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On the stability of string theory vacua

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Abstract: Vacuum compactifications may suffer from instabilities under small perturbations or tunnel effects; both are difficult to analyze. In this paper we consider the issue from a higher-dimensional perspective. We first look at how stability works for supersymmetric vacua, where it is widely expected to hold. We first show that the nucleation of brane bubbles in type II AdS compactifications is forbidden in the probe approximation by a simple argument involving pure spinors and calibrations. We then adapt familiar positive-energy theorems directly to M-theory and type II supergravity, rather than to their effective lower-dimensional reductions, also showing how to consistently include localized sources. We finally initiate an analysis of how these arguments might be extended to non-supersymmetric vacua. In M-theory, at the lower-derivative level, we find that the most natural modifications fail to stabilize the skew-whiffed and Englert vacua.

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1 Introduction and summary

In a gravitational theory, spacetime becomes dynamical. This creates the possibility that it might change drastically, perhaps with a catastrophic change in vacuum energy. For the Minkowski vacuum of general relativity, such worries were put to rest long ago by mathematical theorems [1, 2]. If matter satisfies the dominant energy condition, an appropriately defined total energy, computed in terms of the boundary behavior of the metric, is non-negative, and can only be zero for Minkowski space itself; so not even a quantum tunneling event can trigger spacetime decay.

This result was later generalized in various directions [3–6]. One of the lessons was that gravity can help stabilize scalar potentials that in field theory would lead to vacuum decay, confirming considerations from small fluctuations [7] and instantons [8]. Another is that supersymmetry often plays a crucial role, ultimately because the Hamiltonian is the square of the supercharges. (Indeed even the argument in [2] for pure gravity relies on introducing auxiliary spinors which are covariantly constant at infinity, inspired by the vacuum supersymmetric spinors for the simplest supergravity models.) In particular, it was shown in various $d = 4$ examples that supersymmetric vacua are stable even if the model does not satisfy the dominant energy condition [4].

In theories with extra dimensions such as string theory, however, new decay channels open up. This is demonstrated in a spectacular fashion by the $\mathrm{Mink}_4 \times S^1$ Kaluza-Klein (KK) vacuum of pure five-dimensional gravity, which can decay via the nucleation of a bubble of different topology, created by recombining internal and external directions; from the four-dimensional perspective this appears as a *bubble of nothing* where spacetime ends [9]. (The supersymmetry-inspired argument in [2] does not apply, because on the new topology there exist no spinors that are asymptotically covariantly constant.) In string theory, many other similar decay channels are provided by brane bubbles, where again spacetime can end or across which the flux quanta can change (with early examples in [10–12]). While an effective four-dimensional approach might go a long way towards establishing vacuum stability, it cannot capture such processes, which involve the dynamics of higher KK modes.[1]

[1]Effective theories of compactifications often not only violate the dominant energy condition, but naively seem to have a scalar potential that is unbounded from below, coming from the internal integral of the scalar curvature [13]. This issue was however resolved in [14].

This issue is made all the more pressing by the absence of a scale separation between the KK scale and the vacuum energy in many AdS vacua. While in some theories there was recent progress (beginning with [15]) in computing the full infinite KK mass spectrum, this still only covers small fluctuations.

In this paper, we attack this problem directly in ten or eleven dimensions. For supersymmetric $\mathrm{Mink}_4 \times M_d$ or $\mathrm{AdS}_4 \times M_d$ vacua, $d = 6, 7$, stability is of course widely expected for the reasons we recalled earlier; but quite surprisingly, it was never explicitly shown before. Our original motivation came from vacua that break supersymmetry: de Sitter vacua are expected to be metastable at best, while anti-de Sitter (AdS) vacua have been conjectured to be either unstable or metastable by a generalization of the weak gravity conjecture [16, 17] — see for instance [18–28] for a sample of recent papers discussing non-perturbative (in)stability in string theory.

In the supersymmetric case, as an appetizer we first consider brane bubbles in the probe approximation. Such processes have been shown to destabilize various non-supersymmetric vacua, beginning with [11, 12]; we show why these can never happen in the supersymmetric case, with a simple argument involving the type II pure spinor equations [29].

We then attack the problem in full generality by adapting the strategy of the positive energy theorem [2–6, 30] to compactifications. We consider a spinor ϵ that is *asymptotically supersymmetric*: namely, it is asymptotic to a supersymmetric spinor of the vacuum. From this we define a notion of *BPS energy* $I(\epsilon)$ by integrating the Hodge-dual of a certain two-form E_2, bilinear in ϵ, on the $S^2 \times M_d = \partial S$ asymptotic boundary of a spatial slice S. Using Stokes' theorem, $I(\epsilon) = \int_S \mathrm{d} * E_2$. Our main result is a formula for the divergence $* \mathrm{d} * E_2$, which we obtained both for eleven-dimensional and type II supergravity. In the former case, for example, we show that on-shell and in absence of branes the divergence satisfies the simple identity $\nabla_M E^{MN} = \overline{\mathcal{D}_M \epsilon} \Gamma^{MPN} \mathcal{D}_P \epsilon$, where $\delta\psi_P = \mathcal{D}_P \epsilon$ is the gravitino supersymmetry transformation; several flux and curvature terms conspire on the right-hand side to reconstruct the bulk equations of motion, which we can then set to zero. (The type II result is similar, but also contains the operator $\mathcal{O}$ in the dilatino transformations, $\delta\lambda = \mathcal{O}\epsilon$.)

The divergence $\nabla_M E^{MN}$ has the same form as in $d = 4$, so the remaining steps are standard. Namely, let ϵ be asymptotically supersymmetric and such that the *Witten condition* $\Gamma^a \mathcal{D}_a \epsilon = 0$ holds, where a is a flat index tangent to S. A gamma matrix identity now implies $I(\epsilon) = \int_S (\mathcal{D}^a \epsilon)^\dagger (\mathcal{D}_a \epsilon) \geq 0$, which is non-negative and only vanishes for supersymmetric vacua, as required.

In presence of branes, the bulk equations of motion produce delta-like contributions to $I(\epsilon)$; using supersymmetry and the theory of calibrations as generalized in [31], we show that these contributions are still all positive. (This is similar in spirit to how fields in multiplets not containing the graviton give extra contributions to $\nabla_\mu E^{\mu\nu}$ in $d = 4$ supersymmetric models [4].)

This proves stability in the supersymmetric case, up to a couple of technical details which are expected to hold and were checked in similar situations in existing literature. First, for AdS vacua one needs to check that $I(\epsilon)$ is independent of the boundary of the spatial slice ∂S; or in other words, that the BPS energy is conserved. In $d = 4$ this can be

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done for example by relating it to the energy defined in the covariant phase space formalism, which is conserved [32]. Second, one needs to prove that an ϵ that is asymptotically supersymmetric and satisfies the Witten condition $\Gamma^a \mathcal{D}_a \epsilon = 0$ always exists. Here we check a standard formal argument; a rigorous analytical proof was given in [33] for the $d = 4$ case (with an AdS_4 generalization in [34]), and in [35, 36] for Calabi-Yau compactifications.

Having worked out the argument for stability of supersymmetric vacua, we finally proceed to non-supersymmetric ones. The idea is now to look for an operator $\mathcal{D}'_M$ different from the one appearing in supersymmetry, trying to i) solve $\mathcal{D}'_M \epsilon = 0$ and ii) to make sure that the positivity argument still holds. In $d = 4$ models [6] this leads to the idea of *fake superpotential*: a W which is related to the scalar potential by the usual supergravity formula $V = 2(\partial_\phi W)^2 - 3W^2$, but which isn't the supersymmetric one. There is no direct analogue of this in our higher-dimensional setting; in M-theory, the most general gauge-invariant operator $\mathcal{D}'_M$ at one-derivative level is obtained by changing the coefficients of the flux terms in $\mathcal{D}_M$, and adding a new term proportional to Γ_M.

We are able to satisfy condition i) above on some simple AdS_4 vacua: the *skew-whiffed* ones [37, 38], obtained from supersymmetric ones by reversing the sign of G, and the *Englert* vacua [39], where internal flux components are added. In both of these cases we find multiple violations to the stability argument. One might want to conclude from this absence of protection that these vacua are in fact unstable. Of course it is possible, however, that a more sophisticated modification of the supersymmetric argument does work for them, or that a completely different protection mechanism is at play. We hope our methods can be refined in the future to settle the issue one way or another.

In section 2 we warm up, as we mentioned, by giving an argument that forbids brane bubbles in the probe approximation for AdS supersymmetric vacua. In section 3 we switch gears and review positive-energy theorems and their applications to stability in $d = 4$ theories. This provides a blueprint for subsequent developments: in section 4 we show positivity for M-theory, and in section 5 for type II supergravity. Finally we consider supersymmetry-breaking in section 6, mainly in M-theory. Many technical aspects of the computations are provided in detail in the appendices.

2 Probe brane bubbles

A vacuum may be unstable under tunneling effects. The probability of such a decay can be computed in Euclidean signature, by looking for bubbles that connect the old vacuum at infinity with a new one at their core. In this section we consider the simple case of decays mediated by branes in AdS vacua, in the *probe approximation*, namely disregarding their back-reaction. This is of course not always appropriate, but we consider it here as a warm-up: focusing for concreteness on D-brane bubbles in type II AdS_4 vacua, we are able to give a very simple argument that such bubbles cannot be nucleated. The argument is generalizable to other settings and directly exploits the ten-dimensional geometrical structure of the background. In other words, it does not rely on any effective four-dimensional description, which may be problematic in absence of scale separation.

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2.1 Preliminary general remarks

We start by reviewing AdS_d tunneling mediated by a probe $(d-2)$-brane charged under a d-form field strength $F_d = \mathrm{d}A_{d-1}$.[2] We first set up the formalism in a d-dimensional theory, and then discuss the modifications in the presence of extra dimensions.

In global coordinates, the Euclidean AdS_d metric is

$$\mathrm{d}s^2_{\mathrm{EAdS}_d} = L^2(\mathrm{d}r^2 + \sinh^2 r\, \mathrm{d}s^2_{S^{d-1}})\,, \tag{2.1}$$

where L is the AdS radius. This metric has an $\mathrm{SO}(1,d)$ isometry group. As in [8, 41], the dominant instantonic bubbles are expected to be invariant under the largest possible amount of symmetries. Hence, without loss of generality we can focus on Euclidean bubbles wrapping the S^{d-1} in (2.1), which are invariant under the isometry subgroup $\mathrm{SO}(d)$.[3]

The semiclassical decay rate is controlled by a $(d-1)$-dimensional Euclidean action of the form

$$S = \tau \int \mathrm{d}^{d-1}\sigma\sqrt{-g} - q\int A_{d-1}\,, \tag{2.2}$$

with some τ and q representing the tension and charge of the brane. By spacetime symmetry, the d-form flux in a vacuum will be of the form

$$\begin{aligned} F_d = \mathrm{d}A_{d-1} = L^{-1}h\,\mathrm{vol}_{\mathrm{EAdS}_d} &= L^{d-1}h\sinh^{d-1} r\,\mathrm{d}r\wedge \mathrm{vol}_{S^{d-1}}\,;\\ A_{d-1} = L^{d-1}h\,c(r)\,\mathrm{vol}_{S^{d-1}}\,, &\qquad c'(r) = \sinh^{d-1} r\,, \end{aligned} \tag{2.3}$$

for some constant h, which in a quantum gravity theory is expected to satisfy an appropriate quantization condition. Now (2.2) becomes

$$S = L^{d-1}\mathrm{vol}_{S^{d-1}}\left[\tau\sinh^{d-1} r - qhc(r)\right]\,. \tag{2.4}$$

This is extremized at $r = r_0$ if an only if

$$\tanh r_0 = \frac{(d-1)\tau}{qh}\,. \tag{2.5}$$

This condition can be satisfied, and then the tunnel effect is allowed, only if

$$qh > (d-1)\tau\,. \tag{2.6}$$

Intuitively, this means that the WZ term, which wants to expand the bubble, wins over the gravitational term, which wants to crush it.

We can now uplift these simple d-dimensional arguments to higher D-dimensional vacua of the (possibly warped) product form $\mathrm{AdS}_d \times M$, where M is a compact $(D-d)$-dimensional space, having in mind the string/M-theory models ($D = 10, 11$). We assume that the $(d-1)$-dimensional bubble corresponds to a microscopic p-brane of the form

[2] In the simplest d-dimensional setting in which only the metric and A_{d-1} appear as dynamical bulk fields, it is relatively easy to take into account backreaction effects [40]. This is much harder for generic string/M-theory AdS vacua, in particular in absence of scale separation.

[3] The bubbles considered in [12] would superficially not look spherically symmetric, but in fact they are; see ([21], section 4.1.2).

$S^{d-1} \times \Sigma$, where $\Sigma \subset M$ is an internal closed $(p-d+2)$-dimensional submanifold, possibly supporting non-trivial world-volume fields. By invoking again the SO(d) symmetry, the internal cycle Σ as well as the world-volume fields cannot depend on the position along the external S^{d-1}. Hence, they must satisfy some internal equations of motion, decoupled from the external ones, which fix them to some particular configuration. One is then reduced to a $(d-1)$-dimensional action of the form (2.2), with τ and q fixed by the on-shell internal p-brane configuration. After having computed τ and q in this way, one can then check whether the instability condition (2.6) is satisfied or not. Of course we expect that (2.6) never holds for supersymmetric vacua. In the following section we will highlight the microscopic geometric origin of this stability in the case of D-brane bubbles in AdS_4-vacua.

We also remark that (2.6) admits an alternative interpretation. Put the Lorentzian version of the same $(d-2)$-brane along a $\mathbb{R}^{1,2}$ slice of the Lorentzian AdS_d in Poincaré coordinates. If (2.6) holds, the brane will experience a run-away potential, which will push it to the AdS boundary. This brane ejection effect is indeed often adopted as an alternative criterion to detect the instability of an AdS vacuum; it was argued for holographically in [22, 42] and by using (2.6) in ([21], section 5.1). In the last two references, the role of the dependence of τ, q on the internal coordinates was emphasized.

2.2 Brane bubbles in type II AdS_4 vacua

We now focus on D-brane bubbles in type II vacua of the form $\mathrm{AdS}_4 \times_{\mathrm{W}} M_6$, with the metric given by $\mathrm{d}s^2 = e^{2A}\,\mathrm{d}s^2_{\mathrm{AdS}_4} + \mathrm{d}s^2_{M_6}$. We will work in string units $2\pi\sqrt{\alpha'} = 1$. We begin by recalling that the conditions for such a vacuum to have $\mathcal{N}=1$ supersymmetry can be written as [29]

$$\mathrm{d}_H(\mathrm{e}^{2A-\phi}\mathrm{Re}\Phi_\mp) = 0\,, \tag{2.7a}$$

$$\mathrm{d}_H(\mathrm{e}^{3A-\phi}\Phi_\pm) = \frac{2}{L}\mathrm{e}^{2A-\phi}\mathrm{Re}\Phi_\mp\,, \qquad {}^{\mathrm{IIA}}_{\mathrm{IIB}} \tag{2.7b}$$

$$\mathrm{d}_H(\mathrm{e}^{4A-\phi}\mathrm{Im}\Phi_\mp) = \frac{3}{L}\mathrm{e}^{3A-\phi}\mathrm{Im}\Phi_\pm - \mathrm{e}^{4A} * \lambda f\,, \tag{2.7c}$$

where $*$ is the Hodge-star along M_6. The $\Phi_\pm$ are polyforms that obey a certain algebraic *purity* condition, defined as bilinears of the internal supersymmetry parameters η^a. We use the normalization $\|\eta^a_+\|^2 = 8\mathrm{e}^A$, and $\Phi_\pm$ such that $(\bar\Phi_\pm, \Phi_\pm) = 8\mathrm{i}$; this is convenient but not necessary. $f = \sum_k f_k$ is a polyform collecting the internal RR fluxes f_k, H is the NSNS flux, $\lambda\alpha_k \equiv (-1)^{\lfloor k/2 \rfloor}\alpha_k$ and $\mathrm{d}_H \equiv \mathrm{d} - H\wedge$. The total RR flux is related to the internal one by

$$F = f + \mathrm{vol}_4 \wedge \mathrm{e}^{4A} * \lambda f\,. \tag{2.8}$$

Crucially for us, the equations (2.7) can be interpreted as closure of the (generalized) calibrations for D-branes extended in AdS_4 along time plus one, two, and three space coordinates respectively [43, 44].

Consider now a Euclidean Dp-brane bubble of the form $S^3 \times \Sigma$, where $\Sigma \subset M_6$ is a $(p-2)$-cycle. Σ can support a world-volume two-form flux $\mathcal{F}$, satisfying the Bianchi identity $\mathrm{d}\mathcal{F} = H|_\Sigma$, where $|_\Sigma$ denotes the pull-back to Σ. By reducing the usual D-brane DBI action on Σ, we easily get

$$\tau = 2\pi \int_\Sigma \mathrm{d}^{p-2}\xi\, \mathrm{e}^{3A}\sqrt{\det(h+\mathcal{F})}\,. \tag{2.9}$$

Here ξ^i are world-volume coordinates along Σ and h_{ij} is the pull-back of the string frame metric along M_6. The factor of e^{3A} comes about because the brane is extended along three directions in EAdS_4.

In order to obtain qh, we must analogously reduce the D-brane CS-term

$$-2\pi \int_{S^3\times\Sigma} C\wedge \mathrm{e}^{\mathcal{F}} \tag{2.10}$$

where C is the polyform of RR potentials, such that $F = \mathrm{d}_H C$. We can take

$$C = c_{\rm int} + L^4 c(r)\mathrm{vol}_{S^3}\wedge \mathrm{e}^{4A} * \lambda f \tag{2.11}$$

with $\mathrm{d}c_{\rm int} = f$ (which will not play any role) and $c(r)$ as in (2.3). By reducing (2.10) along Σ and matching it against (2.2) with A_{d-1} as in (2.3), we get

$$qh = 2\pi L\int_\Sigma e^{4A} * \lambda f\wedge \mathrm{e}^{\mathcal{F}} = 6\pi\int_\Sigma e^{3A-\phi}\mathrm{Im}\,\Phi_\pm\wedge \mathrm{e}^{\mathcal{F}}\,. \tag{2.12}$$

In the second step, we used (2.7c).

We now recall that $\Phi_\pm$ can be regarded as D-brane calibrations [43, 44]. This allows us to write the bound:

$$\mathrm{d}^{p-2}\xi\,\sqrt{\det(h+\mathcal{F})} \geq \left[\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta}\Phi_\pm)|_\Sigma\wedge \mathrm{e}^{\mathcal{F}}\right]_{\rm top} \qquad (\text{for any } \theta) \tag{2.13}$$

where the inequality applies to the coefficients of the local top-form $\mathrm{d}^{p-2}\xi$ on Σ. By using (2.13) inside (2.12) and recalling (2.9), we get the bound

$$qh \leq 3\tau \tag{2.14}$$

which shows that (2.6) (with the correct numerical coefficient $d-1=3$) is indeed never attained in these vacua.

Note also that the calibration appearing in (2.12) corresponds to the phase $e^{\mathrm{i}\theta} = -\mathrm{i}$ in (2.13). Precisely with such a choice $\mathrm{Re}(e^{\mathrm{i}\theta}\Phi_\pm)$ is the calibration for a domain-wall-like D-brane filling the Poincaré $\mathbb{R}^{1,2}$ slice of the Lorentzian AdS_4. Such a configuration is supersymmetric precisely when (2.13) is saturated. In this case $qh = 3\tau$ and then we have threshold stability under the brane ejection mechanism discussed in the previous subsection.

This argument clearly illustrates how the non-perturbative stability of supersymmetric AdS vacua under nucleation of probe Euclidean brane bubbles is controlled by calibrations. Hence, it is natural to guess that any more general stability argument involving backreacting branes, which possibly contribute to both the classical background and the semiclassical nucleated bubbles, must crucially depend on the existence of appropriate calibration structures. The following sections will show how this guess is precisely realized.

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3 Stability in four-dimensional (super)gravity

In order to extend the above stability argument beyond the probe regime, we will adapt methods successfully used in gravity without extra dimensions.

The method stems from the expectation [45, 46] that a supersymmetric state in supergravity saturates a quantum BPS bound of the schematic form

$$I \equiv \langle \{Q, Q\} \rangle \geq 0\,, \tag{3.1}$$

where Q is any of the real supercharges preserved by the supersymmetric state. One may then loosely conclude that the supersymmetric state must be stable, or at most threshold-decay to other supersymmetric states saturating the same bound.

A more precise realization of this formal positivity argument was given by Witten [2] and Nester [30]. The first application in these papers was to the Minkowski vacuum in Einstein gravity, resulting in a new proof of the result in [1]. Indeed the argument is sufficiently robust that it may be applied to theories without supersymmetry, in which case the spinors involved in the proof can just be taken to be auxiliary objects. The proof was later extended to non-zero cosmological constant Λ in gravity and supergravity [3–5], and later to various models in different dimensions, which may also arise as EFTs or consistent truncations of string/M-theory compactifications. However, somewhat surprisingly, no successful attempt has so far been made to either apply this approach directly to 10/11-dimensional string/M-theory models, or to include local backreacting brane sources in the argument.

In the next sections we will explicitly fill these gaps. Since the discussion will inevitably be quite technical, in this section we will first provide a self-contained review of some key points of this formalism in simpler four-dimensional settings.

3.1 Mink_4 stability from supergravity

Consider first the minimal four-dimensional supergravity action

$$\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(R + \mathrm{i}\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho\right) \tag{3.2}$$

where the gravitino ψ_μ is in the Majorana representation, and we work in Planck units $8\pi G = M_\mathrm{P}^{-2} = 1$. As reviewed in appendix B, by applying the standard Noether procedure we obtain the following formula for the supercharge associated with a given supersymmetry generator ε:

$$Q(\varepsilon) = -\mathrm{i}\int_{\partial\Sigma}\bar{\varepsilon}\,\gamma_5\gamma_\mu\psi_\nu\,\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \tag{3.3}$$

where $\gamma_5 \equiv \mathrm{i}\gamma^{\underline{0123}}$ is the four-dimensional chiral operator and $\partial\Sigma$ is the boundary of a three-dimensional non-timelike hypersurface. Now, the Dirac brackets of two such generators must give $\{Q(\varepsilon'), Q(\varepsilon)\} = \delta_{\varepsilon'}Q(\varepsilon)$, which can be computed from $\delta_\varepsilon\psi_\mu = D_\mu\varepsilon$. Hence, the usual quantum argument suggests that the positive definite quantity associated with $Q(\varepsilon)$ can be identified with $I(\varepsilon) \equiv \delta_\varepsilon Q(\varepsilon)$, with ε now considered as a commuting spinor. This logic leads to the following quantity:

$$I(\varepsilon) = \int_{\partial\Sigma} *E_2\,, \tag{3.4}$$

where

$$E_2 \equiv -\frac{1}{2}\,\bar{\varepsilon}\gamma_{\mu\nu}{}^{\rho}D_{\rho}\varepsilon\,\mathrm{d}x^{\mu}\wedge\mathrm{d}x^{\nu}\,. \tag{3.5}$$

Note that in (3.3) and $\delta_\varepsilon\psi_\mu = D_\mu\varepsilon$ the spinor ε is anticommuting, while in (3.5) it must be considered as a commuting spinor.

On asymptotically Minkowski spacetimes, one can choose Σ to reduce asymptotically to a plane with radial coordinate r, and identify $\partial\Sigma$ with a sphere of constant radius $R\to\infty$. If one imposes that ε asymptotically reduces to a constant spinor ε_0 up to $\mathcal{O}(1/r)$ terms, then $I(\varepsilon)$ coincides with the Nester-Witten energy [30], which is still valid in the presence of additional matter coupled to the minimal gravity multiplet through an energy-momentum tensor $T^{(\mathrm{mat})}_{\mu\nu}$. More precisely, in an appropriate normalization, we can make the identification

$$I(\varepsilon) = -k_0^{\mu}P_{\mu}\,, \tag{3.6}$$

where P_μ is the four-momentum of the system and

$$k^{\mu}\equiv\bar{\varepsilon}\gamma^{\mu}\varepsilon \quad\Rightarrow\quad k_0^{\mu}\equiv\bar{\varepsilon}_0\gamma^{\mu}\varepsilon_0\,. \tag{3.7}$$

In particular, $I(\varepsilon)=I(\varepsilon_0)$: it only depends on the asymptotic value of the spinor. Note that here we are using a Majorana spinor ε (as for instance in [6]), while in [2, 30] a complex ε is used. This implies that in our case k^μ is a future-pointing null vector. However, since ε_0 is arbitrary, k_0^μ spans the entire future light-cone of the asymptotic flat space. This means that, if $I(\varepsilon)\geq 0$ for any ε_0, then P^μ must be necessarily non-spacelike and future-pointing. In particular, the energy is non-negative in any frame.

In order to prove that $I(\varepsilon)\geq 0$ if $T^{(\mathrm{mat})}_{\mu\nu}$ satisfies the dominant energy condition, we now assume that Σ is a regular space-like surface as in [2, 30]. (Horizons may be included as in [47] and the discussion may be generalized to null surfaces Σ following [48].) Then by using Stokes' theorem and the identity $\bar{\varepsilon}\gamma_{\mu\nu\rho}D^{\nu}D^{\rho}\varepsilon = G_{\mu\nu}k^{\nu}$ (which we will show in detail in section 4.1), we can write (3.4) as follows:

$$\begin{aligned} I(\varepsilon) &= \int_\Sigma \mathrm{d}*E_{(2)} = \int_\Sigma \nabla^{\nu}E_{\mu\nu}*\mathrm{d}x^{\mu} = \int_\Sigma\left(D^{\nu}\bar{\varepsilon}\gamma_{\mu\nu\rho}D^{\rho}\varepsilon+\frac{1}{2}G_{\mu\nu}k^{\nu}\right)*\mathrm{d}x^{\mu} \\ &\overset{\text{on-shell}}{=} \int_\Sigma D^{\nu}\bar{\varepsilon}\gamma_{\mu\nu\rho}D^{\rho}\varepsilon\, n^{\mu}\,\mathrm{vol}_\Sigma + \frac{1}{2}\int_\Sigma T^{(\mathrm{mat})}_{\mu\nu}k^{\nu}n^{\mu}\,\mathrm{vol}_\Sigma\,. \end{aligned} \tag{3.8}$$

In the last line we have rewritten $*\mathrm{d}x^{\nu}|_\Sigma = \mathrm{vol}_\Sigma n^{\nu}$, where n^ν is a time-like unit vector orthogonal to the surface Σ and vol_Σ is the induced volume form on Σ, and we have imposed Einstein's equations $G_{\mu\nu}=T^{(\mathrm{mat})}_{\mu\nu}$. The last term of (3.8) is manifestly non-negative if $T^{(\mathrm{mat})}_{\mu\nu}k^{\nu}n^{\mu}\geq 0$ for any null future-directed vector k^μ and then, by linearity, for any non-space-like vector k^μ. This is precisely the definition of the dominant energy condition, which we are assuming. In order to prove that $I(\varepsilon)\geq 0$, it then remains to show that also the first term in the last line of (3.8) is non-negative.

At this point it is convenient to pick an adapted vielbein $e^A = e^A_\mu \mathrm{d}x^\mu = (e^{\underline{0}}, e^a)$ (with spatial flat indices $a\in\{\underline{1},\underline{2},\underline{3}\}$) and the dual frame $e_A = e^\mu_A\partial_\mu = (e_{\underline{0}}, e_a)$ such that $e^{\underline{0}}|_\Sigma = 0$ and then $n^\mu\equiv e^\mu_{\underline{0}}$. The first term in the last line of (3.8) can then be decomposed into:

$$\int_\Sigma D^a\bar{\varepsilon}\gamma_{\underline{0}ab}D^b\varepsilon\mathrm{vol}_\Sigma = \int_\Sigma (D^a\varepsilon)^\dagger\,(D_a\varepsilon)\,\mathrm{vol}_\Sigma - \int_\Sigma|\gamma^a D_a\varepsilon|^2\mathrm{vol}_\Sigma. \tag{3.9}$$

The first contribution is manifestly positive; the second vanishes upon imposing the *Witten condition*:

$$\gamma^a D_a \varepsilon = 0. \tag{3.10}$$

This has a natural interpretation from the supergravity viewpoint [5]: it can be regarded as following from a "transverse" gauge choice $\gamma^a \psi_a = 0$ [49, 50] on the gravitino. In other words, the last term in (3.9) can be associated with "longitudinal" unphysical degrees of freedom, which can be gauged away.

Let us for the moment assume that (3.10) can be solved for any choice of the spatial surface Σ and asymptotic ε_0. Combining (3.8) and (3.9), we can then conclude that $I(\varepsilon) \geq 0$. Moreover, if $I(\varepsilon) = 0$, $D_a\varepsilon$ needs to vanish; if this is true for any ε_0, by varying Σ we find four constant spinors on our spacetime, which implies that it is Minkowski. Since the only spacetime where $I(\varepsilon) = 0$ for any ε_0 is Minkowski, and $I(\varepsilon)$ is conserved (it does not depend on Σ), Minkowski space cannot evolve into anything else: it is stable.

The remaining key step is then to prove that (3.10) always admits a solution for the given boundary condition $\varepsilon \to \varepsilon_0 + \mathcal{O}(1/r)$. This question can be addressed at various levels of mathematical rigor [2, 33, 47], but for our purposes we will just focus on the most crucial requirement: the elliptic operator $\gamma^a D_a$ must have no normalizable zero modes. If this holds, $\gamma^a D_a$ can be inverted into a Green's operator. We can then start with any trial ε_1 asymptotic to ε_0 and find a normalizable solution ε_2 to $\gamma^a D_a \varepsilon_2 = -\gamma^a D_a \varepsilon_1$; now $\varepsilon_1 + \varepsilon_2$ solves (3.10) [2, 33]. The absence of a normalizable zero mode $\varepsilon_{\rm zm}$ of $\gamma^a D_a$ can be understood as follows. Such an $\varepsilon_{\rm zm}$ would decrease sufficiently fast to make $I(\varepsilon_{\rm zm})$ as defined in (3.4) vanish. But then, by rewriting $I(\varepsilon_{\rm zm})$ as in (3.8) and using $\gamma^a D_a \varepsilon_{\rm zm} = 0$ inside the identity (3.9), we would get

$$\int_\Sigma (D^a \varepsilon_{\rm zm})^\dagger (D_a \varepsilon_{\rm zm}) + \frac{1}{2}\int_\Sigma T^{(\rm mat)}_{\mu\nu} k^\nu_{\rm zm} n^\mu \,{\rm vol}_\Sigma = 0\,. \tag{3.11}$$

This identity is clearly impossible unless $D_a \varepsilon_{\rm zm} = 0$ for any Σ, and then $\varepsilon_{\rm zm} \equiv 0$ (since $\varepsilon_{\rm zm}$ must vanish asymptotically). We will also apply the same kind of argument to more complicated settings.

3.2 AdS$_4$ stability

The logic outlined in the previous subsection can be adapted to AdS vacua in theories of gravity and supergravity [3–5]. The simplest, "minimal" model is obtained by adding a constant superpotential W_0, which generates a negative cosmological constant $\Lambda = -3|W_0|^2$ (in Planck units). Without loss of generality, we assume that W_0 is real and positive, so that $W_0 = \sqrt{-\Lambda/3} \equiv 1/L$, where L is the AdS radius. In global coordinates, the AdS metric reads

$$ds^2_{\rm AdS_4} = -\left(1 + \frac{\rho^2}{L^2}\right) dt^2 + \left(1 + \frac{\rho^2}{L^2}\right)^{-1} d\rho^2 + \rho^2 d\Omega^2\,; \tag{3.12}$$

at $\rho \to \infty$, the rescaled $\rho^{-2} ds^2_{\rm AdS_4}$ induces on the boundary the metric of $\mathbb{R} \times S^2$. The formula (3.3) for the supercharge is unchanged while the gravitino transformation is modified into $\delta_\varepsilon \psi_\mu = \mathcal{D}_\mu \varepsilon$, where

$$\mathcal{D}_\mu \equiv D_\mu + \frac{1}{2} W_0 \gamma_\mu\,. \tag{3.13}$$

Recall that an ε annihilated by $\mathcal{D}_\mu$ is said to be a *Killing spinor*. AdS has the maximal number of such spinors: they behave asymptotically as ([7], (3.17)) $\varepsilon \sim \rho^{1/2}\varepsilon_0$, with ε_0 projecting on the boundary to a conformal Killing spinor of $\mathbb{R} \times S^2$.

Changing $D \to \mathcal{D}$ everywhere in the argument (3.8)–(3.10) in the previous subsection now again establishes in this minimal model that $I(\varepsilon) \geq 0$, and that $I(\varepsilon) = 0$ only in the AdS vacuum. The same procedure can be repeated for non-minimal gauged supergravities [4]: the only difference is the appearance of additional contributions to $I(\varepsilon)$, also manifestly non-negative, containing the supersymmetry variations of additional fermions in the supersymmetric multiplets. (See also [51, 52] for applications to the calculation of BPS bounds for non-vacuum states in $\mathcal{N} = 2$ gauged supergravities.) However, the different global structure of AdS changes the rest of the argument. The presence of the $\mathbb{R} \times S^2$ time-like boundary makes the spacetime not *globally hyperbolic* — one cannot predict the future from the data on a Cauchy slice alone, but rather one also needs to know what happens at spatial infinity. A priori this might make it unclear whether $I(\varepsilon)$ is conserved. A natural boundary condition for the metric is to impose that $\rho^{-2}\mathrm{d}s^2$ be conformal to $\mathbb{R} \times S^2$ at the boundary, as happens for the AdS vacuum. In [32] it was shown (in a pure gravity model) that with this boundary condition, $I(\varepsilon)$ is equal to the conserved charge $\mathcal{E}(k_0)$ associated by the covariant phase space formalism [53] to the asymptotic isometry $k_0^\mu = \bar{\varepsilon}_0\gamma^\mu\varepsilon_0$. The latter are conformal Killing vectors of $\mathbb{R} \times S^2$, and together they generate its conformal isometry group $\mathrm{SO}(2,3)$. Moreover, $\mathcal{E}(k_0)$ does not depend on the space slice Σ (i.e., it is conserved).[4]

That the asymptotically supersymmetric ε can satisfy the Witten condition (now $\gamma^a\mathcal{D}_a\varepsilon = 0$) was shown in a slightly more general context in ([34], section V). So we have in fact several non-negative conserved quantities, to which we can apply the argument in the previous subsection.

The covariant phase space method can also be applied to supersymmetry itself, in which case it reproduces once again the expression (3.3) for the supercharge $Q(\varepsilon)$ [55]. The formalism is built so that the conserved charge is a Hamiltonian generator for the associated symmetry; from the (super-)Jacobi identity it follows that $Q(\varepsilon)$ and the $\mathrm{SO}(2,3)$ generators J_{AB} together form a superalgebra, as expected. In particular [3–5]

$$I(\varepsilon) = \frac{1}{2} J_{AB}\, \bar{\varepsilon}_0 \sigma^{AB} \varepsilon_0 \,, \tag{3.14}$$

where $\sigma^{ab} \equiv \frac{1}{2}\gamma^{ab}$ and $\sigma^{a4} = \frac{1}{2}\gamma^a$. The AdS energy $E_{\mathrm{AdS}} \equiv J_{04}$ is obtained by tracing (3.14) over a basis of ε_0; it vanishes only when both $T^{(\mathrm{mat})}_{\mu\nu} \equiv 0$ and $D_\mu\varepsilon \equiv 0$, i.e., maximally supersymmetric AdS is the unique $E_{\mathrm{AdS}} = 0$ configuration of these models.

3.3 Breaking supersymmetry

As we mentioned at the beginning, the arguments in this section are robust enough that they don't need supersymmetry: the spinors ε can be auxiliary variables, unrelated to any fermionic symmetry. Indeed, even the first application [2, 30] was to pure Einstein gravity.

[4] A detailed comparison of $I(\varepsilon)$ to alternative notions of energy in asymptotically AdS spacetimes is also given in [32]; see also [54] for a review.

One can follow this idea to give stability arguments both for vacua in non-supersymmetric theories, and for supersymmetry-breaking vacua in supersymmetric ones.

A systematic analysis was initiated in [6] (and generalized to $d > 4$ in [56]). To illustrate the idea, consider a model with a single scalar ϕ and action

$$\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\,(R - \partial_\mu\phi\partial^\mu\phi - 2V(\phi))\ . \tag{3.15}$$

There is no supersymmetry, but we may nevertheless consider spinors ε that satisfy $\mathcal{D}'_\mu\varepsilon = 0$, with

$$\mathcal{D}'_\mu \equiv D_\mu + \frac{1}{2}W(\phi)\gamma_\mu\,, \tag{3.16}$$

generalizing (3.13). Trying to prove an analogue of (3.8), one finds that it works if

$$V = 2(\partial_\phi W)^2 - 3W^2\,. \tag{3.17}$$

This has the same structure of an $\mathcal{N} = 1$ model; but one may also look for such a W in absence of supersymmetry, in which case it is known as a *fake* superpotential.

The discussion of boundary conditions for ϕ is quite intricate; see for example [34]. As is well known, near the boundary it behaves like $\alpha_-\rho^{-\Delta_-} + \alpha_+\rho^{-\Delta_+}$, where $\Delta_- \leq \Delta_+$ are the two roots of $\Delta(\Delta - d + 1) = m^2$. For "fast" boundary conditions $\alpha_- = 0$, the BPS energy again is equal to the conserved energy from the covariant phase space formalism, but for choices of the type $\alpha_+ = f(\alpha_-)$ the two differ. In fact there are in general two local solutions $W_- \leq W_+$ to (3.17) for each given V around the vacuum; W_+ always exists globally, but when $\alpha_- \neq 0$ the BPS energy associated to it is infinite. So one has to use W_-, but its global existence is guaranteed only for some potentials.

3.4 Some comments

We close this section with some remarks.

- Even though the identification of the positive definite $I(\varepsilon)$ exploits the supersymmetric structure, the stability argument regards only purely bosonic configurations. This is not really an issue at the classical level, since non-vanishing fermionic profiles do not admit a classical interpretation. Hence, the above stability arguments are complete as long as we are interested in classical instabilities or semiclassical ones mediated by Coleman-de Luccia instantonic bubbles [8, 41]. Indeed, the portion of spacetime created after nucleation of a bubble can be considered as a localized classical excitation of the Mink_4 or AdS_4 vacuum, preserving the appropriate boundary conditions.

- These four-dimensional arguments can be applied to string/M-theory models as long as the four-dimensional theory is a consistent EFT and the possible decay processes are describable within the EFT regime of validity. But many examples of string/M-theory compactifications to AdS_4 do not admit such an EFT description, since for instance the AdS scale $\sqrt{-\Lambda}$ and the KK scale are of the same order. In such cases, there often exist some consistent truncations which lead to extended gauged

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supergravities of the kind considered in [4]. However, any conclusion on the stability of the vacuum based on these consistent truncations is inconclusive.

- To our knowledge, the models considered so far in the literature do not include possible charged membranes, which for instance catalyse the kinds of decays discussed in section 2 (in the probe approximation). Such membranes could be incorporated into a four-dimensional $\mathcal{N}=1$ EFT as in [57, 58] and should be taken into account. Moreover, in the absence of scale separation, such membranes should actually be regarded as higher dimensional branes as in section 2.2.

In the next section we will address some of the open issues raised in the second and third items by explicitly showing how the above stability arguments can be uplifted to string/M-theory. Moreover, we will see how various (backreacting) branes can be taken into account, hence providing a first important set of examples on how such extended charged objects can be consistently incorporated into these kinds of arguments.

4 Positivity and stability in M-theory

We now move on to the (expected) stability of supersymmetric string/M-theory supergravity backgrounds, including possible localized sources. In this section we will consider the M-theory case, while in the next we will discuss type II theories. The extension of our results to other string theories should be straightforward.

Some useful properties of eleven-dimensional supergravity are reviewed in appendix B, in which we also fix our conventions. Here we just recall that the bulk fields are the metric g_{MN}, a three-form potential C with closed four-form field-strength $G=\mathrm{d}C$, and a Majorana gravitino ψ_M. Furthermore, we will denote the eleven-dimensional gamma matrices by Γ_M, and we will work in units $\ell_{\rm P}=1$, in which the supergravity Einstein-Hilbert term is $2\pi\int R*1$ and the M2-brane tension is 2π.

Let us follow the same steps of section 3. First, one can again apply the Noether procedure reviewed in appendix B.2, to get the following expression for the supercharge:

$$Q(\epsilon)=-\int_{\partial S}\bar\epsilon\,\Gamma_{(8)}\wedge\psi\,, \tag{4.1}$$

where (commuting) Majorana spinor ϵ defines an arbitrary supersymmetry transformation and ∂S is the boundary of a ten-dimensional hypersurface S. We are using the intrinsic notation where $\psi\equiv\psi_M\mathrm{d}x^M$ and

$$\Gamma_{(p)}\equiv\frac{1}{p!}\Gamma_{M_1\cdots M_p}\mathrm{d}x^{M_1}\wedge\cdots\wedge\mathrm{d}x^{M_p}\,. \tag{4.2}$$

The supersymmetry transformation of the gravitino reads

$$\delta\psi_M=\mathcal{D}_M\epsilon+\mathcal{O}(\psi^2)\,, \tag{4.3}$$

where ϵ is a Majorana spinor and

$$\mathcal{D}_M\equiv D_M+\frac{1}{12}(G\Gamma_M-G_M)=D_M+\frac{1}{24}(-\Gamma_M G+3G\Gamma_M)\,, \tag{4.4}$$

with D_M being the usual spinorial covariant derivative. In this expression, G should actually be written as $G_{/} \equiv \frac{1}{4!} G_{MNPQ} \Gamma^{MNPQ}$ (often also denoted by $\not{G}$), the bispinor associated to the four-form G under the Clifford map $_{/}$ in (A.2). To get cleaner spinorial equations, we will often omit the slash symbols, hoping it will be clear from context whether we are referring to a form or to the associated bispinor. For instance, in (4.4) we also have $G_M \equiv (\iota_M G)_{/} = \frac{1}{3!} G_{MNPQ} \Gamma^{NPQ}$ — again see appendix B for more details on our notation.

The argument outlined in section 3.1 leads to the following natural candidate for a positive definite quantity associated with ϵ:

$$I(\epsilon) \equiv \int_{\partial S} * E_2 \tag{4.5}$$

where

$$E_2 \equiv -\frac{1}{2}\, \bar{\epsilon}\, \Gamma_{MN}{}^{P} \mathcal{D}_P \epsilon \, \mathrm{d}x^M \wedge \mathrm{d}x^N \,. \tag{4.6}$$

This two-form is the eleven dimensional counterpart of (3.5).

These formulas have already been exploited together with the usual quantum supersymmetry argument to formally argue for some BPS bounds as in [59] — see also [60] for a more recent discussion in terms of calibrations which is more directly relevant for what follows. For this reason, we will refer to (4.5) as the *BPS energy*, having in mind a non-negative linear combination of energy and possible central charges that vanishes when supersymmetry is preserved. However, even though it is expected, positivity of the BPS energy (4.5) has not been proved to date. This is precisely one of our main goals.

In order to prove positivity, as in the four-dimensional case we assume that S is a regular space-like hypersurface and use Stokes' theorem

$$I(\epsilon) = \int_S \mathrm{d} * E_2 = \int_S \nabla^M E_{NM} \, * \mathrm{d}x^N \,. \tag{4.7}$$

Horizons and asymptotically null hypersurfaces S may be also considered following [61] and [48] respectively, but we will not do it here. On the other hand, we will allow for M2 and M5 branes, showing how to properly take into account their backreaction.

4.1 The main identity

Our next crucial step is to compute the divergence appearing in (4.7). We start by presenting the final result:

$$\nabla_M E^{MN} = \overline{\mathcal{D}_M \epsilon} \Gamma^{MPN} \mathcal{D}_P \epsilon + \tfrac{1}{2} \mathcal{E}^{NP} K_P + \tfrac{1}{4} \bar{\epsilon} \Big[\mathrm{d}x^N \wedge \Big(-\mathrm{d}G + \mathrm{d} * G + \tfrac{1}{2} G \wedge G \Big) \Big]_{/} \epsilon \,, \tag{4.8}$$

where we have introduced the future-pointing causal vector

$$K \equiv K^M \partial_M = \bar{\epsilon} \Gamma^M \epsilon \, \partial_M \,, \tag{4.9}$$

and $\mathcal{E}_{MN}$ denotes the bulk contribution to the Einstein equations:

$$\begin{aligned} \mathcal{E}_{MN} &\equiv R_{MN} - \frac{1}{2} g_{MN} R - \frac{1}{2} T^{(G)}_{MN} \\ \text{where} \qquad T^{(G)}_{MN} &\equiv G_M \cdot G_N - \frac{1}{2} |G|^2 g_{MN} \,. \end{aligned} \tag{4.10}$$

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Note that (4.8) holds identically for *any off-shell* configuration of g_{MN} and G, in the sense that one does not need to impose either the equations of motion or the Bianchi identities.

While until here our logic followed quite closely the steps in section 3, the proof of (4.8) is quite specific to M-theory. We present here the main steps of the argument, leaving several details to appendix B.3. We begin with

$$\begin{aligned}\nabla_M E^{MN} &= -\nabla_M(\bar{\epsilon}\Gamma^{MNP}\mathcal{D}_P\epsilon) = -\overline{D_M\epsilon}\Gamma^{MNP}\mathcal{D}_P\epsilon + \bar{\epsilon}\,\Gamma^{NMP}D_M\mathcal{D}_P\epsilon \\ &= -\overline{\mathcal{D}_M\epsilon}\Gamma^{MNP}\mathcal{D}_P\epsilon - \frac{1}{24}\bar{\epsilon}\,A^{NP}D_P\epsilon \\ &\quad -\bar{\epsilon}\left[\Gamma^{MNP}\left(D_M D_P + \frac{1}{24}[D_M, -\Gamma_P G + 3G\Gamma_P]\right) + \frac{1}{24^2}Q^N\right]\epsilon\,,\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}A^{NP} &= (-G\Gamma_M + 3\Gamma_M G)\Gamma^{MNP} - \Gamma^{NPM}(-\Gamma_M G + 3G\Gamma_M)\,, \\ Q^N &= (-G\Gamma_M + 3\Gamma_M G)\Gamma^{MNP}(-\Gamma_P G + 3G\Gamma_P)\,.\end{aligned} \tag{4.12}$$

We show in appendix B.3 that $A^{NP} = 0$. The first term on the last line of (4.11) can be evaluated recalling that the commutator of two spinorial covariant derivatives involves the Riemann tensor, and some relatively standard gamma matrix identities:

$$\begin{aligned}\Gamma^{MNP}[D_N, D_P] &= \frac{1}{4}R^{AB}{}_{NP}\Gamma^{MNP}\Gamma_{AB} = \frac{1}{4}R^{AB}{}_{NP}(\Gamma^{MNP}_{AB} + 6\delta^{[M}_A\Gamma^{NP]}_B - 6\delta^{[M}_A\delta^N_B\Gamma^{P]}) \\ &= \left(R^{MN} - \frac{1}{2}Rg^{MN}\right)\Gamma_N\,.\end{aligned} \tag{4.13}$$

The commutator term on the last line in (4.11) is computed in the appendix as

$$\Gamma^{NMP}[D_M, -\Gamma_P G + 3G\Gamma_P] = 6(-\mathrm{d}x^N \wedge \mathrm{d}G + \iota^N * \mathrm{d} * G)_{/}\,. \tag{4.14}$$

Finally we also show in appendix B.3 that Q^N evaluates precisely to the quadratic terms in the equations of motion of the metric and G, leading to (4.8).

4.2 Positivity in the absence of branes

We are now in a position to present our positivity argument, following the four-dimensional case reviewed in section 3. In this subsection we make the simplifying assumption that no M2 or M5 branes are present or can be nucleated. The inclusion of branes will be discussed in the following subsection.

In the absence of branes, the equations of motion and Bianchi identities of the supergravity fields are

$$\mathcal{E}_{MN} = 0\,, \qquad \mathrm{d}G = 0\,, \qquad \mathrm{d} * G + \frac{1}{2}G \wedge G = 0\,. \tag{4.15}$$

Hence, the identity (4.8) reduces on-shell to

$$\nabla_M E^{MN} \overset{\text{on-shell}}{=} \overline{\mathcal{D}_M\epsilon}\Gamma^{MPN}\mathcal{D}_P\epsilon \qquad (\text{no branes})\,, \tag{4.16}$$

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and the BPS energy (4.7) becomes

$$I_0(\epsilon) = \int_S \overline{\mathcal{D}_M\epsilon}\Gamma^{MN}{}_P\mathcal{D}_N\epsilon\, n^P \mathrm{vol}_S\,, \tag{4.17}$$

where vol_S is the volume form induced on S and $n = n^M\partial_M$ is a future-pointing unit vector orthogonal to S. We can now proceed as in four dimensions, by using an adapted vielbein

$$e^A = (e^{\underline{0}}, e^a) \quad \text{with} \quad e^{\underline{0}}|_S = 0\,, \tag{4.18}$$

and the dual frame $e_A = (e_{\underline{0}}, e_a)$ with $e_{\underline{0}} = n$. Now (4.17) can be rewritten as

$$I_0(\epsilon) = \int_S \mathrm{vol}_S\, \left[(\mathcal{D}^a\epsilon)^\dagger(\mathcal{D}_a\epsilon) - |\Gamma^a\mathcal{D}_a\epsilon|^2\right]\,. \tag{4.19}$$

By choosing an ϵ that satisfies the M-theory version of the Witten condition

$$\Gamma^a\mathcal{D}_a\epsilon = 0\,, \tag{4.20}$$

we see that the BPS energy is manifestly non-negative: $I_0(\epsilon) \geq 0$. Furthermore, $I_0(\epsilon) = 0$ if and only if $\mathcal{D}_a\epsilon = 0$. Since S can be freely deformed, this condition actually requires that $\mathcal{D}_M\epsilon = 0$. Hence, the BPS energy is vanishing if and only if ϵ satisfies the supersymmetry equations.

Note that, as in the four-dimensional case, we can again associate (4.20) with a transverse gauge-fixing condition $\Gamma^a\psi_a = 0$ for the gravitino. Furthermore, one can adapt the argument around (3.11) to argue that for any S the operator $\Gamma^a\mathcal{D}_a$ does not admit any normalizable zero mode ϵ_{zm}. Indeed, by combining (4.19) with $I_0(\epsilon_{\mathrm{zm}}) = 0$ and $\Gamma^a\mathcal{D}_a\epsilon_{\mathrm{zm}} = 0$ one gets $\mathcal{D}_a\epsilon_{\mathrm{zm}} = 0$. Since ϵ_{zm} must vanish as one approaches the boundary ∂S, we then expect that the conditions $\mathcal{D}_a\epsilon_{\mathrm{zm}} = 0$ impose that $\epsilon_{\mathrm{zm}} \equiv 0$, at least for reasonable asymptotic structures. This supports the expectation that (4.20) indeed admits a solution, although an explicit proof of this result would require a more detailed case by case investigation which is beyond the scope of the present paper.

4.3 Inclusion of M2 branes

We would now like to allow M2 or M5 branes to be present in the background configuration, or to be generated by possible fluctuations around it.

We first assume the presence of only M2-branes. This means that the Bianchi identity $\mathrm{d}G = 0$ is still satisfied, while the bulk equations of motion are modified. The bosonic action of an M2-brane is

$$S_{(\mathrm{M2})} = -2\pi\int_{\mathcal{C}} \mathrm{d}^3\sigma\sqrt{-h} + 2\pi\int_{\mathcal{C}} C\,, \tag{4.21}$$

where $h_{\alpha\beta} \equiv g_{MN}\partial_\alpha X^M\partial_\beta X^N$ is the metric induced on the world-volume $\mathcal{C}$ by the embedding $\sigma^\alpha \mapsto X^M(\sigma)$. The WZ term in (4.21) provides a localized source term to the C equations of motion:

$$\mathrm{d} * G + \frac{1}{2}G\wedge G = \delta^{(8)}(\mathcal{C})\,, \tag{4.22}$$

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where $\delta^{(8)}(\mathcal{C})$ is a delta-like eight-form such that $\int_{\mathcal{C}} C \equiv \int C\wedge\delta^{(8)}(\mathcal{C})$ for any three-form C.[5] Analogously, the Nambu-Goto term of (4.21) enters the Einstein equations by a localized source term $T^{MN}_{(\mathrm{M2})}$, such that:

$$\mathcal{E}^{MN} = \frac{1}{2} T^{MN}_{(\mathrm{M2})}\,, \tag{4.23}$$

where $\mathcal{E}^{MN}$ is defined in (4.10). In order to describe $T^{MN}_{(\mathrm{M2})}$, it is convenient to pick an adapted vielbein

$$e^A = (e^{\underline{\alpha}}, e^{\tilde{a}}) \qquad \text{with } e^{\tilde{a}}|_{\mathcal{C}} = 0\,, \tag{4.24}$$

and the corresponding dual frame $e_A = e_A^M\partial_M = (e_{\underline{\alpha}}, e_{\tilde{a}})$, with $A = 0,\dots,10$, $\alpha = 0,1,2$ and $\tilde{a} = 3,\dots,8$. Then one can verify that

$$T^{MN}_{(\mathrm{M2})} = \eta^{\underline{\alpha\beta}} e^M_{\underline{\alpha}} e^N_{\underline{\beta}} * [e^{\underline{012}}\wedge\delta^{(8)}(\mathcal{C})]\,, \tag{4.25}$$

see appendix C. Note that the right-hand side of (4.22) and (4.23) can be immediately generalized to multiple M2-branes by summing over the corresponding contributions. Hence, we can focus on a single M2-brane without loss of generality.

By using (4.22) and (4.23) (together with $\mathrm{d}G = 0$) in the identity (4.8), we now get

$$\begin{aligned}\nabla_M E^{MN} &\overset{\text{on-shell}}{=} \overline{\mathcal{D}_M\epsilon}\Gamma^{MPN}\mathcal{D}_P\epsilon + \frac{1}{4}T^{NP}_{(\mathrm{M2})}K_P + \frac{1}{4}\bar{\epsilon}[\mathrm{d}x^N\wedge\delta^{(8)}(\mathcal{C})]\epsilon \\ &= \overline{\mathcal{D}_M\epsilon}\Gamma^{MPN}\mathcal{D}_P\epsilon + \frac{1}{4}T^{NP}_{(\mathrm{M2})}K_P + \frac{1}{4}\big(*[\mathrm{d}x^N\wedge\delta^{(8)}(\mathcal{C})]\big)\cdot\Omega^{(\mathrm{M2})}\,,\end{aligned} \tag{4.26}$$

where we have introduced the two-form

$$\Omega^{(\mathrm{M2})} \equiv \bar{\epsilon}\Gamma_{(2)}\epsilon = \frac{1}{2}\bar{\epsilon}\Gamma_{MN}\epsilon\,\mathrm{d}x^M\wedge\mathrm{d}x^N\,. \tag{4.27}$$

By using (4.26) inside (4.7) we find that the total BPS energy can be split into bulk and brane contributions:

$$I(\epsilon) = I_0(\epsilon) + I_{(\mathrm{M2})}(\epsilon)\,. \tag{4.28}$$

Here $I_0(\epsilon)$ is defined in (4.17) and represents the bulk contribution. On the other hand, the brane contribution

$$I_{(\mathrm{M2})}(\epsilon) = \frac{1}{4}\int_{\mathcal{C}\cap S}\left(K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap\Sigma} - \Omega^{(\mathrm{M2})}\right) \tag{4.29}$$

comes from the last two terms in (4.26) — see appendix C for more details. In (4.29) $K^{\underline{0}} = K^M e^{\underline{0}}_M$ is the zeroth component of K in the local frame (4.18), that is, $K^{\underline{0}}$ is the component of K normal to S.

In appendix C we prove, following [31], that the integrated forms in (4.29) satisfy the algebraic bound

$$K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap S} \geq \Omega^{(\mathrm{M2})}|_{\mathcal{C}\cap S}\,, \tag{4.30}$$

[5]More generically, for any q-dimensional submanifold Σ of a d-dimensional space, $\delta^{(d-q)}(\Sigma)$ is defined by $\int_\Sigma \omega = \int \omega\wedge\delta^{(d-q)}(\Sigma)$ for any q-form ω.

which can be regarded as a local BPS bound.[6] Hence we can conclude that

$$I_{(\mathrm{M2})}(\epsilon) \geq 0 \tag{4.31}$$

and then the inclusion of fully backreacting M2-branes preserves the positivity of BPS energy:

$$I(\epsilon) \geq 0\,. \tag{4.32}$$

Furthermore, remembering the conclusion of subsection 4.2, $I(\epsilon) = 0$ if and only if ϵ satisfies the supersymmetry equations and the bound (4.30) is saturated:

$$K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap S} = \Omega^{(\mathrm{M2})}|_{\mathcal{C}\cap S}\,. \tag{4.33}$$

As explained in appendix C, this condition is equivalent to requiring that the M2-brane does not break the supersymmetry generated by ϵ.

As in [31], the local BPS bound (4.30) provides a generalization to non-static settings of the ordinary calibration bounds [62], which instead apply only to static backgrounds. These local bounds are usually exploited to study the energetics of *probe* branes on fixed supersymmetric vacua. Our result (4.28) makes it clear how, even after the brane backreaction is taken into account and more general on-shell bulk deformations are allowed, the combination entering the local bound (4.30) still determines the M2 contribution $I_{(\mathrm{M2})}(\epsilon)$ to the positive BPS energy $I(\epsilon)$. Furthermore, we explicitly see how the inclusion of M2-branes cannot be represented just by their energy-momentum tensor (4.25) (which satisfies the dominant energy condition), since this would not take into account the second (possibly negative) contribution in (4.29). We will explicitly show how these observations also hold for other branes in string/M-theory, and we indeed expect them to be universal properties of supergravity models including charged branes.

4.4 Inclusion of M5-branes

Conceptually, the inclusion of M5-branes should proceed as for the M2 case. In particular, we expect an M5 to contribute to the BPS energy with a non-negative contribution $I_{(\mathrm{M5})}(\epsilon)$ analogous to (4.29). This contribution may be computed along the lines of what we did for M2-branes, starting from the M5 action of [63]. One technical difficulty comes from the presence of the self-dual three-form on the M5-brane, which induces an M2-charge on the M5.

In order to alleviate the presentation, we start by simply ignoring the self-dual three-form contribution. Then the bosonic action associated with an M5-brane wrapping a six-dimensional submanifold $\mathcal{C}$ reduces to

$$S_{(\mathrm{M5})} = -2\pi \int_{\mathcal{C}} \mathrm{d}^6\sigma\sqrt{-h} + 2\pi\int_{\mathcal{C}} \tilde{C} + \ldots \tag{4.34}$$

[6] The bound (4.30) involves top-forms on $\mathcal{C}\cap S$ and should more precisely be read as a bound for the corresponding coefficients with respect to a given reference top-form, fixing the overall sign ambiguity by requiring that $\mathrm{vol}_{\mathcal{C}\cap S}$ is positive.

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where $\tilde{C}$ is the dual six-form potential, locally defined by $\mathrm{d}\tilde{C} = -\left(*G + \frac{1}{2}C \wedge G\right)$, and the ellipsis refers to the omitted terms depending on the world-volume self-dual 3-form. The WZ term contributes to the Bianchi identity for G, which becomes:

$$\mathrm{d}G = \delta^{(5)}(\mathcal{C})\,. \tag{4.35}$$

Also the Einstein equations get modified, by a contribution similar to the M2-brane one appearing on the right-hand side of (4.23). By repeating the same steps followed for the M2 case, we arrive at the following M5 contribution to BPS energy:

$$I_{(\mathrm{M5})}(\epsilon) = \frac{1}{4}\int_{\mathcal{C}\cap\Sigma}\left(K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap\Sigma} - \Omega^{(\mathrm{M5})}\right)\,, \tag{4.36}$$

where

$$\Omega^{(\mathrm{M5})} \equiv \bar{\epsilon}\,\Gamma_{(5)}\epsilon\,. \tag{4.37}$$

We have again a calibration bound

$$K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap\Sigma} \geq \Omega^{(\mathrm{M5})}|_{\mathcal{C}\cap\Sigma}\,, \tag{4.38}$$

which is saturated by the supersymmetric configuration. This implies the positivity bound

$$I_{(\mathrm{M5})}(\epsilon) \geq 0\,, \tag{4.39}$$

for the M5 contribution to the BPS energy.

Coming back to the self-dual 3-form, it induces an M2-charge on the M5-brane. Hence, it affects the C equations of motion and contributes to the Einstein equations by further terms localized on the M5-brane. These can be straightforwardly computed starting from the action of [63], but we will not do this exercise here, since in section 5.4 we will do it for D-branes supporting general world-volume fluxes. Since M5-branes and D-branes are related by dualities, the results of section 5.4 are sufficient to conclude that (4.39) indeed still holds once the M5 self-dual three-form flux is taken into account, and is saturated only by supersymmetric configurations.

In summary, the total BPS energy (4.5) can be written as the sum of three non-negative terms

$$I(\epsilon) = I_0(\epsilon) + I_{(\mathrm{M2})}(\epsilon) + I_{(\mathrm{M5})}(\epsilon)\,, \tag{4.40}$$

where $I_0(\epsilon)$ becomes manifestly non-negative if we impose the Witten condition (4.20). Furthermore $I(\epsilon) = 0$ if and only if the three terms separately vanish. This can happen only if the complete configuration is supersymmetric, that is, if ϵ is a bulk supercharge and the branes preserve it.

4.5 Positivity and stability of G_2-compactifications

We can illustrate the above general results by discussing energy positivity and stability in M-theory compactifications on special holonomy spaces. As we have mentioned, this stability is widely expected, but checking it is a good exercise towards attacking vacua without supersymmetry. The energy positivity associated with these kinds of configurations has been already discussed in [35, 36], which however focused on the gravitational sector (plus a possible energy-momentum tensor satisfying the dominant energy condition). Our results allow us to include the effects of the G flux and of the M2/M5 branes as well.

For concreteness we focus on vacua of the form $M_0 = \mathbb{M}_4 \times Y$, where $\mathbb{M}_4$ is four-dimensional Minkowski space and Y admits a given G_2-holonomy metric. (Compactifications to $\mathbb{M}_{d\neq 4}$, or on Y with special holonomy $\neq G_2$, are analogous.) In these purely geometrical vacua, the eleven-dimensional metric is a direct sum $\mathrm{d}s^2_0 = \mathrm{d}s^2_{\mathbb{M}_4} + \mathrm{d}s^2_Y$, and preserves a supercharge

$$\epsilon_0 = \varepsilon_0 \otimes \eta_0 \tag{4.41}$$

where ε_0 is a constant four-dimensional Majorana spinor, while η_0 is the covariantly constant Majorana spinor along the G_2-manifold Y, which we choose to have unit norm $\eta_0^\dagger \eta_0 = 1$. We are interested in general deformations which asymptotically reduce to the $M_0 = \mathbb{M}_4 \times Y$ vacuum. On M_0 we can introduce some adapted coordinates (x^μ, y^m), where y^m are coordinates along Y and $x^\mu = (t, r, \theta, \phi)$ are standard spherical coordinates over $\mathbb{M}_4$. Asymptotically, we can use these coordinates also on the deformed backgrounds and choose the nine-dimensional space-like surface S to take the form $\Sigma \times Y$, where Σ is an asymptotically flat three-dimensional surface of constant time t. Correspondingly, the nine-dimensional boundary ∂S appearing in the BPS energy (4.5) takes the form $\partial S = \partial\Sigma \times Y$, where $\partial\Sigma$ is the two-sphere S^2_r of constant radius r, in the limit $r \to \infty$.

We impose boundary conditions where the eleven-dimensional metric and spinor of the deformed background are asymptotically $g = g_0 + \Delta g$ and $\epsilon = \epsilon_0 + \Delta\epsilon$, with Δg and $\Delta\epsilon$ of order $\mathcal{O}(r^{-1})$ and such that $\mathcal{D}^0_M \epsilon \equiv \nabla_M \Delta\epsilon \sim \mathcal{O}(r^{-2})$ (in the natural asymptotically flat frame). In order to guarantee that (4.5) is finite, we also assume that the variation of the spin-connection $\Delta\omega^{AB} \equiv \Delta\omega_M{}^{AB}\mathrm{d}x^M$ and of the field-strength $\Delta G \equiv G$ are at least of order $\mathcal{O}(r^{-2})$. From the identity

$$* \, E_2 = -\bar{\epsilon}\Gamma_{(8)} \wedge \mathcal{D}\epsilon \,, \tag{4.42}$$

with $\mathcal{D}\epsilon \equiv \mathcal{D}_M\epsilon \, \mathrm{d}x^M$, we see that the integrand appearing in (4.5) reduces asymptotically to

$$- \bar{\epsilon}_0 \Gamma_{0(8)} \wedge (\mathcal{D} - \mathcal{D}_0)\epsilon_0 + \mathrm{d}(\ldots) + \mathcal{O}(r^{-3}) \,. \tag{4.43}$$

It follows that $I(\epsilon)$ can be written in the following form — see also ([60], eq. (5.18)):

$$\begin{aligned} I(\epsilon) =& \frac{1}{4}\int_{\partial\Sigma\times Y} *(e^A \wedge e^B \wedge K^\flat) \wedge \Delta\omega_{AB} \\ &+ \frac{1}{4}\int_{\partial\Sigma\times Y} \left(\Omega^{(\mathrm{M2})} \wedge *\Delta G + \Omega^{(\mathrm{M5})} \wedge \Delta G \, - 2 * \Omega^{(\mathrm{M5})} \wedge \Delta\omega\right) \,, \end{aligned} \tag{4.44}$$

where we have introduced the one-form $K^\flat \equiv K_M \mathrm{d}x^M$, obtained by lowering the index of (4.9), and the three-form $\Delta\omega \equiv \frac{1}{2}\Delta\omega_{AB} \wedge e^A \wedge e^B$, while $\Omega^{(\mathrm{M2})}$ and $\Omega^{(\mathrm{M5})}$ are defined in (4.27) and (4.37) respectively. These can be expressed in terms of the associative form ϕ and the co-associative form $*_7\phi$ on the G_2 manifold as:

$$K^\flat = v, \quad \Omega^{(\mathrm{M2})} = w, \quad \Omega^{(\mathrm{M5})} = v \wedge *_7\phi + *_4 w \wedge \phi, \quad *\Omega^{(\mathrm{M5})} = w \wedge *_7\phi + *_4 v \wedge \phi, \tag{4.45}$$

where $v_\mu \equiv \bar{\varepsilon}\gamma_\mu\varepsilon$, $w_{\mu\nu} \equiv \bar{\varepsilon}\gamma_{\mu\nu}\varepsilon$, and ε is a four dimensional Majorana spinor.

Note that in (4.44), the forms $K^\flat$, $\Omega^{(\mathrm{M2})}$ and $\Omega^{(\mathrm{M5})}$, as well as the Hodge-star operator, can in fact be computed by using the supersymmetric vacuum metric g_0 and supercharge ϵ_0. We can then identify the first two terms appearing in the second line of (4.44) as measuring the asymptotic central charges associated to M2-branes stretching along $\mathbb{M}_4$ and M5-branes wrapping internal three- and four-cycles. Since we want to consider fluctuations around the vacuum configurations, we can further restrict the boundary conditions in such a way that these terms vanish.

Similarly, the three-form $\Delta\omega$ may be interpreted as measuring the flux of a KK6-brane [59], and $*\Omega^{(\mathrm{M5})}$ can be considered as a corresponding calibration — see section 5 of [60]. Then the last term of (4.44) may be interpreted as measuring the central charge of a KK6-branes wrapping internal four-cycles and does not contribute if we assume vacuum boundary conditions.

On the other hand, the first term on the right-hand side of (4.44) can be interpreted as defining the conserved component $-K^M P_M$ of the total momentum of the system [30, 59]. By using the fact that $K^M = (k_0^\mu, 0)$, we see that

$$I(\epsilon) = I(\varepsilon_0) = -k_0^\mu P_\mu \tag{4.46}$$

as in (3.6). Our previous discussion on the positivity of $I(\epsilon)$ then translates into a fully eleven-dimensional proof of the positivity of the energy of this class of compactifications.

Moreover, if $I(\epsilon) = 0$ for any ε_0, spacetime coincides with the vacuum. To see this, first observe that, just as in the pure four-dimensional case, the combination of (4.19), (4.20) and $I(\epsilon) = 0$ imply (when applied to all possible Σ) that there are four independent spinors ϵ such that $\mathcal{D}_M\epsilon = 0$. This in turn implies that there are four independent Killing vectors $\bar{\epsilon}\Gamma^M\epsilon$. These coincide at infinity with the four translations of Minkowski space; so their orbits need to be four-dimensional. Since the finite transformation generated by a Killing vector is an isometry, the metric is invariant along these four-dimensional orbits; so on the union of these orbits the metric is in fact equal to its value at infinity $\mathbb{M}_4 \times Y$.

We now use the usual logic: since $I(\epsilon)$ does not depend on Σ, it is conserved; but since it vanishes for any ε_0 only on the vacuum, the latter is fully stable. As usual, this conclusion assumes the existence of a solution of the Witten condition (4.20). In the absence of fluxes and branes, this has been rigorously proven in [35, 36]. As at the end of subsections 3.1 and 4.2, one could argue that $\Gamma^a\mathcal{D}_a$ does not have any normalizable zero-mode ϵ_{zm}.

Finally, note that we may in fact relax the above restrictions on the asymptotic behaviour of G and $*G$ and allow for possible non-trivial asymptotic fluxes thereof. In this case the positivity of (4.44) would correspond to an extremality bound as in [61, 64], which would include the contribution of four-dimensional string and membrane central charges.

4.6 Stability of AdS_4 vacua

Again in the spirit of gaining experience that might be later useful for supersymmetry-breaking vacua, we can adapt the discussion of the previous subsection to M-theory compactifications down to AdS_d. For $d = 4$, the vast majority of such supersymmetric vacua are of the famous Freund-Rubin (FR) type:

$$\mathrm{d}s_0^2 = \mathrm{d}s^2_{\mathrm{AdS}_4} + 4L^2\mathrm{d}s^2_Y\,, \qquad G_4 = \frac{3}{L}\mathrm{vol}_{\mathrm{AdS}_4}\,. \tag{4.47}$$

The AdS radius is $L^6 = \frac{N}{3\cdot 2^7\mathrm{Vol}(Y)}$, where N is a positive integer; the internal space Y admits an internal Killing spinor η, satisfying $(D_m - \frac{\mathrm{i}}{2}m\gamma_m)\eta = 0$; these include weak G_2 ($\mathcal{N}=1$), Sasaki-Einstein manifolds ($\mathcal{N}=2$), 3-Sasaki ($\mathcal{N}=3$). The total supercharge is then $\epsilon_0 = \hat{\varepsilon}\otimes\eta$, where $\hat{\varepsilon}(x) = S(x)\varepsilon_0$ is one of AdS Killing spinors mentioned after (3.13). We will modify these solutions in section 6.2 to break supersymmetry.

In order to address the stability issue, one again needs to specify appropriate boundary conditions. The discussion in sections 3.2 and 3.3 suggests that we should require the metric at infinity to factorize as $\mathrm{d}s_4^2 + \mathrm{d}s_7^2$, with $\rho^{-2}\mathrm{d}s_4^2$ inducing on the boundary a metric conformal to $\mathbb{R}\times S^2$, and $\mathrm{d}s_7^2 = 4L^2\mathrm{d}s_Y^2 + \mathcal{O}(\rho^{-2})$. Moreover, the spinors ϵ are such that $K^M = \bar{\epsilon}\Gamma^M\epsilon$ is asymptotically $(k_0^\mu = \bar{\varepsilon}\gamma^\mu\varepsilon, 0)$. These boundary conditions should be confirmed by uplifting the analysis in [32, 34, 55] to the case with extra dimensions. In particular one should check that the BPS energy is still conserved with this choice, and that the Witten condition can be imposed; we hope to return to this in the future.

The logic then proceeds along lines that are by now familiar. We choose a space-like surface S that asymptotically takes the form $\Sigma\times Y$, with Σ being a space-like three-dimensional slice with boundary $\partial\Sigma\simeq S^2$. The BPS energy (4.5) is defined by an integral over $\partial S = \partial\Sigma\times Y \simeq S^2\times Y$. We may also apply (4.44) and provide a more explicit geometrical formula for $I(\epsilon)$, but the supersymmetry argument which led from (4.1) to (4.5) anyway guarantees that $I(\epsilon) = I(\varepsilon_0)$ must take the form (3.14) in terms of the $\mathrm{SO}(2,3)$ charges J_{AB}. As in subsection 3.2, this implies the positivity of $E\equiv J_{04}\geq 0$. A version of the argument in the previous subsection shows that the only zero-energy configuration is the vacuum $\mathrm{AdS}_4\times Y$. As usual, if the energy is conserved it now follows that such a vacuum is stable.[7]

5 Positivity and stability in type II

In this section we will show that the main points of the above M-theory discussion hold also for ten-dimensional type II string theories, up to some technical but interesting details.

The type II fields are: the string frame metric g_{MN}; a two-form potential B with closed three-form field-strength $H = \mathrm{d}B$; RR potentials C_k with field-strengths $F_k = \mathrm{d}C_k - H\wedge C_{k-2}$; two Majorana-Weyl (MW) gravitinos ψ^a_M; two MW dilatinos λ^a. It will be convenient to use the *democratic* formalism, where all RR field-strengths F_k are included for all $0\leq k\leq 10$, even in IIA and odd in IIB, and the formal total sum $F\equiv\sum_k F_k$ satisfies

$$F = *\lambda F\,. \tag{5.1}$$

Here λ acts on a form by reversing the order its indices, i.e., $\lambda F_k = (-)^{\frac{k(k-1)}{2}}F_k$, and the Hodge-star is computed in the string frame, which is more natural for the democratic

[7] It was found in [65] that small perturbations in AdS can refocus until they collapse into a black hole; this is a different notion of stability from the one of the vacuum we consider in this paper.

formulation. The theory is defined by a pseudo-action [66], whose equations of motion must be supplemented by the self-duality conditions (5.1).

The supersymmetry parameters are two MW spinors ϵ_a, $a = 1, 2$. It is convenient to collect all the spinors into doublets: $\psi_M \equiv \binom{\psi_{1M}}{\psi_{2M}}$, and so on. The supersymmetry transformations of the fermions, with fermions set to zero, now read

$$\delta_\epsilon \psi_M = \mathcal{D}_M \epsilon\,, \qquad \delta_\epsilon \lambda = \mathcal{O}\epsilon\,, \tag{5.2}$$

where[8]

$$\begin{aligned} \mathcal{D}_M &\equiv D_M \otimes \mathbf{1}_2 - \frac{1}{4} H_M \otimes \sigma_3 + \mathcal{F}\,\Gamma_M\,, \\ \mathcal{O} &\equiv \mathrm{d}\phi \otimes \mathbf{1}_2 - \frac{1}{2} H \otimes \sigma_3 + \Gamma^M \mathcal{F}\,\Gamma_M\,, \\ \text{with} \quad \mathcal{F} &\equiv \frac{\mathrm{e}^\phi}{16} \begin{pmatrix} 0 & F \\ \pm\lambda(F) & 0 \end{pmatrix} \ \text{in} \ \ {}^{\text{IIA}}_{\text{IIB}}\,. \end{aligned} \tag{5.3}$$

Here and in what follows, the upper/lower sign will refer to IIA/IIB. We work in string units $2\pi\sqrt{\alpha'} = 1$.

5.1 BPS energy

In order to identify the BPS energy, we will start from the supercharge generator, following the same strategy used in M-theory. The explicit form of the supercharge can be obtained by various means. For instance, it can be identified in IIA by dimensional reduction of the M-theory supercharge (4.1) — see appendix E — and then extrapolated to IIB. The result is

$$Q(\epsilon) = \int_{\partial S} \mathrm{e}^{-2\phi} \left(\mathrm{d}x^M \wedge \bar\epsilon\,\Gamma\Gamma_{(7)}\psi_M + \bar\epsilon\,\Gamma\Gamma_{(8)}\lambda \right)\,, \tag{5.4}$$

again in the notation of (4.2). In (5.4), S is a nine-dimensional spacelike surface and we have introduced the chiral operator $\Gamma \equiv \Gamma^{\underline{01\ldots 9}}$.[9]

We can then compute $\delta_\epsilon Q(\epsilon) = \{Q(\epsilon), Q(\epsilon)\}$ by using (5.2), and obtain in this way the BPS energy:

$$I(\epsilon) = \int_{\partial S} * E_2 \tag{5.5}$$

where we have introduced the Nester-like two form

$$E_2 \equiv -\frac{1}{2} \mathrm{e}^{-2\phi} \bar\epsilon (\Gamma_{MN}{}^P \mathcal{D}_P - \Gamma_{MN}\mathcal{O})\epsilon\, \mathrm{d}x^M \wedge \mathrm{d}x^N\,. \tag{5.6}$$

More explicitly, we can write the components of $E_2 = \frac{1}{2} E_{MN} \mathrm{d}x^M \wedge \mathrm{d}x^N$ as follows:

$$E_{MN} = -\mathrm{e}^{-2\phi} \bar\epsilon\, \Gamma_{MN}{}^P (D_P + \mathcal{A}_P)\epsilon\,, \tag{5.7}$$

[8]Our conventions are as in [60]. They differ from those in [67] by a sign change $H \to -H$.

[9]The supercharge (5.4) admits a simpler form in terms of the Einstein frame gravitino. Denoting the Einstein frame quantities with a hat symbol $\hat{}$, they are related to those in the string frame by $g_{MN} = \mathrm{e}^{\phi/2}\hat g_{MN}$, $\epsilon = \mathrm{e}^{\phi/8}\hat\epsilon$, $\lambda = \mathrm{e}^{-\phi/8}\hat\lambda$, $\psi_M = \mathrm{e}^{\phi/8}\left(\hat\psi_M + \frac{1}{8}\hat\Gamma_M\hat\lambda\right)$. Then (5.4) takes the form $Q(\hat\epsilon) = \int_{\partial S} \mathrm{d}x^M \wedge \bar{\hat\epsilon}\,\Gamma\hat\Gamma_{(7)}\hat\psi_M$.

where D_M is the ordinary spinor covariant derivative and $\mathcal{A}_M$ is defined by the relation

$$\mathcal{D}_M - \frac{1}{8}\Gamma_M\mathcal{O} = D_M + \mathcal{A}_M\,. \tag{5.8}$$

In the following we will also need the identity

$$\Gamma^P{}_{MN}\mathcal{A}_P = \frac{1}{4}\Gamma_{[M}H\Gamma_{N]}\otimes\sigma_3 - \Gamma_{MN}\mathrm{d}\phi - 2\Gamma_{[M}\mathcal{F}\Gamma_{N]}\,. \tag{5.9}$$

This identification can be obtained from (5.7) by using (A.1) with $k=2$ and $H_M = \frac{1}{2}\{\Gamma_M, H\}$.

5.2 Main identity

Once again we need to compute the divergence of E_2. We begin with

$$\begin{aligned}\mathrm{e}^{2\phi}\nabla_M E^{MN} &= -2\partial_M\phi E^{MN} - \overline{D_M\epsilon}\,\Gamma^{MNP}\mathcal{A}_P\epsilon - \bar\epsilon\, D_M(\Gamma^{MNP}\mathcal{A}_P\epsilon) \\ &= -\overline{D_M\epsilon}\Gamma^{MNP}(D_P\epsilon + 2\mathcal{A}_P\epsilon) - \bar\epsilon\,\Gamma^{MNP}(D_M D_P - 2\partial_M\phi\mathcal{A}_P + [D_M, \Gamma^{MNP}\mathcal{A}_P])\epsilon\,.\end{aligned} \tag{5.10}$$

We have skipped a few steps similar to (4.11), and we have used (A.14) for $k=2$ to derive $\bar\epsilon\,\Gamma^{MNP}\mathcal{A}_P D_M\epsilon = \overline{D_M\epsilon}\Gamma^{MNP}(\mathcal{A}_P + 2\partial_P\phi)\epsilon$.

The term $\bar\epsilon\,(\cdots)D_M\epsilon$ now does not vanish automatically (unlike A^{NP} in (4.12)). But by (5.8) we expect the appearance of the square of this operator. The remainder looks at first quite complicated, but a little experimentation shows that it simplifies if we subtract a further $\overline{\mathcal{O}\epsilon}\Gamma^N\mathcal{O}\epsilon$. In order to write the final result in the most transparent way, we first use the spinor ϵ to construct the future-pointing time-like or null vector

$$K \equiv \frac{1}{2}\,\bar\epsilon\,\Gamma^M\epsilon\,\partial_M \tag{5.11}$$

and the (poly)forms:

$$\Omega^{(\mathrm{F1})} \equiv \frac{1}{2}\,\bar\epsilon\,\Gamma_{(1)}\otimes\sigma_3\epsilon\,, \qquad \Omega^{(\mathrm{NS5})} \equiv \frac{1}{2}\mathrm{e}^{-2\phi}\,\bar\epsilon\,\Gamma_{(5)}\otimes\sigma_3\epsilon\,, \tag{5.12a}$$

$$\Omega^{(\mathrm{D})} = \sum_{k\ \text{even/odd}}\Omega^{(\mathrm{D})}_k \equiv \sum_k \mathrm{e}^{-\phi}\,\bar\epsilon_1\Gamma_{(k)}\epsilon_2\,. \tag{5.12b}$$

Then, with some further computations presented in appendix D, we are able to show that

$$\begin{aligned}\nabla_M E^{MN} = {}&\mathrm{e}^{-2\phi}\overline{\left(\mathcal{D}_M - \frac{1}{8}\Gamma_M\mathcal{O}\right)\epsilon}\,\Gamma^{MPN}\left(\mathcal{D}_P - \frac{1}{8}\Gamma_P\mathcal{O}\right)\epsilon - \frac{1}{8}\mathrm{e}^{-2\phi}\overline{\mathcal{O}\epsilon}\,\Gamma^N\mathcal{O}\epsilon \\ &+ \mathcal{E}^{NP}K_P + \frac{1}{2}\mathcal{H}^{NP}\Omega^{(\mathrm{F1})}_P - \frac{1}{2}(\mathrm{d}H\wedge\mathrm{d}x^N)\cdot\Omega^{(\mathrm{NS5})} + \frac{1}{2}(\mathrm{d}_H F\wedge\mathrm{d}x^N)\cdot\Omega^{(\mathrm{D})}\,.\end{aligned} \tag{5.13}$$

Here $\mathrm{d}_H \equiv \mathrm{d} - H\wedge$ and the tensors $\mathcal{E}_{MN}$ and $\mathcal{H}_{MN}$ are defined by

$$\mathrm{e}^{2\phi}\mathcal{E}_{MN} \equiv R_{MN} - \frac{1}{2}g_{MN}R + 2[(\nabla_M\nabla_N - g_{MN}\nabla^2)\phi + g_{MN}|\mathrm{d}\phi|^2] - \frac{1}{2}T^{(HF)}_{MN}\,, \tag{5.14a}$$

$$\mathcal{H}_2 = \frac{1}{2}\mathcal{H}_{MN}\mathrm{d}x^M\wedge\mathrm{d}x^N \equiv *\left[\mathrm{d}(\mathrm{e}^{-2\phi}*H) - \frac{1}{2}(F\wedge\lambda F)_8\right]\,, \tag{5.14b}$$

where $T^{(HF)}_{MN} \equiv H_M \cdot H_N - \frac{1}{2} g_{MN}|H|^2 + \frac{1}{2}\mathrm{e}^{2\phi} F_M \cdot F_N$. Here we are using a notation similar to that in (4.10), e.g. $H_M \equiv \iota_M H$ and $F_M \equiv \iota_M F$. Furthermore $(\cdots)_8$ denotes taking the eight-form part of the polyform.

We emphasize that (5.13) is valid for both type II theories and holds identically for any (possibly off-shell) configuration.

5.3 Positivity in the absence of branes

As in section 4.2, we first consider the positivity of the BPS energy (5.5) in the absence of branes (and orientifolds). In this case the Einstein and B-field equations of motion require that $\mathcal{E}_{MN} = 0$ and $\mathcal{H}_{MN} = 0$, respectively. Furthermore the B-field Bianchi identity is $\mathrm{d}H = 0$ and the democratic RR equations of motion/Bianchi identities read $\mathrm{d}_H F = 0$.[10] Then only the first line in (5.13) survives; the second vanishes by the theory's equations of motion and Bianchi identities.

If we again introduce an adapted vielbein $e^A = (e^{\underline{0}}, e^a)$ such that $e^{\underline{0}}|_S = 0$ and use Stokes' theorem, we can write (5.5) as $I_0(\epsilon) = \int_S \mathrm{d} * E_2 = \int_S \mathrm{vol}_S \nabla_M E^{\underline{0}M}$. By using (5.13) we then get the following on-shell BPS energy:

$$I_0(\epsilon) = \int_S \mathrm{vol}_S \left[\left(\mathcal{D}^a \epsilon - \frac{1}{8}\Gamma^a \mathcal{O}\epsilon \right)^\dagger \left(\mathcal{D}_a \epsilon - \frac{1}{8}\Gamma_a \mathcal{O}\epsilon \right) + \frac{1}{8}(\mathcal{O}\epsilon)^\dagger \mathcal{O}\epsilon - \left| \left(\Gamma^a \mathcal{D}_a - \frac{9}{8}\mathcal{O} \right) \epsilon \right|^2 \right] . \tag{5.15}$$

So, the BPS energy $I_0(\epsilon)$ can be made manifestly non-negative if we can choose an ϵ satisfying the modified Witten identity:

$$\left(\Gamma^a \mathcal{D}_a - \frac{9}{8}\mathcal{O} \right) \epsilon = 0 . \tag{5.16}$$

Hence, under this assumption we get $I_0(\epsilon) \geq 0$. Furthermore this bound is saturated if and only if the background is supersymmetric.

Note that, as in the previous cases, the condition (5.16) can be related to a gauge-fixing condition for the gravitino [5], which now reads $\Gamma^a \psi_a = \frac{9}{8}\lambda$. This does not look like a standard transverse gauge, but only because we are working in the string frame, in which the gravitino and dilatino kinetic terms are not diagonal. Indeed the condition $\Gamma^a \psi_a = \frac{9}{8}\lambda$ corresponds to the transverse gauge $\Gamma^a \hat{\psi}_a = 0$ in the Einstein frame — see footnote 9. Furthermore, one can argue for the absence of normalizable zero modes of the operator appearing in (5.16), and then for the existence of a solution of (5.16), by adapting the arguments proposed at the end of sections 3.1 and 4.2.

5.4 Inclusion of localized sources

As in the M-theory case, one can include branes, which can be part of the background configuration or can contribute to the fluctuations around it. As we will see, the presence of O-planes can be similarly taken into account.

[10] In order to get an on-shell configuration, these equations must be supplemented by the dilaton equation of motion $R - \frac{1}{2}|H|^2 - 4\mathrm{e}^{\phi}\nabla^2 \mathrm{e}^{\phi} = 0$.

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The simplest example is provided by F1-strings, which works very similarly to the M2-brane case of section 4.3. In particular, the presence of an F1-string along a two-dimensional world-sheet $\mathcal{C}$ modifies the B-field equations of motion to

$$\mathcal{H}_2 = *\delta^{(8)}(\mathcal{C})\,, \tag{5.17}$$

as well as the Einstein equations. The bottom line is that the first two terms in the second line of (5.13) do not vanish anymore, but provide the following localized contribution to $I(\epsilon) = \int_S \mathrm{d} * E_2 = \int_S \mathrm{vol}_S \nabla_M E^{\underline{0}M}$:

$$I_{(\mathrm{F1})}(\epsilon) = \frac{1}{2}\int_{\mathcal{C}\cap S}\left(K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap\Sigma} - \Omega^{(\mathrm{F1})}\right)\,. \tag{5.18}$$

One can adapt the calculation of appendix C.2 for M2-branes to rewrite the local bound found in ([31], eq. (3.25)) in the form

$$K^{\underline{0}}\mathrm{vol}_{\mathcal{C}\cap S} \geq \Omega^{(\mathrm{F1})}|_{\mathcal{C}\cap S}\,. \tag{5.19}$$

This shows that (5.18) is non-negative. Furthermore, (5.19) is saturated if and only if the F1-string does not break the supersymmetry generated by ϵ [31]. Hence, we get the bound $I(\epsilon) = I_0(\epsilon) + I_{(\mathrm{F1})}(\epsilon) \geq 0$ for the total BPS energy, which is saturated if and only if the complete configuration is supersymmetric. For comparison with the D-brane discussion below, we observe that we can rewrite (5.18) in the form

$$I_{(\mathrm{F1})}(\epsilon) = \frac{1}{2}\int_{\mathcal{C}\cap S}\left[V(K)\mathrm{vol}_{\mathcal{C}\cap S} - \Omega^{(\mathrm{F1})}\right], \tag{5.20}$$

where we have introduced the following one-form defined along $\mathcal{C}$:

$$V \equiv h^{\alpha\beta} g_{MN} e^{\underline{0}}_{\alpha}\partial_\beta X^M \mathrm{d}x^N\,, \ \text{section of } T^*M|_{\mathcal{C}}\,, \tag{5.21}$$

with M denoting the ten-dimensional spacetime. (Recall that, according to our general notation, $h_{\alpha\beta}$ denotes the induced metric on the brane.) This equivalence can be understood by picking a $\mathcal{C}$ adapted vielbein $e^A = (e^{\underline{\alpha}}, e^{\hat{a}})$, with $e^{\hat{a}}|_{\mathcal{C}} = 0$ (and still imposing $e^{\underline{0}}|_S = 0$).

As with the discussion for M5-branes in section 4.4, the incorporation of D-branes and NS5-branes is complicated by the presence of additional world-volume fluxes, but these branes are still expected to provide non-negative localized contributions $I_{(\mathrm{D})} \geq 0$ and $I_{(\mathrm{NS5})} \geq 0$ respectively. As a preliminary easier check, let us consider an NS5-brane along a submanifold $\mathcal{C}$ and neglect again its world-volume flux, as we did for M5-branes in section 4.4. The Einstein equations and the B-field Bianchi identity get modified — for instance, the latter becomes $\mathrm{d}H = \delta^{(4)}(\mathcal{C})$ — and the first and third term of the second line of (5.13) produce a non-negative contribution to the BPS energy of the form

$$I_{(\mathrm{NS5})}(\epsilon) = \frac{1}{2}\int_{\mathcal{C}\cap S}\left(e^{-2\phi}K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap S} - \Omega^{(\mathrm{NS5})}\right) \geq 0\,, \tag{5.22}$$

which vanishes if and only if the (fluxless) NS5-branes preserve the bulk supersymmetry generated by ϵ.

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In order to better understand the possible role of world-volume fluxes, we now discuss in more detail the presence of D-branes. Since they are related to NS5-branes and M5-branes by dualities, our conclusions will provide a convincing support to some of our previous claims.

Consider the inclusion of a Dp-brane. In the democratic formulation, it modifies the RR Bianchi identities/equations of motion as follows:

$$\mathrm{d}_H F = -\lambda \left[\delta^{(9-p)}(\mathcal{C})\right] \wedge \mathrm{e}^{-\mathcal{F}} \tag{5.23}$$

where $\mathcal{C}$ now denotes the Dp-brane world-volume and $\mathcal{F}$ is the gauge invariant world-volume field strength, such that $\mathrm{d}\mathcal{F} = -H|_{\mathcal{C}}$. Furthermore, by using the D-brane effective action

$$S_{\rm D} = -2\pi \int_{\mathcal{C}} \mathrm{e}^{-\phi}\sqrt{-\det \mathcal{M}} + 2\pi \int_{\mathcal{C}} C \wedge \mathrm{e}^{\mathcal{F}} \tag{5.24}$$

where $\mathcal{M}_{\alpha\beta} \equiv h_{\alpha\beta} + \mathcal{F}_{\alpha\beta}$ and $C \equiv \sum_k C_k$, one can check that the Einstein equations and the B-field equations of motion get modified into

$$\mathcal{E}^{MN} = \frac{1}{4} e^{-\phi}\sqrt{-\det \mathcal{M}_{\underline{\alpha\beta}}}\, \mathcal{M}^{(\underline{\alpha\beta})} e^M_{\underline{\alpha}} e^N_{\underline{\beta}} * [e^{\underline{0\dots p}} \wedge \delta^{(9-p)}(\mathcal{C})]\,, \tag{5.25a}$$

$$\mathcal{H}^{MN} = -\frac{1}{2} e^{-\phi}\sqrt{-\det \mathcal{M}_{\underline{\alpha\beta}}}\, \mathcal{M}^{[\underline{\alpha\beta}]} e^M_{\underline{\alpha}} e^N_{\underline{\beta}} * [e^{\underline{0\dots p}} \wedge \delta^{(9-p)}(\mathcal{C})]\,, \tag{5.25b}$$

where we are using the adapted vielbein $e^A = (e^{\underline{\alpha}}, e^{\hat{a}})$ such that $e^{\hat{a}}|_{\mathcal{C}} = 0$.

As in [31], it is useful to combine K and $\Omega^{(\rm F1)}$ into the 'generalized vector'

$$\mathcal{K} \equiv K + \Omega^{(\rm F1)}\,, \quad \text{section of } TM \oplus T^*M\,. \tag{5.26}$$

Along $\mathcal{C}$, we can also introduce another generalized vector field:

$$\mathcal{V} \equiv \mathrm{e}^{-\phi}\frac{\sqrt{-\det \mathcal{M}}}{\sqrt{-\det h}} \left(\mathcal{M}^{(\alpha\beta)}\partial_\alpha X^M g_{MN}\mathrm{d}x^N + \mathcal{M}^{[\alpha\beta]}\partial_\alpha X^M \partial_M\right) e^{\underline{0}}_\beta\,, \tag{5.27}$$

which is a section of $(TM \oplus T^*M)|_{\mathcal{C}}$ and can be regarded as the D-brane counterpart of (5.21). By using (5.23) and (5.25), we see that the first, second and fourth terms in the second line of (5.13) are non-vanishing in presence of the Dp-brane, which then contributes to $I(\epsilon) = \int_S \mathrm{vol}_S \nabla_M E^{\underline{0}M}$ by

$$I_{(\rm D)}(\epsilon) = \frac{1}{2}\int_{\mathcal{C}\cap S} \left[\mathcal{V}(\mathcal{K})\,\mathrm{dvol}_{\mathcal{C}\cap S} - \Omega^{(\rm D)} \wedge \mathrm{e}^{\mathcal{F}}\right], \tag{5.28}$$

where $\mathcal{V}(\mathcal{K})$ denotes the natural pairing between the generalized vectors $\mathcal{V}$ and $\mathcal{K}$:

$$\mathcal{V}(\mathcal{K}) = \mathrm{e}^{-\phi}\frac{\sqrt{-\det \mathcal{M}}}{\sqrt{-\det h}} \left(\mathcal{M}^{(\alpha\beta)} K_M \partial_\alpha X^M + \mathcal{M}^{[\alpha\beta]}\Omega^{(\rm F1)}_M \partial_\alpha X^M\right) e^{\underline{0}}_\beta\,. \tag{5.29}$$

Note the formal analogy between (5.28) and (5.20). Now, adapting again the steps of appendix C.2, one can rewrite the local bound (4.33) of [31] in the form

$$\mathcal{V}(\mathcal{K})\,\mathrm{dvol}_{\mathcal{C}\cap S} \geq \left(\Omega^{(\rm D)} \wedge \mathrm{e}^{\mathcal{F}}|_{\mathcal{C}\cap S}\right)_{\rm top}\,, \tag{5.30}$$

which is saturated precisely if the D-brane preserves ϵ. Hence, as expected, the complete D-brane contribution (5.28) is always non-negative and vanishes only if the D-brane preserves ϵ. By duality, this strongly supports our claim that these properties hold also for the complete $I_{(\text{NS5})}$ (and $I_{(\text{M5})}$).

We then conclude that, if we choose an ϵ satisfying (5.16), the total BPS energy

$$I(\epsilon) = I_0(\epsilon) + I_{(\text{F1})}(\epsilon) + I_{(\text{D})}(\epsilon) + I_{(\text{NS5})}(\epsilon) \tag{5.31}$$

satisfies the bound

$$I(\epsilon) \geq 0\,, \tag{5.32}$$

which is saturated if and only if ϵ satisfies the bulk supersymmetry equations and all the branes preserve ϵ, that is, if and only if the complete configuration is supersymmetric.

We would now like to consider the possible presence of orientifolds. Suppose that $\iota : M \to M$ denotes the orientifold involution associated with an Op-plane, located at the corresponding fixed locus $\mathcal{C}$. The Op-plane contributes to the effective action as a Dp-brane with a tension/charge rescaled by a factor -2^{p-5} and a vanishing $\mathcal{F}$, which is consistent with the fact that the B-field is odd under the orientifold involution and then has vanishing pull-back on the Op-plane. So, the Op-plane contribution $I_{(\text{O})}$ to the BPS energy takes the form (5.28) with $\mathcal{F} = 0$, up to an overall -2^{p-5} factor. Furthermore, the orientifold projection on the supersymmetry generator $\epsilon \equiv \binom{\epsilon_1}{\epsilon_2}$ is

$$\iota^*\epsilon_2 = \epsilon_1\,, \qquad \iota^*\epsilon_1 = (-)^{\lfloor\frac{p-1}{2}\rfloor}\epsilon_2\,, \tag{5.33}$$

see for instance [68]. Hence, from (5.12b) we see that $\Omega^{(\text{D})}$ is invariant under the orientifold involution:

$$\iota^*\Omega^{(\text{D})} = \Omega^{(\text{D})}\,. \tag{5.34}$$

Furthermore, since the Op-plane world-volume $\mathcal{C}$ is fixed under the orientifold involution, compatibility with (5.33) requires that along the Op-plane one must impose the boundary condition

$$(\epsilon_1 - \Gamma_{\underline{0\ldots p}}\epsilon_2)|_{\mathcal{C}} = 0\,, \tag{5.35}$$

where we are again using an adapted frame $e^A = (e^{\underline{\alpha}}, e^{\hat{a}})$ with $e^{\hat{a}}|_{\mathcal{C}} = 0$. This condition is precisely equivalent to requiring that the local bound (5.30) (with $\mathcal{F} = 0$) is saturated along the Op-plane. Hence $I_{(\text{O})}(\epsilon) \equiv 0$ and the BPS energy $I(\epsilon)$ is still given by the non-negative combination (5.31).

We close this section by emphasising that so far we have neglected higher order curvature corrections to the brane effective actions, which would modify the form of the corresponding localized BPS energy.[11] For instance, the curvature corrections of [69–71] would modify the second term appearing in the r.h.s. of (5.28) into something of the form $-\frac{1}{2}\int_{\mathcal{C}\cap S}\Omega^{(\text{D})} \wedge \mathrm{e}^{\mathcal{F}} \wedge (\text{curvature})$. The expected local brane supersymmetry suggests that $\mathcal{V}$ is also modified, so that $I_{(\text{D})}(\epsilon) \geq 0$ still holds and is saturated precisely if the D-brane preserves ϵ. It would be interesting to better investigate this point, also in connection with bulk higher-derivative corrections.

[11] World-volume higher curvature corrections play for instance and key role in the D-brane mediated instabilities recently discussed in [27].

6 Supersymmetry breaking

In the previous sections, we clarified the mechanism that makes supersymmetric vacua stable. We will now try to use this knowledge to provide stability arguments for vacua that break supersymmetry. Our attempts will fail to exhibit stable non-supersymmetric vacua, but fail in interesting ways that might point the way to better future strategies.

6.1 An operator in M-theory

The idea is a bit similar to that in section 3.3. A supersymmetric vacuum is defined by having a solution ϵ to $\mathcal{D}_M\epsilon = 0$ in M-theory and $\mathcal{D}_M\epsilon = \mathcal{O}\epsilon = 0$ in type II (recalling (4.4), (5.3)). We would like to find a modification $\mathcal{D}'_M$, $\mathcal{O}'$ of these supersymmetry operators such that

i) an ϵ annihilated by them exists on some non-supersymmetric vacuum, and

ii) the BPS energy $I'(\epsilon)$ defined via the analogues of (4.6), (5.5) is still positive and related to the physical energy.

This is a hard set of requirements to satisfy, but let us try anyway. There are infinitely many operators $\mathcal{D}'_M$ that one might consider, but the problem becomes more tractable if we demand only objects with a single derivative to appear. As reviewed in appendix B.5, the consistency conditions $[\mathcal{D}_M, \mathcal{D}_N]\epsilon = 0$ for supersymmetry imply most of the equations of motion. Having more than one derivative in the modified $\mathcal{D}'_M$ would imply more than two derivatives in the consistency conditions $[\mathcal{D}'_M, \mathcal{D}'_N]\epsilon = 0$, and it might be hard to make these in turn compatible with the equations of motion. Another restriction that seems reasonable is that, in order to avoid gauge dependence, only the field-strengths appear and not the potentials.

In M-theory, the most general allowed operator obtained by deforming D_M would now be[12]

$$\mathcal{D}'_M = D_M + \frac{1}{24}(a_1\Gamma_M G + a_2 G\Gamma_M) + a_3\Gamma_M \, , \tag{6.1}$$

where the a_i are constants. Comparing with (the second expression in) (4.4), we see that the supersymmetric case is recovered for $a_1 = -1$, $a_2 = 3$, $a_3 = 0$.

Notice that in particular we are not considering a modification of the type

$$\mathcal{D}'_M = \mathcal{D}_M + V_M \, . \tag{6.2}$$

This would have the interesting feature that the two-form $(E'_2)_{MN} = -\bar{\epsilon}\Gamma_{MN}{}^P\mathcal{D}'_P\epsilon$ defined with this operator is in fact equal to the E_2 in (4.6); so the positivity argument would still apply. Moreover solving $\mathcal{D}'_M\epsilon = 0$ would be superficially reminiscent of $(D_m + \mathrm{i}A_m)\eta_+ = 0$, which does have a solution on a Kähler manifold (although with a crucial difference of an i).

[12]One additional term we could add here is $a_4\Gamma_M D$, with $D = \Gamma^N D_N$ the Dirac operator. Consider $\Gamma^M\mathcal{D}'_M\epsilon = 0$ with this modification; for $a_4 \neq -1/11$, this equation determines $D\epsilon$, which can be fed back into $\mathcal{D}'_M\epsilon = 0$ to obtain a new equation without the new term. If $a_4 = -1/11$, $\mathcal{D}'_M$ contains the *conformal Killing operator* $D_M - \frac{1}{d}\Gamma_M D$. We will ignore this particular case in what follows.

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However, such a vector V_M cannot be an external piece of data: for the positivity argument to apply, it has to be defined for all possible configurations with a certain boundary condition. So it has to be somehow defined in terms of the fields. Given the requirements we listed at the beginning of this section, there does not seem to be any such vector field, and hence we will ignore the possibility (6.2).

6.2 Skew-whiffed and Englert vacua

We will first look at requirement i) above: namely, whether we can solve $\mathcal{D}'_M\epsilon = 0$ on any non-supersymmetric vacua.

Skew-whiffed vacua are obtained from a supersymmetric FR vacuum (section 4.6) by reversing the orientation of the internal space, or in other words by mapping $G \to -G$ [37, 38]. As reviewed in section 4.6, in a FR solution $\mathrm{AdS}_4 \times Y$ supersymmetry requires solving the internal Killing spinor equation $(D_m - \frac{\mathrm{i}}{2}m\gamma_m)\eta = 0$ on Y. Reversing the sign of G flips the sign of m, and the new spinor equation now has no solution (except when $Y = S^7$); so supersymmetry is broken. But the equations of motion are still satisfied because they are quadratic in G. As noted already in [37, 38], these solutions are automatically stable under small perturbations.[13] On these non-supersymmetric solutions, we can clearly solve $\mathcal{D}'_M\epsilon = 0$ with

$$a_1 = 1\,, \qquad a_2 = -3\,, \qquad a_3 = 0\,; \tag{6.3}$$

in other words, with the operator obtained from the supersymmetric $\mathcal{D}_M$ by reversing the sign of G.

We next consider the *Englert* vacua [39]. Here, Y is a *weak* G_2 manifold, namely one with a G_2-structure ϕ such that[14]

$$\mathrm{d}\phi = -4 *_7 \phi\,. \tag{6.4}$$

This implies that the cone over Y has Spin(7) holonomy, so Y admits a Killing spinor η_0 with $m = 1$. As a consequence, Y is also Einstein: more specifically $R_{mn} = 6g_{mn}$. Taking the fields as

$$\mathrm{d}s^2_{11} = L^2\mathrm{d}s^2_{\mathrm{AdS}_4} + r_0^2\mathrm{d}s^2_Y\,, \qquad G = g_0\mathrm{vol}_{\mathrm{AdS}_4} + g_1 *_7 \phi\,, \tag{6.5}$$

the equations of motion have two branches of solutions: one with $g_1 = 0$, leading back to FR, and one with

$$g_0 = \frac{9}{25}r_0^3\,, \qquad g_1^2 = 4r_0^6\,, \qquad L = \sqrt{\frac{3}{10}}r_0\,. \tag{6.6}$$

A perturbative instability was found for these vacua in [77], but only when Y has more than one Killing spinor. One can check from (6.5) that the brane nucleation condition (2.6) is *not* satisfied for the easiest case of M2-branes (as noted in [28] for some particular cases).

A solution of $\mathcal{D}'_M\epsilon = 0$ is found as $\epsilon = \zeta_+ \otimes \eta + \zeta_- \otimes \eta^{\mathrm{c}}$. $\zeta_- = (\zeta_+)^{\mathrm{c}}$, with $^{\mathrm{c}}$ denoting Majorana conjugation; as usual we assume the Killing spinor equation $D_\mu\zeta_\pm = \frac{1}{2}\gamma_\mu\zeta_\mp$ also along AdS_4; and $\eta = \mathrm{e}^{\mathrm{i}\alpha}\eta_0$, with α a constant and η_0 Killing. With these assumptions,

[13] Further analysis of stability of these solutions under squashing was carried out in [72, 73].

[14] Another class of solutions uses instead the SU(3)-structure on a Sasaki-Einstein manifold [74–76]. It would be interesting to apply our methods to this case as well; we thank N. Bobev for the suggestion.

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both the internal and external components of our equation become algebraic, and can be solved for

$$a_2 = \frac{3 \pm \sqrt{114}}{10}\,, \qquad a_1 = 3 - a_2\,, \qquad a_3 = \pm\frac{21 - 2a_2}{3}\,, \qquad \tan 2\alpha = \frac{3 - 2a_2}{4a_2}\,; \tag{6.7}$$

the two signs are independent.

6.3 Lack of positivity

Having found some examples of solutions to $\mathcal{D}'_M\epsilon = 0$, we now look at requirement ii) above, namely that one can still prove positivity and stability with the modified operator $\mathcal{D}'_M$.

In principle there are various terms that can ruin positivity. In (4.11), all the terms beyond the first should now be re-examined. Already the second term $\bar\epsilon A^{NP} D_P \epsilon$ is worrisome: it is unlikely to have a fixed sign because of the derivative of ϵ, so it had better vanish.[15] Repeating the steps in (B.12) we see that in our present more general setting $2A^{NP} = -3(3a_1 + a_2)[\Gamma^{NP}, G]$. In particular this imposes

$$a_2 = -3a_1\,. \tag{6.8}$$

We see from (6.3), (6.7) that the skew-whiffed vacua do satisfy this, while the Englert vacua do not. While it is conceivable that a positivity property might be proven by more ingenious methods without assuming (6.8), in the rest of this subsection we will assume that it must hold.

With this assumption, defining $E'_{MN} = -\bar\epsilon\Gamma_{MN}{}^P\mathcal{D}'_P\epsilon$ similar to (4.6), (4.8) is modified to

$$\begin{aligned}\nabla_M E'^{MN} = \overline{\mathcal{D}'_M\epsilon}\Gamma^{MPN}\mathcal{D}'_P\epsilon + \frac{1}{4}(2\mathcal{G}^{NP} - a_1^2 T^{NP}_{(G)})K_P \\ + \frac{1}{4}\bar\epsilon\left[\mathrm{d}x^N \wedge \left(a_1\mathrm{d}G - a_1\mathrm{d}*G + \frac{1}{2}a_1^2 G\wedge G - 12a_1a_3 G - 360a_3^2\right)\right]\epsilon\,,\end{aligned} \tag{6.9}$$

where $\mathcal{G}_{MN} \equiv R_{MN} - \frac{1}{2}g_{MN}R$ is the usual Einstein tensor; see appendix B.4 for the details of this computation. In the absence of branes, the terms multiplying K_P can be reassembled using the equations of motion as $(1 - a_1^2)T^{NP}_{(G)}$. By using (4.7) we see that, if we assume $|a_1| \le 1$, the contribution of this term to $I'(\epsilon)$ is positive because the flux stress-energy tensor obeys the dominant energy condition.

Of more concern are the terms on the second line of (6.9). We can use the equations of motion (4.15) to get rid of $\mathrm{d}G$ and $\mathrm{d}*G$, turning the quadratic flux term into $\frac{1}{2}a_1(a_1 + 1)G\wedge G$. However, there is no reason to believe this term to have a definite sign for all solutions.[16] Furthermore, if $a_3 \neq 0$ we also have a linear G term, which is even more clearly of indefinite sign.

[15] One might want to take care of this term by "completing the square" to reabsorb it in the term $\overline{\mathcal{D}'_M\epsilon}\Gamma^{MPN}\mathcal{D}_P\epsilon$; but this requires looking for a X_M such that $X_M\Gamma^{MNP} = [G, \Gamma^{NP}]$; there appears to be no natural choice of such an object.

[16] Consider for example a solution where K is timelike; here ϵ will define an SU(5)-structure (J_2, Ω_5) in the remaining directions (studied in [78]), in terms of which the term of interest is proportional to $J_2\wedge G\wedge G$, a top-form in $d = 10$. Choosing a holomorphic vielbein h^a, in terms of which $J_2 = \frac{\mathrm{i}}{2}\sum_a h^a \wedge \overline{h^a}$, it is easy to see that the forms $G = 2\mathrm{Re}(h^1\wedge h^2\wedge h^3\wedge \overline{h^4})$ and $G = J_2\wedge J_2$ result in opposite signs for $J_2\wedge G\wedge G$.

It is also interesting to consider the effect of localized sources. The same steps leading to (4.29) now give a localized contribution to the BPS-energy

$$I'_{(\mathrm{M2})}(\epsilon) = \frac{1}{4}\int_{\mathcal{C}\cap S}\left(K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap\Sigma} - |a_1|\Omega^{(\mathrm{M2})}\right) \tag{6.10}$$

where the overall sign of the term including $\Omega^{(\mathrm{M2})}$ has been fixed without loss of generality, since it can be changed by inverting the M2 orientation, that is, by swapping the M2 for an anti-M2 brane, or viceversa.

The sign of $\int_{\mathcal{C}\cap S}\Omega^{(\mathrm{M2})}$ depends on the M2 orientation. Clearly $I'_{(\mathrm{M2})}(\epsilon)$ is manifestly positive if $\int_{\mathcal{C}\cap S}\Omega^{(\mathrm{M2})} \le 0$. If instead $\int_{\mathcal{C}\cap S}\Omega^{(\mathrm{M2})} > 0$, the second term on the r.h.s. of (6.10) is negative. However, we can still use the algebraic local bound (4.30), which implies that

$$I'_{(\mathrm{M2})}(\epsilon) \ge \frac{1}{4}(1-|a_1|)\int_{\mathcal{C}\cap S}\Omega^{(\mathrm{M2})}\,. \tag{6.11}$$

We then see that $I'_{(\mathrm{M2})}(\epsilon)$ has a definite sign if $|a_1| \le 1$, which is the same condition that we found above. This suggests that vacua can be unstable under M2 nucleation precisely if $|a_1| > 1$. Note that this conclusion does not apply to the skew-whiffed case (in which $a_1 = 1$). A similar argument could be repeated for the non-supersymmetric generalization of the M5 contribution (4.36). However, in this case the neglected world-volume fluxes could affect the final conclusion.

We saw earlier that the Englert vacua don't satisfy (6.8). Even if we ignore this issue and repeat (6.9) with $a_2 \neq -3a_1$, it turns out that the sign of the $T^{NP}_{(G)}$ is negative; so this class of solutions is unlikely to enjoy a positivity theorem by the spinorial strategy we are considering.

The skew-whiffed vacua seem more promising in this respect, but they still have a non-vanishing $G\wedge G$ term, which as we saw earlier does not have a definite sign. So we again conclude that no positivity theorem exists with the present spinorial method. This does not prove that these vacua are unstable, but certainly seems to give evidence in that direction.

6.4 Type II

The supersymmetry transformations in M-theory already contain all the possible terms that one can write with at most one derivative, except for one term; this led us to (6.1).

In contrast, in the type II operators (5.3) it would be possible to add several new terms. For example one could change the 2×2 matrices acting on the index a of the spinor doublet ϵ_a. Or one could add entirely new terms, such as $H\Gamma_M$, $\Gamma_M\mathcal{F}$, $\partial_M\phi$ to $\mathcal{D}_M$ or $\mathcal{F}$ to $\mathcal{O}$, tensored by any 2×2 matrix compatible with the chirality of the spinors. Even more drastically one could violate the democratic structure of the operator and add a term only involving a single RR-form degree. This gives rise to a bewildering array of possibilities, that we will not explore here. For example it would be interesting to try to reproduce for example the GKP supersymmetry breaking [79], or the supersymmetry-breaking mechanisms discussed in [67, 80], from this point of view.

Instead of attempting a general analysis, it would be reasonable to add one such term at a time. The simplest possibilities would be

$$\mathcal{D}'_M = \mathcal{D}_M + \Gamma_M \otimes a\,, \qquad \mathcal{O}' = \mathcal{O} + 1 \otimes b\,, \tag{6.12}$$

where a, b are some 2×2 matrices. Notice however that this only makes sense in IIA provided a and b are off-diagonal, since in IIB $\mathcal{D}_M\epsilon$ and $\Gamma_M \otimes a\epsilon$ have opposite chiralities, and so do $\mathcal{O}\epsilon$ and $1\otimes b\epsilon$. It would be interesting to apply to (6.12) in IIA the same program outlined earlier in this section for M-theory.

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A Useful spinorial identities

Here we recall some standard definitions and techniques useful in any number d of dimensions; we assume Lorentz signature.

The easiest gamma matrix identities are

$$\begin{aligned} \Gamma_M \Gamma^{N_1\cdots N_k} &= \Gamma_M{}^{N_1\cdots N_k} + k\delta_M^{[N_1}\Gamma^{\cdots N_k]}\,,\\ \Gamma^{N_1\cdots N_k}\Gamma_M &= (-1)^k(\Gamma_M{}^{N_1\cdots N_k} - k\delta_M^{[N_1}\Gamma^{\cdots N_k]})\,. \end{aligned} \tag{A.1}$$

Viewing $\Gamma^{N_1\cdots N_k}$ as the image of the Clifford map $_{/}$

$$[\mathrm{d}x^{M_1}\wedge\cdots\wedge \mathrm{d}x^{M_k}]_{/} = \Gamma^{M_1\cdots M_k}\,, \tag{A.2}$$

equations (A.1) can be thought of as operator identities:

$$\overrightarrow{\Gamma^M} = g^{MN}\iota_N + \mathrm{d}x^M\wedge\,, \qquad \overleftarrow{\Gamma^M} = (-g^{MN}\iota_N + \mathrm{d}x^M\wedge)(-1)^{\mathrm{deg}}\,. \tag{A.3}$$

The arrows denote action from the left and right: $\Gamma^M \alpha_{/} = (\overrightarrow{\Gamma^M}\alpha)_{/}$, $\alpha_{/}\Gamma^M = (\overleftarrow{\Gamma^M}\alpha)_{/}$. Moreover $\deg \alpha_k \equiv k\alpha_k$, with the subscript denoting form degree; $\iota_M(\mathrm{d}x^{N_1}\wedge\cdots\wedge \mathrm{d}x^{N_k}) \equiv k\delta_M^{[N_1}\mathrm{d}x^{N_2}\wedge\cdots\wedge \mathrm{d}x^{N_k]}$ is the contraction operator.

A useful identity following from (A.3) is

$$\Gamma_M \alpha_k \Gamma^M = (-1)^k(d-2k)\alpha_k\,. \tag{A.4}$$

One particular consequence of this and (A.1) for $k=2$ is

$$\Gamma_M\Gamma^{MNP} = \Gamma^{MNP}\Gamma_M = (d-2)\Gamma^{NP}\,. \tag{A.5}$$

Sometimes, in order to avoid ambiguities, we underline flat indices. For instance, the chiral matrix is $\Gamma = c\Gamma^{\underline{01\ldots d-1}}$. In general c is chosen such that $\Gamma^2=1$, and in both $d=10$ and

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11 we can just take $c = 1$. Under the Clifford map (A.2), left multiplication by Γ is related to the Hodge star:

$$\overrightarrow{\Gamma} = c * \lambda\,, \tag{A.6}$$

where $\lambda\alpha_k \equiv (-1)^{\lfloor k/2 \rfloor}\alpha_k$. Our Hodge-star operator $*$ is defined by

$$* \left(e^{A_1} \wedge \ldots \wedge e^{A_k}\right) = \frac{1}{(d-k)!}\epsilon_{B_1 \cdots B_{d-k}}{}^{A_1 \cdots A_k} e^{B_1} \wedge \cdots \wedge e^{B_{d-k}} \tag{A.7}$$

where $e^A = e^A_M \mathrm{d}x^M$ is a vielbein and the totally antisymmetric $\epsilon_{A_1 \cdots A_d}$ is such that $\epsilon_{\underline{012\cdots}} = 1$. Notice that $(*\lambda)^2 = -(-)^{\lfloor d/2 \rfloor}$ in Lorentzian signature. Using this and (A.3) we also obtain that the Hodge operator exchanges wedges with contractions:

$$* \lambda\, \mathrm{d}x^M \wedge = (-1)^{d-1} \iota^M * \lambda\,, \qquad *\lambda\, \iota^M = (-1)^{d-1} \mathrm{d}x^M \wedge *\lambda\,. \tag{A.8}$$

In the slightly different notation introduced in (4.2), we can also show

$$\Gamma \Gamma_{(k)} = c(-1)^{\lfloor d/2 \rfloor} * \lambda(\Gamma_{(d-k)})\,. \tag{A.9}$$

The natural inner product and norm-squared of forms are:

$$\alpha_k \cdot \beta_k \equiv \frac{1}{k!}\alpha_{M_1 \cdots M_k}\beta^{M_1 \cdots M_k}\,, \qquad |\alpha_k|^2 \equiv \alpha_k \cdot \alpha_k\,. \tag{A.10}$$

There is also a natural inner product among bispinors, related under (A.2) to the one among forms:

$$\mathrm{Tr}((\lambda\alpha_k)_{/}(\beta_k)_{/}) = 2^{\lfloor d/2 \rfloor}\alpha_k \cdot \beta_k\,; \qquad (\lambda\alpha_k)_{/} = (-1)^k \Gamma_{\underline{0}}(\alpha_{k/})^\dagger \Gamma^{\underline{0}}\,. \tag{A.11}$$

Repeated application of (A.3) gives the other useful identities

$$(\iota^M \alpha_k)\cdot(\iota^N \beta_k) - \frac{1}{2}\alpha_k \cdot \beta_k\, g^{MN} = \frac{(-1)^{k+1}}{2 \cdot 2^{\lfloor d/2 \rfloor}} \mathrm{Tr}\left((\lambda\alpha_k)_{/}\Gamma^{(M} \beta\!\!\!/_k \Gamma^{N)}\right)\,, \tag{A.12a}$$

$$(\iota^M \alpha_k)\cdot(\mathrm{d}x^N \wedge \alpha_k) = \frac{(-1)^k}{2 \cdot 2^{\lfloor d/2 \rfloor}} \mathrm{Tr}\left((\lambda\alpha_k)_{/}\Gamma^{[M} \alpha\!\!\!/_k \Gamma^{N]}\right)\,. \tag{A.12b}$$

Notice that the left-hand side of (A.12a) has the form of a stress-energy tensor associated to a k-form field-strength.

We will sometimes use Fierz identities. These come about by expanding a bispinor along the $\Gamma^{N_1 \cdots N_k}$. When d =even, the set of such objects $k = 0, \cdots, d$ is a basis; when d = odd, there are redundancies relating k to $d-k$, so we can expand along only $k = 0, \cdots, (d-1)/2$.

The coefficients of such expansions often involve inner products $\bar{\epsilon}\epsilon'$, where

$$\bar{\epsilon} \equiv \epsilon^\dagger \Gamma_{\underline{0}}\,. \tag{A.13}$$

We also recall the intertwiner property $\Gamma_M^\dagger \Gamma^{\underline{0}} = -\Gamma^{\underline{0}}\Gamma_M$. In both $d = 10$ and 11, one can work in a basis where all gamma matrices are real; a Majorana spinor is then such that $\epsilon^* = \epsilon$, and for two Majorana spinors one then easily sees

$$\bar{\epsilon}'\, \Gamma_{M_1 \cdots M_k}\epsilon = (-1)^{\lfloor (k-1)/2 \rfloor}\bar{\epsilon}\, \Gamma_{M_1 \cdots M_k}\epsilon'\,. \tag{A.14}$$

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It is also useful to note that

$$\overline{\Gamma_{M_1 \cdots M_k}\epsilon} = (-1)^k \bar{\epsilon}\,\Gamma_{M_k \cdots M_1} = (-1)^{\lfloor (k-1)/2 \rfloor}\bar{\epsilon}\,\Gamma_{M_1 \cdots M_k}\,. \tag{A.15}$$

The Clifford map (A.2) also works well at the differential level. One can extend the spinorial covariant derivative to bispinors C as $D_M C \equiv \partial_M C + \frac{1}{4}\omega_M^{AB}[\Gamma_{AB}, C]$; this is then related to the ordinary bosonic covariant derivative ∇_M by

$$D_M(\alpha_/) = (\nabla_M \alpha)_/\,. \tag{A.16}$$

B Some details on M-theory

The two derivative action of 11 dimensional supergravity reads:

$$\begin{aligned} S = \frac{2\pi}{\ell_{\rm P}^9}\int \Bigg(& R * 1 - 2\mathrm{i}\overline{\psi} \wedge \Gamma_{(8)} \wedge D\psi - \frac{1}{2} G \wedge *G - \frac{1}{6} G \wedge G \wedge C \\ & - \frac{i}{6}\overline{\psi} \wedge \Gamma_{(8)} \wedge \left(\Gamma_{(1)} G\!\!\!/ - 3\left(\iota_{(1)} G\right)_/\right)\psi + \mathcal{O}\left(\psi^4\right)\Bigg), \end{aligned} \tag{B.1}$$

where D is the spin covariant derivative, and $G = \mathrm{d}C$, and $G\!\!\!/$ and $(\iota_N G)_/$ are defined below (4.4) in agreement with the general (A.2). The supersymmetry variation of the 3-form gauge field and the gravitini are given by:

$$\delta_\epsilon C_{MNP} = -3\bar{\epsilon}\,\Gamma_{[MN}\psi_{P]}, \qquad \delta_\epsilon \psi_M = \mathcal{D}_M \epsilon\,, \tag{B.2}$$

where $\mathcal{D}_M$ is defined in (4.4).

B.1 Supercharge from the Noether theorem

Let us now briefly summarize a derivation of the supercurrent following an infinitesimal variation under local supersymmetry, along the lines of [81].

Given a Lagrangian $L\,(X, \partial X)$ depending on fields X and their derivatives $\partial_M X$, its variation is given by

$$\begin{aligned} \delta L &= \sum_X \left(\frac{\delta L}{\delta X}\delta X + \frac{\delta L}{\delta \partial_M X}\partial_M \delta X\right) \\ &= \sum_X \left(\frac{\delta L}{\delta X}\delta X - \left(\partial_M \frac{\delta L}{\delta \partial_M X}\right)\delta X\right) + \sum_X \partial_M \left(\frac{\delta L}{\delta \partial_M X}\delta X\right) \\ &\equiv \sum_X \left(\mathcal{E}_X\,\delta X + \partial_M N^M\right), \end{aligned} \tag{B.3}$$

where $\mathcal{E}_X$ represents the equations of motion for X. Now, the variation of the Lagrangian under an infinitesimal supersymmetry transformation gives $\delta_\epsilon L \equiv \partial_M V^M$. Identifying this with the above gives

$$\partial_M \underbrace{\left(V^M - N^M\right)}_{\equiv J^M} = \sum_X \mathcal{E}_X\,\delta X \overset{\text{on-shell}}{=} 0, \tag{B.4}$$

which gives the supercurrent $J^M = V^M - N^M$ that is conserved on shell. The corresponding supercharge is $Q = \int \mathrm{d}^{d-1}x J^0$, which also generates the supersymmetry transformations of the fields: $\delta_\epsilon X = \{Q, X\}$.

B.2 Noether charge in M-theory

Applying to M-theory the Noether procedure outlined in B.1 we get,

$$N^M = \frac{\delta L}{\delta\left(\partial_M \psi_N\right)}\delta_\epsilon \psi_N + \frac{1}{3!}\frac{\delta L}{\delta\left(\partial_M C_{NPQ}\right)}\delta_\epsilon C_{NPQ} + \cdots\,, \tag{B.5}$$

where the ellipsis represents variations with respect to the vielbien and the spin connection. These eventually cancel out and so we will not write them here. Up to an overall rescaling by $\ell_{\mathrm{P}}^9/(8\pi)$, this gives

$$\begin{aligned} N^M =& -\frac{\mathrm{i}e}{2}\overline{\psi}_P \Gamma^{PMN}\mathcal{D}_N\epsilon + \frac{3e}{4\cdot 4!}G^{MNPQ}\overline{\epsilon}\,\Gamma_{[NP}\psi_{Q]} \\ &+ \frac{1}{4\cdot 12^3}\epsilon^{MNPQM_5\ldots M_{11}}G_{M_5\ldots M_8}C_{M_9 M_{10} M_{11}}\overline{\epsilon}\,\Gamma_{[NP}\psi_{Q]} + \cdots\,, \end{aligned} \tag{B.6}$$

where again the ellipsis represents, in addition to the variation with respect to e^A_M and ω above, quadratic terms in ψ. The surface terms arising from an infinitesimal variation of the action under the local supersymmetry transformation give $\delta_\epsilon L = \frac{8\pi}{\ell_{\mathrm{P}}^9}\partial_M V^M$, where

$$\begin{aligned} V^M =& \frac{\mathrm{i}e}{2}\overline{\psi}_P \Gamma^{PMN}\mathcal{D}_N\epsilon + \frac{3e}{4\cdot 4!}G^{MNPQ}\overline{\epsilon}\,\Gamma_{[NP}\psi_{Q]} \\ &+ \frac{1}{4\cdot 12^3}\epsilon^{MNPQM_5\ldots M_{11}}G_{M_5\ldots M_8}C_{M_9 M_{10} M_{11}}\overline{\epsilon}\,\Gamma_{[NP}\psi_{Q]} + \cdots\,, \end{aligned} \tag{B.7}$$

and the ellipsis has the same meaning as above. Together with (B.6), this gives the supercurrent

$$J^M = V^M - N^M = \mathrm{i}e\overline{\psi}_P\Gamma^{PMN}\mathcal{D}_N\epsilon. \tag{B.8}$$

For our purposes, it is convenient to use the fermionic real supercurrent obtained from (B.8) by considering ϵ to be commuting and dropping the overall i. The corresponding fermionic supercharge $Q(\epsilon)$ is then given by the integral:

$$Q = \int_S \overline{\psi}_P \Gamma^{PMN}\mathcal{D}_N\epsilon * \mathrm{d}x_M = \int_{\partial S}\overline{\psi}_P\Gamma^P{}_{MN}\epsilon * \mathrm{d}x^{MN} = -\int_{\partial S}\overline{\epsilon}\,\Gamma_{(8)}\wedge\psi\,. \tag{B.9}$$

In the second step, we have used the gravitino equations of motion $\Gamma^{MNP}\mathcal{D}_N\psi_P = 0 + \mathcal{O}(\psi^3)$.

B.3 The divergence identity

We will now give some more details about the computations leading to (4.8). For simplicity in the rest of this appendix, as in most of the paper, we mostly drop the Clifford map symbol $/$, whose implicit presence should be clear from the context.

As a warm-up we first note that, from (A.4):

$$\Gamma^M G\Gamma_M = 3G\,, \qquad \Gamma^M(-\Gamma_M G + 3G\Gamma_M) = (-11+3\cdot 3)G = -2G\,. \tag{B.10}$$

We then evaluate the commutator term in (4.11):

$$\begin{aligned} &\Gamma^{NMP}[D_M, -\Gamma_P G + 3G\Gamma_P] \overset{(A.16)}{=} \Gamma^{NMP}(-\Gamma_P\nabla_M G + 3\nabla_M G\Gamma_P) \\ &\overset{(A.5),\,(A.1),\,(B.10)}{=} 6\Gamma^{[M}\nabla_M G\Gamma^{N]} \overset{(A.3)}{=} 6[(\mathrm{d}x\wedge +\iota)(\mathrm{d}x\wedge -\iota)]^{[MN]}\nabla_M G \\ &= 6(\mathrm{d}x^M\wedge\mathrm{d}x^N\wedge - \iota^M\iota^N)\nabla_M G \overset{(A.8)}{=} 6(-\mathrm{d}x^N\wedge\mathrm{d}G + \iota^N * \mathrm{d} * G)\,. \end{aligned} \tag{B.11}$$

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We now look at the terms in (4.11) that are quadratic in G. Using (A.1) for $k = 2$, it is now easy to evaluate A^{NP} in (4.12):

$$\begin{aligned} A^{NP} =& (-G\Gamma_M + 3\Gamma_M G)(\Gamma^M\Gamma^{NP} - 2g^{M[N}\Gamma^{P]}) \\ &- (\Gamma^{NP}\Gamma^M - 2\Gamma^{[N}g^{P]M})(-\Gamma_M G + 3G\Gamma_M) \\ =& -2G\Gamma^{NP} - 2(-G\Gamma^{[N} + 3\Gamma^{[N}G)\Gamma^{P]} + 2\Gamma^{NP}G + 2\Gamma^{[N}(-\Gamma^{P]}G + 3G\Gamma^{P]}) = 0\,. \end{aligned} \tag{B.12}$$

Another useful consequence of (A.1) is:

$$\Gamma^{MNP} = \Gamma^M\Gamma^N\Gamma^P - g^{MN}\Gamma^P - \Gamma^M g^{NP} + \Gamma^N g^{MP}\,. \tag{B.13}$$

Using this and (B.10), we also get

$$Q^N = 9(G\Gamma^N G + \Gamma_M G\Gamma^N G\Gamma^M)\,. \tag{B.14}$$

For our purposes it is now useful to apply the Fierz identity, i.e., to expand (B.14) in the basis $\Gamma^{N_1\cdots N_k}$, $k = 0, \cdots, (d-1)/2 = 5$. This works out to

$$Q^N = um_{k=0}^{5}\frac{1}{32k!}\mathrm{Tr}(Q^N\Gamma_{N_k\cdots N_1})\Gamma^{N_1\cdots N_k}\,. \tag{B.15}$$

Actually, it is easy to see that $\Gamma_{\underline{0}}(Q^N)^\dagger\Gamma^{\underline{0}} = -Q^N$, and $\Gamma_{\underline{0}}(\Gamma^{N_1\cdots N_k})^\dagger\Gamma^{\underline{0}} = (-1)^{\lfloor k+1/2\rfloor}\Gamma^{N_1\cdots N_k}$; so in fact only $k = 1, 2, 5$ appear in (B.15). The traces can now be simplified using (A.4) again. The $k = 1$ and $k = 2$ terms give the quadratic G contributions to the equations of motion (4.10), (4.15), while in $k = 5$ the two terms cancel each other:

$$\begin{aligned} Q^N &= \frac{1}{32}\left(\mathrm{Tr}(Q^N\Gamma^P)\Gamma_P + \frac{1}{2}\mathrm{Tr}(Q^N\Gamma^{QP})\Gamma_{PQ}\right) \\ &= 36\left(T^{NP}_{(G)}\Gamma_P - \frac{1}{4(4!)^2}\epsilon^{NPQM_1\cdots M_8}G_{M_1\cdot M_4}G_{M_5\cdot M_8}\Gamma^{PQ}\right) \\ &= 36T^{NP}_{(G)}\Gamma_P - 18\iota^N * (G\wedge G)\,. \end{aligned} \tag{B.16}$$

Thus as promised, this gives the quadratic terms in the equations of motion in (4.8).

B.4 The supersymmetry-breaking case

Here we will show briefly how the computation is modified for the operator (6.1) with $a_2 = -3a_1$, namely

$$\mathcal{D}'_M = D_M + \frac{a_1}{24}(\Gamma_M G - 3G\Gamma_M) + a_3\Gamma_M\,. \tag{B.17}$$

Following the same steps as in (4.11), the explicit G terms there are rescaled by $-a_1$; (4.12) are replaced by

$$A'^{NP} = -a_1 A^{NP} + 24a_3(\Gamma_M\Gamma^{MNP} - \Gamma^{MNP}\Gamma_M) \overset{(B.12),\,(A.5)}{=} 0 \tag{B.18}$$

and Q'^N given by

$$\begin{aligned}\frac{Q'^N + a_1 Q^N}{24a_3} &= -a_1\Gamma_M\Gamma^{MNP}(-\Gamma_P G + 3G\Gamma_P) - a_1(-G\Gamma_M + 3\Gamma_M G)\Gamma^{MNP}\Gamma_P \\ &\quad + 24a_3\Gamma_M\Gamma^{MNP}\Gamma_P \\ &\overset{(A.5)}{=} -9a_1(\Gamma^N\Gamma^P - g^{NP})(-\Gamma_P G + 3G\Gamma_P) \\ &\quad - 9a_1(-G\Gamma_M + 3\Gamma_M G)(\Gamma^M\Gamma^N - g^{MN}) \\ &\quad + 90\cdot 24a_3\Gamma^N \overset{(B.10)}{=} 36a_1\{\Gamma^N, G\} + 90\cdot 24a_3\Gamma^N \\ &\overset{(A.3)}{=} 12(3a_1 \mathrm{d}x^N \wedge G + 180a_3\mathrm{d}x^N)_{/}\,. \end{aligned} \tag{B.19}$$

Putting it all together, one arrives at (6.9).

B.5 Equations of motion from supersymmetry

Finally we show here how the Einstein equations are (almost completely) implied by supersymmetry and the G equations. This was shown in ([78], appendix B), but some results we saw in this appendix allow us to derive it more quickly. This is useful for us in the context of modifying $\mathcal{D}_M$ to break supersymmetry, in section 6.

We consider the operator

$$\Gamma^M[\mathcal{D}_M, \mathcal{D}_N]\,. \tag{B.20}$$

The purely gravitational term $\Gamma^M[D_M, D_N]$ is well-known to be $\frac{1}{2}R_{NP}\Gamma^P$; this is how a manifold with a covariantly constant spinor in Euclidean signature is shown to be Ricci-flat. The remaining terms can again be separated into those involving G linearly, and those involving it quadratically.

The linear term proceeds similar to (B.11):

$$\begin{aligned}\Gamma^M[D_{[M}, -\Gamma_{N]}G + 3G\Gamma_{N]}] &= -\Gamma^M\Gamma_{[N}\nabla_{M]}G + 3\Gamma^M\nabla_{[M}G\Gamma_{N]} \\ \overset{(A.4)}{=} \frac{1}{2}\Gamma_N\Gamma^M\nabla_M G + \frac{3}{2}\Gamma^M\nabla_M G\Gamma_N &\overset{(A.3)}{=} 2(-\mathrm{d}x_N \wedge + 2\iota_N)(\mathrm{d}G + *\mathrm{d}*G)\,. \end{aligned} \tag{B.21}$$

For the quadratic term, we first use (B.10) repeatedly to obtain:

$$\begin{aligned}&\Gamma^M[-\Gamma_M G + 3G\Gamma_M, -\Gamma_N G + 3G\Gamma_N] \\ &= -9(G\Gamma_N G + \Gamma^M G\Gamma_N G\Gamma_M) + 9\Gamma_N G^2 - 3\Gamma_N\Gamma^M G^2\Gamma_M\,. \end{aligned} \tag{B.22}$$

The parenthesis on the right-hand side is just Q^N from (B.14), so we can evaluate it as in (B.16). For the other two terms, we first notice that $G^2 = \frac{1}{2}\{G, G\} = (G^2)_0 + (G^2)_4 + (G^2)_8$: it only has zero-, four- and eight-form parts. From (A.4) we then have

$$3G^2 - \Gamma^M G^2\Gamma_M = -8(G^2)_0 - 8(G^2)_8 = -8(|G|^2 + G\wedge G)\,. \tag{B.23}$$

Recalling now the first of (A.1) and putting everything together, we arrive at

$$\Gamma^M[\mathcal{D}_M, \mathcal{D}_N] = \frac{1}{2}\mathcal{E}_{NP}\Gamma^P + \frac{1}{12}(\mathrm{d}x_N \wedge -2\iota_N)\left(\mathrm{d}G + *\left(\mathrm{d}*G + \frac{1}{2}G\wedge G\right)\right)\,, \tag{B.24}$$

where recall that $\mathcal{E}_{NP} = 0$ is the Einstein equation of motion from (4.10).

The argument is now standard, and we only repeat it here for completeness. If the G Bianchi identity and equations of motion hold (away from M2 and M5 branes), acting with (B.24) on a supercharge ϵ we get $\mathcal{E}_{NP}\Gamma^P\epsilon = 0$. For a Majorana spinor ϵ in $d = 11$, the bilinear $K_M = \bar{\epsilon}\Gamma_M\epsilon$ is either time-like or null. If it is timelike, ϵ has no one-form that annihilates it; so all components of $\mathcal{E}_{NP}$ are zero. If K_M is null, it is the only one-form that annihilates ϵ. In an adapted vielbein such that $K = e^+$, all components of $\mathcal{E}_{NP}$ except $\mathcal{E}_{N-}$ are zero; since $\mathcal{E}$ is symmetric, in fact all components except $\mathcal{E}_{--}$ are zero.[17]

C Inclusion of M2-branes

In this appendix we provide details of the derivation of (4.29) and of the bound (4.30).

C.1 M2 BPS-energy

We first derive the second term appearing on the r.h.s. of (4.29), which comes from the contribution of the last term of (4.26) inside (4.7):

$$\begin{aligned}&\frac{1}{4}\int_S *\mathrm{d}x_N\left(*\left[\mathrm{d}x^N\wedge\delta^{(8)}(\mathcal{C})\right]\right)\cdot\Omega^{(\mathrm{M2})} = -\frac{1}{4}\int_S *e^{\underline{0}}\left(*\left[e^{\underline{0}}\wedge\delta^{(8)}(\mathcal{C})\right]\right)\cdot\Omega^{(\mathrm{M2})}\\&= -\frac{1}{4}\int_S \mathrm{dvol}_S\left(*_S\,\delta^{(8)}(\mathcal{C})|_\Sigma\right)\cdot\Omega^{(\mathrm{M2})} = -\frac{1}{4}\int_S \Omega^{(\mathrm{M2})}\wedge\delta^{(8)}(\mathcal{C}) = -\frac{1}{4}\int_{\mathcal{C}\cap S}\Omega^{(\mathrm{M2})}\,,\end{aligned}\qquad\text{(C.1)}$$

where in the second line we have used the adapted vielbein $e^A = (e^{\underline{0}}, e^a)$ introduced in (4.18). Notice that instead the final result does not depend on such a choice.

In order to obtain the first term on the r.h.s. of (4.29), we first derive the formula (4.25) of $T^{MN}_{(\mathrm{M2})}$. This can be extracted from the variation of the M2 action (4.21) under a metric deformation

$$\begin{aligned}&\frac{1}{2}\int\delta g_{MN}T^{MN}_{(\mathrm{M2})}*1 \equiv \frac{1}{2\pi}\delta S_{(\mathrm{M2})} = -\frac{1}{2}\int_{\mathcal{C}}\mathrm{d}^3\sigma\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\delta g_{MN}\\&= -\frac{1}{2}\int_{\mathcal{C}}e^{\underline{012}}\,\eta^{\underline{\alpha\beta}}e_{\underline{\alpha}}(X^M)e_{\underline{\beta}}(X^N)\delta g_{MN} = -\frac{1}{2}\int\delta g_{MN}e^M_{\underline{\alpha}}\,e^N_{\underline{\beta}}\,\eta^{\underline{\alpha\beta}}\,e^{\underline{012}}\wedge\delta^{(8)}(\mathcal{C})\\&= \frac{1}{2}\int\delta g_{MN}e^M_{\underline{\alpha}}\,e^N_{\underline{\beta}}\,\eta^{\underline{\alpha\beta}}\,*\left[e^{\underline{012}}\wedge\delta^{(8)}(\mathcal{C})\right]*1\,,\end{aligned}\qquad\text{(C.2)}$$

where we have used the adapted vielbein (4.24).

By using (4.25) in (4.26), we see that it contributes to (4.7) by the term

$$\begin{aligned}&\frac{1}{4}\int K_M e^M_{\underline{\alpha}}\,e^N_{\underline{\beta}}\,\eta^{\underline{\alpha\beta}}\,*\left[e^{\underline{012}}\wedge\delta^{(8)}(\mathcal{C})\right]*\mathrm{d}x_N = \frac{1}{4}\int K_{\underline{\alpha}}*e^{\underline{\alpha}}\,*\left[e^{\underline{012}}\wedge\delta^{(8)}(\mathcal{C})\right]\\&= \frac{1}{4}\int K^{\underline{\alpha}}\,\iota_{\underline{\alpha}}\left[e^{\underline{012}}\wedge\delta^{(8)}(\mathcal{C})\right] = \frac{1}{4}\int K^{\underline{0}}\,e^{\underline{12}}\wedge\delta^{(8)}(\mathcal{C}) = \frac{1}{4}\int_{\mathcal{C}\cap S}K^{\underline{0}}\,\mathrm{vol}_{\mathcal{C}\cap S}\,,\end{aligned}\qquad\text{(C.3)}$$

where we have used a further adapted vielbein combining the properties of (4.18) and (4.24). This completes our derivation of the M2 BPS energy (4.29).

[17] If there is at least another supercharge with a different K (which is the case for vacuum compactifications with a $d \geq 3$ external spacetime) then all components $\mathcal{E}_{NP} = 0$.

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C.2 M2 bound

Consider a spacetime foliated by space-like leaves S parametrized by a 'time' t. By using adapted coordinates $x^M = (t, x^m)$, the line element can be written as

$$\mathrm{d}s^2 = \mathrm{e}^{2D}\mathrm{d}t^2 + \hat{g}_{mn}(\mathrm{d}x^m + \hat{V}^m\mathrm{d}t)(\mathrm{d}x^n + \hat{V}^n\mathrm{d}t)\,, \tag{C.4}$$

where $\hat{g}_{mn}$ is the metric induced on S. The adapted vielbein (4.18) is then

$$e^{\underline{0}} = \mathrm{e}^D\mathrm{d}t\,, \qquad e^a = \hat{e}^a_m(\mathrm{d}x^m + \hat{V}^m\mathrm{d}t)\,, \tag{C.5}$$

where $\hat{e}^a = \hat{e}^a_m\mathrm{d}x^m$ is a vielbein on S. Note also that the dual frame $e_A = (e_{\underline{0}}, e_a)$ is such that

$$e_{\underline{0}} = \mathrm{e}^{-D}(\partial_t - \hat{V}^m\partial_m)\,, \qquad e_a = \hat{e}_a\,. \tag{C.6}$$

Now consider an M2-brane world-volume $\mathcal{C}$. One can impose a partial static gauge and use t as time coordinate along $\mathcal{C}$. The world-volume coordinates $\sigma^\alpha = (t, \sigma^i)$ induce a decomposition of the world-volume line-element

$$\mathrm{d}s^2_{\mathcal{C}} = \mathrm{e}^{2D}\mathrm{d}t^2 + \hat{h}_{ij}(\mathrm{d}\sigma^i + \hat{v}^i\mathrm{d}t)(\mathrm{d}\sigma^j + \hat{v}^j\mathrm{d}t)\,, \tag{C.7}$$

where $\hat{h}_{ij}$ is the induced metric on $\mathcal{C} \cap S$. Furthermore, we can pick an adapted bulk vielbein $e^A = (e^{\underline{0}}, e^a) = (e^{\underline{\alpha}}, e^{\tilde{a}})$ satisfying the properties of (4.18) and (4.24). Hence, the pull-back of $e^{\underline{\alpha}}$ to $\mathcal{C}$ gives a vielbein $\mathfrak{e}^{\underline{\alpha}}$ of $\mathrm{d}s^2_{\mathcal{C}}$. Note also that on $\mathcal{C}$ we can identify the push-forward of the dual frame $\mathfrak{e}_{\underline{\alpha}}$ with the bulk $e_{\underline{\alpha}}$.

By following [31], we can introduce the world-volume momentum density

$$\mathcal{P}^M = -\sqrt{-h}\,h^{t\alpha}\partial_\alpha X^M\,. \tag{C.8}$$

Notice that $\mathcal{P}^M$ and $\bar{\chi}\Gamma^M\chi$ are both causal and future-pointing, for any spinor χ. Hence

$$-\mathcal{P}^M\bar{\chi}\Gamma_M\chi \geq 0\,, \tag{C.9}$$

where the inequality is saturated if and only if $\chi = 0$. By choosing $\chi = (\mathbf{1} - \Gamma_{\text{M2}})\epsilon$ with

$$\Gamma_{\text{M2}} \equiv \frac{\epsilon^{\alpha\beta\gamma}}{3!\sqrt{-h}}\partial_\alpha X^M\partial_\beta X^N\partial_\gamma X^P\Gamma_{MNP}\,, \tag{C.10}$$

and recalling that $K^M = \bar{\epsilon}\Gamma^M\epsilon$, we get the inequality

$$-\mathcal{P}^M K_M \geq -\mathcal{P}^M\bar{\epsilon}\Gamma_M\Gamma_{\text{M2}}\epsilon\,. \tag{C.11}$$

This is saturated iff $\Gamma_{\text{M2}}\epsilon = \epsilon$.

By using the adapted vielbein above, we can write

$$\begin{aligned}
-\mathcal{P}^M\bar{\epsilon}\Gamma_M\Gamma_{\text{M2}}\epsilon &= -\frac{1}{3!}\mathcal{P}^M e^{\underline{\alpha}}_M\,\epsilon^{\underline{\beta\gamma\delta}}\,\bar{\epsilon}\Gamma_{\underline{\alpha}}\Gamma_{\underline{\beta\gamma\delta}}\epsilon = -\frac{1}{2}\mathcal{P}^M e^{\underline{\alpha}}_M\eta_{\underline{\alpha\beta}}\,\epsilon^{\underline{\beta\gamma\delta}}\,\bar{\epsilon}\Gamma_{\underline{\gamma\delta}}\epsilon \\
&= -\frac{1}{2}\mathcal{P}^M e^{\underline{\alpha}}_M\eta_{\underline{\alpha\beta}}\,\epsilon^{\underline{\beta\gamma\delta}}\,\Omega^{(\text{M2})}_{\underline{\gamma\delta}} = -\frac{1}{2\sqrt{-h}}\mathcal{P}^M e^{\underline{\alpha}}_M\eta_{\underline{\alpha\beta}}\mathfrak{e}^{\underline{\beta}}_\beta\epsilon^{\beta\gamma\delta}\Omega^{(\text{M2})}_{\gamma\delta} \\
&= \frac{1}{2}h^{t\alpha}h_{\alpha\beta}\epsilon^{\beta\gamma\delta}\Omega^{(\text{M2})}_{\gamma\delta} = \frac{1}{2}\epsilon^{t\alpha\beta}\Omega^{(\text{M2})}_{\beta\gamma}\,,
\end{aligned} \tag{C.12}$$

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and then

$$-\mathcal{P}^M\bar{\epsilon}\Gamma_M\Gamma_{\text{M2}}\epsilon\,\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2=\Omega^{(\text{M2})}|_{\mathcal{C}\cap S}\,. \tag{C.13}$$

Hence, with respect to the oriented two-form $\mathrm{d}^2\sigma\equiv\mathrm{d}\sigma^1\wedge\mathrm{d}\sigma^2>0$ along $\mathcal{C}\cap S$, we can write (C.11) in the form

$$-\mathcal{P}^M K_M\mathrm{d}^2\sigma\geq\Omega^{(\text{M2})}|_{\mathcal{C}\cap S}\,. \tag{C.14}$$

This is the M2 counterpart of ([31], eq. (3.25)) for F1 strings. On the other hand

$$\begin{aligned}-\mathcal{P}^M K_M\mathrm{d}^2\sigma&=\sqrt{-h}\,h^{t\alpha}\partial_\alpha X^M e^A_M K_A\mathrm{d}^2\sigma=\sqrt{-h}\,h^{t\alpha}\mathfrak{e}^{\underline{\alpha}}_\alpha K_{\underline{\alpha}}\mathrm{d}^2\sigma\\&=\sqrt{-h}\,\mathfrak{e}^t_{\underline{\alpha}}K^{\underline{\alpha}}\,\mathrm{d}^2\sigma=K^{\underline{0}}\,\mathrm{dvol}_{\mathcal{C}\cap S}\,,\end{aligned} \tag{C.15}$$

where in the last step we have used $\mathfrak{e}^t_{\underline{\alpha}}=\delta^{\underline{0}}_{\underline{\alpha}}\mathrm{e}^{-D}$ and

$$\sqrt{-h}\,\mathrm{d}t\wedge\mathrm{d}^2\sigma=\mathfrak{e}^{\underline{0}}\wedge\mathfrak{e}^{\underline{1}}\wedge\mathfrak{e}^{\underline{2}}=\mathrm{e}^D\sqrt{\hat{h}}\,\mathrm{d}t\wedge\mathrm{d}^2\sigma=\mathrm{e}^D\mathrm{d}t\wedge\mathrm{dvol}_{\mathcal{C}\cap S}\,. \tag{C.16}$$

Hence (C.14) is equivalent to

$$K^{\underline{0}}\,\mathrm{dvol}_{\mathcal{C}\cap S}\geq\Omega^{(\text{M2})}|_{\mathcal{C}\cap S}\,, \tag{C.17}$$

as stated in (4.30). Furthermore, supersymmetry is preserved if and only if the bound is saturated.

D Details on type II

In this appendix we describe the computation leading from (5.10) to (5.13). We will be less detailed than in section B.3; we advise the reader to read that first.

The first step is subtracting the terms on the first line of (5.13); this is lengthy and tedious, but in principle straightforward. Useful identities for this include:

$$\Gamma_M H\Gamma^H=-4H\,,\qquad\Gamma_M H\Gamma^{MN}=-4H\Gamma^N-\Gamma^N H\,, \tag{D.1}$$

which follow from (A.4) and from $\Gamma^M\Gamma^N=\Gamma^{MN}+g^{MN}1$; the usual definition $\{\Gamma^M,\Gamma^N\}=2g^{MN}1$ of Clifford algebra; and (A.15). The result is

$$\begin{aligned}\mathrm{e}^{2\phi}\nabla_M E^{MN}=&\,\overline{\left(\mathcal{D}_M-\frac{1}{8}\Gamma_M\mathcal{O}\right)\epsilon}\,\Gamma^{MPN}\left(\mathcal{D}_P-\frac{1}{8}\Gamma_P\mathcal{O}\right)\epsilon-\frac{1}{8}\overline{\mathcal{O}\epsilon}\,\Gamma^N\mathcal{O}\epsilon\\&-\bar{\epsilon}\,\Gamma^{MNP}(D_M D_P+[D_M,\Gamma^{MNP}\mathcal{A}_P])\epsilon-\bar{\epsilon}\left[\sum_{\alpha=1}^{6}Q^N_\alpha\right]\epsilon\end{aligned} \tag{D.2}$$

where the Q^N_α are terms quadratic in the fields, which we will write below; they will need further, less trivial processing.

D.1 Linear terms

We start from the linear terms in (D.2).

The $\bar{\epsilon}\Gamma^{MNP}D_M D_P\epsilon$ is evaluated with (4.13). For the remaining terms, we recall (5.7) and (A.16):

$$[D_M, \Gamma^{MNP}\mathcal{A}_P] = +\frac{1}{4}\Gamma^{[M}\nabla_M H\Gamma^{N]}\otimes\sigma_3 - \Gamma^{MN}\nabla_M \mathrm{d}\phi - 2\Gamma^{[M}\nabla_M\mathcal{F}\,\Gamma^{N]}\,. \tag{D.3}$$

The dilaton term gives

$$-\bar{\epsilon}\,\Gamma^{MN}\nabla_M\mathrm{d}\phi\,\epsilon = -\bar{\epsilon}(\Gamma^{MNP}+2\Gamma^{[M}g^{N]P})\nabla_M\partial_P\phi\epsilon = -(\nabla^P\nabla^N - g^{NP}\nabla^2)\phi\,\bar{\epsilon}\Gamma_P\epsilon\,. \tag{D.4}$$

The H term simplifies similar to (B.11):

$$\begin{aligned}\frac{1}{4}\bar{\epsilon}\,\Gamma^{[M}\nabla_M H\Gamma^{N]}\otimes\sigma_3\epsilon &= -\frac{1}{4}\bar{\epsilon}[(\mathrm{d}x\wedge+\iota)(\mathrm{d}x\wedge-\iota)]^{[MN]}\nabla_M H\otimes\sigma_3\epsilon\\ &= \frac{1}{4}\bar{\epsilon}(\mathrm{d}x^N\wedge\mathrm{d}H - \iota^N * \mathrm{d}*H)\otimes\sigma_3\epsilon\,.\end{aligned} \tag{D.5}$$

Following the same steps, recalling the definition of $\mathcal{F}$ in (5.3) and the self-duality property (5.1), the RR term becomes

$$-2\bar{\epsilon}\Gamma^{[M}\nabla_M\mathcal{F}\,\Gamma^{N]}\epsilon = \pm\frac{1}{4}\bar{\epsilon}_1(\mathrm{d}x^N\wedge\mathrm{d}(\mathrm{e}^\phi F))\epsilon_2 + \frac{1}{4}\bar{\epsilon}_2(\mathrm{d}x^N\wedge\mathrm{d}(\mathrm{e}^\phi\lambda F))\epsilon_1\,. \tag{D.6}$$

By (A.15), the two terms on the right-hand side are in fact equal to each other.

D.2 Quadratic tems

We now turn to the Q^N_α in (D.2). We organized them depending on what fields they contain.

We first give some useful identities. One is obtained by applying the Fierz identities to the bispinors $\epsilon_a\otimes\bar{\epsilon}_a$. As in (B.16), the only non-zero terms in the sum are those for $k = 1$, 5 and 9. Moreover the last is dual to the first. This results in

$$\begin{aligned}32\epsilon_1\otimes\bar{\epsilon}_1 &= (\bar{\epsilon}_1\,\Gamma_M\epsilon_1)(1+\Gamma)\Gamma^M + \frac{1}{5!}(\bar{\epsilon}_1\,\Gamma_{M_5\cdots M_1}\epsilon_1)\Gamma^{M_1\cdots M_5}\,,\\ 32\epsilon_2\otimes\bar{\epsilon}_2 &= (\bar{\epsilon}_2\,\Gamma_M\epsilon_2)(1\mp\Gamma)\Gamma^M + \frac{1}{5!}(\bar{\epsilon}_2\,\Gamma_{M_5\cdots M_1}\epsilon_2)\Gamma^{M_1\cdots M_5}\,.\end{aligned} \tag{D.7}$$

The first is purely quadratic in H:

$$\begin{aligned}64\bar{\epsilon}\,Q^N_1\epsilon &\equiv \bar{\epsilon}\,\Gamma_M H\Gamma^N H\Gamma^M\epsilon = \mathrm{Tr}(\Gamma_M H\Gamma^N H\Gamma^M(\epsilon_1\otimes\bar{\epsilon}_1+\epsilon_2\otimes\bar{\epsilon}_2))) \\ &\overset{(D.7),\,(A.4)}{=} -\frac{1}{4}\bar{\epsilon}_1\Gamma_M\epsilon_1\mathrm{Tr}(H\Gamma^N H(1-\Gamma)\Gamma^M) - \frac{1}{4}\bar{\epsilon}_2\Gamma_M\epsilon_2\mathrm{Tr}(H\Gamma^N H(1\pm\Gamma)\Gamma^M)\\ &\overset{(A.12a)}{=} 16T^{MN}_{(H)}\bar{\epsilon}\,\Gamma_M\epsilon\,.\end{aligned} \tag{D.8}$$

With similar steps,

$$\bar{\epsilon}\,Q^N_2\epsilon \equiv \bar{\epsilon}\,(\partial\!\!\!/\phi\Gamma^N\partial\!\!\!/\phi - 2\nabla^N\phi\partial\!\!\!/\partial\!\!\!/\phi)\epsilon = -g^{MN}|\mathrm{d}\phi|^2\bar{\epsilon}\,\Gamma_M\epsilon\,; \tag{D.9}$$

$$\begin{aligned}\bar{\epsilon}\,Q^N_3\epsilon &\equiv \bar{\epsilon}\,\Gamma_M\mathcal{F}\Gamma^N\mathcal{F}\Gamma^M\epsilon\\ &= \frac{\mathrm{e}^{2\phi}}{8}\Big((\bar{\epsilon}_1\,\Gamma_M\epsilon_1)(T^{MN}_F - *(F\wedge\lambda F)^{MN}) + (\bar{\epsilon}_2\,\Gamma_M\epsilon_2)(T^{MN}_F + *(F\wedge\lambda F))^{MN}\Big)\,.\end{aligned} \tag{D.10}$$

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In (D.10) we have also used (A.12b), and

$$(\mathrm{d}x^M \wedge \mathrm{d}x^N \wedge F \wedge \lambda F)_{10} = -(\iota^M F) \cdot (\mathrm{d}x^N \wedge F)\,, \tag{D.11}$$

which follows from repeated use of (A.8) and self-duality (5.1).

The remaining three Q^N_α require use of (A.3). We have

$$\begin{aligned} 8Q^N_4 &= [\{H, \Gamma^N\}, \partial\!\!\!/\phi] = \partial_M\phi(\overleftarrow{\Gamma}^M\overleftarrow{\Gamma}^N - \overrightarrow{\Gamma}^M\overrightarrow{\Gamma}^N - 2\,\overrightarrow{\Gamma}^{[M}\overleftarrow{\Gamma}^{N]})H \\ &= \partial_M\phi\left(-(\mathrm{d}x\wedge -\iota)^2 - (\mathrm{d}x\wedge +\iota)^2 + 2(\mathrm{d}x\wedge -\iota)(\mathrm{d}x\wedge +\iota)\right)^{[MN]} H \\ &= -4\partial_M\phi\iota^M\iota^N H = -4\iota_{\mathrm{d}\phi}\iota^N H = 4\iota^N * \mathrm{d}\phi \wedge *H\,, \end{aligned} \tag{D.12}$$

recalling (A.8) in the last step. A similar computation, skipping a few steps, gives

$$\begin{aligned} Q^N_5 &= \partial\!\!\!/\phi\mathcal{F}\Gamma^N - \Gamma^N\mathcal{F}\partial\!\!\!/\phi = 2\partial_M\phi\,\overrightarrow{\Gamma}^{[M}\overleftarrow{\Gamma}^{N]}\,\mathcal{F} \\ &= \frac{\mathrm{e}^\phi}{8}\left(\mp(1-*\lambda)\mathrm{d}x^N\wedge\mathrm{d}\phi\wedge F\otimes b^\dagger - (1-\lambda*)\mathrm{d}x^N\wedge\mathrm{d}\phi\wedge\lambda F\otimes b\right)\,. \end{aligned} \tag{D.13}$$

Finally, the most complicated term is

$$\begin{aligned} 8Q^N_6 &= [\Gamma_M, H\Gamma^N]\mathcal{F}\Gamma^M + \Gamma^M\mathcal{F}[\Gamma^N H, \Gamma_M] \\ &= \mp 8((H\wedge +\iota_H)(\mathrm{d}x-\iota)^N + (H\wedge -\iota_H)(\mathrm{d}x+\iota)^N)\mathcal{F} = \mp 16(H\wedge \mathrm{d}x^N - \iota_H\iota^N)\mathcal{F} \\ &= \mp\mathrm{e}^\phi(1-*\lambda)H\wedge\mathrm{d}x^N\wedge F\otimes b^\dagger - \mathrm{e}^\phi(1-\lambda*)H\wedge\mathrm{d}x^N\wedge\lambda F\otimes b\,; \end{aligned} \tag{D.14}$$

we defined $\iota_H \equiv \frac{1}{3!}H_{MNP}\iota^M\iota^N\iota^P$, and we used

$$\begin{aligned} \overrightarrow{H} + \overrightarrow{\Gamma}^M\overleftarrow{H}_M &= 4(H\wedge +\iota_H)\,, \\ \overleftarrow{H} + \overleftarrow{\Gamma}^M\overrightarrow{H}_M &= 4(H\wedge -\iota_H)(-1)^{\deg}\,, \end{aligned} \tag{D.15}$$

which can in turn be derived from (A.3).

Collecting all the results in this subsection finally takes (D.2) to (5.13).

E From M-theory to IIA

Consider the compactification of M-theory on a circle with periodic variable $y \simeq y+1$ with $\ell_\mathrm{s} = \ell_\mathrm{P} = 1$. Hatted quantities will refer to M-theory. The 11d coordinates are $x^{\hat M} = (x^M, y)$ and the $d = 11$ vielbein $\hat e^{\hat A} = (\hat e^A, \hat e^{\underline{0}})$ is related to the 10d one e^A by:

$$\hat e^A = \mathrm{e}^{-\phi/3}e^A\,, \qquad \hat e^{\underline{10}}\,\mathrm{e}^{2\phi/3} = (\mathrm{d}y - C_1)\,. \tag{E.1}$$

Correspondingly, $\hat\epsilon = \mathrm{e}^{-\phi/6}\epsilon \equiv \mathrm{e}^{-\phi/6}(\epsilon_1+\epsilon_2)$ while the 11d gravitino splits as follows into string frame gravitino and dilatino:

$$\hat\psi = \hat\psi_{\hat M}\mathrm{d}x^{\hat M} = \mathrm{e}^{-\phi/6}\left(\psi - \frac{1}{6}\Gamma_{(1)}\lambda\right) + \frac{1}{3}e^{5\phi/6}\Gamma^{\underline{10}}\lambda\,(\mathrm{d}y - C_1)\,. \tag{E.2}$$

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Consider now a foliation into space-like slices $\hat{S} = S \times S^1$ and the corresponding asymptotic boundaries $\partial\hat{S} = \partial S \times S^1$. By identifying $\Gamma^{\underline{10}} \equiv \Gamma$, the corresponding conserved supercharge reduces as follows

$$\begin{aligned}
\hat{Q}(\hat{\epsilon}) &= -\int_{\partial\hat{S}} \bar{\hat{\epsilon}}\hat{\Gamma}_{(8)} \wedge \hat{\psi} \\
&= -\int_{\partial\hat{S}} \mathrm{e}^{-2\phi}\bar{\epsilon}[e^{-\phi}\Gamma_{(8)} + \Gamma_{(7)}\Gamma \wedge (\mathrm{d}y - C_1)] \wedge \left[\psi - \frac{1}{6}\Gamma_{(1)}\lambda + \frac{1}{3}e^{\phi}\Gamma\lambda(\mathrm{d}y - C_1)\right] \\
&= -\int_{\partial\hat{S}} \mathrm{e}^{-2\phi}\bar{\epsilon}\Gamma\Gamma_{(7)} \wedge \left(\psi - \frac{1}{6}\Gamma_{(1)}\lambda\right) \wedge (\mathrm{d}y - C_1) + \frac{1}{3}\int_{\hat{S}} \mathrm{e}^{-2\phi}\bar{\epsilon}\Gamma\Gamma_{(8)}\lambda \wedge (\mathrm{d}y - C_1) \\
&= -\int_{\partial\hat{S}} \mathrm{e}^{-2\phi}\bar{\epsilon}\Gamma(\Gamma_{(7)} \wedge \psi - \Gamma_{(8)}\lambda) \wedge (\mathrm{d}y - C_1) \\
&= -\int_{\partial S} \mathrm{e}^{-2\phi}\bar{\epsilon}\Gamma\left(\Gamma_{(7)} \wedge \psi - \Gamma_{(8)}\lambda\right) \equiv Q(\epsilon)\,.
\end{aligned} \tag{E.3}$$

So we indeed get the type IIA supercharge (5.4).

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