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Well Posedness and Characterization of Solutions to Non Conservative Products in Non Homogeneous Fluid Dynamics Equations

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Abstract
Consider a balance law where the flux depends explicitly on the space variable. At jump discontinuities, modeling considerations may impose the defect in the conservation of some quantities, thus leading to non conservative products. Below, we deduce the evolution in the smooth case from the jump conditions at discontinuities. Moreover, the resulting framework enjoys well posedness and solutions are uniquely characterized. These results apply, for instance, to the flow of water in a canal with varying width and depth, as well as to the inviscid Euler equations in pipes with varying geometry.

Keywords Fluid flows in canals and pipes · Non conservative products in balance laws · Nonhomogeneous Balance laws with measure source term

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1 Introduction

The flow of water in a canal of smoothly varying width and smoothly varying bed elevation is described by the following balance law

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$$\begin{cases} \partial_t a + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{a} + \frac{1}{2} g \frac{a^2}{\sigma} \right) = \frac{1}{2} g \frac{a^2}{\sigma^2} \partial_x \sigma - g \, a \, \partial_x b \,, \end{cases} \tag{1.1}$$

see [19, Formula (1.1)]. Here g is gravity, t is time, x is the longitudinal coordinate along the canal, $a = a(t, x)$ is the wetted cross sectional area, $q = q(t, x)$ is the water flow, $\sigma = \sigma(x)$ is the canal width and $b = b(x)$ is the height of the bottom.

The presence of discontinuities in the channel width σ or in the bed elevation b prevents the application of standard theorems to (1.1). Indeed, discontinuities arise in the flux and non conservative products appear in the source term. As is well know the latter terms lack a unique way to be defined. As a reference to non conservative products, we refer to [12, 18].

In the present work, we construct a framework where (1.1) has a meaning and is well posed, requiring σ and b to be merely of bounded variation.

Whenever σ and b are piecewise constant with jumps at, say, $\bar{x}_1, \ldots, \bar{x}_N$, equation (1.1) fits into the non–homogeneous system of conservation laws

$$\partial_t u + \partial_x f\left(\zeta(x), u\right) = 0 \qquad x \in \mathbb{R} \setminus \{\bar{x}_1, \ldots, \bar{x}_N\},$$

equipped with suitable conditions

$$\Psi\left(\zeta(\bar{x}_i+), u(t, \bar{x}_i+), \zeta(\bar{x}_i-), u(t, \bar{x}_i-)\right) = 0 \quad \text{for a.e. } t > 0 \text{ and } i = 1, \ldots, N \tag{1.2}$$

where ζ is as in (1.9).

This junction condition, thanks to to the assumptions below, by [8, Lemma 4.1] and by an immediate extension of [8, Lemma 4.2], can be reformulated as

$$f\left(\zeta(\bar{x}+), u(t, \bar{x}+)\right) - f\left(\zeta(\bar{x}-), u(t, \bar{x}-)\right) = \Xi\big(\zeta(\bar{x}+), \zeta(\bar{x}-), u(t, \bar{x}-)\big) \quad \text{for a.e. } t > 0 \tag{1.3}$$

where $\bar{x}$ is any point of jump and Ξ measures the defect in the conservation of u at $\bar{x}$.

We show that choosing (1.3) actually singles out the source term in (1.5) below, which accounts both for the smooth changes as well as for the points of jump in ζ. In the case of (1.1), this amounts to show that a careful choice of Ξ allows to extend (1.1) to the case of σ and b in **BV**.

More precisely, when $\zeta \in \mathbf{BV}(\mathbb{R}; \mathbb{R}^p)$ and given a piecewise constant approximation ζ^h of ζ with finite number of jumps located at $\bar{x} \in \mathcal{I}(\zeta^h)$, we obtain the following balance law with measure-valued source term

$$\begin{cases} \partial_t u + \partial_x f(\zeta^h, u) = \sum\limits_{\bar{x} \in \mathcal{I}(\zeta^h)} \Xi\left(\zeta^h(\bar{x}+), \zeta^h(\bar{x}-), u(\cdot, \bar{x}-)\right) \delta_{\bar{x}} \\ u(0, x) = u_o(x), \end{cases} \tag{1.4}$$

where $\delta_{\bar{x}}$ denotes the Dirac measure at $\bar{x}$.

In the general - non characteristic - setting established below, solutions to (1.4) are shown to converge as ζ^h converges to ζ in a suitable - strong - sense, to solutions to

$$\begin{cases} \partial_t u + \partial_x f(\zeta, u) = \sum\limits_{\bar{x} \in \mathcal{I}(\zeta)} \Xi\left(\zeta(\bar{x}+), \zeta(\bar{x}-), u(\cdot, \bar{x}-)\right) \delta_{\bar{x}} + D_v^+ \Xi(\zeta, \zeta, u) \left\|\mu\right\| \\ u(0, x) = u_o(x) \,. \end{cases} \tag{1.5}$$

The terms in the singular source term above are defined as follows. Since $\zeta \in \mathbf{BV}(\mathbb{R}; \mathbb{R}^p)$, the right and left limits $\zeta(\bar{x}+)$ and $\zeta(\bar{x}-)$ are well defined and the distributional derivative $D\zeta$ can be split in a discrete part and a non discrete one, which may contain a Cantor part:

$$D\zeta = \sum_{\bar{x} \in \mathcal{I}(\zeta)} \left(\zeta(\bar{x}+) - \zeta(\bar{x}-)\right) \delta_{\bar{x}} + v \left\|\mu\right\| , \tag{1.6}$$

where the function v is Borel measurable with norm 1 and μ is the non atomic part of $D\zeta$. In (1.5) we also used the (one sided) directional derivative

$$D_v^+ \Xi(z, z, u) = \lim_{t \to 0+} \frac{\Xi(z + t\, v, z, u) - \Xi(z, z, u)}{t}. \tag{1.7}$$

A preliminary result was obtained in [8], where a sequence of solutions to (1.4) is shown to converge to a solution to (1.5). Here, we extend the framework in [8] considering space dependent fluxes, prove that (1.4) generates a Lipschitz semigroup, say S^h, and show the convergence of S^h to a semigroup whose orbits solve (1.5). Moreover, we provide a full characterization of the solutions to (1.5) in terms of integral inequalities, in the spirit of [4].

The present results comprise the case of balance laws with a space dependent flux and a non conservative source term of the type

$$\partial_t u + \partial_x f(\zeta, u) = D_\zeta G(\zeta, u)\, D\zeta \tag{1.8}$$

see [8, § 3.4]. Setting $p = 2$, $\mathcal{Z} =]0, +\infty[\times \mathbb{R}$ and

$$\zeta(x) = \begin{bmatrix} 1/\sigma(x) \\ b(x) \end{bmatrix} \quad \text{and} \quad G\,(z, (a, q)) = \begin{bmatrix} 0 \\ -\frac{1}{2}\, g\, a^2\, z_1 - g\, a\, z_2 \end{bmatrix} \tag{1.9}$$

we see that (1.1) fits into (1.8):

$$\begin{cases} \partial_t a + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{a} + \dfrac{1}{2}\, g\, \zeta_1\, a^2 \right) = -\dfrac{1}{2}\, g\, a^2\, \partial_x \zeta_1 - g\, a\, \partial_x \zeta_2 \end{cases} \tag{1.10}$$

and hence our main result, Theorem 2.3, applies setting, for instance,

$$\Xi(z^+, z^-, u^-) = G(z^+, u^-) - G(z^-, u^-).$$

As noted in [8, Section 3], different choices of Ξ may yield different solutions emanating from discontinuities in ζ while giving the same solutions wherever ζ is smooth.

Moreover, all the applications considered in [8, Section 3] fall within the scope of Theorem 2.3. They are the classical p-system, i.e., isentropic gas dynamics, in a pipe with varying section or with bends, see also [17], as well as the full Euler compressible system in pipes, see also [15].

Thus, in addition to the existence of solutions proved in [8], here we also ensure the Lipschitz continuous dependence of the solutions on the initial data. Further, we provide a characterization of the solutions by means of the integral relations *(i)* and *(ii)* in Theorem 2.3. These results hold under assumptions on the source terms that are strictly weaker than those in [1]. Moreover, the present construction encompasses fluxes explicitly depending on the space variable.

2 Hypotheses and Main Theorem

Here, for a real number x, $|x|$ is its absolute value, while $\|v\|$ is the Euclidean norm of a vector v and $\|\mu\|$ is the total variation of a measure μ. The open ball in $\mathbb{R}^n$ centered at u with radius δ is denoted by $B(u; \delta)$, its closure is $\overline{B(u; \delta)}$. We also use the following notation for left/right limits and for differences at a point:

$$F(x-) = \lim_{\xi \to x^-} F(\xi), \quad F(x+) = \lim_{\xi \to x^+} F(\xi) \quad \text{and} \quad \Delta F(x) = F(x+) - F(x-).$$

Throughout, we choose the *left*–continuous representatives of **BV** functions.

The problem we tackle is defined by the flow f and by the functions Ξ and ζ. Here we detail the key assumptions, Ω being an open convex subset of $\mathbb{R}^n$ and $\mathcal{Z}$ a convex open subset of $\mathbb{R}^p$:

(f.1) $f \in \mathbf{C}^4(\mathcal{Z} \times \Omega; \mathbb{R}^n)$;
(f.2) the Jacobian matrix $D_u f(z, u)$ is strictly hyperbolic for every $z \in \mathcal{Z}$ and $u \in \Omega$;
(f.3) each characteristic field is either genuinely nonlinear or linearly degenerate for all $z \in \mathcal{Z}$.

In the latter assumption we refer to the classical definitions by Lax [16], see also [11, § 7.5].

By **(f.1)** and **(f.2)** we know that, possibly restricting Ω, the eigenvalues $\lambda_1(z, u), \ldots, \lambda_n(z, u)$ of $D_u f(z, u)$ depend smoothly on z and can be indexed so that, for all $u \in \Omega$ and $z \in \mathcal{Z}$,

$$\lambda_1(z, u) < \lambda_2(z, u) < \cdots < \lambda_n(z, u).$$

We thus require the usual non resonance condition

(f.4) there exists $i_o \in \{1, \ldots, n-1\}$ such that $\lambda_{i_o}(z, u) < 0 < \lambda_{i_o+1}(z, u)$ for all $z \in \mathcal{Z}$ and all $u \in \Omega$.

Note that both the cases of characteristic speeds being either all positive or all negative are simpler.

On the function Ξ in (1.3), used to rewrite the coupling condition induced by Ψ, we require:

(Ξ.1) $\Xi \colon \mathcal{Z} \times \mathcal{Z} \to \mathbf{C}^1(\overline{\Omega}; \mathbb{R}^n)$ is a Lipschitz continuous map and $\Xi \colon \mathcal{Z} \times \mathcal{Z} \to \mathbf{C}^2(\overline{\Omega}; \mathbb{R}^n)$;
(Ξ.2) $\sup_{z^+, z^- \in \mathcal{Z}} \left\| \Xi(z^+, z^-, \cdot) \right\|_{\mathbf{C}^2(\Omega; \mathbb{R})} < +\infty$;
(Ξ.3) $\Xi(z, z, u) = 0$ for every $z \in \mathcal{Z}$ and $u \in \Omega$;
(Ξ.4) there exists a non decreasing map $\sigma \colon [0, \bar{t}[\to \mathbb{R}$ with $\lim_{t \to 0} \sigma(t) = 0$ such that for all $(z, v, u) \in \mathcal{Z} \times \overline{B(0; 1)} \times \Omega$

$$\left\| \Xi(z + t\, v, z, u) - D_v^+ \Xi(z, z, u)\, t \right\| \leq \sigma(t)\, t$$

and moreover the map $(z, v, u) \to D_v^+ \Xi(z, z, u)$ is Lipschitz continuous.

In the latter condition, recall the definition (1.7) of the Dini right derivative. Our requiring this low regularity, i.e. the mere existence of the Dini derivative rather than differentiability, is motivated by the example of a pipe with angles, where Ξ depends on $\left\| z^+ - z^- \right\|$, see [8, Section 3.1].

In Problem (1.5) we require that $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$. Throughout, the map ζ is assumed to be left continuous and the set of jump discontinuities in ζ is denoted by $\mathcal{I}(\zeta)$, with $\mathcal{I}(\zeta) \subset \mathbb{R}$.

We now precisely state what we mean by *solution* to (1.5).

Definition 2.1 Let $u_o \in \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n)$. A map $u \in \mathbf{C}^0([0, +\infty[; \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n))$ with $u(t) \in \mathbf{BV}(\mathbb{R}; \mathbb{R}^n)$ and left continuous for all $t \in \mathbb{R}_+$, is a *solution* to (1.5) if for all test functions $\varphi \in \mathbf{C}^1_c(]0, +\infty[\times \mathbb{R}; \mathbb{R})$,

$$\begin{aligned} & - \int_0^{+\infty} \int_{\mathbb{R}} \left(u(t, x)\, \partial_t \varphi(t, x) + f\left(\zeta(x), u(t, x)\right)\, \partial_x \varphi(t, x) \right) \mathrm{d}x \mathrm{d}t \\ = & \sum_{\bar{x} \in \mathcal{I}(\zeta)} \int_0^{+\infty} \Xi\left(\zeta(\bar{x}+), \zeta(\bar{x}), u(t, \bar{x})\right) \varphi(t, \bar{x}) \mathrm{d}t \\ & + \int_0^{+\infty} \int_{\mathbb{R}} D^+_{v(x)} \Xi\left(\zeta(x), \zeta(x), u(t, x)\right) \varphi(t, x)\, \mathrm{d}\|\mu\|(x)\, \mathrm{d}t \end{aligned} \tag{2.1}$$

where v, μ are as in (1.6), and moreover $u(0) = u_o$.

In the last integral in (2.1), the integrand is Borel measurable in (t, x) since, for instance, by the above assumptions on u, we have at *every* $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$

$$u(t, x) = \lim_{h \to 0} \frac{1}{h} \int_{x-h}^{x} u(t, y)\, \mathrm{d}y.$$

Moreover, Borel measurability on $\mathbb{R}^2$ ensures measurability with respect to the product measure.

Note that the value of the integrand in the first line in (2.1) is independent of changes of the integrand on sets of *Lebesgue* measure 0 in $\mathbb{R}^2$, while the latter integrand is integrated with respect to the product measure $\|\mu\| \otimes \mathrm{d}t$. Nevertheless, (2.1) is meaningful, since u is prescribed *pointwise*, at *every* point and not merely almost everywhere.

The above definition is known not to guarantee uniqueness. Nevertheless, Theorem 2.3 below does guarantee uniqueness, relying on an extension to the case of (1.5) the precise characterization originally provided in [4] for homogeneous systems of conservation laws.

Definition 2.2 By *Generalized Riemann Problem* we mean the Cauchy Problem (1.5) with ζ and the initial datum u_o as follows:

$$\zeta(x) = z^- \chi_{]-\infty,0[}(x) + z^+ \chi_{]0,+\infty[}(x) \quad \text{and} \quad u_o(x) = u^\ell \chi_{]-\infty,0[}(x) + u^r \chi_{]0,+\infty[}(x). \tag{2.2}$$

For $z \in \mathcal{Z}$ and $u \in \Omega$, call $\sigma_i \to H_i(z, \sigma_i)(u)$ the Lax curve of the i–th family w.r.t. $f(z, \cdot)$ exiting u, see [5, § 5.2] or [11, § 9.3]. For $\boldsymbol{\sigma} \equiv (\sigma_1, \ldots, \sigma_n)$, we use below the notation

$$H(z, \boldsymbol{\sigma}) = H_n(z, \sigma_n) \circ H_{n-1}(z, \sigma_{n-1}) \circ \cdots \circ H_2(z, \sigma_2) \circ H_1(z, \sigma_1)\,(u)\,. \tag{2.3}$$

Introduce recursively the states $w_0, \ldots, w_{n+1} \in \Omega$ with $w_0 = u^\ell$, $w_{n+1} = u^r$ and

$$\begin{cases} w_{i+1} = H_{i+1}(z^+, \sigma_{i+1})(w_i) & \text{if } i = 0, \ldots, i_o - 1, \\ f(z^+, w_{i_o+1}) - f(z^-, w_{i_o}) = \Xi(z^+, z^-, w_{i_o}) & \\ w_{i+1} = H_i(z^-, \sigma_i)(w_i) & \text{if } i = i_o + 1, \ldots, n\,. \end{cases} \tag{2.4}$$

We thus define as *Admissible Solution* to the Generalized Riemann Problem (1.5)–(2.2) the gluing along $x = 0$ of the Lax solutions to the (standard) Riemann Problems

$$\begin{cases} \partial_t u + \partial_x f(z^-, u) = 0 \\ u(0, x) = u^\ell \chi_{]-\infty,0[}(x) + w_{i_o} \chi_{]0,+\infty[}(x), \end{cases} \quad \begin{cases} \partial_t u + \partial_x f(z^+, u) = 0 \\ u(0, x) = w_{i_o+1} \chi_{]-\infty,0[}(x) + u^r \chi_{]0,+\infty[}(x). \end{cases}$$

Throughout, we refer to the stationary jump discontinuities due to jumps in z as to *zero waves*.

Below, Lemma 3.3 ensures that, with the above definition, the Generalized Riemann Problem (1.5)–(2.2) turns out to be well posed.

Aiming at the characterization of solutions to (1.5), we now extend to the present case the general definitions introduced in [4], see also [5, Chapter 9]. Fix $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ and a function $u = u(t, x)$ with $u(t) \in \mathbf{BV}(\mathbb{R}; \Omega)$ for all t and a point $(\tau, \xi) \in [0, +\infty[\times \mathbb{R}$. Define the function $U^\sharp_{(u;\tau,\xi)}$ as the solution to the generalized Riemann Problem

$$\begin{cases} \partial_t U + \partial_x f\left(\zeta(\xi), U\right) = \Xi\left(\zeta(\xi+), \zeta(\xi), u(t, \xi-)\right) \delta_\xi \\ U(0, x) = \begin{cases} u(\tau, \xi-) & x < \xi\,; \\ u(\tau, \xi+) & x > \xi\,. \end{cases} \end{cases} \tag{2.5}$$

Note that if $\xi \notin \mathcal{I}(\zeta)$, then the right hand side in (2.5) vanishes due to **(Ξ.2)** and the above definition of $U^\sharp_{(u;\tau,\xi)}$ reduces to the classical one in [4, Chapter 9] related to the homogeneous flow $u \to f(\zeta(\xi), u)$.

We define the function $U^\flat_{(u;\tau,\xi)}$ as the *unique* solution, see Lemma 3.17, to the following linear hyperbolic problem with constant coefficients and measure-valued source term

$$\begin{cases} \partial_t U + A\,\partial_x U = g \\ U(0,x) = u(\tau, x) \end{cases} \tag{2.6}$$

with $A = D_u f(\zeta(\xi), u(\tau,\xi))$ and g is the stationary vector measure such that for any Borel subset E of $\mathbb{R}$,

$$\begin{aligned} g(E) = & \sum_{\bar x \in \mathcal{I}(\zeta)} \left(\Xi(\zeta(\bar x+), \zeta(\bar x), u(\tau,\xi)) - f(\zeta(\bar x+), u(\tau,\xi)) + f(\zeta(\bar x), u(\tau,\xi))\right) \delta_{\bar x}(E) \\ & + \int_E \left(D^+_{v(x)} \Xi(\zeta(x), \zeta(x), u(\tau,\xi)) - D_z f(\zeta(x), u(\tau,\xi))\; v(x)\right) \mathrm{d}\|\mu\|(x). \end{aligned} \tag{2.7}$$

where we used the same notation as in (1.6) and (1.7).

We are now ready to state the main result of this work.

Theorem 2.3 *Let f satisfy* **(f.1)–(f.4)**, *Ξ satisfy* **(Ξ.1)–(Ξ.4)**. *Fix $\bar z \in \mathcal{Z}$, $\bar u \in \Omega$. Then, there exist positive δ and L such that for any $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ with $\mathrm{TV}(\zeta) < \delta$ and $\|\zeta(x) - \bar z\| < \delta$ there exists a domain $\mathcal{D}^\zeta \subseteq \bar u + \mathbf{L}^1(\mathbb{R}; \Omega)$ containing all functions u in $\bar u + \mathbf{L}^1(\mathbb{R}; \Omega)$ with $\mathrm{TV}(u) < \delta$ and a semigroup $S^\zeta : \mathbb{R}_+ \times \mathcal{D}^\zeta \to \mathcal{D}^\zeta$ such that*

1. *For all $u_o \in \mathcal{D}^\zeta$, the orbit $t \to S^\zeta_t u_o$ solves (1.5) in the sense of Definition 2.1.*
2. *S^ζ is $\mathbf{L}^1$–Lipschitz continuous, i.e. for all $u_o, u^1_o, u^2_o \in \mathcal{D}^\zeta$ and for all $t, t_1, t_2 \in \mathbb{R}_+$*

$$\begin{aligned} \left\| S^\zeta_t u^1_o - S^\zeta_t u^2_o \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} &\le L \left\| u^1_o - u^2_o \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}; \\ \left\| S^\zeta_{t_1} u_o - S^\zeta_{t_2} u_o \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} &\le L\, |t_1 - t_2|. \end{aligned}$$

3. *If $\zeta \in \mathbf{PC}(\mathbb{R}; \mathcal{Z})$ and $u_o \in \mathbf{PC}(\mathbb{R}; \Omega)$, then for t sufficiently small, the map $(t,x) \to (S^\zeta_t u_o)(x)$ coincides with the gluing of Admissible Solutions, in the sense of Definition (2.2), to Generalized Riemann Problems at the points of jumps of u_o and of ζ.*

Moreover, let $\hat\lambda$ be an upper bound for the (moduli of) characteristic speeds and define $u(t,x) = (S^\zeta_t u_o)(x)$. Then, for every $(\tau, \xi) \in \mathbb{R}_+ \times \mathbb{R}$,

(i)

$$\lim_{\vartheta \to 0} \frac{1}{\vartheta} \int_{\xi - \vartheta\hat\lambda}^{\xi + \vartheta\hat\lambda} \left| u(\tau+\vartheta, x) - U^\sharp_{(u;\tau,\xi)}(\vartheta, x) \right| \mathrm{d}x = 0.$$

(ii) *There exists a constant C such that for every $a, b \in \mathbb{R}$ with $a < \xi < b$ and for every $\vartheta \in]0, (b-a)/(2\hat\lambda)[$,*

$$\begin{aligned} & \frac{1}{\vartheta} \int_{a+\vartheta\hat\lambda}^{b-\vartheta\hat\lambda} \left| u(\tau+\vartheta, x) - U^\flat_{(u;\tau,\xi)}(\vartheta, x) \right| \mathrm{d}x \\ & \le C \left[\mathrm{TV}\left(u(\tau),]a,b[\right) + \mathrm{TV}\left(\zeta,]a,b[\right)\right]^2. \end{aligned}$$

If $u : [0,T] \to \mathcal{D}^\asymp$ is $\mathbf{L}^1$–Lipschitz continuous and satisfies (i) and (ii) for almost every time τ and for all $\xi \in \mathbb{R}$, then $t \to u(t, \cdot)$ coincides with an orbit of the semigroup S^ζ.

Note that whenever ζ is piecewise constant, the properties 1., 2. and 3. above uniquely characterize the semigroup S^{ζ}, see Lemma 3.14.

3 Proofs

Below, $\mathcal{O}(1)$ denotes a constant depending exclusively on f, Ξ and on a neighborhood of $\bar{u}$. By $\hat{\lambda}$ we denote an upper bound for (the moduli) of characteristic speeds.

3.1 Preliminary Results

First, we recall a Lipschitz-type estimate on the map Ξ, of use throughout this paper.

Lemma 3.1 ([8, Lemma 4.3]) *Let $W \subset \mathbb{R}^m$ be non empty, open, bounded and convex. Let $\varphi\colon \mathcal{Z} \times \mathcal{Z} \to \mathbf{C}^1(\overline{W}; \mathbb{R}^n)$ be Lipschitz continuous and such that $\varphi(z, z, w) = 0$ for every $z \in \mathcal{Z}$ and $w \in W$. Then,*

$$\begin{aligned} \left\|\varphi(z^+, z^-, w)\right\| &\le \mathcal{O}(1) \left\|z^+ - z^-\right\| \\ \left\|\varphi(z^+, z^-, w_2) - \varphi(z^+, z^-, w_1)\right\| &\le \mathcal{O}(1) \left\|z^+ - z^-\right\| \|w_2 - w_1\| . \end{aligned} \tag{3.1}$$

Proof Since $\varphi(z^-, z^-, w) = 0$, we have $\left\|\varphi(z^+, z^-, w)\right\| = \left\|\varphi(z^+, z^-, w) - \varphi(z^-, z^-, w)\right\|$ and the first inequality in (3.1) follows by the global Lipschitz continuity of φ with respect to the z variables.

Observe that $D_w\varphi(z^-, z^-, w) = 0$. Hence, using again the Lipschitz continuity of φ,

$$\begin{aligned} &\left\|\varphi(z^+, z^-, w_2) - \varphi(z^+, z^-, w_1)\right\| \\ &= \left\|\int_0^1 D_w\varphi\left(z^+, z^-, w_2 + \varsigma(w_1 - w_2)\right)(w_1 - w_2)\mathrm{d}\varsigma\right\| \\ &= \left\|\int_0^1 \left[D_w\varphi\left(z^+, z^-, w_2 + \varsigma(w_1 - w_2)\right)\right.\right. \\ &\quad \left.\left. - D_w\varphi\left(z^-, z^-, w_2 + \varsigma(w_1 - w_2)\right)\right](w_1 - w_2)\mathrm{d}\varsigma\right\| \\ &\le \mathcal{O}(1) \left\|z^+ - z^-\right\| \|w_2 - w_1\| . \end{aligned}$$

□

Note that **(Ξ.1)** and **(Ξ.3)** are stronger than the assumptions in Lemma 3.1, so that Ξ satisfies (3.1).

Introduce a map T related to the generalized Riemann Problem.

Lemma 3.2 *Let f satisfy* **(f.1)–(f.4)** *and Ξ satisfy* **(Ξ.1)**, **(Ξ.3)**. *Then, for any $\bar{z} \in \mathcal{Z}$ and $\bar{u} \in \Omega$, there exists $\delta > 0$ and a Lipschitz map $T\colon B(\bar{z}; \delta)^2 \to \mathbf{C}^2\left(B(\bar{u}; \delta); \Omega\right)$ such that*

$$\begin{cases} f(z^+, u^+) - f(z^-, u^-) = \Xi(z^+, z^-, u^-) \\ z^+, z^- \in B(\bar{z}; \delta) \\ u^+, u^- \in B(\bar{u}; \delta) \end{cases} \iff u^+ = T(z^+, z^-)(u^-) . \tag{3.2}$$

Furthermore,

1. *$T(z, z)(u) = u$ and the map $(z^+, z^-, u) \to T(z^+, z^-)(u) - u$ satisfies the assumptions of Lemma 3.1.*

2. *The following expansion holds:*

$$\begin{aligned} & f(z^+, u^*) - f(z^-, u^*) - \Xi(z^+, z^-, u^*) + D_u f(z^*, u^*) \left(T(z^+, z^-)(u) - u\right) \\ & = \mathcal{O}(1) \left\| z^+ - z^- \right\| \left(\left\| z^+ - z^* \right\| + \left\| z^+ - z^- \right\| + \left\| u - u^* \right\| \right) \end{aligned}$$

Proof (This Lemma is an extension of [8, Lemma 4.4] to the case f dependent on z, too.)
Since $\bar{u} \in \Omega$, **(f.1)** and **(f.2)** ensure that the function $u \to f(z, u)$ has a local $\mathbf{C}^2$ inverse φ, in the sense that $\varphi(z, f(z, u)) = u$, for z sufficiently close to $\bar{z}$. Define

$$T(z^+, z^-)(u^-) = \varphi\left(z^+, f(z^-, u^-) + \Xi(z^+, z^-, u^-)\right). \tag{3.3}$$

T enjoys the required Lipschitz regularity and moreover

$$T(z, z)(u) = \varphi(z, f(z, u) + \Xi(z, z, u)) = \varphi(z, f(z, u)) = u.$$

To prove 2., rewrite

$$f(z^+, u^*) - f(z^-, u^*) - \Xi(z^+, z^-, u^*) + D_u f(z^*, u^*) \left(T(z^+, z^-)(u) - u\right) = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \tag{3.4}$$

where we used the definition of T and set

$$\begin{aligned} \mathcal{E}_1 &= f(z^+, u^*) - f\left(z^+, T(z^+, z^-)(u^*)\right) + D_u f(z^+, u^*) \left(T(z^+, z^-)(u^*) - u^*\right) \\ \mathcal{E}_2 &= -D_u f(z^+, u^*) \left(T(z^+, z^-)(u^*) - u^*\right) + D_u f(z^*, u^*) \left(T(z^+, z^-)(u^*) - u^*\right) \\ \mathcal{E}_3 &= -D_u f(z^*, u^*) \left(T(z^+, z^-)(u^*) - u^*\right) + D_u f(z^*, u^*) \left(T(z^+, z^-)(u) - u\right) \end{aligned}$$

By a Taylor expansion, we have:

$$\begin{aligned} \|\mathcal{E}_1\| &\le \mathcal{O}(1) \left\| T(z^+, z^-)(u^*) - u^* \right\|^2 \\ &= \mathcal{O}(1) \left\| z^+ - z^- \right\|^2. \end{aligned}$$

Concerning $\mathcal{E}_2$,

$$\begin{aligned} \|\mathcal{E}_2\| &= \left\| D_u f(z^+, u^*) - D_u f(z^*, u^*) \right\| \left\| T(z^+, z^-)(u^*) - u^* \right\| \\ &\le \mathcal{O}(1) \left\| z^+ - z^* \right\| \left\| z^+ - z^- \right\|. \end{aligned}$$

Finally, by Lemma 3.1

$$\begin{aligned} \|\mathcal{E}_3\| &= \left\| D_u f(z^*, u^*) \right\| \left\| \left(T(z^+, z^-)(u^*) - u^*\right) - \left(T(z^+, z^-)(u) - u\right) \right\| \\ &\le \mathcal{O}(1) \left\| z^+ - z^- \right\| \left\| u - u^* \right\|, \end{aligned}$$

completing the proof. □

As a consequence, we also prove the well posedness of the Generalized Riemann Problem (1.5)–(2.2).

Lemma 3.3 *Let f satisfy* **(f.1)–(f.4)** *and Ξ satisfy* **(Ξ.1)**. *Then, there exists a positive δ such that if $u_\ell, u_r \in \Omega$ and $z^+, z^- \in \mathcal{Z}$ satisfy*

$$\|u_\ell - u_r\| \le \delta, \qquad \left\| z^+ - z^- \right\| \le \delta,$$

then, the Generalized Riemann Problem (1.5)–(2.2) admits a unique solution in the sense of Definition 2.2. Moreover, the waves' sizes $(\sigma_1, \ldots, \sigma_n)$ and the states $(w_1, \ldots, w_n)$ in (2.4) exist, are uniquely defined and are Lipschitz continuous functions of z^+, z^-, u_r, u_ℓ.

Proof Simply rewrite (2.4) by means of (3.2) to use [1, Lemma 3]. □

The following notation is of use below:

$$(\sigma_1, \ldots, \sigma_n) = E(z^+, z^-, u_r, u_\ell)\,. \tag{3.5}$$

We separate the waves with negative ($\boldsymbol{\sigma}'$) or positive ($\boldsymbol{\sigma}''$) propagation speed as follows:

$$\begin{aligned} &\boldsymbol{\sigma}' = (\sigma_1, \ldots, \sigma_{i_o}, 0, \ldots, 0), \qquad \boldsymbol{\sigma}'' = (0, \ldots, 0, \sigma_{i_o+1}, \ldots, \sigma_n), \\ &\boldsymbol{\sigma} = \boldsymbol{\sigma}' + \boldsymbol{\sigma}'' \in \mathbb{R}^n. \end{aligned} \tag{3.6}$$

Given two n-tuples of waves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the waves i with size $\alpha_i \neq 0$ and j with size $\beta_j \neq 0$ are *approaching* whenever $i > j$ or $i = j$, the i–th family is genuinely nonlinear and $\min\left\{\alpha_i, \beta_j\right\} < 0$, see [5, § 7.3] or [11, § 9.9]. Call $\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ the set of these pairs (i, j).

Lemma 3.4 ([21, Theorem p. 30]) *Let* $\varphi \in \mathbf{C}^{2,1}(\overline{B(0, \bar\delta)} \times \overline{B(0, \bar\delta)}; \mathbb{R}^m)$ *be such that*

$$\varphi(\boldsymbol{\alpha}, \boldsymbol{\beta}) = 0 \quad \textit{for all} \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \quad \textit{with} \quad \mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \emptyset\,. \tag{3.7}$$

Then, for all $\boldsymbol{\alpha}, \boldsymbol{\beta}$

$$\|\varphi(\boldsymbol{\alpha}; \boldsymbol{\beta})\| \leq \mathcal{O}(1) \sum_{(i,j)\colon i>j} \left|\alpha_i\, \beta_j\right| + \mathcal{O}(1)\, (\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|) \sum_{\substack{i\colon \min\{\alpha_i,\beta_i\}<0 \\ gen.\, nonl.}} |\alpha_i\, \beta_i|.$$

Proof Observe that for all $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $\overline{B(0, \bar\delta)}$, we have $\mathcal{A}_{\boldsymbol{\alpha},0} = \mathcal{A}_{0,\boldsymbol{\beta}} = \emptyset$. Hence,

$$\varphi(\boldsymbol{\alpha}; 0) = \varphi(0; \boldsymbol{\beta}) = 0 \quad \text{and} \quad \partial_{\alpha_i}\varphi(\boldsymbol{\alpha}; 0) = \partial_{\beta_j}\varphi(0; \boldsymbol{\beta}) = 0$$

for all $i, j = 1, \ldots, n$. Following [21], we have

$$\begin{aligned} &\|\varphi(\boldsymbol{\alpha}; \boldsymbol{\beta})\| \\ &\quad = \|\varphi(\boldsymbol{\alpha}; \boldsymbol{\beta}) - \varphi(\boldsymbol{\alpha}; 0)\| \\ &\quad \leq \sum_{i=1}^n \|\varphi(\alpha_1, \ldots, \alpha_i, 0, \ldots, 0; \boldsymbol{\beta}) - \varphi(\alpha_1, \ldots, \alpha_{i-1}, 0, \ldots, 0; \boldsymbol{\beta})\| \\ &\quad \leq \sum_{i=1}^n \int_0^{\alpha_i} \left\|\partial_{\alpha_i}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0, \ldots, 0; \boldsymbol{\beta})\right\| \mathrm{d}a \\ &\quad = \sum_{i=1}^n \int_0^{\alpha_i} \left\|\partial_{\alpha_i}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0, \ldots, 0; \boldsymbol{\beta})\right. \\ &\qquad \left. -\partial_{\alpha_i}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0, \ldots, 0; 0)\right\| \mathrm{d}a \\ &\quad \leq \sum_{i=1}^n \sum_{j=1}^n \int_0^{\alpha_i} \left\|\partial_{\alpha_i}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0, \ldots, 0; 0, \ldots, 0, \beta_j, \ldots, \beta_n)\right. \\ &\qquad \left. -\partial_{\alpha_i}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0, \ldots, 0; 0, \ldots, 0, \beta_{j+1}, \ldots, \beta_n)\right\| \mathrm{d}a \\ &\quad \leq \sum_{i=1}^n \sum_{j=1}^n \int_0^{\alpha_i} \int_0^{\beta_j} \left\|\partial_{\alpha_i}\partial_{\beta_j}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0,\right. \\ &\qquad \left. \ldots, 0; 0, \ldots, 0, b, \beta_{j+1}, \ldots, \beta_n)\right\| \mathrm{d}b\mathrm{d}a \\ &\quad \leq \sum_{(i,j)\in\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \int_0^{\alpha_i} \int_0^{\beta_j} \left\|\partial_{\alpha_i}\partial_{\beta_j}\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0,\right. \end{aligned}$$

$$\ldots, 0; 0, \ldots, 0, b, \beta_{j+1}, \ldots, \beta_n)\big\| \,\mathrm{d}b\mathrm{d}a$$
$$\leq \left\|D^2\varphi\right\|_{\mathbf{C}^0} \sum_{i>j} \left|\alpha_i\, \beta_j\right| + Lip(D^2\varphi)\,(\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|) \sum_{\substack{i:\ \min\{\alpha_i,\beta_i\}<0\\ gen.\,nonl.}} \left|\alpha_i\, \beta_j\right|.$$

Above, we noted that some terms in the latter double sum vanish by (3.7), since

$$(i,j) \notin \mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \implies \mathcal{A}_{\alpha_1,\ldots,\alpha_{i-1},a,0,\ldots,0;0,\ldots,0,b,\beta_{j+1},\ldots,\beta_n} = \emptyset \quad \text{for all } \begin{array}{l} a \text{ between } 0 \text{ and } \alpha_i; \\ b \text{ between } 0 \text{ and } \beta_j. \end{array}$$

In the terms with $i > j$, we use a standard estimate bounding the integral by means of the $\mathbf{C}^0$ norm. We are left with the terms with $i = j$, the i–th field is genuinely nonlinear and $\min\{\alpha_i, \beta_i\} < 0$. In this case, (3.7) ensures that

$$\varphi(\alpha_1, \ldots, \alpha_{i-1}, a, 0, \ldots, 0; 0, \ldots, 0, b, \beta_{i+1}, \ldots, \beta_n) = 0 \quad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

Hence $\partial_{\alpha_1}\partial_{\beta_i}\varphi(0;0) = 0$ and $\left\|\partial_{\alpha_i}\partial_{\beta_i}\varphi(\boldsymbol{\alpha},\boldsymbol{\beta})\right\| \leq Lip(D^2\varphi)\,(\|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\|)$. □

Lemma 3.5 ([1, Lemma 4] and [8, Lemma 4.8]) *Let f satisfy* **(f.1)**–**(f.4)**, *Ξ satisfy* **(Ξ.1)** *and* **(Ξ.3)**. *Then, there exists a positive δ such that if $u_\ell, u_r \in \Omega$ and $z^+, z^- \in \mathcal{Z}$ are such that*

$$\|u_\ell - u_r\| \leq \delta, \qquad \left\|z^+ - z^-\right\| \leq \delta$$

and if $\boldsymbol{\sigma} = E(z^+, z^-, u_r, u_\ell)$ is as in (3.5), we have

$$\|u_r - u_\ell\| = \mathcal{O}(1)\left(\|\boldsymbol{\sigma}\| + \left\|z^+ - z^-\right\|\right) \quad \textit{and} \quad \|\boldsymbol{\sigma}\| = \mathcal{O}(1)\left(\|u_r - u_\ell\| + \left\|z^+ - z^-\right\|\right).$$

Lemma 3.6 *Let f satisfy* **(f.1)**–**(f.4)**. *For all $z \in \mathcal{Z}$, $u \in \Omega$ and for all sufficiently small $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$*

$$\|H(z,\boldsymbol{\beta}) \circ H(z,\boldsymbol{\alpha})(u) - H(z,\boldsymbol{\alpha}+\boldsymbol{\beta})(u)\| \leq \mathcal{O}(1) \sum_{(\alpha_i,\beta_i)\in\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} |\alpha_i\, \beta_i| \tag{3.8}$$

$$\left\|H(z^+,\boldsymbol{\alpha})(u) - H(z^-,\boldsymbol{\alpha})(u)\right\| \leq \mathcal{O}(1)\left\|z^+ - z^-\right\| \sum_{i=1}^n |\alpha_i| \tag{3.9}$$

Proof The classical Glimm interaction estimate (3.8) follows from Lemma 3.4 with $f(\boldsymbol{\alpha},\boldsymbol{\beta}) = H(z,\boldsymbol{\beta}) \circ H(z,\boldsymbol{\alpha})(u) - H(z,\boldsymbol{\alpha}+\boldsymbol{\beta})(u)$.

To obtain the second, apply Lemma 3.1 with $w_2 = \boldsymbol{\alpha}$, $w_1 = 0$ and $\varphi(z^+, z^-, \boldsymbol{\alpha}) = H(z^+,\boldsymbol{\alpha})(u) - H(z^-,\boldsymbol{\alpha})(u)$. □

The following lemma comprises the interaction estimates necessary below.

Lemma 3.7 *Let f satisfy* **(f.1)**–**(f.4)** *and Ξ satisfy* **(Ξ.1)**, **(Ξ.3)**. *Then, there exists a positive δ such that if $u_\ell, u_r \in \Omega$; $z^+, z^- \in \mathcal{Z}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are such that*

$$\|u_\ell - u_r\| \leq \delta, \qquad \left\|z^+ - z^-\right\| \leq \delta, \qquad \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| \leq \delta$$

with reference to Fig. 1, the following general interaction estimate holds:

$$\|u_* - u_r\| \leq \mathcal{O}(1)\left(\sum_{(i,j)\in\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left|\alpha_i\, \beta_j\right| + \left\|z^+ - z^-\right\| \sum_{i>i_o} |\alpha_i|\right). \tag{3.10}$$

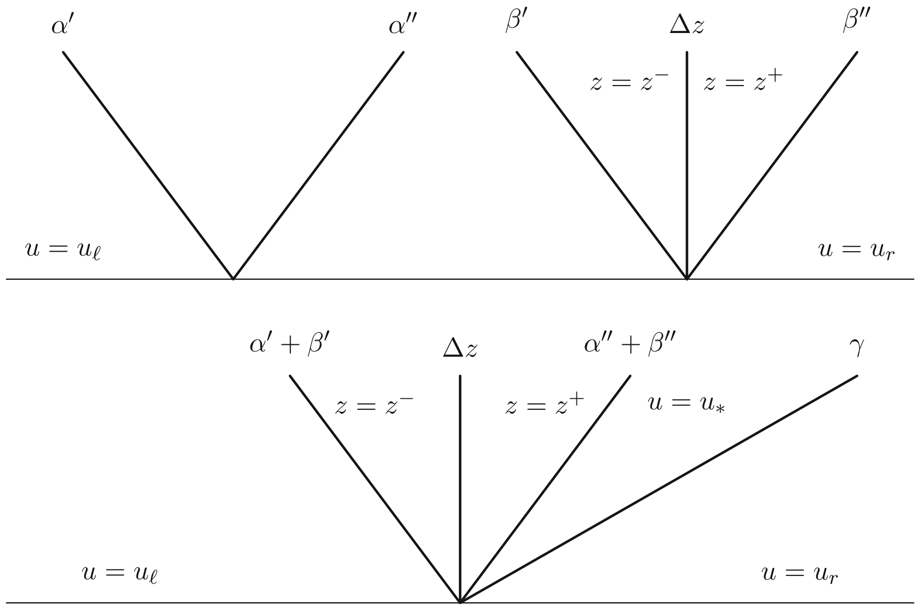

Fig. 1 Notation used in Lemma 3.7. γ denotes a fictitious wave separating the states u_*, as defined in (3.11), and u_r. Δz denotes the zero wave between z^- and z^+

Proof Referring to Fig. 1, we have:

$$\begin{aligned} u_r &= H(z^+, \boldsymbol{\beta}'') \circ T(z^+, z^-) \circ H(z^-, \boldsymbol{\beta}') \circ H(z^-, \boldsymbol{\alpha}'') \circ H(z^-, \boldsymbol{\alpha}')(u_\ell) \\ u_* &= H(z^+, \boldsymbol{\alpha}'' + \boldsymbol{\beta}'') \circ T(z^+, z^-) \circ H(z^-, \boldsymbol{\alpha}' + \boldsymbol{\beta}')(u_\ell). \end{aligned} \tag{3.11}$$

Introduce

$$\begin{aligned} \check{u} &= H(z^+, \boldsymbol{\beta}'') \circ T(z^+, z^-) \circ H(z^-, \boldsymbol{\alpha}'') \circ H(z^-, \boldsymbol{\beta}') \circ H(z^-, \boldsymbol{\alpha}')(u_\ell) \\ \hat{u} &= H(z^+, \boldsymbol{\beta}'') \circ H(z^+, \boldsymbol{\alpha}'') \circ T(z^+, z^-) \circ H(z^-, \boldsymbol{\beta}') \circ H(z^-, \boldsymbol{\alpha}')(u_\ell) \end{aligned}$$

so that

$$\|u_r - u_*\| \le \left\|u_r - \check{u}\right\| + \left\|\check{u} - \hat{u}\right\| + \left\|\hat{u} - u_*\right\|$$

and by Lemma 3.7, setting $\tilde{u} = H(z^-, \boldsymbol{\alpha}')(u_\ell)$,

$$\begin{aligned} \left\|u_r - \check{u}\right\| &\le \mathcal{O}(1) \left\|H(z^-, \beta') \circ H(z^-, \alpha'')(u) - H(z^-, \alpha'') \circ H(z^-, \beta')(u)\right\| \\ &= \mathcal{O}(1) \left\|H(z^-, \beta') \circ H(z^-, \alpha'')(u) - H(z^-, \alpha'' + \beta')(u)\right\| \\ &\le \mathcal{O}(1) \sum_{(i,j)\in\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left|\alpha_i\, \beta_j\right|. \end{aligned}$$

Similarly, setting now $\tilde{u} = H(z^-, \beta') \circ H(z^-, \boldsymbol{\alpha}')(u_\ell)$,

$$\left\|\check{u} - \hat{u}\right\| \le \mathcal{O}(1)\left\|T(z^+, z^-) \circ H(z^-, \alpha'')(\tilde{u}) - H(z^-, \alpha'') \circ T(z^+, z^-)(\tilde{u})\right\|$$

apply Lemma 3.1 with $w_2 = \alpha''$, $w_1 = 0$ and $\varphi(z^+, z^-, \alpha'') = T(z^+, z^-) \circ H(z^-, \alpha'')(\tilde{u}) - H(z^-, \alpha'') \circ T(z^+, z^-)(\tilde{u})$ to obtain

$$\left\|\check{u} - \hat{u}\right\| \le \mathcal{O}(1) \left\|z^+ - z^-\right\| \sum_{i>i_o} |\alpha_i|\,. \tag{3.12}$$

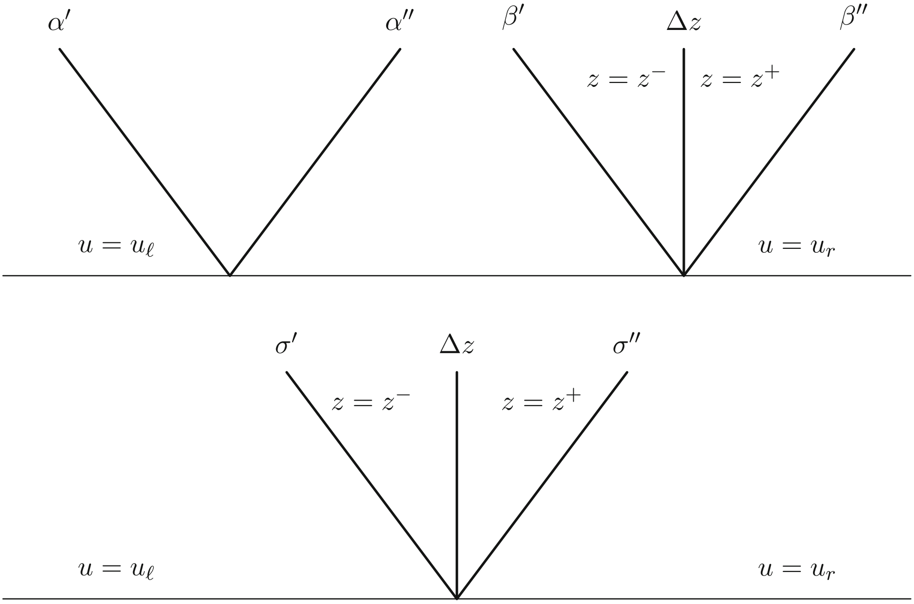

Fig. 2 Notation used in Lemma 3.8

Finally, using (3.8) in Lemma 3.6,

$$\left\| \hat{u} - u_* \right\| \le \mathcal{O}(1) \sum_{(i,j)\in\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left|\alpha_i \, \beta_j\right|,$$

completing the proof. □

Note that entirely similar estimates apply to the case where the $\boldsymbol{\alpha}$ waves are on the *right* of the zero wave, i.e., in the region where z attains the value z^+.

Lemma 3.8 *Let f satisfy* **(f.1)–(f.4)** *and Ξ satisfy* **(Ξ.1)**, **(Ξ.3)**. *Then, there exists a positive δ such that if $u_\ell, u_r \in \Omega$; $z^+, z^- \in \mathcal{Z}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are such that*

$$\|u_\ell - u_r\| \le \delta, \qquad \left\|z^+ - z^-\right\| \le \delta, \qquad \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| \le \delta$$

with reference to Fig. 2, the following general interaction estimate holds:

$$\|\boldsymbol{\sigma} - (\boldsymbol{\alpha} + \boldsymbol{\beta})\| \le \mathcal{O}(1) \left(\sum_{(i,j)\in\mathcal{A}_{\boldsymbol{\alpha},\boldsymbol{\beta}}} \left|\alpha_i \, \beta_j\right| + \left\|z^+ - z^-\right\| \sum_{i>i_o} |\alpha_i| \right)$$

Proof Let u_* be defined as in (3.11) and use the notation (3.5) to obtain:

$$\|\boldsymbol{\sigma} - (\boldsymbol{\alpha} + \boldsymbol{\beta})\| \le \left\| E(z^+, z^-, u_r, u_\ell) - E(z^+, z^-, u_*, u_\ell) \right\| \le \mathcal{O}(1) \, \|u_r - u_*\|.$$

An application of Lemma 3.7 completes the proof. □

Lemmas 3.7 and 3.8 suggest that the quantity $\left\|z^+ - z^-\right\|$ is a convenient way to measure the strength of the zero–waves associated to the coupling condition. More precisely, we define the strength of the zero–wave at a junction with parameters $z^+, z^- \in \mathcal{Z}$ as $\sigma = \left\|z^+ - z^-\right\|$.

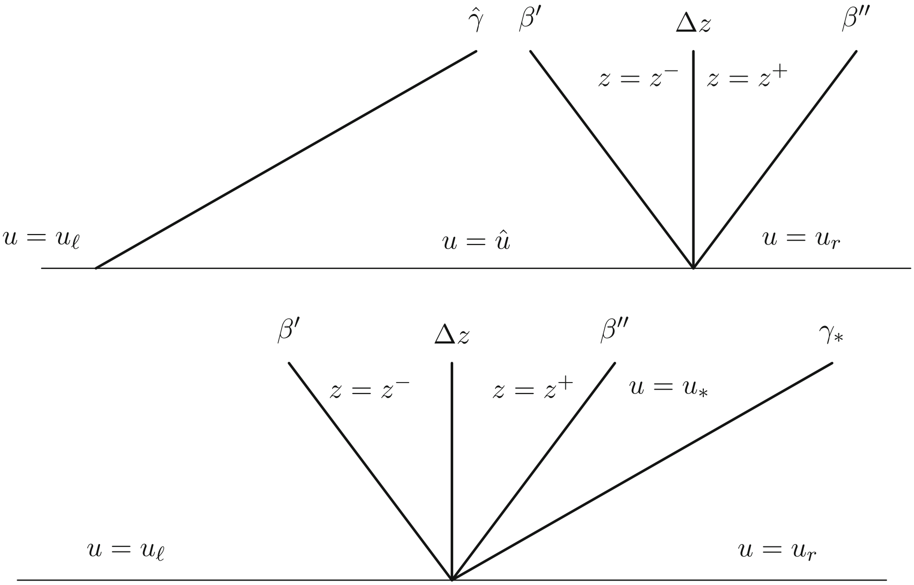

Fig. 3 Notation used in Lemma 3.9

Lemma 3.9 *Let f satisfy* **(f.1)–(f.4)** *and Ξ satisfy* **(Ξ.1)**, **(Ξ.3)**. *Then, there exists a positive δ such that if $u_\ell, u_r \in \Omega$; $z^+, z^- \in \mathcal{Z}$ and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n$ are such that*

$$\|u_\ell - u_r\| \leq \delta, \qquad \left\|z^+ - z^-\right\| \leq \delta, \qquad \|\boldsymbol{\alpha}\| + \|\boldsymbol{\beta}\| \leq \delta$$

with reference to Fig. 3, the following general interaction estimate holds:

$$\left|\|u_r - u_*\| - \left\|\hat{u} - u_\ell\right\|\right| \leq \mathcal{O}(1) \left(\sum_{j=1}^{n} \left|\beta_j\right| + \left\|z^+ - z^-\right\|\right) \left\|\hat{u} - u_\ell\right\|$$

Proof Referring to Fig. 3, straightforward computations lead to:

$$\begin{aligned}
&\left|\|u_r - u_*\| - \left\|u_\ell - \hat{u}\right\|\right| \\
&\quad\leq \left\|(u_r - u_*) - (u_\ell - \hat{u})\right\| \\
&\quad= \left\|\left(H(z^+, \beta'') \circ T(z^+, z^-) \circ H(z^-, \beta')(u_\ell) - u_\ell\right)\right. \\
&\qquad \left. - \left(H(z^+, \beta'') \circ T(z^+, z^-) \circ H(z^-, \beta')(\hat{u}) - \hat{u}\right)\right\| && (3.13)\\
&\quad\leq \left\|\left(H(z^+, \beta'') \circ T(z^+, z^-) \circ H(z^-, \beta')(u_\ell) - T(z^+, z^-)(u_\ell)\right)\right. && (3.14)\\
&\qquad \left. - \left(H(z^+, \beta'') \circ T(z^+, z^-) \circ H(z^-, \beta')(\hat{u}) - T(z^+, z^-)(\hat{u})\right)\right\| \\
&\qquad + \left\|\left(T(z^+, z^-)(u_\ell) - u_\ell\right) - \left(T(z^+, z^-)(\hat{u}) - \hat{u}\right)\right\| \\
&\quad\leq \mathcal{O}(1) \left(\left\|u_\ell - \hat{u}\right\| \sum_{j=1}^{n} \left|\beta_j\right| + \left\|u_\ell - \hat{u}\right\| \left\|z^+ - z^-\right\|\right) && (3.15)
\end{aligned}$$

completing the proof. Above we used the fact that the term in the norm (3.13)–(3.14) is a smooth function that vanishes for $u_\ell = \hat{u}$ as well as for $\beta = 0$, see [5, § 2.9]. Moreover, Lemma 3.1 can be applied to the term (3.15), with $\varphi(z^+, z^-, u) = T(z^+, z^-)(u) - u$. □

Lemma 3.10 *Let f satisfy* **(f.1)–(f.4)**, Ξ *satisfy* **(Ξ.1)–(Ξ.3)**. *Then, there exists a $\delta > 0$ such that if $\hat{u}^l, \hat{u}^r, \check{u}^l, \check{u}^r \in \Omega$, $\hat{z}^-, \hat{z}^-, \check{z}^+, \check{z}^+ \in \mathcal{Z}$ and*

$$\left\|\hat{z}^+ - \hat{z}^-\right\| + \left\|\hat{u}^l - \hat{u}^r\right\| < \delta, \qquad \left\|\check{z}^+ - \check{z}^-\right\| + \left\|\check{u}^l - \check{u}^r\right\| < \delta$$

the solutions $\hat{u}$ and $\check{u}$ to the corresponding Generalized Riemann Problems (1.5) *with data*

$$\begin{array}{ll} \hat{u}_o(x) = \begin{cases} \hat{u}^l & x < \xi \\ \hat{u}^r & x \geq \xi \end{cases} & \hat{\zeta}(x) = \begin{cases} \hat{z}^- & x < \xi \\ \hat{z}^+ & x \geq \xi \end{cases} \\ \check{u}_o(x) = \begin{cases} \check{u}^l & x < \xi \\ \check{u}^r & x \geq \xi \end{cases} & \check{\zeta}(x) = \begin{cases} \check{z}^- & x < \xi \\ \check{z}^+ & x \geq \xi \end{cases} \end{array} \tag{3.16}$$

satisfy the estimate

$$\begin{aligned} &\frac{1}{h}\int_{\xi-\hat{\lambda}h}^{\xi+\hat{\lambda}h} \left\|\hat{u}(h,x) - \check{u}(h,x)\right\| \mathrm{d}x \\ &\leq \mathcal{O}(1)\left(\left\|\hat{u}^l - \check{u}^l\right\| + \left\|\hat{u}^r - \check{u}^r\right\| + \left\|\hat{u}^r - \hat{u}^l\right\| \left(\left\|\hat{z}^- - \check{z}^-\right\| + \left\|\hat{z}^+ - \check{z}^+\right\|\right)\right. \\ &\quad \left. + \min\left\{\left\|\hat{z}^+ - \hat{z}^+\right\| + \left\|\check{z}^+ - \check{z}^-\right\|,\ \left\|\hat{z}^- - \check{z}^-\right\| + \left\|\hat{z}^+ - \check{z}^+\right\|\right\}\right). \end{aligned}$$

Proof The self similarity of the solutions to Riemann Problems ensures that

$$\frac{1}{h}\int_{\xi-\hat{\lambda}h}^{\xi+\hat{\lambda}h} \left\|\hat{u}(h,x) - \check{u}(h,x)\right\| \mathrm{d}x = \int_{\xi-\hat{\lambda}}^{\xi+\hat{\lambda}} \left\|\hat{u}(1,\xi+\lambda) - \check{u}(1,\xi+\lambda)\right\| \mathrm{d}\lambda.$$

Recall that both $\lambda \mapsto \hat{u}(1, \xi+\lambda)$ and $\lambda \mapsto \check{u}(1, \xi+\lambda)$ consist of a sequence of constant states, jump discontinuities and Lipschitz continuous rarefaction profiles. Call $\hat{p}_1, \hat{p}_2, \ldots \hat{p}_{2n+2}$ the positions of waves in $\hat{u}$, in the sense that $\hat{p}_{2l-1} = \hat{p}_{2l}$ when a shock, a contact discontinuity or a zero wave in $\hat{u}$ is supported there; while $\hat{p}_{2l-1} < \hat{p}_l$ whenever a (non trivial) rarefaction in $\hat{u}$ is supported on $[\hat{p}_{2l-1}, \hat{p}_l]$. Define $\check{p}_1, \check{p}_2, \ldots \check{p}_{2n+2}$ similarly, with reference to $\check{u}$. The map $(z^-, z^+, u^l, u^r) \mapsto p$ is Lipschitz in the z variables and smooth in the u variables.

Set $\hat{p}_0 = \check{p}_0 = \xi - \hat{\lambda}$ and $\hat{p}_{2n+3} = \check{p}_{2n+3} = \xi + \hat{\lambda}$. Then,

$$\begin{aligned} &\int_{\xi-\hat{\lambda}}^{\xi+\hat{\lambda}} \left\|\hat{u}(1,\xi+\lambda) - \check{u}(1,\xi+\lambda)\right\| \mathrm{d}\lambda \\ &\leq \mathcal{O}(1)\left(\sum_{i=1}^{2n+2}\left|\hat{p}_i - \check{p}_i\right| + \sum_{j=0}^{n+1}\left\|\hat{u}\left(1, \frac{\hat{p}_{2j}+\hat{p}_{2j+1}}{2}\right) - \check{u}\left(1, \frac{\check{p}_{2j}+\check{p}_{2j+1}}{2}\right)\right\|\right. \\ &\quad \left. + \sum_{j=1}^{n+1}\int_{[\hat{p}_{2j-1},\hat{p}_{2j}]\cap[\check{p}_{2j-1},\check{p}_{2j}]} \left\|\hat{u}(1,x) - \check{u}(1,x)\right\| \mathrm{d}x\right). \end{aligned}$$

Above, each of the quantities $\hat{p}_i - \check{p}_i$, $\hat{u}\left(1, \frac{\hat{p}_{2j}+\hat{p}_{2j+1}}{2}\right) - \check{u}\left(1, \frac{\check{p}_{2j}+\check{p}_{2j+1}}{2}\right)$ and $\hat{u}(1,x) - \check{u}(1,x)$ can be written as a difference $G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \check{u}^l, \check{u}^r)$, the function G being Lipschitz continuous in z and smooth in u. Hence,

$$\begin{aligned} &\left\|G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \check{u}^l, \check{u}^r)\right\| \\ &\leq \left\|G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^r)\right\| + \left\|G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \check{u}^l, \check{u}^r)\right\| \\ &\leq \left\|G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^r)\right\| + \mathcal{O}(1)\left(\left\|\check{u}^l - \hat{u}^l\right\| + \left\|\check{u}^r - \hat{u}^r\right\|\right). \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\left\| G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^r) \right\| \\
&\quad \le \left\| \left(G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^r) - G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^r) \right) - \left(G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^l) - G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^l) \right) \right\| \\
&\qquad + \left\| \left(G(\hat{z}^-, \hat{z}^+, \hat{u}^l, \hat{u}^l) - G(\check{z}^-, \check{z}^+, \hat{u}^l, \hat{u}^l) \right) - \underbrace{\left(G(\hat{z}^-, \hat{z}^-, \hat{u}^l, \hat{u}^l) - G(\check{z}^-, \check{z}^-, \hat{u}^l, \hat{u}^l) \right)}_{=0} \right\| \\
&\quad \le \left| \int_{\hat{u}^l}^{\hat{u}^r} \left\| D_4 G(\hat{z}^-, \hat{z}^+, \hat{u}^l, w) - D_4 G(\check{z}^-, \check{z}^+, \hat{u}^l, w) \right\| \mathrm{d}w \right| \\
&\qquad + \mathcal{O}(1) \min \left\{ \left\|\hat{z}^+ - \hat{z}^+\right\| + \left\|\check{z}^+ - \check{z}^-\right\|,\ \left\|\hat{z}^- - \check{z}^-\right\| + \left\|\hat{z}^+ - \check{z}^+\right\| \right\} \\
&\quad \le \mathcal{O}(1) \left\| \hat{u}^r - \hat{u}^l \right\| \left(\left\|\hat{z}^- - \check{z}^-\right\| + \left\|\hat{z}^+ - \check{z}^+\right\| \right) \\
&\qquad + \mathcal{O}(1) \min \left\{ \left\|\hat{z}^+ - \hat{z}^+\right\| + \left\|\check{z}^+ - \check{z}^-\right\|,\ \left\|\hat{z}^- - \check{z}^-\right\| + \left\|\hat{z}^+ - \check{z}^+\right\| \right\},
\end{aligned}
$$

completing the proof. □

3.2 The Case $\zeta \in \mathbf{PC}(\mathbb{R}; \mathbb{R}^p)$

3.2.1 Wave Front Tracking

Fix a $\zeta \in (\mathbf{PC} \cap \mathbf{BV})(\mathbb{R}; \mathbb{R}^p)$, $\mathcal{I}(z)$ being the set of points of jump in z. Let $u \in \mathbf{PC}(\mathbb{R}; \Omega)$ and call $\mathcal{I}(u)$ the set of points of jump in u. Let $\sigma_{x,i}$ be the (signed) strength of the i–th wave in the solution to the Riemann problem for (1.5) with data $u(x-)$ and $u(x+)$, i.e. $(\sigma_{x,1}, \ldots, \sigma_{x,n}) = E\left(\zeta(x+), \zeta(x-), u(x+), u(x-)\right)$ as in (3.5). Define

$$
\mathcal{A}^\zeta(u) = \left\{ \begin{array}{l} ((x,i),(y,j)) \in ((\mathcal{I}(u) \cup \mathcal{I}(\zeta)) \times \{1, \ldots, n\})^2 : \\ x < y \text{ and either } i > j \text{ or } i = j, \text{ the } i--\text{th field is} \\ \text{genuinely non linear and } \min\{\sigma_{x,i}, \sigma_{y,i}\} < 0 \end{array} \right\} \tag{3.17}
$$

Extending what introduced in [7], the linear and the interaction potential are

$$
\begin{aligned}
\mathbf{V}^\zeta(u) &= \sum_{x \in \mathcal{I}(u) \cup \mathcal{I}(\zeta)} \sum_{i=1}^n \left|\sigma_{x,i}\right| + \sum_{x \in \mathcal{I}(\zeta)} \|\Delta\zeta(x)\| \\
\mathbf{Q}^\zeta(u) &= \sum_{((x,i),(y,j)) \in \mathcal{A}^\zeta(u)} \left|\sigma_{x,i}\sigma_{y,j}\right| \\
&\qquad + \sum_{x \in \mathcal{I}(\zeta)} \|\Delta\zeta(x)\| \left(\sum_{y \in \mathcal{I}(u),\, y<x} \sum_{j>i_0} \left|\sigma_{y,j}\right| + \sum_{y \in \mathcal{I}(u),\, y>x} \sum_{j \le i_0} \left|\sigma_{y,j}\right| \right) \\
\mathbf{\Upsilon}^\zeta(u) &= \mathbf{V}^\zeta(u) + C_0 \cdot \mathbf{Q}^\zeta(u)
\end{aligned} \tag{3.18}
$$

where C_0 is a suitable positive constant. For $\delta > 0$ sufficiently small, we define

$$
\mathcal{D}_\delta^\zeta = \mathrm{cl}\left\{ u \in \bar{u} + \mathbf{L}^1(\mathbb{R}, \Omega) : u \text{ is piecewise constant and } \mathbf{\Upsilon}^\zeta(u) < \delta \right\} \tag{3.19}
$$

where the closure is in the strong $\mathbf{L}^1$–topology.

We adapt the wave-front tracking techniques from [1, 5, 8, 9, 13] to construct a sequence of approximate solutions to the Cauchy problem (1.5) and prove uniform **BV**-estimates in space. The approximate solutions converge towards a solution to the Cauchy problem with finitely many junctions. First, we define the approximations.

Definition 3.11 Let $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ be piecewise constant. For $\varepsilon > 0$, a continuous map

$$u^\varepsilon \colon [0, +\infty[\to \mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n)$$

is an ε-approximate solution to (1.5) if the following conditions hold:

- u^ε, as a function of (t, x), is piecewise constant with discontinuities along finitely many straight lines in the (t, x)-plane. There are only finitely many wave-front interactions and at most two waves interact with each other. There are four types of discontinuities: shocks (or contact discontinuities), rarefaction waves, non–physical waves and zero–waves. We distinguish these waves' indexes in the sets $\mathcal{J} = \mathcal{S} \cup \mathcal{R} \cup \mathcal{NP} \cup \mathcal{ZW}$, the generic index in $\mathcal{J}$ being α.
- At a shock (or contact discontinuity) $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{S}$, the traces $u^+ = u^\varepsilon(t, x_\alpha+)$ and $u^- = u^\varepsilon(t, x_\alpha-)$ are related by $u^+ = H_{i_\alpha}(\zeta(x_\alpha), \sigma_\alpha)(u^-)$ for $1 \le i_\alpha \le n$, see (2.3). If the i_α–th family is genuinely nonlinear, the admissibility condition $\sigma_\alpha < 0$ holds and
$$\left|\dot{x}_\alpha - \lambda_{i_\alpha}(\zeta(x_\alpha), u^+, u^-)\right| \le \varepsilon, \tag{3.20}$$
where $\lambda_{i_\alpha}(\zeta(x_\alpha), u^+, u^-)$ is the wave speed described by the Rankine-Hugoniot conditions w.r.t. $u \mapsto f(\zeta(x_\alpha), u)$.
- On the sides of a rarefaction wave $x_\alpha = x_\alpha(t)$, $\alpha \in \mathcal{R}$ in a genuinely nonlinear family, the traces are related by $u^+ = H_{i_\alpha}(\zeta(x_\alpha), \sigma_\alpha)(u^-)$ where $1 \le i_\alpha \le n$ and $0 < \sigma_\alpha \le \varepsilon$. Moreover,
$$\left|\dot{x}_\alpha - \lambda_{i_\alpha}(\zeta(x_\alpha), u^+)\right| \le \varepsilon.$$
- All non–physical fronts $x = x_\alpha(t)$, $\alpha \in \mathcal{NP}$ travel at the same speed $\dot{x}_\alpha = \hat{\lambda}$ with $\hat{\lambda} > \sup_{z,u,i} |\lambda_i(z, u)|$. The total strength of all non–physical fronts is uniformly bounded by
$$\sum_{\alpha \in \mathcal{NP}} \left\| u^\varepsilon(t, x_\alpha+) - u^\varepsilon(t, x_\alpha-) \right\| \le \varepsilon \quad \text{for all } t > 0\,.$$
- Zero–waves are located at the discontinuities $x_\alpha \in \mathcal{I}(\zeta)$. At a zero–wave x_α, $\alpha \in \mathcal{ZW}$, the traces are related by the coupling condition $u^+ = T\left(\zeta(x_\alpha+), \zeta(x_\alpha)\right)(u^-)$ for all $t > 0$, see (3.2), except at the interaction times.
- At time $t = 0$, u^ε satisfies $\|u^\varepsilon(0, \cdot) - u_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le \varepsilon$.

Proposition 3.12 ([8, Theorem 4.11]) *Let f satisfy* **(f.1)**–**(f.4)** *and Ξ satisfy* **(Ξ.1)**–**(Ξ.3)**. *Fix $\bar{z} \in \mathcal{Z}$ and $\bar{u} \in \Omega$. Then, there exist $\delta, K > 0$ such that for all piecewise constant $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ with*
$$\zeta(\mathbb{R}) \subseteq B(\bar{z}; \delta) \quad \textit{and} \quad \mathrm{TV}(\zeta) < \delta \tag{3.21}$$
and for all initial data $u_o \in \mathcal{D}^\zeta_\delta$, for every ε sufficiently small there exists an ε–approximate solution to (1.5) in the sense of Definition 3.11. Moreover, the total variation in space $\mathrm{TV}(u^\varepsilon(t, \cdot))$ and the total variation in time $\mathrm{TV}(u^\varepsilon(\cdot, x))$ are bounded uniformly for ε sufficiently small, i.e., for all $t > 0$ and for all $x \in \mathbb{R}$
$$\Upsilon^\zeta(u^\varepsilon(t, \cdot)) \le \delta + K\,\varepsilon \quad \textit{and} \quad \mathrm{TV}(u^\varepsilon(\cdot, x)) \le K.$$

Proof of Proposition 3.12 We use here the well known wave front tracking algorithm originally introduced in [10] and adapted to the present situation in [1], see also [2, 5, 6, 11, 14]. Indeed, waves supported in the points of jump of ζ, that is the *zero* waves indexed in $\mathcal{ZW}$, behave as linearly degenerate waves from the point of view of the wave front tracking algorithm developed in [2], to which we refer also for the terminology. Remark that Lemma 3.7, Lemma 3.8 and Lemma 3.9 allow to extend to interactions involving zero waves estimates of the same form as those typically used in standard wave front tracking procedures.

We refer to [8, Theorem 4.11] for more details. □

3.2.2 An Extended *Almost–Decreasing* Functional

To prove the Lipschitz continuous dependence of solutions on the initial datum, we introduce a functional uniformly equivalent to the $\mathbf{L}^1(\mathbb{R}, \mathbb{R}^n)$–distance [6]. We follow the considerations in [1, Section 4.2].

Let u, respectively v, be an ε–approximate, respectively ε'–approximate, solutions as in Proposition 3.12 with the same piecewise constant $\zeta \in \mathbf{BV}(\mathbb{R}; \mathcal{Z})$ as in (3.41). The functions $u(0, \cdot)$ and $v(0, \cdot)$ do not necessarily coincide. Introduce the concatenation of shock curves

$$S(z, \boldsymbol{q})(u) = S_n(z, q_n) \circ \cdots \circ S_1(z, q_1)(u) \tag{3.22}$$

where $q_i \mapsto S_i(z, q_i)$ are the shock curves with respect to the flux function $u \mapsto f(z, u)$ possibly violating the admissibility condition. We define $\mathbf{q}(z, u, v) \equiv (\mathrm{q}_1, \ldots, \mathrm{q}_n)(z, u, v)$ implicitly by

$$v = S\left(z, \mathbf{q}(z, u, v)\right)(u) \tag{3.23}$$

and the i–th shock speed, with the same notation as in (3.20),

$$\begin{aligned}\Lambda_i(z, u, v) = \lambda_i\, \big(z, S_i\left(z, \mathrm{q}_i(z, u, v)\right) \circ \cdots \circ S_1\left(z, \mathrm{q}_1(z, u, v)\right)(u),\\ S_{i-1}\left(z, \mathrm{q}_{i-1}(z, u, v)\right) \circ \cdots \circ S_1\left(z, \mathrm{q}_1(z, u, v)\right)(u)\big)\,.\end{aligned} \tag{3.24}$$

For sufficiently small $\mathbf{q}(z, u, v)$ and for z in a small neighborhood of $\bar{z}$, we have

$$\frac{1}{C}\,|u - v| \le \sum_{i=1}^{n} |\mathrm{q}_i(z, u, v)| \le C\,|u - v|$$

for a constant $C > 1$. We define the following functional equivalent to the $\mathbf{L}^1(\mathbb{R}; \mathbb{R}^n)$ distance:

$$q_i(t, x) = \mathrm{q}_i\left(\zeta(x), u(t, x), v(t, x)\right) \qquad i = 1, \ldots, n,$$

$$\Phi(u, v)(t) = \sum_{i=1}^{n} \int_{\mathbb{R}} |q_i(t, x)|\, W_i(t, x) \mathrm{d}x, \tag{3.25}$$

$$W_i(t, x) = 1 + \kappa_1\, B_i(t, x) + \kappa_2\left(Q(u, t) + Q(v, t)\right), \tag{3.26}$$

$$B_i(t, x) = A_i(t, x) + \begin{cases} \sum_{x_\alpha < x, \alpha \in \mathcal{ZW}} |\sigma_\alpha| & \text{if } i \le i_o,\\ \sum_{x_\alpha > x, \alpha \in \mathcal{ZW}} |\sigma_\alpha| & \text{if } i > i_o,\end{cases} \tag{3.27}$$

with positive κ_1, κ_2, chosen below and with A_i defined as in [1, Formula (4.9)], [5, Formulæ (8.8)–(8.9)] or [6, Formulæ (2.17)–(2.18)] and Q is the usual Glimm interaction potential for piecewise constant approximate solutions, see [5, Formula (7.54)], also including all zero waves. We follow [5, 6] and ensure that whatever the values of the constants κ_1, κ_2, the parameter $\delta > 0$ in (3.21) can be reduced so that $1 \le W_i(t, x) \le 2$.

We obtain the following result by the same procedure as in the proof of [1, Lemma 9]. Observe that all arguments hold for $f(\zeta(x), \cdot)$ instead of f by **(f.1)** and by the smallness of $\mathrm{TV}(\zeta)$.

Lemma 3.13 *Let f satisfy* **(f.1)**–**(f.4)** *and Ξ satisfy* **(Ξ.1)**–**(Ξ.3)**. *Fix $\bar{z} \in \mathcal{Z}$ and $\bar{u} \in \Omega$. There exist suitable positive $\kappa_1, \kappa_2, \delta$ such that if ζ is piecewise constant and satisfies* (3.21)*, u is an ε–approximate solution and v is an ε'–approximate solution as in Theorem 3.12, both corresponding to ζ, with $u(0,\cdot)$ and $v(0,\cdot)$ in $\mathcal{D}^{\zeta}_{\delta}$, as defined in* (3.19)*, then the functional Φ satisfies for all $0 \le s \le t$*

$$\Phi(u, v)(t) - \Phi(u, v)(s) \le \mathcal{O}(1)\ \max\{\varepsilon, \varepsilon'\}\ (t - s)\,.$$

Proof At any interaction time t, the same computations as in [1, 5, 6] ensure that Φ strictly decreases, thanks to the term $\kappa_2\,(Q(u,t) + Q(v,t))$ in (3.26).

At a time t between any two interaction times, use the set $\mathcal{J}$ to index the discontinuities in $u(t,\cdot)$, $v(t,\cdot)$ and in ζ at time t. The same procedure used in [1, 5, 6], to which we refer also for the standard notation employed below, allows to compute the derivative of Φ with respect to time as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(u, v)(t) = \sum_{\nu\in\mathcal{J}} \sum_{i=1}^{n} \left(\left|q_i^{\nu+}\right| W_i^{\nu+}(\lambda_i^{\nu+} - \dot{x}_\nu) - \left|q_i^{\nu-}\right| W_i^{\nu-}(\lambda_i^{\nu-} - \dot{x}_\nu)\right).$$

The standard procedure in [1, 5, 6] ensures that the above sum restricted to physical or non–physical waves is bounded as follows:

$$\sum_{\nu\in\mathcal{J}\setminus\mathcal{ZW}} \sum_{i=1}^{n} \left(\left|q_i^{\nu+}\right| W_i^{\nu+}(\lambda_i^{\nu+} - \dot{x}_\nu) - \left|q_i^{\nu-}\right| W_i^{\nu-}(\lambda_i^{\nu-} - \dot{x}_\nu)\right) \le \mathcal{O}(1)\,\varepsilon,$$

where, as in Definition 3.11, $\mathcal{ZW}$ groups the indexes referring to zero waves.

Now consider zero waves: $\nu \in \mathcal{ZW}$. Then, $\dot{x}_\nu = 0$ and

$$\begin{aligned}&\left|q_i^{\nu+}\right| W_i^{\nu+}(\lambda_i^{\nu+} - \dot{x}_\nu) - \left|q_i^{\nu-}\right| W_i^{\nu-}(\lambda_i^{\nu-} - \dot{x}_\nu)\\ &\quad = W_i^{\nu+}\left(\left|q_i^{\nu+}\right| - \left|q_i^{\nu-}\right|\right)\lambda_i^{\nu+} + W_i^{\nu+}\left|q_i^{\nu-}\right|\left(\lambda_i^{\nu+} - \lambda_i^{\nu-}\right) + \left(W_i^{\nu+} - W_i^{\nu-}\right)\left|q_i^{\nu-}\right|\lambda_i^{\nu-}. \end{aligned} \tag{3.28}$$

Now we bound the latter three summands separately. First, we use Lemma 3.1 with $w \equiv (q_1^{\nu-}, \ldots, q_n^{\nu-})$ and $\varphi(z^+, z^-, w) = q_i^{\nu+} - q_i^{\nu-}$, obtaining

$$\left|\left|q_i^{\nu+}\right| - \left|q_i^{\nu-}\right|\right| \le \left|q_i^{\nu+} - q_i^{\nu-}\right| = \left|\varphi(z^+, z^-, w) - \varphi(z^+, z^-, 0)\right| \le \mathcal{O}(1)\,\|\Delta z\| \sum_{i=1}^{n} \left|q_i^{\nu-}\right|.$$

Second, by the Lipschitz continuity of λ_i,

$$\left|q_i^{\nu-}\right| \left|\lambda_i^{\nu+} - \lambda_i^{\nu-}\right| \le \mathcal{O}(1)\|\Delta z\| \sum_{i=1}^{n} \left|q_i^{\nu-}\right|.$$

To bound the third term, introduce the sets $\hat{I} = \left\{i \in \{1, \ldots, n\} \colon q_i^{\nu+}\, q_i^{\nu-} > 0\right\}$ and $\check{I} = \left\{i \in \{1, \ldots, n\} \colon q_i^{\nu+}\, q_i^{\nu-} \le 0\right\}$. For $i \in \hat{I}$ we have $A_i^+ = A_i^-$. If $i \le i_o$ then $\lambda_i < 0$ and by (3.27) $\Delta B_i = \|\Delta z\|$ while if $i \ge i_o + 1$ then $\lambda_i > 0$ and $\Delta B_i = -\|\Delta z\|$. In both cases, the third summand in (3.28) satisfies

$$(W_i^{\nu+} - W_i^{\nu-})\left|q_i^{\nu-}\right|\lambda_i^{\nu-} < -c\,\kappa_1\,\|\Delta z\|\,\left|q_i^{\nu-}\right|.$$

On the other hand, if $i \in \check{I}$, A_i^+ and A_i^- are not directly related, but

$$\left|q_i^{\nu+}\right| + \left|q_i^{\nu-}\right| = \left|q_i^{\nu+} - q_i^{\nu-}\right| \leq \mathcal{O}(1)\, \|\Delta z\| \sum_{i=1}^{n} \left|q_i^{\nu-}\right| \tag{3.29}$$

so that

$$(W_i^{\nu+} - W_i^{\nu-})\left|q_i^{\nu-}\right|\lambda_i^{\nu-} \leq \mathcal{O}(1)\, \|\Delta z\| \sum_{i=1}^{n} \left|q_i^{\nu-}\right|, \tag{3.30}$$

concluding the estimates on the three terms in (3.28). Moreover, by (3.29),

$$\sum_{i=1}^{n} \left|q_i^{\nu-}\right| = \sum_{i\in\hat{I}} \left|q_i^{\nu-}\right| + \sum_{i\in\check{I}} \left|q_i^{\nu-}\right| \leq \sum_{i\in\hat{I}} \left|q_i^{\nu-}\right| + \mathcal{O}(1)\, \|\Delta z\| \sum_{i=1}^{n} \left|q_i^{\nu-}\right|.$$

Hence, for Δz sufficiently small,

$$\sum_{i=1}^{n} \left|q_i^{\nu-}\right| \leq 2 \sum_{i\in\hat{I}} \left|q_i^{\nu-}\right|.$$

Adding the different estimates obtained, we bound the term in (3.28) by

$$\begin{aligned}
&\sum_{i=1}^{n} \left(\left|q_i^{\nu+}\right| W_i^{\nu+}(\lambda_i^{\nu+} - \dot{x}_\nu) - \left|q_i^{\nu-}\right| W_i^{\nu-}(\lambda_i^{\nu-} - \dot{x}_\nu)\right) \\
&\quad \leq -c\,\kappa_1\, \|\Delta z\| \sum_{i\in\hat{I}} \left|q_i^{\nu-}\right| + \mathcal{O}(1)\, \|\Delta z\| \sum_{i=1}^{n} \left|q_i^{\nu-}\right| \\
&\quad \leq (\mathcal{O}(1) - c\,\kappa_1)\|\Delta z\| \sum_{i\in\hat{I}} \left|q_i^{\nu-}\right| \\
&\quad < 0,
\end{aligned} \tag{3.31}$$

assuming κ_1 sufficiently large.

The proof is now completed by means of standard arguments, refer to [1, 5, 6]. $\square$

3.2.3 Proof of Theorem 2.3 in the Case $\zeta \in \mathbf{PC}(\mathbb{R}; \mathbb{R}^p)$

Let f satisfy **(f.1)–(f.4)**, Ξ satisfy **(Ξ.1)–(Ξ.4)**. Fix $\bar{z} \in \mathcal{Z}$, $\bar{u} \in \Omega$ and δ as defined in Lemma 3.13. Choose $\zeta \in \mathbf{PC}(\mathbb{R}; \mathcal{Z})$ with $\mathrm{TV}(\zeta) < \delta$, $\|\zeta - \bar{z}\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^p)} < \delta$ and let $\mathcal{D}^\zeta = \mathcal{D}^\zeta_\delta$ be as in (3.19). Note that $\mathcal{D}^\zeta \subseteq \bar{u} + \mathbf{L}^1(\mathbb{R}; \Omega)$ contains all functions u in $\bar{u} + \mathbf{L}^1(\mathbb{R}; \Omega)$ with $\mathrm{TV}(u) < \delta$.

Lemma 3.14 *There exist positive δ, L and a unique semigroup $S^\zeta : \mathbb{R}_+ \times \mathcal{D}^\zeta \to \mathcal{D}^\zeta$ such that $\mathcal{D}^\zeta \supseteq \left\{u \in \bar{u} + \mathbf{L}^1(\mathbb{R}; \Omega) : \mathrm{TV}(u) < \delta\right\}$ and points 1., 2. and 3. in Theorem 2.3 hold. Moreover, S^ζ is obtained as limit of wave front tracking ε–approximate solutions.*

Proof Since $\mathcal{D}^\zeta_\delta$ is separable, the existence of a Lipschitz continuous semigroup S^ζ enjoying properties 1., 2. and 3. can be obtained through the limit of (subsequences of) wave front tracking ε–approximations in Definition 3.11 following standard arguments, see for instance, [1] or [5–8].

To prove uniqueness, let Σ^ζ be any Lipschitz continuous semigroups satisfying 1., 2. and 3.. Fix an initial datum $u_o \in \mathcal{D}^\zeta_\delta$. Call u^ε a wave front tracking ε–approximate solution approaching the orbit $t \mapsto S^\zeta u_o$ as $\varepsilon \to 0$. Then, by the Lipschitz continuity of Σ^ζ and Lemma 3.13,

$$\left\| \Sigma_t^\zeta u_o - S_t^\zeta u_o \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le \left\| \Sigma_t^\zeta u^\varepsilon(0) - u^\varepsilon(t) \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} + \mathcal{O}(1)\,\varepsilon\,(1+t)\,. \tag{3.32}$$

We now use [5, Theorem 2.9] to bound the first term in the right hand side above:

$$\left\| \Sigma_t^\zeta u^\varepsilon(0) - u^\varepsilon(t) \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \le \mathcal{O}(1) \int_0^t \liminf_{h\to 0} \frac{\left\| \Sigma_h^\zeta u^\varepsilon(t) - u^\varepsilon(t+h) \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)}}{h} \mathrm{d}\tau\,. \tag{3.33}$$

Using the notation as in Definition 3.11 and the classical estimates on physical and non–physical waves in [5, Lemma 9.1], for h so small that solutions to adjacent Riemann Problems do not overlap, we have:

$$\begin{aligned}
&\left\| \Sigma_h^\zeta u^\varepsilon(t) - u^\varepsilon(t+h) \right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^n)} \\
&\le \sum_{\alpha\in\mathcal{J}} \int_{x_\alpha - \hat\lambda h}^{x_\alpha+\hat\lambda h} \left\| \left(\Sigma_h^\zeta u^\varepsilon(t)\right)(x) - u^\varepsilon(t+h,x) \right\| \mathrm{d}x \\
&\le \mathcal{O}(1)\,\varepsilon\,h + \sum_{\alpha\in\mathcal{ZW}} \int_{x_\alpha - \hat\lambda h}^{x_\alpha+\hat\lambda h} \left\| \left(\Sigma_h^\zeta u^\varepsilon(t)\right)(x) - u^\varepsilon(t+h,x) \right\| \mathrm{d}x \\
&= \mathcal{O}(1)\,\varepsilon\,h
\end{aligned}$$

since for all zero waves $\left(\Sigma_h^\zeta u^\varepsilon(t)\right)(x) = u^\varepsilon(t+h,x)$ for a.e. t and wave front tracking ε–approximation solves Riemann Problems at zero waves exactly.

Insert the latter bound in (3.33), so that in the limit $\varepsilon \to 0$, (3.32) and the arbitrariness of u_o yield the equality of S^ζ and Σ^ζ. □

Lemma 3.15 *Fix $\xi \in \mathbb{R}$ and define*

$$\tilde\zeta(x) = \begin{cases} \zeta(\xi) & x \le \xi, \\ \zeta(\xi+) & x > \xi. \end{cases} \tag{3.34}$$

Choose $u \in \mathcal{D}^\zeta \cap \mathcal{D}^{\tilde\zeta}$. Then, for all $\vartheta > 0$,

$$\frac{1}{\vartheta} \int_{\xi-\hat\lambda\vartheta}^{\xi+\hat\lambda\vartheta} \left\| S_\vartheta^\zeta u(x) - S_\vartheta^{\tilde\zeta} u(x) \right\| \mathrm{d}x \le \mathcal{O}(1)\ \mathrm{TV}\Big(\zeta; \,]\xi - 2\hat\lambda\vartheta, \xi[\cup]\xi, \xi+2\hat\lambda\vartheta[\Big)\,.$$

Proof Let $\hat u^\varepsilon$ be an ε-wave front tracking approximation of $S^\zeta u$ so that

$$\int_{\xi-\hat\lambda\vartheta}^{\xi+\hat\lambda\vartheta} \left\| S_\vartheta^\zeta u(x) - S_\vartheta^{\tilde\zeta} u(x) \right\| \mathrm{d}x \le \mathcal{O}(1)\,\varepsilon + \int_{\xi-\hat\lambda\vartheta}^{\xi+\hat\lambda\vartheta} \left\| u^\varepsilon(\vartheta,x) - \left(S_\vartheta^{\tilde\zeta} u^\varepsilon(0)\right)(x) \right\| \mathrm{d}x.$$

By [5, Theorem 2.9],

$$
\begin{aligned}
&\int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| u^\varepsilon(\vartheta, x) - \left(S^{\tilde{\zeta}}_\vartheta u^\varepsilon(0)\right)(x)\right\| \mathrm{d}x \\
&\le \mathcal{O}(1) \int_0^\vartheta \liminf_{h\to 0+} \frac{1}{h} \int_{\xi-2\hat{\lambda}\vartheta+\hat{\lambda}(t+h)}^{\xi+2\hat{\lambda}\vartheta-\hat{\lambda}(t+h)} \left\| u^\varepsilon(t+h, x) - \left(S^{\tilde{\zeta}}_h u^\varepsilon(t)\right)(x)\right\| \mathrm{d}x\,\mathrm{d}t \\
&\le \mathcal{O}(1) \int_0^\vartheta \liminf_{h\to 0+} \frac{1}{h} \int_{\xi-2\hat{\lambda}\vartheta+\hat{\lambda}(t+h)}^{\xi+2\hat{\lambda}\vartheta-\hat{\lambda}(t+h)} \left\| u^\varepsilon(t+h, x) - \left(S^{\zeta}_h u^\varepsilon(t)\right)(x)\right\| \mathrm{d}x\,\mathrm{d}t \qquad (3.35)\\
&\quad + \mathcal{O}(1) \int_0^\vartheta \liminf_{h\to 0+} \frac{1}{h} \int_{\xi-2\hat{\lambda}\vartheta+\hat{\lambda}(t+h)}^{\xi+2\hat{\lambda}\vartheta-\hat{\lambda}(t+h)} \left\| \left(S^{\zeta}_h u^\varepsilon(t)\right)(x) - \left(S^{\tilde{\zeta}}_h u^\varepsilon(t)\right)(x)\right\| \mathrm{d}x\,\mathrm{d}t. \qquad (3.36)
\end{aligned}
$$

The integral in (3.35) is bounded by $\mathcal{O}(1)\,\varepsilon$ since u^ε is a piecewise constant ε–approximation of the trajectory $t \mapsto S^\zeta_t\,(u^\varepsilon(0))$. The map $x \mapsto u^\varepsilon(t,x)$ is piecewise constant, hence the integral in (3.36) can be computed estimating the differences in the local solutions to Riemann Problems arising from the discontinuities in $u^\varepsilon(t)$ using Lemma 3.10 in the case $\check{u}_o = \hat{u}_o$. Thus, the term in (3.36) is estimated as

$$
\begin{aligned}
&\liminf_{h\to 0+} \frac{1}{h} \int_{\xi-2\hat{\lambda}\vartheta+\hat{\lambda}(t+h)}^{\xi+2\hat{\lambda}\vartheta-\hat{\lambda}(t+h)} \left\| \left(S^{\zeta}_h u^\varepsilon(t)\right)(x) - \left(S^{\tilde{\zeta}}_h u^\varepsilon(t)\right)(x)\right\| \mathrm{d}x \\
&\le \sum_{\substack{x_\alpha \in \mathcal{I}(u^\varepsilon(t)) \\ x_\alpha \in]\xi-2\hat{\lambda}\vartheta+\hat{\lambda}t, \xi+2\hat{\lambda}\vartheta-\hat{\lambda}t[\\ x_\alpha \neq \xi,\ x_\alpha \notin \mathcal{I}(\zeta)}} \liminf_{h\to 0+} \frac{1}{h} \int_{x_\alpha-\hat{\lambda}h}^{x_\alpha+\hat{\lambda}h} \left\| \left(S^{\zeta}_h u^\varepsilon(t)\right)(x) - \left(S^{\tilde{\zeta}}_h u^\varepsilon(t)\right)(x)\right\| \mathrm{d}x \\
&\quad + \sum_{\substack{x_\alpha \in \mathcal{I}(\zeta) \\ x_\alpha \in]\xi-2\hat{\lambda}\vartheta+\hat{\lambda}t, \xi+2\hat{\lambda}\vartheta-\hat{\lambda}t[\\ x_\alpha \neq \xi}} \liminf_{h\to 0+} \frac{1}{h} \int_{x_\alpha-\hat{\lambda}h}^{x_\alpha+\hat{\lambda}h} \left\| \left(S^{\zeta}_h u^\varepsilon(t)\right)(x) - \left(S^{\tilde{\zeta}}_h u^\varepsilon(t)\right)(x)\right\| \mathrm{d}x \\
&\quad + \liminf_{h\to 0+} \frac{1}{h} \int_{\xi-\hat{\lambda}h}^{\xi+\hat{\lambda}h} \underbrace{\left\| \left(S^{\zeta}_h u^\varepsilon(t)\right)(x) - \left(S^{\tilde{\zeta}}_h u^\varepsilon(t)\right)(x)\right\|}_{=0} \mathrm{d}x \\
&\le \mathcal{O}(1) \sum_{\substack{x_\alpha \in \mathcal{I}(u^\varepsilon(t)) \\ x_\alpha \in]\xi-2\hat{\lambda}\vartheta+\hat{\lambda}t, \xi+2\hat{\lambda}\vartheta-\hat{\lambda}t[\\ x_\alpha \neq \xi,\ x_\alpha \notin \mathcal{I}(\zeta)}} \left\| \Delta u^\varepsilon(t, x_\alpha)\right\| \left\| \zeta(x_\alpha) - \tilde{\zeta}(x_\alpha)\right\| \\
&\quad + \mathcal{O}(1) \sum_{\substack{x_\alpha \in \mathcal{I}(\zeta) \\ x_\alpha \in]\xi-2\hat{\lambda}\vartheta+\hat{\lambda}t, \xi+2\hat{\lambda}\vartheta-\hat{\lambda}t[\\ x_\alpha \neq \xi}} \left(\left\| \Delta u^\varepsilon(t, x_\alpha)\right\| \left\| \zeta(x_\alpha) - \tilde{\zeta}(x_\alpha)\right\| + \left\| \Delta\zeta(x_\alpha)\right\| \right) \\
&\le \mathcal{O}(1)\,\mathrm{TV}\left(\zeta;]\xi - 2\hat{\lambda}\vartheta, \xi[\cup]\xi, \xi + 2\hat{\lambda}\vartheta[\right),
\end{aligned}
$$

and, in the limit $\varepsilon \to 0$, the proof of Lemma 3.15 follows. □

Lemma 3.16 *Fix $u_o \in \mathcal{D}^\zeta$. For a $\xi \in \mathbb{R}$ define*

$$\tilde{u}(x) = \begin{cases} u_o(\xi-) & x \in]\xi - \delta, \xi[\\ u_o(\xi+) & x \in]\xi, \xi + \delta[\\ u_o(x) & x \in]-\infty, \xi - \delta[\cup]\xi + \delta, +\infty[\end{cases} \tag{3.37}$$

and assume that $\tilde{u} \in \mathcal{D}^\zeta$. Then, for $\vartheta \in]0, \delta/(2\hat{\lambda})[$,

$$\frac{1}{\vartheta} \int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| S^\zeta_\vartheta u_o(x) - S^\zeta_\vartheta \tilde{u}(x) \right\| dx \le \mathcal{O}(1)\ \mathrm{TV}\left(u_o;]\xi - 2\hat{\lambda}\vartheta, \xi[\cup]\xi, \xi + 2\hat{\lambda}\vartheta[\right).$$

Proof Use the Lipschitz continuity of S^ζ, see Lemma 3.14, on the dependency domain,

$$\begin{aligned} \frac{1}{\vartheta} \int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| S^\zeta_\vartheta u_o(x) - S^\zeta_\vartheta \tilde{u}(x) \right\| dx &\le \frac{L}{\vartheta} \int_{\xi-2\hat{\lambda}\vartheta}^{\xi+2\hat{\lambda}\vartheta} \| u_o(x) - \tilde{u}(x) \| dx \\ &\le \frac{L}{\vartheta} \int_{\xi-2\hat{\lambda}\vartheta}^{\xi+2\hat{\lambda}\vartheta} \mathrm{TV}\left(u_o;]\xi - 2\hat{\lambda}\vartheta, \xi[\cup]\xi, \xi + 2\hat{\lambda}\vartheta[\right) dx \end{aligned}$$

and the proof follows. □

We are now ready to complete the proof of *(i)* in Theorem 2.3 for a piecewise constant ζ. Use $\tilde{u}$ as defined in (3.37) with $u_o = u(\tau)$ and $\tilde{\zeta}$ as in (3.34), so that $U^\sharp_{(u,\tau,\xi)}(\vartheta, x) = (S^{\tilde{\zeta}}_\vartheta \tilde{u})(x)$ for ϑ in a right neighborhood of 0 and x near ξ:

$$\begin{aligned} &\int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| u(\tau + \vartheta, x) - U^\sharp_{(u,\tau,\xi)}(\vartheta, x) \right\| dx \\ &= \int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| (S^\zeta_\vartheta u(\tau))(x) - (S^{\tilde{\zeta}}_\vartheta \tilde{u}(\tau))(x) \right\| dx \\ &\le \int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| (S^\zeta_\vartheta u(\tau))(x) - (S^\zeta_\vartheta \tilde{u}(\tau))(x) \right\| dx + \int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| (S^\zeta_\vartheta \tilde{u}(\tau))(x) - (S^{\tilde{\zeta}}_\vartheta \tilde{u}(\tau))(x) \right\| dx \end{aligned}$$

and the latter two terms are estimated by means of Lemmas 3.16 and 3.15, obtaining

$$\begin{aligned} &\frac{1}{\vartheta} \int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta} \left\| u(\tau + \vartheta, x) - U^\sharp_{(u,\tau,\xi)}(\vartheta, x) \right\| dx \\ &\le \mathcal{O}(1) \left(\mathrm{TV}\left(u,]\xi - 2\hat{\lambda}\vartheta, \xi[\cup]\xi, \xi + 2\hat{\lambda}\vartheta[\right) + \mathrm{TV}\left(\zeta,]\xi - 2\hat{\lambda}\vartheta, \xi[\cup]\xi, \xi + 2\hat{\lambda}\vartheta[\right)\right) \end{aligned} \tag{3.38}$$

and the statement follows passing to the limit $\vartheta \to 0$.

We now head towards the proof of *(ii)* in Theorem 2.3. Preliminary is the following result.

Lemma 3.17 *Let A be an $n \times n$ non singular matrix with n real eigenvalues $\lambda_1, \dots, \lambda_n$, n linearly independent right, respectively left, eigenvectors $r_1, \dots, r_n$, respectively $l_1, \dots, l_n$, and let m be a fixed finite vector measure. Then, the equation*

$$\partial_t u + A\, \partial_x u = m$$

generates the $\mathbf{L}^1$–Lipschitz semigroup

$$\begin{aligned} L_t : \mathbf{L}^1(\mathbb{R}; \mathbb{R}^n) &\to \mathbf{L}^1(\mathbb{R}; \mathbb{R}^n) \\ u &\to \sum_{i=1}^n l_i \cdot \left(u(x - \lambda_i t) + \frac{1}{\lambda_i} \int_{x-\lambda_i t}^x dm \right) r_i. \end{aligned}$$

The proof relies on a direct computation and is omitted. Note for later use that

$$L_t u = \sum_{i=1}^{n} l_i \cdot u(x - \lambda_i\, t)\, r_i + A^{-1} \sum_{i=1}^{n} l_i \cdot \int_{x-\lambda_i t}^{x} \mathrm{d}m\ r_i$$

The next Lemma proves *(ii)* in Theorem 2.3 in the case $x \mapsto u(\tau, x)$ is piecewise constant.

Lemma 3.18 *Under the same assumptions of Theorem 2.3, assume moreover that* $x \to u(\tau, x)$ *is piecewise constant. Then,* (ii) *in Theorem 2.3 holds.*

Proof With the notation in Lemma 3.17, recalling the definition (2.6) of $U^\flat$ in the case of a piecewise constant ζ,

$$U^\flat_{(u;\tau,\xi)}(\vartheta, \cdot) = L_\vartheta u(\tau) \text{ if } \begin{cases} A = D_u f\,(\zeta(\xi), u(\tau,\xi)) \\ k(\bar{x}) = \Xi\,(\zeta(\bar{x}+), \zeta(\bar{x}), u(\tau,\xi)) \\ \qquad -f\,(\zeta(\bar{x}+), u(\tau,\xi)) + f\,(\zeta(\bar{x}), u(\tau,\xi)) \\ m = \sum_{\bar{x}\in\mathcal{I}(\zeta)} k(\bar{x})\,\delta_{\bar{x}}\,. \end{cases} \tag{3.39}$$

Below, set for simplicity $u_o = u(\tau)$ and assume that $\tau = 0$. Use [5, Theorem 2.9] with the notation in the statement of Theorem 2.3, recalling that $x \mapsto L_t u_o(x)$ is piecewise constant, since so is u_o and L is linear.

$$\begin{aligned}
&\frac{1}{\vartheta}\int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| S_\vartheta u_o - U^\flat_{(u_o;0,\xi)} \right\| \mathrm{d}x \\
&= \frac{1}{\vartheta}\int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \| S_\vartheta u_o - L_\vartheta u_o \| \mathrm{d}x \\
&\le \mathcal{O}(1)\,\frac{1}{\vartheta}\int_0^\vartheta \liminf_{h\to 0+} \frac{1}{h}\int_{a+\hat{\lambda}(t+h)}^{b-\hat{\lambda}(t+h)} \| S_h L_t u_o - L_h L_t u_o \| \mathrm{d}x\mathrm{d}t \\
&\le \mathcal{O}(1)\,\frac{1}{\vartheta}\int_0^\vartheta \sum_{\bar{x}\in\mathcal{I}(L_t u_o)\cup\mathcal{I}(\zeta)} \liminf_{h\to 0+} \frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \| S_h L_t u_o - L_h L_t u_o \| \mathrm{d}x\mathrm{d}t.
\end{aligned}$$

To compute the latter sum, we distinguish the two cases $\bar{x} \notin \mathcal{I}(\zeta)$ or $\bar{x} \in \mathcal{I}(\zeta)$.

In the former case, in a neighborhood of $\bar{x}$ we have that the semigroup L locally coincide with that generated by the homogeneous equation $\partial_t u + A\,\partial_x u = 0$. Hence, by [3, Formula (3.8)], Lemma 3.18 and (3.39), we have

$$\begin{aligned}
&\frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \| S_h L_t u_o - L_h L_t u_o \| \mathrm{d}x \\
&\le \mathcal{O}(1) \sup_{x\in]a,b[} \| A - D_u f\,(\zeta(x), (L_t u_o)(x)) \| \, \| \Delta(L_t\, u_o)(\bar{x}) \| \\
&\le \mathcal{O}(1) \sup_{x\in]a,b[} \| D_u f\,(\zeta(\xi), u_o(\xi)) - D_u f\,(\zeta(x), (L_t u_o)(x)) \| \, \| \Delta(L_t\, u_o)(\bar{x}) \| \\
&\le \mathcal{O}(1) \sup_{x\in]a,b[} \left(\| \zeta(x) - \zeta(\xi) \| + \| u_o(\xi) - (L_t u_o)(x) \| \right) \| \Delta(L_t\, u_o)(\bar{x}) \| \\
&\le \mathcal{O}(1)\, \| \Delta(L_t u_o)(\bar{x}) \| \left(\mathrm{TV}\,(u_o,]a,b[) + \mathrm{TV}\,(\zeta,]a,b[) \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\bar{x}\in\mathcal{I}(L_t u_o), \bar{x}\notin\mathcal{I}(\zeta)} \frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \|S_h L_t u_o - L_h L_t u_o\| \mathrm{d}x \\
&\le \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}(L_t u_o), \bar{x}\notin\mathcal{I}(\zeta)} \|\Delta(L_t u_o)(\bar{x})\| \left(\mathrm{TV}\,(u_o,]a, b[) + \mathrm{TV}\,(\zeta,]a, b[)\right) \\
&\le \mathcal{O}(1) \left(\mathrm{TV}\,(u_o,]a, b[) + \mathrm{TV}\,(\zeta,]a, b[)\right)^2 .
\end{aligned}$$

Assume now that $\bar{x} \in \mathcal{I}(\zeta)$. Using the map T defined in Lemma 3.2, define

$$\begin{aligned}
\tilde{u} &= (L_t u_o)(\bar{x}-) \\
w(x) &= \begin{cases} \tilde{u} & x < \bar{x} \\ T\,(\zeta(\bar{x}+), \zeta(\bar{x}-))\,(\tilde{u}) & x > \bar{x} \end{cases}
\end{aligned}$$

and note that $(L_t u_o)(\bar{x}+) = \tilde{u} + A^{-1}\,k(\bar{x})$. Then, $S_h w = w$ and $L_h L_t u_o = L_t u_o$ in a neighborhood of $\bar{x}$. Moreover, recall the Lipschitz continuity of S_h restricted to dependency domains:

$$\int_{\bar{x}-\bar{\lambda}h}^{\bar{x}+\bar{\lambda}h} \|S_h L_t u_o - S_h w\| \mathrm{d}x \le \int_{\bar{x}-2\bar{\lambda}h}^{\bar{x}+2\bar{\lambda}h} \|L_t u_o - w\| \mathrm{d}x,$$

and, using the notation in (3.39), proceed

$$\begin{aligned}
&\frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \|S_h L_t u_o - L_h L_t u_o\| \mathrm{d}x \\
&\le \frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \|S_h L_t u_o - S_h w\| \mathrm{d}x + \frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \|S_h w - L_h L_t u_o\| \mathrm{d}x \\
&\le \mathcal{O}(1) \left\| T\,(\zeta(\bar{x}+), \zeta(\bar{x}))\,(\tilde{u}) - \left(\tilde{u} + A^{-1}\,k(\bar{x})\right) \right\| \\
&\le \mathcal{O}(1) \left\| A\,T\,(\zeta(\bar{x}+), \zeta(\bar{x}))\,(\tilde{u}) - (A\tilde{u} + k(\bar{x})) \right\| \\
&= \mathcal{O}(1) \,\| D_u f\,(\zeta(\xi), u_o(\xi))\,(T\,(\zeta(\bar{x}+), \zeta(\bar{x}))\,(\tilde{u}) - \tilde{u}) \\
&\quad - \Xi\,(\zeta(\bar{x}+), \zeta(\bar{x}), u_o(\xi)) + f\,(\zeta(\bar{x}+), u_o(\xi)) - f\,(\zeta(\bar{x}), u_o(\xi)) \| \\
&\le \mathcal{O}(1) \,\|\Delta\zeta(\bar{x})\| \left(\|\zeta(\bar{x}+) - \zeta(\xi)\| + \|\Delta\zeta(\bar{x})\| + \|u_o(\xi) - \tilde{u}\|\right) \\
&\le \mathcal{O}(1) \,\|\Delta\zeta(\bar{x})\| \left(\mathrm{TV}\,(u_o,]a, b[) + \mathrm{TV}\,(\zeta,]a, b[)\right),
\end{aligned}$$

where we used 2. in Lemma 3.2. Now, we add over $\bar{x} \in \mathcal{I}(\zeta)$:

$$\sum_{x\in\mathcal{I}(\zeta)} \frac{1}{h}\int_{\bar{x}-\hat{\lambda}h}^{\bar{x}+\hat{\lambda}h} \|S_h L_t u_o - L_h L_t u_o\| \mathrm{d}x \le \mathcal{O}(1) \left(\mathrm{TV}\,(u_o,]a, b[) + \mathrm{TV}\,(\zeta,]a, b[)\right)^2,$$

completing the proof of Lemma 3.18. □

To complete the proof of *(ii)* in Theorem 2.3, consider the case of $x \to u(\tau, x)$ not necessarily piecewise constant. We keep using the equalities $u(\tau + \vartheta) = S_\vartheta\,(u(\tau))$ and $U^\flat_{(u;\tau,\xi)} = L_\vartheta\,(u(\tau))$, the linear operator L being defined in Lemma 3.17 with A and k as in (3.39). Then, for $\varepsilon > 0$ call u^ε a piecewise constant approximation of $u(\tau)$ with

$\mathrm{TV}(u^\varepsilon) \leq \mathrm{TV}\,(u(\tau))$. By the Lipschitz continuity of S_ϑ and L_t, we have

$$\begin{aligned}
&\frac{1}{\vartheta}\int_{a-\hat\lambda\vartheta}^{b+\hat\lambda\vartheta}\left\|u(\tau+\vartheta,x)-U^\flat_{(u;\tau,\xi)}(\vartheta,x)\right\|\mathrm{d}x\\
&\leq \mathcal{O}(1)\frac{\varepsilon}{\vartheta}+\frac{1}{\vartheta}\int_{a-\hat\lambda\vartheta}^{b+\hat\lambda\vartheta}\left\|(S_\vartheta u^\varepsilon)(x)-(L_\vartheta u^\varepsilon)(x)\right\|\mathrm{d}x\\
&\leq \mathcal{O}(1)\frac{\varepsilon}{\vartheta}+\mathcal{O}(1)\left(\mathrm{TV}\,(u(\tau),]a,b[)+\mathrm{TV}\,(\zeta,]a,b[)\right)^2.
\end{aligned}$$

Passing first to the limit $\varepsilon \to 0$ and then to the limit $\vartheta \to 0$, we complete the proof.

3.3 The General Case

Consider now the case $\zeta \in \mathbf{BV}(\mathbb{R};\mathbb{R}^p)$.

Lemma 3.19 *Assume f satisfies* **(f.1)–(f.4)** *and Ξ satisfies* **(Ξ.1)–(Ξ.4)**. *Then, for all $z \in \mathcal{Z}$, $v \in \mathbb{R}^p$. $\omega \in \Omega$, if $\delta > 0$ is sufficiently small,*

$$\begin{aligned}
&\Xi(z+\delta\,v,z,\omega)-f(z+\delta\,v,\omega)\\
&\quad+f(z,\omega)=\delta\left(D_v^+\,\Xi(z,z,\omega)-D_z f(z,\omega)\,v\right)+\mathcal{O}(1)\,\sigma(\delta)\,\delta.
\end{aligned}$$

The proof directly follows from–**(Ξ.4)** and from the Taylor expansion of f.

Proof of Theorem 2.3 Let $\zeta \in \mathbf{BV}(\mathbb{R};\mathcal{Z})$. Call $\mathcal{I}(\zeta)$ the, at most countable, set of points of jump in ζ. Recall that $D\zeta$ is a finite vector measure. Let μ and v be as in (1.6). By Lusin Theorem [20, Theorem 2.24], for any $h > 0$, there exists a $\tilde v^h \in \mathbf{C}^0_c(\mathbb{R};\mathbb{R}^p)$ such that

$$\left\|\tilde v^h(x)\right\| \leq 1 \quad \text{and} \quad \|D\zeta\|\left(\left\{x \in \mathbb{R}\colon \tilde v^h(x) \neq v(x)\right\}\right) < h\,. \tag{3.40}$$

Following [8, Step 1, § 4.3], introduce points $\{x_1,\ldots,x_{N_h-1}\} \in \mathbb{R}$ such that:

- **(i)** $x_0 = -\infty, x_1 < -1/h, x_{i-1} < x_i$ for $i = 2,\ldots,N_h-1, x_{N_h-1} > 1/h$ and $x_{N_h} = +\infty$.
- **(ii)** $\sum_{x\in\mathcal{I}(\zeta)\setminus\mathcal{I}^h}\|\Delta\zeta(x)\| < h$ for a suitable set of points $\mathcal{I}^h$ contained in $\{x_1,x_2,\ldots,x_{N_h-1}\}$.
- **(iii)** Whenever[1] $x_i \in \mathcal{I}^h$, $\mathrm{TV}\,(\zeta,[x_{i-1},x_i[) < h/(1+\sharp\mathcal{I}^h)$.
- **(iv)** $\mathrm{TV}\,(\zeta,]x_{i-1},x_i[) < h$ for all $i = 1,\ldots,N_h$.
- **(v)** $\left\|\tilde v^h\left(x'\right)-\tilde v^h\left(x''\right)\right\| < h$ for $x',\ x'' \in]x_{i-1},x_i[$, $i = 1,\ldots,N_h$.
- **(vi)** $x_i - x_{i-1} \in]0,h[$ for all $i = 2,\ldots,N_h-1$.

Points satisfying **(i)** are easily constructed. Then, adding more points, one fulfills also **(ii)** and this condition fully defines $\mathcal{I}^h$ and, hence, $\sharp\mathcal{I}^h$. Iteratively continuing to add points, thus increasing N_h, we satisfy also **(iii)**, **(iv)**, **(v)** and **(vi)**, in this order. Define the piecewise constant map

$$\zeta^h = \zeta\,(-\infty)\,\chi_{]-\infty,x_1]}+\sum_{i=2}^{N_h-1}\zeta\,(x_{i-1}+)\;\chi_{]x_{i-1},x_i]}+\zeta\left(x_{N_h-1}+\right)\chi_{]x_{N_h-1},+\infty[} \tag{3.41}$$

and note that

$$\zeta^h(x_i)=\zeta(x_{i-1}+) \quad \text{and} \quad \zeta^h(x_i+)=\zeta(x_i+)\,. \tag{3.42}$$

[1] Everywhere, $\sharp A$ stands the (finite) cardinality of the set A.

The approximations ζ^h converge to ζ uniformly on $\mathbb{R}$ as $h \to 0$. Indeed, fix h and for any $x \in \mathbb{R}$, by **(i)** we have $x \in]x_{i-1}, x_i]$ (obviously excluding $+\infty$) and for $i \in \{1, \dots, N_h\}$, by **(iv)**,

$$\left\| \zeta^h(x) - \zeta(x) \right\| \le \mathrm{TV}\,(\zeta,]x_{i-1}, x_i[) \le h.$$

Call $S^{\zeta^h} : [0, +\infty[\times \mathcal{D}_\delta^{\zeta^h} \to \mathcal{D}_\delta^{\zeta^h}$ the semigroup whose existence is proved in the piecewise constant case in § 3.2.3, provided δ is sufficiently small. We prove that as $h \to 0$ the semigroups S^{ζ^h} converge to a semigroup S^ζ in $\mathbf{L}^1$.

Using the notation (3.19), introduce the sets:

$$\check{\mathcal{D}}_\delta^\zeta = \bigcap_{h>0} \mathcal{D}_\delta^{\zeta^h} \qquad \hat{\mathcal{D}}_\delta^\zeta = \overline{\bigcup_{h>0} \mathcal{D}_\delta^{\zeta^h}} \tag{3.43}$$

the latter closure is understood in the strong $\mathbf{L}^1$ topology. If ζ has sufficiently small total variation then suitably choosing positive δ and δ'

$$\check{\mathcal{D}}_\delta^\zeta \subseteq \hat{\mathcal{D}}_\delta^\zeta \subseteq \check{\mathcal{D}}_{\delta'}^\zeta$$

and all these sets are not empty since they contain all u with sufficiently small total variation.

Since $\check{\mathcal{D}}_{\delta'}^\zeta$ is separable with respect to the strong $\mathbf{L}^1$ topology, by a diagonalization process there exists a sequence h_i such that for all $u \in \check{\mathcal{D}}_{\delta'}^\zeta$ and for all $t \in [0, +\infty[$, the sequence $S_t^{\zeta^{h_i}} u$ converges in $\mathbf{L}^1_{\mathbf{loc}}(\mathbb{R}; \mathbb{R}^n)$ to a limit which we define as $S_t^\zeta u$. Clearly, $S_t^\zeta u \in \hat{\mathcal{D}}_{\delta'}^\zeta$. Moreover, whenever $S_t^\zeta u \in \check{\mathcal{D}}_{\delta'}^\zeta$, thanks to the Lipschitz continuity of $u \mapsto S_t^{\zeta h_i} u$, the semigroup property holds in the limit $h_i \to 0$, i.e., $S_s^\zeta S_t^\zeta u = S_{s+t}^\zeta u$ for all $s \ge 0$.

Define now

$$\mathcal{D}_\delta^\zeta = \left\{ u \in \hat{\mathcal{D}}_\delta^\zeta : \exists\, t \in \mathbb{R}_+ \text{ and } \exists w \in \check{\mathcal{D}}_\delta^\zeta \text{ such that } S_t^\zeta w = u \right\}. \tag{3.44}$$

Note that

$$\check{\mathcal{D}}_\delta^\zeta \subseteq \mathcal{D}_\delta^\zeta \subseteq \hat{\mathcal{D}}_\delta^\zeta.$$

For all $t \in \mathbb{R}_+$, the domain $\mathcal{D}_\delta^\zeta$ is invariant with respect to S_t^ζ, in the sense that $(S_t^\zeta \mathcal{D}_\delta^\zeta) \subseteq \mathcal{D}_\delta^\zeta$.

Following the lines of [8, Theorem 2.2], the above construction proves 1. in Theorem 2.3. Condition 2. in Theorem 2.3 also follows, since the semigroup $S^{\zeta^{h_i}}$ admits a Lipschitz constant independent of h_i. In statement 3. of Theorem 2.3 ζ is required to be piecewise constant and the results in § 3.2.3 apply, since in this case for h sufficiently small ζ^h coincides with ζ.

To prove 4. in Theorem 2.3, we consider first *(i)*.

Proof of *(i)*. To simplify the notation, we denote h_i by h. By $U^{\sharp h}_{(u;\tau,\xi)}$ denote the solution to the Riemann Problem (2.5) with ζ replaced by ζ^h. Clearly,

$$\begin{aligned}
&\frac{1}{\vartheta} \int_{\xi-\hat\lambda\vartheta}^{\xi+\hat\lambda\vartheta} \left\| \left(S_\vartheta^\zeta u(\tau)\right)(x) - U^\sharp_{(u;\tau,\xi)}(\vartheta, x) \right\| \mathrm{d}x \\
&\quad \le \frac{1}{\vartheta} \int_{\xi-\hat\lambda\vartheta}^{\xi+\hat\lambda\vartheta} \left\| \left(S_\vartheta^\zeta u(\tau)\right)(x) - \left(S_\vartheta^{\zeta^h} u(\tau)\right)(x) \right\| \mathrm{d}x
\end{aligned} \tag{3.45}$$

$$
\begin{aligned}
&+\frac{1}{\vartheta}\int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta}\left\|\left(S_{\vartheta}^{\zeta^h}u(\tau)\right)(x)-U^{\sharp h}_{(u;\tau,\xi)}(\vartheta,x)\right\|\mathrm{d}x\\
&+\frac{1}{\vartheta}\int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta}\left\|U^{\sharp h}_{(u;\tau,\xi)}(\vartheta,x)-U^{\sharp}_{(u;\tau,\xi)}(\vartheta,x)\right\|\mathrm{d}x. \qquad (3.46)
\end{aligned}
$$

The first integral in the right hand side above vanishes in the limit $h \to 0$. To estimate the second integral, use (3.38) and [8, Formula (4.29)] to get

$$
\begin{aligned}
&\frac{1}{\vartheta}\int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta}\left\|\left(S_{\vartheta}^{\zeta^h}u(\tau)\right)(x)-U^{\sharp h}_{(u;\tau,\xi)}(\vartheta,x)\right\|\mathrm{d}x\\
&\le \mathcal{O}(1)\left(\mathrm{TV}\left(u(\tau),\,]\xi-2\hat{\lambda}\vartheta,\xi[\cup]\xi,\xi+2\hat{\lambda}\vartheta[\right)\right.\\
&\quad\left.+\mathrm{TV}\left(\zeta^h,\,]\xi-2\hat{\lambda}\vartheta,\xi[\cup]\xi,\xi+2\hat{\lambda}\vartheta[\right)\right)\\
&\le \mathcal{O}(1)\left(\mathrm{TV}\left(u(\tau),\,]\xi-2\hat{\lambda}\vartheta,\xi[\cup]\xi,\xi+2\hat{\lambda}\vartheta[\right)\right.\\
&\quad\left.+\mathrm{TV}\left(\zeta,\,]\xi-2\hat{\lambda}\vartheta,\xi[\cup]\xi,\xi+2\hat{\lambda}\vartheta[\right)+h\right).
\end{aligned}
$$

In the limits, first for $h \to 0$ and then for $\vartheta \to 0$, the latter term above vanishes.

Concerning the third term (3.46), use Lemma 3.10 and obtain

$$
\begin{aligned}
&\frac{1}{\vartheta}\int_{\xi-\hat{\lambda}\vartheta}^{\xi+\hat{\lambda}\vartheta}\\
&\left\|U^{\sharp h}_{(u;\tau,\xi)}(\vartheta,x)-U^{\sharp}_{(u;\tau,\xi)}(\vartheta,x)\right\|\mathrm{d}x \le \mathcal{O}(1)\left(\left\|\zeta^h(\xi)-\zeta(\xi)\right\|+\left\|\zeta^h(\xi+)-\zeta(\xi+)\right\|\right)
\end{aligned}
$$

which vanishes as $h \to 0$ by the uniform convergence of ζ^h to ζ, completing the proof of *(i)*.

Proof of *(ii)*. We now pass to *(ii)* in item 4. of Theorem 2.3. the following definitions and preliminary results are of use below.

Recall the notation in (1.6). For $i = 1, \ldots, N_h$, let

$$
\delta_i = \|\mu\|\left(]x_{i-1},x_i[\right); \qquad (3.47)
$$

$$
v_i = \begin{cases} \frac{1}{\delta_i}\,\mu\left(]x_{i-1},x_i[\right) & \delta_i \neq 0;\\ 0 & \delta_i = 0;\end{cases} \qquad (3.48)
$$

$$
v^h = \sum_{i=1}^{N_h-1} v_i\,\chi_{]x_{i-1},x_i]} + v_{N_h}\,\chi_{]x_{N_h-1},+\infty[}. \qquad (3.49)
$$

Note also that for $\delta_i \neq 0$, $v_i = \frac{1}{\delta_i}\int_{]x_{i-1},x_i[} v\,\mathrm{d}\|\mu\|$.

Claim: We have the convergence

$$
\lim_{h\to 0}\int_{\mathbb{R}}\left\|v^h - v\right\|\mathrm{d}\|\mu\| = 0. \qquad (3.50)
$$

Indeed, recalling (3.40),

$$
\begin{aligned}
&\int_{\mathbb{R}} \left\| v^h(x) - v(x) \right\| \mathrm{d}\|\mu\|(x) \\
&= \sum_{i=1}^{N_h} \int_{]x_{i-1},x_i[} \|v(x) - v_i\| \mathrm{d}\|\mu\|(x) \\
&= \sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \frac{1}{\delta_i} \int_{(]x_{i-1},x_i[)^2} \|v(x) - v(y)\| \mathrm{d}(\|\mu\| \otimes \|\mu\|)(x,y) \\
&= \sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \frac{1}{\delta_i} \int_{(]x_{i-1},x_i[)^2} \left[\left\| v(x) - \tilde{v}^h(x) \right\| + \left\| \tilde{v}^h(y) - v(y) \right\| \right] \mathrm{d}(\|\mu\| \otimes \|\mu\|)(x,y)
\end{aligned} \tag{3.51}
$$

$$
+ \sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \frac{1}{\delta_i} \int_{(]x_{i-1},x_i[)^2} \left\| \tilde{v}^h(x) - \tilde{v}^h(y) \right\| \mathrm{d}(\|\mu\| \otimes \|\mu\|)(x,y). \tag{3.52}
$$

The two terms in the integral in (3.51) are estimated in the same way, using (3.40), as

$$
\begin{aligned}
&\sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \int_{(]x_{i-1},x_i[)^2} \frac{1}{\delta_i} \left\| v(x) - \tilde{v}^h(x) \right\| \mathrm{d}(\|\mu\| \otimes \|\mu\|)(x,y) \\
&= \sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \int_{]x_{i-1},x_i[} \left\| v(x) - \tilde{v}^h(x) \right\| \mathrm{d}\|\mu\|(x) \\
&= \int_{\mathbb{R}} \left\| v(x) - \tilde{v}^h(x) \right\| \mathrm{d}\|\mu\|(x) \\
&\leq \int_{\{x \in \mathbb{R} \colon v(x) \neq \tilde{v}^h(x)\}} \left(\|v(x)\| + \left\| \tilde{v}^h(x) \right\| \right) \mathrm{d}\|\mu\|(x) \\
&\leq 2h \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}
$$

We now estimate the term (3.52) by means of **(v)**:

$$
\begin{aligned}
&\sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \frac{1}{\delta_i} \int_{(]x_{i-1},x_i[)^2} \left\| \tilde{v}^h(x) - \tilde{v}^h(y) \right\| \mathrm{d}(\|\mu\| \otimes \|\mu\|)(x,y) \\
&\leq h \sum_{\substack{i=1,N_h \\ \delta_i \neq 0}} \frac{1}{\delta_i} \int_{(]x_{i-1},x_i[)^2} \mathrm{d}(\|\mu\| \otimes \|\mu\|)(x,y) \\
&\leq h \sum_{i=1}^{N_h} \delta_i \\
&\leq h \, \|\mu\|(\mathbb{R}) \\
&\to 0 \quad \text{as } h \to 0,
\end{aligned}
$$

completing the proof of the Claim.

Apply Lemma 3.17 with $A = Df\,(\zeta(\xi), u(\tau, \xi))$, first with $m = g$ as defined in (2.7), then with $m = g^h$ where g^h is defined, for all Borel subset E of $\mathbb{R}$, by

$$g^h(E) = \sum_{y \in \mathcal{I}(\zeta^h)} \left(\Xi\left(\zeta^h(y+), \zeta^h(y), u(\tau, \xi)\right) \right.\\ \left. - f\left(\zeta^h(y+), u(\tau, \xi)\right) + f\left(\zeta^h(y), u(\tau, \xi)\right)\right) \delta_y(E)$$

and write

$$\begin{aligned} U^\flat_{(u;\tau,\xi)}(\vartheta, x) &= \sum_{i=1}^n l_i \cdot \left(u(\tau, x - \lambda_i\,\vartheta) + \frac{1}{\lambda_i} \int_{x-\lambda_i\vartheta}^x \mathrm{d}g \right) r_i, \\ U^{\flat h}_{(u;\tau,\xi)}(\vartheta, x) &= \sum_{i=1}^n l_i \cdot \left(u(\tau, x - \lambda_i\,\vartheta) + \frac{1}{\lambda_i} \int_{x-\lambda_i\vartheta}^x \mathrm{d}g^h \right) r_i. \end{aligned} \tag{3.53}$$

Similarly to (3.45), fix a, b, ξ in $\mathbb{R}$ with $a < \xi < b$, let $\vartheta \in]0, (b-a)/\hat{\lambda}[$ and compute

$$\frac{1}{\vartheta} \int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| \left(S^\zeta_\vartheta u(\tau)\right)(x) - U^\flat_{(u;\tau,\xi)}(\vartheta, x) \right\| \mathrm{d}x$$

$$\leq \frac{1}{\vartheta} \int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| \left(S^\zeta_\vartheta u(\tau)\right)(x) - \left(S^{\zeta^h}_\vartheta u(\tau)\right)(x) \right\| \mathrm{d}x \tag{3.54}$$

$$+ \frac{1}{\vartheta} \int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| \left(S^{\zeta^h}_\vartheta u(\tau)\right)(x) - U^{\flat h}_{(u;\tau,\xi)}(\vartheta, x) \right\| \mathrm{d}x \tag{3.55}$$

$$+ \frac{1}{\vartheta} \int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| U^{\flat h}_{(u;\tau,\xi)}(\vartheta, x) - U^\flat_{(u;\tau,\xi)}(\vartheta, x) \right\| \mathrm{d}x. \tag{3.56}$$

The first term (3.54) vanishes as $h \to 0$ by the above construction of S.

Since ζ^h is piecewise constant, to bound (3.55) we can use (ii) in Theorem 2.3 as proved in § 3.2.3 in the piecewise constant case:

$$\begin{aligned} [(3.55)] &\leq C \left[\mathrm{TV}\,(u(\tau),]a, b[) + \mathrm{TV}\left(\zeta^h,]a, b[\right)\right]^2 \\ &\leq C\,[\mathrm{TV}\,(u(\tau),]a, b[) + h + \mathrm{TV}\,(\zeta,]a, b[)]^2 \end{aligned}$$

where we used [8, Formula (4.29)]. In the limit $h \to 0$ we obtain the desired estimate.

Compute (3.56) by means of (3.53) as

$$\begin{aligned} [(3.56)] &= \int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| \sum_{i=1}^n \frac{1}{\lambda_i} l_i \cdot \int_{x-\lambda_i\vartheta}^x \left(\mathrm{d}g - \mathrm{d}g^h\right) \right\| \mathrm{d}x \\ &\leq \mathcal{O}(1) \sum_{i=1}^n \int_{a+\hat{\lambda}\vartheta}^{b-\hat{\lambda}\vartheta} \left\| \int_{x-\lambda_i\vartheta}^x \left(\mathrm{d}g - \mathrm{d}g^h\right) \right\| \mathrm{d}x. \end{aligned}$$

We now estimate the latter integrals, assuming that neither x nor $x - \lambda_i \vartheta$ are discontinuity points for ζ or ζ^h. Fix i and call J the real interval with extreme points x and $x - \lambda_i \vartheta$.

$$\begin{aligned}
&\left\| \int_J \left(\mathrm{d}g - \mathrm{d}g^h\right) \right\| \\
&= \Bigg\| \sum_{\bar{x} \in \mathcal{I}(\zeta) \cap J} \left(\Xi\left(\zeta(\bar{x}+), \zeta(\bar{x}), u(\tau,\xi)\right) - f\left(\zeta(\bar{x}+), u(\tau,\xi)\right) + f\left(\zeta(\bar{x}), u(\tau,\xi)\right)\right) \\
&\quad + \int_J \left(D^+_{v(x)} \Xi\left(\zeta(x), \zeta(x), u(\tau,\xi)\right) - D_z f\left(\zeta(x), u(\tau,\xi)\right) v(x)\right) \mathrm{d}\|\mu\|(x) \\
&\quad - \sum_{\bar{x} \in \mathcal{I}(\zeta^h) \cap J} \Big(\Xi\left(\zeta^h(\bar{x}+), \zeta^h(\bar{x}), u(\tau,\xi)\right) \\
&\quad - f\left(\zeta^h(\bar{x}+), u(\tau,\xi)\right) + f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right)\Big)\Bigg\| \\
&\le \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 + \mathcal{E}_5 + \mathcal{E}_6.
\end{aligned}$$

The terms $\mathcal{E}_1, \ldots, \mathcal{E}_6$ are defined below.

In the first term, using the definition of $\mathcal{I}^h$ in **(ii)**, we show that the sum of all jumps in ζ not in $\mathcal{I}^h$ is $\mathcal{O}(1)\, h$:

$$\begin{aligned}
\mathcal{E}_1 &= \left\| \sum_{\bar{x} \in (\mathcal{I}(\zeta) \setminus \mathcal{I}^h) \cap J} \left(\Xi\left(\zeta(\bar{x}+), \zeta(\bar{x}), u(\tau,\xi)\right) - f\left(\zeta(\bar{x}+), u(\tau,\xi)\right) + f\left(\zeta(\bar{x}), u(\tau,\xi)\right)\right) \right\| \\
&\le \mathcal{O}(1) \sum_{\bar{x} \in \mathcal{I}(\zeta) \setminus \mathcal{I}^h} \|\Delta\zeta(\bar{x})\| && \text{[By } (\mathbf{f.1}), (\boldsymbol{\Xi}.\mathbf{1}) \text{ and } (\boldsymbol{\Xi}.\mathbf{3})] \\
&\le \mathcal{O}(1)\, h && \text{[By } (\mathbf{ii})] \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Now we estimate the effect of passing from $\zeta^h(\bar{x})$ to $\zeta(\bar{x})$ in the jumps $\bar{x}$ in $\mathcal{I}^h$, calling $\bar{\bar{x}}$ the point in $\mathcal{I}(\zeta^h)$ that precedes $\bar{x}$ and using (3.42):

$$\begin{aligned}
\mathcal{E}_2 = &\Bigg\| \sum_{\bar{x} \in \mathcal{I}(\zeta) \cap \mathcal{I}^h \cap J} \Big[\Xi\left(\zeta^h(\bar{x}+), \zeta(\bar{x}), u(\tau,\xi)\right) - f\left(\zeta^h(\bar{x}+), u(\tau,\xi)\right) + f\left(\zeta(\bar{x}), u(\tau,\xi)\right) \\
&- \Xi\left(\zeta^h(\bar{x}+), \zeta^h(\bar{x}), u(\tau,\xi)\right) + f\left(\zeta^h(\bar{x}+), u(\tau,\xi)\right) - f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right)\Big]\Bigg\| \\
&\le \mathcal{O}(1) \sum_{\bar{x} \in \mathcal{I}(\zeta) \cap \mathcal{I}^h \cap J} \left\|\zeta(\bar{x}) - \zeta^h(\bar{x})\right\| && \text{[By } (\mathbf{f.1}) \text{ and } (\boldsymbol{\Xi}.\mathbf{1})] \\
&\le \mathcal{O}(1) \sum_{\bar{x} \in \mathcal{I}(\zeta) \cap \mathcal{I}^h \cap J} \left\|\zeta(\bar{x}) - \zeta(\bar{\bar{x}}+)\right\| && \text{[By (3.42)]} \\
&\le \mathcal{O}(1) \sum_{\bar{x} \in \mathcal{I}(\zeta) \cap \mathcal{I}^h \cap J} \mathrm{TV}\left(\zeta;]\bar{\bar{x}}, \bar{x}[\right) && \text{[By 3.41]} \\
&\le \mathcal{O}(1) \sum_{\bar{x} \in \mathcal{I}(\zeta) \cap \mathcal{I}^h \cap J} \frac{h}{1 + \sharp\mathcal{I}^h} && \text{[By } (\mathbf{iii})] \\
&\le \mathcal{O}(1)\, h \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Call $\bar{\bar{x}}$ the point in $\mathcal{I}(\zeta^h)$ that precedes $\bar{x}$. Out of $\mathcal{I}^h$, the measure μ approximates the measure $D\zeta$, so that $\zeta^h(\bar{x}) + \bar{\delta}\,\bar{v}$ approximates $\zeta^h(\bar{x}+)$ as in (3.48)–(3.49):

$$\begin{aligned}
\mathcal{E}_3 = & \left\| \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \Big[\Xi\left(\zeta^h(\bar{x}+), \zeta^h(\bar{x}), u(\tau,\xi)\right) \right. \\
& \quad -f\left(\zeta^h(\bar{x}+), u(\tau,\xi)\right) + f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right) \\
& \quad - \Xi\left(\zeta^h(\bar{x}) + \bar{\delta}\bar{v}, \zeta^h(\bar{x}), u(\tau,\xi)\right) \\
& \quad \left. +f\left(\zeta^h(\bar{x}) + \bar{\delta}\bar{v}, u(\tau,\xi)\right) - f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right)\Big] \right\| \\
\leq & \ \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \left\| \zeta^h(\bar{x}+) - \zeta^h(\bar{x}) - \mu\left(]\bar{\bar{x}}, \bar{x}[\right) \right\| \\
= & \ \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \left\| \zeta^h(\bar{x}+) - \zeta^h(\bar{\bar{x}}+) - \mu\left(]\bar{\bar{x}}, \bar{x}[\right) \right\| \\
= & \ \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \left\| D\zeta\left(]\bar{\bar{x}}, \bar{x}]\right) - \mu\left([\bar{\bar{x}}, \bar{x}]\right) \right\| \\
\leq & \ \mathcal{O}(1) \sum_{\bar{x}\notin\mathcal{I}^h} \|\Delta\zeta(\bar{x})\| \\
\leq & \ \mathcal{O}(1)\, h \\
\to & \ 0 \quad \text{as } h \to 0.
\end{aligned}$$

Using Lemma 3.19, the differences at the jumps in ζ^h out of $\mathcal{I}^h$ are approximated by means of derivatives:

$$\begin{aligned}
\mathcal{E}_4 = & \left\| \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \Big[\Xi\left(\zeta^h(\bar{x}) + \bar{\delta}\bar{v}, \zeta^h(\bar{x}), u(\tau,\xi)\right) \right. \\
& \quad -f\left(\zeta^h(\bar{x}) + \bar{\delta}\bar{v}, u(\tau,\xi)\right) + f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right) \\
& \quad \left. -\bar{\delta}\left(D^+_{\bar{v}}\Xi\left(\zeta^h(\bar{x}), \zeta^h(\bar{x}), u(\tau,\xi)\right) - D_z f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right)\bar{v}\right)\Big] \right\| \\
\leq & \ \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \sigma(\bar{\delta})\,\bar{\delta} \\
\leq & \ \mathcal{O}(1)\,\sigma(h)\,\mathrm{TV}(\zeta) \\
\to & \ 0 \quad \text{as } h \to 0.
\end{aligned}$$

If $\bar{x} \in \mathcal{I}^h$, $\|\mu\|\left(]\bar{\bar{x}}, \bar{x}[\right) = \bar{\delta}$ is negligible, so that by **(iii)**, (3.47), (3.48) and (3.49),

$$\begin{aligned}
\mathcal{E}_5 = & \left\| \sum_{\bar{x}\in\mathcal{I}(\zeta^h)\cap J\setminus\mathcal{I}^h} \Big[\bar{\delta}\left(D_{\bar{v}}\Xi\left(\zeta^h(\bar{x}), \zeta^h(\bar{x}), u(\tau,\xi)\right) - D_z f\left(\zeta^h(\bar{x}), u(\tau,\xi)\right)\bar{v}\right) \right. \\
& \left. - \int_J \left(D^+_{v^h(x)}\Xi\left(\zeta^h(x), \zeta^h(x), u(\tau,\xi)\right) - D_z f\left(\zeta^h(x), u(\tau,\xi)\right) v^h(x)\right) \mathrm{d}\|\mu\|(x)\Big] \right\| \\
\leq & \ \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}^h} \bar{\delta} + \mathcal{O}(1)\, h
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{O}(1) \sum_{\bar{x}\in\mathcal{I}^h} \frac{h}{1+\sharp\mathcal{I}^h} + \mathcal{O}(1)\, h \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}
$$

We now use the Lipschitz continuity of $D_v\,\Xi$, see **(Ξ.4)**, and of $D_z f$, see **(f.1)**, the uniform convergence of $\zeta^h \to \zeta$ and the convergence $v^h \to v$ by (3.50):

$$
\begin{aligned}
\mathcal{E}_6 &= \left\| \int_J \left(D^+_{v^h(x)}\,\Xi\left(\zeta^h(x), \zeta^h(x), u(\tau,\xi)\right) - D_z f\left(\zeta^h(x), u(\tau,\xi)\right) v^h(x)\right) \mathrm{d}\|\mu\|(x) \right. \\
&\quad \left. - \int_J \left(D^+_{v(x)}\,\Xi\left(\zeta(x), \zeta(x), u(\tau,\xi)\right) - D_z f\left(\zeta(x), u(\tau,\xi)\right) v(x)\right) \mathrm{d}\|\mu\|(x) \right\| \\
&\leq \mathcal{O}(1) \int_J \left(\left\|\zeta(x) - \zeta^h(x)\right\| + \left\|v(x) - v^h(x)\right\| \right) \mathrm{d}\|\mu\|(x) \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}
$$

The proof of *(ii)* is completed.

We now prove that a $\mathbf{L}^1$–Lipschitz continuous map u satisfying *(i)* and *(ii)* for a.e. t is actually an orbit of S^ζ. Using [5, Theorem 2.9] as in [5, § 9.2], for any $a, b \in \mathbb{R}$ with $a < b$

$$
\begin{aligned}
&\left\| u(t) - S^\zeta_t u(0) \right\|_{\mathbf{L}^1([a+\hat\lambda\, t, b-\hat\lambda t];\mathbb{R}^n)} \\
&\quad \leq L \int_0^t \liminf_{h\to 0+} \frac{1}{h} \left\| u(\tau+h) - S^\zeta_h u(\tau) \right\|_{\mathbf{L}^1([a+\hat\lambda\,(\tau+h), b-\hat\lambda(\tau+h)];\mathbb{R}^n)} \mathrm{d}\tau.
\end{aligned}
$$

Let τ be such that *(i)* and *(ii)* hold. Fix $\varepsilon > 0$ and choose $x_0 = a + \hat\lambda\tau$, $x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N$, $x_N = b - \hat\lambda\tau$ such that, for $i = 1, \ldots, N$,

$$
\mathrm{TV}(u(\tau);]x_{i-1}, x_i[) + \mathrm{TV}(\zeta;]x_{i-1}, x_i[) < \varepsilon.
$$

Then, for $h > 0$ sufficiently small, and for $\xi_i \in]x_{i-1}, x_i[$

$$
\begin{aligned}
&\left\| u(\tau+h) - S^\zeta_h u(\tau) \right\|_{\mathbf{L}^1([a+\hat\lambda\,(\tau+h), b-\hat\lambda(\tau+h)];\mathbb{R}^n)} \\
&= \sum_{i=1}^{N} \int_{x_{i-1}+\hat\lambda\, h}^{x_i-\hat\lambda\, h} \left\| u(\tau+h, x) - \left(S^\zeta_h u(\tau)\right)(x) \right\| \mathrm{d}x \\
&\quad + \sum_{i=1}^{N-1} \int_{x_i-\hat\lambda\, h}^{x_i+\hat\lambda\, h} \left\| u(\tau+h, x) - \left(S^\zeta_h u(\tau)\right)(x) \right\| \mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{x_{i-1}+\hat\lambda\, h}^{x_i-\hat\lambda\, h} \left\| u(\tau+h, x) - U^\flat_{(u;\tau,\xi_i)}(h,x) \right\| \mathrm{d}x \\
&\quad + \sum_{i=1}^{N} \int_{x_{i-1}+\hat\lambda\, h}^{x_i-\hat\lambda\, h} \left\| U^\flat_{(u;\tau,\xi_i)}(h,x) - \left(S^\zeta_h u(\tau)\right)(x) \right\| \mathrm{d}x \\
&\quad + \sum_{i=1}^{N-1} \int_{x_i-\hat\lambda\, h}^{x_i+\hat\lambda\, h} \left\| u(\tau+h, x) - U^\sharp_{(u;\tau,x_i)}(h,x) \right\| \mathrm{d}x \\
&\quad + \sum_{i=1}^{N-1} \int_{x_i-\hat\lambda\, h}^{x_i+\hat\lambda\, h} \left\| U^\sharp_{(u;\tau,x_i)}(h,x) - \left(S^\zeta_h u(\tau)\right)(x) \right\| \mathrm{d}x.
\end{aligned}
$$

Since both u and S^ζ satisfy *(ii)*, we get

$$\begin{aligned}
&\frac{1}{h}\left\|u(\tau+h)-S^\zeta_h u(\tau)\right\|_{\mathbf{L}^1([a+\hat\lambda\, t,b-\hat\lambda t];\mathbb{R}^n)}\\
&\leq \mathcal{O}(1)\sum_{i=1}^{N}\left(\mathrm{TV}\left(u(\tau);\,]x_{i-1},x_i[\right)+\mathrm{TV}\left(\zeta;\,]x_{i-1},x_i[\right)\right)^2\\
&\quad+\sum_{i=1}^{N-1}\frac{1}{h}\int_{x_i+\hat\lambda\, h}^{x_i-\hat\lambda\, h}\left\|u(\tau+h,x)-U^\sharp_{(u;\tau,x_i)}(h,x)\right\|\mathrm{d}x\\
&\quad+\sum_{i=1}^{N-1}\frac{1}{h}\int_{x_i+\hat\lambda\, h}^{x_i-\hat\lambda\, h}\left\|U^\sharp_{(u;\tau,x_i)}(h,x)-\left(S_h u(\tau)\right)(x)\right\|\mathrm{d}x.
\end{aligned}$$

Both u and S^ζ satisfy *(i)*, hence in the $\liminf_{h\to 0}$ the latter two terms vanish. Thus,

$$\begin{aligned}
&\liminf_{h\to 0}\frac{1}{h}\left\|u(\tau+h)-S^\zeta_h u(\tau)\right\|_{\mathbf{L}^1([a+\hat\lambda\, t,b-\hat\lambda t];\mathbb{R}^n)}\\
&\leq \mathcal{O}(1)\sum_{i=1}^{N}\left(\mathrm{TV}\left(u(\tau);\,]x_{i-1},x_i[\right)+\mathrm{TV}\left(\zeta;\,]x_{i-1},x_i[\right)\right)^2\\
&\leq \mathcal{O}(1)\,\varepsilon\,\left(\mathrm{TV}\left(u(\tau)\right)+\mathrm{TV}(\zeta)\right).
\end{aligned}$$

Since ε is arbitrary, the term in the left hand side above vanishes. The arbitrariness of a and b allows to complete the proof. □

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Data Availability Not applicable.

Declarations

Ethical Approval Not applicable.

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