Optimal improvement of communication network congestion via nonlinear programming with generalized Nash equilibrium constraints

Mauro Passacantando and Fabio Raciti

Abstract We consider a popular model of congestion control in communication networks within the theory of generalized Nash equilibrium problems with shared constraints, where each player is a user who has to send his/her flow over a path in the network. The cost function of each player consists of two parts: a pricing and a utility term. Within this framework we assume that the network system manager can invest a given amount of money to improve the network by enhancing the capacity of its links and, because of limited financial resources, has to make a choice as to which of the links to improve. This choice is made with the help of a performance function which is computed for each set of improvements under consideration and has the property that, once the equilibrium has been reached, maximizes the aggregate utility and minimizes the sum of delays at the links. We model this problem as a nonlinear knapsack problem with generalized Nash equilibrium constraints and show some preliminary numerical experiments.

Key words: Generalized Nash equilibrium; congestion control; investment optimization.

1 Introduction

Routing and congestion control problems have been two crucial aspects in the use of the Internet from its beginning and have gained even more importance in the recent years due to the huge increase of flows to be processed in this *big data* era.

Mauro Passacantando (corresponding author)
Department of Computer Science, University of Pisa, Largo B. Pontecorvo 3, 56127 Pisa, Italy, e-mail: mauro.passacantando@unipi.it

Fabio Raciti
Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125 Catania, Italy, e-mail: fraciti@dmi.unict.it

In this respect, the use of game theory has proved to be a useful tool and a large number of papers have been devoted to model the above mentioned problems within the *cadre* of Nash equilibrium problems (see, e.g., [1, 2, 11, 13]). In this note, we focus on the congestion control framework put forward in [1], where each network user is considered as a player endowed with a cost function which is the difference of a pricing and a utility term. The pricing term has the role of congestion control, while the utility term expresses the user satisfaction. The bandwidth is the main resource of the system and players compete to send their flow from a given origin to a certain destination node. Because users share some network links, the strategy space of each player also depends on the variables of all the other players. Nash games of this kind have been introduced a long time ago by Rosen in his influential paper [12] and have been reformulated more recently by using the powerful tools of variational inequalities (see, e.g., [3, 4, 8, 10]). In the recent literature, they are termed as generalized Nash equilibrium problems (GNEPs) with shared constraints.

In this note, we adopt the model in [1] with some modifications in the pricing part of the players' cost functions, which give rise to multiple solutions of the game, but to only one *variational equilibrium*, which is considered a particularly recommended kind of equilibrium from the socio-economic standpoint (see, e.g., [3]). We then consider the possibility that the system manager can make an investment in order to improve the network performance by enhancing the capacity of the links. However, because of budget constraints not all the capacities can be enhanced and he/she has to make a decision as to which links is better to improve. The decision process is made according to its impact on a network cost function associated to each set of improvements, which has the role of maximizing the aggregate utility while minimizing the total delay at the links. The computation of the network cost function requires the knowledge of the variational Nash equilibrium in each case. Once the set of variational equilibria is known, for all the scenarios under consideration, we are then faced with a knapsack-type problem which, for instances of reasonable dimensions, can be solved by classifying all the solutions according to their corresponding relative variation of the above mentioned function.

The paper is structured as follows. In the following Sect. 2 we summarize the congestion control model proposed in [1], along with our modifications, and briefly recall some results about generalized Nash equilibrium problems with shared constraints and the variational inequality approach. In Sect. 3 we present our investment optimization model, while Sect. 4 is devoted to some illustrative numerical experiments. We conclude the paper with a small section where we touch on some possible extensions.

2 The congestion control model and its variational inequality formulation

Throughout the paper, vectors of $\mathbb{R}^n$ are thought of as rows but in matrix operations they will be considered as columns and the superscript ^{T} will denote transposition. We now describe the network topology which consists of a set of links $\mathcal{L} = \{l_1, \dots, l_L\}$ connecting the nodes in the set $\mathcal{N} = \{n_1, \dots, n_N\}$. The set of network users (players) is denoted by $\{1, \dots, M\}$. A route R in the network is a set of consecutive links and each user i wishes to send a flow x_i between a given pair $O_i - D_i$ of origin-destination nodes; $x \in \mathbb{R}^M$ is the (route) flow of the network and the useful notation $x = (x_i, x_{-i})$ will be used in the sequel when it is important to distinguish the flow component of player i from all the others. We assume that the routing problem has already been solved and that there is only one route R_i assigned to user i. Each link l has a fixed capacity $C_l > 0$, so that user i cannot send a flow greater than the minimum capacity of the links of his/her route, and we group these capacities into a vector $C \in \mathbb{R}^L$. To describe the link structure of each route, it is useful to introduce the link-route matrix whose entries are given by:

$$A_{li} = \begin{cases} 1, & \text{if link } l \text{ belongs to route } R_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Using the link-route matrix, the set of feasible flows can be written in compact form as

$$X := \left\{ x \in \mathbb{R}^M \, : \; x \geq 0, \;\; Ax \leq C \right\}.$$

Because users share some links, the possible amount of flow x_i depends on the flows sent by the other users and is bounded from above by the quantity

$$m_i(x_{-i}) = \min_{l \in R_i} \left\{ C_l - \sum_{j=1,\; j \neq i}^{M} A_{lj} x_j \right\} \geq 0.$$

In this model the cost function of each player i has the following structure:

$$J_i(x) = P_i(x) - U_i(x_i), \tag{2}$$

where U_i represents the utility function of player i which only depends on the flow that he/she sends through the network, while P_i is a pricing term which represents some kind of toll that user i pays to exploit the network resources and depends on the flows of the players with common links to i. Players compete in a non-cooperative manner, as it is assumed that they do not communicate, and act selfishly to increase their flow. With these assumptions, the solution concept adopted is the Nash equilibrium *à la Rosen* [12], which in the modern literature is known as generalized Nash equilibrium (with shared constraints). More precisely, we say that $x^* \in \mathbb{R}^M$ is a generalized Nash equilibrium if for each $i \in \{1, \dots, M\}$:

$$J_i(x_i^*, x_{-i}^*) = \min_{0 \leq x_i \leq m_i\left(x_{-i}^*\right)} J_i(x_i, x_{-i}^*). \tag{3}$$

It is well known (see, e.g., [5]) that, under standard differentiability and convexity assumptions, the above problem is equivalent to a quasivariational inequality and that a particular subset of solutions (called variational equilibria) can be found by solving the variational inequality $VI(F, X)$, where X is the feasible set previously defined and F is the so-called *pseudogradient* of the game, defined by:

$$F(x) = \left(\nabla_{x_1} J_1(x), \dots, \nabla_{x_M} J_M(x)\right). \tag{4}$$

In this note we do not posit assumptions on general functions U_i and P_i, but instead consider the specific functional form treated in [1], with a slight modification, and show the existence of a unique variational equilibrium of the game. Due to our modification, it is possible that some of the capacities are saturated at equilibrium and the case that some users have zero flow at equilibrium is not ruled out. Moreover, we provide examples where also non variational equilibria are possible. In these regards our results are in contrast with the ones in [1].

Specifically, the utility function U_i of player i is given by:

$$U_i(x_i) = u_i \log(x_i + 1), \tag{5}$$

where u_i is a parameter, while the route price function P_i of player i is the sum of the price functions of the links associated to route R_i:

$$P_i(x) = \sum_{l \in R_i} P_l \left(\sum_{j=1}^{M} A_{lj} x_j \right). \tag{6}$$

Let us notice that P_l is modeled so as to only depend on the variables of players who share the link l, namely:

$$P_l \left(\sum_{j=1}^{M} A_{lj} x_j \right) = \frac{k}{C_l - \sum_{j=1}^{M} A_{lj} x_j + e}, \tag{7}$$

where $k > 0$ is a network parameter, and e is a small positive number which we introduce to allow capacity saturation, while obtaining a well behaved function. The price function of player i is thus given by:

$$P_i(x) = \sum_{l \in R_i} \frac{k}{C_l - \sum_{j=1}^{M} A_{lj} x_j + e}, \tag{8}$$

and the resulting expression of the cost for player i is:

$$J_i(x) = \sum_{l \in R_i} \frac{k}{C_l - \sum_{j=1}^{M} A_{lj} x_j + e} - u_i \log(x_i + 1). \tag{9}$$

The following properties of the above functions are easy to check:
(i) U_i is twice continuously differentiable, non-decreasing and strongly concave on any compact interval $[0, b]$ (the last condition means that there exists $\tau > 0$ such that $\partial^2 U_i(x_i)/\partial x_i^2 \le -\tau$ for any $x_i \in [0, b]$);
(ii) P_i is twice continuously differentiable, convex and $P_i(\cdot, x_{-i})$ is non-decreasing.

These properties of U_i and P_i entail an important monotonicity property of the pseudogradient F defined in (4), as the following theorem shows.

Theorem 1 *Let U_i and P_i be given as in* (5) *and* (8)*, then F is strongly monotone on X, i.e., there exists $\alpha > 0$ such that*

$$(F(x) - F(y))^\top (x - y) \geq \alpha \|x - y\|^2, \qquad \forall\, x, y \in X.$$

Proof Similarly to [1], it can be shown that the Jacobian matrix of F is positive definite on X, uniformly with respect to x, thus F is strongly monotone on X. □

The unique solvability of $VI(F, X)$ is based on standard arguments, as the following theorem shows.

Theorem 2 *There exists a unique variational equilibrium of the GNEP.*

Proof The variational equilibria of the GNEP are the solutions of $VI(F, X)$. Existence of solutions of $VI(F, X)$ follows from the continuity of F and the compactness and convexity of X. The solution is unique because F is strongly monotone on X. □

We now introduce a function f which describes a global property of the game:

$$f(x) = \sum_{l \in \mathcal{L}} P_l \left(\sum_{j=1}^{M} A_{lj} x_j \right) - \sum_{i=1}^{M} U_i(x_i), \tag{10}$$

which represents the aggregate delay at the links minus the sum of the utilities of all players. The function f turns out to be the potential of the GNEP, as the following theorem shows.

Theorem 3 *The unique variational equilibrium of the GNEP coincides with the optimal solution of the system problem: $\min_{x \in X} f(x)$.*

Proof Since both f and X are convex, $\bar{x}$ is a minimizer of f on X if and only if

$$\nabla f(\bar{x})^\top (y - \bar{x}) \geq 0, \quad \forall y \in X.$$

Since $\nabla f = F$, the expression above is nothing else that the variational inequality $VI(F, X)$ which gives the variational equilibrium. □

3 The optimal network improvement model

We now suppose that the network system manager has a budget B available to improve the network performance. He/she can only increase the capacity of a subset $\widetilde{\mathcal{L}} \subseteq \mathcal{L}$ of links and knows that I_l is the investment required to enhance the capacity of link l by a given ratio γ_l. Since the available budget is generally not sufficient to enhance the capacities of all the links of $\widetilde{\mathcal{L}}$, he/she has to decide which subset of links to invest in, in order to improve as much as possible the system cost f computed at the variational equilibrium of the game with new link capacities, while satisfying the budget constraint. This problem can be formulated as an integer nonlinear program.

To this end, we define a binary variable y_l, for any $l \in \widetilde{\mathcal{L}}$, which takes on the value 1 if the investment is actually carried out on link l, and 0 otherwise. A vector $y = (y_l)_{l \in \widetilde{\mathcal{L}}}$ is feasible if the budget constraint $\sum_{l \in \widetilde{\mathcal{L}}} I_l y_l \leq B$ is satisfied. Given a feasible vector y, the new capacity of each link $l \in \widetilde{\mathcal{L}}$ is equal to

$$C'_l(y) := \gamma_l C_l y_l + (1 - y_l) C_l,$$

i.e., $C'_l(y) = \gamma_l C_l$ if $y_l = 1$ and $C'_l(y) = C_l$ if $y_l = 0$. The network manager aims to maximize the percentage relative variation of the system cost defined as

$$\varphi(y) = 100 \cdot \frac{f(\bar{x}(0)) - f(\bar{x}(y))}{|f(\bar{x}(0))|},$$

where f is defined in (10), $\bar{x}(0)$ is the variational equilibrium of the GNEP before the investment, while $\bar{x}(y)$ is the variational equilibrium of the GNEP on the improved network according to y. Therefore, the proposed optimization model is

$$\begin{aligned} \max \quad & \varphi(y) \\ \text{subject to} \quad & \sum_{l \in \widetilde{\mathcal{L}}} I_l y_l \leq B, \\ & y_l \in \{0,1\} \quad l \in \widetilde{\mathcal{L}}. \end{aligned} \tag{11}$$

The above model can be considered a generalized knapsack problem since the computation of the nonlinear function φ at a given y requires to find the variational equilibria of the GNEPs both for the original and the improved network.

We remark that, since the variational equilibria of the GNEPs are the minimizers of f (see Theorem 3), problem (11) can be reformulated as the following mixed integer nonlinear program:

$$\begin{aligned} \min \quad & \sum_{l \in \widetilde{\mathcal{L}}} \frac{k}{\gamma_l C_l y_l + (1 - y_l) C_l - \sum_{i=1}^{M} A_{li} x_i + e} + \sum_{l \in \mathcal{L} \setminus \widetilde{\mathcal{L}}} \frac{k}{C_l - \sum_{i=1}^{M} A_{li} x_i + e} \\ & - \sum_{i=1}^{M} u_i \log(x_i + 1) \\ \text{subject to} \quad & \sum_{i=1}^{M} A_{li} x_i \leq \gamma_l C_l y_l + (1 - y_l) C_l \qquad \forall\, l \in \widetilde{\mathcal{L}}, \\ & \sum_{i=1}^{M} A_{li} x_i \leq C_l \qquad \forall\, l \in \mathcal{L} \setminus \widetilde{\mathcal{L}}, \\ & \sum_{l \in \widetilde{\mathcal{L}}} I_l y_l \leq B, \end{aligned}$$

$$x_i \geq 0, \qquad \forall\, i = 1, \ldots, M,$$
$$y_l \in \{0, 1\} \qquad \forall\, l \in \widetilde{\mathcal{L}}.$$

4 Numerical tests

This section is devoted to some preliminary numerical experiments on two test networks. The numerical computation of the solutions of the GNEPs was performed by using Matlab 2018a and its optimization toolbox.

Example 1. We consider the network shown in Fig. 1 (see also [1]) with nine nodes and nine links. The origin-destination pairs of the users and their routes are described in Table 1.

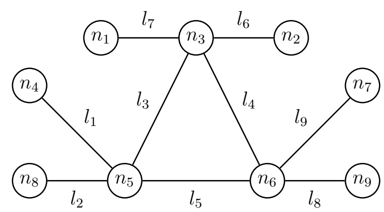

Fig. 1 Network topology of Example 1.

Table 1 Origin-Destination pairs and routes (sequence of links) of the users in Example 1.

User	Origin	Destination	Route
1	n_8	n_2	l_2, l_3, l_6
2	n_8	n_7	l_2, l_5, l_9
3	n_4	n_7	l_1, l_5, l_9
4	n_2	n_7	l_6, l_4, l_9
5	n_9	n_7	l_8, l_9

First, we show three instances of the considered GNEP where the variational equilibrium (i) belongs to the interior of the feasible region X; (ii) has some components equal to 0; (iii) saturates the capacity constraint of some links:

(i) If we set parameters $e = 0.01$, $k = 1$, $u_i = 10$ for any $i = 1, \ldots, 5$, and $C_l = 10$ for any $l \in \mathcal{L}$, then the variational equilibrium is $\bar{x} = (6.70,\ 2.02,\ 2.66,\ 2.03,\ 2.70)$ with the corresponding link flow equal to $(2.66,\ 8.72,\ 6.70,\ 2.03,\ 4.67,\ 8.73,\ 0,\ 2.70,\ 9.40)$, hence $\bar{x}$ belongs to the interior of X.

(ii) If we set $e = 0.01$, $k = 1$, $u_1 = 0.01$, $u_i = 10$ for any $i = 2, \ldots, 5$ and $C_l = 10$ for any $l \in \mathcal{L}$, then the variational equilibrium is $(0,\ 2.34,\ 2.34,\ 2.36,\ 2.38)$.

(iii) If we set $e = 0.1$, $k = 0.01$, $u_i = 100$ for any $i = 1, \ldots, 5$, and $C_l = 10$ for any $l \in \mathcal{L}$, then the variational equilibrium is $(7.8429,\ 2.1571,\ 2.8429,\ 2.1571,\ 2.8430)$ and the corresponding link flow is equal to $(2.84, 10, 7.84, 2.16, 5.00, 10, 0, 2.84, 10)$, where the capacity constraints of links l_2, l_6 and l_9 are saturated.

Moreover, we notice that there can be (infinitely) many generalized Nash equilibria unlike the unique variational one. For example, in the instance (iii) a non-variational equilibrium is (8.0944, 1.9056, 3.0944, 1.9056, 3.0944). It is a so-called normalized equilibrium [12], which has been computed by solving the variational inequality $VI(F', X)$, where $F'_i = w_i F_i$, for $i = 1, \dots, 5$, and the vector of weights $w = (1/3, 1/6, 1/6, 1/6, 1/6)$ (see [10]). Similarly, other (normalized) generalized Nash equilibria can be obtained by appropriately modifying the vector w.

We now show some numerical results for the proposed optimal network improvement model. We set $e = 0.01$, $k = 1$, $u_i = 10$ for any $i = 1, \dots, 5$, and $C_l = 10$ for any $l \in \mathcal{L}$. We assume that the available budget $B = 20$ k€, the set of links to be maintained is $\widetilde{\mathcal{L}} = \mathcal{L}$, while the values of γ_l and I_l are shown in Table 2.

Table 2 Capacity enhancement factors and investments for links of Example 1.

Links	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9
γ_l	1.2	1.5	1.1	1.6	1.3	1.4	1.1	1.7	1.3
I_l (k€)	3	8	2	10	4	5	2	12	4

Table 3 shows the ten best feasible solutions together with the percentage of total cost improvement $\varphi(y)$ and the corresponding investment $I(y) = \sum_{l \in \widetilde{\mathcal{L}}} I_l y_l$. It is interesting noting that the fifth to tenth solutions have very similar values but the tenth one needs a much lower investment than the others.

Table 3 The ten best feasible solutions for the optimal network improvement model in Example 1.

Ranking	y	$\varphi(y)$	$I(y)$
1	(0,1,1,0,0,1,0,0,1)	17.7381	19
2	(1,1,0,0,0,1,0,0,1)	16.8796	20
3	(0,1,0,0,0,1,1,0,1)	16.8426	19
4	(0,1,0,0,0,1,0,0,1)	16.8285	17
5	(0,1,1,0,1,0,1,0,1)	13.2289	20
6	(0,1,1,0,1,0,0,0,1)	13.2148	18
7	(1,0,1,0,1,1,1,0,1)	13.1952	20
8	(1,0,1,0,1,1,0,0,1)	13.1811	18
9	(0,0,1,0,1,1,1,0,1)	13.1396	17
10	(0,0,1,0,1,1,0,0,1)	13.1255	15

Example 2. We now consider the network shown in Fig. 2 with 10 nodes and 13 links. The O-D pairs of the users and their routes are described in Table 4.

We now show some numerical results for the proposed optimal network improvement model. We set $e = 0.01$, $k = 1$, $u_i = 10$ for any $i = 1, \dots, 10$, and $C_l = 10$ for

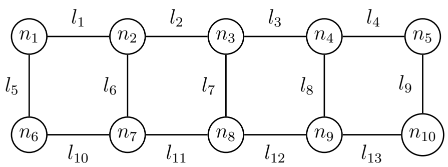

Fig. 2 Network topology of Example 2.

Table 4 Origin-Destination pairs and routes (sequence of links) of the users in Example 2.

User	Origin	Destination	Route	User	Origin	Destination	Route
1	n_1	n_5	l_1, l_2, l_3, l_4	6	n_5	n_1	l_4, l_3, l_2, l_1
2	n_6	n_{10}	$l_{10}, l_{11}, l_{12}, l_{13}$	7	n_{10}	n_6	$l_{13}, l_{12}, l_{11}, l_{10}$
3	n_2	n_{10}	$l_6, l_{11}, l_{12}, l_{13}$	8	n_5	n_8	l_9, l_{13}, l_{12}
4	n_8	n_5	l_7, l_3, l_4	9	n_4	n_6	$l_8, l_{12}, l_{11}, l_{10}$
5	n_6	n_5	l_5, l_1, l_2, l_3, l_4	10	n_8	n_1	l_7, l_2, l_1

any $l \in \mathcal{L}$. We assume that the available budget $B = 20$ k€, the set of links to be maintained is $\widetilde{\mathcal{L}} = \mathcal{L}$, while the values of γ_l and I_l are shown in Table 5.

Table 5 Capacity enhancement factors and investments for links of Example 2.

Links	l_1	l_2	l_3	l_4	l_5	l_6	l_7	l_8	l_9	l_{10}	l_{11}	l_{12}	l_{13}
γ_l	1.2	1.5	1.1	1.6	1.3	1.4	1.1	1.7	1.3	1.5	1.1	1.8	1.3
I_l (k€)	3	8	2	10	4	5	2	12	4	8	2	13	4

Table 6 shows the ten best feasible solutions together with the value of φ and the corresponding investment I.

Table 6 The ten best feasible solutions for the optimal network improvement model in Example 2.

Ranking	y	$\varphi(y)$	$I(y)$
1	(0,0,0,0,0,0,0,0,0,0,1,1,1)	14.2823	19
2	(1,0,0,0,0,0,0,0,0,0,0,1,1)	13.3226	20
3	(0,0,1,0,0,0,0,0,0,0,0,1,1)	13.1370	19
4	(0,0,0,0,0,0,1,0,0,0,0,1,1)	12.7595	19
5	(0,0,0,0,0,0,0,0,0,0,0,1,1)	12.6185	17
6	(1,0,1,0,0,0,0,0,0,0,1,1,0)	11.0705	20
7	(1,0,0,0,0,0,1,0,0,0,1,1,0)	10.6949	20
8	(1,0,0,0,0,0,0,0,0,0,1,1,0)	10.5456	18
9	(0,0,1,0,0,0,1,0,0,0,1,1,0)	10.5079	19
10	(0,0,1,0,0,0,0,0,0,0,1,1,0)	10.3600	17

5 Conclusions and future directions

In this note we investigated a game theoretic model of congestion control in communication networks which is widely used in the literature on this topic. After introducing some modifications in the model, we studied an investment optimization problem that the network system manager faces in order to improve the capacity of links.

An interesting extension of this model could be considering the possibility that some of the data are not deterministic but random. Since we used the variational inequality approach to GNEP, the inclusion of such random data should be performed within the framework of stochastic variational inequalities (see, e.g., [6, 7, 9]).

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