Regularization of Brézis pseudomonotone variational inequalities

M. Bianchi[*], G. Kassay[†], R. Pini [‡]

Abstract

In this paper we prove the existence of solutions of regularized set-valued variational inequalities involving Brézis pseudomonotone operators in reflexive and locally uniformly convex Banach spaces. By taking advantage of this result, we approximate a general set-valued variational inequality with suitable regularized set-valued variational inequalities, and we show that their solutions weakly converge to a solution of the original one. Furthermore, by strengthening the coercivity conditions, we can prove the strong convergence of the approximate solutions.

Keywords. set-valued variational inequality; B-pseudomonotonicity; approximate solutions; equilibrium problem; Navier-Stokes operator

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1 Introduction

Let X be a real, reflexive Banach space, with dual X^*. We will assume that both X and X^* are renormed so that they are locally uniformly convex. Denote by $C \subseteq X$ a nonempty, convex and closed set. Given a set-valued map $T : X \rightrightarrows X^*$ with $C \subseteq \mathrm{dom}(T)$, the set-valued variational inequality $\mathrm{VI}(T, C)$ is to find $\overline{x} \in C$ such that

$$\sup_{x^* \in T(\overline{x})} \langle x^*, y - \overline{x} \rangle \geq 0 \quad \text{for all } y \in C. \tag{1.1}$$

Whenever T is weakly-compact valued for all $x \in C$, then $\mathrm{VI}(T, C)$ is equivalent to finding $\overline{x} \in C$ such that, for every $y \in C$, there exists $x^*(y) \in T(\overline{x})$ satisfying

$$\langle x^*(y), y - \overline{x} \rangle \geq 0.$$

In case $\overline{x}$ satisfies the inequality above, $\overline{x}$ is said to be a *weak solution* of $\mathrm{VI}(T, C)$. If $x^*(y)$ can be chosen independently of $y \in C$, then $\overline{x}$ is said to be a *strong solution* of $\mathrm{VI}(T, C)$.

[*]Università Cattolica del Sacro Cuore, Milano (monica.bianchi@unicatt.it)

[†]Babes-Bolyai University Cluj (kassay@math.ubbcluj.ro)

[‡]Università degli Studi di Milano-Bicocca (rita.pini@unimib.it)

Note that, by the Sion minimax theorem (see [?]), every weak solution is a strong solution in case T is weakly-compact and convex valued on C (see, for instance, [?]).

Several problems in applied mathematics can be formulated via set-valued variational inequalities and many authors have contributed to the study of this problem. A great number of direct applications of set-valued variational inequalities can be found, for instance, in Economics, Transportation, and Operations Research ([?], [?]).

Most of the existence results require some kind of continuity properties together with suitable assumptions of monotonicity of the operator T. One is the notion of pseudomonotonicity introduced by Karamardian and Schaible, arising from and extending the classical notion of monotonicity for operators; in this framework some existence results can be found, for instance, in [?] and the references therein. A different kind of pseudomonotonicity for operators, known as Brézis pseudomonotonicity (B-pseudomonotonicity, for short), was introduced by Brézis when dealing with integral equations and partial differential equations. As a matter of fact, the theory of B-pseudomonotone operators has played an important role in the study of solvability of operator equations and quasi–linear elliptic equations. To this purpose, the reader could refer for instance to [?], Ch. 27.

This notion of Brézis pseudomonotonicity is not directly related to any classical monotonicity property, but the operators are required to satisfy a different topological condition. However, under some kind of continuity, the classical monotone operators are B-pseudomonotone (see Proposition 1 below). Recently some authors provided existence of solutions for set-valued variational inequalities under B-pseudomonotonicity of the operator T (see, for instance, [?], [?], [?]).

Our purpose is to prove the solvability of $\mathrm{VI}(T, C)$ by approximating the pair (T, C) with a sequence of more regular operators $T_k : X \rightrightarrows X^*$, and sets $C_k \subseteq \mathrm{dom}(T_k)$. We study the existence of solutions $\overline{x}_k \in C_k$ of the regularized variational inequality $\mathrm{VI}(T_k + \alpha_k J, C_k)$

$$\sup_{x^* \in T_k(\overline{x}_k) + \alpha_k J(\overline{x}_k)} \langle x^*, y - \overline{x}_k \rangle \geq 0 \quad \text{for all } y \in C_k, \tag{1.2}$$

where J denotes the duality map. The solvability of $\mathrm{VI}(T_k + \alpha_k J, C_k)$ is proved in terms of standard assumptions which can be on the whole referred to results on equilibrium problems that can be found in some papers by Brézis et al. ([?], [?]). The main concern will be to relate the solutions of the regularized problems with possible solutions of the original $\mathrm{VI}(T, C)$. As a matter of fact, we will show that, under suitable closeness of the images of T_k and T, and good behaviour of C_k with respect to C, every weak cluster point of any sequence of strong solutions of $\mathrm{VI}(T_k + \alpha_k J, C_k)$ is a weak (strong) solution of $\mathrm{VI}(T, C)$. Under more restrictive assumptions, the strong convergence can be proved.

The motivation of our study can be summarized as follows. It is well known that the problem of finding a solution of $\mathrm{VI}(T, C)$ is often ill-posed and small perturbations of the data T and C can lead to significant changes in the solution set (see [?] and the references therein). This ill-posedness creates difficulties in application where often only approximations T_k and C_k of the data are available. To overcome this issue Tikhonov [?] and Browder [?] introduced the notion of regularized variational inequality, which turns out to be usually well-posed and, thus, less sensitive to data perturbations. To broaden the applicability of their results, it is important to study the convergence of the solutions of the regularized variational inequalities to the solution of the original problem for other classes of operators, like Brézis pseudomonotone operators, which, as we have already pointed out, are useful in the study of operator equations.

The paper is organized as follows: In Section 2 some preliminaries on set-valued B-pseudomonotone operators and notations are recalled. Noting that to every set-valued operator $T : X \rightrightarrows X^*$ one can naturally associate the bifunction $G_T : X \times X \to \mathbb{R} \cup \{+\infty\}$, given by

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle,$$

in Section 3 we study existence results for a set-valued variational inequality via the solvability of the equilibrium problem related to G_T. In addition, taking advantage of the nice properties of the duality map J, we provide an existence result for VI$(T + \alpha J, C)$, under mild conditions on T. In Section 4 we face the main problem, i.e., we prove an existence result for the original variational inequality following two steps: first, we show that every sequence of strong solutions of VI$(T_k + \alpha_k J, C_k)$ has a weak cluster point in C. Then, we show that every weak cluster point is indeed a weak (strong) solution for VI(T, C). Besides, more restrictive assumptions will lead to strong convergence.

2 Preliminaries and notations

In order to investigate the existence of solutions of VI(T, C), we will focus on a notion of monotonicity property for the operator T that was introduced by Brézis for single-valued operators in 1968 (see [?]), called in the sequel *B-pseudomonotonicity*. Let us emphasize that this kind of monotonicity has no direct relation with the monotonicity of mappings with respect to an ordering as the one introduced by Karamardian and Schaible in [?] and commonly known as *algebraic pseudomonotonicity* (see for instance [?]). In [?] the author defined a mapping $A : X \to X^*$ to be B-pseudomonotone on a set $D \subseteq X$ if, whenever $\{x_n\} \subset D$, $x_n \rightharpoonup x$ and $\limsup_n \langle A(x_n), x_n - x \rangle \leq 0$, then,

$$\langle A(x), x - y \rangle \leq \liminf_n \langle A(x_n), x_n - y \rangle, \quad \forall y \in D,$$

where $\rightharpoonup$ denotes the weak convergence.

This definition can be extended to set-valued maps as follows (see, for instance, [?], Definition 1 (c), or [?], Definition 6.1-(c), p.365):

Definition 1. We say that $T : X \rightrightarrows X^*$ is B-pseudomonotone on a nonempty subset D of dom(T) if, for every $\{x_n\}$ in D such that $x_n \rightharpoonup x \in D$, and for every $x_n^* \in T(x_n)$, with $\limsup_n \langle x_n^*, x_n - x \rangle \leq 0$, one has that for every $y \in D$, there exists $x^*(y) \in T(x)$ such that $\langle x^*(y), x - y \rangle \leq \liminf_n \langle x_n^*, x_n - y \rangle$.

In [?] Brézis showed that radially continuous monotone single-valued mappings are B-pseudomonotone. In order to extend this result to set-valued mappings, we need the following properties of set-valued mappings:

Definition 2. Let $T : X \rightrightarrows X^*$ and $D \subseteq \mathrm{dom}(T)$. We say that T is *radially continuous* on D if $\mathrm{gph}(T)$ satisfies the condition: for all $x, y \in D$, $t_k \in (0,1]$, $t_k \to 0$, if

$$((1-t_k)x + t_k y, z_k^*) \in \mathrm{gph}(T), \quad \text{and} \quad z_k^* \rightharpoonup z^*,$$

then $(x, z^*) \in \mathrm{gph}(T)$.

T is said to be *s-w-closed* on D if, for any $(x_k, x_k^*) \in \mathrm{gph}(T|_D)$, if $x_k \to x$ and $x_k^* \rightharpoonup x^*$, then $(x, x^*) \in \mathrm{gph}(T|_D)$.

Note that if T is s-w-closed on D, then T is radially continuous on D.

By extending a result due to Brézis in [?] for the case of single-valued operators, we prove that monotone operators satisfying a suitable continuity condition are B-pseudomonotone. Let us recall that an operator $T : X \rightrightarrows X^*$ is said to be *monotone* on $D \subseteq \mathrm{dom}(T)$ if

$$\langle x^* - y^*, x - y\rangle \geq 0, \quad \forall x, y \in D, \quad \forall x^* \in T(x), \forall y^* \in T(y).$$

The following proposition holds:

Proposition 1. Let $C \subseteq \mathrm{int}(\mathrm{dom}(T))$, C closed and convex, and $T : X \rightrightarrows X^*$ be monotone, and radially continuous on C. Then T is B-pseudomonotone on C.

Proof: Take any $\{x_n\} \subset C$, $x_n^* \in T(x_n)$, such that

$$x_n \rightharpoonup x \in C, \quad \liminf_n \langle x_n^*, x - x_n\rangle \geq 0. \tag{2.1}$$

By the monotonicity,

$$\langle x_n^* - x^*, x_n - x\rangle \geq 0 \quad \forall x^* \in T(x),$$

i.e.,

$$\langle x_n^*, x - x_n\rangle \leq \langle x^*, x - x_n\rangle.$$

Since $\langle x^*, x - x_n\rangle \to 0$, we have that

$$\limsup_n \langle x_n^*, x - x_n\rangle \leq 0.$$

Therefore, from (**??**),

$$\langle x_n^*, x - x_n\rangle \to 0. \tag{2.2}$$

For every $y \in C$, let $z_k = (1 - \frac{1}{k})x + \frac{1}{k}y \in C$, and note that $z_k \to x$ in the norm topology. For every $k \in \mathbb{N}$, from the monotonicity,

$$\langle x_n^* - z_k^*, x_n - z_k\rangle \geq 0, \qquad \forall z_k^* \in T(z_k),$$

i.e.,

$$\frac{1}{k}\langle x_n^*, x-y\rangle \geq \langle x_n^*, x-x_n\rangle + \langle z_k^*, x_n - x\rangle + \frac{1}{k}\langle z_k^*, x-y\rangle.$$

From (**??**) and the assumption $x_n \rightharpoonup x$, taking the lower limit of both sides, we get, for every k,

$$\liminf_n \langle x_n^*, x-y\rangle \geq \langle z_k^*, x-y\rangle.$$

Taking into account that every monotone operator is locally bounded in the interior of its domain, there exists $k_0 \in \mathbb{N}$, and $M > 0$ such that

$$\bigcup_{k\geq k_0} T(z_k) \subseteq \overline{B}_{X^*}(0, M),$$

where $\overline{B}_{X^*}(0, M)$ denotes the closed ball in X^* centred at 0 and with radius M. Therefore $\{z_k^*\} \subset \overline{B}_{X^*}(0, M)$ for $k \geq k_0$, and, by the reflexivity of X^*, there exists $\{k_m\}$ such that $z_{k_m}^*$ weakly converges to $\overline{x}^* \in \overline{B}_{X^*}(0, M)$. From the radial continuity of T we get that $\overline{x}^* \in T(x)$. Thus,

$$\liminf_n \langle x_n^*, x-y\rangle \geq \langle \overline{x}^*, x-y\rangle.$$

Since $\langle x_n^*, x_n - y\rangle = \langle x_n^*, x_n - x\rangle + \langle x_n^*, x-y\rangle$, by (**??**)

$$\liminf_n \langle x_n^*, x_n - y\rangle = \liminf_n \langle x_n^*, x-y\rangle \geq \langle \overline{x}^*, x-y\rangle,$$

thereby proving the assertion.

□

Let us finally recall some notions and notations that will appear in the forthcoming sections. Given a sequence $\{C_k\}$ of subsets of X, we set

$$s - \liminf C_k = \{x \in X : \exists\{x_k\},\, x_k \in C_k : x_k \to x\},$$

$$w - \limsup C_k = \{x \in X : \exists\{x_{n_k}\},\, x_{n_k} \in C_{n_k} : x_{n_k} \rightharpoonup x\}.$$

In particular, the sequence $\{C_k\}$ is said to be *Mosco convergent* to C if

$$s - \liminf C_k = w - \limsup C_k = C.$$

Moreover, given two subsets A, B of a metric space (E, d_E), the excess from A to B is defined as

$$e(A, B) := \sup_{a\in A} d_E(a, B) = \sup_{a\in A} \inf_{b\in B} d_E(a, b),$$

under the convention $e(\emptyset, B) := 0$, and $e(A, \emptyset) := +\infty$, for $A \neq \emptyset$. The *Hausdorff distance* between A and B is defined as

$$\mathrm{Haus}(A, B) := \max\{e(A, B), e(B, A)\}.$$

The duality mapping $J : X \rightrightarrows X^*$ is defined as follows

$$J(x) := \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}.$$

The map J has nonempty, closed, convex and bounded values. In our setting of local uniform convexity of X, the duality mapping turns out to be single-valued, coercive (i.e., $\lim_{\|x\|\to+\infty} \frac{\langle J(x),x\rangle}{\|x\|} = +\infty$), demicontinuous (i.e., for every $w \in X$, the map $x \mapsto \langle J(x), w\rangle$ is continuous, or, equivalently, J is w^*-continuous), and B-pseudomonotone.

3 Equilibrium problems and existence results for set-valued variational inequalities

The main aim of this section is to provide existence results for the so-called *regularized variational inequality* $\mathrm{VI}(T + \alpha J, C)$ defined as follows: find $\overline{x} \in C$ such that

$$\sup_{\overline{x}^* \in T(\overline{x})} \langle \overline{x}^* + \alpha J(\overline{x}), y - \overline{x}\rangle \geq 0, \quad \forall y \in C, \tag{3.1}$$

where $T : X \rightrightarrows X^*$, C is a closed and convex subset of $\mathrm{dom}(T)$ and α is a positive number.

We start our analysis by investigating existence results for the initial problem $\mathrm{VI}(T, C)$ associated to the operator T. Note that, by introducing the bifunction $G_T : C \times C \to \mathbb{R} \cup \{+\infty\}$ given by

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle,$$

the $\mathrm{VI}(T, C)$ corresponds to the equilibrium problem (EP): find $x \in C$ such that

$$G_T(x, y) \geq 0, \quad \forall y \in C.$$

The equilibrium problems were formulated by Blum and Oettli in [**?**], and in recent years several authors have investigated existence results for (EP), but the first ones date back to the seventies ([**?**], [**?**]).

Let us recall a well-known existence result for (EP) that holds in the more general setting of Hausdorff topological vector spaces:

Theorem 1. (see [**?**], Theorem 1) Let C be a nonempty, closed and convex subset of a Hausdorff topological vector space E, and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

i. $f(x, x) \geq 0$ for all $x \in C$;

ii. for every $x \in C$, the set $\{y \in C : f(x, y) < 0\}$ is convex;

iii. for every $y \in C$, the function $f(\cdot, y)$ is upper semicontinuous on the intersection of C with any finite dimensional subspace Z of E;

iv. whenever $x, y \in C$, $x_n \in C$, $x_n \to x$ and $f(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0,1]$ and for all n, then $f(x,y) \geq 0$;

v. if C is unbounded, there exists a compact subset K of E, and $y_0 \in K \cap C$ such that $f(x, y_0) < 0$ for every $x \in C \setminus K$.

Then, there exists $\overline{x} \in C \cap K$ such that

$$f(\overline{x}, y) \geq 0 \quad \text{for all } y \in C.$$

Note that condition iv. applied to the bifunction G_T is equivalent to a property of the operator T which is termed *C-pseudomonotonicity* in [**?**].

By applying the previous theorem to the bifunction G_T, we can prove an existence result for VI(T, C) (for related results, see also [**?**], [**?**], [**?**] and [**?**]). Let us first recall the following Berge-type result:

Lemma 1. (see [**?**], Proposition 3.3, p. 83) Let E_1, E_2 be Hausdorff topological spaces, $u : E_1 \times E_2 \to \overline{\mathbb{R}}$ be an upper semicontinuous function, and $F : E_2 \rightrightarrows E_1$ be an upper semicontinuous map with nonempty, compact values. Then, the value function $v : E_2 \to \overline{\mathbb{R}}$ given by $v(y) = \sup_{x \in F(y)} u(x,y)$ is upper semicontinuous.

In our setting, $E_1 = X^*$ will be endowed with the weak topology, and $E_2 = C \cap Z$, with the strong topology, where Z is a finite dimensional subspace of X. We get the following result:

Theorem 2. Let $T : X \rightrightarrows X^*$ and C be a nonempty, closed and convex subset of dom(T). Suppose that:

i. $T(x)$ is bounded, closed and convex for every $x \in C$;

ii. T satisfies the following property: for every finite dimensional subspace Z of X, for every $\{x_k\} \subset C \cap Z$, $x_k \to x$, and $x_k^* \in T(x_k)$, there is a subsequence $\{x_{k_n}^*\}$ converging in the weak topology to some point in $T(x)$;

iii. T is B-pseudomonotone on C;

iv. if C is unbounded, there exists a weakly compact subset K of X, and $y_0 \in K \cap C$ such that $G_T(x, y_0) < 0$ for every $x \in C \setminus K$.

Then, VI(T, C) is solvable, and all weak solutions of this problem are strong.

Proof. We will show that all the assumptions of Theorem **??** are satisfied by the bifunction $G_T(x,y) = \sup_{x^* \in T(x)} \langle x^*, y - x \rangle$:

- i. and ii. of Theorem **??** follow easily from the definition of G_T.

- Under assumptions i. and ii., and the continuity in the weak $\times$ norm topology on $X^* \times (C \cap Z)$ of the function $(x^*, x) \to \langle x^*, y - x\rangle$, by applying Lemma **??** we get that, for every $y \in C$, the function $G_T(\cdot, y)$ is upper semicontinuous on the intersection of C with any finite dimensional space Z of X; therefore iii. of Theorem **??** is fulfilled.

- Let us now prove that iv. in Theorem **??** is implied by i. and iii. This could be done by adapting Theorem 13 in [**?**] to our case. For the reader's convenience, we provide below a direct proof. Let $x, y \in C$, $x_n \in C$, $x_n \rightharpoonup x$ and $G_T(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0, 1]$, that is
$$\inf_{t \in [0,1]} \sup_{x_n^* \in T(x_n)} \langle x_n^*, (1-t)x + ty - x_n\rangle \geq 0.$$
First, note that, from i.,
$$\sup_{x_n^* \in T(x_n)} \langle x_n^*, (1-t)x + ty - x_n\rangle = \max_{x_n^* \in T(x_n)} \langle x_n^*, (1-t)x + ty - x_n\rangle.$$
Moreover, by the Sion minimax theorem,
$$\inf_{t \in [0,1]} \left(\max_{x_n^* \in T(x_n)} \langle x_n^*, (1-t)x + ty - x_n\rangle \right) = \max_{x_n^* \in T(x_n)} \left(\inf_{t \in [0,1]} \langle x_n^*, (1-t)x + ty - x_n\rangle \right) \geq 0.$$
Thus there exists $\bar{x}_n^* \in T(x_n)$ such that
$$\langle \bar{x}_n^*, (1-t)x + ty - x_n\rangle \geq 0 \quad \text{for all } t \in [0, 1]. \tag{3.2}$$
In particular, for $t = 0$, we get $\langle \bar{x}_n^*, x - x_n\rangle \geq 0$ and thus
$$\limsup_n \langle \bar{x}_n^*, x_n - x\rangle \leq 0.$$
From the B-pseudomonotonicity we get in particular that, for every $y \in C$, there exists $x^*(y) \in T(x)$ such that
$$\langle x^*(y), x - y\rangle \leq \liminf_n \langle x_n^*, x_n - y\rangle.$$
Thus
$$G_T(x, y) \geq \limsup_n \langle x_n^*, y - x_n\rangle \geq 0,$$
where the last inequality follows from (**??**) for $t = 1$.

$\square$

Taking into account Theorem **??** we are now in the position to prove an existence result for VI$(T + \alpha J, C)$:

Theorem 3. Let $T : X \rightrightarrows X^*$ be an operator and C be a nonempty, closed and convex subset of $\mathrm{dom}(T)$. Suppose that T satisfies i.-iii. of Theorem **??**. Moreover, in case C is unbounded, assume that there exists $y_0 \in C$ such that

$$\limsup_{\|x\|\to+\infty,\, x\in C} \frac{G_T(x, y_0)}{\|x - y_0\|^{2-\epsilon}} < +\infty, \tag{3.3}$$

for some positive ϵ. Then, $\mathrm{VI}(T + \alpha J, C)$ is solvable, and all weak solutions of this problem are strong.

Proof: Let us verify that the set–valued mapping $T_\alpha = T + \alpha J$ satisfies the assumptions of Theorem **??**, for any $\alpha > 0$. Indeed, the images $T_\alpha(x)$ are bounded, closed and convex for all $x \in C$. From the demicontinuity of J, easy computations show that T_α satisfies the assumption ii. in Theorem **??**. Moreover, since the sum of B-pseudomonotone operators is B-pseudomonotone, (see [**?**], Proposition 6.15, p.368), by the properties of J, we get that T_α is B-pseudomonotone.

Finally, the coercivity assumption (**??**), together with the coercivity property of J, entails that T_α satisfies iv. in Theorem **??** with respect to y_0. In fact, note that

$$G_{T_\alpha}(x, y_0) = G_T(x, y_0) + \alpha\langle J(x), y_0 - x\rangle.$$

Therefore, taking into account (**??**), we get

$$\begin{aligned} &\limsup_{\|x\|\to+\infty,\, x\in C} \frac{G_{T_\alpha}(x, y_0)}{\|x - y_0\|^{2-\epsilon}} \\ &\qquad = \limsup_{\|x\|\to+\infty,\, x\in C} \left(\frac{G_T(x, y_0)}{\|x - y_0\|^{2-\epsilon}} + \alpha \frac{\langle J(x), y_0 - x\rangle}{\|x - y_0\|^{2-\epsilon}} \right) \\ &\qquad \le \limsup_{\|x\|\to+\infty,\, x\in C} \frac{G_T(x, y_0)}{\|x - y_0\|^{2-\epsilon}} + \alpha \limsup_{\|x\|\to+\infty,\, x\in C} \frac{\langle J(x), y_0 - x\rangle}{\|x - y_0\|^{2-\epsilon}} \\ &\qquad = -\infty. \end{aligned}$$

This implies that there exists $M > \|y_0\|$, such that

$$G_{T_\alpha}(x, y_0) < 0, \quad \forall x \in C,\ \|x\| > M.$$

Therefore, $G_{T_\alpha}(x, y_0) < 0$ for every $x \in C\backslash K$, where K is the weakly compact set $\overline{B}_X(0, M)$. □

From Theorem **??** we can recover the next result, proved in [**?**], Corollary 2.2:

Corollary 1. Let $T : X \rightrightarrows X^*$, and $C \subseteq \mathrm{int}(\mathrm{dom}(T))$ be a nonempty, closed and convex set. If T is monotone, convex-valued and s-w-closed on C, $\mathrm{VI}(T + \alpha J, C)$ has a unique strong solution.

Proof: Let us show that Theorem **??** can be applied. From the monotonicity, T is locally bounded at every point of C; moreover, $T(x)$ is closed and convex for $x \in C$. By applying the Closed Graph Theorem (see Section 16.12, p. 529 in [**?**]), the s-w-closedness of the graph, together with the local boundedness of T imply condition ii. of Theorem **??**. Note that, from Proposition **??**, T is B-pseudomonotone on C. Finally, let us prove that the coercivity condition (**??**) holds. Take any $x_0 \in C$, and $x_0^* \in T(x_0)$. For every $x \in C$, and $x^* \in T(x)$, the monotonicity of T implies that

$$\langle x^* - x_0^*, x - x_0 \rangle \geq 0.$$

Thus

$$\langle x^*, x - x_0 \rangle \geq \langle x_0^*, x - x_0 \rangle \geq -\|x_0^*\| \|x - x_0\|$$

and eventually

$$\frac{\langle x^*, x_0 - x \rangle}{\|x_0 - x\|} \leq \|x_0^*\|, \quad \forall x^* \in T(x).$$

This implies that (**??**) holds. Furthermore, the strong monotonicity of the operator $T + \alpha J$ entails the uniqueness of the solution. □

4 Existence of solutions via approximate problems

The problem of finding solutions for VI(T, C) occurs in many practical fields and often is ill-posed in the sense that small perturbation of the data may affect seriously the set of solutions. Our aim in the sequel is to find conditions for avoiding ill-posedness in the sense above. To this purpose we will introduce a sequence of approximate problems VI$(T_k + \alpha_k J, C_k)$.

Let $T_k, T : X \rightrightarrows X^*$ be operators, and $C_k \subseteq \text{dom}(T_k)$, $C_k \neq \emptyset$, closed and convex, for every k. Let $\alpha_k > 0$, $\alpha_k \to 0$, and denote by $x_k \in C_k$ a strong solution of the regularized variational inequality VI$(T_k + \alpha_k J, C_k)$, i.e., there exists $x_k^* \in T_k(x_k)$ such that

$$\langle x_k^* + \alpha_k J(x_k), y - x_k \rangle \geq 0, \qquad \forall y \in C_k. \tag{4.1}$$

Beside the fact that solving the regularized problem VI$(T_k + \alpha_k J, C_k)$ is more convenient from computational point of view, it is important to guarantee that any sequence $\{x_k\}$, where x_k is a solution of VI$(T_k + \alpha_k J, C_k)$, is approximating in some sense a solution of the initial problem VI(T, C). For this reason, in this section we will investigate suitable conditions leading on one hand to the existence of weak cluster points for a sequence of strong solutions of VI$(T_k + \alpha_k J, C_k)$, and, on the other hand, to show that these weak cluster points are indeed weak (strong) solutions of the original VI(T, C).

Let us start by analysing the first question. In other words, let x_k be a strong solution of VI$(T_k + \alpha_k J, C_k)$: under what conditions does there exist a bounded subsequence of $\{x_k\}$? Note first that, in case there exists a subsequence $\{C_{n_k}\}$ such that $\cup_{n_k} C_{n_k}$ is a bounded subset of X, this is trivially true. In the general case, the following result holds:

Theorem 4. Let C_k, C be nonempty, closed and convex subsets of X, and $T_k, T : X \rightrightarrows X^*$ be operators, such that

i. $C_k \subseteq \mathrm{dom}(T_k)$, and $C \cup \{\cup_k C_k\} \subseteq \mathrm{dom}(T)$;

ii. $\mathrm{Haus}(T_k(x), T(x)) \leq \frac{\beta_k}{\|x\|+1}$, for every $x \in C_k$, where $\beta_k > 0$ and $\beta_k \to 0$;

iii. if $\cup_{n_k} C_{n_k}$ is unbounded for every $\{n_k\}$, the following coercivity condition holds: $\exists \tilde{x} \in \cap_k C_k$ such that
$$\limsup_{\|x\|\to+\infty,\, x\in\cup_k C_k} G_T(x, \tilde{x}) < 0.$$

Suppose that x_k is a strong solution of $\mathrm{VI}(T_k + \alpha_k J, C_k)$, for every k. Then, $\{x_k\}$ is bounded.

Proof: Suppose by contradiction that there exists an unbounded subsequence of $\{x_k\}$, which will be denoted, by simplicity, also by $\{x_k\}$. From $\|x_k\| \to +\infty$, the coercivity condition iii. entails that
$$\limsup_k \sup_{x^*\in T(x_k)} \langle x^*, \tilde{x} - x_k\rangle < 0. \tag{4.2}$$

From the assumptions, for any fixed k there exists $x_k^* \in T_k(x_k)$ such that
$$\langle x_k^* + \alpha_k J(x_k), y - x_k\rangle \geq 0, \quad \forall y \in C_k.$$

In particular,
$$\langle x_k^* + \alpha_k J(x_k), \tilde{x} - x_k\rangle \geq 0.$$

From a well-known fact related to the notion of Hausdorff distance (see, for instance, Lemma 3.62 in [**?**]) given $x_k^* \in T_k(x_k)$, for every $\epsilon > 0$ there exists $\eta_k^* \in T(x_k)$ such that
$$\|\eta_k^* - x_k^*\| < \mathrm{Haus}(T_k(x_k), T(x_k)) + \epsilon.$$

Since $\|x_k\| \to +\infty$, we may suppose $x_k \neq \tilde{x}$ and, by taking $\epsilon_k = \frac{\beta_k}{\|x_k - \tilde{x}\|}$, from ii. we get that there exists $\eta_k^* \in T(x_k)$ such that
$$\|\eta_k^* - x_k^*\| \leq \frac{\beta_k}{\|x_k\|+1} + \frac{\beta_k}{\|x_k - \tilde{x}\|}.$$

We have that
$$\begin{aligned}\langle \eta_k^*, x_k - \tilde{x}\rangle &\leq \langle x_k^* - \eta_k^*, \tilde{x} - x_k\rangle + \alpha_k \langle J(x_k), \tilde{x} - x_k\rangle \\ &\leq \|x_k^* - \eta_k^*\| \cdot \|\tilde{x} - x_k\| + \alpha_k \|x_k\| (\|\tilde{x}\| - \|x_k\|) \\ &\leq \left(\frac{\beta_k}{\|x_k\|+1} + \frac{\beta_k}{\|x_k - \tilde{x}\|}\right) \cdot \|\tilde{x} - x_k\| + \alpha_k \|x_k\| (\|\tilde{x}\| - \|x_k\|).\end{aligned}$$

Since $(\|\tilde{x}\| - \|x_k\|)$ is negative for k big enough, we have that
$$\langle \eta_k^*, x_k - \tilde{x}\rangle \leq \beta_k \left(\frac{\|x_k - \tilde{x}\|}{\|x_k\|+1} + 1\right).$$

Since the right hand side goes to 0, we have that

$$\limsup_k \langle \eta_k^*, x_k - \tilde{x} \rangle \le 0,$$

thereby contradicting (**??**). Indeed, from (**??**), we get that, for k big enough,

$$\sup_{x^* \in T(x_k)} \langle x^*, \tilde{x} - x_k \rangle \le c < 0,$$

and therefore, in particular, for k big enough,

$$\langle \eta_k^*, \tilde{x} - x_k \rangle \le c < 0.$$

□

Let us now face the second question, that is we look for sufficient conditions guaranteeing that every weak cluster point of strong solutions of $\mathrm{VI}(T_k + \alpha_k J, C_k)$ is indeed a weak (strong) solution of the initial $\mathrm{VI}(T, C)$.

We recall first that an operator $T : X \rightrightarrows X^*$ is said to be *bounded* if it maps bounded subsets of the domain into bounded sets (see, for instance, Definition 1.4-(c), p.302 in [**?**]). The following result holds:

Theorem 5. Let C_k, C be nonempty, closed and convex subsets of X, and let $T_k, T : X \rightrightarrows X^*$. Assume that $C_k \subseteq \mathrm{dom}(T_k)$, $C \cup \{\cup_k C_k\} \subseteq \mathrm{dom}(T)$, and either condition 1 or 2 is satisfied:

1. i. $C \subseteq s - \liminf C_k$;
   ii. T is bounded on $\cup_k C_k$;
   iii. T is B-pseudomonotone on $C \cup \{\cup_k C_k\}$;
2. i. $C \subseteq C_k$, for every k;
   ii. T is B-pseudomonotone on $\cup_k C_k$.

Furthermore, suppose that

$$\mathrm{Haus}(T_k(x), T(x)) \le \beta_k q(x), \qquad \forall x \in C_k, \ \forall k,$$

where $q : X \to \mathbb{R}_+$ is bounded on bounded sets, and $\beta_k > 0$, $\beta_k \to 0$. Then, every point in C which is a weak cluster point of a sequence of strong solutions of $\mathrm{VI}(T_k + \alpha_k J, C_k)$ is a weak solution of $\mathrm{VI}(T, C)$. If, in addition, T has weakly compact and convex values on C, then the above solution is strong.

Proof: We will denote by S_k the operator $T_k + \alpha_k J : X \rightrightarrows X^*$. Suppose that $x_k^* \in S_k(x_k)$ satisfies

$$\langle x_k^*, z - x_k \rangle \ge 0, \qquad \forall z \in C_k. \tag{4.3}$$

Fix any $y \in C$. From the assumptions 1.i., or 2.i., there exists $y_k \in C_k$, $y_k \to y$; in particular, in case 2.i. we can trivially take $y_k = y$. Suppose that $\overline{x} \in C$ is a weak cluster point of $\{x_k\}$; then, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \rightharpoonup \overline{x}$. The assumption about the Hausdorff distance implies that the following condition holds:

$$\mathrm{Haus}(S_k(x), T(x)) \leq \mathrm{Haus}(T_k(x), T(x)) + \alpha_k \|x\|, \quad \forall x \in C_k,\, \forall k.$$

Therefore,

$$\mathrm{Haus}(S_{n_k}(x_{n_k}), T(x_{n_k})) \leq (\alpha_{n_k} + \beta_{n_k})(q(x_{n_k}) + \|x_{n_k}\|), \quad \forall k.$$

Take $\eta^*_{n_k} \in T(x_{n_k})$ such that

$$\|\eta^*_{n_k} - x^*_{n_k}\| \to 0. \tag{4.4}$$

Let us now distinguish case 1. and case 2.

Case 1.: Since $\eta^*_{n_k} \in T(x_{n_k})$ and $\{x_{n_k}\}$ is bounded, it follows by ii. that also the sequence $\{\eta^*_{n_k}\}$ is bounded. We have

$$0 \leq \langle x^*_{n_k}, y_{n_k} - x_{n_k}\rangle = \langle x^*_{n_k} - \eta^*_{n_k}, y_{n_k} - x_{n_k}\rangle + \langle \eta^*_{n_k}, y_{n_k} - x_{n_k}\rangle,$$

or, equivalently,

$$\langle \eta^*_{n_k}, y - x_{n_k}\rangle \geq \langle \eta^*_{n_k}, y - y_{n_k}\rangle + \langle \eta^*_{n_k} - x^*_{n_k}, y_{n_k} - x_{n_k}\rangle.$$

Taking into account the boundedness of $\{\eta^*_{n_k}\}$, the strong convergence of $\{y_k\}$ to y, (**??**) and the boundedness of $\{y_{n_k} - x_{n_k}\}$, we easily get that

$$\liminf_k \langle \eta^*_{n_k}, y - x_{n_k}\rangle \geq 0. \tag{4.5}$$

Case 2.: Since $y \in C_{n_k}$ for all k, by (**??**) we have

$$\langle \eta^*_{n_k}, y - x_{n_k}\rangle = \langle \eta^*_{n_k} - x^*_{n_k}, y - x_{n_k}\rangle + \langle x^*_{n_k}, y - x_{n_k}\rangle \geq -\|\eta^*_{n_k} - x^*_{n_k}\| \cdot \|y - x_{n_k}\|. \tag{4.6}$$

This implies (**??**).

The rest of the proof can now be carried on in both cases in the same way. By choosing in (**??**) $y = \overline{x}$, we can apply the $B-$pseudomonotonicity of T. This implies that, for every $z \in C$, there exists $x^*(z) \in T(\overline{x})$ such that

$$\langle x^*(z), \overline{x} - z\rangle \leq \liminf_k \langle \eta^*_{n_k}, x_{n_k} - z\rangle.$$

From (**??**), by setting this time $y = z$, we have that $\liminf_k \langle \eta^*_{n_k}, z - x_{n_k}\rangle \geq 0$, i.e. $\limsup_k \langle \eta^*_{n_k}, x_{n_k} - z\rangle \leq 0$. Therefore, $\langle x^*(z), z - \overline{x}\rangle \geq 0$, i.e. $\overline{x}$ is a weak solution of $\mathrm{VI}(T, C)$. In addition, if T has weakly compact and convex values, by the Sion minimax theorem every weak solution is actually a strong solution. □

Remark 1. Note that a weak cluster point of a sequence of strong solutions of VI($T_k + \alpha_k J, C_k$) belongs to C if we assume that

$$C \supseteq w - \limsup C_k. \tag{4.7}$$

This is true, in particular, if the sequence C_k Mosco-converges to C.

Remark 2. The condition $w - \limsup C_k \subseteq C \subseteq C_k$ is satisfied in case $\emptyset \neq C = \cap_k C_k$, and $C_{k+1} \subseteq C_k$, for all k. Indeed, take any sequence $\{x_{n_k}\}$, such that $x_{n_k} \in C_{n_k}$ and $x_{n_k} \rightharpoonup \overline{x}$. We have, in particular, that $x_{n_k} \in \cap_{i \leq n_k} C_i$, for all k. Since $\cap_{i \leq n_k} C_i$ is weakly closed, we have that $\overline{x} \in \cap_{i \leq n_k} C_i$, for every k; in particular, $\overline{x} \in \cap_i C_i = C$.

For the reader's convenience we collect the assumptions providing the solvability of VI(T, C). In the sequel, T_k, T will denote, as before, operators from X to X^*, as well as C_k, C will be subsets of X.

(A) i. C_k is nonempty, closed and convex for every k, and $\{C_k\}$ Mosco converges to C;

ii. C_k is nonempty, closed and convex for every k, $w - \limsup C_k \subseteq C \subseteq C_k$, for all k;

(B) dom(T_k) $\supseteq C_k$, $T_k(x)$ is bounded, closed and convex for every $x \in C_k$, T_k is B-pseudomonotone on C_k and satisfies the following property: for every finite dimensional subspace Z of X, for every $\{x_n\} \subset C_k \cap Z$, $x_n \to x$, and $x_n^* \in T_k(x_n)$, there is a subsequence $\{x_{n_j}^*\}$ converging in the weak topology to some point in $T_k(x)$, and, in addition, if C_k is unbounded, there exists $\tilde{y}_k \in C_k$ such that

$$\limsup_{\|x\| \to +\infty,\, x \in C_k} \frac{G_{T_k}(x, \tilde{y}_k)}{\|x - \tilde{y}_k\|} < +\infty; \tag{4.8}$$

(C) i. dom(T) $\supseteq C \cup \{\cup_k C_k\}$, T is bounded on $\cup_k C_k$, and B-pseudomonotone on $C \cup \{\cup_k C_k\}$, Haus($T_k(x), T(x)$) $\leq \frac{\beta_k}{\|x\|+1}$, for every $x \in C_k$, where $\beta_k > 0$ and $\beta_k \to 0$, and, in case $\cup_{n_k} C_{n_k}$ is unbounded for every $\{n_k\}$, there exists $\tilde{x} \in \cap_k C_k$ such that

$$\limsup_{\|x\| \to +\infty,\, x \in \cup_k C_k} G_T(x, \tilde{x}) < 0.$$

ii. dom(T) $\supseteq C \cup \{\cup_k C_k\}$, T is B-pseudomonotone on $\cup_k C_k$, Haus($T_k(x), T(x)$) $\leq \frac{\beta_k}{\|x\|+1}$, for every $x \in C_k$, where $\beta_k > 0$ and $\beta_k \to 0$, and, in case $\cup_{n_k} C_{n_k}$ is unbounded for every $\{n_k\}$, there exists $\tilde{x} \in \cap_k C_k$ such that

$$\limsup_{\|x\| \to +\infty,\, x \in \cup_k C_k} G_T(x, \tilde{x}) < 0.$$

Taking into account what was proved in this and in the previous section, we get the following final existence theorem that extends to B-pseudomonotone operators a result proved by Alber et al. in the monotone framework (see [**?**], Theorem 3.1):

Theorem 6. Let X be a reflexive Banach space, $C \subseteq X$ be a nonempty, closed and convex set, and $T : X \rightrightarrows X^*$ be an operator such that $\operatorname{dom}(T) \supseteq C$. Given the sequence $\{C_k, T_k\}$, where $C_k \subseteq X$, $T_k : X \rightrightarrows X^*$, under the assumptions: (A)-i, (B), (C)-i, or (A)-ii, (B), (C)-ii, $\mathrm{VI}(T, C)$ has a weak solution. If in addition T has weakly compact and convex values, all weak solutions of this problem are strong.

Let us stress that, in fact, under the assumptions of Theorem 6 we have proved more; namely, that the regularized problem $\mathrm{VI}(T_k + \alpha_k J, C_k)$ admits a strong solution x_k for all $k \in \mathbb{N}$, the sequence $\{x_k\}$ is bounded and that each weak cluster point of it is a solution of $\mathrm{VI}(T, C)$.

5 Strong convergence of the approximating solutions

In the last section we focused on conditions leading to the weak convergence of a sequence of strong approximate solutions of $\mathrm{VI}(T_k + \alpha_k J, C_k)$ to a weak (strong) solution of the original variational inequality $\mathrm{VI}(T, C)$. The aim of this section is to find out results that guarantee more, that is, the *strong convergence* to a strong solution of $\mathrm{VI}(T, C)$. Taking into account that, from the beginning, the Banach space X is assumed to be locally uniformly convex, we have that the Kadeč–Klee property is automatically satisfied (see, for instance, [**?**], p.28). This means that every sequence $\{x_n\}$ satisfying the two conditions

$$x_n \rightharpoonup \overline{x}, \qquad \|x_n\| \to \|\overline{x}\|,$$

is strongly convergent to $\overline{x}$, i.e., $x_n \to \overline{x}$. Consequently, since we already have a weakly convergent sequence, we look for conditions guaranteeing the convergence of the norms.

In the sequel we will denote by $S(T, C)$ the set of strong solutions of $\mathrm{VI}(T, C)$ and we will assume that the set $S(T, C)$ is nonempty.

Proposition 2. Let $T : X \rightrightarrows X^*$ be an operator satisfying the conditions

i. T is B-pseudomonotone on $C \subseteq \operatorname{dom}(T)$;

ii. $T(x)$ is weakly compact and convex, for every $x \in C$.

Then $S(T, C)$ is weakly closed.

Proof: Let $\{x_n\}$ be a sequence in $S(T, C)$, i.e., there exists $x_n^* \in T(x_n)$ such that

$$\langle x_n^*, y - x_n \rangle \geq 0, \quad \forall y \in C.$$

If $x_n \rightharpoonup \overline{x} \in C$, taking $y = \overline{x}$ we have that

$$\limsup_n \langle x_n^*, x_n - \overline{x} \rangle \leq 0.$$

From i., for every $y \in C$ there exists $x^*(y) \in T(\overline{x})$ such that

$$\langle x^*(y), y - \overline{x} \rangle \geq \limsup_n \langle x_n^*, y - x_n \rangle \geq 0,$$

therefore $\overline{x}$ is a weak solution of VI(T, C). From ii., it is also a strong solution. □

From the result above, since the function $x \mapsto \|x\|$ is weakly lower semicontinuous on X, there exists at least one strong solution in C with minimal norm. Let $\overline{r} \geq 0$ be the minimal norm.

Lemma 2. If in Theorem **??** we replace the coercivity condition (**??**) by the following: there exist $r > 0$ and $y_0 \in C$, $\|y_0\| \leq r$, such that

$$G_T(x, y_0) \leq \alpha r(r - \|y_0\|), \quad \forall x \in C, \|x\| > r,$$

then all solutions $\overline{x}$ of VI$(T + \alpha J, C)$ will satisfy $\|\overline{x}\| \leq r$.

Proof: If there is no $x \in C$ such that $\|x\| > r$, there is nothing to prove. Hence, suppose that such x exists. Note that, if C is unbounded, then obviously (**??**) is satisfied, and therefore VI$(T + \alpha J, C)$ admits solutions due to Theorem **??**. For arbitrary $x \in C$, $\|x\| > r$, we have:

$$\begin{aligned} G_{T_\alpha}(x, y_0) &= G_T(x, y_0) + \alpha\langle J(x), y_0 - x\rangle \\ &\leq \alpha r(r - \|y_0\|) - \alpha\|x\|^2 + \alpha\langle J(x), y_0\rangle \\ &\leq \alpha r(r - \|y_0\|) - \alpha\|x\|^2 + \alpha\|x\| \cdot \|y_0\| \\ &= \alpha(r - \|x\|)(r - \|y_0\| + \|x\|) \\ &< 0. \end{aligned}$$

Let now $\overline{x}$ be an arbitrary solution of VI$(T + \alpha J, C)$. Then $G_{T_\alpha}(\overline{x}, y) \geq 0$, for every $y \in C$, thus $G_{T_\alpha}(\overline{x}, y_0) \geq 0$ implies $\|\overline{x}\| \leq r$. □

Let us consider the following coercivity assumption, that strengthens the previous (**??**):

if C_k is unbounded, there exists $\tilde{y}_k \in C_k$ such that

$$G_{T_k}(x, \tilde{y}_k) < \alpha_k \overline{r}(\overline{r} - \|\tilde{y}_k\|), \quad \forall x \in C_k, \|x\| > \overline{r}. \tag{5.1}$$

The next theorem provides conditions entailing that a suitable subsequence of approximate solutions strongly converges to a minimal norm solution of VI(T, C).

Theorem 7. Let X be a reflexive and locally uniformly convex Banach space. Let $C \subseteq X$ be a nonempty, closed and convex set, and $T : X \rightrightarrows X^*$ be an operator with weakly compact and convex values such that dom$(T) \supseteq C$. Set $\overline{r} = \min\{\|x\| : x \in S(T, C)\}$. Given the sequence $\{C_k, T_k\}$, where $C_k \subseteq X$, $T_k : X \rightrightarrows X^*$, suppose that (**??**) is fulfilled, x_k is a strong solution of VI$(T_k + \alpha_k J, C_k)$ for every k, and $\{x_k\}$ is bounded. Under the assumptions of Theorem **??**, together with condition (**??**), $\{x_k\}$ admits a strong cluster point which belongs to $S(T, C)$.

Proof: Let us denote by $\{x_{n_k}\}$ a subsequence of $\{x_k\}$ converging weakly to $\overline{x}$. Note that, by (**??**), $\overline{x} \in C$. By Theorem **??**, we have that $\overline{x} \in S(T, C)$. Furthermore, by applying the previous lemma for any k, one obtains that $\|\overline{x}_{n_k}\| \leq \overline{r}$. Since $\|\cdot\|$ is weakly l.s.c.,

$$\overline{r} \leq \|\overline{x}\| \leq \liminf_k \|x_{n_k}\| \leq \limsup_k \|x_{n_k}\| \leq \overline{r},$$

thus

$$\overline{r} = \lim_k \|x_{n_k}\| = \|\overline{x}\|. \tag{5.2}$$

Therefore, by the Kadeč-Klee property, $\{x_{n_k}\}$ converges strongly to a minimal-norm solution of VI(T, C). $\square$

6 The Navier-Stokes operator

In the following we provide an example of an operator T which is B-pseudomonotone, but not monotone, and for this reason it does not fit into the framework of Alber et al. ([**?**]). The Navier-Stokes operator is indeed an interesting example of B-pseudomonotone and bounded operator.

Let us recall that an operator $N : X \to X^*$ is called a *Navier-Stokes operator* if

$$Nu = Au + B[u],$$

where

i. $A : X \to X^*$ is a linear, continuous and symmetric operator such that

$$\langle Au, u\rangle \geq \alpha\|u\|^2, \text{ for all } u \in X$$

for a suitable positive α;

ii. $B[u] = B(u, u)$, where $B : X \times X \to X^*$ is a bilinear continuous operator satisfying the conditions

  a. $\langle B(u, v), v\rangle = 0$, for $u, v \in X$,
  b. the map $B[\cdot] : X \to X^*$ is weakly continuous.

In [**?**] it was proved that N is B-pseudomonotone. Concerning the boundedness, note that A is obviously bounded, since it is linear and continuous, while the boundedness of $B[\cdot]$ follows from ii.b. Indeed, suppose by contradiction that $B[\cdot]$ is not bounded. Thus there exists a bounded set C, which can be assumed convex and closed without loss of generality, such that $B[C] = \cup_{u\in C} B[u]$ is unbounded. Hence there exists $\{u_n\}, u_n \in C$ such that $\|B[u_n]\| \to +\infty$. Let $\{u_{n_k}\}$ be a weakly convergent subsequence of $\{u_n\}$, $u_{n_k} \rightharpoonup u \in C$. By the weak continuity of $B[\cdot]$, we get $B[u_{n_k}] \rightharpoonup B[u]$, and thus $\|B[u_{n_k}]\|$ is bounded, a contradiction.

Let us prove that for some suitable operators B such that $B[\cdot]$ is not identically 0, the Navier-Stokes operator is not monotone. Take a functional $b : X \times X \times X \to \mathbb{R}$ such that

i. b is linear with respect to each of its variables;

ii. $b(u, v, w) = -b(u, w, v)$, for all $u, v, w \in X$;

iii. there exists $\overline{w} \in X$ such that the function $u \to b(\overline{w}, u, \overline{w})$ is not identically 0;

iv. whenever $u_n \rightharpoonup u$, then $b(u_n, u_n, v) \to b(u, u, v)$ for all $v \in X$.

Such a functional does exist (see [**?**] for a specific example in Sobolev spaces), and we can define via b the operator

$$B : X \times X \to X^*, \quad \langle B(u, v), w\rangle = b(u, v, w).$$

From iii., the operator $B[\cdot]$ is not identically 0. The operator N turns out to be a Navier-Stokes operator (see Lemma 2 in [**?**]). Let us now prove that N is not monotone. For any $u, v \in X$ by simple computation, taking into account the properties of b, we get

$$\begin{aligned}\langle N(v) - N(u), v - u\rangle &= \langle A(v - u), v - u\rangle + \langle B[v] - B[u], v - u\rangle \\ &= \langle A(v - u), v - u\rangle - \langle B(v, v), u\rangle - \langle B(u, u), v\rangle \\ &= \langle A(v - u), v - u\rangle - b(v, v, u) + b(u, v, u).\end{aligned}$$

Fix $\overline{w} \in X$ such that $b(\overline{w}, \cdot, \overline{w})$ is not identically 0, and set $v = u + \overline{w}$. Then,

$$\langle N(u + \overline{w}) - N(u), \overline{w}\rangle = \langle A\overline{w}, \overline{w}\rangle + b(\overline{w}, u, \overline{w}).$$

From the linearity in each component, $\inf_{u \in X} b(\overline{w}, u, \overline{w}) = -\infty$, and thereby we reach a contradiction.

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