Comparative dynamics analysis of the environmental policy efficacy in a nonlinear Cournot duopoly with differentiated goods and emission charges

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Abstract

The price and the exchanged quantity volatility observed in real-world markets may be explained, according to the existing empirical literature, in terms of the endogenous fluctuations generated by the presence of nonlinearities. We then replace with a sigmoid adaptive best response mechanism the linear partial adjustment best response rule considered in Mamada and Perrings (2020), where the effect produced by quadratic emission charges on the dynamics of a Cournot duopoly model with homogeneous goods was investigated. Moreover, the sigmoid nonlinearity, in addition to being well suited to describe the bounded output variations caused by physical, historical and institutional constraints, makes the model able to generate interesting, non-divergent dynamic outcomes, despite the linearity of the demand function and of marginal costs. Additionally, following the suggestion in Mamada and Perrings (2020), we deal with the more general case of differentiated products. Beyond analytically studying the stability of the unique steady state, coinciding with the Nash equilibrium, and the effect produced by the main parameters on the stability region, we perform two comparative dynamics investigations which allow to evaluate the environmental policy efficacy when the Nash equilibrium is not stable and thus the standard

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comparative statics technique does not fit for the purpose. In particular, the former analysis, which is based on a comparison of emissions for different levels of charges, shows that, also in case the Nash equilibrium is not stable, the considered environmental policy may be effective both with complements and with substitutes. The latter investigation, consisting in a comparison of emissions along non-stationary trajectories and along the equilibrium path, in the proposed experiments highlights that emissions are larger along non-stationary trajectories. This gives us the opportunity to show how to act on the level of the asymptotes of the sigmoid adjustment mechanism to reduce output variations, reaching at one time a complete stabilization of the system and limiting pollution.
Keywords: Cournot duopoly, emission charges, environmental policy efficacy, comparative dynamics

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1. Introduction

According to the existing empirical literature (see e.g. [1, 2, 3]) the main variables, i.e., good prices and exchanged quantities, connected with real-world markets, especially those for agricultural commodities, display chaotic and erratic behaviors, including volatility. In particular, those empirical studies suggest that the therein identified dynamic phenomena may be explained in terms of the endogenous fluctuations generated by the presence of nonlinearities. Also the experimental literature (cf. for instance [4]) concerning the real markets dynamics underlines the emergence of oscillatory behaviors in good prices and exchanged quantities. Hence, in proposing a model to describe those contexts, also in connection with ecological issues, firstly we have to guarantee that it is able to generate interesting, i.e., non-stationary, non-divergent, dynamic outcomes, so as to be "*up to the task of adequately addressing the implications of these complex system dynamics and the unpredictability which seems to be their hallmark*", quoting [5] in regard to ecological economics. Secondly, if we are interested in investigating the effect produced by an environmental policy scheme on the generated pollution, we have to explain how the environmental policy efficacy can be evaluated in the case of non-stationary trajectories. Namely, the classical comparative statics technique, applied to the system equilibrium, which is a steady state, is not empirically grounded in such a context, in which the steady state is rarely stable. Thus, the need to develop alternative, dynamical solutions arises, based for instance on the behavior of the time series of the cumulative aggregate emission. In this manner, the environmental policy efficacy, to be measured in relation to an emission reduction, could be implied by a negative variation of cumulative emissions over the considered time interval as a consequence of an increased strictness of the environmental policy scheme. We tackle both issues by revisiting the framework in [6], where the effects produced by emission charges on the dynamics of a Cournot duopoly model were investigated. In more detail, motivated by the two points described above, we replace the linear partial output adjustment rule considered therein, whose linearity causes a discrepancy between the simulative outcomes and the empirical data, with a gradual sigmoid version of the best response mechanism, characterized by the presence of two horizontal asymptotes, which help avoid diverging trajectories and negativity issues. The same sigmoid formulation has been used in different macro contexts e.g. in [7, 8], but, to the best of our knowledge, this is the first time that such nonlinearity is introduced in the decisional mechanism within a game theoretical framework. We stress that the sigmoid output adjustment rule, in addition to opening the door to complex dynamics and to giving us the opportunity to perform the above mentioned investigations to test the environmental policy efficacy, is also sensible from an economic viewpoint, since it is well-suited to describe the bounded output variations caused by physical, historical and institutional constraints. Namely, when the difference between the best response and the current output level of a firm is positive, capacity constraints will bound the increase in the production volume, because of the limited expansion from time to time of capital and labor stock. When instead the difference between the best response and the current output level of a firm is negative, capital cannot be destroyed proportionally to that difference as the only factors that may lower productivity are attrition of machines from wear, time, and obsolescence. Additionally, the labor factor imposes constraints, too: indeed, due to the presence of trade unions, it is difficult, or impossible, to reduce employment below a certain threshold level. Notice that the proposed sigmoid adjustment mechanism is suitable to describe also the gradual output variations deriving from the limits imposed by an environmental policy scheme aiming at containing pollution by bounding production, due to the direct proportionality linking them. In fact, acting on the levels of the horizontal asymptotes we obtain a further tool to contain pollution, which stabilizes the dynamics, too. The latter aspect turns out to be particularly relevant when emissions are larger along non-stationary trajectories than along equilibrium paths. In such cases, reducing output variations by lowering the distance between the asymptotes allows at one time to decrease the system dynamic complexity and to limit pollution. Moreover, with respect to the original setting in [6], where the goods produced by the two firms were homogeneous, we assume that firms produce differentiated goods, following the suggestion contained in the concluding section of their work. On the other hand, in regard to emission charges, we stick to the quadratic formulation considered therein.

As concerns the existing literature, we stress that in [9] we extended the setting in [6] by introducing differentiated goods, but without altering the output adjustment rule, while the effect played by the introduction of differentiated goods and of a nonlinear output adjustment mechanism in the framework in [6] has been investigated in [10]. However, in the latter work goods can be just substitutes, not complements, and the quadratic emission charges can only be convex, without the linear term considered in [6], so that the environmental policy is always effective in the setting in [10]. Furthermore, in their context the nonlinearity is represented by output dependent factors that replace the constant adjustment coefficients in the best reply mechanism in [6], implying that the system admits two boundary equilibria, in addition to the internal equilibrium corresponding to the one in [6]. Moreover, the authors in [10] deal with the case in which marginal production costs do not coincide across firms and analyze, among other issues, the conditions for market transitions between duopoly and monopoly.

Turning back to the here considered setting, in studying it we start by analyzing the stability of the unique steady state, which coincides with the Nash equilibrium, common to the framework in [9], as well as its bifurcations and the role played by the main model parameters. We find that the equilibrium is stable when the two goods are nearly independent, while complex dynamics can arise when the interdependence degree between the two goods is strong enough, depending also on the intensity of the reactivity of the adjustment mechanism and on the distance between the asymptotes of the sigmoid map, despite the linearity of the demand function and of marginal costs. We recall that when the steady state is unstable in [6] the output quantities produced by the two firms tend instead to become unbounded, positive or negative. Furthermore, due to the introduction of the sigmoid adjustment mechanism, the equilibrium stability region is reduced in the here analyzed context with respect to [9]. On the other hand, as long as the Nash equilibrium is stable in our framework, and we can thus rely on the classical comparative statics approach, we find a confirmation of the static results obtained in [9], which showed that the considered environmental policy becomes detrimental when emission charges increase too slowly with production. In order to deal with the cases in which the Nash equilibrium is not stable, we perform two alternative, comparative dynamics investigations to evaluate the environmental policy efficacy, based either on a comparison of emissions for different levels of charges or on a comparison of emissions along non-stationary trajectories and along the equilibrium path. The proposed solutions are mainly numerical in nature, involving non-stationary orbits. The former analysis shows that, also when the Nash equilibrium is not stable, the considered environmental policy may be effective in reducing pollution, both with complements and substitutes. In regard to the latter investigation, since in the proposed experiments it happens that emissions are larger along non-stationary trajectories than along the equilibrium path, we explain how to act on the levels of the asymptotes of the sigmoid in view of reducing output variations, reaching a complete stabilization of the system and limiting pollution, without modifying the equilibrium position.

Looking at the numerical simulations that we perform to illustrate our analytical results about the steady state stability loss and the bifurcations that occur on varying the main model parameters, we notice that the attractors obtained for certain parameter configurations lie on the diagonal of the phase space, but this is not always the case. In order to try to understand why such phenomenon may occur, we study the one-dimensional dynamical system corresponding to the restriction of the planar system to the diagonal, which is invariant under its action, by the symmetry of the setting we propose. Namely, if we assume that the initial conditions for the output of the two firms coincide, then their future choices will be identical in each time period. The analysis we perform highlights that there are both analogies and differences between the 1D and the 2D frameworks. In particular, it can happen that the steady state of the one-dimensional setting is stable when the dynamics of the planar model are quasiperiodic or chaotic. This fact has important consequences from a policy viewpoint. Indeed, focusing just on the behavior of the steady state via comparative statics exercises prevents the consideration and the understanding of the system out-of-equilibrium dynamics, while, as explained above, a comparative statics result is economically grounded if it concerns an equilibrium which is asymptotically stable and thus orbits converge towards it after a transient period. Moreover, if agents are completely homogeneous, so that also their past choices, summarized by the output initial conditions, coincide, even an environmental policy maker who were aware of the limits of the equilibrium analysis could erroneously believe that the equilibrium is stable, thus relying on comparative statics tools, while the steady state is actually unstable in the 2D framework. Hence, in addition to being implausible that firms are identical from an evolutionary viewpoint, the assumption that initial conditions for their output coincide can make the use of the comparative statics technique apparently grounded, when it is not. Thus, in the comparative dynamics investigations that we propose to evaluate the environmental policy efficacy when the Nash equilibrium is not stable,

we will always consider heterogeneous initial conditions for the output of the two firms. In this respect, we recall the concept of "path dependence" in [11], according to which the initial conditions, representing a summary of agents' history, matter in determining the evolution of the system. Still in regard to the role played by symmetric vs asymmetric initial conditions, as well as by identical vs quasi-identical parameter values, we refer the interested reader to [12], where the authors find very different dynamic outcomes in consequence of small parameter variations, in contrast to the argument, widespread in the economic literature [13], that small heterogeneities among agents can be neglected, which is often employed to justify the representative agent assumption (see for instance [14, 15, 16] for further discussion on the topic).
We conclude by underlying that, to the best of our knowledge, the existing literature on ecological economics is either static [17, 18, 19, 20, 21, 22, 23, 24, 25, 26] or, even when the proposed models are dynamical in nature, the focus is on the behavior of the steady state [6, 27, 28, 29, 30] and the investigation of the efficacy of the considered environmental policy in non-stationary regimes is neglected. A partial exception is represented by [10], where the authors study the dynamic outcomes of their nonlinear model, while the environmental policy efficacy is granted by the assumptions made therein. Notice that the nonlinearities considered in the present work and in [10] concern the decisional mechanism, even if in a different manner, as explained in Section 2, where we present the setting that we consider. In Section 3 we perform the stability and bifurcation analysis, both for the planar model and for its one-dimensional restriction to the diagonal of the phase space. In Section 4 we investigate the efficacy of the environmental policy from a dynamic viewpoint. In Section 5 we briefly discuss our results and describe possible developments of the here studied framework.

2. The model

The extension to differentiated goods of the context in [6] has been briefly presented in [9]. For the reader's convenience and in order to add some important aspects in its derivation, in what follows we describe the main steps related to its static part, turning then to illustrating the sigmoid best response mechanism, which allows us to keep the same Nash equilibrium found in [9], solving at the same time the issue with diverging trajectories when equilibrium stability is lost in the linear framework. Namely, the new formulation choice allows for more realistic dynamic outcomes, suitable to mimic the volatility displayed by the variables involved in real-world markets. Moreover, from the modeling viewpoint, the sigmoid adjustment mechanism is appropriate to describe the gradual output variations caused by material, historical and institutional constraints in the production side of an economy, as well as by the limits imposed by an environmental policy scheme on production levels, due to their direct proportionality with emissions.[3]

Denoting by $q_{i,t+1}$ the output level of firm i at time $t+1$ and by $q^e_{j,t+1}$ the output level of firm j at time $t+1$ expected by firm i at the end of period t, with $i \neq j \in \{1,2\}$, we assume that in time period $t+1 \in \mathbb{N} \setminus \{0\}$ firm $i \in \{1,2\}$ maximizes the expected profit function

$$\pi^e_{i,t+1} = (p - \beta_i q_{i,t+1} - \gamma q^e_{j,t+1}) q_{i,t+1} - c q^2_{i,t+1} - C_{i,t+1} \tag{2.1}$$

where p, c are positive parameters and $C_{i,t+1}$ is the emission charge, faced at time $t+1$ by firm i, that we will describe in (2.2). For the parameters β_i and γ, as usual in the case of differentiated goods, we suppose that $|\gamma| < \beta_i$, with $i \in \{1,2\}$.

We recall that, according to [31], the expression for the inverse demand function entering (2.1) can be derived by assuming that in an economy with a monopolistic sector with two firms, each one producing a differentiated good, and a competitive numeraire sector, there is a continuum of consumers of the same

[3]Indeed, in Section 4 we shall show that acting on the position of the sigmoid map asymptotes may represent a form of *direct* intervention on output and emission levels through a modification in the bound to the strategy variation between one period and the following one, in contrast to the *indirect* nature of the pollution control obtained by means of emission charges in (2.2). Furthermore, by varying the position of the asymptotes we will make the system converge toward the Nash equilibrium in (2.8) - whose position remains unchanged - starting from a situation characterized by the presence of a different (periodic or complex) attractor, so that comparative statics results become economically grounded.

type with a utility function U separable and linear in the numeraire good, so that there are no income effects on the monopolistic sector, and it is possible to perform partial equilibrium analysis. In symbols, the representative consumer has to maximize in each time period $U(q_1, q_2) - \rho_1 q_1 - \rho_2 q_2$, i.e., the utility function subject to the budget constraint, where ρ_i is the price of good i and $U(q_1, q_2) = p_1 q_1 + p_2 q_2 - \frac{1}{2}\left(\beta_1 q_1^2 + \beta_2 q_2^2 + 2\gamma q_1 q_2\right)$, with p_i and β_i positive, $\beta_1\beta_2 - \gamma^2 > 0$ and $p_i\beta_j - p_j\gamma > 0$ for $i \neq j \in \{1, 2\}$. Dealing, like in [6], with the simplified case in which $p_1 = p_2 = p$ and $\beta_1 = \beta_2 = \beta$, the utility function reads as $U(q_1, q_2) = p(q_1 + q_2) - \frac{\beta}{2}\left(q_1^2 + q_2^2\right) - \gamma q_1 q_2$, with p and β positive, and $|\gamma| < \beta$, as supposed above. In particular, if $\gamma > 0$ utility decreases when consuming the two goods together, i.e., they are substitutes; if $\gamma < 0$ utility increases when consuming the two goods together, i.e., they are complements; if $\gamma = 0$ utility is not affected by a joint consumption of the two goods, i.e., they are independent. The homogeneous good framework is obtained in the limit case $\gamma = \beta = k$, where k is the price-depressing effect of oligopoly. Taking the FOC of $U(q_1, q_2) - \rho_1 q_1 - \rho_2 q_2$, it is straightforward to obtain $\rho_i = p - \beta q_i - \gamma q_j$ for $i \neq j \in \{1, 2\}$ as inverse demand functions. Further details can be found in [31, 32].

Concerning $C_{i,t+1}$, in [6], followed by [9], the authors propose the quadratic formulation for emission charges

$$C_{i,t+1} = bu_{i,t+1} + \frac{1}{2}du_{i,t+1}^2, \tag{2.2}$$

with $b > 0$, $d \in \mathbb{R}$ and where, denoting by $\varepsilon > 0$ the emissions per unit output[4], $u_{i,t+1} = \varepsilon q_{i,t+1}$ are emissions by firm $i \in \{1, 2\}$ at time $t + 1$. In [6] d is used as bifurcation parameter, finding that it has a stabilizing effect on the system equilibrium in the case of homogeneous goods. Moreover, the sign of d

[4]In the analysis performed in the present paper, also in view of comparing our results with those obtained in [9], we assume, as in [6], that emissions per unit output coincide across firms. On the other hand, in order to test the robustness of our findings, in a future work we will deal with firm-specific emissions per unit output, similar to what done e.g. in [21] in a homogeneous good framework with non-point source (NPS) pollution.

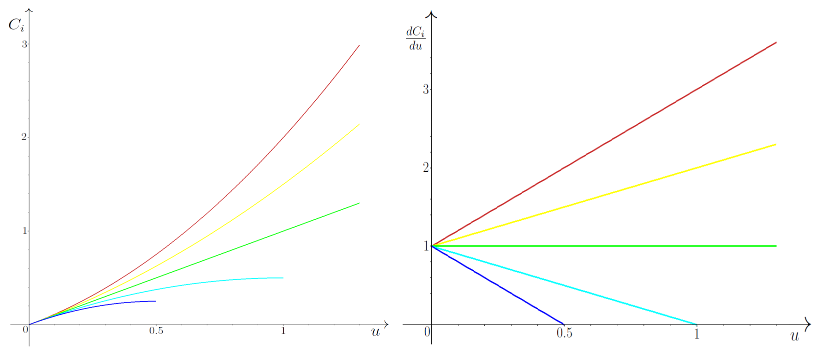
determines if the marginal emission charge

$$\frac{dC_{i,t+1}}{du_{i,t+1}} = b + du_{i,t+1}$$

is positive or negative. Since the marginal emission charge cannot be negative, when $d < 0$ the constraint

$$0 < q_{i,t+1} < \frac{-b}{\varepsilon d} \tag{2.3}$$

emerges. Notice that, *ceteris paribus*, an increase in d produces a raise in emission charges in (2.2) both when starting from positive or negative values for d. However, when d is positive the marginal emission charge increases with u, while when d is negative it decreases with u. In such eventuality, according to (2.3), the maximum value u can assume is given by $-b/d$. See Fig. 1 for a graphical illustration.



(A) (B)

Figure 1: In (A) we draw the graph of C_i in (2.2) as a function of u for $b = 1$, and $d = -2$ in blue, $d = -1$ in cyan, $d = 0$ in green, $d = 1$ in yellow, $d = 2$ in orange. In (B) we represent the corresponding marginal emission charges $\frac{dC_i}{du}$ as a function of u, using the same color distribution as in (A).

Turning back to (2.1), since

$$\frac{\partial \pi^e_{i,t+1}}{\partial q_{i,t+1}} = p - b\varepsilon - (2(\beta + c) + d\varepsilon^2)q_{i,t+1} - \gamma q^e_{j,t+1} \tag{2.4}$$

for $i \neq j \in \{1, 2\}$, expected profits are strictly concave when

$$2(\beta + c) + d\varepsilon^2 > 0. \tag{2.5}$$

This condition is satisfied when C_i in (2.2) is convex for $i \in \{1,2\}$, i.e., for $d \geq 0$, as well as for $d \in \left(-\frac{2(\beta+c)}{\varepsilon^2}, 0\right]$, in which case C_i is concave, but production variations lead to emission charge variations close to those that we would have in the linear case, corresponding to $d = 0$. Assumption (2.5) will be maintained along the manuscript. Moreover, setting $\partial \pi^e_{i,t+1}/\partial q_{i,t+1}$ in (2.4) equal to 0 we obtain

$$R_{i,t+1}(q^e_{j,t+1}) = \frac{p - b\varepsilon - \gamma q^e_{j,t+1}}{2(\beta + c) + d\varepsilon^2} \tag{2.6}$$

as best response function for $i \neq j \in \{1,2\}$, representing the optimal strategy for firm i in period $t+1$, given the strategy for firm j expected by firm i for that same period. Notice that $R_{i,t+1}(q^e_{j,t+1})$ is well defined under (2.5).

Imposing like in [6] that firms have static expectations, it holds that $q^e_{j,t+1} = q_{j,t}$, so that (2.6) can be rewritten as

$$R_{i,t+1}(q^e_{j,t+1}) = R_{i,t+1}(q_{j,t}) = \frac{p - b\varepsilon - \gamma q_{j,t}}{2(\beta + c) + d\varepsilon^2}\,. \tag{2.7}$$

Hence, calling (q_1^*, q_2^*) the solution to the system

$$\begin{cases} q_1 = R_1(q_2) \\ q_2 = R_2(q_1) \end{cases}$$

which arises by supposing that both firms simultaneously produce the best response output to their opponent's strategy, we find like in [9] that the unique (symmetric) Nash equilibrium is given by

$$(q_1^*, q_2^*) = \left(\frac{p - b\varepsilon}{2(\beta + c) + d\varepsilon^2 + \gamma}, \frac{p - b\varepsilon}{2(\beta + c) + d\varepsilon^2 + \gamma}\right). \tag{2.8}$$

In order to avoid negativity issues, we can assume that $p > b\varepsilon$ and that $2(\beta + c) + d\varepsilon^2 + \gamma > 0$, similar to what done in [6] in the case of homogeneous products, or we can suppose that $p < b\varepsilon$ and $2(\beta + c) + d\varepsilon^2 + \gamma < 0$. Like in [9], taking into account also (2.5), we will need to split the model analysis in Section 3 according to those two scenarios in the case of complements, while with substitutes the numerator and the denominator of the Nash equilibrium can just be positive.

As mentioned at the beginning of the section, we will deal with a sigmoid best response mechanism, which on the one hand allows for nontrivial and realistic erratic dynamic outcomes, and, on the other hand, is suitable to describe the gradual output variations deriving both from the limits imposed by an environmental policy scheme aiming at containing pollution by bounding production, as well as from the technical and institutional constraints that pertain the production side of an economy. Namely, when the difference between the best response and the current output level is positive, capacity constraints will bound the increase in the production volume, due to the limited expansion from time to time of capital and labor stock; when instead the difference between the best response and the current output level is negative, capital cannot be destroyed proportionally to that difference as the only factors that may reduce productivity are attrition of machines from wear, time, and obsolescence. Furthermore, also the labor factor imposes limits: indeed, due to the presence of trade unions, it is difficult, or impossible, to reduce employment below a certain threshold level. In more detail, firms, due to an adjustment capacity constraint, in [6, 9] modify their output level according to the size and the extent of the difference between the best response and the current output level in just a partial way. However, due to the above recalled reasons related to dynamic outcomes and economic significance of a sigmoid adaptive best response mechanism, rather than the linear formulation adopted in [6, 9]

$$q_{i,t+1} = q_{i,t} + \lambda(R_i(q_{j,t}) - q_{i,t}), \tag{2.9}$$

where the reactivity parameter λ varies in $(0, 1)$ and, to lighten notation, $R_i(q_{j,t})$ stands for $R_{i,t+1}(q_{j,t})$, i.e., the best response function in (2.7) of firm i at time $t+1$ to the output $q_{j,t}$ produced by firm j at time t, for $i \neq j \in \{1, 2\}$, we will now consider

$$q_{i,t+1} = q_{i,t} + \delta \left(\frac{\upsilon + \delta}{\upsilon\, e^{-\sigma(R_i(q_{j,t}) - q_{i,t})} + \delta} - 1 \right). \tag{2.10}$$

Moving $q_{i,t}$ to the left-hand side of (2.10), we obtain that the output variation of firm $i \neq j \in \{1, 2\}$ between the next period and the current one is described by the sigmoid map

$$g(x) := \delta \left(\frac{\upsilon + \delta}{\upsilon\, e^{-\sigma x} + \delta} - 1 \right), \tag{2.11}$$

with x measuring the difference between the next period optimal strategy and the current output volume, so that (2.10) can be rewritten as

$$q_{i,t+1} - q_{i,t} = g(R_i(q_{j,t}) - q_{i,t}). \tag{2.12}$$

Before looking at the features of g, let us introduce the concept of *relative variation*, that we will denote by $\mathcal{RV}$, and which at time t for firm i is defined as

$$\mathcal{RV}_{i,t} := \frac{q_{i,t+1} - q_{i,t}}{R_i(q_{j,t}) - q_{i,t}}, \tag{2.13}$$

with $i \neq j \in \{1,2\}$, i.e., as the ratio between the output variation in a given period and the difference between the optimal output and the current strategy. Thanks to such notion, we will be able to compare in a more formal manner our adjustment mechanism in (2.10) with that in (2.9), considered in [6], as well as with the nonlinear updating rule adopted in [10], which reads as

$$q_{i,t+1} = q_{i,t} + Kq_{i,t}(R_i(q_{j,t}) - q_{i,t}), \tag{2.14}$$

for $i \neq j \in \{1,2\}$ and $K \in (0,+\infty)$.

Computing the relative variation in the three different scenarios, we obtain $\mathcal{RV}_{i,t} = \lambda$ in relation to (2.9), $\mathcal{RV}_{i,t} = Kq_{i,t}$ in relation to (2.14), while $\mathcal{RV}_{i,t}$ has no explicit formulation in connection with (2.10), but we will derive below some qualitative properties for it. The common feature for the relative variation in the three settings is that it is always linked with the reactivity, given just by λ for (2.9), by K for (2.14), and by σ for (2.10). Of course, the connection between the relative variation and the reactivity is different in each case. Indeed, in [6] the two notions coincide, the relative variation being constant, while in the setting in [10], that may be considered as an extension of the former framework, the relative variation proportionally increases with the current production volume, linked with the firm size, the proportionality factor being given by the reactivity. Also in regard to the sigmoid adjustment rule in (2.10), the relative variation is not constant, depending on the current production level.

Turning back to the mechanism in (2.12), we notice that g in (2.11) is increasing and that it passes through the origin. Moreover, it is bounded from below by

$-\delta$ and from above by υ. The presence of the two horizontal asymptotes helps avoid diverging trajectories and negativity issues. In particular, by raising (resp. lowering) υ and δ we obtain an increase (resp. decrease) in the possible output variations, which have to be contained in the interval $(-\delta, \upsilon)$. Accordingly, we can call δ and υ *potential reactivities* as they determine the size of the interval in which output variations have to lie in each time period. In more detail, notice that acting on υ (resp. δ) produces an effect when the best response is above (resp. below) the current production level. We show the graph of the sigmoid function g for different values of the reactivity σ in Fig. 2, where we denote by y the output difference between tomorrow and today output levels by firm i. Namely, σ is a non-negative parameter describing the intensity with which the difference between the best response and the current output level determines the output variation. For $\sigma = 0$, firms are completely insensitive to that difference and they keep their output unchanged in time, so that $q_{i,t+1} - q_{i,t} = 0$ for every t. In the limit $\sigma \to +\infty$, the sigmoid approaches a piecewise constant function, taking just the lowest and the highest possible values: indeed, the value of g coincides with $-\delta$ for negative values of the signal $R_i(q_{j,t}) - q_{i,t}$ and with υ for positive values of $R_i(q_{j,t}) - q_{i,t}$. Moreover, as observed above, σ influences, together with υ and δ, the relative variation in connection with (2.12), that, recalling (2.13), is given by

$$\mathcal{RV}_{i,t} = \frac{g(R_i(q_{j,t}) - q_{i,t})}{R_i(q_{j,t}) - q_{i,t}}. \tag{2.15}$$

Notice that, looking at (2.10) and recalling the meaning of variable x introduced after (2.11), $x = 0$ corresponds to the Nash equilibrium (q_1^*, q_2^*) in (2.8), as the best response function formulation is still the one in (2.7). Since g is an increasing map passing through the origin, $x = 0$ coincides also with the unique steady state for (2.10). Some features of $\mathcal{RV}_{i,t}$ easily follow from (2.11) and (2.15), observing that: (I) $g(x)/x \to 0^+$ when $x \to \pm\infty$;

(II) $g(x)/x$ is an even map when $\upsilon = \delta$;

$$\text{(III)} \qquad \frac{g(x)}{x} \to \widetilde{\sigma} := \frac{\upsilon\delta\sigma}{\upsilon + \delta} \quad \text{when} \quad x \to 0. \tag{2.16}$$

Namely, (I) implies that for us there is no direct proportionality between $\mathcal{RV}_{i,t}$ and $q_{i,t}$, contrary to what occurs in the setting in [10], where with (2.14) the relative variation proportionally increases with the current production volume. Indeed, with (2.10) $\mathcal{RV}_{i,t}$ tends to vanish when the production volume is far from the current optimal output level. In particular, due to (II), such decrease in $\mathcal{RV}_{i,t}$ is symmetric with respect to positive or negative values of the signal $R_i(q_{j,t}) - q_{i,t}$ that have the same modulus if the upper and lower asymptotes of g are at the same distance from the horizontal axis. Finally, by (III) in (2.16) the relative variation at the steady state coincides with $\widetilde{\sigma}$, which, taking into account the joint effect of σ, υ and δ, will be called joint reactivity in what follows. Notice that σ influences $\widetilde{\sigma}$, without modifying the value of the asymptotes and that, like in the linear framework proposed in [6] it would be possible to consider different values of the reactivity parameter λ for the two firms, also in the present nonlinear setting we could assume personalized values of the sensitivity parameter σ. However, since this hypothesis would overburden the analysis, making the interpretation of the results less neat, similar to [6], where the reactivity parameter is homogeneous for the two firms, we prefer to deal with the case in which the value of σ coincides for firms.

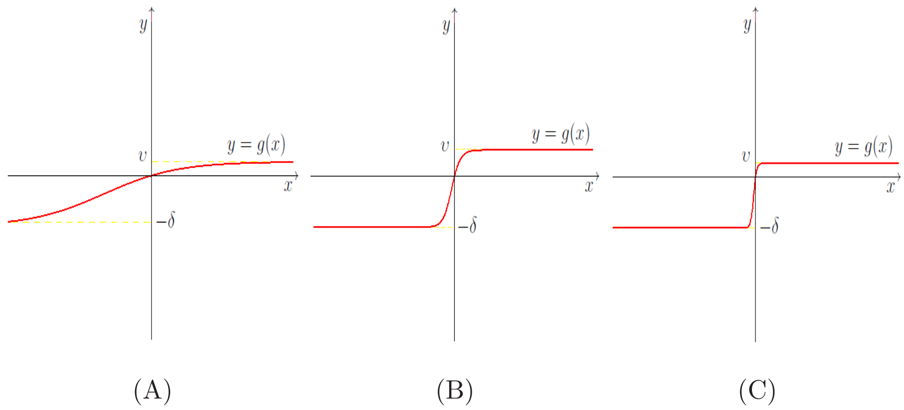


Figure 2: The graph of the sigmoid function g in (2.11), for a low (in (A)), intermediate (in (B)) and high (in (C)) value of σ.

3. Local stability and bifurcation analysis

Let us start by investigating how the presence of the sigmoid adjustment mechanism influences, with respect to the linear framework with differentiated goods considered in [9], the local stability of the Nash equilibrium for (2.10), which for $i \neq j \in \{1, 2\}$ may be explicitly written as

$$\begin{cases} q_{1,t+1} = q_{1,t} + \delta \left(\frac{\upsilon+\delta}{\upsilon\, e^{-\sigma\left(R_1(q_{2,t}) - q_{1,t}\right)} + \delta} - 1 \right) = q_{1,t} + \delta \left(\frac{\upsilon+\delta}{\upsilon\, e^{-\sigma\left(\frac{p-b\varepsilon-\gamma q_{2,t}}{2(\beta+c)+d\varepsilon^2} - q_{1,t}\right)} + \delta} - 1 \right) \\ q_{2,t+1} = q_{2,t} + \delta \left(\frac{\upsilon+\delta}{\upsilon\, e^{-\sigma\left(R_2(q_{1,t}) - q_{2,t}\right)} + \delta} - 1 \right) = q_{2,t} + \delta \left(\frac{\upsilon+\delta}{\upsilon\, e^{-\sigma\left(\frac{p-b\varepsilon-\gamma q_{1,t}}{2(\beta+c)+d\varepsilon^2} - q_{2,t}\right)} + \delta} - 1 \right) \end{cases} \tag{3.1}$$

Calling $F : (0, +\infty)^2 \to (0, +\infty)^2$ the planar map associated with the above dynamical system, in Subsection 3.1 we will deal with the cases of substitutable and independent goods, in which $\gamma \in [0, \beta)$, so that the framework with homogeneous goods, corresponding to $\gamma = \beta$, is encompassed as limit case, while in Subsection 3.2 we will focus on complements, with $\gamma \in (-\beta, 0)$.

3.1. Substitutable and independent goods

In the present subsection, we deal with the case $\gamma \in [0, \beta)$. Since (2.5) has to be satisfied, it follows that $2(\beta + c) + d\varepsilon^2 + \gamma > 0$. Hence, the positivity of the Nash equilibrium (q_1^*, q_2^*) in (2.8) requires that

$$p - b\varepsilon > 0, \tag{3.2}$$

similar to [6]. Consequently, the following result, which highlights the destabilizing role of the joint reactivity $\widetilde{\sigma} = \frac{\upsilon\delta\sigma}{\upsilon+\delta}$ introduced in (2.16), as well as the stabilizing effect of parameter d entering the formulation for emission charges in (2.2), holds true:

Proposition 3.1. *When* $\gamma \geq 0$*, under* (2.5) *and* (3.2)*,* (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $d > -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$*. If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *when* $d > -\frac{2\beta+2c-\gamma}{\varepsilon^2}$ *and* $\widetilde{\sigma} < \frac{2}{1+\frac{\gamma}{2(\beta+c)+d\varepsilon^2}}$.

Proof. We are going to derive the local stability conditions for our system at the steady state by using Jury conditions in [33]

$$\begin{aligned} &(i) \quad \det(J) < 1; \\ &(ii) \quad 1 + \operatorname{tr}(J) + \det(J) > 0; \\ &(iii) \quad 1 - \operatorname{tr}(J) + \det(J) > 0, \end{aligned} \tag{3.3}$$

where $J = J_F(q_1^*, q_2^*)$ is the Jacobian matrix for F computed at (q_1^*, q_2^*), which reads as

$$J_F(q_1^*, q_2^*) = \begin{bmatrix} 1-\widetilde{\sigma} & -\frac{\widetilde{\sigma}\gamma}{2(\beta+c)+d\varepsilon^2} \\ -\frac{\widetilde{\sigma}\gamma}{2(\beta+c)+d\varepsilon^2} & 1-\widetilde{\sigma} \end{bmatrix} \tag{3.4}$$

and that is well defined by (2.5). Since as expressions for the determinant and for the trace of J we respectively find

$$\det(J) = \widetilde{\sigma}^2 \left(1 - \frac{\gamma^2}{(2(\beta+c)+d\varepsilon^2)^2}\right) - 2\widetilde{\sigma} + 1, \qquad \operatorname{tr}(J) = 2 - 2\widetilde{\sigma},$$

condition (iii) reads as

$$1 - \frac{\gamma^2}{(2(\beta+c)+d\varepsilon^2)^2} > 0, \tag{3.5}$$

so that, setting $A := 1 - (\gamma^2/(2(\beta+c)+d\varepsilon^2)^2)$, we have that A has to lie in the interval $(0,1)$. Condition (i) is then equivalent to

$$\widetilde{\sigma} < \frac{2}{A}, \tag{3.6}$$

while condition (ii) holds if and only if

$$A\widetilde{\sigma}^2 - 4\widetilde{\sigma} + 4 > 0. \tag{3.7}$$

It is straightforward to check that (3.6) and (3.7) are jointly satisfied when

$$\widetilde{\sigma} < \frac{2}{1 + \frac{\gamma}{2(\beta+c)+d\varepsilon^2}}. \tag{3.8}$$

Moreover, since (3.5) can be equivalently rewritten as

$$(2(\beta+c)+d\varepsilon^2-\gamma)(2(\beta+c)+d\varepsilon^2+\gamma) > 0 \tag{3.9}$$

and by (2.5) it holds that $2(\beta + c) + d\varepsilon^2 + \gamma > 0$, then (3.5) is equivalent to

$$2(\beta + c) + d\varepsilon^2 - \gamma > 0 \tag{3.10}$$

as well, which can be rewritten as

$$d > -\frac{2\beta + 2c - \gamma}{\varepsilon^2}\,. \tag{3.11}$$

Since, by (2.5), the conditions in (2.3) lead to

$$d > -\frac{b(2\beta + 2c + \gamma)}{\varepsilon p}, \tag{3.12}$$

the desired conclusion follows from (3.8), (3.11) and (3.12). □

Comparing the above result with the findings obtained in [9], Proposition 3.1 highlights an overall reduction in the stability region for (q_1^*, q_2^*) with respect to Proposition 3 therein due to the introduction of the sigmoid adjustment mechanism in (2.10). Indeed, under (3.2) in the nonlinear context we observe stricter conditions for stability than in its linear counterpart, where just (3.11) was present[5], describing the stabilizing role of d. Namely, when d is positive, or at least not too negative, i.e., when the policy intensity degree is sufficiently high, production is reduced, as it is evident from the best response functions in (2.6), and this may favor the equilibrium stability. On the other hand, with the sigmoid adjustment mechanism in (2.10), in order to have (q_1^*, q_2^*) in (2.8) locally asymptotically stable for System (3.1), in addition to (3.11), we need a further condition (cf. (3.8)), arising from the destabilizing role played by the joint reactivity $\widetilde{\sigma}$. In this respect, we stress that all parameters encompassed in the expression of $\widetilde{\sigma}$ in (2.16) positively affect the joint reactivity, as a raise in δ, υ or σ makes $\widetilde{\sigma}$ increase.[6] Hence, δ, υ and σ have a destabilizing effect on

[5]We recall that in [6] the only found stability condition, i.e., $d > -\frac{k+2c}{\varepsilon^2}$, coincides with (3.11) in the limit case $\gamma = \beta$, corresponding to homogeneous goods.

[6]Namely, dividing e.g. by δ the numerator and the denominator of $\widetilde{\sigma}$, we obtain for the latter the reformulation $\widetilde{\sigma} = (\upsilon\sigma)/\left(\frac{\upsilon}{\delta} + 1\right)$, which shows that the joint reactivity is directly proportional to δ. We can proceed in an analogous manner with υ, while the direct proportionality between $\widetilde{\sigma}$ and σ is even more apparent. Hence, in the main dynamic results along the manuscript (cf. Propositions 3.1, 3.2 and 3.3) we analyze the role played on the

the steady state, fact which can be easily explained by observing that a raise in any of them increases the possible output variations $q_{i,t+1} - q_{i,t}$ between two consecutive periods. Namely, since from (2.10) it follows that

$$q_{i,t+1} - q_{i,t} = \delta \left(\frac{\upsilon + \delta}{\upsilon\, e^{-\sigma(R_i(q_{j,t}) - q_{i,t})} + \delta} - 1 \right), \tag{3.13}$$

the left-hand side is increasing in σ, bounded from below by $-\delta$ and from above by υ. Thus, in the present context, not only the reactivity σ, which plays a destabilizing role also in the case of the linear formulations considered, with minor differences, e.g. in [10, Section 4] and in [34], but even the parameters regulating the level of the two asymptotes, i.e., δ and υ, favor the equilibrium destabilization, through a violation of (3.8). Specifically, when (3.8) is violated, a period-doubling bifurcation occurs at the steady state,[7] possibly opening the door to interesting dynamic phenomena. We illustrate two different scenarios, according to the sign of d, in Fig. 3, where we let σ vary.

More precisely, for a positive value of d, if (3.2) holds true, the only condition in Proposition 3.1 that can be violated is the last one, i.e., that in (3.8). Indeed, for the parameter configuration considered in Fig. 3 (A), where $d = 0.1$, (3.8) reads as $\widetilde{\sigma} < 1.413$ or, equivalently, as $\sigma < 4.176$, since $\delta = 0.4$ and $\upsilon = 2.2$. In fact, in that bifurcation diagram σ varies in $(0, 6.4)$ and (q_1^*, q_2^*) in (2.8) is stable for low values of σ, while above the stability threshold we observe a cascade of period-doubling bifurcations leading to chaos. The presence of chaotic dynamics for $\sigma > 5.675$ is confirmed both by the plot of the maximal

equilibrium stability by the joint reactivity, both in view of simplifying computations and due to its representativeness, as such analysis clarifies also the effect of the parameters defining $\widetilde{\sigma}$. However, in the bifurcation diagrams illustrating those results we will use σ, i.e., the reactivity, since this is the parameter most commonly considered in the literature in relation to adjustment rules (see e.g. [10, 34]), even if choosing δ or υ as bifurcation parameters would produce analogous numerical outcomes.

[7] According to [35], page 249, this is a consequence of the fact that a violation of (3.8) occurs when condition (*ii*) in (3.3) becomes an equality, so that the Jacobian matrix in (3.4) admits a real eigenvalue equal to -1. Analogously, a period-doubling bifurcation occurs at the steady state with complements when (3.18) in the proof of Proposition 3.2 is violated. Of course, the same is still true when (3.8) and (3.18) are rewritten in order to make explicit a parameter different from $\widetilde{\sigma}$, such as d or γ. In this respect, we remark that a period-doubling bifurcation occurs on *increasing* $\widetilde{\sigma}$ or $|\gamma|$ (cf. Corollaries 3.5 and 3.6 for γ), while a period-doubling bifurcation occurs on *decreasing* d (see Corollaries 3.1 and 3.3). We can then equivalently say that a period-halving bifurcation occurs on increasing d.

Lyapunov exponent against σ in Fig. 3 (B), as well as by the non-periodic attractor that we witness in the phase plane for $\sigma = 6.38$ in Fig. 3 (C). Namely, in regard to the former we recall that there exists evidence for periodic behavior when the maximal Lyapunov exponent is zero, while it becomes slightly positive with quasiperiodic motions and assumes higher positive values in the presence of chaos. As concerns the attractor in (C), we notice instead that, despite the asymmetric initial conditions, it lies on the diagonal of the phase plane, i.e. on $\Delta := \{(q_1, q_2) \in (0, +\infty)^2 : q_1 = q_2\}$, due to the symmetry of the model, but this is not always the case, as we shall see for different sets of parameters in Fig. 5 (C) and (F), as well as in Fig. 7 (C) and (G). We will investigate why these dissimilar outcomes arise by analyzing in Subsection 3.4 the one-dimensional system obtained by setting $q_1 = q_2$ in (3.1), which describes the behavior of the restriction of the planar system to Δ. Indeed, due to the symmetry of (3.1), if also initial conditions for the output of the two firms coincide, then their future choices will be identical in each time period, and thus the planar model behaves like the one-dimensional setting in (3.21).

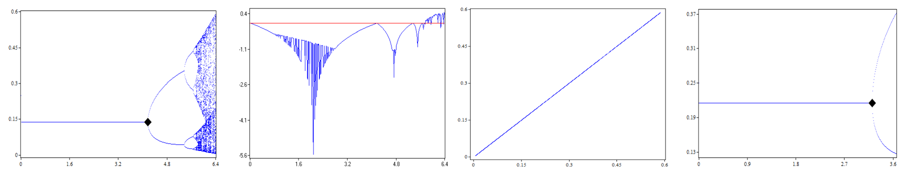
On the other hand, when dealing with negative values for d, no condition in Proposition 3.1 is granted. In particular, in Fig. 3 (D) we fix $d = -0.4$ and, for the chosen parameter set, conditions (2.5) and (3.2) are fulfilled, as well as the stability condition in (3.11), which reads as $d > -0.480$, while (3.8) reads as $\widetilde{\sigma} < 1.089$, or equivalently $\sigma < 3.217$. Hence, like in (A), also in this case the steady state is stable for low values of σ. On the other hand, since the admissibility condition in (2.3) leads to[8] $q_i < 0.370$ for $i \in \{1, 2\}$, it is fulfilled just for $\sigma \in (0, 3.660)$. This is indeed the interval for σ depicted in Fig. 3 (D), where we have to interrupt the bifurcation diagram before the cascade of period-doubling bifurcations (cf. also Footnote 7). We stress that, both in (A) and in (B), orbits do not diverge when the steady state is not stable, thanks to the introduction of the sigmoid adjustment mechanism, differently

[8]Notice that, when the Nash equilibrium is unstable, (2.3) has to be satisfied by all production levels, not just by those computed at the equilibrium.

from what happens with the linear adjustment rule considered in [6, 9]. Notice that for the parameter configurations in Fig. 3 (A) and (D), disregarding the parameters not encompassed in the model linear formulation, (q_1^*, q_2^*) in (2.8) is stable for the dynamical system analyzed in [9] since the stability condition in (3.11), common to that framework (cf. Proposition 3 therein), is fulfilled. We also stress that considering $\gamma = \beta = 3.1$ in Fig. 3 (A) and (D), so that goods are homogeneous, we obtain dynamics analogous to the ones detected in those bifurcation diagrams, where the interdependence degree between goods is already very high. In particular, for the parameter configurations in Fig. 3 (A) and (D) except for $\gamma = \beta = 3.1$, (q_1^*, q_2^*) in (2.8), which in the case of homogeneous goods reads as

$$(q_1^*, q_2^*) = \left(\frac{p - b\varepsilon}{3\beta + 2c + d\varepsilon^2}, \frac{p - b\varepsilon}{3\beta + 2c + d\varepsilon^2} \right),$$

is stable for the dynamical system considered in [6] since the stability condition in (3.11), which becomes $d > -\frac{\beta+2c}{\varepsilon^2}$, is fulfilled for those parameter values.



(A) (B) (C) (D)

Figure 3: The bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to σ and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = 3$, and $d = 0.1$ in (A), $d = -0.4$ in (D). For the parameter configuration considered in (A), we plot in (B) the maximal Lyapunov exponent against σ and in (C) the attractor in the phase plane for $\sigma = 6.38$.

In regard to comparative statics, we remark that the same conclusions contained in Propositions 1 and 2 in [9], showing that, with substitutable and independent goods, under (3.2), the components of the Nash equilibrium in (2.8) decrease when b, d or ε increase in the model linear formulation, hold true with the sigmoidal adjustment mechanism, too. Hence, the efficacy of the environmental policy described by the emission charges in (2.2) would seem not to be affected by the nonlinear output adjustment rule introduced in the present work. On

the other hand, as underlined in [9], a comparative statics result is economically grounded if it concerns an equilibrium which is asymptotically stable and thus orbits converge towards it after a transient period. Due to the above highlighted destabilizing role played by $\widetilde{\sigma}$, that concerns the case of complements, too (cf. Subsection 3.2), we can then say that the significance of the comparative statics analysis is reduced when dealing with the model nonlinear formulation in (3.1), rather than with its linear counterpart. Accordingly, in Section 4 we will perform alternative, comparative dynamics investigations to evaluate the environmental policy efficacy, based either on a comparison of emissions along non-stationary trajectories and along the equilibrium path or on a comparison of emissions for different levels of charges in (2.2), described by increasing values of d.
Still in regard to d, we observe that it is possible to rewrite the statement of Proposition 3.1 in order to make its role explicit starting from (3.8) as follows: [9]

Corollary 3.1. *When* $\gamma \geq 0$, *under* (2.5) *and* (3.2), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $d > -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$. *If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *for* $\widetilde{\sigma} < 2$ *and*

$$d > \max\left\{\frac{-2(\beta+c)}{\varepsilon^2} + \frac{\gamma}{\varepsilon^2}, \frac{-2(\beta+c)}{\varepsilon^2} + \frac{\gamma}{\varepsilon^2\left(\frac{2}{\widetilde{\sigma}} - 1\right)}\right\}.$$

Hence, similar to what happened with the model linear formulation in [6], in the case of homogeneous goods, and in [9], with substitutable and independent products, under (2.5) and (3.2) parameter d plays just a stabilizing role on the Nash equilibrium - since by raising it we make the policy intensity degree higher, hence favoring a decrease in production - when it influences its stability. Namely, it can also happen that, under (2.5) and (3.2), the equilibrium in (2.8) is stable for any admissible value for d according to (2.3). However, with respect to [6, 9], due to the sigmoid adjustment mechanism in (2.10), we now have one extra stability condition, involving the joint reactivity $\widetilde{\sigma}$, which has to be not

[9] We stress that, similar to what occurs in [9] (cf. Footnote 3 therein), also in the present framework b, differently from d, does not influence the system stability, being not present in the Jacobian matrix in (3.4).

too large due to its destabilizing role (see (3.13) and the related comments), and thus more frameworks may arise. We represent the three main possibilities[10] in Fig. 4, where we take d as bifurcation parameter and the various values of p and σ allow for a different ordering among the thresholds contained in Corollary 3.1. In particular, calling d_1, d_2 the stability thresholds therein, d_a the admissibility threshold coming from (2.3) and d_c the threshold coming from (2.5), i.e.,

$$\begin{aligned} d_1 &:= \tfrac{-2(\beta+c)}{\varepsilon^2} + \tfrac{\gamma}{\varepsilon^2}, \qquad d_2 := \tfrac{-2(\beta+c)}{\varepsilon^2} + \tfrac{\gamma}{\varepsilon^2\left(\frac{2}{\widetilde{\sigma}}-1\right)}, \\ d_a &:= -\tfrac{b(2\beta+2c+\gamma)}{\varepsilon p}, \qquad d_c := -\tfrac{2(\beta+c)}{\varepsilon^2}, \end{aligned} \tag{3.14}$$

for the parameter configuration in Fig. 4 (A) it holds that $d_c = -0.892 < d_2 = -0.754 < d_1 = -0.480 < d_a = -0.469$. Since the stability thresholds d_1 and d_2 are below the admissibility threshold d_a, the Nash equilibrium is stable for all values for which it is admissible, as highlighted by the bifurcation diagram in Fig. 4 (A), that we draw for $d \in (-0.469, 0)$. Moreover, the steady state is decreasing with d, in agreement with Proposition 1 in [9]. On the other hand, for the parameter values considered in Fig. 4 (B) it holds that $d_a = -1.005 < d_c = -0.892 < d_2 = -0.754 < d_1 = -0.480$. Hence, although this time both stability thresholds d_1 and d_2 are larger than d_a and satisfy (2.5), which imposes $d > d_c$, also the admissibility condition in (2.3), that leads to $q_i < -\frac{0.4}{2.7d}$ for $i \in \{1, 2\}$, since d varies, has to be taken into account. It is straightforward to check that (q_1^*, q_2^*) is admissible at the stability threshold $d = d_1$, since $q_i^* = 0.053 < -\frac{0.4}{2.7d_1} = 0.309$ for $i \in \{1, 2\}$. Hence, by continuity, it should be possible to represent the corresponding bifurcation diagram for values of d in a left neighborhood of d_1. However, due to a monotone divergence phenomenon which occurs as soon as d is below d_1, in Fig. 4 (B) we draw the bifurcation diagram for $d \in [-0.480, -0.450)$ only, where the Nash equilibrium is

[10] Notice indeed that no crucial differences emerge when the smaller between the two stability thresholds d_1 and d_2 in (3.14) lies above or below the admissibility threshold d_a. In regard to the comparison between d_1 and d_2, with substitutes we have that $d_1 < d_2$ for $\widetilde{\sigma} > 1$. In particular, this remark applies to the limit case in which $\gamma = \beta$, i.e., when we are dealing with homogeneous goods. We finally stress that d_1 and d_2 coincide when products are independent.

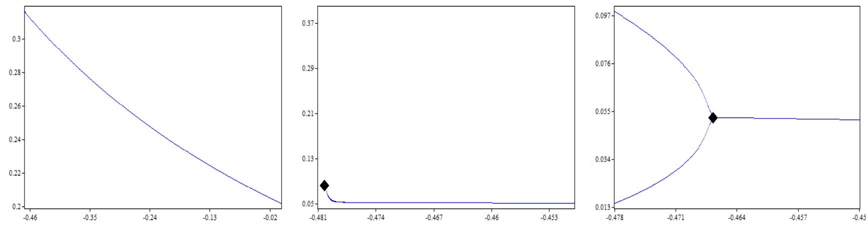
locally asymptotically stable.[11] We stress that at $d = d_1 = -0.480$ a saddle-node bifurcation occurs.[12] In regard to the divergence issue, we recall that by (3.13) the sigmoid adjustment mechanism bounds in each period the output variation and thus lowers the speed of divergence of the orbits, sometimes completely preventing divergence, like e.g. in Figs. 3 and 7. However, this is not always the case, as shown by Fig. 4 (B). Finally, in Fig. 4 (C) we have $d_a = -1.005 < d_c = -0.892 < d_1 = -0.480 < d_2 = -0.467$, so that also in such framework both stability thresholds d_1 and d_2 are larger than d_a and satisfy (2.5), too. Recalling that, in the considered framework, the admissibility condition in (2.3) reads as $q_i < -\frac{0.4}{2.7d}$ for $i \in \{1, 2\}$, a numerical check shows that we can represent the corresponding bifurcation diagram e.g. for $d \in (-0.478, -0.450)$. This is indeed the choice made in Fig. 4 (C), which confirms that the Nash equilibrium is stable for $d > d_2$, whereas for values of d slightly smaller than d_2 we observe a cyclic behavior, since at $d = d_2$ a period-halving bifurcation occurs (cf. Footnote 7). Similar to (B), also in this case it would be possible to consider a wider range of values of d. However, since the values of q_1 raise very rapidly for decreasing values of $d \approx -0.480$, we focus on a small variation interval for d in Fig. 4 (C), in order to better focus on the steady state stability recovery.

Notice that, differently from d, parameter b plays no role on the stability of the Nash equilibrium, being not present in the Jacobian matrix in (3.4).

Starting again from Proposition 3.1 and making this time explicit the effect of γ on the stability of the Nash equilibrium, we obtain the following result, which

[11] Actually, it would be possible to represent the bifurcation diagram in Fig. 4 (B) e.g. for $d \in [-0.480, 0)$, but, in order to better highlight what occurs in the proximity of the stability threshold $d = d_1$, we focus on a smaller interval of values for d. A similar remark applies to the choice of the range for d in the bifurcation diagram in Fig. 4 (C).

[12] According to [35], page 249, when condition (iii) in (3.3) becomes an equality, and thus the Jacobian matrix in (3.4) admits a real eigenvalue equal to $+1$, then a saddle-node, a pitchfork or a transcritical bifurcation occurs. Since condition (iii) in (3.3) becomes an equality just when (3.11) is violated, we are in one of those three cases for $d = d_1$. In particular, we can infer that a saddle-node bifurcation occurs for the parameter configuration considered in Fig. 4 (B), since, according to the chosen initial conditions, the production of one firm positively diverges, while the production of the other firm negatively diverges. As we shall see in Subsection 3.2, with complements, depending on the parameter values we deal with, (iii) in (3.3) is either always or never fulfilled and thus no fold, pitchfork or transcritical bifurcations can occur in that framework.



(A) (B) (C)

Figure 4: The bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to d and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, for $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = 3$, and $p = 3$, $\sigma = 1.478$ in (A), $p = 1.4$, $\sigma = 1.478$ in (B), $p = 1.4$, $\sigma = 3$ in (C).

shows that an increasing degree of interdependence between goods, measured by $|\gamma|$, is destabilizing when we deal with substitutes:

Corollary 3.2. *When* $\gamma \geq 0$, *under* (2.5) *and* (3.2), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $\gamma > -\left(2(\beta + c) + \frac{d\varepsilon p}{b}\right)$. *If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *when* $\widetilde{\sigma} < 2$ *and*

$$0 \leq \gamma < \min\left\{2(\beta + c) + d\varepsilon^2, \left(\frac{2}{\widetilde{\sigma}} - 1\right)\left(2(\beta + c) + d\varepsilon^2\right)\right\}. \tag{3.15}$$

Hence, in our duopoly setting with emission charges we obtain a conclusion contrasting with the destabilizing role of the degree of substitutability between commodities found in [34], in a Cournot duopoly framework with differentiated goods and nonlinear demand functions. Namely, in [34], where goods can be just substitutes, the degree of substitutability/differentiation between the commodities is denoted by α and it is assumed to vary in the interval $(0, 1)$, with goods that are perfect substitutes for $\alpha = 1$ and independent for $\alpha = 0$, similar to what occurs with our γ, which however varies in $(0, \beta)$ in the case of substitutes (cf. Section 2). Since the dynamic results in [34] are interpreted for increasing values of the product heterogeneity, i.e., for *decreasing*[13] values of α, the destabilizing role of the degree of substitutability between commodities found in [34]

[13] We stress that a similar choice is made e.g. in [36], in a Cournot duopoly framework with differentiated goods, in which commodities can be substitutes or complements. We shall compare our findings with those in [36] in Subsection 3.3, where we will deal with complements, too. Namely, like in [36], in the case of complements we will interpret our stability results by looking at what happens for decreasing values of γ, i.e., when the interdependence degree

on the unique steady state is in contrast to Corollary 3.2, according to which the equilibrium destabilization is obtained on raising γ, so that decreasing values of γ would be stabilizing. Our result can be easily explained by recalling that $|\gamma|$ for us represents the interdependence degree between goods and by noticing that, when the two goods are almost independent, a price variation in one of them does not influence much the demand of the other. If however their interdependence degree raises, the demand for each commodity becomes more heavily affected by a price variation in the other good, making fluctuations increase and leading, for large enough values of $\widetilde{\sigma}$, to the system instability, due also to the already observed destabilizing effect of the joint reactivity. In this respect, we more precisely stress that, according to Corollary 3.2, if $\widetilde{\sigma}$ is too high, exceeding 2, then the equilibrium is stable for no values of $\gamma \in (0, \beta)$, while the threshold value $\gamma_2 := \left(\frac{2}{\widetilde{\sigma}} - 1\right)\left(2(\beta + c) + d\varepsilon^2\right)$ in (3.15) lies in the interval $(0, \beta)$, thus becoming effective, for large enough values of $\widetilde{\sigma} \in (0, 2)$. Moreover, calling $\gamma_1 := 2(\beta + c) + d\varepsilon^2$ the other threshold value in (3.15), we notice that $\gamma_1 < \gamma_2$ if and only if $\widetilde{\sigma} \in (0, 1)$, while the opposite inequality holds true for $\widetilde{\sigma} \in (1, 2)$. Rather than illustrating Corollary 3.2, we will show in Fig. 7 possible bifurcation diagrams drawn for positive and negative values of γ, in relation to the findings in the more general Corollary 3.5, which encompasses both the case of substitutes and (Scenario I) of complements.

3.2. The case of complements

In the present subsection, we deal with the case $\gamma < 0$. Accordingly, under (2.5), two different scenarios ensure the positivity of the Nash equilibrium in (2.8), i.e.,

$$p - b\varepsilon > 0, \qquad 2(\beta + c) + d\varepsilon^2 + \gamma > 0, \tag{3.16}$$

or

$$p - b\varepsilon < 0, \qquad 2(\beta + c) + d\varepsilon^2 + \gamma < 0. \tag{3.17}$$

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between goods, measured by $|\gamma|$, raises. Indeed, the dependence of the stability/instability of the steady state on the absolute value of γ will more clearly emerge in Subsection 3.3, where we will combine our results for substitutes, obtained in the present subsection, with those for complements, that we shall obtain in Subsection 3.2.

The former scenario, in which (2.5) is granted and that occurs either when $d \geq 0$ or for not too negative values for d and γ, leads to findings similar to, but not coinciding with, those described in Subsection 3.1. In the latter scenario, that occurs for negative enough values for d and γ, (2.5) needs to be imposed, being not guaranteed, and outcomes will be drastically different from those detected in the other cases. In order to easily refer to the scenarios related to (3.16) and (3.17), in what follows we will call them Scenario I and Scenario II, respectively, and we will analyze them separately.

3.2.1. Analysis of Scenario I

The dynamic result, which represents the counterpart to Proposition 3.1, reads as follows:

Proposition 3.2. *When* $\gamma < 0$, *under* (3.16), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $d > -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$. *If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *when* $\widetilde{\sigma} < \frac{2}{1-\frac{\gamma}{2(\beta+c)+d\varepsilon^2}}$.

Proof. Using Jury conditions like in the proof of Proposition 3.1, we find again (3.5), equivalent to (3.9), as well as (3.6) and (3.7). However, this time, since γ is negative, (3.6) and (3.7) are jointly fulfilled when

$$\widetilde{\sigma} < \frac{2}{1 - \frac{\gamma}{2(\beta+c)+d\varepsilon^2}}\,. \tag{3.18}$$

Moreover, by (3.16), (3.9) is equivalent to (3.10) and to (3.11). Since, still by (3.16), the conditions in (2.3) lead to (3.12) and it holds that

$$-\frac{b(2\beta+2c+\gamma)}{\varepsilon p} > -\frac{2\beta+2c+\gamma}{\varepsilon^2} > -\frac{2\beta+2c-\gamma}{\varepsilon^2},$$

the assertion follows by (3.12) and (3.18). □

Similar to what happened with substitutes, we notice that although the same conclusions about comparative statics contained in Propositions 4 and 5 in [9], according to which the components of the Nash equilibrium in (2.8) decrease when b, d or ε increase, hold true with the sigmoidal adjustment mechanism, too, the significance of the comparative statics analysis is reduced when dealing with the model nonlinear formulation in (3.1), since the stability region in Proposition 3.2 for (q_1^*, q_2^*) in (2.8) is reduced with respect to Proposition 6 in [9], obtained for the model linear formulation. Indeed, in that setting the Nash equilibrium is stable under (3.16) anytime it is admissible according to (2.3), i.e., when (3.12) holds true, while with the sigmoidal adjustment mechanism the extra condition in (3.18), highlighting the destabilizing role played by the joint reactivity $\widetilde{\sigma}$, is required for stability. As explained in relation to Proposition 3.1, this occurs because raising the value of any of the parameters defining $\widetilde{\sigma}$, i.e., δ, υ and σ, increases the possible output variations between two consecutive periods (see (3.13)). Concerning Proposition 3.2, two different scenarios, according to the sign of d, are illustrated in Fig. 5, where we let σ vary, finding outcomes which bear resemblance to those detected in Fig. 3, as well as some differences.

Namely, for the parameter configuration considered in Fig. 5 (A), where $d = 0.1$, since (3.16) holds true and (3.12) is guaranteed by the positivity of d, the only condition in Proposition 3.2 that can be violated is the one in (3.18), which reads as $\widetilde{\sigma} < 1.413$ or equivalently as $\sigma < 4.176$, since $\upsilon = 2.2$ and $\delta = 0.4$. Indeed, in that bifurcation diagram σ varies in $(0, 20)$ and (q_1^*, q_2^*) in (2.8) is stable for low values of σ, while after the period-doubling bifurcation occurring at $\sigma = 4.176$ (cf. Footnote 7), we observe a secondary Neimark-Sacker bifurcation at $\sigma = 5.112$, at which the period-two cycle loses stability and quasiperiodic dynamics emerge, followed by chaotic motions for larger values of σ. The presence of complex dynamics is confirmed both by the plot of the maximal Lyapunov exponent against σ in Fig. 5 (B), in which values are positive for $\sigma > 5.67$, except for the periodicity windows that we observe in (A), where the maximal Lyapunov exponent becomes null or negative, as well as by the chaotic attractor that we observe in the phase plane for $\sigma = 9.9$ in Fig. 5 (C).

When considering negative values for d, no condition in Proposition 3.2 is granted. In particular, in Fig. 5 (D) we fix $d = -0.1$ and, for the chosen parameter set, condition (3.16) is fulfilled, while (3.18) reads as $\widetilde{\sigma} < 1.316$, or equivalently $\sigma < 3.888$. Hence, like in (A), also in this case the steady state is stable for low values of σ. On the other hand, since the admissibility condition in (2.3) leads to $q_i < 1.481$ for $i \in \{1, 2\}$, it is fulfilled just for $\sigma \in (0, 5.334)$. This is the interval for σ depicted in Fig. 5 (D), which is sufficient to witness the secondary Neimark-Sacker bifurcation of the period-two cycle, followed by quasiperiodic dynamics,[14] whose presence is confirmed by the positive values for $\sigma > 5.156$ in the plot of the maximal Lyapunov exponent against σ in Fig. 5 (E) and by the shape of the attractor that we observe in the phase plane for $\sigma = 5.247$ in Fig. 5 (F). Differently from Fig. 5 (B), the maximal Lyapunov exponent against σ in (E) never vanishes or becomes negative for $\sigma > 5.156$, due to the lack of periodicity windows in (D). Moreover, we notice that, unlike what occurs in Fig. 3 (C), despite the symmetry of the model, the attractors in Fig. 5 (C) and (F) do not lie on the diagonal of the phase plane. We will try to clarify the reason behind such discrepancy by means of the 1D analysis that we shall perform in Subsection 3.4.

We also remark that both in Fig. 5 (A) and (D) orbits do not diverge after the steady state stability loss, differently from the linear framework considered in [6, 9], thanks to the introduction of the sigmoid adjustment mechanism.

We stress that the threshold values in Fig. 3 (A) and in Fig. 5 (A) for σ coincide because in the two bifurcation diagrams we considered the same parameter values, except for $\gamma = 3$ in the former, and $\gamma = -3$ in the latter, so that $|\gamma| = 3$ in either case.[15] Namely, we will see in Corollary 3.6 in Subsection 3.3, which encompasses the frameworks of substitutes and of complements, that, since $d > 0$ in Figs. 3 (A), 5 (A) and (3.2) holds true in both frameworks, then

[14] In this respect we remark that choosing a more negative value of d would allow us to draw the bifurcation diagram for a smaller interval of values for σ. For instance, for $d = -0.15$ we should interrupt the diagram in Fig. 5 (D) just after the secondary Neimark-Sacker bifurcation. The same remark applies to Fig. 3. Namely, in (D) therein we have to truncate the bifurcation diagram just after the period-doubling bifurcation with $d = -0.4$, while if we chose $d = -0.1$ in Fig. 3 (D), we would obtain a bifurcation diagram more similar to the one in Fig. 3 (A), encompassing complex dynamics.

[15] Notice that in Fig. 5 (D) we had to consider a different value for $d < 0$ with respect to Fig. 3 (D) due to the admissibility condition (3.12), with consequent different threshold values for σ in Figs. 3 (D) and 5 (D) although having $|\gamma| = 3$ in both bifurcation diagrams.

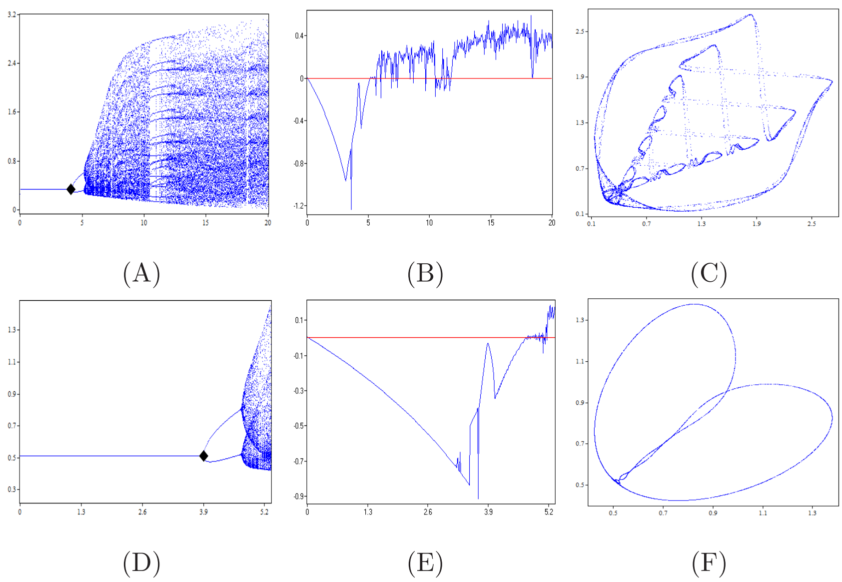


Figure 5: The bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to σ and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = -3$, and $d = 0.1$ in (A), $d = -0.1$ in (D). For the parameter configuration considered in (A), we plot in (B) the maximal Lyapunov exponent against σ and in (C) the attractor in the phase plane for $\sigma = 9.9$. Similarly, for the parameter configuration considered in (D), we plot in (E) the maximal Lyapunov exponent against σ and in (F) the attractor in the phase plane for $\sigma = 5.247$.

$|\gamma|$ determines the stability threshold value for $\widetilde{\sigma}$, as it follows by making the latter parameter explicit (cf. (3.20)).

Let us now state the analogues of Corollaries 3.1 and 3.2, which are obtained by highlighting in the statement of Proposition 3.2 the role of d and of γ, respectively.

In regard to the former, making d explicit in (3.18), we obtain the following:

Corollary 3.3. *When* $\gamma < 0$, *under* (3.16), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $d > -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$. *If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *for* $\widetilde{\sigma} < 2$ *and*

$$d > \frac{-2(\beta + c)}{\varepsilon^2} - \frac{\gamma}{\varepsilon^2 \left(\frac{2}{\widetilde{\sigma}} - 1\right)}.$$

Hence, similar to what happened with substitutes in Corollary 3.1, also with complements under (3.16) we find that d plays a stabilizing role on the Nash equilibrium, as long as the joint reactivity $\widetilde{\sigma}$ is not excessive, otherwise the equilibrium is never stable, independently on d. Namely, on the one hand, a raise in d, making emission charges higher, favors a decrease in production, but this effect may be neutralized by too large values of the joint reactivity, recalling that by (3.13) a raise in $\widetilde{\sigma}$ makes output variations increase.

In more detail, when illustrating the different dynamic scenarios compatible with Corollary 3.3, starting from the threshold values for d in (3.14) related to Corollary 3.1, we notice that d_c and d_1 are no more involved, while the admissibility condition d_a is still necessary, and, in place of d_2, we need to consider $d_2' := \frac{-2(\beta+c)}{\varepsilon^2} - \frac{\gamma}{\varepsilon^2\left(\frac{2}{\widetilde{\sigma}}-1\right)}$. Hence, the only two possibilities are given by $d_2' < d_a$ and $d_a < d_2'$. We represent them in Fig. 6 (A) and (B), respectively, where we take d as bifurcation parameter, for different values of p and σ. In more detail, with the parameter configuration in Fig. 6 (A) (coinciding with that in Fig. 4 (A), except for the opposite value for γ) it holds that $d_2' = -0.754 < d_a = -0.173$. Since the stability threshold d_2' is below the admissibility threshold d_a, the Nash equilibrium is stable for all values for which it is admissible, as confirmed by the bifurcation diagram in Fig. 6 (A), that we draw for $d \in (-0.173, 0)$. Notice that the steady state is decreasing with d, in agreement with Proposition 4 in [9]. On the other hand, for the parameter values considered in Fig. 6 (B)[16] it holds that $d_a = -0.471 < d_2' = -0.467$. Thus, although this time the stability threshold d_2' is larger than d_a, in order to take into account the admissibility condition in (2.3), that leads to $q_i < -\frac{0.4}{2.7d}$ for $i \in \{1, 2\}$, since d varies, we can represent the corresponding bifurcation diagram just for $d \in (-0.469, 0)$. However, in Fig. 6 (B) we focus on $d \in (-0.469, -0.420)$, to better highlight that (q_1^*, q_2^*) is locally asymptotically stable for (3.1) when $d > d_2'$, whereas for values of d slightly smaller than d_2' we observe a cyclic behavior, that is replaced by monotone divergence for lower values of d.

[16] We stress that with the parameter configuration in Fig. 4 (B) and $\gamma = -3$ we would have found again the scenario with $d_2' < d_a$ in Fig. 6 (B). Hence, in this case, unlike in (A), we had to consider a different parameter configuration with respect to Fig. 4 (B).

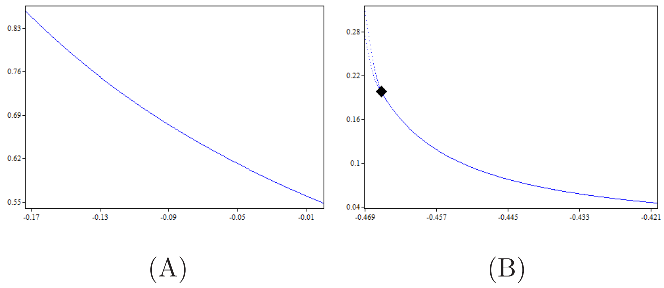


Figure 6: The bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to d and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, for $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = -3$, and $p = 3$, $\sigma = 1.478$ in (A), $p = 1.1$, $\sigma = 3$ in (B).

Making explicit in the statement of Proposition 3.2 the role of γ rather than that of d, we obtain the next:

Corollary 3.4. *When* $\gamma < 0$, *under* (3.16), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $\gamma > -\left(2(\beta + c) + \frac{d\varepsilon p}{b}\right)$. *If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *when* $\widetilde{\sigma} < 2$ *and*

$$0 > \gamma > \left(1 - \frac{2}{\widetilde{\sigma}}\right)\left(2(\beta + c) + d\varepsilon^2\right).$$

Although this result may seem to provide opposite conclusions with respect to Corollary 3.2 about the effect played on the Nash equilibrium by γ, looking destabilizing in that context and stabilizing here, we recall that the interdependence degree between the two goods is measured by $|\gamma|$. Namely, since two goods are independent for $\gamma = 0$ and their interdependence degree raises when γ becomes either positive or negative, the correct way to interpret our findings is on increasing γ *in absolute value*. Hence, while discussing Corollary 3.2, where we dealt with positive values for γ, we considered its role when it raises, in commenting Corollary 3.4, where we deal with complements, we focus on its effect when it *decreases*. Accordingly, we can conclude that, similar to what occurred in Corollary 3.2, $|\gamma| = -\gamma$ is destabilizing in Corollary 3.4, too, since by raising $|\gamma|$ the demand of each good becomes more influenced by a price variation in the other, favoring oscillations in the produced quantities. However, like with substitutes, also in Scenario I of complements the effect of γ may be observed

just for high enough, though not excessive, values for $\widetilde{\sigma}$. Namely, for $\widetilde{\sigma}$ close to 0 (resp., for $\widetilde{\sigma} \geq 2$), the destabilizing role of the joint reactivity becomes so weak (resp., so strong) that it hides the effect played by the interdependence degree between goods, with the equilibrium being stable (resp., unstable) for every value of $\gamma \in (-\beta, \beta)$.
We will return on the role of $|\gamma|$ in Subsection 3.3 where, dealing both with substitutes and with complements in Scenario I, its effect on the equilibrium stability will be better highlighted in Corollary 3.5.
Before stating that result, we complete the investigation of what occurs with complements by focusing on Scenario II.

3.2.2. Analysis of Scenario II

Even under (3.17), the conclusions about comparative statics found in [9] (cf. Propositions 7 and 8 therein) still hold true with the model nonlinear formulation, showing that the environmental policy described by the emission charges C_i in (2.2) is not effective in reducing pollution when d is negative enough and thus emission charges increase too slowly with production. Namely, under (3.17), which can be fulfilled just by values of d that are much lower than 0, the components of the Nash equilibrium in (2.8) increase with b, d and ε.
In this scenario however we find confirmation of the dynamic result in [9] (cf. Proposition 9 therein), too, according to which the steady state is never stable, when it is admissible. Namely, the counterpart to Propositions 3.1 and 3.2 reads as follows:

Proposition 3.3. *When $\gamma < 0$, under* (2.5) *and* (3.17), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $d < -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$. *If this is the case,* (q_1^*, q_2^*) *is always unstable for System* (3.1).

Proof. Using Jury conditions, we find again (3.5), equivalent to (3.9), as well as (3.6) and (3.7). However, since we are now supposing that $2(\beta+c)+d\varepsilon^2+\gamma < 0$, this time from (3.9) it follows that

$$2(\beta + c) + d\varepsilon^2 - \gamma < 0,$$

which contradicts (2.5) with $\gamma < 0$. Hence, Jury conditions are never satisfied for System (3.1) at (q_1^*, q_2^*) , leading to the assertion. □

The reason behind the lack of stability for the equilibrium in Scenario II of complements lies in the fact that we are in such framework for negative enough values for d and γ. Indeed, due to the negativity in d, emission charges C_i in (2.2) increase too slowly with production, being thus not effective in bounding the output by firms in a sufficient manner, and the strong negativity condition on γ leads to fluctuations, being $|\gamma|$, i.e., the interdependence degree between goods, high. Combining such two aspects, we obtain an explanation of why, even for very low values of $\widetilde{\sigma}$, the Nash equilibrium is never stable in Scenario II. Namely, differently from the previously considered frameworks (cf. Propositions 3.1 and 3.2 and the relative comments), in this case the destabilization is not caused, or at least amplified, by the sigmoid mechanism, as confirmed by the fact that in Scenario II of complements stability never occurs, also with the linear adjustment rule formulation considered in [6, 9].

We will not illustrate the dynamics arising on increasing the main model parameters in this scenario, in which no stability thresholds are present, since the numerical simulations we performed provided divergent outcomes, with no emerging attractors.

In order to conclude the local stability analysis, we shall better highlight the role of γ, as discussed after Corollary 3.4. This will be done in the next subsection.

3.3. The role of the interdependence degree between goods

In view of drawing conclusions valid both for the case of substitutes and of complements under (3.16), we merge Corollaries 3.2 and 3.4, so as to obtain the following:

Corollary 3.5. *Under* (2.5) *and* (3.16), (q_1^*, q_2^*) *in* (2.8) *is admissible according to* (2.3) *for* $\gamma > -\left(2(\beta+c)+\frac{d\varepsilon p}{b}\right)$. *If this is the case,* (q_1^*, q_2^*) *is locally asymptotically stable for System* (3.1) *when* $\widetilde{\sigma} < 2$ *and*

$$\left(1-\frac{2}{\widetilde{\widetilde{\sigma}}}\right)\left(2(\beta+c)+d\varepsilon^2\right) < \gamma < \min\left\{2(\beta+c)+d\varepsilon^2, \left(\frac{2}{\widetilde{\widetilde{\sigma}}}-1\right)\left(2(\beta+c)+d\varepsilon^2\right)\right\}. \tag{3.19}$$

Although containing heavy conditions, Corollary 3.5 shows that increasing the absolute value of γ has a destabilizing effect on System (3.1), in agreement with the discussion following Corollary 3.4. Indeed, as stressed there, the role of γ can be witnessed just for intermediate values for the joint reactivity $\widetilde{\sigma}$, neither too low nor too high, and, under such condition, the equilibrium destabilization occurs on raising $|\gamma|$, because in this manner the demand of each good becomes more influenced by a price variation in the other, favoring oscillations in the produced quantities.

In order to compare Corollary 3.5 with the results in [36] where the authors, like us, deal with a duopoly framework with differentiated goods in which commodities can be either substitutes or complements, we recall that therein a parameter d is introduced, named degree of horizontal product differentiation, varying in the interval $(-1, 1)$ after a normalization. In particular, d is negative in the case of complements and positive in the case of substitutes, with goods being perfect substitutes for $d = 1$ and independent when $d = 0$, similar to what occurs with our $\gamma \in (-\beta, \beta)$ (cf. Section 2). On the other hand, while the dynamic results in [36] are interpreted for increasing values of the product heterogeneity, i.e., for decreasing values of d, we think of $|\gamma|$ as interdependence degree between goods, and we interpret our results on increasing it. Given this starting point, the authors in [36] find in principle up to two stability thresholds for their parameter d at the equilibrium and the unique steady state is stable for values of d in between. However, in [36] it is shown that at most the lower between the two stability thresholds can lie in the feasible interval $(-1, 1)$, so that an increasing degree of product differentiation may just destabilize the market equilibrium. Such result is in partial contrast to Corollary 3.5, which encompasses both the destabilizing effect proven for γ in Corollary 3.2 for the case of substitutes and its stabilizing effect shown in Corollary 3.4 for the case of complements in Scenario I. Namely, recalling the different meaning attached to d in [36] and to $|\gamma|$ in the present work, the findings in [36] are in disagreement with Corollary 3.5, and thus with Corollary 3.2, if goods are substitutes, but are in agreement with Corollary 3.5, and thus with Corollary 3.4, if goods are complements and we are in Scenario I.

The statement of Corollary 3.5 is much simplified under the assumption that $d > 0$, in which case, being the emission charges in (2.2) strictly convex, there is no need for the admissibility condition in (2.3), (2.5) is always fulfilled and (3.16) reduces to (3.2). Moreover, the stability condition (3.10) derived in Proposition 3.1 is granted, and thus we easily obtain the next:

Corollary 3.6. *Assuming that $d > 0$, under (3.2) it holds that (q_1^*, q_2^*) in (2.8) is locally asymptotically stable for System (3.1)*

- *for every value of $|\gamma| < \beta$ if $\widetilde{\sigma} \leq \eta := \frac{4\beta+4c+2d\varepsilon^2}{3\beta+2c+d\varepsilon^2}$;*
- *for every value of $|\gamma| < \left(\frac{2}{\widetilde{\sigma}} - 1\right)\left(2(\beta + c) + d\varepsilon^2\right)$ if $\eta < \widetilde{\sigma} < 2$;*
- *for no values of γ if $\widetilde{\sigma} \geq 2$.*

Proof. For $d > 0$, (3.19) reads as

$$|\gamma| < \left(\frac{2}{\widetilde{\sigma}} - 1\right)\left(2(\beta + c) + d\varepsilon^2\right), \tag{3.20}$$

which can be fulfilled by some values of γ just when $\widetilde{\sigma} < 2$, since $2(\beta+c)+d\varepsilon^2 > 0$. Recalling that $\gamma \in (-\beta, \beta)$, (3.20) imposes stricter bounds on γ exclusively when $\left(\frac{2}{\widetilde{\sigma}} - 1\right)\left(2(\beta + c) + d\varepsilon^2\right) < \beta$, i.e., when $\widetilde{\sigma} > \eta := \frac{4\beta+4c+2d\varepsilon^2}{3\beta+2c+d\varepsilon^2} \in (1, 2)$. The proof is complete. □

Such result clearly shows that, as already observed, the effective role played by the interdependence degree $|\gamma|$ between goods depends also on the strength of the joint reactivity, due to the destabilizing effect of $\widetilde{\sigma}$ on the Nash equilibrium (cf. (3.13) and the related comments). Namely, for very low values of $\widetilde{\sigma}$, the steady state is stable for every admissible value of γ. On raising the joint reactivity, the larger the value of $\widetilde{\sigma}$, the smaller the equilibrium stability region with respect to γ, which becomes empty for $\widetilde{\sigma} \geq 2$. Focusing on intermediate values for the joint reactivity, the steady state is stable for not excessive values of the interdependence degree between goods, i.e., when goods are substitutes, but competition is not extremely intense, commodities being far from being perfect substitutes, or when goods are complements, but close to the case in which they are independent.

The three scenarios described in Corollary 3.6 can be obtained for different, increasing values of σ. For brevity's sake, in Fig. 7, where in (A) and (E) we show the bifurcation diagram of $q_{1,t+1}$ in (3.1) for $\gamma \in (-\beta, \beta)$, we focus just on the second and the third scenarios[17], which are the most interesting from a dynamic viewpoint.

In more detail, for the parameter values considered in Fig. 7 it holds that $d > 0$ and $p > b\varepsilon$, so that (3.2) is fulfilled and it is immediate to check that, similar to the results about comparative statics contained in the above recalled Propositions 1, 2, 4 and 5 in [9], where the role of b, d and ε was investigated, (q_1^*, q_2^*) in (2.8) is decreasing for $\gamma \in (-\beta, \beta) = (-3.1, 3.1)$. Moreover, since for the parameter η introduced in Corollary 3.6 it holds that $\eta = 1.400$, when fixing $\sigma = 5.8$ in Fig. 7 (A) we have $\eta < \widetilde{\sigma} = 1.963 < 2$, so that the Nash equilibrium is locally asymptotically stable just for $\gamma \in (-0.136, 0.136)$, according to (3.20), while when fixing $\sigma = 6$ in Fig. 7 (E) we have $\widetilde{\sigma} = 2.031 > 2$, so that the Nash equilibrium is stable for no values of γ. We finally observe that, as highlighted by the bifurcation diagrams in Fig. 7 (A) and (E), when the steady state loses stability, via period-doubling bifurcations (cf. Footnote 7), because of a high interdependence degree between goods, complex dynamics may emerge due to the presence of chaotic or quasiperiodic attractors, but no divergence issues arise thanks to the bounds imposed by the sigmoidal function in (3.1).

The presence of complex dynamics in Fig. 7 (A) for $\gamma < -2.824$ and for $\gamma > 2.834$ is confirmed both by the positive values in the plot of the maximal Lyapunov exponent against γ in Fig. 7 (B), as well as by the non-periodic attractors that we observe in the phase plane for $\gamma = -3.038$ in (C) and for $\gamma = 3.038$ in (D). Similarly, the positive values in the plot of the maximal Lya-

[17]We stress that the first scenario in Corollary 3.6, in which the steady state is locally asymptotically stable for every $\gamma \in (-\beta, \beta)$, can be obtained e.g. for $\sigma = 2$ in the parameter configuration considered in Fig. 7, in which case $\widetilde{\sigma} = 0.677 < \eta = 1.400$, with η defined as in Corollary 3.6.

punov exponent against γ in Fig. 7 (F) and the non-periodic attractors that we observe in the phase plane for $\gamma = -3.038$ in (G) and for $\gamma = 3.038$ in (H) confirm the presence of complex dynamics in Fig. 7 (E) for $\gamma < -2.563$ and for $\gamma > 2.590$. In more detail, we notice that the attractors in (D) and (H) lie on the diagonal of the phase plane, while those in (B) and (F) do not. We will try to explain such difference by investigating in Subsection 3.4 the features of the 1D setting in (3.21).

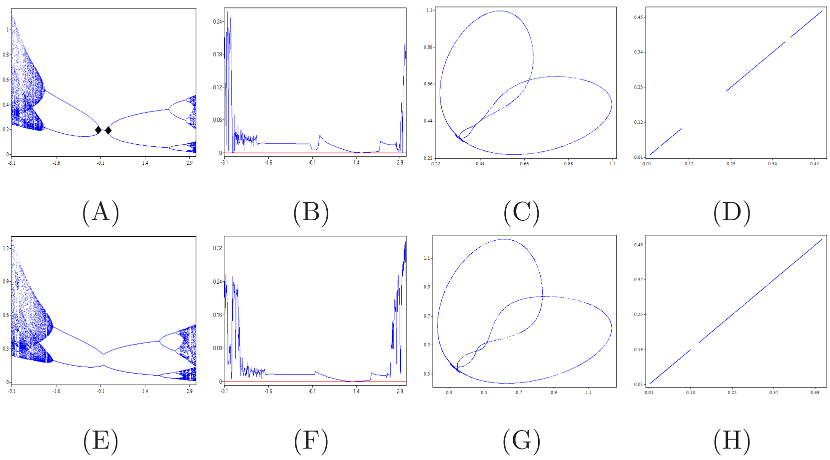


Figure 7: The bifurcation diagram of $q_{1,t+1}$ in (3.1) for $\gamma \in (-3.1, 3.1)$ and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $d = 0.1$, $\varepsilon = 2.7$, and $\sigma = 5.8$ in (A), $\sigma = 6$ in (E). For the parameter configuration considered in (A), we plot in (B) the maximal Lyapunov exponent against γ, and in (C) and (D) the attractor in the phase plane for $\gamma = -3.038$ and $\gamma = 3.038$, respectively. Similarly, for the parameter configuration considered in (E), we plot in (F) the maximal Lyapunov exponent against γ, and in (G) and (H) the attractor in the phase plane for $\gamma = -3.038$ and $\gamma = 3.038$, respectively.

3.4. Dynamics on the diagonal of the phase plane

In order to try to understand why in the numerical simulations performed so far some of, but not all, the attractors lie on the diagonal

$$\Delta := \{(q_1, q_2) \in (0, +\infty)^2 : q_1 = q_2\}$$

of the phase space, we study the one-dimensional dynamical system corresponding to the restriction of the planar model in (3.1) to Δ, i.e.,

$$q_{t+1} = q_t + \delta \left(\frac{\upsilon + \delta}{\upsilon\, e^{-\sigma\left(\frac{p - b\varepsilon - \gamma q_t}{2(\beta + c) + d\varepsilon^2} - q_t\right)} + \delta} - 1 \right), \tag{3.21}$$

obtained by setting $q_{1,t} = q_{2,t}$ in (3.1). Namely, calling $f : (0,+\infty) \to (0,+\infty)$ the 1D map associated with (3.21), for the restriction of F to Δ, i.e., for $F|_\Delta : \Delta \to \Delta$, it holds that $F(q,q) = (f(q), f(q))$, where $F : (0,+\infty)^2 \to (0,+\infty)^2$ is the planar map associated with (3.1). We notice indeed that $\Delta \subset (0,+\infty)^2$ is invariant for F, that is, $F(\Delta) \subseteq \Delta$, because, by the symmetry of (3.1), if also initial conditions for the output of the two firms coincide, then their future choices will be identical in each time period.

We start our study of the 1D setting by observing that the only steady state of (3.21) is given by

$$q^* = \frac{p - b\varepsilon}{2(\beta + c) + d\varepsilon^2 + \gamma} \tag{3.22}$$

so that for the Nash equilibrium in (2.8) it holds that $(q_1^*, q_2^*) = (q^*, q^*) \in \Delta$.

Although in principle the subdivision of the stability analysis depending on whether goods are substitutes or complements and according to the sign of the numerator and of the denominator of q^* should remain unchanged with respect to the planar setting, Proposition 3.4 shows that in the 1D model the stability conditions with substitutes and in Scenario I of complements coincide, although differing from the findings in both the corresponding two-dimensional frameworks, while Proposition 3.5 confirms that, like in the planar setting, the steady state is always unstable in Scenario II of complements also for (3.21).

Namely, recalling also the admissibility condition in (2.3), our first stability result for (3.21) reads as follows:

Proposition 3.4. *Under* (2.5) *and* (3.16)*,* q^* *in* (3.22) *is admissible according to* (2.3) *for* $d > -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$*. If this is the case,* q^* *is locally asymptotically stable for System* (3.21) *when* $\widetilde{\sigma} < \frac{2}{1+\frac{\gamma}{2(\beta+c)+d\varepsilon^2}}$.

Proof. We have that q^* in (3.22) is locally asymptotically stable for System (3.21) when

$$-1 < f'(q^*) = 1 - \widetilde{\sigma}\left(\frac{2(\beta + c) + d\varepsilon^2 + \gamma}{2(\beta + c) + d\varepsilon^2}\right) < 1. \tag{3.23}$$

The right inequality in (3.23) leads to

$$\frac{2(\beta + c) + d\varepsilon^2 + \gamma}{2(\beta + c) + d\varepsilon^2} > 0, \tag{3.24}$$

which is always fulfilled under (2.5) and (3.16).

The left inequality in (3.23) leads to

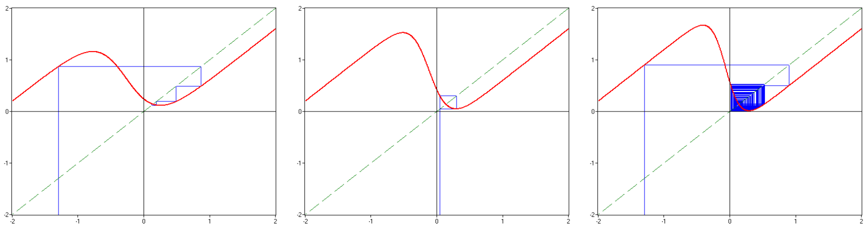
$$\widetilde{\sigma} < \frac{2}{1 + \frac{\gamma}{2(\beta+c)+d\varepsilon^2}},$$

concluding the proof. $\square$

Comparing the above result for the case of substitutes with Proposition 3.1, we notice that condition (3.11) on d is no more necessary, while condition (3.8) is still needed in order to have the equilibrium locally asymptotically stable in the 1D framework. Since in Fig. 3 (A) and (D) condition (3.11) is always fulfilled, we find that those same bifurcation diagrams with respect to σ describe the dynamics for System (3.21), too, as confirmed by the fact that the attractor in Fig. 3 (C), obtained for the 2D system for the parameter set in (A), lies on the diagonal of the phase plane. Namely, for the same parameter configuration considered in Fig. 3 (A), i.e., $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = 3$ and $d = 0.1$, we show in Fig. 8 how the graph of the first iterate of the map f associated with (3.21) looks like for increasing values of σ. In more detail, recalling that for such parameter values (3.8) reads as $\widetilde{\sigma} < 1.413$ or, equivalently, as $\sigma < 4.176$, for $\sigma = 3$ we obtain in Fig. 8 (A) that q^* in (3.22) is locally (actually, globally) asymptotically stable, for $\sigma = 4.8$ we witness in Fig. 8 (B) a stable period-two cycle, composed by the points $q' = 0.049$ and $q'' = 0.299$, while in Fig. 8 (C), for $\sigma = 6.1$, we find a chaotic attractor.

As concerns Fig. 4, we notice that the bifurcation diagrams with respect to d in (A) and (C) remain unchanged in the one-dimensional setting, while the stability threshold that we observe in Fig. 4 (B), arising from (3.11), does not exist anymore in relation to System (3.21).

On the other hand, focusing on Scenario I of complements, we notice that the stability condition in Proposition 3.4 is different from the one in (3.18), present in Proposition 3.2, fact which makes particularly interesting the comparison between the dynamics generated by Systems (3.1) and (3.21). Indeed, focusing e.g. on the parameter configuration considered in Fig. 5 (A), i.e., $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = -3$ and $d = 0.1$, we



(A) (B) (C)

Figure 8: The graph of the function f associated with (3.21) for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = 3$, $d = 0.1$, and $\sigma = 3$ in (A), $\sigma = 4.8$ in (B), $\sigma = 6.1$ in (C), with initial condition $q_0 = -1.3$ in (A) and (C), and $q_0 = 0.049$ in (B).

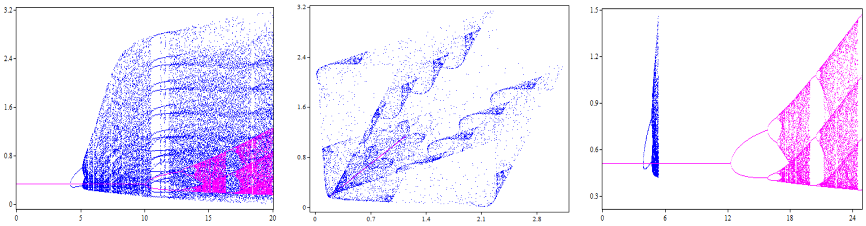
draw in Fig. 9 the bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to σ (in blue), already shown in Fig. 5 (A), together with the bifurcation diagram of q_{t+1} in (3.21) with respect to σ (in magenta). Since σ varies in $(0, 20)$ therein, we observe that the stability threshold for (q_1^*, q_2^*) in (2.8), which, in agreement with Proposition 3.2, is given by $\widetilde{\sigma} < 1.413$ or, equivalently, by $\sigma < 4.176$, does not coincide with the stability threshold for q^* in (3.22). Namely, the latter, in agreement with Proposition 3.4, is given by $\widetilde{\sigma} < 3.419$ or, equivalently, by $\sigma < 10.101$. The different stability thresholds for the 1D and the 2D frameworks imply that q^* in (3.22) is still stable when (q_1^*, q_2^*) in (2.8) has lost stability. As we shall discuss below, such discrepancy between Systems (3.1) and (3.21) has important consequences from the policy viewpoint. Even when the steady state has become unstable in both settings, and chaotic dynamics arise in each framework, different attractors emerge for the two systems. In order to highlight such fact, we juxtapose in Fig. 9 (B) for $\sigma = 19$ in the phase plane the 2D attractor for System (3.1), colored in blue, and the one-dimensional attractor for System (3.21), colored in magenta, that is contained in Δ.

Also when dealing with the parameter values used in Fig. 5 (C), i.e., $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = -3$ and $d = -0.1$, we observe that the stability threshold for (q_1^*, q_2^*) in (2.8) differs from the stability threshold for q^* in (3.22). Namely, according to Proposition 3.2, the former is given by $\widetilde{\sigma} < 1.316$ or, equivalently, by $\sigma < 3.888$, while, according to

Proposition 3.4, the latter is given by $\widetilde{\sigma} < 4.165$ or, equivalently, by $\sigma < 12.307$, as confirmed by Fig. 9 (C), where we draw together the bifurcation diagrams, with respect to σ, of $q_{1,t+1}$ in (3.1) in blue and of q_{t+1} in (3.21) in magenta. Notice that, taking into account the admissibility condition in (2.3), which leads to $q_i < 1.481$ for $i \in \{1, 2\}$ in relation to System (3.1), and to $q < 1.481$ in relation to System (3.21), we can draw the former bifurcation diagram just for $\sigma \in (0, 5.334)$, and the latter for $\sigma \in (0, 24.849)$, only. In relation to both frameworks, we witness in Fig. 9 (C) the emergence of chaotic dynamics after the steady state stability loss. However, the destabilization of q^* in (3.22) occurs when the admissibility condition in (2.3) for System (3.1) is no more satisfied, so that q^* in (3.22) is stable as long as the dynamics of System (3.1) are admissible. Hence, we find deep differences between the one-dimensional and the planar settings for the parameter configuration used in Fig. 5 (C), too, that extend to Fig. 6 (B), as well. Indeed, in that figure we found that (q_1^*, q_2^*) in (2.8) is stable for System (3.1) for $d > -0.467$, but for the same parameter values used therein we have that q^* in (3.22) is stable for System (3.21) whenever it is admissible,[18] i.e., for $d > -0.471$. Also for the parameter configuration considered in Fig. 6 (A), q^* in (3.22) is stable for System (3.21) whenever it is admissible, i.e., for $d > -0.173$, this time in agreement with what we observe in Fig. 6 (A) for (q_1^*, q_2^*) in (2.8). For brevity's sake, we omit the bifurcation diagrams of q_{t+1} in (3.21) with respect to d for the parameter sets used in Fig. 6.

Crucial differences arise between the 1D and the 2D frameworks with the parameter configurations used in Fig. 7 (A) and (E), as well. In order to highlight them, we juxtapose in Fig. 10 the bifurcation diagrams with respect to $\gamma \in (-3.1, 3.1)$ of $q_{1,t+1}$ in (3.1) (in blue) and of q_{t+1} in (3.21) (in magenta)

[18] Namely, for the parameter configuration considered in Fig. 6 (B), (2.3) reads as $q < -\frac{0.4}{2.7d}$ in relation to System (3.21), similar to what occurred for System (3.1), where (2.3) led to $q_i < -\frac{0.4}{2.7d}$ for $i \in \{1, 2\}$. On the other hand, the stability condition with respect to d for q^* in (3.22) in Scenario I of complements is given by $d > \frac{-2(\beta+c)}{\varepsilon^2} + \frac{\gamma}{\varepsilon^2\left(\frac{2}{\sigma}-1\right)}$, which differs from the condition in Corollary 3.3, and that reads as $d > -1.316$ for the parameter values used in Fig. 6 (B).



(A) (B) (C)

Figure 9: For $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\gamma = -3$, and $d = 0.1$ in (A), $d = -0.1$ in (C), we plot in blue the bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to σ and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, and in magenta the bifurcation diagram of q_{t+1} in (3.21) with respect to σ and initial condition $q_0 = 0.25$. For the parameter configuration considered in (A), we juxtapose in (B) for $\sigma = 19$ the attractors in the phase plane for System (3.1) in blue and for System (3.21) in magenta.

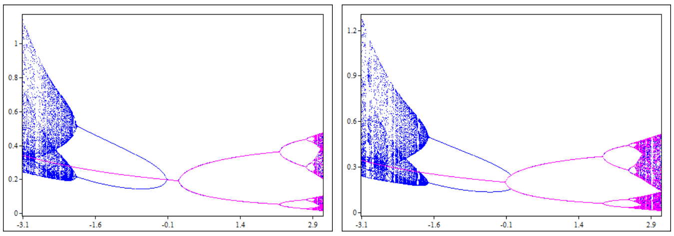
for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $d = 0.1$, $\varepsilon = 2.7$, and $\sigma = 5.8$ in (A), $\sigma = 6$ in (B). Moreover, we notice that, as an immediate consequence of Proposition 3.4, the stability condition with respect to γ for q^* in (3.22) is given by

$$\gamma < \left(\frac{2}{\widetilde{\sigma}} - 1\right)\left(2(\beta + c) + d\varepsilon^2\right), \tag{3.25}$$

both with substitutes - in which case also the assumption $\widetilde{\sigma} < 2$ is needed - and in Scenario I of complements, thus differing from the conditions derived in Corollary 3.5, and in Corollary 3.6 for the case in which d is positive. In particular, since the stability threshold in (3.25) can also be negative, this means that increasing values of γ, rather than of $|\gamma|$, play a destabilizing role for System (3.21).

Namely, in Fig. 10 (A) we observe that q^* in (3.22) is stable for low enough values of γ, when it is positive, and anytime goods are complements or independent, unlike what occurs for System (3.1), in which the equilibrium is stable when the interdependence degree between goods, measured by $|\gamma|$, is sufficiently weak. However, focusing on the case of substitutes, according to Corollary 3.6 and (3.25), both Systems (3.1) and (3.21) are stable for $\gamma \in (0, 0.136)$, and in Fig. 10 (A) we witness that the dynamic behavior for the 1D and 2D frameworks coincides for all values of $\gamma \in (0, \beta) = (0, 3.1)$.

The difference between Systems (3.1) and (3.21) becomes stronger for the parameter configuration in Fig. 10 (B), where, due to a larger value for σ with respect to (A), (q_1^*, q_2^*) in (2.8) is stable for no values of $\gamma \in (-\beta, \beta) = (-3.1, 3.1)$, since $\widetilde{\sigma} = 2.031$ now exceeds 2 (cf. Corollary 3.6), whereas q^* in (3.22) is stable for $\gamma < -0.110$. We stress that the latter dynamic outcome, in addition to being not coincident with the one that we observe in Fig. 10 (B) for (q_1^*, q_2^*) in (2.8), can not occur for System (3.1). Namely, when d is positive like in Fig. 10 (B), according to Corollary 3.6 there are two symmetric stability thresholds for System (3.1), with q^* in (3.22) being stable for values of γ in between, rather than just a negative stability threshold. Despite such crucial difference between the 1D and 2D frameworks, once that System (3.21) becomes unstable in Fig. 10 (B), its dynamics are identical to those of System (3.1).



(A) (B)

Figure 10: For $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $d = 0.1$, $\varepsilon = 2.7$, and $\sigma = 5.8$ in (A), $\sigma = 6$ in (B), we plot in blue the bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to $\gamma \in (-3.1, 3.1)$ and initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$, and in magenta the bifurcation diagram of q_{t+1} in (3.21) with respect to $\gamma \in (-3.1, 3.1)$ and initial condition $q_0 = 0.25$.

We conclude our local stability analysis for (3.21) by focusing on Scenario II of complements, so as to obtain the next result:

Proposition 3.5. *When* $\gamma < 0$, *under* (2.5) *and* (3.17), q^* *in* (3.22) *is admissible according to* (2.3) *for* $d < -\frac{b(2\beta+2c+\gamma)}{\varepsilon p}$. *If this is the case,* q^* *is always unstable for System* (3.21).

Proof. The desired conclusion immediately follows by observing that the right inequality in (3.23) still leads to (3.24), which is however never fulfilled under (2.5) and (3.17). □

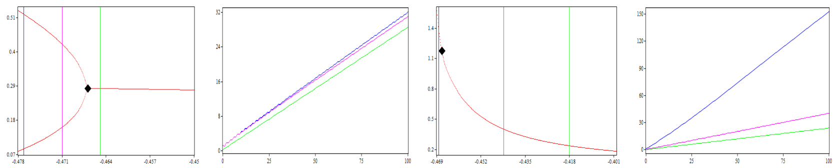
Hence, according to Propositions 3.3 and 3.5, in Scenario II of complements, i.e., when emission charges in (2.2) increase too slowly with production and the interdependence degree between the two goods is high, the steady state is always unstable for both Systems (3.1) and (3.21).

The above investigation has shown that there are both analogies and differences between the 1D and the 2D frameworks. In particular, as highlighted by Fig. 9 (A) and (C), as well as by Fig. 10 (A) and (B), it may happen that the steady state of the one-dimensional setting is stable when the dynamics of the planar model are quasiperiodic or chaotic. This fact has important consequences from the policy viewpoint. Namely, focusing just on the behavior of the steady state via comparative statics exercises prevents the consideration and the understanding of the system out-of-equilibrium dynamics, while, as we have argued in Subsections 3.1 and 3.2, a comparative statics result is economically grounded if it concerns an equilibrium which is asymptotically stable and thus orbits converge towards it after a transient period. Things worsen when assuming that agents are completely homogeneous, so that their past choices coincide, too. Indeed, in such case, even an environmental policy maker that were aware of the limits of the equilibrium analysis could erroneously believe that the equilibrium is stable, thus relying on comparative statics tools, while the steady state is actually unstable in the 2D framework. Hence, in addition to being implausible that firms are identical from an evolutionary viewpoint, assuming coinciding initial conditions for their output can make the use of the comparative statics technique apparently grounded, when it is not. Thus, in the figures of Section 4, illustrating the comparative dynamics investigations that we propose to evaluate the environmental policy efficacy when the Nash equilibrium is not stable, we will always consider heterogeneous initial conditions for the output of the two firms. In this respect, we recall the concept of "path dependence" in [11], according to which the initial conditions, representing a summary of agents' history, matter in determining the evolution of the system.

4. Comparative dynamics analysis of the environmental policy efficacy

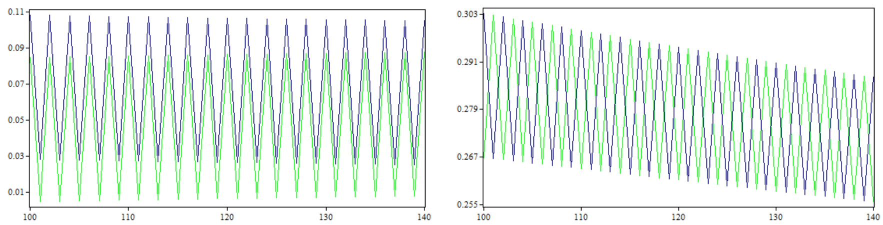
We now discuss how to evaluate the environmental policy efficacy when the steady state is not stable. Namely, in introducing our model we stressed that a growing empirical literature (see e.g. [1, 2, 3]) highlights the chaotic behavior of the main variables in various markets, and in particular in agricultural markets. In order to be realistic, our model has to be able to reproduce the dynamic phenomena identified by those empirical studies, according to which what we see is the result of the action of underlying nonlinear mechanisms. Starting then from the framework in [6], we replaced the linear output adjustment rule considered therein with a sigmoid mechanism in view of obtaining interesting, i.e., non-stationary, non-divergent, dynamic outcomes. The introduction of the sigmoid mechanism, in addition to allowing for nontrivial dynamics, shrinks the steady state stability region. Indeed, comparing Propositions 3.1, 3.2 and 3.3 with the corresponding results in [9], we observe a reduction in the stability region for (q_1^*, q_2^*) both in the case of substitutes and of complements in Scenario I - in the latter framework the Nash equilibrium in [9] was always stable when admissible - and a confirmation of its unconditional instability in Scenario II, in consequence of the introduction of the sigmoid adjustment mechanism in (2.10). Therefore, it is important to understand how the environmental policy efficacy can be evaluated in the case of non-stationary trajectories. Namely, as long as the Nash equilibrium is stable, such as in Figs. 4 (A) and 6 (A), the efficacy of the environmental policy can be assessed using the standard tool, which is represented by the comparative statics analysis. In this respect, we recall that, as mentioned in Section 3, according to Propositions 1 and 4 in [9], which hold true with the nonlinear adjustment mechanism in (2.10), too, the environmental policy described by the emission charges C_i in (2.2) is effective in reducing pollution, i.e., the equilibrium pollution level falls with an increase in b or d, with substitutes or under (3.16), while in agreement with Proposition 7 in [9] it is detrimental under (3.17), since in such case the equilibrium pollution level raises with an increase in b or d, due to the fact that, under (3.17), emission charges increase too slowly with production. Nonetheless, when the steady state is not stable, or when the considered scenario is characterized by the presence of an attractor different from the Nash equilibrium, the comparative statics technique is neither economically, nor empirically grounded. In such cases, we then need to perform alternative, comparative dynamics investigations, based for instance on the behavior, for different values of d, of the time series of the cumulative emissions, defined as the sum, over a certain time interval $[0, T]$, of the aggregate emissions $U_t := u_{1,t} + u_{2,t} = \varepsilon(q_{1,t} + q_{2,t})$ produced in time period $t \in [0, T]$ by both firms, i.e., in symbols $CE_T := \sum_{t=0}^{T} U_t = \varepsilon \sum_{t=0}^{T}(q_{1,t} + q_{2,t})$. In this manner, the environmental policy efficacy could be implied by a negative variation of cumulative emissions over the chosen time interval as a consequence of an increase in d. We can use such strategy to investigate the environmental policy efficacy e.g. in the contexts considered in Figs. 4 (C) and 6 (B), where d was the bifurcation parameter.

To that aim, for the parameter values used therein we reproduce the two bifurcation diagrams for aggregate emissions U_{t+1} in Fig. 11 (A) and (C), respectively, where we fix three different values of d (colored in blue, red and green), some of which lie in the interval where the steady state is unstable, while the remaining ones belong to the stability interval of the Nash equilibrium. We show the corresponding time series of the cumulative emissions CE_T for $T \in [0, 100]$ in Fig. 11 (B) and (D), by using the same colors as in (A) and (C). In particular, in Fig. 11 (A) and (B) the blue color refers to $d = -0.477$, the red color to $d = -0.471$ and the green color to $d = -0.465$, while in Fig. 11 (C) and (D) the blue color refers to $d = -0.468$, the red color to $d = -0.443$ and the green color to $d = -0.418$. The initial conditions in (B) are $U_0 = \varepsilon(q_{1,0} + q_{2,0}) = 2.7 * 0.45 = 1.215$ for the blue and the red time series, and $2\varepsilon q_1^* = 2 * 2.7 * 0.052 = 0.281$ for the green time series, while in (D) the initial conditions are $U_0 = 2.7 * 0.45 = 1.215$ for the blue time series, $2\varepsilon q_1^* = 2 * 2.7 * 0.074 = 0.399$ for the red time series, and $2\varepsilon q_1^* = 2 * 2.7 * 0.044 = 0.238$ for the green time series. Since in (B) and (D) the cumulative emissions for $T \in [0, 100]$ are larger for lower values of d, this means that increasing emission charges in (2.2) reduce pollution, and thus the considered environmental policy is effective. We would find the same conclusion by performing the just described investigation in the framework in Fig. 4 (B), too, in agreement with the comparative statics result for Fig. 4 (A) (see Proposition 1 in [9]). Contrasting Fig. 11 (B) and (D), we observe that, although in both (A) and (C) the considered values of d are equidistant, pollution decreases in (B) rapidly for higher values of d, while in (D) the efficacy of the environmental policy, although raising with d, slows down when d increases. In this sense, an intense increase in d is more useful in (B) than in (D).



(A) (B) (C) (D)

Figure 11: In (A) and (C) we report the bifurcation diagrams of U_{t+1} with respect to d for the same parameter values used in Figs. 4 (C) and 6 (B), respectively. In (B) and (D) we show the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to the values of d marked with different colors in (A) and (C), respectively. The initial conditions in (B) are $U_0 = \varepsilon(q_{1,0} + q_{2,0}) = 2.7 * 0.45 = 1.215$, connected with $q_{1,0} = 0.25$ and $q_{2,0} = 0.2$, for the blue and the red time series, and $2\varepsilon q_1^* = 2 * 2.7 * 0.052 = 0.281$ for the green time series, while in (D) the initial conditions are $U_0 = 2.7 * 0.45 = 1.215$, connected with $q_{1,0} = 0.25$ and $q_{2,0} = 0.2$, for the blue time series, $2\varepsilon q_1^* = 2 * 2.7 * 0.074 = 0.399$ for the red time series, and $2\varepsilon q_1^* = 2 * 2.7 * 0.044 = 0.238$ for the green time series.



(A) (B)

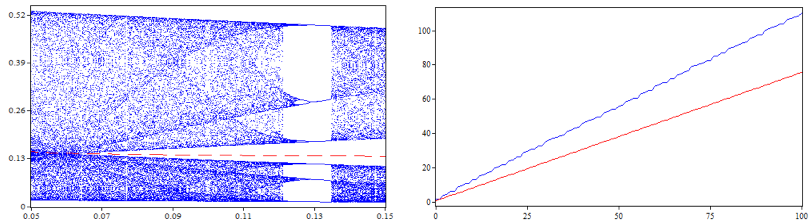
Figure 12: In (A) and (B) we show the time series of $q_{1,t}$ in dark blue and of $q_{2,t}$ in green for $t \in [101, 140]$, corresponding to the values of d marked in blue in Fig. 11 (A) and (C), respectively.

In view of better comparing and understanding Fig. 11 (A) and (C), we draw in Fig. 12 the time series of $q_{1,t}$ in dark blue and of $q_{2,t}$ in green for $t \in [101, 140]$, corresponding to the values of d marked in blue in Fig. 11 (A) and (C), i.e., $d = -0.477$ and $d = -0.468$, respectively. We find that, although for the considered values of d both the bifurcation diagrams of $q_{1,t+1}$ in Figs. 4 (C) and 6 (B) highlight the presence of a stable period-two cycle, in Fig. 12 (A) we witness an agreement between the periods of high/low production strategies for the two firms, so that their outputs give a concordant contribution to aggregate emissions in Fig. 11 (A), while in Fig. 12 (B) there is discordance between the high/low output choice timing for the two firms, still giving rise to a decreasing trend. Such difference between Fig. 12 (A) and (B) is the reason why in the bifurcation diagram in Fig. 11 (A) we witness a period-two cycle for U_{t+1} for low values for d, like it was in Fig. 4 (C), while in Fig. 11 (C) we do not see oscillations for U_{t+1} even before the stability threshold value, i.e., $d = -0.467$. The symmetry that we observe in Fig. 12 (A), and partially in (B), is caused by the low heterogeneity degree in the model, concerning both the demand side and the technology, since we are assuming that $\beta_1 = \beta_2 = \beta$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon$, respectively.

We stress that in the time series in Fig. 12 (A) and (B) we introduced a transient of 100 periods in order to show the asymptotic behavior of the production of the two firms. We also remark that the choice of considering $T \in [0, 100]$ in our experiments in Fig. 11 (B) and (D) has no effect on the behavior of time series for cumulative emissions. Namely, considering a larger time interval, the distance among the found time series would increase, but their ordering would not change. Moreover, we underline that the proposed investigation can be performed in more general frameworks, in which looking at the corresponding bifurcation diagram with respect to d is not clear what is the effect generated by an increase in emission charges on produced quantities, and consequently on emissions.

Hence, thanks to our first analysis based on cumulative emissions we checked in Fig. 11 the efficacy of the environmental policy introduced in (2.2) for the parameter configurations considered in Subsection 3.1 and in Scenario I in Subsection 3.2, in agreement with the comparative statics results obtained for substitutes and complements under (3.16) in Propositions 1 and 4 in [9]. In regard to Scenario II in Subsection 3.2, in which the Nash equilibrium is always unstable when it is admissible, we can say that if the system reached the steady state and remained on it despite the equilibrium instability, we would find that emissions raise with an increase in d, in agreement with Proposition 7 in [9], i.e., the comparative statics result valid for the case of complements under (3.17). On the other hand, since in Scenario II in Subsection 3.2 we are always in an instability regime and the numerical simulations we performed display divergent outcomes, with no emerging attractors, it is not possible to draw conclusions about the environmental policy efficacy in that scenario. Namely, related comments can be made just when orbits visit an attractor.

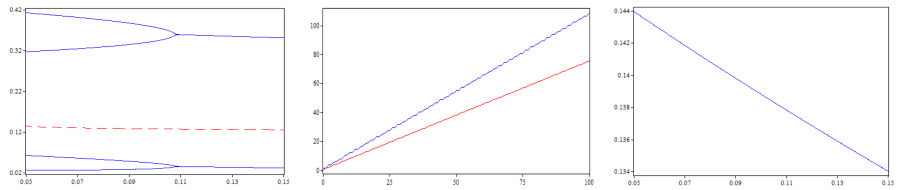
A different extension of the classical comparative statics analysis to the frameworks in which the steady state is not stable consists in a comparison, for the given parameter configuration and over a certain time interval, of cumulative emissions, starting from the unstable Nash equilibrium and from a different point in the basin of attraction of the stable periodic or complex attractor. We show what happens in this respect both with substitutes, in Figs. 13 and 14, and with complements under (3.16), in Figs. 15 and 16, starting in both cases from Fig. 7 (C) and dealing with positive and negative values for γ, respectively.



(A) (B)

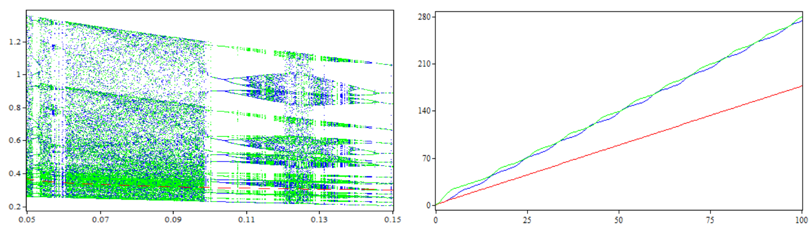
Figure 13: In (A) the bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to $d \in (0.05, 0.15)$ with initial conditions $q_{1,0} = 0.1$, $q_{2,0} = 0.5$, for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\sigma = 6$ and $\gamma = 3$. In (B) we show the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to $d = 0.1$, with initial condition $u_{1,0} + u_{2,0} = 1.62$, connected with $q_{1,0} = 0.1$ and $q_{2,0} = 0.5$, for the blue points and $2\varepsilon q_1^* = 2 * 2.7 * 0.139 = 0.751$ for the red points.

Namely, in Fig. 13 (A) we draw the bifurcation diagram of $q_{1,t+1}$ obtained for the same parameter values used in Fig. 7 (C) but fixing $\gamma = 3$ and letting d vary in $(0.05, 0.15)$. Since the steady state (drawn in red, dashed line) is always unstable for the considered parameter values and, according to the value of $d \in (0.05, 0.15)$, we observe a periodic or a chaotic attractor (in blue), we contrast in Fig. 13 (B) the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to $d = 0.1$, with initial condition $u_{1,0} + u_{2,0} = \varepsilon(q_{1,0} + q_{2,0}) = 2.7*0.6 = 1.62$ for the blue points and $u_{1,0}+u_{2,0} = 2\varepsilon q_1^* = 2*2.7*0.139 = 0.751$ for the red points. We find that the cumulative emissions in the considered time interval are larger along the non-stationary trajectory than along the equilibrium path. Hence, we could try to contain emissions and to stabilize the system by acting on the sigmoid adjustment mechanism, and in particular on the position of its horizontal asymptotes. In this respect, we recall the bounding role played by the horizontal asymptotes, whose level, as explained in Section 2, is controlled by parameters υ and δ. Indeed, reducing υ lowers the upper asymptote, which plays a role when the best response is above the current production level, while decreasing δ raises the lower asymptote, which intervenes when the best response is below current production level. Starting from the framework in Fig. 13 and acting for instance on υ, we obtain the effect illustrated in Fig. 14, where in (A) and (C) we show that, as desired, the complexity of the dynamics decreases by lowering υ. In more detail, fixing the remaining parameters as in Fig. 13 (A), in Fig. 14 (A) for $\upsilon = 1.35$ we obtain a periodic attractor (in blue), i.e., a period-four or a period-two cycle for $d \in (0.05, 0.15)$, while the steady state (drawn in red, dashed line) is always unstable for such values of d. Drawing in Fig. 14 (B) the time series of cumulative emissions for $T \in [0, 100]$ corresponding to $d = 0.1$, with initial condition $u_{1,0} + u_{2,0} = \varepsilon(q_{1,0} + q_{2,0}) = 2.7 * 0.6 = 1.62$ for the blue points and $u_{1,0} + u_{2,0} = 2\varepsilon q_1^* = 2 * 2.7 * 0.139 = 0.751$ for the red points, we find, like in Fig. 13 (B), that cumulative emissions are larger along the non-stationary trajectory than along the equilibrium path. Reducing υ further to 0.5 in Fig. 14 (C), we finally obtain the complete stabilization of the system. This shows that the sigmoid adjustment mechanism can be effective in reducing pollution, by acting on the maximum allowed production variation. As argued above, in consequence of the system stabilization, the comparative statics analysis becomes economically grounded. We recall that, for the case of substitutes, the corresponding comparative statics result in [9] (cf. Proposition 1 therein) states that the equilibrium pollution level falls with an increase in emission charges. We stress that the outcome about the system stabilization is independent from the choice of considering $T \in [0, 100]$ in regard to the time frame, as well as from the choice of $d = 0.1$, since for any value of $d \in (0.05, 0.15)$ we would obtain the same conclusion, whether in Fig. 13 (A) we observe a periodic or a chaotic attractor.



(A) (B) (C)

Figure 14: In (A) and (C) we report the bifurcation diagrams of $q_{1,t+1}$ in (3.1) with respect to $d \in (0.05, 0.15)$ with initial conditions $q_{1,0} = 0.1$, $q_{2,0} = 0.5$, for $p = 2.5$, $\delta = 0.4$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\sigma = 6$, $\gamma = 3$, and $\upsilon = 1.35$ in (A), $\upsilon = 0.5$ in (C), respectively. In (B) we show the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to (A) with $d = 0.1$, with initial conditions $u_{1,0} + u_{2,0} = 1.62$, connected with $q_{1,0} = 0.1$ and $q_{2,0} = 0.5$, for the blue points and $2\varepsilon q_1^* = 2 * 2.7 * 0.139 = 0.751$ for the red points.

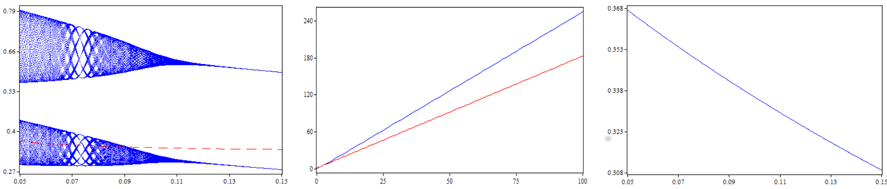


(A) (B)

Figure 15: In (A) the bifurcation diagram of $q_{1,t+1}$ in (3.1) with respect to $d \in (0.05, 0.15)$ with initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$ for the blue points and $q_{1,0} = 0.1$, $q_{2,0} = 0.5$ for the green points, for $p = 2.5$, $\delta = 0.4$, $\upsilon = 2.2$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\sigma = 6$ and $\gamma = -3$. In (B) we show the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to $d = 0.1185$, with initial condition $u_{1,0} + u_{2,0} = 1.215$, connected with $q_{1,0} = 0.25$ and $q_{2,0} = 0.2$, for the blue points, $u_{1,0} + u_{2,0} = 1.62$, connected with $q_{1,0} = 0.1$ and $q_{2,0} = 0.5$, for the green points, and $2\varepsilon q_1^* = 2 * 2.7 * 0.325 = 1.757$ for the red points.

In fact, we shall reach analogous conclusions also with complements, when (3.16) holds true. In this case, starting again from Fig. 7 (C), we draw in Fig. 15 (A) the bifurcation diagram of $q_{1,t+1}$ obtained for the same parameter values used therein but fixing $\gamma = -3$ and letting d vary in $(0.05, 0.15)$, which highlights a multistability phenomenon. Since the steady state (drawn in red, dashed line) is always unstable for the considered parameter values and we observe two coexisting chaotic attractors (in blue and in green), interrupted just by some periodicity windows, we contrast in Fig. 15 (B) the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to $d = 0.1185$, with initial condition $u_{1,0} + u_{2,0} = \varepsilon(q_{1,0} + q_{2,0}) = 2.7 * 0.45 = 1.215$ for the blue points, $u_{1,0} + u_{2,0} = 2.7 * 0.6 = 1.62$ for the green points, and $u_{1,0} + u_{2,0} = 2\varepsilon q_1^* = 2 * 2.7 * 0.325 = 1.757$ for the red points. We find again that the cumulative emissions in the considered time interval are larger along the non-stationary trajectories than along the equilibrium path. Hence, also in this case we could try to contain emissions and to stabilize the system by acting on the sigmoid adjustment mechanism, and in particular by lowering the upper asymptote. Reducing υ we obtain the effect illustrated in Fig. 16, where in (A) and (C) we show that, as desired, the complexity of the dynamics decreases when υ becomes smaller. In more detail, fixing the remaining parameters as in Fig. 15 (A), in Fig. 16 (A) for $\upsilon = 1.28$ we find (in blue) a quasiperiodic attractor in two pieces which disappears for increasing values of $d \in (0.05, 0.15)$ and a stable period-two cycle emerges via a reverse Neimark-Sacker bifurcation, while the steady state (in red, dashed line) is always unstable. Drawing in Fig. 16 (B) the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to $d = 0.1$ with initial condition $u_{1,0} + u_{2,0} = \varepsilon(q_{1,0} + q_{2,0}) = 2.7 * 0.45 = 1.215$ for the blue points and $u_{1,0} + u_{2,0} = 2\varepsilon q_1^* = 2 * 2.7 * 0.336 = 1.814$ for the red points, we find that the cumulative emissions are larger along the quasiperiodic, non-stationary trajectory than on the Nash equilibrium. Lowering υ further to 0.5 in Fig. 16 (C), we finally reach the stabilization of the system. This shows that the sigmoid adjustment mechanism is effective in reducing pollution, by acting on the maximum allowed production variation, also with complements

under (3.16). Notice that such outcome is in agreement with the corresponding comparative statics result in [9] (cf. Proposition 4 therein), stating that the equilibrium pollution level falls with an increase in emission charges, which becomes economically grounded when υ is low enough, so that the steady state is stable. Again, the stabilization of the Nash equilibrium is independent from the choice of dealing with $T \in [0, 100]$ and $d \in (0.05, 0.15)$.
Summarizing, through our first analysis, based on a comparison of emissions for different levels of charges, we have found that increasing values for d raise the dynamic efficacy of the considered environmental policy, while the second study, by means of a comparison of emissions along non-stationary trajectories and along the equilibrium path, has highlighted that, in order to reduce pollution, guaranteeing the convergence to the Nash equilibrium is preferable to allowing for complex or periodic behavior in the firms' output, and that acting on the asymptotes may correspond to a direct control of emissions, in contrast to the *indirect* nature of the pollution control obtained via emission charges in (2.2). In this respect, we stress that the direct control exerted by acting on the sigmoid asymptotes stabilizes the Nash equilibrium without inducing any variation in the output level, contrary to the indirect control described by (2.2) which, according to Propositions 1 and 4 in [9], induces a negative variation in output. The above described conclusions have been reached for the parameter configurations considered in Section 3 both with substitutes and in Scenario I therein with complements under (3.16). On the other hand, due to the fact that, according to Proposition 3.3, the Nash equilibrium is never stable under (3.17) and that divergence issues arise in the numerical simulations we performed for Scenario II in Section 3, our investigations do not allow us to draw conclusions about the efficacy of the environmental policy when dealing with complements under (3.17).



(A) (B) (C)

Figure 16: In (A) and (C) we report the bifurcation diagrams of $q_{1,t+1}$ in (3.1) with respect to $d \in (0.05, 0.15)$ with initial conditions $q_{1,0} = 0.25$, $q_{2,0} = 0.2$ for $p = 2.5$, $\delta = 0.4$, $\beta = 3.1$, $c = 0.15$, $b = 0.4$, $\varepsilon = 2.7$, $\sigma = 6$, $\gamma = -3$, and $\upsilon = 1.28$ in (A), $\upsilon = 0.5$ in (C), respectively. In (B) we show the time series of cumulative emissions CE_T for $T \in [0, 100]$ corresponding to (A) with $d = 0.1$, with initial conditions $u_{1,0} + u_{2,0} = 1.215$, connected with $q_{1,0} = 0.25$ and $q_{2,0} = 0.2$, for the blue points, and $2\varepsilon q_1^* = 2 * 2.7 * 0.336 = 1.814$ for the red points.

5. Conclusion

In agreement with the results of the growing empirical and experimental literature (see e.g. [1, 2, 3, 4]), which highlights the chaotic behavior of the main variables involved in various markets, and in particular in agricultural commodity markets, we proposed a model able to generate interesting, erratic dynamic outcomes. In more detail, starting from the Cournot duopoly framework with quadratic emission charges and homogeneous goods in [6], we replaced the linear partial adjustment best response mechanism considered therein with a sigmoid adaptive best response rule, which, in addition to help avoid diverging trajectories and negativity issues, is also sensible from an economic viewpoint, being suitable to describe the gradual output variations caused by material, historical and institutional constraints in the production side of an economy, as well as by the limits imposed by an environmental policy scheme on production levels, due to their direct proportionality with emissions. Moreover, following the suggestion contained in the concluding section of [6], we assumed that firms produce differentiated goods. Beyond analytically studying the stability of the unique steady state, which coincides with the Nash equilibrium, and the effect produced by the main parameters on the stability region, we performed two comparative dynamics investigations which allow to evaluate the environmental policy efficacy when the Nash equilibrium is not stable and thus the standard comparative statics approach does not fit for the purpose. Involving non-stationary orbits, the proposed solutions are mainly numerical in nature. In particular, the first investigation, which is based on a comparison of emissions for different levels of charges, showed that, also when the Nash equilibrium is not stable, the considered environmental policy may be effective both with complements and substitutes. The second study, consisting in a comparison of emissions along non-stationary trajectories and along the equilibrium path, in the proposed experiments highlighted the presence of larger emissions along non-stationary trajectories. Hence, it gave us the opportunity to illustrate how an intervention on the sigmoid asymptotes may correspond to a direct control of emissions - in contrast to the indirect nature of the pollution control obtained by means of the considered emission charges - that also allows for a complete stabilization of the system, so that comparative statics results become economically grounded, starting from a situation characterized by the presence of a different attractor. In more detail, in making our numerical experiments, we have not only seen that the position of the asymptotes of the sigmoid is crucial in determining the system dynamics, but also that small variations in other parameters, such as the interdependence degree between goods, may generate important differences in the outcomes. In this respect we mention the work [37], which suggests that a particularly careful choice of the (e.g. fiscal or environmental) policy to implement is needed when dealing with nonlinear models in which complex dynamics and bifurcation phenomena can emerge.

We believe that the analyzed setting can be the starting point for other research works.

At first, we deem it essential to fully develop all dynamical aspects hidden inside the proposed model, in order to make it more realistic. Two possible extensions of the studied framework in such direction are represented respectively by the description of the environment as a sector interacting with the economic sphere and by the possibility of describing the transition among different market structures via an evolutive approach based on relative profitability of markets.

Regarding the former extension, in agreement with the seminal work [38], where the environment and its neglect are expressed through a dynamic equation, we could enrich the model by the introduction of one or more dynamic equations describing the evolution of the environment and its mutual interactions with the economic sector. In this manner, differently from the standard approach which depicts the environment in a parametric manner, it would be possible to deal with dynamical models consisting of coupled equations, in order to make explicit the effect of the economic activities on the evolution of the environment, as well as the impact of the environmental features on the economic activities, both in a direct manner, through consumption and production choices, and in an indirect way, through environmental policies. The addition of the dynamic equation(s) describing the environment evolution would make our “semi-dynamic” model fully dynamic and nonlinear. Usually, in that kind of models complex phenomena emerge, such as bifurcations, chaotic behavior, coexistence among different attractors. According to [39, 40], the environmental policy efficacy should be evaluated in those dynamic nonlinear models. In this respect, we stress that along the paper we measured the efficacy of the considered environmental policy in terms of its effectiveness in reducing emissions. Of course, a reduction in emissions is a consequence of an output decrease. A more general evaluation of an environmental policy scheme would require to deal with an oligopoly model that takes into account further variables, i.e., the factors of production, such as the employed labor (see e.g. [41]).

In regard to the latter extension, concerning the transition among different market structures, we start by recalling that [6] tackle the issue of the market structure endogeneity, focusing in particular on the conditions that may lead from duopoly to monopoly, investigated also in [10] under the assumption that marginal production costs do not coincide across firms. An alternative approach to the problem of the market structure endogeneity could be evolutive[19] in nature, with firms deciding whether to operate or not in a given market on the basis of a profitability signal, such as the comparison between the profitability

[19] See [42, 43] for an evolutive approach to environmental policy issues.

of the market with respect to the average profitability of other markets. In this manner the number of firms operating in a market would become an endogenous variable. Such approach would allow to more generally investigate the conditions which lead, possibly in a reversible manner, from a market structure to another one.
Different extensions of the proposed framework, which would be useful in view of testing the robustness of the here obtained results, could concern the formulation of the demand functions of firms and their technology heterogeneity in regard to emissions. In regard to the first point, following e.g. [34, 44], we might deal with nonlinear demand functions deriving from an underlying CES utility function. Regarding instead the second point, we recall that emissions per unit output not coinciding across firms have been considered e.g. in [20, 29]. We will investigate in a future work the effects of such more realistic assumption, not only in regard to the local stability analysis, but also from a policy viewpoint, so as to understand for instance whether the symmetries that we witnessed in the time series in Fig. 12 persist or not.

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