Inter-order relations between equivalence for L_p-quantiles of the Student 's t distribution

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Abstract

In the statistical and actuarial literature, L_p-quantiles, $p \in [1, +\infty)$, represent an important class of risk measures defined through an asymmetric p-power loss function that generalize the classical (L_1-)quantiles. By exploiting inter-order relations between partial moments, we show that for a Student's t distribution with $\nu \in [1, +\infty)$ degrees of freedom the $L_{\nu-j}$-quantile and the L_{j+1}-quantile always coincide for any $j \in [0, \nu - 1]$. For instance, for a Student's t distribution with 4 degrees of freedom, the L_4-quantile and L_1-quantile are equal and the same holds for the L_3-quantile and L_2-quantile; for this distribution, closed form expressions for the L_p-quantile, $p = 1, 2, 3, 4$ are provided. Explicit formulas for the central moments are also established. The usefulness of exact formulas is illustrated on real-world financial data.

Keywords: expectiles; generalized quantiles; partial moments; quantiles; risk measures
JEL: C1; C21; G22; G32

1 Introduction

Literature on financial risk management and actuarial science is always in need for new techniques to evaluate and quantify downside risk. For a long time the Value-at-Risk (VaR), that is the quantile of the loss distribution, has been the benchmark measure for financial risk calculation. Its lack of subadditivity and blindness to the tail beyond the quantile, brought out the class of coherent risk measures, of which the Expected Shortfall, is the most well-known representative, see Artzner et al. (1999) and Acerbi and Tasche (2002). In the recent Fundamental Review of the Trading Book, the Basel Committee on Banking Supervision recommended the use of ES in place of VaR for the calculation of market risk in the Basel III acords (see BCBS, 2019), and a similar move is considered by the European Commission for the insurance regulatory framework Solvency II/III. The debate between academics and practitioners has then moved to the risk measures statistical properties, which are important when ES and VaR are implemented in practice, see for instance, Emmer et al. (2015) and Embrechts et al. (2014). Concepts such as robustness to small perturbations of the loss distribution (Cont et al., 2010), backtestability and elicitability emerged (Gneiting, 2011; Nolde and Ziegel, 2017). A risk measure that is elicitable is defined as the minimizer of an expected loss function that can be used to evaluate and compare the accuracy of risk measure estimates. Due to its tail sensitivity and lack of elicitability, ES has been classified as less robust and more difficult to backtest than its rival VaR. In this framework, expectiles attracted attention as the only example of coherent elicitable risk measure, see for instance Delbaen et al. (2016). Expectiles are defined as the unique minimizer of an expected asymmetric square function. They can be interpreted as the amount of money that, added to the financial position keeps the ratio between gains and losses below a fixed threshold. Their link with the Omega Ratio has been observed in Abdous and Rémillard (1995) and Bellini and Di Bernardino (2017), that also studied their applicability for financial purposes. Each one of these risk measures, VaR, ES and expectiles have advantages and weaknesses that

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must be carefully weighted depending on the context of application. It is also important to investigate the impact that different risk measures have on the regulation, i.e. which one is more/less conservative than the others. In this direction, an important branch of literature focused on the PELVE (Probability Equivalent Level of VaR and ES), that is a distributional index that provides the equivalence of VaR and ES for different distributions, see Li and Wang (2023); Fiori and Rosazza Gianin (2023) extended this index to different risk measures. For expectiles, Koenker (1992, 1993) established that the Student's t distribution with 2 degrees of freedom is the only one for which quantiles and expectiles are equivalent for any level $\tau \in (0, 1)$. Zou (2014) extended this result characterizing distributions for which the expectile at level $\omega(\tau)$ corresponds the τ quantile, for suitable functions $\omega(\cdot)$, while Jones (1994) obtained that an expectile can be defined as the quantile of another distribution.

For $p = 1$ and $p = 2$ respectively, quantiles and expectiles can be embedded in the class of L_p-quantiles, defined as the minimizer of an asymmetric p-power loss function, $p \in [1, +\infty)$. The level $\tau \in (0, 1)$ of the (L_p-)quantile represents the level of asymmetry in the loss function. This class of risk measures was introduced by Chen (1996) and investigated in terms of risk measures in Bellini et al. (2014). Recently, there has been a growing interest in L_p-quantiles for quantitative risk assessment as one of the main applications; see, e.g., Daouia et al. (2019); Gardes et al. (2020); Usseglio-Carleve (2018). In particular, Daouia et al. (2019) showed that L_p-quantiles steer an advantageous middle course between ordinary quantiles and expectiles, which allows to cover a large class of heavy-tailed distributions trading robustness to the influence of potential outliers for estimation efficiency by varying the order p within the range $p \in (1, 2)$. Secondly, L_p-quantiles are more informative than quantiles for $p > 1$ since they rely on higher order partial moments, taking into account the whole tail information about the underlying distribution. This is a major advantage over the latter which use only the information on whether an observation is below or above the predictor. Arab et al. (2022) compared L_p-quantiles of different order p and established their link with the Omega ratio of order p.

Here we focus on the special role that the Student's t distribution plays for the class of L_p-quantiles. We extend the results in Koenker (1993) and Bernardi et al. (2017) showing that for a Student's t distribution with $\nu \in [1, +\infty)$ the $L_{\nu-j}$-quantile and the L_{j+1}-quantile are equivalent for any level $\tau \in (0, 1)$, and $j \in [0, \nu - 1]$. This result shows the special symmetry of the Student's t distribution that balances the right tail with the left one and contradicts the general feeling that L_p-quantiles are closer to the centre of the distribution when p increases. As a side result, we also obtain central, upper and lower partial moments of the Student's t of order j in terms of the central, upper and lower partial moment of order $\nu - j - 1$. Finally, we obtain a closed form for the L_p-quantile for $p = 1, 2, 3, 4$ of a Student's t distribution with 4 degrees of freedom. This represents one of the few cases where an analytical solution is available and complements the result in Shaw (2005) where closed form solution for the quantile of a Student's t are provided for $\nu = 1, 2, 4$. We note, however, that our solution for the quantile is in a polynomial form and do not require the evaluation of trigonometric functions. Recently, Daouia et al. (2023) provided closed form expressions for the L_2-quantile (expectiles) of the Student's t distribution with $\nu = 4, 6$ (among other special cases).

The paper is organized as follows: Section 2 introduces L_p-quantiles and their main properties. Section 3 presents our results on the partial moments of the Student's t and the equality of L_p-quantiles. Closed form solutions for the L_p-quantiles of the Student's t distribution with 4 degrees of freedom are provided in Section 4. Finally, the last section illustrates the usefulness of the derived exact formulas to Microsoft stock logarithmic returns.

2 On the L_p-quantiles

The L_p-quantiles represent an important class of statistical functionals. They arise as a generalization of quantiles and expectiles and were introduced by Chen (1996) for the testing of symmetry in non-parametric regression. They are also embedded in the class of M-quantiles defined by Breckling and Chambers (1988). Consider a convex asymmetric power loss function $\ell_{p,\tau} : \mathbb{R} \to \mathbb{R}^+$, $\ell_{p,\tau}(x) := \tau(x_+)^p + (1 - \tau)(x_-)^p$, where $\tau \in (0, 1)$, $p \in [1, +\infty)$ and $x_+ = \max(x, 0)$, $x_- = \max(-x, 0)$ represent respectively the positive and negative part of $x \in \mathbb{R}$. The L_p-quantile of a random variable Y at level τ, denoted $\rho_{p,\tau}(Y)$, is defined as:

$$\rho_{p,\tau}(Y) := \arg\min_{m \in \mathbb{R}} \mathbb{E}[\ell_{p,\tau}(Y - m)] = \arg\min_{m \in \mathbb{R}} \mathbb{E}[\tau((Y - m)_+)^p + (1 - \tau)((Y - m)_-)^p], \quad (1)$$

provided the expectation exists. L_p-quantiles can also be defined as the solution of the following first order condition:

$$\tau\mathbb{E}\left[((Y-m)_+)^{p-1}\right]=(1-\tau)\mathbb{E}\left[((Y-m)_-)^{p-1}\right],\quad \tau\in(0,1). \tag{2}$$

The solution is always unique for $p>1$. For $p=1$, $\rho_{1,\tau}(Y)$ corresponds to the quantile at level τ of Y and (1) has a unique solution only if the distribution of Y is strictly increasing in a neighborhood of τ. When $p=2$, $\rho_{2,\tau}(Y)$ corresponds to the τ level expectile introduced by Newey and Powell (1987). The main advantage of using (2) is that it is well defined for random variables with $(p-1)^{th}$ finite moments, while (1) requires also the p^{th} moment to be finite.

When $\tau=\frac{1}{2}$, the L_p-quantile is the minimizer of the L_p-distance, indeed

$$\begin{aligned}\rho_{p,\frac{1}{2}}(Y)&=\arg\min_{m\in\mathbb{R}}\mathbb{E}[\frac{1}{2}((Y-m)_+)^p+(1-\frac{1}{2})((Y-m)_-)^p]\\&=\arg\min_{m\in\mathbb{R}}\mathbb{E}[\mid(Y-m)^p\mid]^{\frac{1}{p}}.\end{aligned} \tag{3}$$

From (3), it is easy to see that quantiles and expectiles with $\tau=\frac{1}{2}$ correspond respectively to the median and the mean of Y. L_p-quantiles naturally belong to the class of elicitable statistical functionals, i.e. functionals that can be written as minimizers of a suitable expected loss function, see Gneiting (2011). In terms of risk measures, L_p-quantiles were investigated by Bellini et al. (2014) in the context of generalized quantiles. The following proposition summarizes the main properties of L_p-quantiles and is taken from Proposition 1 in Chen (1996) and Proposition 5 in Bellini et al. (2014):

Proposition 2.1. *For any X,Y, with $(p-1)^{th}$ finite moment, $p\geq1$ and $\tau\in(0,1)$:*

1. *$\rho_{p,\tau}$ is translation invariant, i.e. $\rho_{p,\tau}(X+c)=\rho_{p,\tau}(X)+c$ for any $c\in\mathbb{R}$;*
2. *$\rho_{p,\tau}$ is monotone, i.e. for any $X\leq Y$ $\mathbb{P}$-a.s., then $\rho_{p,\tau}(X)\leq\rho_{p,\tau}(Y)$;*
3. *$\rho_{p,\tau}$ is positive homogeneous, i.e. $\rho_{p,\tau}(\lambda X)=\lambda\rho_{p,\tau}(X)$, for any $\lambda\geq0$;*
4. *$\rho_{p,\tau}$ is convex if and only if $p=2$, i.e. $\rho_{2,\tau}(\lambda X+(1-\lambda)Y)\leq\lambda\rho_{2,\tau}(X)+(1-\lambda)\rho_{2,\tau}(Y)$ for any $\lambda\in[0,1]$;*
5. *If the distribution of the random variable X is symmetric about its mean m, then $\rho_{p,\frac{1}{2}}(X)=m$ and $\rho_{p,\tau}(X)+\rho_{p,1-\tau}(X)=2m$;*
6. *$\rho_{p,\tau}$ is law-invariant, i.e. if $X\stackrel{d}{=}Y$, then $\rho_{p,\tau}(X)=\rho_{p,\tau}(Y)$.*

In particular, properties $1.-4.$ show that the expectiles are the only example of coherent risk measure in the class of L_p-quantiles. In terms of interpretability, if we consider the random variable Y as a financial loss, then $\rho_{p,\tau}(Y)$ represents the minimum amount of capital such that:

$$\begin{aligned}\rho_{p,\tau}(Y):&=\inf\left\{m\in\mathbb{R}:\tau\mathbb{E}\left[((Y-m)_+)^{p-1}\right]-(1-\tau)\mathbb{E}\left[((Y-m)_-)^{p-1}\right]\leq0\right\}\\&=\inf\left\{m\in\mathbb{R}:\frac{\mathbb{E}\left[((Y-m)_+)^{p-1}\right]}{\mathbb{E}\left[((Y-m)_-)^{p-1}\right]}\leq\frac{1-\tau}{\tau}\right\}.\end{aligned} \tag{4}$$

Equation (4), establishes the link between L_p-quantiles and Omega ratio of order $p\geq1$ defined as:

$$\Omega_p(x)=\frac{\mathbb{E}\left[((Y-x)_+)^{p-1}\right]}{\mathbb{E}\left[((Y-x)_-)^{p-1}\right]},\qquad x\in\mathbb{R}. \tag{5}$$

Therefore L_p-quantiles, provides the minimum amount of capital that keeps the p Omega ratio, under a given threshold; for $p=2$ we retrieve the usual Omega ratio used as risk-return performance measure in portfolio management. Omega ratios of order higher than 2, studied, among others, in Bi et al. (2019); Lu et al. (2022) and Arab et al. (2022), could be employed for investment decision making when investors care not only about the mean and variance but also higher moments (Bi et al. 2019). In particular, for

$p = 3$, the L_3-quantile at level τ ensures that the ratio between the semi-variances of returns above and below the target x is no greater than $\frac{1-\tau}{\tau}$.

One of the first questions concerning expectiles, and L_p-quantiles in general, is whether they are more or less conservative than quantiles. Arab et al. (2022) provided conditions to compare L_p-quantiles of different degrees and showed that, under some conditions, L_p-quantiles become less dispersed as the degree p increases. However, their results do not apply to the Student's t distribution and to heavy-tailed models in general. Bellini and Di Bernardino (2017) showed that, for distributions with Paretian tail and tail index $\alpha < 2$ (finite mean, infinite variance) the expectiles are more conservative than the corresponding quantiles, vice versa for tail index $\alpha > 2$. The Student's t distribution seems to play a key role in this framework, indeed already Koenker (1993) showed that the only distribution for which quantiles and expectiles coincides is a Student's t distribution with 2 degrees of freedom. The goal of the next section is to explore the behavior of L_p-quantiles for the Student's t distribution. It is worth noting that, differently from VaR and ES, the expectiles and L_p-quantiles must generally be calculated numerically, notable exceptions are the uniform distribution and the Student's t with 2 degrees of freedom. In Section 4 we provide explicit solutions for the Student's t distribution with 4 degrees of freedom that has been widely adopted in non-life insurance and financial mathematics.

3 Equivalence of L_p-quantiles for the Student's t distribution

In this section, we present our main contribution on the evaluation of L_p-quantiles for the Student's t distribution. Specifically, we show that for a random variable Y having a Student's t distribution with $\nu \in [1, +\infty)$ degrees of freedom the $L_{\nu-j}$-quantile and the L_{j+1}-quantile coincide for any level $\tau \in (0, 1)$, and $j \in [0, \nu - 1]$.

For $j = 1$ and $\nu = 2$, this result was proved by Koenker (1993) and then extended to the case $j = 1$ and $\nu \in \mathbb{N}$ by Bernardi et al. (2017). We expand their results considering real valued degrees of freedom $\nu \in [1, +\infty)$ and $j \in [0, \nu - 1]$.

As a first step, we present innovative results for the partial and complete moments of the Student's t distribution that we obtained as side contributions. The density function of a Student's t distribution with $\nu \in \mathbb{R}^+$ degrees of freedom is given by:

$$f_Y(y) = K_\nu \left(1 + \frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad y \in \mathbb{R},$$

where $K_\nu = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma\left(\frac{\nu}{2}\right)}$. The behavior of the distribution is characterized by the parameter ν. In particular, for $\nu = 1$ and $\nu \to +\infty$, we retrieve the well known Cauchy distribution and the standard Normal distribution, respectively. The raw moments are well defined for any order $j \in [0, \nu - 1]$ with the property that moments of odd order all vanish because f_Y is symmetric around zero.

In Figure 1 we show the behavior of the quantiles for a Student's t distribution with different degrees of freedom $\nu = 1, 2, 3, 4, 6, 30, +\infty$ (yellow, light green, dark green, blue, purple, violet, red) with respect to $\tau \in (0, 1)$ (left) and to $\tau \in [0.9, 1)$ (right). The plots show that as ν increases towards infinity the image distribution becomes less fat-tailed reducing to the quantile function of the Normal density for $\nu \to +\infty$, and the behavior is monotonically increasing in ν with a symmetry point for $\tau = \frac{1}{2}$. With a focus on the right tail (right plot), the quantiles are much higher for Student's t distributions with low degrees of freedom than for the Normal distribution and their difference decreases in ν.

Theorem 3.1. *Let Y be a random variable with Student's t distribution with $\nu \in [1, +\infty)$ degrees of freedom. For $j \in [0, \nu - 1]$ and $m \in \mathbb{R}$, the following equations hold:*

$$\mathbb{E}[((Y - m)_+)^j] = (m^2 + \nu)^{\frac{2j-\nu+1}{2}} \mathbb{E}[((Y - m)_+)^{\nu-1-j}]$$

and

$$\mathbb{E}[((Y - m)_-)^j] = (m^2 + \nu)^{\frac{2j-\nu+1}{2}} \mathbb{E}[((Y - m)_-)^{\nu-1-j}]. \tag{6}$$

Proof. Starting from the definition of upper partial moment of order j we apply a change of variable

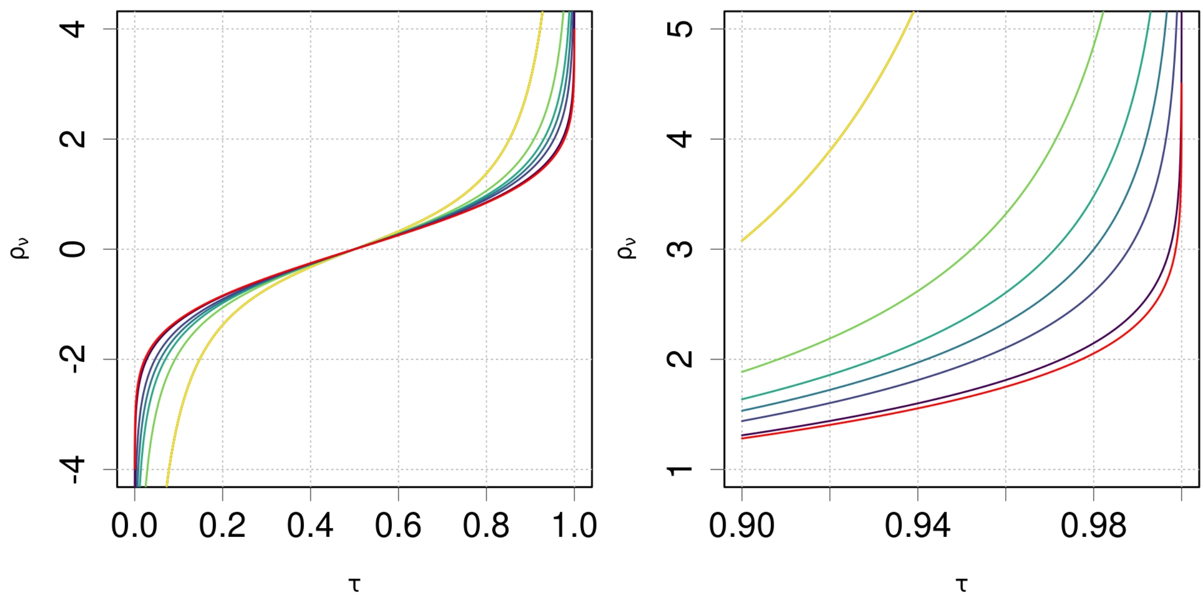

Figure 1: Quantiles of the Student's t distribution with $\nu = 1, 2, 3, 4, 6, 30, +\infty$ (yellow, light green, dark green, blue, purple, violet, red) degrees of freedom with respect to $\tau \in (0, 1)$ (left) and $\tau \in [0.9, 1)$ (right).

$x = m + \frac{\nu+m^2}{y-m}$:

$$\begin{aligned}
\mathbb{E}[((Y-m)_+)^j] &= \int_m^\infty (y-m)^j K_\nu \left(1+\frac{y^2}{\nu}\right)^{-\frac{\nu+1}{2}} \mathrm{d}y \\
&= K_\nu \int_m^\infty \frac{(\nu+m^2)^{j+1}}{(x-m)^{j+2}} \left(\frac{\nu+m^2}{\nu} + \frac{(\nu+m^2)^2}{\nu(x-m)^2} + \frac{2m(\nu+m^2)}{\nu(x-m)}\right)^{-\frac{\nu+1}{2}} \mathrm{d}x \\
&= K_\nu \int_m^\infty \frac{(\nu+m^2)^{\frac{-\nu+2j+1}{2}}}{(x-m)^{j+2}} \left(\frac{1}{\nu} + \frac{\nu+m^2}{\nu(x-m)^2} + \frac{2m}{\nu(x-m)}\right)^{-\frac{\nu+1}{2}} \mathrm{d}x \\
&= K_\nu \int_m^\infty (\nu+m^2)^{\frac{-\nu+2j+1}{2}} \frac{(x-m)^{\nu-j-1}}{\left(\frac{(x-m)^2}{\nu} + \frac{\nu+m^2}{\nu} + \frac{2m(x-m)}{\nu}\right)^{-\frac{\nu+1}{2}}} \mathrm{d}x \\
&= (\nu+m^2)^{\frac{-\nu+2j+1}{2}} \int_m^\infty \frac{(x-m)^{\nu-j-1}}{(1+\frac{x^2}{\nu})^{-\frac{\nu+1}{2}}} K_\nu \mathrm{d}x \\
&= (\nu+m^2)^{\frac{-\nu+2j+1}{2}} \mathbb{E}[((Y-m)_+)^{\nu-j-1}].
\end{aligned}$$

A similar argument provides the result for the lower partial moment. □

The following corollary, shows the above formula in case ν and j are non-negative integers.

Corollary 3.2. *Let Y be a random variable with Student's t distribution with ν degrees of freedom. For $\nu \in \mathbb{N}$, $j \in \{0, 1, \ldots, \nu-1\}$ and $m \in \mathbb{R}$, the following equation for the central moments about the point m holds:*

$$\mathbb{E}[(Y-m)^j] = \begin{cases} (m^2+\nu)^{\frac{2j-\nu+1}{2}} \mathbb{E}[(Y-m)^{\nu-j-1}] & \text{for } \nu \text{ odd,} \\ (m^2+\nu)^{\frac{2j-\nu+1}{2}} \mathbb{E}[|Y-m|^{\nu-j-1}] & \text{for } \nu, j \text{ even,} \\ (m^2+\nu)^{\frac{2j-\nu+1}{2}} \left[\mathbb{E}[((Y-m)_+)^{\nu-j-1}] - \mathbb{E}[((Y-m)_-)^{\nu-j-1}]\right] & \text{for } \nu \text{ even, } j \text{ odd.} \end{cases}$$

Proof. We first note that

$$\begin{aligned}
\mathbb{E}[(Y-m)^j] &= \mathbb{E}[[(Y-m)_+ - (Y-m)_-]^j] = \mathbb{E}[((Y-m)_+)^j] + (-1)^j \mathbb{E}[((Y-m)_-)^j] \\
&= (\nu+m^2)^{\frac{-\nu+2j+1}{2}} [\mathbb{E}[((Y-m)_+)^{\nu-j-1}] + (-1)^j \mathbb{E}[((Y-m)_-)^{\nu-j-1}]]
\end{aligned}$$

If ν is odd $(-1)^{\nu-j-1} = (-1)^j$, from which follows

$$\mathbb{E}[((Y-m)_+)^{\nu-j-1}] + (-1)^j \mathbb{E}[((Y-m)_+)^{\nu-j-1}] = \mathbb{E}[((Y-m))^{\nu-j-1}].$$

Vice versa, for ν even $(-1)^{\nu-j-1} = -(-1)^j$, which provides

$$\begin{aligned}\mathbb{E}[(Y-m)^j] &= (\nu+m^2)^{\frac{-\nu+2j+1}{2}} \left[\mathbb{E}[((Y-m)_+)^{\nu-j-1}] - (-1)^{\nu-j-1}\mathbb{E}[((Y-m)_-)^{\nu-j-1}]\right] \\ &= \begin{cases} (m^2+\nu)^{\frac{2j-\nu+1}{2}} \mathbb{E}[|Y-m|^{\nu-j-1}] & \text{for } \nu, j \text{ even,} \\ (m^2+\nu)^{\frac{2j-\nu+1}{2}} \left[\mathbb{E}[((Y-m)_+)^{\nu-j-1}] - \mathbb{E}[((Y-m)_-)^{\nu-j-1}]\right] & \text{for } \nu \text{ even, } j \text{ odd.}\end{cases}\end{aligned}$$

□

Remark 3.3. Corollary 3.2 also establishes an expression for the raw moments of Y. Indeed, for $m = 0$, we have that

$$\mathbb{E}[Y^j] = \begin{cases} \nu^{\frac{2j-\nu+1}{2}} \mathbb{E}[Y^{\nu-j-1}] & \text{if } \nu \text{ is odd and } j \text{ even,} \\ \nu^{\frac{2j-\nu+1}{2}} \mathbb{E}[|Y|^{\nu-j-1}] & \text{if } \nu \text{ is even and } j \text{ is even}\end{cases}$$

and 0 otherwise.

The next result provides our main contribution on the equivalence between the L_p-quantiles for the Student's t distribution.

Theorem 3.4. *Let Y be a Student's t random variable with $\nu \in [1, +\infty)$ degrees of freedom. For $j \in [0, \nu-1]$ the following equation holds:*

$$\rho_{\nu-j,\tau}(Y) = \rho_{j+1,\tau}(Y) \qquad \text{for any } \tau \in (0,1).$$

Proof. For $\nu = 1$ there is nothing to prove. For $\nu > 1$ and any $\tau \in (0,1)$, the first order condition (2) says that $m = \rho_{\nu-j,\tau}(Y)$ is the unique solution to

$$\tau\mathbb{E}\left[((Y-m)_+)^{\nu-j-1}\right] = (1-\tau)\mathbb{E}\left[((Y-m)_-)^{\nu-j-1}\right]$$

which, thanks to Theorem 3.1 reduces to

$$\tau\mathbb{E}\left[((Y-m)_+)^{j}\right] = (1-\tau)\mathbb{E}\left[((Y-m)_-)^{j}\right].$$

Last equation is satisfied if and only if $m = \rho_{j+1,\tau}(Y)$, from which we obtain the desired result. □

From Theorem 3.4 follows an important property of the L_p-quantiles of a Student's t distribution. Specifically, the $L_{\nu-j}$-quantile not only minimizes the expected asymmetric power loss function of order $\nu - j$ but it also minimizes the one of order $j+1$ for $j \in [0, \nu-1]$ and for any $\tau \in (0,1)$. For degrees of freedom $\nu \in \mathbb{N}$, the (L_1-)quantile and L_ν-quantile coincide, the L_2-quantile (expectile) tallies with the $L_{\nu-1}$-quantile, and so forth for every pair of L_{j+1}- and $L_{\nu-j}$-quantiles.

The next corollary extends Theorem 3.4 to any affine transformation of the Student's t distribution.

Corollary 3.5. *Let X be any affine transformation of a random variable Y with Student's t distribution with $\nu \in [1, +\infty)$ degrees of freedom, that is $X \overset{d}{=} a + bY, a \in \mathbb{R}, b > 0$. For $j \in [0, \nu-1]$, $\rho_{\nu-j,\tau}(X) = \rho_{j+1,\tau}(X)$ for any $\tau \in (0,1)$.*

Proof. From Properties 1., 3. and 6. in Proposition 2.1, $\rho_{\nu-j,\tau}(X) = \rho_{\nu-j,\tau}(a+bY) = a + b\rho_{\nu-j,\tau}(Y)$ for any $a \in \mathbb{R}$, $b > 0$, $\tau \in (0,1)$ and $j \in [0, \nu-1]$ and the result follows immediately. □

Theorem 3.4 and Corollary 3.5 extend the work of Koenker (1992) who originally showed that the class of distributions for which the L_2-quantiles and L_1-quantiles coincide, corresponds to a rescaled Student's t distribution with 2 degrees of freedom. Moreover, in the special case $j = 1$, the equality in Theorem 3.4 reduces to the expression derived in Bernardi et al. (2017), which shows that the L_ν-quantile and the quantile coincide for any level $\tau \in (0,1)$ and degrees of freedom $\nu \in \mathbb{N}$. Our result implies that for $\nu \to +\infty$, L_ν-quantiles have the same tail behavior as those of the Normal distribution quantile, as shown in Figure 1.

We conclude this section with the following corollary which shows that Omega ratios of order p for the Student's t distribution enjoy the same equivalence property of L_p-quantiles.

Corollary 3.6. *Let Y be a Student's t random variable with $\nu \in [1, +\infty)$ degrees of freedom. For $j \in [0, \nu - 1]$ the following equation holds:*

$$\Omega_{\nu-j}(x) = \Omega_{j+1}(x) \qquad \textit{for any } x \in \mathbb{R}.$$

From Corollary 3.6 it is easy to see that, for a Student's t distribution with $\nu \in \mathbb{N}$ degrees of freedom, the standard Omega ratio Ω_2 always coincides with the Omega ratio of order $\nu - 1$, $\Omega_{\nu-1}$. Moreover, this means that all Omega ratios of arbitrarily high order can be obtained readily from those of lower order.

4 Closed form L_p-quantiles for a Student's t distribution with $\nu = 4$ degrees of freedom

In this section we use Theorem 3.4 to obtain a closed form expression of the L_p-quantile of a Student's t distribution with $\nu = 4$ degrees of freedom for $p = 1, 2, 3, 4$. The relevance of this result is two-fold. On one side, in the current literature L_p-quantiles can be expressed in closed form only for very few basic distributions, such as the uniform distribution and the Student's t with 2 degrees of freedom. Shaw (2005) also provides an expression for the quantile of the Student's t with 4 degrees of freedom, that clearly coincides with ours, but in a more involved form that requires the computation of trigonometric functions. On the other side, the Student's t distribution with 4 degrees of freedom is particularly interesting in real-world applications for the analysis of financial and non-life insurance loss claims data ; see also Section 5.

We first establish the analytic solution of the quantile and L_4-quantile of the distribution, that thanks to Theorem 3.4 we know coincide. Specifically, the first order condition in (2) for the L_4-quantile can be rewritten as

$$\tau \mathbb{E}\left[(Y-m)^3\right] = (1-2\tau)\mathbb{E}\left[((Y-m)_-)^3\right], \quad \tau \in (0,1) \tag{7}$$

where we used $\mathbb{E}\left[(Y-m)^3\right] = \mathbb{E}\left[((Y-m)_+)^3\right] - \mathbb{E}\left[((Y-m)_-)^3\right]$. By developing the left-hand side and applying (6) to the right-hand side of (7), one has:

$$\tau\left(\mathbb{E}[Y^3] - 3m\mathbb{E}[Y^2] + 3m^3\mathbb{E}[Y] - m^3\right) = (1-2\tau)(m^2+4)^{\frac{3}{2}}\mathbb{E}\left[((Y-m)_-)^0\right], \quad \tau \in (0,1).$$

Given that $\mathbb{E}[Y] = \mathbb{E}[Y^3] = 0, \mathbb{E}[Y^2] = \frac{\nu}{\nu-2} = 2$, and $\mathbb{E}\left[((Y-m)_-)^0\right] = F(m)$, the first order condition is

$$\tau\left(-6m - m^3\right) = (1-2\tau)(m^2+4)^{\frac{3}{2}}F(m), \quad \tau \in (0,1);$$

From Theorem 3.4, the unique solution to this equation is $m = \rho_{4,\tau}(Y) = \rho_{1,\tau}(Y)$, so that $F(m) = F(\rho_{1,\tau}(Y)) = \tau$, which gives:

$$m\left(6+m^2\right)(m^2+4)^{-\frac{3}{2}} = 2\tau - 1, \quad \tau \in (0,1). \tag{8}$$

Eq. (8) has two real solutions for $\tau \in (0,1)$. These are:

$$m_1 = -\sqrt{-\frac{1}{H_\tau} + \frac{H_\tau}{\tau(\tau-1)} - 4}$$

and

$$m_2 = \sqrt{-\frac{1}{H_\tau} + \frac{H_\tau}{\tau(\tau-1)} - 4},$$

where $H_\tau = \sqrt[3]{-2\tau^2(\tau^2 - 2\tau + 1) + \tau\sqrt{\tau(4\tau^5 - 16\tau^4 + 25\tau^3 - 19\tau^2 + 7\tau - 1)}}$.

Since we are looking for solutions that minimize the definition of L_p-quantiles, we have that $\rho_{4,\tau}(Y) = \rho_{1,\tau}(Y) = m_1$ for $\tau \leq \frac{1}{2}$ and $\rho_{4,\tau}(Y) = \rho_{1,\tau}(Y) = m_2$ for $\tau > \frac{1}{2}$.

The first order condition in (2) for the L_2-quantile can be rewritten as

$$\tau\mathbb{E}\left[(Y-m)\right] = (1-2\tau)\mathbb{E}\left[((Y-m)_-)\right], \quad \tau \in (0,1),$$

from which one gets:

$$\mathbb{E}\left[((Y-m)_-)\right] = \frac{\tau m}{2\tau - 1}, \quad \tau \in (0,1). \tag{9}$$

The first order condition in (2) for the L_3-quantile can be rewritten as

$$\tau \mathbb{E}\left[(Y-m)^2\right] = \mathbb{E}\left[((Y-m)_-)^2\right], \quad \tau \in (0,1). \tag{10}$$

By developing the left-hand side and applying Eq. (6) to the right-hand side of Equation (10), one has:

$$2\tau + \tau m^2 = (m^2+4)^{\frac{1}{2}} \mathbb{E}\left[(Y-m)_-\right], \quad \tau \in (0,1). \tag{11}$$

Since the unique solution to (9) and (11) is $m = \rho_{2,\tau}(Y) = \rho_{3,\tau}(Y)$, we obtain:

$$2 + m^2 = (m^2+4)^{\frac{1}{2}} \frac{m}{2\tau - 1}, \quad \tau \in (0,1). \tag{12}$$

Eq. (12) has the following two real solutions for $\tau \in (0,1)$:

$$m_1 = -\sqrt{-2 - \frac{\sqrt{-(\tau-1)\tau}}{\tau^2 - \tau}}$$

and

$$m_2 = +\sqrt{-2 - \frac{\sqrt{-(\tau-1)\tau}}{\tau^2 - \tau}}.$$

Since we are looking for solutions that minimize the definition of L_p-quantiles, we have that: the minimum of (12) is given by m_1 for $\tau \leq \frac{1}{2}$ and by m_2 for $\tau > \frac{1}{2}$.

From a graphical perspective, Figure 2 illustrates the behavior of the L_p-quantile of a Student's t distribution with $\nu = 3$ (left) and $\nu = 4$ (right) degrees of freedom for $p = 1, \ldots, \nu$ with respect to $\tau \in (0,1)$. It is worth noticing that the L_p-quantiles are monotonically increasing, with a symmetry point for $\tau = \frac{1}{2}$ due to the symmetry of the Student's t centered around zero. Moreover, by looking at the plot on the left, it is possible to see that $\rho_{1,\tau}$ (green) coincides with $\rho_{3,\tau}$ (red) for all $\tau \in (0,1)$. Similarly, in the right-hand picture we have that $\rho_{1,\tau}$ (green) is equal to $\rho_{4,\tau}$ (pink) and $\rho_{2,\tau}$ (blue) coincides with $\rho_{3,\tau}$ (red) for all $\tau \in (0,1)$.

5 Application to Microsoft stock logarithmic returns

In this section we use the results reported in Section 4 for analyzing real-world financial data. Specifically, we use the weekly logarithmic returns for Microsoft (MSFT) stock collected from Yahoo Finance over the period January 1, 1995 through December 31, 2022, which corresponds to 1460 returns. Without loss of generality, before carrying out the analysis returns were centered so to have zero mean. Figure 4 reports the logarithmic returns (left) and the histogram of the data with the density of the Student's t distribution under the estimated degrees of freedom rounded to the nearest integer $\widehat{\nu} = \lceil 3.93 \rceil = 4$ (right). Using formulas derived in Section 4 we calculated the L_p-quantiles of a Student's t distribution with 4 degrees of freedom for $p = 1, 2, 3, 4$ as a function of $\tau \in (0,1)$. We also estimated the empirical L_p-quantiles for the same values of p by minimizing the asymmetric p-power loss function in (1). In Figure 4 (left) we plotted the curves of L_p-quantiles of the Student's t distribution (solid lines: $p = 1, 4$ (deep violet), $p = 2, 3$ (light blue)) against $\tau \in (0,1)$ and superimpose the empirical ones (dashed lines: $p = 1$ (pink), $p = 2$ (orange), $p = 3$ (red), $p = 4$ (black)).
As it can be observed, given the equivalence result in Theorem 3.4, we can only see two solid lines because the L_3-quantile and L_4-quantile coincide with the L_2-quantile and L_1-quantile, respectively. It is also worth noting that the L_p-quantiles lines of the Student's t distribution and their empirical counterparts are overlapped with each other, especially for $p \leq 2$. When $p = 3$ and $p = 4$, the curves are slightly far apart which could be due to the presence of outliers in the tails of the distribution of returns.
We conclude the analysis reporting the Omega ratios of order $p = 1, 2, 3, 4$ for the MSFT returns. That

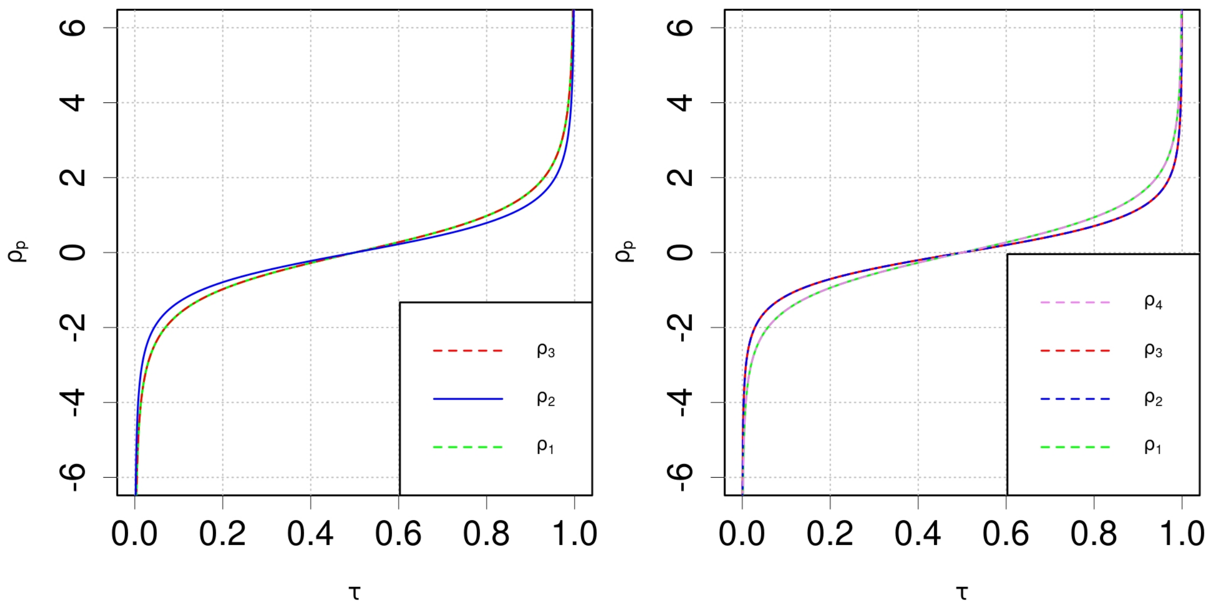

Figure 2: L_p-quantiles for $p = 1, \dots, \nu$ of a Student's t distribution with $\nu = 3$ (left) and $\nu = 4$ (right) degrees of freedom with respect to $\tau \in (0, 1)$. In both plots, the bi-colored dashed lines indicate that the $L_{\nu-j}$-quantile coincides with the L_{j+1}-quantile for $j \in [0, \nu - 1]$ and for all $\tau \in (0, 1)$.

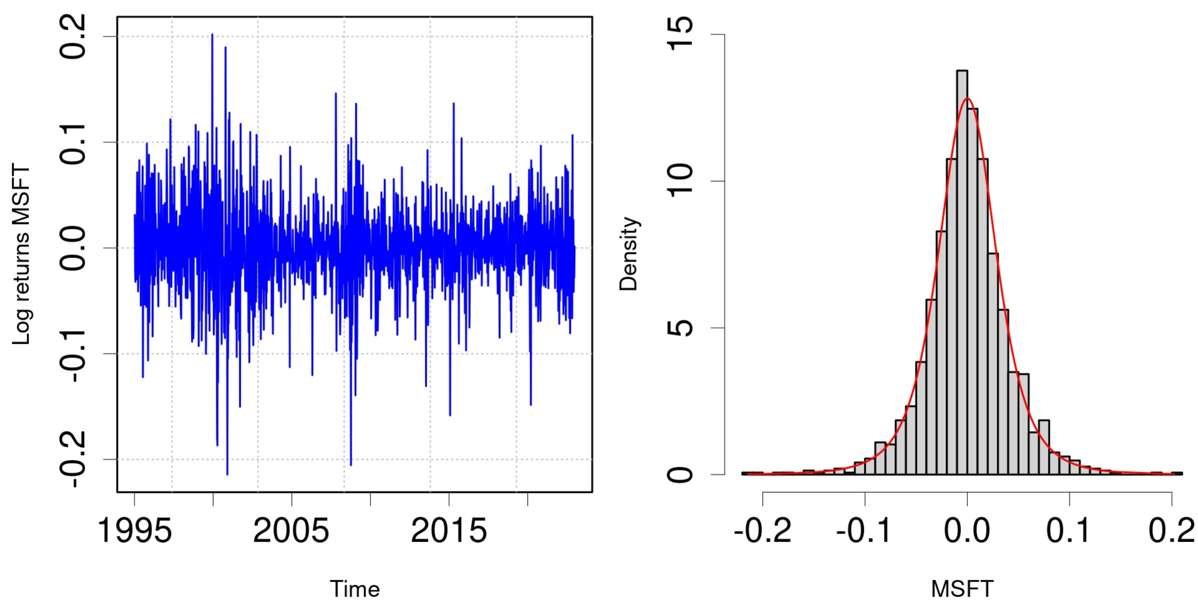

Figure 3: Logarithmic returns (left) and histogram (right) of the MSFT stock.

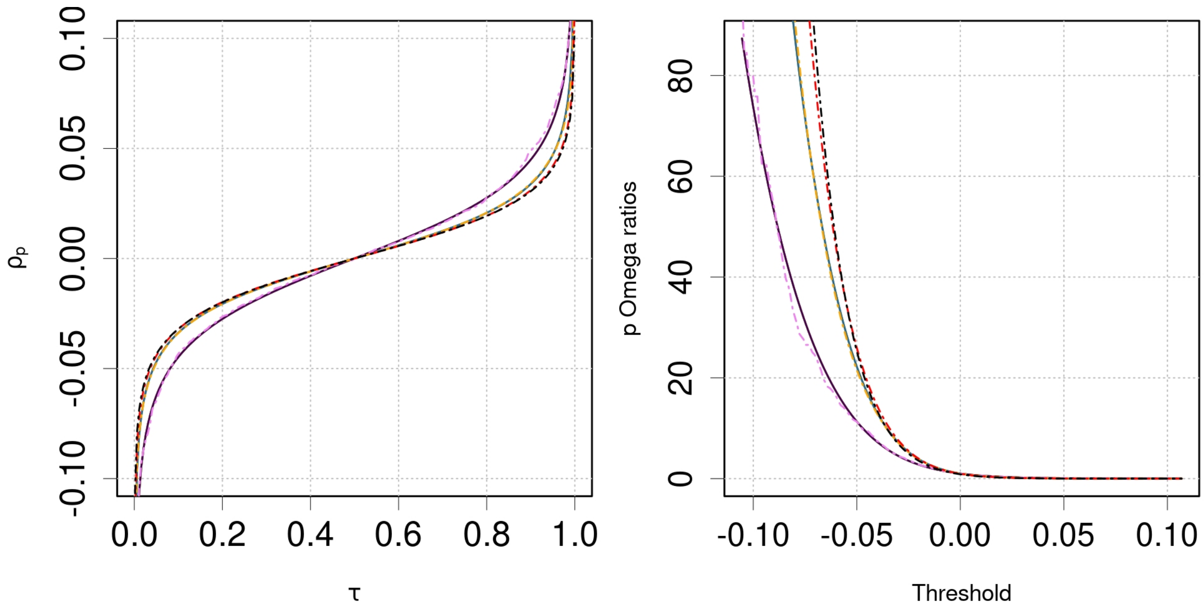

Figure 4: L_p-quantiles of a Student's t distribution with 4 degrees of freedom (solid lines: $p = 1, 4$ (deep violet), $p = 2, 3$ (light blue)) with superimposed the empirical ones (dashed lines: $p = 1$ (pink), $p = 2$ (orange), $p = 3$ (red), $p = 4$ (black)) as a function of $\tau \in (0, 1)$ (left) and p Omega ratios against the threshold (right) of the MSFT logarithmic returns.

is, we computed the Ω_p's using (5) from the fitted Student's t distribution with 4 degrees of freedom, meanwhile the empirical ones are calculated as the ratio of the two empirical partial moments of order $p - 1$. The right-hand plot of Figure 4 illustrates the obtained p Omega ratios over an equispaced grid of thresholds from the 0.01-th percentile to the 0.99-th percentile of the MSFT logarithmic returns. Two comments are in order here. First, the standard Ω_2 is identical to Ω_3, which follows from Corollary 3.6. Second, consistently with the plot on the left, the p Omega ratio curves have significant overlap for $p \leq 2$, while they are somewhat separated when $p > 3$.

Statements and Declarations

Declaration of competing interest: The authors declare that there is no competing interest.

Data availability: No dataset was generated or analysed in this work.

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