$, $n \geq 18$	
$n \equiv 4 \bmod 5$, $n \geq 14$	

Table 2. Permutation representation graphs for the upper bound in Theorem 1.1.

2 Preliminaries, notation and basic results

2.1 String groups generated by involutions and their friends

We collect here some basic information and notation.

In the case that the group G is a permutation group, we introduce an auxiliary gadget for graphically representing an SGGI. Let G be a permutation group of degree n, and let $\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$ be an SGGI. The *permutation representation graph* of Γ is the r-edge-labeled multigraph with vertex set the domain of G, and with an edge $\{a, b\}$ having color i, for each a, b such that $a \neq b$ and $a\rho_i = b$, and for each $i \in \{0, \dots, r-1\}$. For instance, when

$$\Gamma = (\langle (1\,2)(3\,4), (1\,3)(2\,4)\rangle, \{(1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\}),$$

the permutation representation graph is the complete graph on 4 vertices, where to each perfect matching is assigned a distinct color. The permutation representation graph has n vertices and $\sum_{i=0}^{r-1} c_i$ edges, where c_i is the number of cycles of length 2 in ρ_i.

Since some of our arguments are inductive, we need some ad hoc notation for our work. Let

$$\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$$

be an SGGI and let $i, i_1, \dots, i_k \in \{0, \dots, r-1\}$. We let

$$\begin{aligned}
G_{i_1,\dots,i_k} &= \langle \rho_j \mid j \notin \{i_1, \dots, i_k\}\rangle, \\
\Gamma_{i_1,\dots,i_k} &= (G_{i_1,\dots,i_k}, \{\rho_j \mid j \notin \{i_1, \dots, i_k\}\}), \\
G_{\{i_1,\dots,i_k\}} &= \langle \rho_j \mid j \in \{i_1, \dots, i_k\}\rangle, \\
\Gamma_{\{i_1,\dots,i_k\}} &= (G_{\{i_1,\dots,i_k\}}, \{\rho_j \mid j \in \{i_1, \dots, i_k\}\}), \\
G_{<i} &= \langle \rho_0, \dots, \rho_{i-1}\rangle, \\
\Gamma_{<i} &= (G_{<i}, \{\rho_0, \dots, \rho_{i-1}\}), \qquad i \neq 0, \\
G_{>i} &= \langle \rho_{i+1}, \dots, \rho_{r-1}\rangle, \\
\Gamma_{>i} &= (G_{>i}, \{\rho_{i+1}, \dots, \rho_{r-1}\}), \quad i \neq r-1.
\end{aligned}$$

Finally, since our work builds upon some of the results in [6], we recall the definition of *string C-group*. A string C-group is an SGGI $(G, \{\rho_0, \dots, \rho_{r-1}\})$ satisfying the *intersection property*

$$G_I \cap G_J = G_{I \cap J} \quad \text{for each } I, J \subseteq \{0, \dots, r-1\}.$$

When Γ is a string C-group, S is an independent generating set for G and hence Γ is an SGGI of rank $|S|$. For instance, when G is the symmetric group $\mathrm{Sym}(n)$ and $S = \{(1\,2), (2\,3), \dots, (n-1\,n)\}$, we see that (G, S) is a string C-group and that the permutation representation graph is the usual Coxeter diagram.

2.2 Fracture graphs and 2-fracture graphs

Let

$$\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$$

be an SGGI, where G is a permutation group with domain $\{1, \dots, n\}$.

Here, we define fracture graphs when G is transitive and G_i is intransitive for every $i \in \{0, \dots, r-1\}$. Therefore, we assume these conditions hold throughout this section and whenever fracture graphs are used.

For each i, since G_i is intransitive, ρ_i has at least one cycle (of length 2) containing points from different G_i-orbits. Choosing one such cycle for each i, we construct a graph having vertex set $\{1, \dots, n\}$, where each selected cycle defines an edge. The resulting graph, which has r edges, is called a *fracture graph* for Γ. In general, fracture graphs are not unique, as they depend on the choice of cycles.

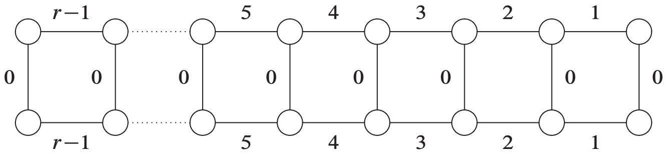
Additionally, under the assumption that, for each i, ρ_i has at least two cycles containing points from different G_i-orbits, then taking (for each i) two i-edges between each of these pairs of points, we obtain a graph on n vertices with $2r$ edges that we call a 2-*fracture graph* for Γ. Observe that, when G is contained in the alternating group $\mathrm{Alt}(n)$, each ρ_i does have at least two cycles (of length 2).

In [6], the authors proved the following two results. Although [6] focuses on string C-groups, none of the proofs in [6, Section 4] rely on the intersection property. Consequently, these results remain valid for SGGIs as well.

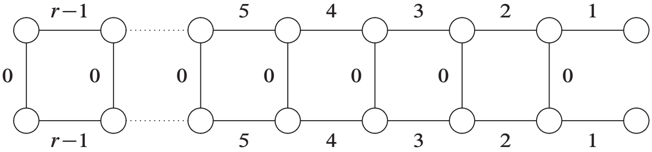
Proposition 2.1 ([6, Proposition 4.12]). *Let $\Gamma = (G, S)$ be an SGGI for a permutation group G of degree n admitting at least one 2-fracture graph. If Γ has no connected 2-fracture graph, then there exists a 2-fracture graph having at least one component that is a tree, while all other components are either trees or have at most one cycle (which is an alternating square).*

Corollary 2.2 ([6, Corollary 4.19]). *Assume $n \geq 9$. Let $\Gamma = (G, S)$ be an SGGI for a permutation group G of degree n. If Γ has a connected 2-fracture graph that is not a tree, then n is even, Γ has rank $n/2$ and, up to duality, Γ falls into one of the following two cases:*

(a) *G is permutation isomorphic to* $\mathrm{Sym}(2) \times \mathrm{Sym}(n/2)$*, and the permutation representation graph is*



(b) *G is permutation isomorphic to* $\mathrm{Sym}(2) \wr \mathrm{Sym}(n/2)$ *when $n/2$ is even, and G is permutation isomorphic to* $\mathrm{Sym}(2)^{n/2-1} \rtimes \mathrm{Sym}(n/2)$ *(which is a subgroup of index* 2 *in* $\mathrm{Sym}(2) \wr \mathrm{Sym}(n/2)$*) when $n/2$ is odd. The permutation representation graph is*



2.3 Splits and perfect splits

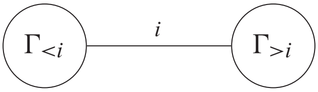
Let G be a permutation group of degree n, and let $\Gamma = (G, S)$ be an SGGI with $S = \{\rho_0, \dots, \rho_{r-1}\}$. Suppose Γ has a fracture graph.

Suppose that, for some $i \in \{0, \dots, r-1\}$,

- G_i has exactly two orbits; observe that, under this condition, there exists a partition of $\{1, \dots, n\}$ into two sets O_1 and O_2, of sizes n_1 and $n_2 = n - n_1$ such that ρ_i is the unique permutation of S swapping elements in different parts of this partition of $\{1, \dots, n\}$; and
- there is exactly one pair of points $(a, b) \in O_1 \times O_2$ such that $a\rho_i = b$.

Then $\{a, b\}$ is called a *split* or *i-split* of Γ. We refer to i as the *label* of a split.

For $j \in \{0, \dots, r-1\} \setminus \{i\}$, let α_j be the restriction of ρ_j to O_1, and let β_j be the restriction of ρ_j to O_2. If, up to duality, $\alpha_j = 1$ for each $j \in \{i+1, \dots, r-1\}$ and $\beta_j = 1$ for each $j \in \{0, \dots, i-1\}$, then we say that the i-split $\{a, b\}$ is *perfect*. The following illustrates the situation when we have a perfect split having label i:



If j is the label of an edge adjacent to a i-split, then $j \in \{i-1, i+1\}$. Hence if $\Gamma_{<i}$ (resp. $\Gamma_{>i}$) is nontrivial, then it must have a pendent edge with label $i-1$ (resp. $i+1$).

3 SGGIs admitting a 2-fracture graph

In [4, Proposition 4.9], the authors classify string C-groups $\Gamma = (G, S)$ of rank at least $(n-1)/2$ that admit a 2-fracture graph, where G is a permutation group of degree n. Since their focus was on string C-groups, they excluded SGGIs that do not satisfy the intersection property from their classification. However, the proof of

[4, Proposition 4.9] inherently provides a classification of SGGIs of a permutation group of degree n with rank at least $(n-1)/2$. Therefore, in the following, we revisit some of the steps in their proof to recover the permutation representation graphs of the SGGIs omitted from their classification.

Proposition 3.1. *Let G be a transitive permutation group of degree n, and let $\Gamma = (G, \{\rho_0, \dots, \rho_{r-1}\})$ be an SGGI of rank r. If Γ has a 2-fracture graph, then $r \leq n/2$. Moreover, if $r \geq (n-1)/2$, then up to duality, Γ has a permutation representation graph isomorphic to one of the graphs in Figure 1.*

Proof. Assume first that $n \geq 9$. Let $\mathcal{G}$ be the permutation representation graph of Γ. By assumption, Γ has a 2-fracture graph. If Γ admits a connected 2-fracture graph that is not a tree, then Corollary 2.2 shows that n is even, $r = n/2$ and $\mathcal{G}$ is one of the graphs described in parts (a) and (b). We have reported these two graphs in the first two rows of Figure 1. Therefore, we may suppose that either Γ admits no connected 2-fracture graph, or that each connected 2-fracture graph is a tree.

In this case, from Proposition 2.1, we deduce that Γ admits a 2-fracture graph having at least one component that is a tree, while all other components are either trees or have at most one cycle. Let $n_1, \dots, n_\kappa$ be the cardinalities of the connected components of this 2-fracture graph. From Euler's formula, the number of edges in each connected component is at most n_j, and there exists a j such that the number of edges is exactly $n_j - 1$. Since the total number of edges in the 2-fracture graph is $2r$, we deduce that $2r \leq \sum_{i=1}^{\kappa} n_i - 1 = n - 1$. This shows that $r \leq n/2$ in all cases, and hence it remains to prove that if $r \geq (n-1)/2$, then $\mathcal{G}$ is isomorphic to one of the graphs in Figure 1. As $r \geq (n-1)/2$, we deduce that $r = (n-1)/2$.

When $n \leq 8$, the proof in [4, Proposition 4.9] is via a computer search; a similar computer search shows that the possible graphs $\mathcal{G}$ are listed in Figure 1. Therefore, we may suppose $n \geq 9$.

At this point, in the proof of [4, Proposition 4.9], the authors consider a 2-fracture graph $\mathcal{F}$ with a minimal number of squares, denoted by s. The proof is then divided into two cases: $s > 0$ and $s = 0$. When $s > 0$, following the proof of [4, Proposition 4.9], we see that the graph $\mathcal{G}$ corresponds to graph (3) in Figure 1.

Now, suppose $s = 0$, meaning that $\mathcal{F}$ is a tree. If every pair of adjacent edges in $\mathcal{F}$ carries consecutive labels, then by the proof of [4, Proposition 4.9], the graph obtained is a linear graph whose sequence of labels is given by

$$(0, 1, 0, 1, 2, 3, 2, 3, \dots, r-2, r-1, r-2, r-1).$$

The resulting graph $\mathcal{G}$ corresponds to graph (4) in Figure 1.

Finally, suppose that $\mathcal{F}$ contains adjacent edges with nonconsecutive labels. Then, by the proof of [4, Proposition 4.9], the resulting graphs $\mathcal{G}$ correspond to graphs (5), (6), (7), (8), (9), (10) and (11) in Figure 1. □

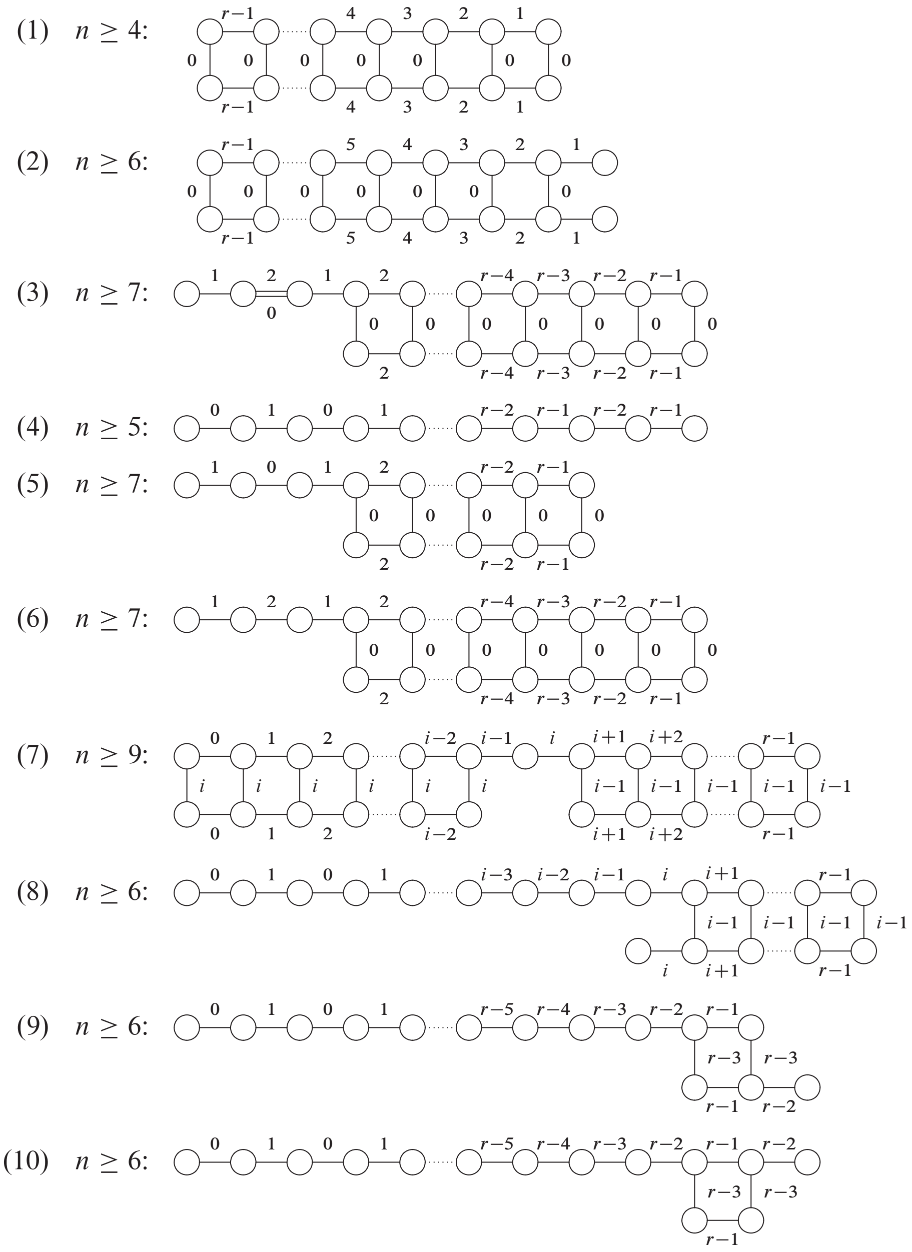


Figure 1. Permutation representation graphs relevant for Proposition 3.1.

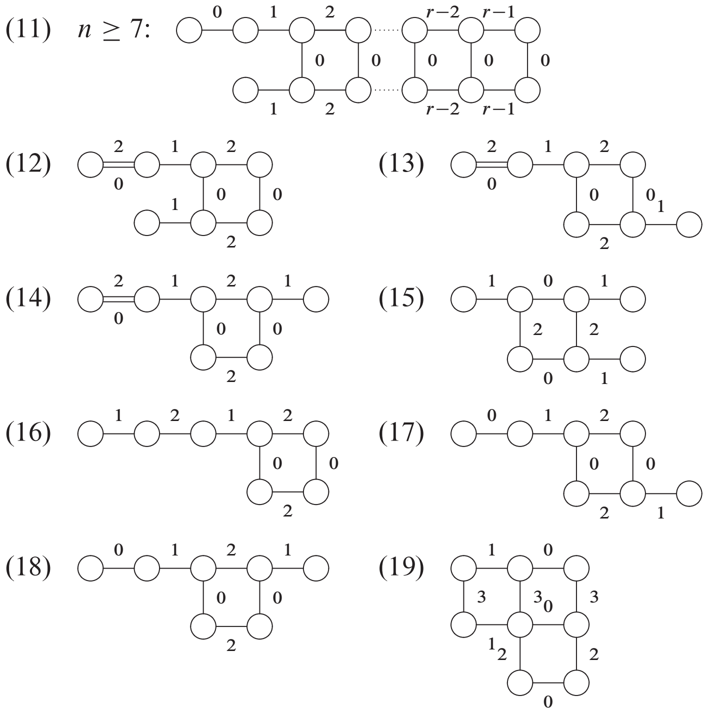


Figure 1 (continued)

4 SGGIs having splits: All of which are perfect

Hypothesis 4.1. Let G be a permutation group of degree n, and let $\Gamma = (G, S)$ be an SGGI of rank r with $S = \{\rho_0, \ldots, \rho_{r-1}\}$. Throughout this section, with the only exception of Proposition 4.4, we assume the following conditions.

- G is a transitive subgroup of $\mathrm{Alt}(n)$.
- For each $i \in \{0, \ldots, r-1\}$, the subgroup G_i is intransitive. These two conditions are equivalent to the existence of a fracture graph for Γ.
- The SGGI Γ does not admit a 2-fracture graph. Consequently, by Section 2.3, Γ admits at least one split. Let s denote the number of splits in Γ, so in particular, $s \geq 1$.
- Every split of Γ is a perfect split.

Lemma 4.2. *Assume Hypothesis 4.1. If* $s = 1$, *then* $r \leq (n-1)/2$.

Proof. Let x be the label of the unique split of Γ and let r_1 and r_2 be the ranks of the SGGIs $\Gamma_{<x}$ and $\Gamma_{>x}$, respectively. Let X_1 and X_2 be the G_x-orbits that are fixed pointwise by $G_{>x}$ and $G_{<x}$, respectively, and let n_1 and n_2 denote their cardinalities.

Since Γ has a unique split and this split is perfect, both $\Gamma_{<x}$ and $\Gamma_{>x}$ admit a 2-fracture graph. Furthermore, the groups $G_{<x}$ and $G_{>x}$ are transitive of degree n_1 and n_2, respectively. Thus, by Proposition 3.1, we have

$$r_1 \leq \frac{n_1}{2} \quad \text{and} \quad r_2 \leq \frac{n_2}{2}.$$

Up to duality, we may suppose $x \neq r-1$. Assume also $x \neq 0$. Since a split does not belong to a square, the permutation representation graph of $\Gamma_{<x}$ or of $\Gamma_{>x}$ has a pendent edge labeled $x-1$ or $x+1$, respectively. If $r_1 = n_1/2$ or $r_2 = n_2/2$, then Proposition 3.1 shows that the permutation representation graph of $\Gamma_{<x}$ or of $\Gamma_{>x}$ is one of the graphs in the first two lines of Figure 1; however, neither of these graphs has a pendent edge of label $x-1$ or $x+1$. Therefore,

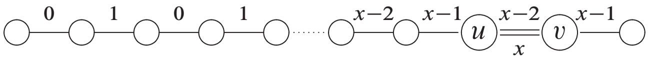
$$r_1 \leq \frac{n_1-1}{2} \quad \text{and} \quad r_2 \leq \frac{n_2-1}{2}.$$

In particular, to conclude the proof, it suffices to exclude the possibility that

$$r_1 = \frac{n_1-1}{2} \quad \text{and} \quad r_2 = \frac{n_2-1}{2}.$$

If $r_1 = (n_1-1)/2$ and $r_2 = (n_2-1)/2$, then by Proposition 3.1, $\Gamma_{<x}$ and $\Gamma_{>x}$ correspond to one of the graphs in Figure 1. The graphs in Figure 1 that contain a pendent edge having label $x-1$ or $x+1$ are (4), (8), (9), (10), (11), (17) and (18). We now consider each of these possibilities in turn. In this case-by-case analysis, we must consider each graph in Figure 1 with a pendent edge, as well as its dual, by relabeling the set of edge labels using either the sequence $(0, \ldots, x-1)$ or $(x+1, \ldots, r-1)$.

As G consists of even permutations, ρ_x has at least two cycles of length 2, and hence there is a pair of distinct vertices $(u, v) \in (X_1 \times X_1) \cup (X_2 \times X_2)$ such that $u\rho_x = v$. Without loss of generality, suppose that $(u, v) \in X_1 \times X_1$. If $\Gamma_{<x}$ is represented by graph (4) with $r_1 = x$, then there is only one possibility for the x-edge (u, v), which is as follows:

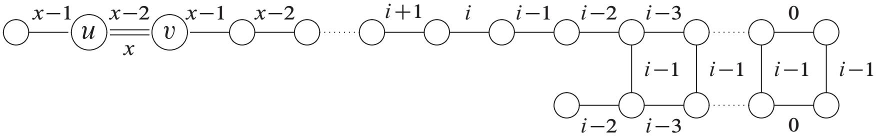


But then Γ has a perfect split with label $x-2$, contradicting the hypothesis that Γ has a unique split and that the label of this split is x.

Suppose that $\Gamma_{<x}$ is graph (8) with the relabeling of the edges given by

$$(0, 1, \dots, r_1 - 1) \mapsto (x - 1, x - 2, \dots, 0).$$

Then the x-edge (u, v) must be as follows:



Then we get the same contradiction as before. For graphs (9) and (10), we get an entirely analogous contradiction.

Now, consider graph (11) with the relabeling

$$(0, 1, \dots, r_1 - 1) \mapsto (x - 1, x - 2, \dots, 0).$$

It is not possible to add the x-edge $\{u, v\}$ to this graph without breaking the commuting property of the generators. The same happens when we consider graphs (17) and (18).

Consequently,

$$r = r_1 + r_2 + 1 \leq \frac{n_1 + n_2 - 3}{2} + 1 = \frac{n - 1}{2}.$$

If $x = 0$, then $\Gamma_{>0}$ has a 2-fracture graph. As before, a case-by-case analysis shows that $\Gamma_{>0}$ cannot be one of the graphs in Figure 1, leading to

$$r - 1 = r_2 \leq \frac{n_2 - 2}{2}.$$

Since in this case $r_1 = 0$ and $n_1 = 1$, it follows that

$$r_1 + r_2 \leq \frac{n_1 + n_2 - 3}{2},$$

as desired. □

Proposition 4.3. *Assume Hypothesis 4.1. If i is the label of a perfect split of Γ, then $i \notin \{1, r - 2\}$.*

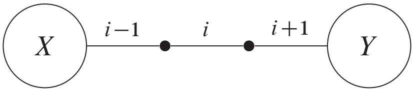
Proof. Up to duality, we may suppose $i = 1$. Let X and Y be the two G_1-orbits. In this case, ρ_0 fixes one of the two G_1-orbits pointwise, say Y, and it is the only permutation acting nontrivially on X. This implies that ρ_0 is a transposition, a contradiction. □

We include here a result from [4] which does not require Hypothesis 4.1.

Proposition 4.4 ([4, Proposition 5.1]). *If G is transitive and Γ has a perfect i-split, then G is primitive.*

Proposition 4.5. *Assume Hypothesis 4.1. If $i-1$, i and $i+1$ are labels of perfect splits, then either $\Gamma_{<i-1}$ or $\Gamma_{>i+1}$ has a 2-fracture graph.*

Proof. Let X be the G_{i-1}-orbit that is fixed by $G_{>i-1}$ pointwise and let Y be the G_{i+1}-orbit that is fixed by $G_{<i+1}$ pointwise. As G consists of even permutations, ρ_i acts nontrivially in one of the two sets X or Y.



Up to duality, we may suppose that ρ_i acts nontrivially on X. Then, as it commutes with all the elements of $G_{<i-1}$, $G_{<i-1}$ is imprimitive. Hence, by Proposition 4.4, $\Gamma_{<i-1}$ has no perfect splits. As all splits of Γ are perfect, $\Gamma_{<i-1}$ has a 2-fracture graph. □

Corollary 4.6. *Assume Hypothesis 4.1. Then Γ cannot have four consecutive perfect splits.*

Proof. If $i-1, i, i+1, i+2$ are labels of perfect splits, then neither $\Gamma_{>i+1}$ nor $\Gamma_{<i}$ has a 2-fracture graph, contradicting Proposition 4.5. □

Proposition 4.7. *Assume Hypothesis 4.1. Then $s \leq 2r/3$.*

Proof. Let $u = (u_i)_{i=0}^{r-1}$ be the sequence defined by

$$u_i = \begin{cases} 1 & \text{if } i \text{ is the label of a perfect split,} \\ 0 & \text{otherwise.} \end{cases}$$

Thus s is the number of ones in the sequence u.

By Proposition 4.3, an edge with label 1 or $r-2$ cannot be a perfect split, meaning that $u_1 = u_{r-2} = 0$. Furthermore, by Corollary 4.6, the sequence u cannot contain four consecutive ones.

Suppose first that u does not contain three consecutive ones. By Proposition 4.3, the following sequence maximizes the number of ones:

$$(1, 0, \underbrace{1, 1, 0, \ldots, 1, 1, 0}_{1,1,0 \text{ is repeated } k \text{ times}}, \underbrace{1, \ldots, 1}_{x \text{ times}}, 0, 1),$$

where

$$x = \begin{cases} 2 & \text{if } r \equiv 0 \pmod 3, \\ 0 & \text{if } r \equiv 1 \pmod 3, \\ 1 & \text{if } r \equiv 2 \pmod 3. \end{cases}$$

In all cases, we obtain $s \leq 2r/3$.

Now suppose that u contains a block of three consecutive ones, say at positions $i-1, i, i+1$. By Proposition 4.5, up to duality, we may assume that $u_k = 0$ for all $k < i-1$. If there are no other triples of consecutive ones, then the following sequence maximizes the number of ones:

$$(0,0,1,1,1,0,\underbrace{1,1,0,\dots,1,1,0}_{1,1,0 \text{ is repeated } k \text{ times}},\underbrace{1,\dots,1}_{x \text{ times}},0,1),$$

where again x depends on $r \pmod 3$. As shown earlier, this gives $s < 2r/3$.

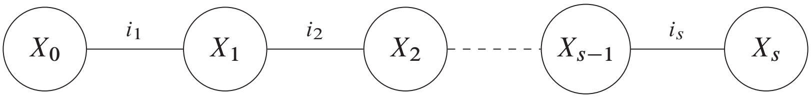
If there is a second triple of consecutive ones, then by Proposition 4.5, u must be constantly zero after this triple. Thus, as before, we conclude that $s < 2r/3$. □

We are finally ready to prove the main result of this section.

Proposition 4.8. *Assume Hypothesis 4.1. Then*

$$r \leq \frac{2n+s-2}{4} \quad \textit{and} \quad r \leq \left\lfloor \frac{3(n-1)}{5} \right\rfloor.$$

Proof. Let $\{i_k \mid k = 1,\dots,s\}$ be the set of labels of the perfect splits of Γ with $i_1 < \cdots < i_s$. Now, let X_0 be the set of points fixed by $G_{>i_1}$, let X_s be the set of points fixed by $G_{<i_s}$ and, for $k \in \{1,\dots,s-1\}$, let X_k be the set of points fixed by both $G_{>i_{k+1}}$ and $G_{<i_k}$. Additionally, let $r_0 = i_1$ be the rank of $G_{<i_1}$, $r_s = r - i_s - 1$ the rank of $G_{>i_s}$ and, for $k \in \{1,\dots,s-1\}$, $r_k = i_{k+1} - i_k - 1$ the rank of $G_{\{i_k+1,\dots,i_{k+1}-1\}}$. Finally, let $n_k = |X_k|$ for $k \in \{0,\dots,s\}$. The following diagram may help illustrate our argument:



By construction, $G_{>i_s}$ and $G_{<i_1}$ are either trivial or transitive SGGIs having a 2-fracture graph. Then, by Proposition 3.1, $r_0 \leq n_0/2$ and $r_s \leq n_s/2$. If $r_0 = n_0/2$, then from Proposition 3.1, we deduce that $\Gamma_{<i_1}$ has permutation representation graph corresponding to graphs (1) or (2) in Figure 1. However, the split with label i_1 must be incident to a pendent edge with label $i_1 - 1$. Since neither (1) nor (2) has such an edge, we deduce $r_0 \leq (n_0-1)/2$. Similarly, $r_s \leq (n_s-1)/2$.

Consider the SGGI arising from the permutation group

$$G^{(k)} = G_{\{i_k+1,\ldots,i_{k+1}-1\}}$$

acting on X_k for each $k \in \{1, \ldots, s-1\}$. Assume that $G^{(k)}$ is intransitive. Observe that this happens when two components of the permutation representation graph of $\Gamma_{i_k+1,\ldots,i_{k+1}-1}$ are connected by an edge with label i_k and an edge with label i_{k+1}. In this case, the group $G_{\{i_k,\ldots,i_{k+1}-1\}}$ acts transitively on $n_k + 1$ points and has a 2-fracture graph. Moreover, this graph has two pendent edges, one with label i_k and another with label $i_{k+1} - 1$. The latter is the edge incident to the split having label i_{k+1}. Hence the permutation representation graph of $\Gamma_{\{i_k,\ldots,i_{k+1}-1\}}$ is neither graph (1) nor graph (2) of Figure 1. We conclude that the rank of $G_{\{i_k,\ldots,i_{k+1}-1\}}$, which is $r_k + 1$, is less than $(n_k + 1)/2$. Thus $r_k + 1 \leq ((n_k + 1) - 1)/2$, which gives $r_k \leq (n_k - 2)/2$.

Let us also denote $G_{<i_1}$ and $G_{>i_s}$ by $G^{(0)}$ and $G^{(s)}$. Let $k \in \{0, \ldots, s-1\}$. Suppose that $G^{(k)}$ and $G^{(k+1)}$ are both transitive and let S_k and S_{k+1} be the ordered generating sets of $G^{(k)}$ and $G^{(k+1)}$, respectively. Consider the SGGI $\Phi = (F, S_k \cup \{\rho_k\} \cup S_{k+1})$, where $F = \langle S_k, \rho_k, S_{k+1}\rangle$. Observe that F is a permutation group of degree $n_k + n_{k+1}$. By construction, this SGGI has exactly one split and this split is perfect. Hence, by Lemma 4.2 applied to Φ,

$$r_k + r_{k+1} \leq \frac{n_k + n_{k+1} - 3}{2}.$$

The previous two paragraphs show that at most $\lceil (s+1)/2 \rceil$ ranks r_k of the set $\{r_0, \ldots, r_s\}$ attain the upper bound $(n_k - 1)/2$. Consequently,

$$\begin{aligned} r = r_0 + \cdots + r_s + s &\leq \frac{(n_0 + \cdots + n_s) - (s+1) - s/2}{2} + s \\ &= \frac{n - s - s/2 - 1}{2} + s. \end{aligned}$$

Thus $r \leq (2n + s - 2)/4$.

Now, the rest of the proof follows from Proposition 4.7. □

5 SGGIs having non-perfect splits

This section is the core of our argument for proving Theorem 1.1 and, as in Section 4, we borrow some of the ideas from [4, 6].

5.1 General notation and basic results

Hypothesis 5.1. Let G be a permutation group of degree n, and let $\Gamma = (G, S)$ be an SGGI of rank r with $S = \{\rho_0, \ldots, \rho_{r-1}\}$. Throughout this section, we assume the following conditions.

- G is a transitive subgroup of $\mathrm{Alt}(n)$.
- For each $i \in \{0, \dots, r-1\}$, the subgroup G_i is intransitive. These two conditions are equivalent to the existence of a fracture graph for Γ.
- The graph Γ does not admit a 2-fracture graph. Consequently, by Section 2.3, Γ admits at least one split.
- There exists some $i \in \{0, \dots, r-1\}$ such that Γ has a split with label i that is not perfect.

In what follows, let $\mathcal{G}$ be the permutation representation graph of Γ. Let $\{a, b\}$ be the split with label i. Let O_a and O_b be the G_i-orbits containing a and b, respectively, and let n_A and n_B denote the sizes of O_a and O_b. From the definition of split in Section 2.3, $O_a \cup O_b$ is equal to the domain of G. Let A and B be the permutation groups induced by G_i in its action on O_a and O_b. Observe that A and B are quotients of G_i, whereas when the split is perfect, A and B are also subgroups of G_i. Note that A and B are SGGIs. For each $l \in \{0, \dots, r-1\} \setminus \{i\}$, let $\rho_l = \alpha_l \beta_l$, where α_l and β_l are the restrictions of ρ_l to O_a and O_b, respectively. Then

$$A = \langle \alpha_l \mid l \in \{0, \dots, r-1\} \setminus \{i\} \rangle,$$
$$B = \langle \beta_l \mid l \in \{0, \dots, r-1\} \setminus \{i\} \rangle.$$

Let

$$J_A = \{l \in \{0, \dots, r-1\} \setminus \{i\} \mid \alpha_l \text{ is not the identity}\},$$
$$J_B = \{l \in \{0, \dots, r-1\} \setminus \{i\} \mid \beta_l \text{ is not the identity}\}.$$

In [6, Section 5.1], the authors prove several results where the intersection property is not required. We recall here some of these results.

Proposition 5.2 ([6, Proposition 5.1]). *If A is primitive, then the set J_A is an interval. The same result holds for B.*

The main result of [6, Section 5.1] gives an upper bound for the rank of Γ, when A and B are both imprimitive.

Proposition 5.3 ([6, Proposition 5.7]). *If A and B are both imprimitive, then $r \leq (n-1)/2$.*

The following result is one of the most important tools in the proofs of this section.

Proposition 5.4 ([6, Proposition 5.18]). *If e is an f-edge of $\mathcal{G}$ not in an alternating square, then any path (not containing another f-edge) from e to an edge with label l, with $l < f$ (resp. $l > f$), contains all labels between l and f. Moreover, there exists a path from e to an l-edge that is fixed by $G_{<l}$ (resp. $G_{>l}$).*

Observe that Proposition 5.4 applies immediately to the edge $e = \{a, b\}$, because a split edge does not belong to any alternating square.

Let us begin with the specific case where $\{0, r-1\} \subseteq J_A$, which, by Proposition 5.4, implies that $J_A = \{0, \dots, r-1\} \setminus \{i\}$.

Proposition 5.5. *If $\{0, r-1\} \subseteq J_A$, then $r \leq (n-1)/2$.*

Proof. Since $\{0, r-1\} \subseteq J_A$, we have $0 \neq i \neq r-1$ and hence it follows that $J_A = \{0, \dots, r-1\} \setminus \{i\}$ is not an interval. Thus, by Proposition 5.2, A is imprimitive. If B is imprimitive, then the result follows from Proposition 5.3. Hence, for the rest of the proof, we may suppose that J_B is an interval and that B is primitive. We consider two cases separately.

Case $G_{<i}$ is transitive on O_a. Let $l > i$. As ρ_l commutes with $G_{<i}$ and $G_{<i}$ is transitive on O_a, it follows that ρ_l is fixed-point-free on O_a. Since $\{a, b\}$ is a split and splits cannot be contained in alternating squares, ρ_{i+1} is the unique involution in S with label $> i$ not fixing a. We conclude that $l = i+1$, $i+1 = r-1$, and $G_i = G_{r-2}$.

Since $G_{i+1} = G_{r-1}$ is intransitive, $\rho_{i+1} = \rho_{r-1}$ acts nontrivially on O_b, that is, $r-1 \in J_B$. As J_B is an interval and

$$J_B \subseteq \{0, \dots, r-1\} \setminus \{i\} = \{0, \dots, r-3\} \cup \{r-1\},$$

we deduce that $J_B = \{r-1\}$. Hence $n_B = 2$ and $n_A = n-2$.

Since ρ_{r-1} acts fixed-point-freely on O_a and commutes with $G_{<r-2}$, we deduce that $G_{<r-2}$ preserves a system of imprimitivity with $n_A/2$ blocks of cardinality 2. Hence we obtain an embedding $G_{<r-2} \leq \mathrm{Sym}(2) \text{ wr } \mathrm{Sym}(n_A/2)$, and ρ_{r-1} swaps all pairs of vertices within the blocks of size 2.

We claim that, for every $j < i$, the group $\langle \rho_0, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_{i-1} \rangle$ acts intransitively on the system of imprimitivity preserved by $G_{<i}$. Assume the contrary, and let $j < i$ be such that $\langle \rho_0, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_{i-1} \rangle$ is transitive on the system of imprimitivity. Now, the group

$$\langle \rho_0, \dots, \rho_{j-1}, \rho_{j+1}, \rho_{j+2}, \dots, \rho_{r-4}, \rho_{r-3}, \rho_{r-1} \rangle$$

acts transitively on the system of imprimitivity of $G_{<i}$ and, via ρ_{r-1}, acts transitively on the two points within each block. Therefore,

$$\langle \rho_0, \dots, \rho_{j-1}, \rho_{j+1}, \dots, \rho_{r-3}, \rho_{r-1} \rangle$$

has orbits O_a and O_b on the domain of G. Finally, since ρ_{r-2} swaps a and b, we deduce that

$$\langle \rho_0, \ldots, \rho_{j-1}, \rho_{j+1}, \ldots, \rho_{r-3}, \rho_{r-2}, \rho_{r-1} \rangle = G_j$$

is transitive, which contradicts Hypothesis 5.1.

The previous claim shows that the set $\{\rho_0, \ldots, \rho_{r-3}\}$ generates the block action independently. The maximal size of an independent set of degree $n_A/2$ is at most $n_A/2 - 1$ (see [10]). Therefore,

$$|\{\rho_0, \ldots, \rho_{r-3}\}| \leq \frac{n_A}{2} - 1 \leq \frac{n-2}{2} - 1.$$

Assume now that $r > (n-1)/2$. Then it follows that

$$|\{\rho_0, \ldots, \rho_{r-3}\}| > \frac{n-1}{2} - 2.$$

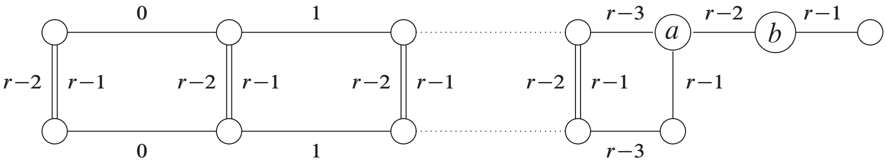
Combining these inequalities forces $n_A = n - 2$ and

$$|\{\rho_0, \ldots, \rho_{r-3}\}| = \frac{n-2}{2} - 1.$$

The independence of $\{\rho_0, \ldots, \rho_{r-3}\}$, together with the string condition, implies that the permutation representation graph of the group generated by $\rho_0, \ldots, \rho_{r-3}$ in its action on the blocks is as in the following picture:

From [10], this implies that $r - 2 \leq n_A/2 - 1$, leading to $r \leq n/2$. If $r < n/2$, then the result follows. Assume then $r = n/2$, that is, $r = n_A/2 + 1$.

Observe that $\langle \rho_0, \ldots, \rho_{r-4} \rangle$ fixes one block, namely the block containing the point a, and acts as the full symmetric group of degree $r - 3$ on the remaining blocks. As $G \leq \mathrm{Alt}(n)$, $\rho_i = \rho_{r-2}$ cannot be a transposition, meaning that it acts nontrivially on O_a. Since $\rho_i = \rho_{r-2}$ commutes with $\langle \rho_0, \ldots, \rho_{r-4} \rangle$, we deduce that ρ_i fixes $r - 3$ blocks setwise. As there are $r - 2$ blocks in total, we conclude that $\rho_i = \rho_{r-2}$ also fixes the remaining block setwise, that is, ρ_i lies in the kernel of the action on the blocks. However, this leads to the contradiction that either ρ_{r-2} or ρ_{r-1} is an odd permutation, as illustrated in the following graph:



Case $G_{<i}$ is intransitive on O_a. Recall that

$$J_A = \{0, \ldots, i-1\} \cup \{i+1, \ldots, r-1\}.$$

Let $A_{<i} = \langle \alpha_l \mid l < i \rangle$ and $A_{>i} = \langle \alpha_l \mid l > i \rangle$. Observe that $A_{<i}$ and $G_{<i}$ induce the same action on O_a. We have $A = \langle A_{<i}, A_{>i} \rangle$ and $A_{<i}$ commutes with $A_{>i}$. Thus $A_{<i} \trianglelefteq A$. Since $G_{<i}$ is intransitive on O_a, so is $A_{<i}$. Hence the orbits of $A_{<i}$ on O_a give rise to a system of imprimitivity for A. In particular, there exists an embedding $A \leq \mathrm{Sym}(k) \mathrm{\ wr\ } \mathrm{Sym}(m)$, with $G_{<i}$ fixing all the blocks setwise.

Let β denote the block containing the vertex a. Since $\{a, b\}$ is an i-split, the only permutation in S with label $> i$ not fixing a is ρ_{i+1}. Therefore, ρ_{i+1} is the only permutation in S with label $> i$ that does not fix the block β. Hence $\beta\rho_{i+1}$ is a $G_{<i}$-orbit disjoint from β. By Proposition 5.4, there exists a path $\mathcal{P}_1$, whose vertices are in β, from a 0-edge to the vertex a, containing all labels from 0 to $i-1$. Similarly, there is a path $\mathcal{P}_2$ in the block $\beta\rho_{i+1}$ containing all labels from 0 to $i-1$.

Subcase: There is an edge with label $l > i$ inside $\beta\rho_{i+1}$ (or β). Since $\beta\rho_{i+1}$ is a block, this implies that ρ_l fixes $\beta\rho_{i+1}$ setwise. Since $G_{<i}$ is transitive on $\beta\rho_{i+1}$ and since $G_{<i}$ centralizes ρ_l, we deduce that ρ_l acts fixed-point-freely on $\beta\rho_{i+1}$. Moreover, since a split cannot belong to an alternating square, we deduce that ρ_{i+2} is the only permutation that acts nontrivially on $a\rho_{i+1} \in \beta\rho_{i+1}$. This implies $l = i+2$ and $r-1 = i+2$. From the definition of the system of imprimitivity, $G_{<i} = \langle \rho_0, \ldots, \rho_{r-3} \rangle$ fixes $\beta\rho_{i+1}$ setwise. Since $\rho_l = \rho_{r-1}$ also fixes $\beta\rho_{i+1}$ setwise, we deduce that G_{i+1} fixes $\beta\rho_{i+1}$ setwise. From this, we deduce that β and $\beta\rho_{i+1}$ are the only two blocks.

From the previous paragraph, $\rho_{i+2} = \rho_{r-1}$ centralizes $G_{<i}$ and acts fixed-point-freely on $\beta\rho_{i+1}$. Therefore, we obtain a homomorphism of $G_{<i}$ into

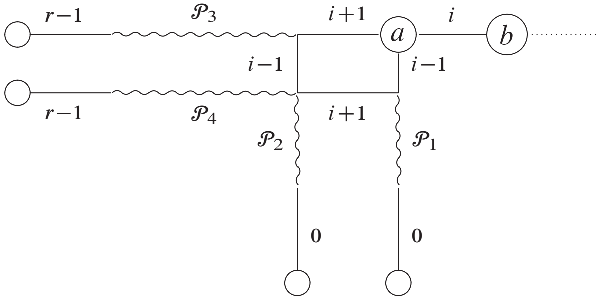
$$\mathrm{Sym}(2) \mathrm{\ wr\ } \mathrm{Sym}(k/2).$$

Moreover, as G_{i+2} is intransitive, $\rho_{i+2} = \rho_{r-1}$ acts nontrivially on O_b. Since J_B is an interval and $i = r-3$, we have $J_B = \{r-2, r-1\}$ and $n_B \geq 3$. In addition, arguing as in the previous case, since G_j is intransitive for every j, the set $\{\rho_0, \ldots, \rho_{i-1}\}$ generates the block action (for the block system with $k/2$ blocks of size 2) independently. Hence, by [10],

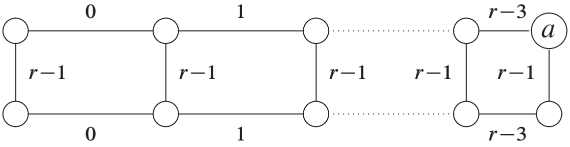
$$i = r-3 \leq \frac{k}{2} - 1 = \frac{n_A}{4} - 1 \leq \frac{n-3}{4} - 1.$$

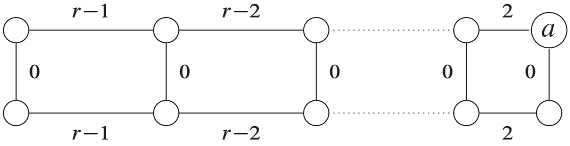
This gives $r \leq (n-1)/2$ (note that n is necessarily at least $4+3=7$).

Subcase: β and $\beta\rho_{i+1}$ do not contain edges with label $l > i$. By Proposition 5.4, there is a path $\mathcal{P}_3$ in O_a containing edges with all labels from $r-1$ to $i+1$. Using the commuting property between ρ_{i-1} and ρ_l with $l > i$, applying ρ_{i-1} to $\mathcal{P}_3$, we obtain a second path $\mathcal{P}_4$ disjoint from $\mathcal{P}_3$ and containing edges with all labels from $r-1$ to $i+1$. Therefore, $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$ have no edge in common, as shown in the following figure:



The inequality $2(r-1) \leq n - n_B$ holds. Equality is achieved if and only if there are no points outside the paths $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$, $\mathcal{P}_4$. This implies that $i \in \{1, r-2\}$, and that the permutation representation graph of A is one of the following:

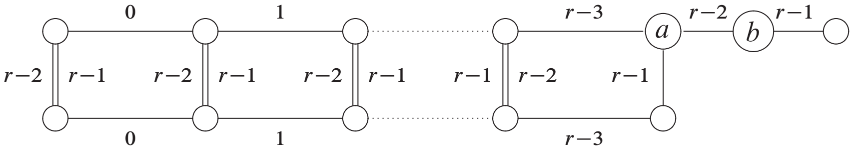




Assume first that $n_B \geq 3$. Then $2(r-1) \leq n-3$, and hence $r \leq (n-1)/2$.

Now assume $n_B = 1$. Suppose first that the equality $2(r-1) = n - n_B = n-1$ holds. Since all G_j are intransitive, ρ_i must be the transposition swapping a and b, contradicting the assumption $\rho_i \in \mathrm{Alt}(n)$. Thus $2(r-1) < n-1$ and therefore $r \leq n/2$. If n is odd, then $r \leq (n-1)/2$. If n is even, then A has odd degree and hence cannot have a block system with blocks of even size or with an even number of blocks. Since $n_B = 1$ and $G \leq \mathrm{Alt}(n)$, we deduce that A consists of even permutations. We conclude that there are at least two vertices not in the paths $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ and $\mathcal{P}_4$. Hence $2(r-1)+1 \leq n-2$, and therefore $r \leq (n-1)/2$.

Finally, assume $n_B = 2$. If $2(r-1) < n - n_B$, then $r \leq (n-1)/2$. Suppose instead that $2(r-1) = n - n_B = n-2$. Assume that A has the permutation representation graph on the left above, that is, $i = r-2$. Then the graph $\mathcal{G}$ is as follows:



Then either ρ_{r-2} or ρ_{r-1} is odd, a contradiction. A similar conclusion is reached if we consider the other possible permutation representation graph for A. $\square$

Hypothesis 5.6. Observe that if A and B are both imprimitive, then Theorem 1.1 follows by Proposition 5.3. In particular, from Proposition 5.2, we may suppose that either J_A or J_B is an interval. Without loss of generality, we may assume that J_B is an interval.

Up to duality and applying Proposition 5.4 with a path from an $(r-1)$-edge in O_b to the i-split $\{a, b\}$, we may suppose that

$$J_B = \{i+1, \dots, r-1\}.$$

Now, applying Proposition 5.4 to the indices in J_A, we deduce that

$$J_A = \{0, \dots, h\} \setminus \{i\}$$

for some $h \in \{0, \dots, r-1\}$. When $i \neq 0$, from Proposition 5.5, we have either $r \leq (n-1)/2$ or $h \neq r-1$. Similarly, when $i = 0$, from [6, Proposition 5.19], we have either $r \leq (n-1)/2$ or $h \neq r-1$. In either case, for the rest of the argument, we may suppose that $h \neq r-1$. Thus $i < h < r-1$.

With this setting, [6, Propositions 5.20–5.23] establish that the permutation representation graph of Γ satisfies the following properties.

- If $i = 0$, there exists a path fixed by $G_{>h}$ that contains the 0-split and that has at least two edges for each label $j \in \{1, \dots, h\}$.
- If $i \neq 0$ and $h \neq i+1$, the group $G_{<i}$ acts intransitively on O_a, and A embeds into a wreath product where the blocks correspond to the $G_{<i}$-orbits. Moreover, if $r > (n-1)/2$, then no generator ρ_j with $j > i$ fixes the $\Gamma_{<i}$-orbits. In this case, the permutation representation graph of $G_{<i}$ contains, in each connected component, a path with all labels in the set $\{0, \dots, i-1\}$; thus there are at least two such paths, $\mathcal{P}_1$ and $\mathcal{P}_2$. Additionally, there exist at least two disjoint paths, $\mathcal{P}_3$ and $\mathcal{P}_4$, each containing all labels in the set $\{i+1, \dots, h\}$, and each of these paths intersects $\mathcal{P}_1 \cup \mathcal{P}_2$ in at most two points.

Further details about these paths can be found in the cited propositions, which together yield the following result.

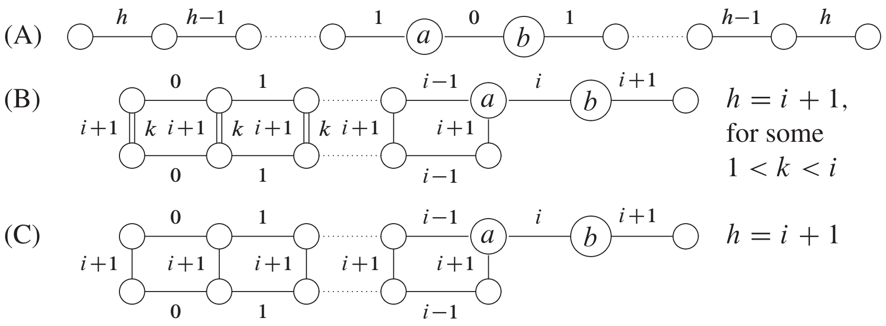


Figure 2. Permutation representation graphs for Proposition 5.7.

Proposition 5.7. *If $r > (n-1)/2$, then for each $\ell \in \{i+1, \dots, h\}$, we have*

$$\ell \leq \frac{n - |X_\ell| - 1}{2}, \quad \textit{where } X_\ell = \{1, \dots, n\} \setminus \mathrm{Fix}(G_{>\ell}).$$

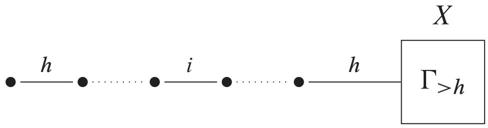
Moreover, if $\ell \neq h$ and $\ell = (n - |X_\ell| - 1)/2$, then $i = 0$ and $\Gamma_{<\ell}$ has a permutation representation graph containing the 0-split, that is, a path, fixed by $G_{>ell}$, with the sequence of labels

$$[\ell-1, \ell-2, \dots, 1, 0, 1, \dots, \ell-2, \ell-1].$$

If $h = (n - |X_h| - 1)/2$, then $\Gamma_{<h+1}$ has permutation representation graph isomorphic to one of the graphs in Figure 2, possibly with some additional edges with label i (the unique i-edge represented is the split $\{a, b\}$).

Together with the notation in Hypotheses 5.1 and 5.6, for the rest of this section, we reserve the letters h and $X = X_h$ for the meanings we have just established.

Note that X is a subset of O_b, as illustrated in the following figure:



Lemma 5.8. *The permutation representation graph of $\Gamma_{>h}$ has a pendent edge labeled $h+1$ which is incident, in $\mathcal{G}$, to an edge labeled h.*

Proof. Let $h+1$ be the label of an edge $e = \{u, v\}$ in $\mathcal{G}$ such that the distance between e and the split $\{a, b\}$ is minimal. By Proposition 5.4, there exists a path connecting e to the split $\{a, b\}$ that is fixed by $G_{>h}$. Let this path be

$$b = u_0, u_1, \dots, u_\ell = u \quad \text{for some } \ell.$$

By minimality and by the fact that edges whose labels differ by two (or more) form an alternating square, e is incident in this path to an edge labeled h, that is, $\{u_{\ell-1}, u_\ell\}$ has label h.

If e is not a pendent edge in the permutation representation graph of $\Gamma_{>h}$, then u_ℓ is on an edge having label $\kappa \geq h+2$. Since $G_{>h}$ fixes the above path, we have

$$u_\ell^{\rho_h} = u_{\ell-1} = u_{\ell-1}^{\rho_\kappa} = u_\ell^{\rho_h \rho_\kappa} = (u_\ell^{\rho_\kappa})^{\rho_h}.$$

Thus $u_\ell^{\rho_\kappa} = u_\ell$, contradicting the fact that u_ℓ is on an edge having label κ. □

Since Proposition 5.7 (applied with $\ell = h$) gives an upper bound on h, it allows us to reduce the problem of determining an upper bound for the rank of $\Gamma = (G, S)$ to that of finding an upper bound for the rank of the SGGI $\Gamma_{>h}$. However, $G_{>h}$ is not necessarily transitive on $X = \{1, \ldots, n\} \setminus \mathrm{Fix}(G_{>h})$. To address this, we require an extension of the definition of fracture graph, as given in Section 2.2, to the case of intransitive SGGIs.

An intransitive SGGI $\Phi = (H, \{\alpha_0, \ldots, \alpha_{w-1}\})$ of rank w is said to have a *fracture graph* if, for each $l \in \{0, \ldots, w-1\}$, the number of orbits of H_l exceeds that of H. An l-edge $\{x, y\}$ of a fracture graph is defined as a pair $\{x, y\}$, where $y = x\rho_l$ and where x and y belong to the same H-orbit but not to the same H_l-orbit.

If, for each $l \in \{0, \ldots, w-1\}$, there are at least two possible choices for the l-edge of a fracture graph, then Φ admits a 2-*fracture graph*. On the other hand, if e is the unique choice for an l-edge of a fracture graph, then e is called a *split*.

5.2 The SGGI $\Gamma_{>h}$ admits a 2-fracture graph

Proposition 5.9 ([6, Proposition 5.27]). *Let $\Gamma = (G, S)$ be a transitive SGGI of rank r. If $t \in \{0, \ldots, r-2\}$ is such that*

- $t \leq (n - |U| - 1)/2$, *with* $U = \{1, \ldots, n\} \setminus \mathrm{Fix}(G_{>t})$,
- $\Gamma_{>t}$ *has a 2-fracture graph,*
- $G_{>t}$ *acts intransitively on U,*

then $r \leq (n-1)/2$.

By combining the previous proposition with the results from Section 3, we complete the analysis of the case when $\Gamma_{>h}$ admits a 2-fracture graph.

Proposition 5.10. *Assume Hypotheses 5.1 and 5.6. If $\Gamma_{>h}$ has a 2-fracture graph, then $r \leq (n-1)/2$.*

Proof. Suppose that $\Gamma_{>h}$ has a 2-fracture graph. If $G_{>h}$ is intransitive on X, then by Propositions 5.7 and 5.9, we have $r \leq (n-1)/2$. If $G_{>h}$ is transitive on X, then by Propositions 3.1 and 5.8, we obtain $r-h-1 \leq (|X|-1)/2$. Now, Proposition 5.7 implies that $r \leq n/2$. Moreover, when equality holds and $r = n/2$, the permutation representation graph of $\Gamma_{>h}$ must be one of the graphs from Figure 1 that contains a pendent edge, and the permutation representation graph of $\Gamma_{<h+1}$ is one of the graphs from Figure 2.

A case-by-case analysis combining all possible graphs for $\Gamma_{<h+1}$ and $\Gamma_{>h}$ shows that there exists a unique i-edge – namely, the split $\{a, b\}$. However, this contradicts the fact that ρ_i is an even permutation. □

5.3 The SGGI $\Gamma_{>h}$ does not admit a 2-fracture graph: Notation

Now assume $\Gamma_{>h}$ does not have a 2-fracture graph. Suppose that j, for $j > h$, is the label of a split $\{c, d\}$ for $\Gamma_{>h}$, which may or may not be perfect. In addition, suppose that i and j are labels of consecutive splits.

For the j-split, we follow the notation in Section 5.1. Let $\{c, d\}$ be the split with label j. Let O_c and O_d be the G_j-orbits containing c and d, respectively, and let n_C and n_D denote the sizes of O_c and O_d. Let C and D be the permutation groups induced by G_j in its action on O_c and O_d. For each $l \in \{0, \ldots, r-1\} \setminus \{j\}$, let $\rho_l = \gamma_l \delta_l$, where γ_l and δ_l are the restrictions of ρ_l to O_c and O_d, respectively. Then

$$C = \langle \gamma_l \mid l \in \{0, \ldots, r-1\} \setminus \{j\} \rangle,$$
$$D = \langle \delta_l \mid l \in \{0, \ldots, r-1\} \setminus \{j\} \rangle.$$

Let

$$J_C = \{l \in \{0, \ldots, r-1\} \setminus \{j\} \mid \gamma_l \text{ is not the identity}\},$$
$$J_D = \{l \in \{0, \ldots, r-1\} \setminus \{j\} \mid \delta_l \text{ is not the identity}\}.$$

In the remainder of this section, we consider two cases, depending on whether the split $\{c, d\}$ is perfect or not.

5.4 The split $\{c, d\}$ is not perfect

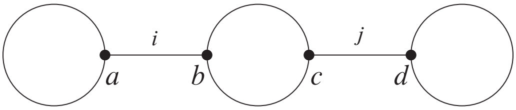
In this section, we make an additional assumption.

Hypothesis 5.11. We assume that i is the maximal label of an l-split satisfying the following property: there exists a permutation ρ_x, with $x > l$, that acts nontrivially on both G_l-orbits. This assumption is compatible with the fact that $\{a, b\}$ is a non-perfect i-split, as well as with Hypotheses 5.1 and 5.6.

Note also that if ρ_g is a permutation that acts nontrivially on both G_j-orbits, then $g < j$. Indeed, since j is the label of a non-perfect split and $j > i$, the maximality of i implies that $g < j$.

Assume further that g is the minimal label of a permutation acting nontrivially on both G_j-orbits.

We may summarize some of the information in Hypotheses 5.1, 5.6 and 5.11 in the following figure:



Lemma 5.12. *Assume Hypotheses* 5.1, 5.6 *and* 5.11. *If* $r > (n-1)/2$, *then*

(a) $J_C = \{0, \dots, j-1\}$ *and* $J_D = \{g, \dots, r-1\} \setminus \{j\}$*;*

(b) $g > 0$*;*

(c) *for each* $\ell \in \{g, \dots, j-1\}$,
$$r - \ell - 1 \leq \frac{n - |Y_\ell| - 1}{2},$$
where $Y_\ell = \{1, \dots, n\} \setminus \mathrm{Fix}(G_{<\ell})$ *and* $Y_g \subseteq O_c$*;*

(d) *if* $r - g - 1 = (n - |Y_g| - 1)/2$, *then* Γ_{g+1} *has permutation representation graph isomorphic to one of the graphs in Figure* 3*, possibly with some additional edges with label* j *(the unique* j*-edge represented is the split* $\{c, d\}$*);*

(e) *if* $\ell \neq g$ *and* $r - \ell - 1 = (n - |Y_\ell| - 1)/2$, *then* $j = r - 1$ *and* $\Gamma_{>\ell}$ *has a permutation representation graph containing the* $(r-1)$*-split, that is, a path, fixed by* $G_{<\ell}$*, with the sequence of labels*
$$[\ell+1, \ell+2, \dots, r-2, r-1, r-2, \dots, \ell+2, \ell+1].$$

Proof. We start by proving (a). As J_C contains $J_A \cup \{i\}$, $0 \in J_C$; see the picture above. If $j = r-1$, then by Proposition 5.4, $J_C = \{0, \dots, r-2\}$; thus J_C is an interval. Assume that $j \neq r-1$. Suppose that J_C is not an interval. Hence, by Propositions 5.2 and 5.3, J_D is an interval. Hence, as g is the minimal label of a permutation acting nontrivially on O_d,
$$J_D = \{g, \dots, j-1\}.$$
Then $\{0, r-1\} \subseteq J_C$, which gives a contradiction by Proposition 5.5, because $r > (n-1)/2$. Hence J_C is an interval and, applying Proposition 5.4, we get $J_C = \{0, \dots, j-1\}$. Now, Proposition 5.4 implies $J_D = \{g, \dots, r-1\} \setminus \{j\}$. This shows part (a).

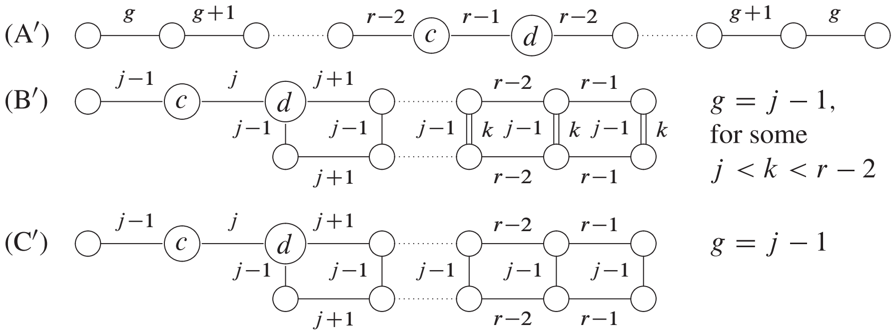


Figure 3. Permutation representation graphs for Proposition 5.12 (d)

Next, we prove part (b). Arguing by contradiction, suppose $g = 0$. In particular, $0 \in J_C$ and $0 \in J_D$. When $j \neq r - 1$, we must have $r - 1 \in J_D$. Hence we obtain a contradiction from Proposition 5.5, because $r > (n - 1)/2$. When $j = r - 1$, by [6, Proposition 5.19] (applied to the dual SGGI), we get $r \leq (n - 1)/2$, which is again a contradiction.

The first statements in (c), (d) and (e) are the dual of Proposition 5.7. The inclusion $Y_g \subseteq O_c$ follows from the minimality of g. □

Lemma 5.13. *Assume Hypotheses 5.1, 5.6, 5.11. We have $n \geq 6$ and, if $r > n/2$, then $g > h$.*

Proof. The existence of two non-perfect splits implies $n \geq 6$.

Suppose that $r > n/2$ and $g \leq h$. By Hypothesis 5.6, ρ_h is defined as the maximal permutation acting nontrivially on both G_i-orbits. In particular, $h \in J_A \cap J_B$. By Hypothesis 5.6, ρ_g is defined as the minimal permutation acting nontrivially on both G_j-orbits. Moreover, by Lemma 5.12 (a),

$$J_C = \{0, \ldots, j - 1\} \quad \text{and} \quad J_D = \{g, \ldots, r - 1\} \setminus \{j\}.$$

In particular, we also have $h \in J_C \cap J_D$. Therefore, ρ_h acts nontrivially in each of the three $G_{i,j}$-orbits. By Proposition 5.7 (applied with $\ell = h$), we have

$$h \leq \frac{n - |X| - 1}{2}, \tag{5.1}$$

where $X = \{1, \ldots, n\} \setminus \mathrm{Fix}(G_{>h})$. We consider separately the cases $h + 1 \neq j$ and $h + 1 = j$.

Case $h + 1 \neq j$. As $g \leq h < h + 1 < j$, from Lemma 5.12 (c), we have

$$r - (h + 1) - 1 \leq \frac{n - |L| - 1}{2}, \tag{5.2}$$

where $L = \{1,\dots,n\} \setminus \mathrm{Fix}(G_{<h+1})$. Let $\bar{L} = \{1,\dots,n\} \setminus L = \mathrm{Fix}(G_{<h+1})$. We claim that $\bar{L} \subseteq X$. Indeed, if $p \in \bar{L}$, then $p \in \mathrm{Fix}(G_{<h+1})$. So

$$p \notin \mathrm{Fix}(G_{\geq h+1}) = \mathrm{Fix}(G_{>h}) = \{1,\dots,n\} \setminus X$$

and hence $p \in X$. Combining (5.1) and (5.2), we get

$$r \leq \frac{n - |X \setminus \bar{L}| + 2}{2}. \tag{5.3}$$

Let us now prove that $X \setminus \bar{L} = \overline{\mathrm{Fix}(G_{>h}) \cup \mathrm{Fix}(G_{<h+1})}$ is nonempty. Observe that the permutation representation graph $\mathcal{G}$ of Γ contains at least one pair of adjacent edges with labels h and $h+1$, for otherwise G would be a direct product. Let p be the meeting point of such a pair of edges. Then $p \notin \mathrm{Fix}(G_{>h}) \cup \mathrm{Fix}(G_{<h+1})$. This implies that $\{1,\dots,n\} \neq \mathrm{Fix}(G_{>h}) \cup \mathrm{Fix}(G_{<h+1})$, and hence $X \neq \bar{L}$. In particular, $|X \setminus \bar{L}| \geq 1$.

Since we are assuming $r > n/2$, from (5.1) and (5.3), we deduce $|X \setminus \bar{L}| = 1$, $r - (h+1) - 1 = (n - |L| - 1)/2$ and $h = (n - |X| - 1)/2$.

From the equality $r - (h+1) - 1 = (n - |L| - 1)/2$, it follows from Lemma 5.12 (e) that $j = r - 1$, and the permutation representation of $\Gamma_{>h}$ is a path $\mathcal{P}$, fixed by $G_{<h+1}$, with the sequence of labels

$$[h+1, h+2, \dots, r-2, r-1, r-2, \dots, h+2, h+1].$$

Since ρ_{r-1} must be even, it must act nontrivially on L. This is only possible if $h+1 = r-2$. Hence $G_{>h} = \langle \rho_{r-2}, \rho_{r-3} \rangle$. As G is even, this implies that $|X| \geq 5$. From the equality $h = (n - |X| - 1)/2$, we then obtain $r \leq n/2$, which is a contradiction.

Case $h + 1 = j$. From the previous case applied dually, we may assume that $g - 1 = i$. This implies $g = h = j - 1 = i + 1$.

There are two possibilities for J_D. Either $J_D = \{r-2\}$ (when $h = r-2$) or $J_D = \{h\} \cup \{h+2, \dots, r-1\}$ (when $h < r-2$).

If $J_D = \{r-2\}$, then $G_{>h} = \langle \rho_{r-1} \rangle$. Since G consists of even permutations, we deduce $|X| \geq 4$. Then (5.1) gives $r - 2 \leq (n - 4 - 1)/2$, which is a contradiction. A dual argument gives $J_A \neq \{1\}$.

It remains to consider the case

$$J_A = \{0, \dots, h-2\} \cup \{h\} \quad \text{and} \quad J_D = \{h\} \cup \{h+2, \dots, r-1\}.$$

Then there are two disjoint paths $\mathcal{P}_1$ and $\mathcal{P}_2$, in O_a, with all labels from 0 to $h-2$. In O_d, there are also two disjoint paths, $\mathcal{P}_3$ and $\mathcal{P}_4$, containing all labels from $h+2$ to $r-1$. We deduce that

$$2(r-3) \leq (|\mathcal{P}_1| - 1) + (|\mathcal{P}_2| - 1) + (|\mathcal{P}_3| - 1) + (|\mathcal{P}_4| - 1).$$

Since $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4 \subseteq \{1, \dots, n\} \setminus (O_b \cap O_c)$, we get

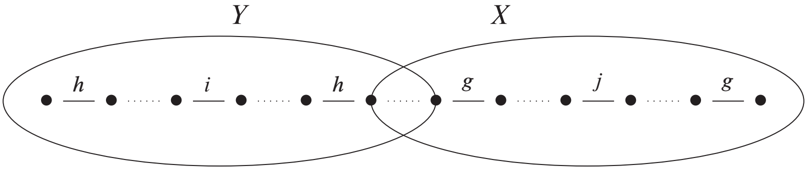
$$2(r-3) \leq n - 4 - |O_b \cap O_c|.$$

Since $|O_b \cap O_c| \geq 2$ (as ρ_h permutes a pair of vertices in this set), we obtain

$$2(r-3) \leq (n-2) - 4.$$

Hence $r \leq n/2$, contradicting the assumption that $r > n/2$. □

In light of Lemma 5.13, we may assume $h < g$ and hence the following figure illustrates the situation we need to address, where $G_{\{h+1,\dots,g-1\}}$ acts on $X \cap Y$, where $X = \{1, \dots, n\} \setminus \mathrm{Fix}(G_{>h})$ and $Y = \{1, \dots, n\} \setminus \mathrm{Fix}(G_{<g})$. In addition, as i and j are labels of consecutive splits, $\Gamma_{\{h+1\dots,g-1\}}$ has the following 2-fracture graph:



Similarly to the proof of Proposition 5.9 given in [6, Proposition 5.27], let us prove the following.

Proposition 5.14. *Let* $\Gamma = (G, S)$ *be a transitive SGGI. Suppose that there exist* $u, v \in \{0, \dots, r-1\}$ *such that*

(a) $0 < u < v < r - 1$*;*

(b) $u \leq (n - |U| - 1)/2$ *with* $U = \{1, \dots, n\} \setminus \mathrm{Fix}(G_{>u})$*;*

(c) $r - v - 1 \leq (n - |V| - 1)/2$ *with* $V = \{1, \dots, n\} \setminus \mathrm{Fix}(G_{<v})$*;*

(d) $\Gamma_{\{u+1,\dots,v-1\}}$ *has a 2-fracture graph and*

(e) $G_{\{u+1,\dots,v-1\}}$ *acts intransitively on* $U \cap V$*.*

Then $r \leq (n-2)/2$*.*

Proof. Suppose that u and v satisfy the hypotheses of this proposition and that $v - u$ is minimal. Define $H = G_{\{u+1,\dots,v-1\}}$, $\Phi = \Gamma_{\{u+1,\dots,v-1\}}$, and let $\mathcal{G}$ be the permutation representation graph of Γ.

In what follows, $X_1, \dots, X_c$ denote the nontrivial H-orbits, while the corresponding group actions of H on each of these sets are denoted by $G^{(1)}, \dots, G^{(c)}$. Furthermore, let $\mathcal{G}^{(1)}, \dots, \mathcal{G}^{(c)}$ be the corresponding permutation representation graphs.

Consider a (simple) fracture graph $\mathcal{F}$ of Φ, i.e., a graph with $|U \cap V|$ vertices, where the number of edges is equal to the rank of Φ. Each l-edge of $\mathcal{F}$ connects vertices in different G_l-orbits within the same H-orbit. For each set X_s, with $s \in \{1, \dots, c\}$, denote by I_s the set of labels of edges in X_s, and by F_s the set of labels of edges of $\mathcal{F}$ within X_s. Clearly, we have $F_s \subseteq I_s$. Choose $\mathcal{F}$ such that it satisfies the following property.

(P) If $l \in F_s$ is the label of the unique l-edge in one component swapping vertices in different G_l-orbits, then no other component has more than one pair of vertices in different G_l-orbits.

When (P) holds, another component must contain exactly one l-edge connecting different G_l-orbits. These l-edges are splits within each G_l-orbit, ensuring the existence of paths with consecutive labels, which are crucial to the proof.

We have that $G^{(s)}$ is generated by a set of involutions (not necessarily independent) with labels in I_s. The subset of involutions with labels in F_s is independent, since F_s corresponds to a subset of labels of edges in $\mathcal{F}$.

Let us bound the set of labels of each component of the fracture graph $\mathcal{F}$. If $\mathcal{G}^{(s)}$ admits a 2-fracture graph with set of labels F_s, then by Proposition 3.1, we have

$$|F_s| \leq \frac{|X_s|}{2}.$$

When two components have exactly the same (labeled) permutation representation graph, since $I_s = I_{s'}$ and $|X_s| = |X_{s'}|$, we have

$$|F_s \cup F_{s'}| = |F_s| \leq |X_s| - 1 = \frac{|X_s| + |X_{s'}|}{2} - 1.$$

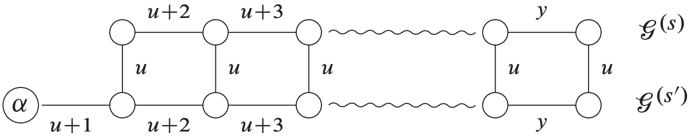
Now, suppose that a component $\mathcal{G}^{(s)}$ is not a copy of any other component and does not admit a 2-fracture graph. Then there exists an edge e with label $l \in F_s$ such that ρ_l swaps only one pair of vertices of X_s, in different G_l-orbits. Let x and y denote, respectively, the minimal and maximal label of an edge in $\mathcal{G}^{(s)}$.

By Proposition 5.4, there exist two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ in X_s such that $\mathcal{P}_1$ contains all labels from $l-1$ to x, and $\mathcal{P}_2$ contains all labels from $l+1$ to y. In particular, we have $I_s = \{x, x+1, \dots, y-1, y\}$. Define P as the set of vertices of $\mathcal{P} = \mathcal{P}_1 \cup \{e\} \cup \mathcal{P}_2$.

Now, let $\mathcal{G}^{(s')}$ be a component adjacent to $\mathcal{G}^{(s)}$ in $\mathcal{G}$. Since $\mathcal{G}^{(s)}$ cannot be a copy of $\mathcal{G}^{(s')}$, using (P), it follows that $x \in \{u+1, u+2\}$ or $y \in \{v-1, v-2\}$.

In what follows, we show that, when $x \in \{u+1, u+2\}$, we get a contradiction. Moreover, a dual argument also leads to a contradiction when $y \in \{v-1, v-2\}$. We analyze the cases $x = u+2$ and $x = u+1$ separately.

Case $x = u + 2$. In this case, $\mathcal{G}^{(s')}$ contains a path $\mathcal{P}'$ that is a copy of $\mathcal{P}$.



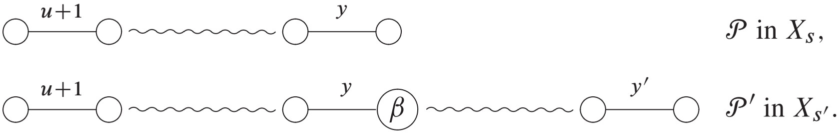
Let C be the set of vertices of $\mathcal{P} \cup \mathcal{P}'$. We have $2(y - (u + 1)) \leq |C| - 2$; hence, by hypothesis (b), we have $y \leq u + |C|/2 \leq (n - |U \setminus C| - 1)/2$.

The vertex α is fixed by $G_{>y}$ and $\{1, \ldots, n\} \setminus \mathrm{Fix}(G_{>y}) \subseteq U \setminus C$ (recall that y is the maximal label of $\mathcal{G}^{(s)}$). This shows that y and v satisfy all the hypotheses of this proposition. Thus we obtain a contradiction with the minimality of $v - u$.

Case $x = u + 1$. Assume that any component containing a unique l-edge between vertices in different G_l-orbits has minimal label $u + 1$. Let y be the maximal label in $\mathcal{G}^{(s)}$ and let $\mathcal{P}$ be as previously defined.

Since Φ has a 2-fracture graph, there exists another component $\mathcal{G}^{(s')}$ (which might not be adjacent to $\mathcal{G}^{(s)}$) containing an l-edge e' between vertices in different G_l-orbits. By (P), this l-edge cannot be in an alternating square inside $X_{s'}$. By assumption, the minimal label in $X_{s'}$ is also $u + 1$. Let y' be the maximal label in $X_{s'}$. Without loss of generality, assume $y' \geq y$. Then, by Proposition 5.4, there exists a path $\mathcal{P}_1'$ containing all labels from $l - 1$ to $u + 1$ and another path $\mathcal{P}_2'$ containing all labels from $l + 1$ to y, both in $\mathcal{G}^{(s')}$. Define $\mathcal{P}' = \mathcal{P}_1' \cup \{e'\} \cup \mathcal{P}_2'$.

Since an l-edge in $X_{s'}$ is not in an alternating square, β (see the picture below) is the unique vertex in $\mathcal{P}'$ that is not fixed by $G_{>y}$. Let C' be the set of vertices of $\mathcal{P} \cup \mathcal{P}'$.



Therefore, $G_{>y}$ fixes the set $C' \setminus \{\beta\}$. Hence $2(y - u) \leq |C'| - 2$, leading to

$$y \leq \frac{n - |U \setminus (C' \setminus \{\beta\})| - 1}{2}.$$

Consequently, $\{1, \ldots, n\} \setminus \mathrm{Fix}(G_{>y}) \subseteq U \setminus (C' \setminus \{\beta\})$, contradicting as in the previous case the minimality of $v - u$.

By the previous two cases, the minimal (respectively maximal) label of a component that does not have a 2-fracture graph is neither $u + 1$ nor $u + 2$ (respectively, neither $v - 1$ nor $v - 2$).

Since $G_{>u}$ fixes $\{1,\dots,n\}\setminus U$, there exists a component $\mathcal{G}^{(s_1)}$ that has a pendent edge with label $u+1$. Hence this component must have a 2-fracture graph; then, by Proposition 3.1,

$$|F_{s_1}| \le \frac{|X_{s_1}|-1}{2}.$$

Similarly, there is another component $\mathcal{G}^{(s_2)}$ that has a pendent edge with label $v-1$, and therefore

$$|F_{s_2}| \le \frac{|X_{s_2}|-1}{2}.$$

Consequently,

$$v-u=\sum_{s=1}^{c}|F_s| \le \frac{|U\cap V|-2}{2}.$$

Therefore, from this inequality and from (b) and (c), we have

$$r \le \frac{2n-(|U|+|V|-|U\cap V|)-2}{2}=\frac{n-2}{2}. \qquad \square$$

Proposition 5.15. *Assume Hypotheses 5.1, 5.6 and 5.11. If $\{c,d\}$ is not perfect, then $r \le n/2$ and $n \ge 6$.*

Proof. The existence of two non-perfect splits implies $n \ge 6$.

We proceed by contradiction and assume that $r > n/2$. Recall from Hypothesis 5.11 that ρ_g is the permutation with the minimal label acting nontrivially on both G_j-orbits. By Lemmas 5.12 and 5.13, we have

$$r-g-1 \le \frac{n-|Y|-1}{2},$$

where $Y=\{1,\dots,n\}\setminus \mathrm{Fix}(G_{<g})$, $Y \subseteq O_c$ and $h<g$.

If $G_{\{h+1,\dots,g-1\}}$ acts intransitively on $X\cap Y$, then by Proposition 5.14, we obtain $r \le (n-2)/2$, contradicting the assumption $r > n/2$. Therefore, we see that $G_{\{h+1,\dots,g-1\}}$ must act transitively on $X\cap Y$.

Since i and j are consecutive splits, the group $\Gamma_{\{h+1,\dots,g-1\}}$ has a fracture graph. Then, by Proposition 3.1, either $g-h-1 < (|X\cap Y|-1)/2$ or the permutation representation graph of $\Gamma_{\{h+1,\dots,g-1\}}$ corresponds to one of the graphs in Figure 1. This graph must contain two pendent edges labeled with the minimal and maximal values, namely $h+1$ and $g-1$, respectively. The only graph in Figure 1 satisfying this condition is graph (4). Hence

$$g-h-1 \le \frac{|X\cap Y|-1}{2}.$$

Furthermore, since $h \le (n-|X|-1)/2$ (from Proposition 5.7) and

$$n=|X|+|Y|-|X\cap Y|,$$

we deduce that $r \leq (n+1)/2$. Hence $r = (n+1)/2$. This implies

$$h = \frac{n - |X| - 1}{2}, \quad r - g - 1 = \frac{n - |Y| - 1}{2}, \quad g - h - 1 = \frac{|X \cap Y| - 1}{2}.$$

However, the permutation representation graph of Γ can be constructed as follows:

- connect a pendent h-edge from the graph in Figure 2 with the pendent $(h+1)$-edge of graph (4) in Figure 1, whose labels are $h+1, \dots, g-1$ and
- connect the pendent $(g-1)$-edge of graph (4) to a pendent edge from one of the graphs in Figure 3.

This construction leads to a contradiction with the evenness of G. Indeed, only graphs (B) and (C) permit the addition of i-edges distinct from a, b, and only graphs (B′) and (C′) permit the addition of j-edges distinct from c, d. Nevertheless, it is impossible for both ρ_i and ρ_{i+1} to be even, as well as for both ρ_j and ρ_{j+1} to be even, a contradiction. □

Corollary 5.16. *Assume Hypotheses 5.1, 5.6 and 5.11. If none of the splits is perfect, then either* $r \leq (n-1)/2$, *or* $r \leq n/2$ *and* $n \geq 6$. *In both cases, we have* $r \leq \lfloor 3(n-1)/5 \rfloor$.

5.5 The split $\{c, d\}$ is perfect

We proceed by induction on the number p of perfect splits of Γ. The base case (that is, $p = 0$) is given by Corollary 5.16.

Corollary 5.17. *Assume Hypotheses 5.1, 5.6 and 5.11. If at least one of the splits is perfect, then* $r \leq \lfloor 3(n-1)/5 \rfloor$.

Proof. From Proposition 5.15, we may assume that $\{c, d\}$ is a perfect split. Thus the number of perfect splits of $\Gamma_{>j}$ is smaller than p. Hence, by induction,

$$r - j - 1 \leq \frac{3n_D - 3}{5}.$$

By construction, $\Gamma_{\{h+1,\dots,j-1\}}$ has a 2-fracture graph, which must have two pendent edges with labels $h+1$ and $j-1$. Hence, by Proposition 5.10,

$$j \leq \frac{n_C - 1}{2}.$$

Therefore,

$$r \leq \frac{6n_D + 5n_C - 1}{10}.$$

Now, note that $r \leq 3(n-1)/5$ whenever $n \geq n_D + 5$, which always holds. Indeed, if $n_C \leq 4$, then ρ_i must be an odd permutation, a contradiction. □

6 The group G_i is transitive for some $i \in \{0, \dots, r-1\}$

In the previous sections, we proved that $r \le 3(n-1)/5$ if Γ has a fracture graph. We now turn to the remaining case, where Γ does not admit a fracture graph. This implies that there exists an index $i \in \{0, \dots, r-1\}$ such that the group G_i is transitive.

We first deal with the case that G_i is primitive. Here, we actually prove something slightly stronger.

Proposition 6.1. *Let G be a finite primitive group of degree n and let r be the maximum rank of an SGGI for G. Then either $r < \lfloor 3(n-1)/5 \rfloor - 1$ or one of the following holds:*

(a) *G contains the alternating group $\mathrm{Alt}(n)$;*

(b) *$n = 5$, $G = D_5$ and $r = 2$;*

(c) *$n = 6$, $G = \mathrm{PSL}_2(5)$ and $r = 3$, or $G = \mathrm{PGL}_2(5)$ and $r = 4$;*

(d) *$n = 7$, $G = D_7$ and $r = 2$;*

(e) *$n = 8$, $G = \mathrm{PGL}_2(7)$ and $r = 3$;*

(f) *$n = 9$, $G = \mathrm{Sym}(3) \mathrm{\ wr\ } \mathrm{Sym}(2)$ or $G = \mathrm{PSL}_2(8)$, and $r = 3$;*

(g) *$n = 10$, $G = \mathrm{Sym}(5)$ and $r = 4$, or $G = \mathrm{P\Sigma L}_2(9)$ and $r = 5$.*

Proof. Our proof is based on the following result of Maróti; see [7]. Let G be a primitive permutation group of degree n. Then one of the following holds:

(a) G is a subgroup of $\mathrm{Sym}(m) \mathrm{\ wr\ } \mathrm{Sym}(t)$ containing the socle $(\mathrm{Alt}(m))^t$, where the action of $\mathrm{Sym}(m)$ is on k-element subsets of $\{1, \dots, m\}$ and the wreath product has the product action of degree $n = \binom{m}{k}^t$;

(b) $G = M_{11}$, M_{12}, M_{23}, or M_{24} with their 4-transitive action;

(c) $|G| \le n \cdot \prod_{i=0}^{\lfloor \log_2(n) \rfloor - 1} (n - 2^i)$.

We deal with each of these cases in turn. Assume that part (b) holds. We have verified, with the auxiliary help of a computer, that M_{11} and M_{23} do not admit SGGIs, that the maximal rank of an SGGI for M_{12} is $4 < 5 = \lfloor 3(12-1)/5 \rfloor - 1$ and that the maximal rank of an SGGI for M_{24} is $5 < 12 = \lfloor 3(24-1)/5 \rfloor - 1$.

Assume that part (c) holds. Since G admits an independent generating set of cardinality r, we deduce that

$$r \le \log_2 |G| \le \log_2 n + \sum_{i=0}^{\lfloor \log_2 n \rfloor - 1} \log_2(n - 2^i).$$

An elementary computation shows that the right-hand side of this inequality is less than $\lfloor 3(n-1)/5 \rfloor - 1$, except when $n \leq 72$. The rest of the proof is computational: for $n \leq 72$, we have selected all the primitive groups G of degree n not containing $\mathrm{Alt}(n)$ and with $\log_2(|G|) \geq \lfloor 3(n-1)/5 \rfloor - 1$, and we have computed the maximal rank of an SGGI for G. The only examples where the rank is at least $\lfloor 3(n-1)/5 \rfloor - 1$ are reported in the statement of the result.

Finally, assume that part (a) holds. As above,

$$\begin{aligned} r &\leq \log_2|G| \leq \log_2(m!^t t!) = \log_2(m!)t + \log_2(t!) \\ &\leq \log_2(m)mt + \log_2(t)t = (\log_2(m)m + \log_2 t)t. \end{aligned}$$

If $r \geq \lfloor 3(n-1)/5 \rfloor - 1$, then

$$(\log_2(m)m + \log_2 t)t \geq \left\lfloor \frac{3\left(\binom{m}{k}^t - 1\right)}{5} \right\rfloor - 1 \geq \left\lfloor \frac{3(m^t - 1)}{5} \right\rfloor - 1.$$

A computation shows that either $t = 1$, or $t = 2$ with $m \leq 12$. Assume first that $t = 2$ and $m \leq 12$. By implementing the exact value of $n = \binom{m}{k}^2$ and refining the upper bound on $|G|$ using

$$|\mathrm{Sym}(m) \text{ wr } \mathrm{Sym}(2)| = m!^2\, 2,$$

we deduce that $r \geq \lfloor 3(n-1)/5 \rfloor - 1$ only when $k = 1$ and $m \in \{5, 6\}$. These two cases can be resolved computationally to determine the exact value of r, and no exceptions arise in this case. Now, assume $t = 1$. Then $\mathrm{Alt}(m) \leq G \leq \mathrm{Sym}(m)$ and $n = \binom{m}{k}$. Moreover, from [10], we have $r \leq m - 1$. The case $t = k = 1$ is the main exception listed in (a); therefore, we may suppose $k \geq 2$. Recall that the binomial coefficient $\binom{m}{k}$ increases with k when $1 \leq k \leq m/2$. The inequality

$$m - 1 \geq \left\lfloor \frac{3\left(\binom{m}{2} - 1\right)}{5} \right\rfloor - 1$$

holds only when $m = 5$. When $G = \mathrm{Sym}(5)$, we obtain one of the exceptions listed in (g). When $G = \mathrm{Alt}(5)$, the rank is $3 < 4 = \lfloor 3(10-1)/5 \rfloor - 1$. □

Proposition 6.2. *Let* $G = \mathrm{Alt}(n)$ *be the alternating group of degree* n *and let* $\Gamma = (G, S)$ *be an SGGI of rank* r. *If* G_i *is primitive for some* $i \in \{0, \ldots, r-1\}$, *then either* $r \leq \lfloor 3(n-1)/5 \rfloor$, *or* $n = 5$ *and* $r = 3$.

Proof. Suppose $r > \lfloor 3(n-1)/5 \rfloor$. As G_i is a proper subgroup of $G = \mathrm{Alt}(n)$, by Proposition 6.1, we deduce that G_i is one of the groups listed in parts (b)–(g). Now, we have verified the correctness of the result using a computer. □

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let $\Gamma = (G, S)$ be an SGGI for G of rank r. If G is primitive, then the result follows from Proposition 6.1. Therefore, for the rest of the proof, we may suppose that G is imprimitive.

As G is imprimitive, it admits a nontrivial system of imprimitivity. We choose a nontrivial system of imprimitivity such that the action of G on the blocks is primitive. Let m be the number of blocks and let k be the cardinality of a block. In particular, G is a transitive subgroup of the wreath product $\mathrm{Sym}(k)$ wr $\mathrm{Sym}(m)$ endowed with its imprimitive action of degree $km = n$. Let $\pi\colon G \to \mathrm{Sym}(m)$ be the projection given by the action of G on the system of imprimitivity and let J be the image of π. Let K be the permutation group induced by the action of the stabilizer of a block on the block. Thus K is a transitive subgroup of $\mathrm{Sym}(k)$ and $G \le K \text{ wr } J$. We now follow closely the argument of Whiston in [10, Section 5], refining the argument by using the fact that S is an independent generating set consisting of involutions. The same idea can also be found in [5, page 471].

Let L be a subset of S forming an independent generating set for the permutation group J, that is, L^π is an independent generating set for J. Let C be the subset of $S \setminus L$ consisting of the elements commuting with L, and let R be $S \setminus (C \cup L)$. Clearly,

$$r = |L| + |C| + |R|. \tag{6.1}$$

Suppose first that J admits two distinct commuting proper normal subgroups A and B with $J = AB$. Since J is primitive, A and B are both transitive. Since A and B centralize each other, we deduce that A and B are both regular. Thus $J = A \times B$ and $m = |A|$. The primitivity of J implies that A and B are non-abelian simple groups. Since L is independent, we get $2^{|L|} \le |J| = m^2$ and hence $|L| \le 2\log_2(m)$, where $m \ge 60$. As the center of J is the identity, $\mathbf{C}_{K \mathrm{wr} J}(\langle J\rangle)$ is contained in the base group K^m of K wr J. Since $\langle L\rangle$ acts transitively on the blocks, $\mathbf{C}_{K^m}(\langle L\rangle)$ is a diagonal subgroup of the base group K^m of K wr J, that is, there exist group automorphisms $\varphi_2, \dots, \varphi_m\colon K \to K$ such that $\mathbf{C}_{K^m}(\langle L\rangle)$ is a subgroup of $\{(k, k^{\varphi_2}, \dots, k^{\varphi_m}) \mid k \in K\}$. As $C \subseteq \mathbf{C}_{K^m}(\langle L\rangle)$, the main result of Whiston [10, Theorem 1] implies

$$|C| \le k - 1. \tag{6.2}$$

Finally, from the structure of J we have described above, R has cardinality at most 4. From (6.1), we get

$$r \le 2\log_2 m + k - 1 + 4 = 2\log_2 m + k + 3.$$

Using $m \ge 60$, it follows from a computation that

$$2\log_2 m + k + 3 < \left\lfloor \frac{3(n-1)}{5} \right\rfloor.$$

For the rest of the argument, we may suppose that J does not admit a decomposition $J = AB$, with A and B as above. In particular, this implies that the set of labels of the elements in L forms an interval, say $L = \{\rho_a, \rho_{a+1}, \dots, \rho_{a+\ell-1}\}$, where $\ell = |L|$ and $(\rho_{a+i}\rho_{a+i+1})^2 \neq 1$ for all $i \in \{0, \dots, \ell - 2\}$. From this, it also follows that

$$|R| \leq 2. \tag{6.3}$$

Applying [10, Theorem 1] to J, we get

$$|L| \leq m - 1, \tag{6.4}$$

where the equality can be attained if and only if $J = \mathrm{Sym}(m)$. Arguing exactly as above with $\mathbf{C}_{K \mathrm{wr} J}(\langle L \rangle)$, we deduce that (6.2) holds true also in this case; moreover, if the equality is attained, then $K = \mathrm{Sym}(k)$ and $\langle C \rangle$ projects surjectively to K.

From (6.1), (6.2), (6.3) and (6.4), we obtain

$$r \leq m + k. \tag{6.5}$$

The inequality $m + k < \lfloor 3(n - 1)/5 \rfloor$ holds, except in the following cases:

- $m = 2$ and $2 \leq k \leq 17$, or $2 \leq m \leq 17$ and $k = 2$, or
- $m = 3$ and $3 \leq k \leq 5$, or $3 \leq m \leq 5$ and $k = 3$.

With the aid of a computer, we have computed the maximum rank of an SGGI for G for every $n \leq 15$. The cases where $r \geq \lfloor 3(n - 1)/5 \rfloor$ are reported in Table 1. In particular, for the remainder of the argument, we may assume that $m = 2$ and $9 \leq k \leq 17$, or $9 \leq m \leq 17$ and $k = 2$.

Case $k = 2$. Here, G is a subgroup of $\mathrm{Sym}(2)^m \rtimes \mathrm{Sym}(m)$. Again, using a computer computation, we have verified that, except for the symmetric group $\mathrm{Sym}(m)$, the maximum rank of an SGGI for a primitive group of degree m is at most $m - 3$. In particular, by refining (6.5), we deduce $r \leq m = n/2 < \lfloor 3(n - 1)/5 \rfloor$, where the last inequality follows from the fact that $m \geq 9$. Therefore, we may assume that $J = \mathrm{Sym}(m)$.

Let $V = \mathrm{Sym}(2)^m$ and regard it as a module for J over the field with two elements. The space V has two natural J-submodules: the 1-dimensional module V_1, consisting of the constant vectors, and the $(m - 1)$-dimensional module V_2, consisting of the vectors whose entries sum to 0. Clearly, $V = V_1 \times V_2$ when m is odd, and $0 < V_1 < V_2 < V$ when m is even. Mortimer [9] shows that V_1 and V_2 are the unique nontrivial J-submodules of V. After this digression, we are ready to conclude the proof of the case $k = 2$.

Recall that $L = \{\rho_a, \rho_{a+1}, \ldots, \rho_{a+\ell-1}\}$ with $\ell = m-1$. Let $\rho \in R \cup C$. As $\langle L \rangle$ projects to J, there exists some $\rho' \in \langle L \rangle$ such that $f = \rho\rho' \in V$. The independence condition implies that $f \neq 1$. When m is even, since the lattice of J-submodules of V forms a chain, the irredundant generating sets of G must contain only one further element besides those in L. Hence $|R \cup C| = 1$ and $r \leq m = n/2 < \lfloor 3(n-1)/5 \rfloor$. The same argument applies when m is odd (leading to the same conclusion), except in the case $G = V \rtimes \mathrm{Sym}(m)$, where an irredundant generating set of G containing L could possibly include two additional elements. Thus, in this case, we obtain $r \leq m+1 = n/2+1$. For odd m, the bound $m+1 < \lfloor 3(n-1)/5 \rfloor$ holds, except when $m \in \{9, 11\}$. Indeed, for these two cases, we have verified by computer that $r = m+1$, and these two examples are recorded in Table 1.

Case $m = 2$. Here $\ell = 1$, $L = \{\rho_a\}$ and $G \leq \mathrm{Sym}(k) \,\mathrm{wr}\, \mathrm{Sym}(2)$. We write the elements of $\mathrm{Sym}(k) \,\mathrm{wr}\, \mathrm{Sym}(2)$ as $(x, y)(1\,2)^{\varepsilon}$, where $x, y \in \mathrm{Sym}(k)$, $\varepsilon \in \{0, 1\}$ and $(1\,2)$ is the generator of J. Replacing ρ_a and G by a suitable conjugate via an element of $\mathrm{Sym}(k) \,\mathrm{wr}\, \mathrm{Sym}(2)$, we may suppose that $\rho_a = (1\,2)$. The set C contains

$$\{\rho_0, \ldots, \rho_{a-2}, \rho_{a+2}, \ldots, \rho_{r-1}\}.$$

For each $j \in \{0, \ldots, a-2, a+2, \ldots, r-1\}$, we may write $\rho_j = (x_j, x_j)(1\,2)^{\varepsilon_j}$, where $x_j \in \mathrm{Sym}(k)$ and $\varepsilon_j \in \{0, 1\}$. Set $V = \langle x_j \mid j \in \{0, \ldots, r-1\} \setminus \{a\} \rangle$.

If $R = \emptyset$, then by refining (6.5), we deduce that $r \leq k = n/2 < \lfloor 3(n-1)/5 \rfloor$. Therefore, for the rest of the argument, we may suppose $R \neq \emptyset$.

Assume V transitive and $V \neq \mathrm{Sym}(k)$. We have determined with Magma the maximum rank of every transitive proper subgroup of $\mathrm{Sym}(k)$ and we have verified that it is at most $k-3$. In particular, if V is transitive, then by refining (6.5), we deduce that $r \leq k = n/2 < \lfloor 3(n-1)/5 \rfloor$.

Assume $V = \mathrm{Sym}(k)$. Suppose $|R| = 2$. In this case, up to duality, we may suppose $L = \{\rho_{r-1}\}$, $R = \{\rho_0, \rho_{r-2}\}$ and $C = \{\rho_1, \ldots, \rho_{r-3}\}$. Now,

$$\rho_0 \in \mathbf{C}_G(\rho_{r-1}) = \langle \rho_{r-1}, \mathrm{Diag}(\mathrm{Sym}(k) \times \mathrm{Sym}(k)) \rangle = \langle L \cup C \rangle,$$

contradicting the fact that S is an independent generating set. Thus $|R| = 1$. Again, up to duality, we may suppose $L = \{\rho_{r-1}\}$, $R = \{\rho_{r-2}\}$ and $C = \{\rho_0, \ldots, \rho_{r-3}\}$. If $|C| = k-2$, then $r = k = n/2 < \lfloor 3(n-1)/5 \rfloor$. Suppose then $|C| = k-1$ and hence $r = k+1$. Thus [3, Theorem 2.1] implies that $x_0, \ldots, x_{r-3}$ are Coxeter generators for $\mathrm{Sym}(k)$. Therefore, without loss of generality, we may suppose that $x_i = (i+1\; i+2)$ for every $i \in \{0, \ldots, r-3\}$. In particular, we have $\langle x_0, \ldots, x_{r-4} \rangle \cong \mathrm{Sym}(k-1)$. Using this explicit description for x_i and hence for ρ_i, we deduce that $\mathbf{C}_G(\langle \rho_0, \ldots, \rho_{r-4} \rangle) = \langle \rho_{r-1} \rangle$. As ρ_{r-2} commutes with $\rho_0, \ldots, \rho_{r-4}$, we get $\rho_{r-2} \in \langle \rho_{r-1} \rangle$, which is clearly a contradiction.

Assume V intransitive. If V has at least three orbits or V does not induce the whole symmetric group in its action on one of its orbits, then [10, Proposition 1] implies that $|C| \leq k-3$ and hence we reach the same conclusion as in the previous paragraphs. Therefore, V has two orbits and V induces the whole symmetric group on both of its orbits. In particular, V is a subdirect subgroup of $\mathrm{Sym}(\kappa) \times \mathrm{Sym}(k-\kappa)$ for some $1 \leq \kappa \leq k/2$. Suppose

$$V \neq \mathrm{Sym}(\kappa) \times \mathrm{Sym}(k-\kappa).$$

With the aid of a computer, we have determined the maximum rank of an SGGI for every proper subdirect subgroup of $\mathrm{Sym}(\kappa) \times \mathrm{Sym}(k-\kappa)$; in all cases, the rank is at most $k-3$. Therefore, in this case, we can argue as above. Thus

$$V = \mathrm{Sym}(\kappa) \times \mathrm{Sym}(k-\kappa).$$

In particular, $|C| \leq k-2$ and hence

$$r = |C| + |L| + |R| \leq k+1.$$

Now, $k+1 < \lfloor 3(n-1)/5 \rfloor$ for $k \in \{14, 15, 16, 17\}$. Therefore, we may suppose $k \in \{9, 10, 11, 12, 13\}$ and $|C| = k-2$. Observe that there exists some $\rho \in C$ such that $R \cup L \subseteq \mathbf{C}_G(\rho)$ and hence $G = \langle C, \mathbf{C}_G(\rho) \rangle$. The proof here again relies on the use of a computer. For every $1 \leq \kappa \leq k/2$, we determined all the SGGI $\bar{\Gamma} = (\bar{G}, \bar{C})$ where

$$\bar{G} = \mathrm{Diag}\big((\mathrm{Sym}(\kappa) \times \mathrm{Sym}(k-\kappa)) \times (\mathrm{Sym}(\kappa) \times \mathrm{Sym}(k-\kappa))\big)$$

and $|\bar{C}| = k-2$. We then selected only the cases where there exists some $\rho \in \bar{C}$ such that $\langle \bar{C}, \mathbf{C}_{\mathrm{Sym}(k)\mathrm{wr}\mathrm{Sym}(2)}(\rho) \rangle$ is transitive. In all such cases, $\kappa = 1$. This implies $V = \mathrm{Sym}(k-1)$ and that the labels of the elements in C form an interval. Therefore, up to duality, we may suppose that

$$C = \{\rho_0, \ldots, \rho_{k-3}\}, \quad R = \{\rho_{k-2}, \rho_k\}, \quad L = \{\rho_{k-1}\}.$$

Since $|C| = k-2$ and $V = \mathrm{Sym}(k-1)$, [3, Theorem 2.1] applied to V implies that $x_0, \ldots, x_{k-3}$ are Coxeter generators for $\mathrm{Sym}(k-1)$. Therefore, without loss of generality, we may suppose that $x_i = (i+1\, i+2)$ for every $i \in \{0, \ldots, k-3\}$. In particular, $\langle x_0, \ldots, x_{k-4} \rangle \cong \mathrm{Sym}(k-2)$. Using this explicit description for x_i and hence for ρ_i, we deduce that

$$\begin{aligned}\langle \rho_{r-2}, \rho_{r-1}, \rho_r \rangle &\leq \mathbf{C}_G(\langle \rho_0, \ldots, \rho_{r-4} \rangle) \\ &\leq \mathbf{C}_{\mathrm{Sym}(k)\mathrm{wr}\mathrm{Sym}(2)}\big(\mathrm{Diag}(\mathrm{Sym}(k-2) \times \mathrm{Sym}(k-2))\big) \cong D_4.\end{aligned}$$

However, this is a contradiction, because the dihedral group of order 4 does not contain any subgroup generated by 3 independent involutions. □

Corollary 6.3. *Let* $G = \mathrm{Alt}(n)$ *be the alternating group of degree* n *and define* $\Gamma = (G, S)$ *to be an SGGI of rank* r. *If* G_i *is transitive for some* $i \in \{0, \dots, r-1\}$, *then* $r \leq \lfloor 3(n-1)/5 \rfloor$, *or* $n = 5$ *and* $r = 3$.

Proof. When G_i is not one of the groups in Table 1, the result follows from Theorem 1.2. When G_i is one of the groups in Table 1 with $G_i \leq \mathrm{Alt}(n)$, we have in particular $n \leq 14$. The result for $n \leq 14$ has been verified with the auxiliary help of a computer. □

7 Proof of Theorem 1.1

We are now ready to bring together the threads of our argument.

Proof of Theorem 1.1. Let $G = \mathrm{Alt}(n)$ and let $\Gamma = (G, S)$ be an SGGI for G of maximum rank. When $n \leq 9$, the proof follows with a computation on small alternating groups. Therefore, for the rest of the argument, we may suppose $n \geq 10$.

Suppose first that Γ has a 2-fracture graph. Then, by Proposition 3.1, we deduce $r \leq n/2$. Using $n \geq 10$, it follows that $n/2 \leq \lfloor 3(n-1)/5 \rfloor$. Suppose next that Γ satisfies Hypothesis 4.1 and hence we may apply the result from Section 4. In this case, the result follows from Proposition 4.8. Suppose next that Γ satisfies Hypothesis 5.1. In this case, the result follows from Corollaries 5.16 and 5.17. Finally, when G_i is transitive for some i, the result follows from Corollary 6.3. □

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Received May 22, 2025; revised November 14, 2025

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