Rendiconti del Circolo Matematico di Palermo Series 2 (2023) 72:2515–2563
https://doi.org/10.1007/s12215-022-00764-5

Conic reductions for Hamiltonian actions of *U*(2) and its maximal torus

Roberto Paoletti[1]

Received: 28 February 2022 / Accepted: 7 April 2022 / Published online: 8 September 2022

Abstract
Suppose given a Hamiltonian and holomorphic action of $G = U(2)$ on a compact Kähler manifold M, with nowhere vanishing moment map. Given an integral coadjoint orbit $\mathcal{O}$ for G, under transversality assumptions we shall consider two naturally associated 'conic' reductions. One, which will be denoted $\overline{M}^G_{\mathcal{O}}$, is taken with respect to the action of G and the cone over $\mathcal{O}$; another, which will be denoted $\overline{M}^T_{\boldsymbol{\nu}}$, is taken with respect to the action of the standard maximal torus $T \leqslant G$ and the ray $\mathbb{R}_+\, \imath\boldsymbol{\nu}$ along which the cone over $\mathcal{O}$ intersects the positive Weyl chamber. These two reductions share a common 'divisor', which may be viewed heuristically as bridging between their structures. This point of view motivates studying the (rather different) ways in which the two reductions relate to the the latter divisor. In this paper we provide some indications in this direction. Furthermore, we give explicit transversality criteria for a large class of such actions in the projective setting, as well as a description of corresponding reductions as weighted projective varieties, depending on combinatorial data associated to the action and the orbit.

Keywords Holomorphic Hamiltonian action · Moment map · Linearization · Contact lift · Unitary group · Maximal torus · Symplectic reduction · Coadjoint orbit · Irreducible representation · Unit circle bundle · Symplectic divisor

Mathematics Subject Classification 53D20 · 17B08 · 32M05 · 57S25

1 Introduction

It is a classical fact in algebraic geometry that the quotient $M//\tilde{G}$ of a complex projective manifold M by the action of a connected and reductive group $\tilde{G}$ may be taken within the setting of Geometric Invariant Theory, by considering the subset $M^{ss} \subseteq M$ of so-called semistable points for the action, and declaring two orbits in M^{ss} to be equivalent if their closures intersect (on the subet of stable points, two orbits are equivalent if and only if they

✉ Roberto Paoletti
roberto.paoletti@unimib.it

[1] Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi 55, 20126 Milan, Italy

coincide). This construction depends on the choice of a linearization of the action, that is, the lifting to an ample line bundle A on M. It is also well-known that there is symplectic counter-part to this construction, which rests on the notion of Hamiltonian action and Marsden-Weinstein reduction. Namely, assuming that $\tilde{G}$ is the complexification of a compact and connected Lie group G that acts preserving a Hermitian metric on A, we can define a moment map for the action of G. One can then characterize semistable points for $\tilde{G}$ as those points in M with the property that the closure of the $\tilde{G}$-orbit intersects $\Phi^{-1}(0)$, and there is a natural identification of $\Phi^{-1}(0)/G$ with $M//\tilde{G}$. The Marsden-Weinstein reduction, or symplectic quotient, comes equipped with both a quotient symplectic structure and a curvature form associated to the principal G-bundle $\Phi^{-1}(0) \to \Phi^{-1}(0)/G$ (assuming that G acts freely on $\Phi^{-1}(0)$). For instance, if $G = S^1$ we obtain a 2-form on the quotient which in many interesting cases is also symplectic; if this is the case, since the latter curvature form is involved in the celebrated Duistermaat-Heckman formula, it seems suggestive to call the resulting symplectic manifold the Duistermaat-Heckman reduction of M. Obviously with no pretense of completeness, for a detailed discussion of the above we refer to [3, 5, 9, 13, 14].

In several circumstances, however, it happens that $\Phi^{-1}(0) = \emptyset$, and the previous approach may not be applied without altering the Hamiltonian structure of the action, i.e., the linearization. An alternative approach to obtaining geometrically interesting quotients consists in replacing, on the symplectic side, the usual Marsden-Weinstein reduction with reduction respect to different coisotropic loci in the coalgebra $\mathfrak{g}^\vee$ of K [6]. A natural choice in this setting is the cone over a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^\vee$; we shall call the corresponding quotients *conic reductions* (one should restrict to so-called integral orbits and impose suitable transversality assumptions to obtain tractable quotients).

For instance, in the special case where G is a compact torus a coadjoint orbit is a point in $\mathfrak{g}^\vee$ and the corresponding cone is the ray through that point. Then the corresponding quotient may interpreted as a Marsden-Weinstein reduction with respect to a certain subtorus of G, and a natural issue is then to describe how these quotients depend on the choice of ray.

The main aim of this paper is to provide a body of examples for this conic construction, and elucidate the geometry of the corresponding quotients, in the special cases where G is either $U(2)$ or its maximal torus. To give a more precise account, some terminology is in order.

Let M be a d-dimensional compact and connected Kähler manifold, with complex structure J, and Kähler form ω. By way of example, M might be complex projective space $\mathbb{P}^d$, and ω the Fubini-Study form.

Let us assume, in addition, that $G = U(2)$ and $\phi : G \times M \to M$ is a holomorphic and Hamiltonian action, with moment map $\Phi : M \to \mathfrak{g}^\vee$, where $\mathfrak{g} = \mathfrak{u}(2)$ is the Lie algebra of G (we refer to [8] for generalities on Hamiltonian actions and moment maps). For example, M might be $\mathbb{P}W$, where W is a complex unitary representation space for G, with the naturally associated G-action. We shall equivariantly identify $\mathfrak{g} \cong \mathfrak{g}^\vee$ by the inner product $\langle \beta_1, \beta_2 \rangle := \mathrm{trace}\big(\beta_1 \, \overline{\beta}_2^t\big)$; hence one can equivalently view Φ as being a $\mathfrak{g}$-valued equivariant map.

An important and ubiquitous geometric construction associated to Hamiltonian actions is the symplectic reduction with respect to an invariant submanifold $\mathcal{R} \subset \mathfrak{g}^\vee$, assuming that Φ is transverse to $\mathcal{R}$; the geometry of the action may lead to different choices of $\mathcal{R}$ ([5, 6]).

Here we shall assume that $\mathbf{0} \notin \Phi(M)$. In this situation, a natural choice for $\mathcal{R}$, suggested by geometric quantization, is the cone $\mathcal{C}(\mathcal{O}) = \mathbb{R}_+ \mathcal{O} \subset \mathfrak{g}^\vee$ over an integral coadjoint orbit $\mathcal{O}$ [6].

Example 1.1 To fix ideas on a specific case, consider the Hamiltonian G-space $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ associated to a unitary representation

$$W_{\mathbf{L},\mathbf{K}} := \bigoplus_{a=1}^{r} \det^{\otimes l_a} \otimes \operatorname{Sym}^{k_a}\left(\mathbb{C}^2\right), \tag{1}$$

where $\mathbf{L} = (l_a) \in \mathbb{Z}^r$, $\mathbf{K} = (k_a) \in \mathbb{N}^r$. Then $\mathbf{0} \notin \Phi(M)$ if and only if either $k_a + 2\,l_a > 0$ for all $a = 1, \ldots, r$, or $k_a + 2\,l_a < 0$ for all $a = 1, \ldots, r$ (see Proposition 2.5).

More explicitly (to be precise, with an extra genericity assumption on $W_{\mathbf{L},\mathbf{K}}$ - see Definition 2.2) the image of Φ is the convex hull of the subsets $\imath\, L_{k_a} + \imath\, l_a\, I_2 \subset \mathfrak{g}$, where L_{k_a} is the set of positive semidefinite Hermitian matrices of trace k_a, for $a = 1, \ldots, r$ (see (23) and Proposition 2.3). Furthermore, if $\boldsymbol{\nu} = (\, \nu_1 \quad \nu_2 \,) \in \mathbb{R}^2$ and $D_{\boldsymbol{\nu}}$ is the diagonal matrix with entries ν_1, ν_2, then $\imath\, D_{\boldsymbol{\nu}}$ belongs to the image of Φ if and only if $\boldsymbol{\nu}$ belongs to the convex hull of the all the vectors $(\, k_a + l_a \quad l_a \,)$ and $(\, l_a \quad k_a + l_a \,)$, for $a = 1, \ldots, r$ (Corollaries 2.8 and 2.9). In addition, if $\nu_1 \neq \nu_2$ then Φ is transverse to the cone over the orbit $\mathcal{O}_{\boldsymbol{\nu}}$ of $\imath\, D_{\boldsymbol{\nu}}$ if and only if $\boldsymbol{\nu}$ does not belong to the one of rays sprayed by the vectors $(\, k_a - j + l_a \quad j + l_a \,)$, for $a = 1, \ldots, r$ and $j = 0, \ldots, k_a$ (Theorem 2.5).

Assume that $\mathbf{0} \notin \Phi(M)$, that $\mathcal{O}$ is an integral orbit, and that Φ is transverse to $\mathcal{C}(\mathcal{O})$; then the (coisotropic, real) hypersurface $M^G_{\mathcal{O}} := \Phi^{-1}\big(\mathcal{C}(\mathcal{O})\big) \subset M$ is compact and connected (Theorem 1.2 of [4]). Let $\sim$ be the equivalence relation given by the null foliation. The symplectic reduction of M with respect to $\mathcal{C}(\mathcal{O})$ is $\overline{M}^G_{\mathcal{O}} := M^G_{\mathcal{O}}/\sim$, together with its naturally induced reduced orbifold symplectic structure $\omega_{\overline{M}^G_{\mathcal{O}}}$. We shall refer to $(\overline{M}^G_{\mathcal{O}},\, \omega_{\overline{M}^G_{\mathcal{O}}})$ as *the conic reduction of M with respect to G and $\mathcal{O}$.*

There are other reductions associated to the integral orbit $\mathcal{O}$ built into this picture. Let $T \leqslant G$ be the maximal torus of diagonal unitary matrices, and $\psi : T \times M \to M$ the restricted action. Then ψ is also Hamiltonian; let $\Psi : M \to \mathfrak{t} \cong \mathfrak{t}^\vee$ be its moment map. We shall identify $\mathfrak{t}$ with $\imath\,\mathbb{R}^2$.

Assume that $\mathbf{0} \notin \Psi(M)$ (this is in principle a stronger hypothesis than $\mathbf{0} \notin \Phi(M)$), and that Ψ is transverse to a ray $\mathbb{R}_+ \cdot \imath\,\boldsymbol{\nu}$, where $\boldsymbol{\nu} = (\, \nu_1 \quad \nu_2 \,) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}$. Let us set $\boldsymbol{\nu}_\perp := (\, -\nu_2 \quad \nu_1 \,) \in \mathbb{Z}^2$. Let $T^1_{\boldsymbol{\nu}_\perp} \leqslant T$ be the subgroup generated by $\imath\,\boldsymbol{\nu}_\perp$. If non-empty, $M^T_{\boldsymbol{\nu}} := \Psi^{-1}(\mathbb{R}_+ \cdot \imath\,\boldsymbol{\nu})$ is then a connected compact hypersurface in M, whose null foliation $\sim'$ is given by the orbits of $T^1_{\boldsymbol{\nu}_\perp}$.

The quotient $\overline{M}^T_{\boldsymbol{\nu}} = M^T_{\boldsymbol{\nu}}/\sim'$ is then also an orbifold, with a reduced Kähler structure $(\overline{M}^T_{\boldsymbol{\nu}}, J_0, \Omega_0)$, which can be viewed as the symplectic quotient (symplectic reduction at 0) for the Hamiltonian action of $T^1_{\boldsymbol{\nu}_\perp}$ on M. We shall refer to $(\overline{M}^T_{\boldsymbol{\nu}}, J_0, \Omega_0)$ as the *conic reduction of M with respect to T and $\boldsymbol{\nu}$.*

The two hypersurfaces $M^G_{\mathcal{O}}$ and $M^T_{\boldsymbol{\nu}}$ meet tangentially along the smooth connected locus $M^G_{\boldsymbol{\nu}} := \Phi^{-1}(\mathbb{R}_+ \cdot \imath\,\boldsymbol{\nu})$ (Theorem 1.2 of [4] - in *loc. cit.* M was assumed to be projective, but Theorem 1.2 holds true in the Kähler setting). Furthermore, the null foliations of $M^G_{\mathcal{O}}$ and $M^T_{\boldsymbol{\nu}}$ are tangent to $M^G_{\boldsymbol{\nu}}$ since the latter is T-invariant, and they actually coincide along it.

Therefore, the quotient $\overline{M}_{\boldsymbol{\nu}}^{G} := M_{\boldsymbol{\nu}}^{G}/\sim$ is an orbifold. $\overline{M}_{\boldsymbol{\nu}}^{G}$ has an intrinsic symplectic structure $\omega_{\overline{M}_{\boldsymbol{\nu}}^{G}}$, and in fact $\left(\overline{M}_{\boldsymbol{\nu}}^{G}, \omega_{\overline{M}_{\boldsymbol{\nu}}^{G}}\right)$ can be interpreted as a symplectic quotient of a symplectic cross section for the G-action, in the sense of [7]. Furthermore, $\left(\overline{M}_{\boldsymbol{\nu}}^{G}, \omega_{\overline{M}_{\boldsymbol{\nu}}^{G}}\right)$ embeds symplectically in both $(\overline{M}_{\boldsymbol{\nu}}^{T}, \Omega_0)$ and $(\overline{M}_{\mathcal{O}}^{G}, \omega_{\overline{M}_{\mathcal{O}}^{G}})$. Hence, $\overline{M}_{\boldsymbol{\nu}}^{G}$ can be viewed as bridging between $\overline{M}_{\mathcal{O}}^{G}$ and $\overline{M}_{\boldsymbol{\nu}}^{T}$. This heuristic point of view motivates investigating $\overline{M}_{\mathcal{O}}^{G}$ and $\overline{M}_{\boldsymbol{\nu}}^{T}$ in relation to $\overline{M}_{\boldsymbol{\nu}}^{G}$.

Regarding $\overline{M}_{\mathcal{O}}^{G}$, we shall prove that in a large class of cases the symplectic orbifold $(\overline{M}_{\mathcal{O}}^{G}, \omega_{\overline{M}_{\mathcal{O}}^{G}})$ factors as the product of $(\overline{M}_{\boldsymbol{\nu}}^{G}, \omega_{\overline{M}_{\boldsymbol{\nu}}^{G}})$ and $\mathbb{P}^1$, endowed with a suitable rescaling of the Fubini-Study form (Theorem 4.1). In the more general situation, $\overline{M}_{\mathcal{O}}^{G}$ is still, in some sense, topologically close to being a product (Theorem 4.2).

Regarding $\overline{M}_{\boldsymbol{\nu}}^{T}$, we shall see that $\overline{M}_{\boldsymbol{\nu}}^{G}$ embeds in it as the zero locus of a transverse section of an orbifold line bundle L; this section is naturally associated to the moment map (Theorem 3.1). The curvature of L is the form Ω_0' introduced in [3] to study the variation of the cohomology class of a symplectic reduction, namely, the curvature to the orbifold S^1-bundle $M_{\boldsymbol{\nu}}^{T} \to \overline{M}_{\boldsymbol{\nu}}^{T}$ (stricly speaking, Ω_0' is not uniquely defined as a form, but in our context there will be a natural choice). If Ω_0' is symplectic and there exists an orbifold complex structure on $\overline{M}_{\boldsymbol{\nu}}^{T}$ compatible with Ω_0', we shall call the triple $\left(\overline{M}_{\boldsymbol{\nu}}^{T}, J_0', \Omega_0'\right)$ the *$\boldsymbol{\nu}$-th DH-conic reduction of M*.

We shall see that this is the case for the spaces $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ in Example 1.1. More precisely, we shall classify the corresponding DH-reductions and explicitly describe them as Kähler weighted projective varieties parametrized by certain combinatoric data depending on $\boldsymbol{\nu}$, $\mathbf{L}$, $\mathbf{K}$. In these cases L is an ample orbifold line bundle on M (Theorem 3.2). Furthermore, for a class of representations that we call *uniform* (Definition 2.3) the complex orbifold $\left(\overline{M}_{\boldsymbol{\nu}}^{T}, J_0'\right)$ remains constant as $\boldsymbol{\nu}$ ranges within one of the fundamental wedges cut out by the 'critical rays' (see Example 1.1).

Finally, we shall focus on the specific case of the irreducible representations $\mathrm{Sym}^k(\mathbb{C}^2)$. We shall see that if $\nu_1 > (k-1)\,\nu_2 > 0$ then $\overline{M}_{\boldsymbol{\nu}}^{T}$ is the weighted projective space $\mathbb{P}(1, 2, \ldots, k)$, and that if $\nu_1 \gg \nu_2 > 0$ (the bounds might be made effective and depend on k) then $\overline{M}_{\boldsymbol{\nu}}^{G}$ is smoothly and symplectically isotopic to $\mathbb{P}(2, \ldots, k) \subset \mathbb{P}(1, 2, \ldots, k)$ (Theorem 3.3).

In closing, we recall that in the usual Marsden-Weinstein setting the relation between the symplectic quotients with respect to a connected compact Lie group and to its maximal torus has been elucidated in a very terse and precise manner by the theory in [12]; in particular, it is proved that the two quotients are related by 'a fibration and an inclusion', and building on this the connection between their topological properties is investigated. Here clearly no comparably general and conclusive results are given, not even in the special case where $G = U(2)$; nonetheless, the present discussion points to a geometric relation of a rather different nature between the corresponding two quotients in the present conic setting, and to the bridging role of the symplectic divisor $\overline{M}_{\boldsymbol{\nu}}^{G}$. In this perspective, the emphasis on the $\boldsymbol{\nu}$-th DH-conic reduction of M is motivated by the fact that $\overline{M}_{\boldsymbol{\nu}}^{G}$ is the zero locus of a $\mathcal{C}^\infty$ section of a complex orbifold line bundle on $\overline{M}_{\boldsymbol{\nu}}^{T}$ with curvature Ω_0'.

2 Transversality criteria

In this section we shall provide some general transversality criteria involving the moment map $\Phi : M \to \mathfrak{g}^\vee$ and a cone $\mathcal{C}(\mathcal{O})$ over a coadjoint orbit in the case of Hamiltonian G-actions associated to unitary G-representations. We shall equivariantly identify $\mathfrak{g} \cong \mathfrak{g}^\vee$ and $\mathfrak{t} \cong \mathfrak{t}^\vee$.

Let $\mathcal{C} := (e_1, e_2)$ be the standard basis of $\mathbb{C}^2$. For any $k = 1, 2, \ldots$, $W_k := \mathrm{Sym}^k(\mathbb{C}^2)$ has an Hermitian structure naturally induced from the standard one of $\mathbb{C}^2$. An orthonormal basis of W_k may be taken $\mathcal{B}_k = (E_{k,j})$, where

$$E_{k,j} := c_{k,j}\, e_1^{k-j}\, e_2^{j}, \quad c_{k,j} := \sqrt{\frac{(k+1)!}{\pi\, j!\,(k-j)!}}, \quad j = 0, 1, \ldots, k. \tag{2}$$

By means of $\mathcal{B}_k$, we shall unitarily identify $W_k \cong \mathbb{C}^{k+1}$, and a point $w = \sum_{j=0}^{k} z_j\, E_{k,j} \in W_k$ with $Z = (z_j)_{j=0}^{k} \in \mathbb{C}^{k+1}$.

Consider the unitary representation $\mu = \mu_1$ of $G = U(2)$ on $W_1 := \mathbb{C}^2$ given by $B \mapsto (B^t)^{-1}$ with respect to $\mathcal{C}$. Then μ_1 naturally induces for every k a unitary representation of G on W_k, which we may regard (given $\mathcal{B}_k$) as a a Lie group homomorphism $\mu_k : G \to U(k+1)$, with derivative $\mathrm{d}\mu_k : \mathfrak{g} \to \mathfrak{u}(k+1)$. Consequently, we have an induced holomorphic Hamiltonian action ϕ_k of G on $\mathbb{P}^k = \mathbb{P}(W_k)$ with respect to $2\,\omega_{FS}$ (here ω_{FS} is the Fubini-Study form); let us compute its moment map $\Phi_k : \mathbb{P}^k \to \mathfrak{g}$.

Let us set for simplicity $E_j = E_{k,j}$. We have for $\alpha \in \mathfrak{g}$

$$\begin{aligned} \mathrm{d}\mu_k(\alpha)(E_j) = &- \sqrt{j\,(k-j+1)}\,\alpha_{21}\, E_{j-1} \\ &- \big[(k-j)\,\alpha_{11} + j\,\alpha_{22}\big]\, E_j \\ &- \sqrt{(k-j)\,(j+1)}\,\alpha_{12}\, E_{j+1}. \end{aligned} \tag{3}$$

Hence the only non-zero entries of $\mathrm{d}\mu_k(\alpha)$ are

$$\begin{aligned} \mathrm{d}\mu_k(\alpha)_{j-1,j} &= - \sqrt{j\,(k-j+1)}\,\alpha_{21}, \\ \mathrm{d}\mu_k(\alpha)_{j,j} &= - \big[(k-j)\,\alpha_{11} + j\,\alpha_{22}\big], \\ \mathrm{d}\mu_k(\alpha)_{j+1,j} &= - \sqrt{(k-j)\,(j+1)}\,\alpha_{12} \end{aligned} \tag{4}$$

for $j = 0, \ldots, k$. For $Z = (z_0, \ldots, z_k)^t \in \mathbb{C}^{k+1}$, let us define the Hermitian matrix $(Z \odot \overline{Z})_{ij} := z_i\, \overline{z}_j$. As is well-known, the moment map for the action of $U(k+1)$ on $(\mathbb{P}^k, 2\,\omega_{FS})$, $\Gamma : \mathbb{P}^k \to \mathfrak{u}(k+1)$, is

$$\Gamma([Z]) := -\frac{\imath}{\|Z\|^2}\, Z \odot \overline{Z}. \tag{5}$$

Given (4) and (5), one obtains by standard arguments that the entries Φ_{ij} are given by

$$
\begin{aligned}
(\Phi_k)_{11}([Z]) &= \frac{\imath}{\|Z\|^2} \sum_{j=0}^{k} (k-j)\,|z_j|^2,\\
(\Phi_k)_{12}([Z]) &= \frac{\imath}{\|Z\|^2} \sum_{j=0}^{k-1} \sqrt{(k-j)\,(j+1)}z_{j+1}\,\overline{z}_j,\\
(\Phi_k)_{21}([Z]) &= \frac{\imath}{\|Z\|^2} \sum_{j=1}^{k} \sqrt{j\,(k-j+1)}\,z_{j-1}\,\overline{z}_j,\\
(\Phi_k)_{22}([Z]) &= \frac{\imath}{\|Z\|^2} \sum_{j=0}^{k} j\,|z_j|^2.
\end{aligned}
\tag{6}
$$

We can reformulate this in a more compact form, as follows. Let us define $F_{k,a} : \mathbb{C}^{k+1} \to \mathbb{C}^k$ for $a = 1,2$ by setting

$$
F_{k,1}(Z) := \begin{pmatrix} \sqrt{k}\,z_0 \\ \sqrt{k-1}\,z_1 \\ \vdots \\ z_{k-1} \end{pmatrix} = \left(\sqrt{k-j+1}\,z_{j-1}\right)_{j=1}^{k}, \tag{7}
$$

$$
F_{k,2}(Z) := \begin{pmatrix} z_1 \\ \sqrt{2}\,z_2 \\ \vdots \\ \sqrt{k}z_k \end{pmatrix} = \left(\sqrt{j}\,z_j\right)_{j=1}^{k}. \tag{8}
$$

Then

$$
\Phi_k([Z]) = \frac{\imath}{\|Z\|^2} \begin{pmatrix} \|F_{k,1}(Z)\|^2 & F_{k,2}(Z)^t\,\overline{F_{k,1}(Z)} \\ F_{k,1}(Z)^t\,\overline{F_{k,2}(Z)} & \|F_{k,2}(Z)\|^2 \end{pmatrix}. \tag{9}
$$

Definition 2.1 Let $k \geq 1$. We shall denote by L'_k the set of all positive semidefinite Hermitian matrices of trace k and rank 1; thus L'_1 is the set of orthogonal projectors onto a 1-dimensional vector subspace of $\mathbb{C}^2$, and $L'_k = k\,L'_1$. Similarly, L_k will denote the set of all 2×2 Hermitian positive semidefinite matrices of trace k.

In particular, L_k is the convex hull of L'_k, and $L_k = k\,L_1$.

Proposition 2.1 $\Phi_1\left(\mathbb{P}^1\right) = \imath\,L'_1$. *If* $k \geq 2$, $\Phi_k\left(\mathbb{P}^k\right) = \imath\,L_k$.

Proof For $k = 1$, (9) specializes to

$$
\Phi_1([Z]) = \frac{\imath}{\|Z\|^2} \begin{pmatrix} |z_0|^2 & z_1\,\overline{z_0} \\ z_0\,\overline{z_1} & |z_1|^2 \end{pmatrix}, \tag{10}
$$

which implies the first statement.

Let us then assume $k \geq 2$. It is evident from (6) and (9) that $\Phi_k(\mathbb{P}^k) \subseteq \imath L_k$. Since $\Phi_k(\mathbb{P}^k)$ is G-invariant in view of the G-equivariance of Φ_k, to prove the reverse inclusion it suffices to show that for any $\lambda \in [0, k]$ we have

$$\imath \begin{pmatrix} \lambda & 0 \\ 0 & k-\lambda \end{pmatrix} \in \Phi_k(\mathbb{P}^k).$$

To this end, we need only set $z_0 = \sqrt{\lambda/k}$, $z_j = 0$ for $j = 1, \ldots, k-1$, $z_k = \sqrt{(k-\lambda)/k}$.

If $\boldsymbol{v} = (\, v_1 \quad v_2 \,)^t \in \mathbb{R}^2$, we shall denote by $D_{\boldsymbol{v}}$ the diagonal matrix with entries v_1, v_2 and by $\mathcal{O}_{\boldsymbol{v}} \subset \mathfrak{g}$ the orbit of $\imath D_{\boldsymbol{v}}$.

Also, let us set

$$J_k := \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \,:\, v_1,\, v_2 \geq 0,\, v_1 + v_2 = k \right\}, \quad J_{k+} := \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in J_k \,:\, v_1 \geq v_2 \right\}.$$

In other words, J_k is the segment joining the points $(\, k \quad 0 \,)^t$, $(\, 0 \quad k \,)^t \in \mathbb{R}^2$.

Corollary 2.1 *In the situation of Proposition 2.1,* $\Phi_1(\mathbb{P}^1) = \mathcal{O}_{\boldsymbol{\epsilon}_1}$, *where* $\boldsymbol{\epsilon}_1 = (\, 1 \quad 0 \,)$, *while for any* $k \geq 2$

$$\Phi_k(\mathbb{P}^k) = \bigcup_{\boldsymbol{v} \in J_k} \mathcal{O}_{\boldsymbol{v}} = \bigcup_{\boldsymbol{v} \in J_{k+}} \mathcal{O}_{\boldsymbol{v}}. \tag{11}$$

In particular, if $\boldsymbol{v} \neq \boldsymbol{0}$ *and* $v_1 \geq v_2$, *then* $\Phi_k(\mathbb{P}^k) \cap \mathcal{C}(\mathcal{O}_{\boldsymbol{v}}) \neq \emptyset$ *if and only if* $v_2 \geq 0$.

The second equality in (11) follows from the fact that if $\boldsymbol{v} = (\, v_1 \quad v_2 \,)^t$ and $\boldsymbol{v}' = (\, v_2 \quad v_1 \,)^t$, then $\mathcal{O}_{\boldsymbol{v}} = \mathcal{O}_{\boldsymbol{v}'}$.

Let us denote by ψ_k the restricted action of T on $\mathbb{P}^k$, and by $\Psi_k : M \to \mathfrak{t}^\vee \cong \mathfrak{t}$ its moment map. Then Ψ_k is the composition of Φ_k with the orthogonal projection $\pi : \mathfrak{g} \to \mathfrak{t}$; the latter amounts to selecting the diagonal component of a matrix in $\mathfrak{g}$.

Corollary 2.2 *For any* $k \geq 1$, $\Psi_k(\mathbb{P}^k) = \imath J_k \subset \imath \mathbb{R}^2$.

Proof of Corollary 2.2 For $k = 1$, this is immediate from (10). Assume then $k \geq 2$. Any matrix in L_k has diagonal part in J_k, hence $\Psi_k(\mathbb{P}^k) \subseteq \imath J_k \subset \imath \mathbb{R}^2$ by Proposition 2.1. Conversely, for any $\boldsymbol{\lambda} := (\, \lambda \quad k-\lambda \,)^t \in J_k$ in the proof of Proposition 2.1 we have found $[Z] \in \mathbb{P}^k$ such that $\Phi([Z]) = \imath D_{\boldsymbol{\lambda}}$. Hence $\Psi_k([Z]) = \imath \boldsymbol{\lambda}$.

Let us notice the following consequence of Proposition 2.1, due to the fact the diagonal part of a matrix in L_k is in L_k:

Corollary 2.3 *For any* $k \geq 2$, $\Psi_k(\mathbb{P}^k) = \Phi_k(\mathbb{P}^k) \cap \mathfrak{t}$.

Proof of Corollary 2.3 Obviously $\Psi_k(\mathbb{P}^k) \supseteq \Phi_k(\mathbb{P}^k) \cap \mathfrak{t}$. Conversely, suppose $\alpha \in \Psi_k(\mathbb{P}^k)$. Viewing α as the diagonal component of a matrix $\alpha' \in \Phi_k(\mathbb{P}^k)$, we conclude that $-\imath\,\alpha$ has non-negative (diagonal) entries and trace k. Hence $\alpha \in \imath L_k = \Phi_k(\mathbb{P}^k)$.

Having characterized the images of Φ_k and Ψ_k, let us determine the orbital cones to which they are transverse. By Corollary 2.1 we may assume $k \geq 2$.

Theorem 2.1 *Assume that $k \geq 2$, $v_1,\ v_2 \geq 0$ and $v_1 \neq v_2$. Then the following conditions are equivalent*:

1. Φ_k is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}})$;
2. $j\,v_1 \neq (k-j)\,v_2$ for all $j \in \{0, 1, \ldots, k\}$.

Remark 2.1 Since $\Phi_k(\mathbb{P}^k) = \imath\,L_k$, if $\boldsymbol{v} = \pm(\,1 \quad -1\,)$ then $\Phi_k(\mathbb{P}^k) \cap \imath\,\mathbb{R}_+ \cdot \boldsymbol{v} = \emptyset$, hence we may assume $v_1 + v_2 \neq 0$. Furthermore, $\Phi_k(\mathbb{P}^k)$ is G-invariant and if $\boldsymbol{v}' := (\,v_2 \quad v_1\,)$ then the matrices the diagonal matrices $\imath\,D_{\boldsymbol{v}}$ and $\imath\,D_{\boldsymbol{v}'}$ belong to the same orbit. We may assume therefore $v_1 \geq v_2$, hence - under the hypothesis of Theorem 2.1 - that $v_1 > v_2$.

Proof of Theorem 2.1 Let $X_k = S^{2k+1}$ be viewed as the unit circle bundle of the tautological line bundle on $\mathbb{P}^k = \mathbb{P}(W_k)$, with projection $\pi_k : X_k \to \mathbb{P}^k$ (the Hopf map), and let us set

$$(X_k)^G_{\boldsymbol{v}} := \pi_k^{-1}\Big(\mathbb{P}(W_k)^G_{\boldsymbol{v}}\Big), \quad (X_k)^G_{\mathcal{O}} := \pi_k^{-1}\Big(\mathbb{P}(W_k)^G_{\mathcal{O}}\Big).$$

Since ϕ_k is induced by the unitary representation μ_k on W_k, there is by restriction of μ_k a natural lift of ϕ_k to an action on X_k, which we shall denote $\tilde{\phi}_k$. We shall also set $\tilde{\Phi}_k := \Phi_k \circ \pi_k : X_k \to \mathfrak{g},\ Z \mapsto \Phi_k([Z])$.

By the discussions in §2.2 of [15] and §4.1.1 of [4], Φ_k is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}})$ if and only if $\tilde{\phi}_k$ is locally free on $(X_k)^G_{\mathcal{O}}$; furthermore, since $(X_k)^G_{\mathcal{O}}$ is the G-saturation of $(X_k)^G_{\boldsymbol{v}}$, the latter condition is in turn equivalent to $\tilde{\phi}_k$ being locally free along $(X_k)^G_{\boldsymbol{v}}$.

For any $\beta \in \mathfrak{g}$, let $\beta_{X_k} \in \mathfrak{X}(X_k)$ denote the associated vector field on X_k. For any $Z \in X_k$, let $\mathfrak{g}_{X_k}(Z) \subseteq T_Z X_k$ denote the vector subspace given by the evaluations of all the β_{X_k}'s at Z, and similarly for $\mathfrak{t}$. Then $\tilde{\phi}$ is locally free at Z if and only if the evaluation map $\mathrm{val}_Z : \mathfrak{g} \to T_Z X_k,\ \beta \mapsto \beta_{X_k}(Z)$, has maximal rank, that is, $\mathfrak{g} \cong \mathfrak{g}_{X_k}(Z)$.

Let us prove that 2.) implies 1.). Let us remark that 2.) can be equivalently reformulated as follows:

$$v_1 \cdot v_2 \neq 0 \quad \text{and} \quad v_1 \neq \frac{k-j}{j}\,v_2, \quad \text{for all} \quad j = 1, \ldots, k-1. \tag{12}$$

Let us consider $Z = (z_0, \ldots, z_k)^t \in (X_k)^G_{\boldsymbol{v}}$, so that

$$\tilde{\Phi}_k(Z) = \imath \begin{pmatrix} \|F_{k,1}(Z)\|^2 & F_{k,2}(Z)^t\,\overline{F_{k,1}(Z)} \\ F_{k,1}(Z)^t\,\overline{F_{k,2}(Z)} & \|F_{k,2}(Z)\|^2 \end{pmatrix} = \imath\,\frac{k}{v_1 + v_2}\begin{pmatrix} v_1 & 0 \\ 0 & v_2 \end{pmatrix}.$$

In particular,

$$v_2\,\|F_{k,1}(Z)\|^2 = v_1\,\|F_{k,2}(Z)\|^2. \tag{13}$$

Lemma 2.1 *Given (12), for any $Z \in (X_k)^G_{\boldsymbol{v}}$ there exist $j, l \in \{0, 1, \ldots, k\}$ with $j \neq l$ and $z_j \cdot z_l \neq 0$.*

Proof of Lemma 2.1 If not, Z has only one non-zero component, say $z_j \in S^1$. Since by (12) and (13) $F_1(Z), F_2(Z) \neq \mathbf{0}$, we need to have $0<j<k$ in view of the definition of F_j. We conclude again by (13) that $v_2\,(k-j) = v_1\,j$ for some $j = 1, \ldots, k-1$, against the assumption.

Let $D \in T \leqslant G$ be a diagonal matrix with entries $e^{\imath\,\vartheta_1}, e^{\imath\,\vartheta_2} \in S^1$. By definition of ϕ and of the E_a's in (2), we have with $Z = (z_a)_{a=0}^k$

$$\tilde{\phi}_D(Z) = \left(e^{-\imath\,[(k-a)\,\vartheta_1 + a\,\vartheta_2]}\,z_a\right).$$

Now suppose that D is close to I_2, so that we may assume $\vartheta_1, \vartheta_2 \sim 0$, and that D fixes Z. Then $e^{\imath\,[(k-a)\,\vartheta_1 + a\,\vartheta_2]}\,z_a = z_a$ for every $a = 0, \ldots, k$ implies in particular $(k-j)\,\vartheta_1 + j\,\vartheta_2 = (k-l)\,\vartheta_1 + l\,\vartheta_2 = 0$, and so $\vartheta_1 = \vartheta_2 = 0$. Hence, there is a neighborhood $T' \subseteq T$ of I_2 such that the only $D \in T'$ that fixes Z is I_2. In other words, T acts locally freely on $(X_k)_{\mathbf{v}}^G$ at Z. In particular, $\mathrm{val}_Z : \mathfrak{t} \to T_Z X_k$ is injective.

By the equivariance of Φ, for any $W \in X_k$ and $\beta \in \mathfrak{g}$ we have

$$\mathrm{d}_W \tilde{\Phi}\left(\beta_{X_k}(W)\right) = \left[\beta, \tilde{\Phi}(W)\right]. \tag{14}$$

Hence if $\beta \in \mathfrak{t} \subset \mathfrak{g}$ and $Z \in (X_k)_{\mathbf{v}}^G$ then $\mathrm{d}_Z \tilde{\Phi}\left(\beta_{X_k}(Z)\right) = 0$; that is,

$$\mathfrak{t}_{X_k}(Z) \subseteq \ker\left(\mathrm{d}_Z \tilde{\Phi}\right) \qquad (Z \in (X_k)_{\mathbf{v}}^G). \tag{15}$$

Now let us define

$$\eta := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \xi := \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix}, \quad \mathfrak{a} := \mathrm{span}(\eta, \xi) \subset \mathfrak{g}, \tag{16}$$

so that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{t}$. By (14) we have at $Z \in (X_k)_{\mathbf{v}}^G$:

$$\mathrm{d}_Z \tilde{\Phi}\left(\xi_{X_k}(Z)\right) = \frac{k\,(v_1 - v_2)}{v_1 + v_2}\,\eta, \quad \mathrm{d}_Z \tilde{\Phi}\left(\eta_{X_k}(Z)\right) = -\frac{k\,(v_1 - v_2)}{v_1 + v_2}\,\xi. \tag{17}$$

Let us set

$$\rho := \begin{pmatrix} \imath & 0 \\ 0 & 0 \end{pmatrix}, \quad \gamma := \begin{pmatrix} 0 & 0 \\ 0 & \imath \end{pmatrix}.$$

Then (ρ, γ) is a basis for $\mathfrak{t}$, and $(\eta, \xi, \rho, \gamma)$ is a basis for $\mathfrak{g}$.

Suppose that for some $x, y, z, t \in \mathbb{R}$ we have $x\,\eta + y\,\xi + z\,\rho + t\,\gamma \in \ker(\mathrm{val}_Z)$:

$$x\,\eta_{X_k}(Z) + y\,\xi_{X_k}(Z) + z\,\rho_{X_k}(Z) + t\,\gamma_{X_k}(Z) = 0. \tag{18}$$

Applying $\mathrm{d}_Z\Phi$, we get by (15) and (17):

$$\begin{aligned} 0 &= x\,\mathrm{d}_Z \tilde{\Phi}\left(\eta_{X_k}(Z)\right) + y\,\mathrm{d}_Z \tilde{\Phi}\left(\xi_{X_k}(Z)\right) \\ &= \frac{k\,(v_1 - v_2)}{v_1 + v_2}\,(-x\,\xi + y\,\eta). \end{aligned} \tag{19}$$

Hence $x = y = 0$, so that $z\,\rho_{X_k}(Z) + t\,\gamma_{X_k}(Z) = 0$. But this means that $z\,\rho + t\,\gamma \in \ker\left(\mathrm{val}_Z|_{\mathfrak{t}}\right) = (0)$; thus we also have $z = t = 0$. We conclude that $\ker(\mathrm{val}_Z) = (0)$ for any $Z \in (X_k)_{\mathbf{v}}^G$, as claimed.

Now let us suppose instead that (12) does not hold. We aim to show that then $\tilde{\phi}$ is not everywhere locally free along $(X_k)^G_{\mathbf{v}}$. If $v_2 = 0$, let $Z := (1 \quad 0 \quad \cdots \quad 0)^t$. Then $\tilde{\Phi} = \imath D$, where D is the diagonal matrix with diagonal entries $(k \quad 0)$; hence $Z \in (X_k)^G_{\mathbf{v}}$. On the other hand, Z is fixed by the 1-dimensional subgroup of G of diagonal matries with diagonal entries $\left(1 \quad e^{\imath\vartheta}\right)$, hence $\tilde{\phi}$ is not free at Z. One argues similarly when $v_1 = 0$, by choosing instead $Z := (0 \quad \cdots \quad 0 \quad 1)^t$. If instead $v_1 \cdot v_2 \neq 0$, then $v_1 = [(k-j)/j]\, v_2$ for some $j = 1, \ldots, k-1$. Let us consider $Z = (z_l)$ with $z_l = \delta_{lj}$, $l = 0, \ldots, k$. Then by (9) $Z \in (X_k)^G_{\mathbf{v}}$. On the other hand now Z is fixed by the 1-dimensional subgroup of diagonal matrices with diagonal entries $\left(e^{-\imath j\vartheta} \quad e^{\imath(k-j)\vartheta}\right)$, hence again $\tilde{\phi}$ is not free at Z.

Let us note in passing that the argument in the proof of Theorem 2.1 can be phrased in slightly more general terms and actally establishes the following criterion.

Lemma 2.2 *Suppose that (M, J) is a complex projective manifold, with ω a Hodge form on it, associated to a positive line bundle (A, h). Let $\phi : G \times M \to M$ be a holomorphic Hamiltonian action on $(M, 2\,\omega)$, with moment map $\Phi : M \to \mathfrak{g}$. Let $X \subset A^\vee$ be the unit circle bundle, with projection $\pi : X \to M$, and assume that there is a contact lift $\tilde{\phi} : G \times X \to X$ of the Hamiltonian action (ϕ, Φ). Suppose $v_1 \neq v_2$, $x \in X$, $\Phi \circ \pi(x) \in \mathbb{R}_+ \cdot \imath D_{\mathbf{v}}$, and that T acts locally freely at x. Then G acts locally freely at x.*

Corollary 2.4 *In the situation of Lemma 2.2, assume in addition that T acts locally freely along the inverse image $X^G_{\mathbf{v}}$ of $M^G_{\mathbf{v}}$ in X. Then Φ is transverse to $\mathcal{C}(\mathcal{O}_{\mathbf{v}})$.*

Next we shall consider the transversality issue for Ψ_k.

Theorem 2.2 *For any $k \geq 1$, Ψ_k is not transverse to a ray $\mathbb{R}_+ \mathbf{v} \subset \imath\,\mathfrak{t} \cong \mathbb{R}^2$ if and only if $\mathbf{v}$ is a positive multiple of $(k-j \quad j)^t$ for some $j = 0, 1, \ldots, k$.*

In other words, the critical rays are those through the points in the intersection $J \cap \mathbb{Z}^2$, up to the factor $\imath$.

Proof of Theorem 2.2 Let $\tilde{\psi}_k$ denote the action of T on X_k. As argued in the proof of Theorem 2.1, $\tilde{\psi}_k$ is not locally free at $Z = (z_l) \in X$ if and only if $|z_l| = \delta_{lj}$ for some $j = 0, \ldots, k$. Hence the rays in $\mathfrak{t}$ to which Ψ is not transverse are those through the images under Ψ if the vectors of the standard basis of $\mathbb{C}^{k+1}$. As we have remarked, their images under $\tilde{\Phi}_k$ form the set

$$\left\{\imath \begin{pmatrix} k-j & 0 \\ 0 & j \end{pmatrix} : j = 0, \ldots, k\right\},$$

and we need only take the diagonal part to reach the claimed conclusion. $\square$

Let us now extend the previous considerations to a general irreducible representation of G (see e.g. §2.3 [18], or §II.5 of [2]). More precisely, we shall denote by $\mu_{k,l}$ the composition of the representation $\det^{\otimes l} \otimes \mathrm{Sym}^k\left(\mathbb{C}^2\right)$ with the Lie group automorphism $B \mapsto (B^t)^{-1}$:

$$(\mu_{k,l})_B(v) := \det(B)^{-l}\, \mu_{k\,(B^t)^{-1}}(v) \quad \left(B \in G,\ v \in W_k \cong \mathbb{C}^{k+1}\right). \tag{20}$$

The induced action $\phi_{k,l}$ on $\mathbb{P}^k$ equals ϕ_k; however, the change in linearization implies a change in the moment map. For any $l \in \mathbb{Z}$, $\mu_{0,l}$ is the representation on $\mathbb{C}$ given by the character $\det^{-l}$. In this case, $\mathbb{P}^0 = \{[1]\}$ is just a point, and we shall take as definition of moment map the function $\Phi_{0,l} : [1] \mapsto \imath\, l\, I_2$. For $k \geq 1$, let us view $\mu_{k,l}$ as a Lie group morphism $G \to U(k+1)$. Then, in place of (3), we have for $\alpha \in \mathfrak{g}$

$$\begin{aligned} \mathrm{d}\mu_{k,l}(\alpha)(E_j) = & -\sqrt{j\,(k-j+1)}\,\alpha_{21}\,E_{j-1} \\ & -\left[l\,\mathrm{trace}(\alpha) + (k-j)\,\alpha_{11} + j\,\alpha_{22}\right] E_j \\ & -\sqrt{(k-j)\,(j+1)}\,\alpha_{12}\,E_{j+1}. \end{aligned} \tag{21}$$

It follows that the new moment map, $\Phi_{k,l} : \mathbb{P}^k \to \mathfrak{g}$, is given by

$$\Phi_{k,l}([Z]) := \Phi_k([Z]) + \imath\, l\, I_2, \tag{22}$$

where Φ_k is as in (9). Therefore, with the notation of Proposition 2.1,

$$\Phi_{1,l}\left(\mathbb{P}^1\right) = \imath\, L_1' + \imath\, l\, I_2, \qquad \Phi_{k,l}\left(\mathbb{P}^k\right) = \imath\, L_k + \imath\, l\, I_2 \quad \forall\, k \geq 2. \tag{23}$$

Let us set

$$\boldsymbol{\zeta} := \begin{pmatrix} 1 & 1 \end{pmatrix}^t, \qquad J_{k,l} := J_k + l\,\boldsymbol{\zeta} \subset \mathbb{R}^2.$$

Thus $J_{k,l}$ is the segment joining $\begin{pmatrix} k+l & l \end{pmatrix}$ and $\begin{pmatrix} l & k+l \end{pmatrix}$. Also, let $\mathcal{C}_{k,l} \subset \mathbb{R}^2 \setminus \{0\}$ be the closed cone through $J_{k,l}$.

Then in place of Corollaries 2.1 and 2.2 we have:

Corollary 2.5 *Under the previous assumptions,*

$$\Phi_{1,l}\left(\mathbb{P}^1\right) = \mathcal{O}_{\boldsymbol{\epsilon}_1 + l\boldsymbol{\zeta}} = \mathcal{O}_{\boldsymbol{\epsilon}_1} + l\, I_2,$$

and for $k \geq 2$

$$\Phi_{k,l}\left(\mathbb{P}^k\right) = \bigcup_{\boldsymbol{v} \in J_k} \mathcal{O}_{\boldsymbol{v} + l\boldsymbol{\zeta}} = \bigcup_{\boldsymbol{v} \in J_{k+}} \mathcal{O}_{\boldsymbol{v} + l\boldsymbol{\zeta}} = \bigcup_{\boldsymbol{v} \in J_{k,l}} \mathcal{O}_{\boldsymbol{v}}. \tag{24}$$

In particular, if $\boldsymbol{v} \neq \mathbf{0}$ then $\Phi_{k,l}\left(\mathbb{P}^k\right) \cap \mathcal{C}(\mathcal{O}_{\boldsymbol{v}}) \neq \emptyset$ if and only if $\boldsymbol{v} \in \mathcal{C}_{k,l}$.

Corollary 2.6 *If $\Psi_{k,l} : \mathbb{P}^k \to \mathfrak{t} \cong \imath\,\mathbb{R}^2$ is the moment map for ψ with respect to $\mu_{k,l}$, then*

$$\Psi_{k,l}\left(\mathbb{P}^k\right) = \imath\, J_{k,l}. \tag{25}$$

Hence $\Psi_{k,l}\left(\mathbb{P}^k\right) \cap \mathbb{R}_+ \cdot \boldsymbol{v} \neq \emptyset$ if and only if $\boldsymbol{v} \in \mathcal{C}_{k,l}$.

The latter Corollary can of course be derived also by the Convexity Theorem in [1] and [7]. Let us also remark the following analogue of Corollary 2.3:

Corollary 2.7 *For any $k \geq 2$ and $l \in \mathbb{Z}$, $\Psi_{k,l}\left(\mathbb{P}^k\right) = \Phi_{k,l}\left(\mathbb{P}^k\right) \cap \mathfrak{t}$.*

Let us now consider the issue of transversality in this case. By Corollary 2.5, we may assume $k \geq 1$. Furthermore, by Proposition 2.1 and (23), $\Phi_{k,l}\left(\mathbb{P}^k\right) \subset V_{k+2l}$, where $V_r \subset \mathfrak{g}$ is the affine subspace of skew-Hermitian matrices of trace $\imath\, r$. If, in particular, $k + 2l = 0$

then $\Phi_{k,l}\left(\mathbb{P}^k\right)$ lies in a proper invariant vector subspace (the kernel of the trace), and is therefore not transverse to any cone $\mathcal{C}(\mathcal{O})$ in $\mathfrak{g}$ intersecting its image. In fact, if $\mathcal{C}(\mathcal{O}) \cap V_0 \neq \emptyset$ then by invariance $\mathcal{C}(\mathcal{O}) \subset V_0$. Thus we assume $k + 2l \neq 0$.

Let us denote by $\tilde{\phi}_{k,l}$ and $\tilde{\psi}_{k,l}$, respectively, the actions of G and T on X_k given by the restrictions of the unitary representation $\mu_{k,l}$. Let $(X'_k)^G_{\mathcal{O}}$, $(X'_k)^G_{\boldsymbol{v}}$ and $(X'_k)^T_{\boldsymbol{v}}$ be defined as $(X_k)^G_{\mathcal{O}}$, $(X_k)^G_{\boldsymbol{v}}$ and $(X_k)^T_{\boldsymbol{v}}$, but in terms of the new moment maps $\Phi_{k,l}$ and $\Psi_{k,l}$. Then, just as before, $\Phi_{k,l}$ is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}})$ if and only if $\tilde{\phi}_{k,l}$ is locally free at every $Z \in (X'_k)^G_{\boldsymbol{v}}$, and $\Psi_{k,l}$ is transverse to $\mathbb{R}_+ \cdot \imath\, \boldsymbol{v}$ if and only if $\tilde{\psi}_{k,l}$ is locally free on $(X'_k)^T_{\boldsymbol{v}}$.

Suppose that $Z \in X_k$. If for some $j = 0, \ldots, k$ we have $z_i = 0$ for all $i \neq j$, then arguing as in the proof of Theorem 2.1 one sees that $\tilde{\psi}_{k,l}$ is not locally free at Z, and therefore neither is $\tilde{\phi}_{k,l}$. In this case we have, with $\tilde{\Phi}_{k,l} := \Phi_{k,l} \circ \pi$:

$$\tilde{\Phi}_{k,l}(Z) := \imath \begin{pmatrix} k - j + l & 0 \\ 0 & j + l \end{pmatrix}. \tag{26}$$

If, conversely, $Z \in X_k$ and $z_l \cdot z_j \neq 0$ for distinct $j,\, l \in \{0, \ldots, k\}$, then a slight adaptation of the previous arguments shows that $\tilde{\psi}_{k,l}$ is locally free at Z. Hence we conclude the following variant of Theorem 2.2:

Theorem 2.3 *Suppose that $k \geq 2$ and $k + 2\,l \neq 0$. Let us define*

$$\boldsymbol{v}_{k,j,l} := \left(k - j + l \quad j + l\right)^t, \qquad j = 0, \ldots, k.$$

Then $\Psi_{k,l}$ is not transverse to $\mathbb{R}_+\, \imath\, \boldsymbol{v}$ if and only if $\boldsymbol{v} \in \mathbb{R}_+ \cdot \boldsymbol{v}_{k,j,l}$ for some $j = 0, \ldots, k$.

The previous argument clearly also shows that $\Phi_{k,l}$ is not transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}_{k,j,l}}) \subset \mathfrak{g}$. In fact, on the one hand if Z is the j-th basis vectors, then $\tilde{\psi}_{k,l}$ is not locally free at Z, and therefore *a fortiori* neither is $\tilde{\phi}_{k,l}$. On the other hand, by (26) we also have $Z \in (X'_k)^G_{\boldsymbol{v}}$.

Let us assume on the other hand that $\boldsymbol{v} \notin \mathbb{R}_+ \boldsymbol{v}_{j,l}$ for every j and that $v_1 \neq v_2$. If $Z \in (X'_k)^G_{\boldsymbol{v}}$, then there exist $l, j \in \{0, \ldots, k\}$ such that $z_l \cdot z_j \neq 0$. If $D \in T$ is a diagonal matrix with diagonal entries $\left(e^{\imath\,\vartheta_1} \quad e^{\imath\,\vartheta_2}\right)$ that fixes Z, then we need to have $e^{\imath\,[l(\vartheta_1 + \vartheta_2) + (k-a)\,\vartheta_1 + a\,\vartheta_2]} = 1$ for $a = j,\, l$. If D is close to I_2, and we assume that $\vartheta_j \sim 0$, we deduce as in the proof of Theorem 2.1 that $\vartheta_1 = \vartheta_2 = 0$. Hence $\tilde{\psi}_{k,l}$ is locally free on $(X'_k)^G_{\boldsymbol{v}}$. To conclude that $\tilde{\phi}_{k,l}$ is also locally free along $(X'_k)^G_{\boldsymbol{v}}$, we may now argue using (14) as in the proof of Theorem 2.1 (the second summand in (22) does not alter commutators). Hence we have the following variant of Theorem 2.1:

Theorem 2.4 *Suppose $k \geq 2$, $k + 2l \neq 0$ and $v_1 \neq v_2$. Then $\Phi_{k,l}$ is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}})$ if and only if $\boldsymbol{v} \notin \mathbb{R}_+ \cdot \boldsymbol{v}_{k,j,l}$ for every $j = 0, \ldots, k$.*

Let us now come to a general representation space of the form

$$W_{\mathbf{L},\mathbf{K}} := \bigoplus_{a=1}^{r} \det{}^{\otimes l_a} \otimes \mathrm{Sym}^{k_a}\left(\mathbb{C}^2\right), \tag{27}$$

where $\mathbf{L} = (l_a) \in \mathbb{Z}^r$, $\mathbf{K} = (k_a) \in \mathbb{N}^r$, as usual composed with the Lie group automorphism $B \mapsto (B^t)^{-1}$ (see (20)). As an abstract vector space,

$$W_{\mathbf{L},\mathbf{K}} \cong \bigoplus_{a=1}^{r} \mathbb{C}^{k_a+1} \cong \mathbb{C}^{|\mathbf{K}|+r} \quad \Rightarrow \quad \mathbb{P}(W_{\mathbf{L},\mathbf{K}}) \cong \mathbb{P}^{|\mathbf{K}|+r-1},$$

where $|\mathbf{K}| = \sum_a k_a$. Hence the corresponding morphism of Lie groups $\mu_{\mathbf{L},\mathbf{K}} : G \to U(|\mathbf{K}| + r)$ is given by

$$\mu_{\mathbf{L},\mathbf{K}}(g) := \begin{pmatrix} \mu_{l_1,k_1}(g) & & \\ & \ddots & \\ & & \mu_{l_r,k_r}(g) \end{pmatrix}.$$

Let us denote by $\phi_{\mathbf{L},\mathbf{K}}$ and $\psi_{\mathbf{L},\mathbf{K}}$, respectively, the induced Hamiltonian actions of G and T on $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$, and by $\Phi_{\mathbf{L},\mathbf{K}} : \mathbb{P}(W_{\mathbf{L},\mathbf{K}}) \to \mathfrak{g}$, $\Psi_{\mathbf{L},\mathbf{K}} : \mathbb{P}(W_{\mathbf{L},\mathbf{K}}) \to \mathfrak{t}$ their moment maps. If, with abuse of notation, we denote the general $Z \in W_{\mathbf{L},\mathbf{K}}$ as $Z = (Z_a)$, with $Z_a = \begin{pmatrix} z_{a,0} & \cdots & z_{a,k_a} \end{pmatrix}^t \in \mathbb{C}^{k_a+1}$, we have

$$\Phi_{\mathbf{L},\mathbf{K}}([Z]) = \frac{\imath}{\|Z\|^2} \sum_{a=1}^{r} \begin{pmatrix} \|F_{k_a,1}(Z_a)\|^2 + l_a \|Z_a\|^2 & F_{k_a,2}(Z_a)^t\, \overline{F_{k_a,1}(Z_a)} \\ F_{k_a,1}(Z_a)^t\, \overline{F_{k_a,2}(Z_a)} & \|F_{k_a,2}(Z_a)\|^2 + l_a \|Z_a\|^2 \end{pmatrix}. \tag{28}$$

Let us first consider the case where $\mathbf{K} = \mathbf{1} := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$, $\mathbf{L} = \mathbf{l} := \begin{pmatrix} l & \cdots & l \end{pmatrix}$. Thus $W_{\mathbf{l},\mathbf{1}} = \det^{\otimes l} \otimes W_1^{\oplus r}$ is isomorphic to $(\mathbb{C}^2)^r$ as a complex vector space. Then the moment map $\Phi_{\mathbf{l},\mathbf{1}} : \mathbb{P}((\mathbb{C}^2)^r) \to \mathfrak{g}$ is as follows. Let us write the general element of $(\mathbb{C}^2)^r$ as $Z = \begin{pmatrix} Z_1 & \cdots & Z_r \end{pmatrix}$ where $Z_j \in \mathbb{C}^2$. Then

$$\Phi_{\mathbf{l},\mathbf{1}}\big([Z_1 : \cdots : Z_r]\big) = \imath \left[\sum_{j=1}^{r} \frac{\|Z_j\|^2}{\|Z\|^2} P_{Z_j} + l I_2 \right], \tag{29}$$

where $P_{\mathbf{0}}$ is the null endomorphism of $\mathbb{C}^2$, while for $Z \neq \mathbf{0}$ we let P_Z denote the orthogonal projector of $\mathbb{C}^2$ on $\mathrm{span}(Z)$.

Let us set $\boldsymbol{v}_{1,j,l} := \begin{pmatrix} 1 - j + l & j + l \end{pmatrix}$, $j = 0, 1$.

Proposition 2.2 *For any $r \geq 2$, the following holds:*

1. $\Phi_{\mathbf{l},\mathbf{1}}\big(\mathbb{P}(W_1^{\oplus r})\big) = \imath L_1 + \imath\, l I_2$;
2. $\Psi_{\mathbf{l},\mathbf{1}}$ is transverse to $\imath \mathbb{R}_+ \cdot \boldsymbol{v}$ if and only if $\boldsymbol{v} \notin \mathbb{R}_+ \cdot \boldsymbol{v}_{1,j,l}$ for $j = 1, 2$;
3. $\Phi_{\mathbf{l},\mathbf{1}}$ is transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}})$ if and only if $\boldsymbol{v} \notin \mathbb{R}_+ \cdot \boldsymbol{v}_{1,j,l}$ for $j = 1, 2$.

Proof of Proposition 2.2 Let us assume $l = 0$; the general case is similar. By (29), the image of $-\imath \Phi_{\mathbf{0},\mathbf{1}}$ consists of all convex linear combinations of $r \geq 2$ orthogonal projectors, and is therefore contained in L_1. Conversely, any matrix in L_1 is a convex linear combination of two such projectors, and so the reverse implication holds.

To prove the second statement, consider $[Z] = [Z_1 : \cdots : Z_r]$, with $\|Z\| = 1$, such that every Z_j is a scalar multiple of $\boldsymbol{\epsilon}_1 := \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then $\Phi_{\mathbf{0},\mathbf{1}}([Z]) = \imath D_{\boldsymbol{\epsilon}_1}$, and on the other hand T does not acts locally freely on S^{4r-1} at Z. Hence $\Psi_{\mathbf{0},\mathbf{1}}$ is not transverse to $\mathbb{R}_+ \imath \boldsymbol{\epsilon}_1$, and $\Phi_{\mathbf{0},\mathbf{1}}$ is not transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{\epsilon}_1})$. The argument for $\boldsymbol{\epsilon}_2$ is similar. If on the other hand the Z_j's are neither all multiples of $\boldsymbol{\epsilon}_1$, nor all multiples of $\boldsymbol{\epsilon}_2$, then T acts locally freely at Z and arguing as in the proof of Theorem 2.1 (or applying Lemma 2.2), one concludes that the same holds of G. This proves the second and third statement. □

Let us return to (27). For the sake of simplicity, we shall consider a slightly restricted class of representation.

Definition 2.2 A representation $W_{\mathbf{L},\mathbf{K}}$ is *generic* if it satisfies the following property. Suppose that for some $l \in \mathbb{Z}$ the pair (l, 1) appears in the sequence $(l_1, k_1), \ldots, (l_r, k_r)$. Then there are $1 \leq a < b \leq r$ such that $(1, r) = (l_a, k_a) = (l_b, k_b)$.

In other words, if $\det^{\otimes l} \otimes \mathbb{C}^2$ appears in the isotypical decomposition of $W_{\mathbf{L},\mathbf{K}}$, then it does so with multiplicity ≥ 2. For example, W_1 and $W_1^{\oplus 2} \oplus (\det^{-2} \otimes W_1) \oplus W_2$ are not generic, while $W_1^{\oplus 2} \oplus W_2$ is.

If $Z_a = \mathbf{0}$ for some a, then the a-th summand in (28) vanishes; therefore, we may restrict the sum to those a's for which $Z_a \neq \mathbf{0}$, and this restricted sum will be indicated by a prime. Hence

$$\begin{aligned}
&\Phi_{\mathbf{L},\mathbf{K}}([Z]) \\
&= \imath \sum\nolimits_{a=1}^{\prime r} \frac{\|Z_a\|^2}{\|Z\|^2} \cdot \frac{1}{\|Z_a\|^2} \begin{pmatrix} \|F_{k_a,1}(Z_a)\|^2 + l_a \|Z_a\|^2 & F_{k_a,2}(Z_a)^t \, \overline{F_{k,1}(Z_a)} \\ F_{k_a,1}(Z_a)^t \, \overline{F_{k_a,2}(Z_a)} & \|F_{k_a,2}(Z_a)\|^2 + l_a \|Z_a\|^2 \end{pmatrix} \\
&= \sum\nolimits_{a=1}^{\prime r} \frac{\|Z_a\|^2}{\|Z\|^2} \, \Phi_{k_a,l_a}([Z_a]).
\end{aligned} \tag{30}$$

Proposition 2.3 *Assume that $W_{\mathbf{L},\mathbf{K}}$ is generic. Then $\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big) \subset \mathfrak{g}$ is the convex hull of the union of the images $\Phi_{k_a,l_a}\big(\mathbb{P}^{k_a}\big)$.*

Proof of Proposition 2.3 Let us denote by $H_{\mathbf{L},\mathbf{K}} \subset \mathfrak{g}$ the convex hull in point. By (28), $\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big) \subseteq H_{\mathbf{L},\mathbf{K}}$. Conversely, suppose $\alpha \in H_{\mathbf{L},\mathbf{K}}$. Then there exist $\lambda_a \geq 0$, $a = 1, \ldots, r$, such that $\sum_a' \lambda_a = 1$, and for each a with $\lambda_a > 0$ there exists $V_a \in \mathbb{C}^{k_a+1}$ of unit norm, such that

$$\alpha = \sum_{a=1}^{\prime r} \lambda_a \Phi_{ka,la}([V_a])$$

Let us set $Z_a := \sqrt{\lambda_a}\, V_a$ if $\lambda_a > 0$, $Z_a = 0 \in \mathbb{C}^{k_a+1}$ if $\lambda_a = 0$, and $Z := (Z_a) \in \mathbb{C}^{|\mathbf{K}|+r}$. Then $\|Z\| = 1$ and $\Phi_{\mathbf{L},\mathbf{K}}([Z]) = \alpha$ by (30), hence $\alpha \in \Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)$.

We can describe $\Psi_{\mathbf{L},\mathbf{K}}$ in a similar manner, and deduce the following:

Proposition 2.4 *$\Psi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big) \subset \mathfrak{t}$ is the convex hull of the union of the images $\Psi_{k_a,l_a}\big(\mathbb{P}^{k_a}\big)$.*

On the other hand, $-\imath\, \Psi_{k_a,l_a}\big(\mathbb{P}^{k_a}\big)$ is the segment joining $(\, k_a + l_a \quad l_a\,)^t$ and $(\, l_a \quad k_a + l_a\,)^t$ for each a. Therefore we conclude the following (which might be also obtained by the Convexity Theorem):

Corollary 2.8 *$-\imath\,\Psi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)\subset\mathbb{R}^2$ is the convex hull of the collection of the points $(\,k_a+l_a \quad l_a\,)^t$ and $(\,l_a \quad k_a+l_a\,)^t$, $a=1,\ldots,r$, or equivalently of the segments J_{k_a,l_a}.*

We have the following analogue of Corollaries 2.3 and 2.7:

Corollary 2.9 *If $W_{\mathbf{L},\mathbf{K}}$ is generic, then $\Psi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)=\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)\cap\mathfrak{t}$.*

Proposition 2.5 *Assume that $W_{\mathbf{L},\mathbf{K}}$ is generic. Then the following conditions are equivalent:*

1. $\mathbf{0}\notin\Psi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)$;
2. $\mathbf{0}\notin\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)$;
3. either $k_a+2\,l_a>0$ for all $a=1,\ldots,r$, or $k_a+2\,l_a<0$ for all $a=1,\ldots,r$.

Proof By Corollary 2.9, 1) and 2) are equivalent. Suppose that 2) holds. By (23), we have $\Phi_{k_a,l_a}\big(\mathbb{P}^{k_a}\big)=\imath\,L_{k_a}+\imath\,l_a\,I_2$ for every a; if $k_a+2\,l_a=0$ for some a, then $l_a\leq 0$ and so

$$(0)=\imath\begin{pmatrix}-l_a & 0\\ 0 & -\,l_a\end{pmatrix}+\imath\,l_a\,I_2\in\Phi_{k_a,l_a}\big(\mathbb{P}^{k_a}\big).$$

Hence assuming 2) we need to have $k_a+2\,l_a\neq 0$ for every $a=1,\ldots,r$. Suppose that $k_a+2\,l_a>0$ and $k_b+2\,l_b<0$ for some $a,\,b=1,\ldots,r$. Then

$$\frac{1}{2}\,(k_a+2\,l_a)\,I_2=\frac{k_a}{2}\,I_2+l_a\,I_2\in\Phi_{k_a,l_a}\big(\mathbb{P}^{k_a}\big),$$

and similarly

$$\frac{\imath}{2}\,(k_b+2\,l_b)\,I_2=\imath\,\frac{k_b}{2}\,I_2+\imath\,l_b\,I_2\in\Phi_{k_b,l_b}\big(\mathbb{P}^{k_b}\big).$$

Hence by the previous dicussion the segment joining these two matrices is contained in $\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)$, and it is obvious that it meets the origin, absurd. Hence 2) implies 3).

Suppose that 3) holds, say with >0. Then for every $a=1,\ldots,r$ and every $\alpha\in\Psi_{l_a,k_a}\big(\mathbb{P}^{k_a}\big)$ we have $-\imath\,\mathrm{trace}(\alpha)=k_a+2\,l_a>0$. Since the convex linear combination of matrices with positive trace has positive trace, 1) also holds by Proposition 2.4. □

Corollary 2.10 *Assume that $W_{\mathbf{L},\mathbf{K}}$ is generic. Then $\mathbf{0}\notin\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)$ if and only if $\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)\subset\mathfrak{g}$ is contained in one of the half-spaces defined by the hyperplane $\mathfrak{su}(2)=\ker(\mathrm{trace})\subset\mathfrak{g}$. In particular, if $\mathbf{0}\notin\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)$ and $\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big)\cap\mathbb{R}_+\cdot\boldsymbol{v}\neq\emptyset$, then $v_1+v_2\neq 0$.*

Definition 2.3 The representation $W_{\mathbf{L},\mathbf{K}}$ will be called *uniform* if it is generic and $k_a+2\,l_a=k_b+2\,l_b$ for all $a,\,b=1,\ldots,r$.

The proof of the following Lemma is left to the reader.

Lemma 2.3 *The following conditions are equivalent*:

1. $W_{\mathbf{L},\mathbf{K}}$ is uniform;

2. $\phi_{\mathbf{L},\mathbf{K}}$ (equivalently, $\psi_{\mathbf{L},\mathbf{K}}$) is trivial on $Z(G)$ (the center of G).

Let us now assume that the equivalent conditions in Proposition 2.5 are satisfied, and consider transversality. Let us denote by $X_{\mathbf{K}} \subset \mathbb{C}^{|\mathbf{K}|+r}$ the unit sphere, by $\pi_{\mathbf{K}} : X_{\mathbf{K}} \to \mathbb{P}^{\mathbf{K}+r-1}$ the Hopf map, and set $\tilde{\Phi}_{\mathbf{L},\mathbf{K}} = \Phi_{\mathbf{L},\mathbf{K}} \circ \pi_{\mathbf{K}} : X_{\mathbf{K}} \to \mathfrak{g}$. Also, let $\tilde{\phi}_{\mathbf{L},\mathbf{K}}$ and $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ denote, respectively, the actions of G and T on $X_{\mathbf{K}}$ by restriction of $\tilde{\phi}_{\mathbf{L},\mathbf{K}}$. These are liftings of the actions $\phi_{\mathbf{L},\mathbf{K}}$ and $\psi_{\mathbf{L},\mathbf{K}}$ on $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$

Let us fix $Z \in X_{\mathbf{K}}$, and denote by $\mathcal{O}^Z \subset \mathfrak{g}$ the orbit through $\tilde{\Phi}_{\mathbf{L},\mathbf{K}}(Z)$. Perhaps after replacing Z with $(\tilde{\phi}_{\mathbf{L},\mathbf{K}})_g(Z)$ for some $g \in G$, without changing $\mathcal{O}^Z$ we may as well assume that $\tilde{\Phi}_{\mathbf{L},\mathbf{K}}(Z) \in \mathfrak{t}$.

Suppose that only one component of Z in non-zero, say z_{aj} for some $a \in \{1,\ldots,r\}$ and $j \in \{0,\ldots,k_a\}$. Then, as in the case $r = 1$, one sees that there is a 1-dimensional torus fixing Z; therefore, neither is $\Phi_{\mathbf{L},\mathbf{K}}$ transverse to $\mathcal{C}(\mathcal{O}^Z)$, nor is $\Psi_{\mathbf{L},\mathbf{K}}$ transverse to $\mathbb{R}_+\,\Psi_{\mathbf{L},\mathbf{K}}(Z)$. In this case, in view of (30) and (26) we have

$$\tilde{\Phi}_{\mathbf{L},\mathbf{K}}(Z) = \Phi_{k_a,l_a}([Z_a]) = \imath \begin{pmatrix} k_a - j + l_a & 0 \\ 0 & j + l_a \end{pmatrix}.$$

Hence, if we set

$$\boldsymbol{v}_{k_a,j,l_a} := \begin{pmatrix} k_a - j + l_a & j + l_a \end{pmatrix}^t \qquad (a = 1,\ldots,r, \quad j = 0,\ldots,k_a), \tag{31}$$

we conclude that $\Phi_{\mathbf{L},\mathbf{K}}$ is not transverse to $\mathcal{C}(\mathcal{O}_{\boldsymbol{v}_{k_a,j,l_a}})$ and that $\Psi_{\mathbf{L},\mathbf{K}}$ is not transverse to $\mathbb{R}_+ \cdot \boldsymbol{v}_{k_a,j,l_a}$ for every a, j.

If, on the other hand, there exist $a \in \{1,\ldots,r\}$ and $j,h \in \{0,\ldots,k_a\}$ with $j \neq h$ and $z_{aj} \cdot z_{ah} \neq 0$, then the arguments used in the proof of Theorems 2.1, 2.4 imply that both $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ and $\tilde{\phi}_{\mathbf{L},\mathbf{K}}$ are locally free at Z.

Thus we reduced to the case where for each $a = 1,\ldots,r$ at most one component of Z_a is non-zero, and $Z_a \neq \mathbf{0}$ for at least two distinct values of a. We shall make this assumption in the following.

So there exist a, $b \in \{1,\ldots,r\}$, $a \neq b$ and $j_a \in \{0,\ldots,k_a\}$, $j_b \in \{0,\ldots,k_b\}$ such that $z_{a,j_a} \cdot z_{b,j_b} \neq 0$, and furthermore $z_{a,j} = 0$ if $j \neq j_a$, $z_{b,j} = 0$ if $j \neq j_b$.

Consider, as before, a diagonal matrix $D \in T$, with diagonal entries $e^{\imath\vartheta_i}$, $i = 1,2$, and suppose that D fixes Z. Also, let us assume that D is in a small neighborhood of I_2, so that without loss $\vartheta_j \sim 0$. Then the condition $(\tilde{\phi}_{\mathbf{L},\mathbf{K}})_D(Z) = Z$ implies that $e^{\imath\,[l_a\,(\vartheta_1+\vartheta_2)+(k_a-j_a)\,\vartheta_1+j_a\,\vartheta_2]}\,z_{a,j_a} = z_{a,j_a}$ and $e^{\imath\,[l_b\,(\vartheta_1+\vartheta_2)+(k_b-j_b)\,\vartheta_1+j_b\,\vartheta_2]}\,z_{b,j_b} = z_{b,j_b}$. Since $\vartheta_j \sim 0$, this forces

$$(l_a + k_a - j_a)\,\vartheta_1 + (l_a + j_a)\,\vartheta_2 = (l_b + k_b - j_b)\,\vartheta_1 + (l_b + j_b)\,\vartheta_2 = 0.$$

This system has non-trivial solutions if and only if the vectors $\boldsymbol{v}_{k_a,j_a,l_a}$ and $\boldsymbol{v}_{k_b,,j_b,l_b}$ are linearly dependent (see (31)); if this is the case, then $\Phi_{k_a,l_a}([Z_a])$ and $\Phi_{k_b,l_b}([Z_b])$ are both scalar multiples of the diagonal matrix $\imath\,D_{\boldsymbol{v}_{k_a,j_a,l_a}}$.

Hence we have the following alternatives.

Let $I \subseteq \{1,\ldots,r\}$ be the non-empty subset of those a's such that $Z_a \neq \mathbf{0}$. If the vectors $\boldsymbol{v}_{k_a,j_a,l_a}$, $a \in I$, are all pairwise linearly dependent, then $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ is not locally free at Z, and therefore neither is $\tilde{\phi}_{\mathbf{L},\mathbf{K}}$. Hence, $\Phi_{\mathbf{L},\mathbf{K}}$ is not transverse to $\mathcal{C}(\mathcal{O}^Z)$ at Z, and similarly $\Psi_{\mathbf{L},\mathbf{K}}$

is not transverse to $\mathbb{R}_+ \cdot \Psi_{\mathbf{L},\mathbf{K}}(Z)$ at Z. Furthermore, in this case we also obtain that $\Phi_{\mathbf{L},\mathbf{K}}([Z])$ is a multiple of $\imath\, D_{\mathbf{v}_{k_a,j_a,l_a}}$, and so $\Psi_{\mathbf{L},\mathbf{K}}([Z])$ is a multiple of $\imath\, \mathbf{v}_{k_a,j_a,l_a}$.

Suppose, on the other hand, that there exist $a, b \in I$ such that $\mathbf{v}_{k_a,j_a,l_a} \wedge \mathbf{v}_{k_b,j_b,l_b} \neq \mathbf{0}$. Then $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ is locally free at Z. Since we are assuming that $\Phi_{\mathbf{L},\mathbf{K}}([Z])$ is diagonal and non-zero, we can apply the argument used in the proof of Theorem 2.1, following (16), to obtain the stronger statement that $\tilde{\phi}_{\mathbf{L},\mathbf{K}}$ is also locally free at Z, and so $\Phi_{\mathbf{L},\mathbf{K}}$ is transverse to $\mathcal{C}(\mathcal{O}^Z)$ at Z.

The outcome of the previous discussion is the following statement. Recall that $\mathbf{v}_{a,j}$ was defined in (31).

Theorem 2.5 *Suppose $v_1 \neq v_2$ and that the equivalent conditions in Proposition 2.5 are satisfied. Then the following conditions are equivalent*:

1. $\Phi_{\mathbf{L},\mathbf{K}}$ is not transverse to $\mathcal{C}(\mathcal{O}_{\mathbf{v}})$;
2. $\Psi_{\mathbf{L},\mathbf{K}}$ is not transverse to $\mathbb{R}_+\, \imath\, \mathbf{v}$;
3. there exist $a \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, k_a\}$, such that $\mathbf{v} = \mathbf{v}_{k_a,j,l_a}$.

If $M \subseteq \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ is a projective submanifold, then the restriction to M of the Fubini-Study form is a Kähler form ω on M. If M is G-invariant, the induced action of G on M is Hamiltonian with respect to $2\,\omega$, with moment map $\Phi_M := \Phi_{\mathbf{L},\mathbf{K}}\big|_M : M \to \mathfrak{g}$. Similar considerations apply to the action of T on M, which is Hamiltonian with respect to $2\,\omega$, with moment map $\Psi_M := \Psi_{\mathbf{L},\mathbf{K}}\big|_M : M \to \mathfrak{t}$.

For $\mathbf{v} = (\,v_1 \quad v_2\,)^t$ with $v_j \geq 0$ and $\mathbf{v} \neq \mathbf{0}$, let us denote by $\mathbb{P}_{\mathbf{v}} \subseteq \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ the locus of those $[Z] = [Z_1 : \ldots : Z_r]$, where $Z_a = (z_{aj}) \in \mathbb{C}^{k_a+1}$, such that $z_{aj} = 0$ if $(\,k_a - j + l_a \quad j + l_a\,)^t$ is not a (positive) multiple of $(\,v_1 \quad v_2\,)^t$. Then $\mathbb{P}_{\mathbf{v}} = \emptyset$ unless $\mathbf{v} = \mathbf{v}_{k_a,j,l_a}$ for some $a = 1, \ldots, r$ and $j = 0, \ldots, k_a$, and each $\mathbb{P}_{\mathbf{v}_{k_a,j,l_a}}$ is a projective subspace. For any (a, j) and (b, j'), either $\mathbb{P}_{\mathbf{v}_{k_a,j,l_a}} = \mathbb{P}_{\mathbf{v}_{k_b,j',l_{a'}}}$, or else $\mathbb{P}_{\mathbf{v}_{k_a,j,l_a}} \cap \mathbb{P}_{\mathbf{v}_{k_b,j',l_{a'}}} = \emptyset$; also, the inverse image in $X_{\mathbf{K},\mathbf{L}}$ of $\bigcup_{a,j} \mathbb{P}_{\mathbf{v}_{k_a,j,l_a}}$ is the locus over which $\Psi_{\mathbf{K},\mathbf{L}}$ is not locally free.

Theorem 2.6 *In the situation of Theorem 2.5, suppose that $M \subseteq \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ is a G-invariant projective manifold. Consider $\mathbf{v} \in \mathbb{N}^2 \setminus \{\mathbf{0}\}$. Then the following conditions are equivalent*:

1) Ψ_M is not transverse to $\mathbb{R}_+ \cdot \imath\, \mathbf{v}$;
2) $\mathbf{v} = \mathbf{v}_{k_a,j,l_a}$ for some (a, j), and $M \cap \mathbb{P}_{\mathbf{v}_{k_a,j,l_a}} \neq \emptyset$.

If, in addition, $v_1 \neq v_2$, *then* 1) and 2) are equivalent to

3) Φ_M is not transverse to $\mathcal{C}(\mathcal{O}_{\mathbf{v}})$.

Proof of Theorem 2.6 Let $X_M \subseteq X$ be the inverse image of M in $X_{\mathbf{L},\mathbf{K}}$; thus, X_M is the circle bundle of the induced polarization. Then $(X_M)^G_{\mathbf{v}} = (X_{\mathbf{L},\mathbf{K}})^G_{\mathbf{v}} \cap X_M$ *etc*. Let us denote by $\tilde{\phi}_M$ and $\tilde{\psi}_M$, respectively, the restrictions of $\tilde{\phi}_{\mathbf{L},\mathbf{K}}$ and $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ to X_M.

Let us prove the equivalence of 1) and 2).

As recalled above, Ψ_M is not transverse to $\mathbb{R}_+ \cdot \imath\, \mathbf{v}$ if and only if there exists $Z \in (X_M)^T_{\mathbf{v}}$ such that $\tilde{\psi}_M$ is not locally free at Z, that is, such that $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ is not locally free at Z. On the other hand, the previous discussion shows that $\tilde{\psi}_{\mathbf{L},\mathbf{K}}$ is not locally free at Z if and only if $[Z] \in \mathbb{P}_{\mathbf{v}_{a,j}}$ for some (a, j), and that if this happens then $\Psi_M([Z]) = \Psi_{\mathbf{L},\mathbf{K}}([Z])$ is a positive multiple of $\imath\,\mathbf{v}_{a,j}$.

Let us assume that $v_1 \neq v_2$, and prove the equivalence with 3).

Suppose that 2) holds, and suppose $Z \in X_M$, $[Z] \in M \cap \mathbb{P}_{a,j}$. Then $\tilde{\psi}_M$ is not locally free at Z, and therefore *a fortiori* neither is $\tilde{\phi}_M$. Furthermore, by the previous discussion $\Phi_M([Z])$ is a positive multiple of $\imath\,D_{\mathbf{v}_{a,j}}$, so $Z \in (X_M)^G_{\mathbf{v}_{a,j}}$. Hence 3) holds.

Conversely, suppose that 3) holds. Then there exists $Z \in (X_M)^G_{\mathcal{O}_{\mathbf{v}}}$ such that $\tilde{\phi}_M$ is not locally free at Z; perhaps after replacing Z in its orbit, we may assume without loss that $\tilde{\Phi}_M(Z)$ is diagonal, that is, $Z \in (X_M)^G_{\mathbf{v}} = (X_M)^T_{\mathbf{v}} \cap (X_M)^G_{\mathcal{O}}$. If $\tilde{\psi}_M$ was locally free at Z, then an argument in the proof of Theorem 2.1 (see (14) and (17)) would imply that $\tilde{\phi}_M$ is itself locally free at Z, absurd. Hence $\tilde{\psi}$ is not locally free at Z, and therefore $[Z] \in \mathbb{P}_{a,j}$ for some a, j, and $\Phi_M([Z])$ is ap positive multiple of $\imath\,D_{\mathbf{v}_{a,j}}$. Hence 1) and 2) hold. $\square$

3 $\overline{M}^T_{\mathbf{v}}$

We shall assume in this section that $\mathbf{0} \notin \Psi(M)$, and that both Ψ and Φ are transverse to $\mathbb{R}_+ \cdot \imath\,\mathbf{v}$, where $v_1 > v_2$. Then $M^T_{\mathbf{v}} \subset M$ is a smooth compact connected T-invariant hypersurface; furthermore, $M^G_{\mathbf{v}} := \Phi^{-1}(\mathbb{R}_+ \cdot \imath\,\mathbf{v}) \subset M^T_{\mathbf{v}}$ is a smooth, compact and connected T-invariant submanifold of real codimension two (three in M) [4]. In §3.1, M is not assumed to be projective.

3.1 The Kähler structure of $\overline{M}^T_{\mathbf{v}}$

The 1-parameter subgroup

$$T^1_{\mathbf{v}_\perp} := \left\{ \kappa_{\mathbf{v}}\left(e^{\imath\vartheta}\right) \,:\, e^{\imath\,\vartheta} \in S^1 \right\}, \quad \kappa_{\mathbf{v}}\left(e^{\imath\vartheta}\right) := \begin{pmatrix} e^{-\imath\,v_2\,\vartheta} & 0 \\ 0 & e^{\imath\,v_1\,\vartheta} \end{pmatrix} \tag{32}$$

acts locally freely on $M^T_{\mathbf{v}}$; its orbits are the leaves of the null foliation of $M^T_{\mathbf{v}}$. If v_1 and v_2 are coprime, as we may assume without loss, $\kappa_{\mathbf{v}} : S^1 \to T^1_{\mathbf{v}_\perp}$ in (32) is a Lie group isomorphism.

Let us set

$$\overline{M}^T_{\mathbf{v}} := M^T_{\mathbf{v}} / T^1_{\mathbf{v}_\perp}, \qquad \overline{M}^G_{\mathbf{v}} := M^G_{\mathbf{v}} / T^1_{\mathbf{v}_\perp} \subset \overline{M}^T_{\mathbf{v}}. \tag{33}$$

Then $\overline{M}^T_{\mathbf{v}}$ is an orbifold of (real) dimension $2\,(d-1)$, and $\overline{M}^G_{\mathbf{v}} \subset \overline{M}^T_{\mathbf{v}}$ is a suborbifold of real codimension two, meaning that the topological embedding $\overline{M}^G_{\mathbf{v}} \subset \overline{M}^T_{\mathbf{v}}$ can be lifted to an embedding of local slices. We shall let $q_{\mathbf{v}} : M^T_{\mathbf{v}} \to \overline{M}^T_{\mathbf{v}}$ denote the projection.

Definition 3.1 $\psi_{\mathbf{v}_\perp}$ is the action of $T^1_{\mathbf{v}_\perp}$ on M given by restriction of ψ.

By means of $\kappa_{\mathbf{v}}$, we shall view $\psi_{\mathbf{v}_\perp}$ as a Hamiltonian S^1-action, with moment map $\Psi_{\mathbf{v}_\perp} := \langle \Psi, \mathbf{v}_\perp \rangle$. The proof of the following is left to the reader:

Lemma 3.1 *Given that* Ψ *is transverse to* $\mathbb{R}_+ \cdot \imath\, \boldsymbol{v}$, 0 *is a regular value of* $\Psi_{\boldsymbol{v}_\perp}$, *and* $M^T_{\boldsymbol{v}} = {\Psi_{\boldsymbol{v}_\perp}}^{-1}(0)$.

As an orbifold, $\overline{M}^T_{\boldsymbol{v}}$ coincides with the symplectic quotient (symplectic reduction at $\mathbf{0}$) $M//T^1_{\boldsymbol{v}_\perp}$. Hence it inherits a reduced Kähler orbifold structure $\left(\overline{M}^T_{\boldsymbol{v}}, J_{\overline{M}^T_{\boldsymbol{v}}}, \omega_{\overline{M}^T_{\boldsymbol{v}}}\right)$.

As mentioned in the introduction, $\overline{M}^G_{\boldsymbol{v}}$ may also be viewed as a symplectic quotient, namely $\overline{M}^G_{\boldsymbol{v}} = Y//T^1_{\boldsymbol{v}_\perp}$, where $Y \subset M$ is the 'symplectic cross section' discussed in [7]. Hence $\overline{M}^G_{\boldsymbol{v}}$ also carries a symplectic orbifold structure $(\overline{M}^G_{\boldsymbol{v}}, \omega_{\overline{M}^G_{\boldsymbol{v}}})$. Since both $\omega_{\overline{M}^G_{\boldsymbol{v}}}$ and $\omega_{\overline{M}^T_{\boldsymbol{v}}}$ are both induced from ω, $(\overline{M}^G_{\boldsymbol{v}}, \omega_{\overline{M}^G_{\boldsymbol{v}}})$ is a symplectic suborbifold of $(\overline{M}^T_{\boldsymbol{v}}, \omega_{\overline{M}^T_{\boldsymbol{v}}})$.

The T-invariant direct sum decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{a}$ determines a splitting $\Phi = \Psi \oplus \Upsilon' : M \to \mathfrak{g}$, where both $\Psi : M \to \mathfrak{t}$ and $\Upsilon' : M \to \mathfrak{a}$ are T-equivariant (notation is as in (16)). By restriction we obtain a T-equivariant smooth map

$$\Upsilon := \Upsilon'|_{M^T_{\boldsymbol{v}}} : M^T_{\boldsymbol{v}} \to \mathfrak{a}. \tag{34}$$

Since

$$\begin{pmatrix} e^{\imath\,\vartheta_1} & 0 \\ 0 & e^{\imath\,\vartheta_2} \end{pmatrix} \imath \begin{pmatrix} a & z \\ \overline{z} & b \end{pmatrix} \begin{pmatrix} e^{-\imath\,\vartheta_1} & 0 \\ 0 & e^{-\imath\,\vartheta_2} \end{pmatrix} = \imath \begin{pmatrix} a & e^{\imath\,(\vartheta_1-\vartheta_2)}\, z \\ e^{-\imath\,(\vartheta_1-\vartheta_2)}\, \overline{z} & b \end{pmatrix}, \tag{35}$$

identifying $\mathfrak{a} \cong \mathbb{C}$ by the parameter z in (35), we may interpret Υ as a map $M^T_{\boldsymbol{v}} \to \mathbb{C}$ with the equivariance property

$$\Upsilon \circ \psi_{D(\vartheta_1,\vartheta_2)^{-1}} = e^{-\imath\,(\vartheta_1-\vartheta_2)}\, \Upsilon, \tag{36}$$

where $D(\vartheta_1, \vartheta_2) \in T$ is the diagonal matrix with entries $e^{\imath\,\vartheta_j}$.

By Theorem 1.2 of [4], $M^T_{\boldsymbol{v}} \cap M^G_{\mathcal{O}} = M^G_{\boldsymbol{v}}$, and the intersection is tangential, that is, $T_m M^T_{\boldsymbol{v}} = T_m M^G_{\mathcal{O}} \subset T_m M$ if $m \in M^G_{\boldsymbol{v}}$. Since $M^G_{\mathcal{O}}$ is G-invariant, for any $\beta \in \mathfrak{g}$ the vector field $\beta_M \in \mathfrak{X}(M)$ induced by β is tangent to $M^G_{\mathcal{O}}$. Hence, if $m \in M^G_{\boldsymbol{v}}$ then $\beta_M(m) \in T_m M^T_{\boldsymbol{v}}$. Therefore, $\mathfrak{a}_M(m) \subset T_m M^T_{\boldsymbol{v}}$ for any $m \in M^G_{\boldsymbol{v}}$. The argument used for (17), and the remark that $M^G_{\boldsymbol{v}} = \Upsilon^{-1}(0)$, imply the following.

Lemma 3.2 *Under the previous assumptions, we have*:

1. $\mathrm{d}_m \Upsilon\big(\mathfrak{a}_M(m)\big) = \mathfrak{a}$, $\forall\, m \in M^T_{\boldsymbol{v}}$;
2. 0 is a regular value of Υ;
3. we have a T-equivariant direct sum decomposition

$$T_m M^T_{\boldsymbol{v}} = T_m M^G_{\boldsymbol{v}} \oplus \mathfrak{a}_M(m), \quad \forall\, m \in M^G_{\boldsymbol{v}}. \tag{37}$$

Lemma 3.3 *The summands on the right hand side of* (37) *are symplectically orthogonal.*

Proof of Lemma 3.3 Let us consider the Hamiltonian functions $\Phi^\eta := \langle \Phi, \eta \rangle$ and $\Phi^\xi := \langle \Phi, \xi \rangle$. Explicitly, if

$$\Phi = \imath \begin{pmatrix} a & z \\ \overline{z} & b \end{pmatrix},$$

where $a, b : M \to \mathbb{C}$ and $z : M \to \mathbb{C}$ are $\mathcal{C}^\infty$, then $\Phi^\eta = -2\,\Im(z)$, $\Phi^\xi = 2\,\Re(z)$.

By definition of $M^G_{\boldsymbol{v}}$, z vanishes identically on $M^G_{\boldsymbol{v}}$; therefore, for any $(m, v) \in TM^G_{\boldsymbol{v}}$ we have

$$0 = \mathrm{d}_m\Phi^\eta(v) = \omega_m\big(\eta_M(m), v\big),$$

and similarly for ξ. □

Corollary 3.1 *$\mathfrak{a}_M(m) \subseteq T_mM$ is a symplectic vector subspace, $\forall\, m \in M^G_{\boldsymbol{v}}$.*

Proof of Corollary 3.1 This follows immediately from Lemma 3.3. Alternatively, we need to show that $\omega_m\big(\eta_M(m), \xi_M(m)\big) \neq 0$. For $m \in M^G_{\boldsymbol{v}}$, we have $\Phi(m) = \imath\,\lambda(m)\,D_{\boldsymbol{v}}$ where $\lambda(m) > 0$. Arguing as for (17) we obtain

$$\omega_m\big(\eta_M(m), \xi_M(m)\big) = \big\langle \mathrm{d}_m\Phi\big(\xi_M(m)\big), \eta\big\rangle = \lambda(m)\,(v_1 - v_2)\,\langle \eta, \eta\rangle > 0. \tag{38}$$

□

Definition 3.2 If $m \in M^T_{\boldsymbol{v}}$, $F_m \leqslant T^1_{\boldsymbol{v}^\perp}$ denotes its stabilizer subgroup for $\psi_{\boldsymbol{v}_\perp}$ (Definition 3.1). Furthermore, $F_{\boldsymbol{v}} \leqslant T^1_{\boldsymbol{v}^\perp}$ denotes the stabilizer for $\psi_{\boldsymbol{v}_\perp}$ of a general $m \in M^T_{\boldsymbol{v}^\perp}$.

Hence, $F_{\boldsymbol{v}} \leqslant F_m$, $\forall\, m \in M^T_{\boldsymbol{v}}$.

Lemma 3.4 *If $m \in M^T_{\boldsymbol{v}} \setminus M^G_{\boldsymbol{v}}$, then $F_m \leqslant T^1_{\boldsymbol{v}^\perp} \cap Z(G)$. In particular, $F_{\boldsymbol{v}} \leqslant T^1_{\boldsymbol{v}^\perp} \cap Z(G)$.*

Proof of Lemma 3.4 By equivariance, if $\phi_g(m) = m$, then $\mathrm{Ad}_g\big(\Phi(m)\big) = \Phi(m) \in \mathfrak{g}$ where Ad is the adjoint action. If $m \in M^T_{\boldsymbol{v}} \setminus M^G_{\boldsymbol{v}}$ then $\Phi(m)$ is not diagonal. The claim then follows from by (35). □

Remark 3.1 For a uniform representation $F_{\boldsymbol{v}} = T^1_{\boldsymbol{v}^\perp} \cap Z(G)$, since $Z(G)$ acts trivially on M (Definition 2.3).

Let us introduce the quotients (isomorphic to S^1)

$$S^1(\boldsymbol{v}) := T^1_{\boldsymbol{v}^\perp}/F_{\boldsymbol{v}}, \qquad T^1(\boldsymbol{v}) := T^1_{\boldsymbol{v}^\perp}/\big(T^1_{\boldsymbol{v}^\perp} \cap Z(G)\big). \tag{39}$$

The induced action $\overline{\psi}_{\boldsymbol{v}_\perp} : S^1(\boldsymbol{v}) \times M^T_{\boldsymbol{v}} \to M^T_{\boldsymbol{v}}$ is locally free and generically free, hence effective. If $\big(M^T_{\boldsymbol{v}}\big)_{sm} \subseteq M^T_{\boldsymbol{v}}$ is the dense open set where $F_m = F_{\boldsymbol{v}}$, then $\big(M^T_{\boldsymbol{v}}\big)_{sm}$ is a principal $S^1(\boldsymbol{v})$-bundle over its image $(\overline{M}^T_{\boldsymbol{v}})_{sm}$.

Given a character $\chi : S^1(\boldsymbol{v}) \to \mathbb{C}^*$ we obtain an Hermitian orbifold line bundle L_χ. Given the CR structure on $M^T_{\boldsymbol{v}}$, L_χ is in fact an holomorphic orbifold line bundle on $\overline{M}^T_{\boldsymbol{v}}$. A smooth function $\Sigma : M^T_{\boldsymbol{v}} \to \mathbb{C}$ such that $\Sigma \circ (\overline{\psi}_{\boldsymbol{v}_\perp})_{g^{-1}} = \chi(g)\,\Sigma$ for any $g \in S^1(\boldsymbol{v})$ determines a smooth section σ_Σ of L_χ.

By Lemma 3.4, we have a short exact sequence

$$0 \to \big(T^1_{\boldsymbol{v}^\perp} \cap Z(G)\big)/F_{\boldsymbol{v}} \to S^1(\boldsymbol{v}) \to T^1(\boldsymbol{v}) \to 0;$$

therefore, any character of $T^1(\boldsymbol{v})$ yields a character of $S^1(\boldsymbol{v})$. In particular, we obtain a character of $S^1(\boldsymbol{v})$ from any character of T with kernel $Z(G)$, whence from the character $e^{-\imath(\theta_1-\theta_2)}$ appearing in (35). Explicitly, evaluating the latter on $T^1_{\boldsymbol{v}} \cong S^1$ we obtain the character $e^{\imath(v_1+v_2)\vartheta}$. We shall denote by χ the corresponding character of $S^1(\boldsymbol{v}_\perp)$.

By (36), Υ determines a section σ_Υ of L_χ. By Lemma 3.2 we conclude the following.

Theorem 3.1 *The symplectically embedded orbifold $\overline{M}^G_{\boldsymbol{v}} \subset \overline{M}^T_{\boldsymbol{v}}$ is the zero locus of the transverse section σ_Υ of L_χ. If $\bar{\imath}_T : \overline{M}^G_{\boldsymbol{v}} \subset \overline{M}^T_{\boldsymbol{v}}$ is the inclusion, there is a direct sum decomposition of orbifold vector bundles*

$$\bar{\imath}_T^*\left(T\overline{M}^T_{\boldsymbol{v}}\right) = T\overline{M}^G_{\boldsymbol{v}} \oplus \bar{\imath}_T^*(L_{\mathfrak{a}}).$$

3.2 The case of $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$

We aim to classify the DH reductions $(\overline{M}^T_{\boldsymbol{v}}, J'_0, \Omega'_0)$ when $M = \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$, assuming that $W_{\mathbf{L},\mathbf{K}}$ is generic (Definition 2.2). In particular, we shall interpret each such Kähler orbifold as a weighted projective variety, related to certain explicit combinatorial data associated to $\mathbf{L}$, $\mathbf{K}$, $\boldsymbol{v}$. Before doing so, in §3.2.1 we shall review a general construction from [16], producing a Kähler orbifold from a homolomorphic Hamiltonian action with positive moment map (see [17] for a generalization to torus actions). We shall apply this procedure first to actions on projective spaces, thus obtaining a class of Kähler forms on weighted projective spaces, and then to actions on products of projective spaces, obtaining a class of Kähler suborbifolds of certain weighted projective spaces. Next, in §3.2.2 we shall describe a family of Hamiltonian circle actions on projective spaces for which the DH reduction can be described in terms of the previous construction, applied to a related Hamiltonian holomorphic action (with positive moment map) on a mixed product $\mathbb{P}^k \times \overline{\mathbb{P}^l}$; it follows that the DH reduction of the original action of projective space can be realized as a Kähler suborbifold of an appropriate weighted projective space. Building on these considerations, in §3.2.3 we shall determine the DH reductions when $M = \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$. Finally, in §3.2.4 we shall focus on the irreducible representation μ_k and give an explicit description of the pair $(\overline{M}^T_{\boldsymbol{v}}, \overline{M}^G_{\boldsymbol{v}})$ in the range $v_1 \gg v_2$.

3.2.1 From Hamiltonian circle actions to orbifolds

The object of this section is to review and slightly extend a general construction from [16], providing a Kähler orbifold from a Hamiltonian circle action with positive moment map. This construction generalizes the one of weighted projective spaces. A wider formulation in the setting of Hamiltonian torus actions is given in [17].

Let R be an r-dimensional connected projective manifold, with complex structure J_R, and let (B, h) be a positive holomorphic line bundle on R, with ∇ the unique compatible covariant derivative. Also, let $Y \subset B^\vee$ be the unit circle bundle, with projection $\pi : Y \to R$; let $\alpha \in \Omega^1(Y)$ be the connection form corresponding to ∇. Hence (by the positivity of (B, h)) $\mathrm{d}\alpha = 2\,\pi^*(\omega_R)$, where ω_R is a Hodge form on R. Thus $(R, J_R, 2\,\omega_R)$ is a Kähler manifold.

Suppose that there is an holomorphic and Hamiltonian circle action $\mu : T^1 \times R \to R$ on $(R, J_R, 2\,\omega_R)$, with (normalized) moment map $\mathcal{M} : R \to \mathbb{R}$. Then there is an infinitesimal 'action' $\mathrm{d}\mu : \mathfrak{t}^1 \to \mathfrak{X}(R)$ at Lie algebra level. These Hamiltonian data determine an infinitesimal contact CR action of T^1 on Y, lifting $\mathrm{d}\mu$ [10]: if $\xi = \partial/\partial r \in Lie(T^1) \cong \mathbb{R}$ then

$$\xi_Y := \xi_R^\sharp - \mathcal{M}\,\partial_\theta \in \mathfrak{X}(Y) \tag{40}$$

is a contact vector field. Here $\upsilon^\sharp \in \mathfrak{X}(Y)$ is the horizontal lift of the vector field $\upsilon \in \mathfrak{X}(R)$ with respect to α, and ∂_θ is the generator of the structure circle action on Y (fiber rotation). Furthermore, we write $\mathcal{M}$ for $\mathcal{M} \circ \pi : Y \to \mathbb{R}$.

Let us make the stronger hypothesis that that there is an actual group action $\tilde{\mu} : T^1 \times Y \to Y$ lifting μ associated to this infinitesimal lift; that is, $\mathrm{d}\tilde{\mu}(\xi) = \xi_Y$. Let us suppose also that $\mathcal{M} > 0$. Then, in view of (40), $\xi_Y(y) \neq 0$ at every $y \in Y$; thus $\tilde{\mu}$ is locally free. Perhaps passing to a quotient group if necessary, we may assume that $\tilde{\mu}$ is effective, whence generically free. Therefore the orbit space $R' := Y/\tilde{\mu}$ is naturally an orbifold, and the projection $\pi' : Y \to R'$ is an orbifold circle bundle on R'.

On Y, we have the following distributions:

1. the vertical tangent space for π, $V(\pi) := \ker(\mathrm{d}\pi) = \mathrm{span}(\partial_\theta)$;
2. the horizontal tangent space for α, $H = \ker(\alpha)$;
3. the vertical tangent space for π', $V(\pi') := \ker(\mathrm{d}\pi') = \mathrm{span}(\xi_Y)$.

For every $y \in Y$, $V(\pi)_y \subset T_yY$ is the tangent space to the S^1-orbit (we denote the circle by S^1 when it acts on Y by the structure rotation action), $V(\pi')_y \subset T_yY$ is the tangent space to the T^1-orbit, and $H(y)$ is isomorphic to $T_{\pi(y)}R$ via $\mathrm{d}_y\pi$, and to the uniformized tangent space $T_{\pi'(y)}R'$ via $\mathrm{d}_y\pi'$. The tangent bundle of Y splits as

$$TY = V(\pi) \oplus H = V(\pi') \oplus H. \tag{41}$$

Let J_H be the complex structure on the vector bundle H given by pull-back of J. Then (H, J_H) is a $\tilde{\mu}$-invariant CR structure on Y, and it descends to an orbifold complex structure $J_{R'}$ on R' (the arguments in [16] were formulated over the smooth locus, but they can be extended to the orbifold case; see also [17]). Thus $(R', J_{R'})$ is a complex orbifold.

Let us set $\beta := \alpha/\mathcal{M} \in \Omega^1(Y)$; then $H = \ker(\beta)$, β is $\tilde{\mu}$-invariant and $\beta(\xi_Y) = -1$. Hence β is a connection form for q. Thus there exists $\omega_{R'} \in \Omega^2(R')$ such that $\mathrm{d}\beta = 2\,(\pi')^*(\omega_{R'})$. Since

$$\mathrm{d}\beta = -\frac{1}{\mathcal{M}^2}\,\mathrm{d}\mathcal{M} \wedge \alpha + \frac{2}{\mathcal{M}}\,\pi^*(\omega_R),$$

$\mathrm{d}\beta$ restricts on each $H(y)$ to a linear symplectic structure compatible with $J_H(y)$; therefore $\omega_{R'}$ is an orbifold Kähler form on $(R', J_{R'})$ (see §2.2 of [16]).

Remark 3.2 The two orbifold fibrations $R \xleftarrow{\pi} Y \xrightarrow{\pi'} R'$ are dual to each other, meaning that $(R')' = R$ as Kähler orbifolds. More precisely, the S^1-action r on Y given by counter-clockise fiber rotation descends to an Hamiltonian action μ' on $(R', \omega_{R'})$, with moment map $1/\mathcal{M}$ (interpreted as a function on R'), of which it is the contact lift. Applying the same procedure to $(R', J_{R'}, \omega_{R'}, \mu')$ we return to (R, J_R, ω_R, μ) (see §2.3 of [16]). In principle, one would need to phrase the previous discussion assuming that R itself is an orbifold, but this won't be needed in the following.

A special case of this construction is given by weighted projective spaces. Let $\mathbf{a} = (\, a_0 \quad \cdots \quad a_k \,)$ be a string of positive integers, and consider the action $\mu^{\mathbf{a}}$ of T^1 on $\mathbb{P}^k$ given by

$$\mu^{\mathbf{a}}_{\vartheta} : [z_0 : \cdots : z_k] \mapsto \left[e^{-\imath\, a_0\, \vartheta}\, z_0 : \cdots : e^{-\imath\, a_k\, \vartheta}\, z_k\right]. \tag{42}$$

Then $\mu^{\mathbf{a}}$ is Hamiltonian with respect to $2\,\omega_{FS}$, with normalized moment map

$$\Phi^{\mathbf{a}}(Z) := \frac{1}{\|Z\|^2} \sum_{j=0}^{k} a_j\, |z_j|^2. \tag{43}$$

Let $H_k = \mathcal{O}_{\mathbb{P}^k}(1)$ be the hyperplane line bundle on $\mathbb{P}^k$, endowed with the standard Hermitian metric; its dual $H_k^\vee$ is the tautological line bundle, and the unit circle bundle in $H_k^\vee$ is the unit sphere $S^{2k+1} \subset \mathbb{C}^{k+1}$, with projection the Hopf map $\pi : S^{2k+1} \to \mathbb{P}^k$. A contact lift of $\mu^{\mathbf{a}}$ is given by the restriction to S^{2k+1} of the unitary representation

$$\tilde{\mu}^{\mathbf{a}}_{\vartheta} : (z_0, \cdots, z_k) \mapsto \left(e^{-\imath\, a_0\, \vartheta}\, z_0, \cdots, e^{-\imath\, a_k\, \vartheta}\, z_k\right). \tag{44}$$

We shall use the same symbol $\tilde{\mu}^{\mathbf{a}}_{\vartheta}$ for both the unitary representation and its restriction to S^{2k+1}. $\tilde{\mu}^{\mathbf{a}}$ is generically free if the a_j's are coprime. The quotient $S^{2k+1}/\tilde{\mu}^{\mathbf{a}}$ is the weighted projective space $\mathbb{P}(\mathbf{a})$. Let $\pi' : S^{2k+1} \to \mathbb{P}(\mathbf{a})$ denote the projection.

The induced orbifold Kähler structure $\eta^{\mathbf{a}} \in \Omega^2(\mathbb{P}(\mathbf{a}))$ is as follows. The vector field generating (44) is $-\tilde{V}_{\mathbf{a}}$, where

$$\tilde{V}_{\mathbf{a}} = \imath \sum_{j=0}^{k} a_j \left(z_j\, \frac{\partial}{\partial z_j} - \bar{z}_j\, \frac{\partial}{\partial \bar{z}_j} \right), \tag{45}$$

viewed as a vector field on S^{2k+1}. $\tilde{V}_{\mathbf{a}}$ is a contact lift of $V_{\mathbf{a}}$, where $-V_{\mathbf{a}}$ is the vector field generating (42). The corresponding moment map (43) can be obtained by pairing $\tilde{V}_{\mathbf{a}}$ with the connection form

$$\alpha = \frac{\imath}{2} \sum_{j=0}^{k} \left(z_j\, \mathrm{d}\bar{z}_j - \bar{z}_j\, \mathrm{d}z_j \right).$$

Hence $\beta^{\mathbf{a}} := \alpha/\Phi^{\mathbf{a}}$ is a connection form for the action generated by $V_{\mathbf{a}}$ on S^{2k+1} (as usual, we write $\Phi^{\mathbf{a}}$ for $\Phi^{\mathbf{a}} \circ \pi$). Then $\eta^{\mathbf{a}}$ is determined by the relation $2\,\pi'^*(\eta^{\mathbf{a}}) = \mathrm{d}\beta^{\mathbf{a}}$.

The Kähler structures on $\mathbb{P}^k$ and $\mathbb{P}(\mathbf{a})$ can be changed by modifying the Hermitian product on $\mathbb{C}^{k+1}$. Let $\mathbf{d} = (d_0, \ldots, d_k)$ be a string of positive integers, and set

$$h_{\mathbf{d}}(Z, Z') := \sum_{j=0}^{k} d_j\, z_j\, \overline{z'_j}, \quad \tilde{\omega}_{\mathbf{d}} := -\Im(h_{\mathbf{d}}) = \frac{\imath}{2} \sum_{j=0}^{k} d_j\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j. \tag{46}$$

The action $r_{-\vartheta} : Z \mapsto e^{-\imath\,\vartheta}\, Z$ of S^1 on $(\mathbb{C}^{k+1}, 2\,\tilde{\omega}_{\mathbf{d}})$ is Hamiltonian, with normalized moment map

$$N_{\mathbf{d}}(Z) := \sum_{j=0}^{k} d_j\, |z_j|^2.$$

Let $S^{2k+1}_{\mathbf{d}} := N_{\mathbf{d}}^{-1}(1) \subset \mathbb{C}^{k+1}$ be the unit sphere for $h_{\mathbf{d}}$. Thus $S^{2k+1}_{\mathbf{d}}$ is the unit circle bundle in $H_k^\vee$ with respect to the line bundle metric induced by $h_{\mathbf{d}}$. The quotient $S^{2k+1}_{\mathbf{d}}/r$ is again $\mathbb{P}^k$, with a new Kähler structure $\omega_{\mathbf{d}}$ (the symplectic reduction of $\tilde{\omega}_{\mathbf{d}}$). More explicitly, let $\pi_{\mathbf{d}} : S^{2k+1}_{\mathbf{d}} \to \mathbb{P}^k$ be the projection, $\iota_{\mathbf{d}} : S^{2k+1}_{\mathbf{d}} \to \mathbb{C}^{k+1}$ the inclusion, and set

$$\alpha_{\mathbf{d}} := \iota_{\mathbf{d}}^*\left(\frac{\imath}{2}\sum_{j=0}^{k} d_j \left(z_j\, \mathrm{d}\overline{z}_j - \overline{z}_j\, \mathrm{d}z_j\right)\right).$$

Then $\alpha_{\mathbf{d}}$ is the connection 1-form on $S^{2k+1}_{\mathbf{d}}$ for $\pi_{\mathbf{d}}$, and

$$\mathrm{d}\alpha_{\mathbf{d}} = 2\,\pi_{\mathbf{d}}^*(\omega_{\mathbf{d}}) = 2\,\iota_{\mathbf{d}}^*(\tilde{\omega}_{\mathbf{d}}).$$

The action $\mu^{\mathbf{a}}$ in (42) is Hamiltonian on $\left(\mathbb{P}^k, 2\,\omega_{\mathbf{d}}\right)$, with normalized moment map

$$\Phi^{\mathbf{a}}_{\mathbf{d}}([Z]) := \frac{\sum_{j=0}^k a_j \cdot d_j\, |z_j|^2}{\sum_{j=0}^k d_j\, |z_j|^2}. \tag{47}$$

The contact lift of $\mu^{\mathbf{a}}$ to $S^{2k+1}_{\mathbf{d}}$ is again functionally given by (44); we still have $S^{2k+1}_{\mathbf{d}}/\tilde{\mu}^{\mathbf{a}} = \mathbb{P}(\boldsymbol{a})$, but with a new Kähler form $\eta^{\mathbf{a}}_{\mathbf{d}}$. Namely, $\beta^{\mathbf{a}}_{\mathbf{d}} := \alpha_{\mathbf{d}}/\Phi^{\mathbf{a}}_{\mathbf{d}}$ is a connection form for $\tilde{\mu}^{\mathbf{a}}$ on $S^{2k+1}_{\mathbf{d}}$, and $\eta^{\mathbf{a}}_{\mathbf{d}}$ is determined by the condition

$$\mathrm{d}\beta^{\mathbf{a}}_{\mathbf{d}} = 2\, q^{\mathbf{a}\,*}_{\mathbf{d}}(\eta^{\mathbf{a}}_{\mathbf{d}}), \tag{48}$$

where $q^{\mathbf{a}}_{\mathbf{d}} : S^{2k+1}_{\mathbf{d}} \to \mathbb{P}(\boldsymbol{a})$ is the projection. The linear automorphism $\tilde{f}_{\mathbf{d}} : \mathbb{C}^{k+1} \to \mathbb{C}^{k+1}$ given by $(z_j)\mapsto(\sqrt{d}_j\, z_j)$ descends to automorphisms $f_{\mathbf{d}} : \mathbb{P}^k \to \mathbb{P}^k$ and $f^{\mathbf{a}}_{\mathbf{d}} : \mathbb{P}(\mathbf{a}) \to \mathbb{P}(\mathbf{a})$, satisfying $f_{\mathbf{d}}^*(\omega_{FS}) = \omega_{\mathbf{d}}$ and $f^{\mathbf{a}\,*}_{\mathbf{d}}(\eta'_{\mathbf{a}}) = \eta^{\mathbf{a}}_{\mathbf{d}}$.

Let us remark in passing the following homogeneity property.

Lemma 3.5 *For any string of positive integers* $\mathbf{d} = (\, d_0 \quad \cdots \quad d_k\,)$ *and* $r = 1, 2, \ldots$, *we have* $\omega_{r\,\mathbf{d}} = \omega_{\mathbf{d}} \in \Omega^2(\mathbb{P}^k)$.

Proof of Lemma 3.5 Let $\pi_{\mathbf{d}} : S^{2k+1}_{\mathbf{d}} \to \mathbb{P}^k$, $\pi_{r\,\mathbf{d}} : S^{2k+1}_{r\,\mathbf{d}} \to \mathbb{P}^k$ be the the Hopf maps. We have, by definition, $h_{r\mathbf{a}} = r\, h_{\mathbf{a}}$; therefore, $S^{2k+1}_{r\,\mathbf{a}} = \delta_{\frac{1}{\sqrt{r}}}\left(S^{2k+1}_{\mathbf{a}}\right)$, where $\delta_s(Z) = s\,Z$. Since $\pi_{\mathbf{a}} = \pi_{r\mathbf{a}} \circ \delta_{\frac{1}{\sqrt{r}}}$, we have

$$\pi_{\mathbf{a}}^*(\omega_{r\,\mathbf{a}}) = \delta^*_{\frac{1}{\sqrt{r}}}\left(\pi^*_{r\mathbf{a}}(\omega_{r\,\mathbf{a}})\right) = \delta^*_{\frac{1}{\sqrt{r}}}(\tilde{\omega}_{r\,\mathbf{a}}) = \tilde{\omega}_{\mathbf{a}} = \pi_{\mathbf{a}}^*(\omega_{\mathbf{a}}).$$

Corollary 3.2 *If* $r = 1, 2, \ldots$ *and* $\mathbf{r} = (\, r \quad \cdots \quad r\,)$, *then* $\omega_{\mathbf{r}} = \omega_{FS}$ (*the standard Fubini-Study form*)*.?*ul " "?>□

Proof ω_{FS} corresponds to $\mathbf{1} = (\,1 \quad \cdots \quad 1\,)$.

The following variant yields a class of weighted projective varieties. Let $\mathbf{b} = (\, b_0 \quad \cdots \quad b_l\,)$ be another string of positive integers. On $\mathbb{P}^k \times \mathbb{P}^l$, consider the Kähler structure $\omega_{\mathbf{a},\mathbf{b}} := \omega_{\mathbf{a}} + \omega_{\mathbf{b}}$ (symbols of pull-back are omitted). $\omega_{\mathbf{a},\mathbf{b}}$ is the Hodge form associated to $H_{k,l} := \mathcal{O}_{\mathbb{P}^k}(1)\boxtimes\mathcal{O}_{\mathbb{P}^l}(1)$ and the tensor product of the Hermitian products $h_{\mathbf{a}}$, $h_{\mathbf{b}}$. The corresponding unit circle bundle $X_{\mathbf{a},\mathbf{b}} \subset H^\vee_{k,l}$ can be identified with the image

$S^{2k+1}_{\mathbf{a}} \otimes_{k,l} S^{2l+1}_{\mathbf{b}} \subset \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ of the map

$$\tau_{\mathbf{a},\mathbf{b}} : (Z, W) \in S^{2k+1}_{\mathbf{a}} \times S^{2l+1}_{\mathbf{b}} \mapsto Z \otimes_{k,l} W \in \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}; \tag{49}$$

we have denoted by $\otimes_{k,l} : \mathbb{C}^{k+1} \times \mathbb{C}^{l+1} \to \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ the tensor product operation. Equivalently, $X_{\mathbf{a},\mathbf{b}}$ is the quotient of $S^{2k+1}_{\mathbf{a}} \times S^{2l+1}_{\mathbf{b}}$ by the S^1-action $(Z, W) \mapsto \left(e^{\imath\,\theta} Z, e^{-\imath\,\theta} W\right)$. The S^1-action on $X_{\mathbf{a},\mathbf{b}}$ given by scalar multiplication (clockwise rotation) is $r_{e^{\imath\vartheta}}(Z \otimes_{k,l} W) := e^{\imath\,\vartheta} Z \otimes_{k,l} W$. The projection $\pi_{\mathbf{a},\mathbf{b}} : X_{\mathbf{a},\mathbf{b}} \to \mathbb{P}^k \times \mathbb{P}^l$ is $\pi_{\mathbf{a},\mathbf{b}}(Z \otimes_{k,l} W) := ([Z], [W])$.

Let $\imath_{\mathbf{a},\mathbf{b}} : S^{2k+1}_{\mathbf{a}} \times S^{2l+1}_{\mathbf{b}} \hookrightarrow \mathbb{C}^{k+1} \times \mathbb{C}^{l+1}$ be the inclusion. The connection 1-form $\alpha_{\mathbf{a},\mathbf{b}}$ on $X_{\mathbf{a},\mathbf{b}}$ is determined by the relation

$$\tau^*_{\mathbf{a},\mathbf{b}}(\alpha_{\mathbf{a},\mathbf{b}}) = \imath^*_{\mathbf{a},\mathbf{b}}(\tilde{\alpha}_{\mathbf{a},\mathbf{b}}), \tag{50}$$

where

$$\tilde{\alpha}_{\mathbf{a},\mathbf{b}} := \frac{\imath}{2}\left[\sum_{j=0}^{k} a_j \left(z_j\, \mathrm{d}\overline{z}_j - \overline{z}_j\, \mathrm{d}z_j\right) + \sum_{j=0}^{l} b_j \left(w_j\, \mathrm{d}\overline{w}_j - \overline{w}_j\, \mathrm{d}w_j\right)\right]. \tag{51}$$

Furthermore, $\mathrm{d}\alpha_{\mathbf{a},\mathbf{b}} = 2\,\pi^*_{\mathbf{a},\mathbf{b}}(\omega_{\mathbf{a},\mathbf{b}})$.

The product T^1-action

$$\begin{aligned} \mu^{\mathbf{a},\mathbf{b}}_{\vartheta}([Z],\, [W]) =& \left(\left[e^{-\imath\, a_0\,\vartheta} z_0 : \cdots : e^{-\imath\, a_k\,\vartheta} z_k\right], \left[e^{-\imath\, b_0\,\vartheta} w_0 : \cdots : e^{-\imath\, b_l\,\vartheta} w_l\right]\right) \\ =& \left(\mu^{\mathbf{a}}_{\vartheta}([Z]),\, \mu^{\mathbf{b}}_{\vartheta}([W])\right) \end{aligned} \tag{52}$$

is clearly Hamiltonian on $\left(\mathbb{P}^k \times \mathbb{P}^l, 2\,\omega_{\mathbf{a},\mathbf{b}}\right)$, with normalized moment map

$$\Phi_{\mathbf{a},\mathbf{b}}([Z], [W]) := \Phi^{\mathbf{a}}_{\mathbf{a}}([Z]) + \Phi^{\mathbf{b}}_{\mathbf{b}}([W]), \tag{53}$$

where $\Phi^{\mathbf{a}}_{\mathbf{a}}$ and $\Phi^{\mathbf{b}}_{\mathbf{b}}$ are as in (47). Its contact lift $\tilde{\mu}^{\mathbf{a},\mathbf{b}}$ is the restriction to $X_{\mathbf{a},\mathbf{b}} = S^{2k+1}_{\mathbf{a}} \otimes_{k,l} S^{2l+1}_{\mathbf{b}}$ of the tensor product representation $\tilde{\mu}^{\mathbf{a}} \otimes \tilde{\mu}^{\mathbf{b}}$ on $\mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$. The latter is the unitary representation $\tilde{\mu}^{\mathbf{c}}_{\vartheta} : (X_{ij}) \mapsto \left(e^{-\imath\, c_{ij}\,\vartheta} X_{ij}\right)$ associated to the string $\mathbf{c} = (c_{ij})$, with $c_{ij} := a_i + b_j > 0$.

We shall set

$$\mathbb{P}(\mathbf{a}, \mathbf{b}) := X_{\mathbf{a},\mathbf{b}}/\tilde{\mu}^{\mathbf{a},\mathbf{b}},$$

with projection $\pi'_{\mathbf{a},\mathbf{b}} : X_{\mathbf{a},\mathbf{b}} \to \mathbb{P}(\mathbf{a}, \mathbf{b})$, orbifold complex structure $K_{\mathbf{a},\mathbf{b}}$, and Kähler form $\eta_{\mathbf{a},\mathbf{b}}$. Explicitly, $\beta_{\mathbf{a},\mathbf{b}} := \alpha_{\mathbf{a},\mathbf{b}}/\Phi_{\mathbf{a},\mathbf{b}}$ is a connection form for $\pi'_{\mathbf{a},\mathbf{b}}$, and $\eta_{\mathbf{a},\mathbf{b}}$ is determined by the relation

$$2\,{\pi'}^*_{\mathbf{a},\mathbf{b}}(\eta_{\mathbf{a},\mathbf{b}}) = \mathrm{d}\beta_{\mathbf{a},\mathbf{b}}. \tag{54}$$

We can interpret $\mathbb{P}(\mathbf{a}, \mathbf{b})$ as a weighted projective variety, as follows. Consider the Segre embedding

$$\sigma_{k,l} : ([Z], [W]) \in \mathbb{P}^k \times \mathbb{P}^l \mapsto [Z \otimes_{k,l} W] \in \mathbb{P}\left(\mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}\right) \cong \mathbb{P}^{kl+k+l}.$$

In coordinates, this is given by $T_{ij} = Z_i\, W_j$. Let $\mathcal{C}_{k,l} \subset \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ be the affine cone over $\sigma_{k,l}(\mathbb{P}^k \times \mathbb{P}^l)$; its ideal $I(\mathcal{C}_{k,l}) \trianglelefteq \mathbb{K}[X_{ij}]$ is generated by the quadratic polynomials $T_{ij}\, T_{ab} - T_{ib}\, T_{aj}$ $(0 \le i, a \le k,\ 0 \le j, b \le l)$.

Let us denote by $\tilde{\mu}^{\mathbf{c}}_{\mathbb{C}^*}$ the extension of $\tilde{\mu}^{\mathbf{c}}$ to $\mathbb{C}^*$, and consider the weighted projective space

$$\mathbb{P}(\mathbf{c}) := \left(\mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1} \setminus \{0\}\right)/\tilde{\mu}^{\mathbf{c}}_{\mathbb{C}^*}.$$

The weighted projective subvarieties of $\mathbb{P}(\mathbf{c})$ are in one-to-one correspondence with the prime ideals of $\mathbb{K}[T_{ij}]$ that are homogeous with respect to the grading $\deg_{\mathbf{c}}(T_{ij}) = c_{ij}$. Since $I(\mathcal{C}_{k,l})$ is generated by $\deg_{\mathbf{c}}$-homogenous elements, it determines a weighted projective subvariety

$$P(\mathcal{C}_{k,l};\mathbf{c}) := \mathcal{C}_{k,l}/\tilde{\mu}^{\mathbf{c}}_{\mathbb{C}^*} \subset \mathbb{P}(\mathbf{c}).$$

Let $\mathbf{d} = (d_{ij})$ be any positive sequence, and let $S^{2(kl+k+l)+1}_{\mathbf{d}} \subset \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ be the unit sphere for the Hermitian product $h_{\mathbf{d}}$. Then $S^{2(kl+k+l)+1}_{\mathbf{d}}$ is $\tilde{\mu}^{\mathbf{c}}$-invariant, and $\mathbb{P}(\mathbf{c}) = S^{2(kl+k+l)+1}_{\mathbf{d}}/\tilde{\mu}^{\mathbf{c}}$. With this description, $\mathbb{P}(\mathbf{c})$ inherits the orbifold Kähler structure $\eta^{\mathbf{c}}_{\mathbf{d}}$. Explicitly, let $\iota_{\mathbf{d}} : S^{2(kl+k+l)+1}_{\mathbf{d}} \hookrightarrow \mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}$ be the inclusion, and set

$$\alpha_{\mathbf{d}} := \iota^*_{\mathbf{d}}\left(\frac{\imath}{2}\sum_{i,j} d_{ij}\left[T_{ij}\,\mathrm{d}\overline{T}_{ij} - \overline{T}_{ij}\,\mathrm{d}T_{ij}\right]\right), \tag{55}$$

$$\Phi^{\mathbf{c}}_{\mathbf{d}}([T]) := \frac{\sum_{i,j} c_{ij}\cdot d_{ij}\,|T_{ij}|^2}{\sum_{i,j} d_{ij}\,|T_{ij}|^2} \qquad \left([T] \in \mathbb{P}\left(\mathbb{C}^{k+1} \otimes \mathbb{C}^{l+1}\right)\right), \tag{56}$$

$$\beta^{\mathbf{c}}_{\mathbf{d}} := \frac{1}{\Phi^{\mathbf{c}}_{\mathbf{d}}}\,\alpha_{\mathbf{d}}, \tag{57}$$

where in the latter relation $\Phi^{\mathbf{c}}_{\mathbf{d}}$ is viewed as a function on $S^{2(kl+k+l)+1}_{\mathbf{d}}$. Then $\beta^{\mathbf{c}}_{\mathbf{d}}$ is a connection 1-form for the projection $q^{\mathbf{c}}_{\mathbf{d}} : S^{2(kl+k+l)+1}_{\mathbf{d}} \to \mathbb{P}(\mathbf{c})$, and $\eta^{\mathbf{c}}_{\mathbf{d}}$ satisfies

$$2\,q^{\mathbf{c}\,*}_{\mathbf{d}}\left(\eta^{\mathbf{c}}_{\mathbf{d}}\right) = \mathrm{d}\beta^{\mathbf{c}}_{\mathbf{d}} \tag{58}$$

(recall (48) and (47)). Hence, $\eta^{\mathbf{c}}_{\mathbf{d}}$ restricts to an orbifold Kähler structure on the complex suborbifold $P(\mathcal{C}_{k,l};\mathbf{c}) \subset \mathbb{P}(\mathbf{c})$.

The following is left to the reader:

Lemma 3.6 *If $d_{ij} = a_i \cdot b_j$, then $\mathcal{C}_{k,l} \cap S^{2(kl+k+l)+1}_{\mathbf{d}} = X_{\mathbf{a},\mathbf{b}}$. Hence $P(\mathcal{C}_{k,l};\mathbf{c}) = \mathbb{P}(\mathbf{a},\mathbf{b})$.*

Lemma 3.7 *Assume $c_{ij} = a_i + b_j$, $d_{ij} = a_i\,b_j$. Let $\jmath : \mathbb{P}(\mathbf{a},\mathbf{b}) \hookrightarrow \mathbb{P}(\mathbf{c})$ be the inclusion, and let $\eta_{\mathbf{a},\mathbf{b}}$ be as in (54). Then $\jmath^*(\eta^{\mathbf{c}}_{\mathbf{d}}) = \eta_{\mathbf{a},\mathbf{b}}$.*

Proof of Lemma 3.7 In view of (54), (57) and (58), we need only prove that $\alpha_{\mathbf{d}}$ and $\Phi^{\mathbf{c}}_{\mathbf{d}}$ pull back on $X_{\mathbf{a},\mathbf{b}}$ to, respectively, $\alpha_{\mathbf{a},\mathbf{b}}$ in (50) and $\Phi_{\mathbf{a},\mathbf{b}}$ in (53). This follows from a straighforward computation by setting $T_{ij} = Z_i\,W_j$ in (55) and (56).

Summing up, we have proved the following .

Proposition 3.1 *Let* $\mathbf{a} = (a_0, \ldots, a_k)$, $\mathbf{b} = (b_0, \ldots, b_l)$ *be sequences of positive integers, and set* $c_{ij} := a_i + b_j$. *Define a grading on* $\mathbb{K}[T_{ij}]$ *by setting* $\deg_{\mathbf{c}}(T_{ij}) = c_{ij}$. *Then the ideal* $I \trianglelefteq \mathbb{K}[T_{ij}]$ *with generators* $T_{ij} T_{ab} - T_{ib} T_{aj}$ *is* $\deg_{\mathbf{c}}$*-homogenous, and* $\mathbb{P}(\mathbf{a}, \mathbf{b}) \subset \mathbb{P}(\mathbf{c})$ *is the corresponding weighted projective variety. Furthermore, if* $d_{ij} := a_i b_j$ *then* $(\mathbb{P}(\mathbf{a}, \mathbf{b}), \eta_{\mathbf{a},\mathbf{b}})$ *is a Kähler suborbifold of* $(\mathbb{P}(\mathbf{c}), \eta^{\mathbf{c}}_{\mathbf{d}})$.

The T^1-action on $\mathbb{P}^k \times \mathbb{P}^l$

$$\begin{aligned} \mu^{\mathbf{a},-\mathbf{b}}_\vartheta([Z],\,[W]) :=& \left(\left[e^{-\imath a_0 \vartheta} z_0 : \cdots : e^{-\imath a_k \vartheta} z_k\right],\, \left[e^{\imath b_0 \vartheta} w_0 : \cdots : e^{\imath b_l \vartheta} w_l\right]\right) \\ =& \left(\mu^{\mathbf{a}}_\vartheta([Z]),\, \mu^{\mathbf{b}}_{-\vartheta}([W])\right) \end{aligned} \tag{59}$$

can be interpreted in terms of the previous case by passing to the opposite Kähler structure on $\mathbb{P}^l$, and noting that $e^{\imath b_j \vartheta} e_j = e^{-\imath b_j \vartheta} \bullet e_j$, where (e_j) is the standard basis and $\bullet$ denotes scalar multiplication in $\overline{\mathbb{C}^{l+1}}$. Namely, let us consider $\mathbb{P}^k \times \overline{\mathbb{P}^l}$, endowed with the Kähler form $\omega_{\mathbf{a},-\mathbf{b}} := \omega_{\mathbf{a}} - \omega_{\mathbf{b}}$. The latter is the Hodge form associated to the holomorphic line bundle $H_{k,\bar{l}} := \mathcal{O}_{\mathbb{P}^k}(1) \boxtimes \mathcal{O}_{\overline{\mathbb{P}^l}}(1)$ and the positive metric on it given by the tensor product of the Hermitian metrics induced by $h_{\mathbf{a}}$ on $\mathbb{C}^{k+1}$ and $\overline{h_{\mathbf{b}}}$ on $\overline{\mathbb{C}^{l+1}}$. The corresponding unit circle bundle $X_{\mathbf{a},-\mathbf{b}} = S^{2k+1}_{\mathbf{a}} \otimes_{k,\bar{l}} S^{2l+1}_{\mathbf{b}}$ is the image of the map

$$\tau_{\mathbf{a},-\mathbf{b}} : (Z, W) \in S^{2k+1}_{\mathbf{a}} \times S^{2l+1}_{\mathbf{b}} \mapsto Z \otimes_{k,\bar{l}} W \in \mathbb{C}^{k+1} \otimes \overline{\mathbb{C}^{l+1}};$$

we have denoted by $\otimes_{k,\bar{l}} : \mathbb{C}^{k+1} \times \mathbb{C}^{l+1} \to \mathbb{C}^{k+1} \otimes \overline{\mathbb{C}^{l+1}}$ the tensor product operation. Thus componentwise $(Z_i) \otimes_{k,\bar{l}} (W_j) = (Z_i \overline{W}_j)$. Equivalently, it is the quotient of $S^{2k+1}_{\mathbf{a}} \times S^{2l+1}_{\mathbf{b}}$ by the S^1-action $(Z, W) \mapsto \left(e^{\imath\theta} Z, e^{\imath\theta} W\right)$. The projection $\pi_{\mathbf{a},-\mathbf{b}} : X_{\mathbf{a},-\mathbf{b}} \to \mathbb{P}^k \times \overline{\mathbb{P}^l}$ is $Z \otimes_{k,\bar{l}} W \mapsto ([Z], [W])$, and the connection form $\alpha_{\mathbf{a},-\mathbf{b}}$ is determined by obvious variants of (50) and (51). We have $2\,\pi^*_{\mathbf{a},-\mathbf{b}}(\omega_{\mathbf{a},-\mathbf{b}}) = \mathrm{d}\alpha_{\mathbf{a},-\mathbf{b}}$.

Then $\mu^{\mathbf{a},-\mathbf{b}}$ in (59) is Hamiltonian with respect to $2\,\omega_{\mathbf{a},-\mathbf{b}}$, with normalized moment map $\Phi_{\mathbf{a},\mathbf{b}}$ in (53). Its contact lift $\tilde{\mu}^{\mathbf{a},-\mathbf{b}}$ to $X_{\mathbf{a},-\mathbf{b}}$ is the tensor product (for $\otimes_{k,\bar{l}}$) of the flows $\tilde{\mu}^{\mathbf{a}}_\vartheta$ and $\tilde{\mu}^{-\mathbf{b}}_\vartheta$. We shall set $\mathbb{P}(\mathbf{a}, -\mathbf{b}) := X_{\mathbf{a},-\mathbf{b}}/\tilde{\mu}^{\mathbf{a},-\mathbf{b}}$, with projection $q_{\mathbf{a},-\mathbf{b}} : X_{\mathbf{a},-\mathbf{b}} \to \mathbb{P}(\mathbf{a}, -\mathbf{b})$, and denote by $\eta_{\mathbf{a},-\mathbf{b}}$ and $K_{\mathbf{a},-\mathbf{b}}$ its (orbifold) symplectic and complex structures, respectively. Thus

$$2\,q^*_{\mathbf{a},-\mathbf{b}}(\eta_{\mathbf{a},-\mathbf{b}}) = \mathrm{d}\beta_{\mathbf{a},-\mathbf{b}}, \quad \text{where} \quad \beta_{\mathbf{a},-\mathbf{b}} := \alpha_{\mathbf{a},-\mathbf{b}}/\Phi_{\mathbf{a},\mathbf{b}}. \tag{60}$$

The Segre embedding

$$\sigma_{k,\bar{l}} : ([Z], [W]) \in \mathbb{P}^k \times \mathbb{P}^l = \mathbb{P}^k \times \overline{\mathbb{P}^l} \mapsto [Z \otimes_{k,\bar{l}} W] \in \mathbb{P}\left(\mathbb{C}^{k+1} \otimes \overline{\mathbb{C}}^{l+1}\right),$$

given in coordinates by $T_{i,\bar{j}} = Z_i \overline{W}_j$, intertwines $\mu^{\mathbf{a}} \times \mu^{-\mathbf{b}}$ with $\mu^{\mathbf{a}} \otimes_{k,\bar{l}} \mu^{-\mathbf{b}} = \mu^{\mathbf{c}}$, where $c_{ij} = a_i + b_j$. The unitary representation $\tilde{\mu}^{\mathbf{c}}$ on $\mathbb{C}^{k+1} \otimes \overline{\mathbb{C}^{l+1}}$ is defined in terms of the identification $\mathbb{C}^{k+1} \otimes \overline{\mathbb{C}^{l+1}} \cong \mathbb{C}^{k\,l+k+l+1}$ given by the basis $e_{i\bar{j}} := e^k_i \otimes_{k,\bar{l}} e^l_j$, where $(e^k_i)^k_{i=0}$ and $(e^l_j)^l_{j=0}$ are, respectively, the standard basis of $\mathbb{C}^{k+1}$ and $\mathbb{C}^{l+1}$. Coordinatewise, $\mu^{\mathbf{c}}_\vartheta([T_{i,\bar{j}}]) = \left[e^{-\imath c_{ij} \vartheta} T_{i,\bar{j}}\right]$. The same argument used above realizes $\mathbb{P}(\mathbf{a}, -\mathbf{b})$ as the weighted projective variety associated to the cone $\mathcal{C}_{k,\bar{l}} \subset \mathbb{C}^{k+1} \otimes \overline{\mathbb{C}}^{l+1}$ over $\sigma_{k,\bar{l}}(\mathbb{P}^k \times \overline{\mathbb{P}^l})$ and the weighting $\mathbf{c}$, with induced orbifold Kähler structure $\eta_{\mathbf{a},-\mathbf{b}}$.

The latter case is equivalent to the previous one, once we use the standard basis to induce a unitary isomorphism $\overline{\mathbb{C}^{l+1}} \cong \mathbb{C}^{l+1}$. The reason for emphasizing the coexistence of the complex structures on $\mathbb{P}^l$ and $\overline{\mathbb{P}^l}$ is the following. Being the quotient of $S^{2k+1}_{\mathbf{a}} \times S^{2l+1}_{\mathbf{b}}$ by the S^1-action $(Z, W) \mapsto \left(e^{\imath\,\theta} Z, e^{\imath\,\theta} W\right)$, $X_{\mathbf{a},-\mathbf{b}}$ is diffeomorphic to the submanifold $Y_{\mathbf{a},-\mathbf{b}} \subset \mathbb{P}^{k+l+1}$ given by

$$Y_{\mathbf{a},-\mathbf{b}} := \left\{[Z : W] \in \mathbb{P}^{k+l+1} \;:\; \|Z\|_{\mathbf{a}} = \|W\|_{\mathbf{b}}\right\}. \tag{61}$$

Explicitly, the diffeomorphism

$$f_{\mathbf{a},-\mathbf{b}} : [Z : W] \in Y_{\mathbf{a},-\mathbf{b}} \mapsto \frac{Z}{\|Z\|_{\mathbf{a}}} \otimes_{k,\bar{l}} \frac{W}{\|W\|_{\mathbf{b}}} \in X_{\mathbf{a},-\mathbf{b}} \tag{62}$$

intertwines the S^1-action

$$r : \left(e^{\imath\,\vartheta}, [Z : W]\right) \in S^1 \times Y_{\mathbf{a},-\mathbf{b}} \mapsto \left[e^{\imath\vartheta/2} Z : e^{-\imath\vartheta/2} W\right] \in Y_{\mathbf{a},-\mathbf{b}} \tag{63}$$

with the structure bundle action on $X_{\mathbf{a},-\mathbf{b}}$ given by scalar multiplication.

As a hypersurface in $\mathbb{P}^{k+l+1}$, $Y_{\mathbf{a},-\mathbf{b}}$ inherits an alternative CR structure. To interpret the latter, notice that $Y_{\mathbf{a},-\mathbf{b}}$ may be identified with the unit circle bundle $Z_{\mathbf{a},-\mathbf{b}} \subset \mathcal{O}_{\mathbb{P}^k}(-1) \boxtimes \mathcal{O}_{\mathbb{P}^l}(1)$. To make this explicit, given a one-dimensional complex vector space L and $\ell \in L$, $\ell \neq 0$, let $\ell^* \in L^\vee$ be the uniquely determined element such that $\ell^*(\ell) = 1$. Then the diffeomorphism

$$g_{\mathbf{a},-\mathbf{b}} : [Z : W] \in Y_{\mathbf{a},-\mathbf{b}} \mapsto \frac{Z}{\|Z\|_{\mathbf{a}}} \otimes_{k,l} \left(\frac{W}{\|W\|_{\mathbf{b}}}\right)^* \in Z_{\mathbf{a},-\mathbf{b}} \tag{64}$$

intertwines the action (63) with the structure bundle action on $Z_{\mathbf{a},-\mathbf{b}}$ given by scalar multiplication. Thus we have two S^1-equivariant diffeomorphisms $X_{\mathbf{a},-\mathbf{b}} \xleftarrow{f_{\mathbf{a},-\mathbf{b}}} Y_{\mathbf{a},-\mathbf{b}} \xrightarrow{g_{\mathbf{a},-\mathbf{b}}} Z_{\mathbf{a},-\mathbf{b}}$ and the composition $f_{\mathbf{a},-\mathbf{b}} \circ g^{-1}_{\mathbf{a},-\mathbf{b}} : Z_{\mathbf{a},-\mathbf{b}} \to X_{\mathbf{a},-\mathbf{b}}$ covers the identity $\mathbb{P}^k \times \mathbb{P}^l \to \mathbb{P}^k \times \overline{\mathbb{P}^l}$.

3.2.2 Application to symplectic reductions

Let be given an Hamiltonian action $\beta : S^1 \times N \to N$ on a symplectic manifold (N, Ω), with normalized moment map $\mathfrak{B} : N \to \mathbb{R}$, such that 0 is a regular value of $\mathfrak{B}$. Then the quotient $N_0 := \mathfrak{B}^{-1}(0)/\beta$ is an orbifold.

Let $\pi : \mathfrak{B}^{-1}(0) \to N_0$ be the projection, and $\imath : \mathfrak{B}^{-1}(0) \hookrightarrow N$ be the inclusion. The reduced orbifold symplectic structure Ω_0 is determined by the condition $\imath^*(\Omega) = \pi^*(\Omega_0)$.

One the other hand, given a connection 1-form α for the S^1-action on $\mathfrak{B}^{-1}(0)$, a closed form Ω_0' on N_0 is determined by the condition $d\alpha = 2\,\pi^*(\Omega_0')$ [3]. $[\Omega_0'] \in H^2(N_0, \mathbb{R})$ is the Chern class of a principal S^1-bundle naturally associated to π (see [3, 19] for a precise discussion).

Let J be a complex structure on N compatible with Ω, so that (N, J, Ω) is a Kähler manifold, and such that β is holomorphic (i.e., $\beta_g : M \to M$ is J-holomorphic for every $g \in S^1$); then J descends to an orbifold complex structure J_0 on N_0 compatible with Ω_0.

Thus (N_0, J_0, Ω_0) is a Kähler orbifold. On the other hand, even if Ω_0' turns out to be symplectic, J_0 needn't be compatible with Ω_0'.

We shall apply the considerations in §3.2.1 to describe a class of Hamiltonian circle actions for which Ω_0' is a symplectic form; furthermore, there is a natural alternative choice of a complex structure J_0' on N_0, compatible with Ω_0'. Therefore, in this situation the triple (N_0, J_0', Ω_0') is a Kähler orbifold, generally different from (N_0, J_0, Ω_0). Since $[\Omega_0'] \in H^2(N_0, \mathbb{R})$ is the class appearing in the Duistermaat-Heckman Theorem on the variation of cohomology in symplectic reduction [3], we shall call (N_0, J_0', Ω_0') the *DH-reduction* of (N, J, Ω) under β.

Given integers k, $l \geq 1$, let $\mathbf{a} = (\,a_0 \quad \cdots \quad a_k\,)$, $\mathbf{b} = (\,b_0 \quad \cdots \quad b_l\,)$ be strings of positive integers, and consider the holomorphic action of T^1 on $\mathbb{P}^{k+l+1}$ given by

$$\begin{aligned} &\gamma^{\mathbf{a},-\mathbf{b}}_{e^{\imath\vartheta}}\big([z_0 : \cdots : z_k : w_0 : \cdots : w_l]\big) \\ &\quad = \big[e^{-\imath\, a_0\,\vartheta}\, z_0 : \cdots : e^{-\imath\, a_k\,\vartheta}\, z_k : e^{\imath\, b_0\,\vartheta}\, w_0 : \cdots : e^{\imath\, b_k\,\vartheta}\, w_l\big]. \end{aligned} \tag{65}$$

Then $\gamma^{\mathbf{a},-\mathbf{b}}$ is Hamiltonian with respect to $\Omega = 2\,\omega_{FS}$, with normalized moment map

$$\Gamma_{\mathbf{a},-\mathbf{b}}\big([Z:W]\big) := \frac{1}{\|Z\|^2 + \|W\|^2}\left(\sum_{j=0}^{k} a_j\,|z_j|^2 - \sum_{j=0}^{l} b_j\,|w|_j^2\right). \tag{66}$$

Hence $\Gamma^{-1}_{\mathbf{a},-\mathbf{b}}(0) = Y_{\mathbf{a},-\mathbf{b}}$ (see (61)), and 0 is a regular value of $\Gamma_{\mathbf{a},-\mathbf{b}}$ [7]. In fact, the diffeomorphism $f_{\mathbf{a},-\mathbf{b}}$ in (62) intertwines $\gamma^{\mathbf{a},-\mathbf{b}}$ and $\tilde{\mu}^{\mathbf{a},-\mathbf{b}}$. Therefore, the Kähler orbifold (N_0, Ω_0', J_0') is in this case isomorphic to $\big(\mathbb{P}(\mathbf{a},-\mathbf{b}), \eta_{\mathbf{a},-\mathbf{b}}\big)$ (hence abstractly to $\big(\mathbb{P}(\mathbf{a},\mathbf{b}), \eta_{\mathbf{a},\mathbf{b}}\big)$).

We can relate the complex structures J_0 and J_0' pointwise, as follows. Let $\pi' := q_{\mathbf{a},-\mathbf{b}} \circ f_{\mathbf{a},-\mathbf{b}} : Y_{\mathbf{a},-\mathbf{b}} \to \mathbb{P}(\mathbf{a},-\mathbf{b})$ be the projection, and consider $[Z:W] \in Y_{\mathbf{a},-\mathbf{b}}$. We may assume $\|Z\|_{\mathbf{a}} = \|W\|_{\mathbf{b}} = 1$, i.e. $Z \in S^{2k+1}_{\mathbf{a}}$, $W \in S^{2l+1}_{\mathbf{b}}$. Let $H_Z(S^{2k+1}_{\mathbf{a}}) \subset T_Z S^{2k+1}_{\mathbf{a}}$ and $H_W(S^{2l+1}_{\mathbf{b}}) \subset T_W S^{2l+1}_{\mathbf{b}}$ be the maximal complex subspaces (with respect to the complex structures of $\mathbb{C}^{k+1}$ and $\mathbb{C}^{l+1}$, respectively), with respective complex structures K_Z and L_W. Then the uniformized tangent space of $\mathbb{P}(\mathbf{a},-\mathbf{b})$ at $\pi'([Z:W])$ is canonically isomorphic to $H_Z(S^{2k+1}_{\mathbf{a}}) \times H_W(S^{2l+1}_{\mathbf{b}})$ as a real vector space. The complex structures J_0 and J_0' at $\pi'([Z:W])$ correspond to $K_Z \times L_W$ and $K_Z \times (-L_W)$, respectively.

The previous considerations extend to the cases $k = 0$, $l > 0$, and $k > 0$, $l = 0$. Consider an action γ of T^1 on $\mathbb{P}^{l+1}$ of the form

$$\gamma_{e^{\imath\vartheta}}\big([z_0 : \cdots : z_k : w_0]\big) := \big[e^{-\imath\, a_0\,\vartheta}\, z_0 : \cdots : e^{-\imath\, a_k\,\vartheta}\, z_k : e^{\imath\, b_0 \vartheta}\, w_0\big],$$

with moment map

$$\Gamma : [z_0 : \cdots : z_k : w_0] \mapsto \frac{1}{\|Z\|^2 + |w_0|^2}\left[\sum_{j=0}^{k} a_j\,|z_j|^2 - b_0\,|w_0|^2\right].$$

Hence $Y := \Gamma^{-1}(0)$ is entirely contained in the affine open set where $w_0 \neq 0$; explicitly,

$$Y = \left\{\left[z_0 : \cdots : z_k : \frac{1}{\sqrt{b_0}}\right] \,:\, \sum_{j=0}^{k} a_j\,|z_j|^2 = 1\right\} \cong S^{2k+1}_{\mathbf{a}}.$$

The diffeomorphism $[\mathbf{z} : 1/\sqrt{b_0}] \in Y \mapsto \mathbf{z} \in S^{2k+1}_{\mathbf{a}}$ intertwines γ with the action $\hat{\gamma}_{e^{\imath\vartheta}} : (z_j) \mapsto \left(e^{-\imath\,(a_j+b_0)\,\vartheta}\, z_j\right)$. Assuming, say, that the integers $a_j + b_0$ are coprime, $Y/\tilde{\gamma}$ may be identified with the weighted projective space $\mathbb{P}(b_0 + a_0, \ldots, b_0 + a_k)$, and under the same identification Ω'_0 is the Kähler form $\eta_{(b_0+a_j)}$. In this case, $J_0 = J'_0$.

3.2.3 The DH-reduction of $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$

We aim to describe the DH-reductions of a general $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ with respect to $T^1_{\boldsymbol{\nu}_\perp}$, when $\boldsymbol{\nu}$ varies in $\mathbb{Z}^2$. We shall call this as the $\boldsymbol{\nu}$*-th DH-reduction* of $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$. Recall that this is the triple (N_0, J'_0, Ω'_0) (in the notation in the preample of §3.2.2) when $N = \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ and $\beta = \psi_{\boldsymbol{\nu}_\perp}$ (the restriction of $\Phi_{\mathbf{L},\mathbf{K}}$ to $T^1_{\boldsymbol{\nu}_\perp} \cong S^1$ - see (32)).

By way of example, let us start with two special cases.

Example 3.1 Consider the representation $\mu_1^{\oplus r}$ of G on $W_1^{\oplus r}$, for some $r \geq 1$, as usual composed with the Lie group automorphism $B \mapsto (B^t)^{-1}$. This corresponds to (27) with $\mathbf{K} = \mathbf{1} := \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix}$, $\mathbf{L} = \mathbf{0}$. Let us assume $\nu_1, \nu_2 > 0$.

By (7) and (8), $F_{1,j} : \mathbb{C}^2 \to \mathbb{C}$ for $j = 1, 2$ are given by $F_{1,1}(Z) = z_0$ and $F_{1,2}(Z) = z_1$, where $Z = \begin{pmatrix} z_0 & z_1 \end{pmatrix}$. Hence by (28) the moment map $\Phi_{\mathbf{0},\mathbf{1}} : \mathbb{P}\left(W_1^{\oplus r}\right) \to \mathfrak{g}$ is

$$\Phi_{\mathbf{0},\mathbf{1}}([Z]) = \frac{\imath}{\|Z\|^2} \begin{pmatrix} \sum_{a=1}^r |z_{a,0}|^2 & \sum_{a=1}^r z_{a,1}\,\overline{z_{a,0}} \\ \sum_{a=1}^r z_{a,0}\,\overline{z_{a,1}} & \sum_{a=1}^r |z_{a,1}|^2 \end{pmatrix}. \tag{67}$$

Here $Z = (Z_1, \ldots, Z_r) \in (\mathbb{C}^2)^r \cong \mathbb{C}^{2r}$, and for each a $Z_a = \begin{pmatrix} z_{a,0} & z_{a,1} \end{pmatrix}$. Therefore, with $M = \mathbb{P}\left(W_1^{\oplus r}\right)$,

$$M^T_{\boldsymbol{\nu}} = \left\{ [Z] \,:\, \nu_2 \sum_{a=1}^r |z_{a,0}|^2 = \nu_1 \sum_{a=1}^r |z_{a,1}|^2 \right\}.$$

Let us define $S_j : (\mathbb{C}^2)^r \to \mathbb{C}^r$ by setting $S_j(Z) := \begin{pmatrix} z_{1j} & \cdots & z_{rj} \end{pmatrix}$ for $j = 0, 1$. With the unitary change of coordinates $Z \in \mathbb{C}^{2r} \mapsto \left(S_1(Z), S_0(Z)\right) \in \mathbb{C}^{2r}$, we can identify $M^T_{\boldsymbol{\nu}}$ with

$$M'^T_{\boldsymbol{\nu}} = \left\{ [S_1 : S_0] \in \mathbb{P}^{2r-1} \,:\, \nu_1 \|S_1\|^2 = \nu_2 \|S_0\|^2 \right\}.$$

Let us identify $T^1_{\boldsymbol{\nu}^\perp}$ with S^1 as in (32). Then the action $\psi_{\boldsymbol{\nu}_\perp}$ of $T^1_{\boldsymbol{\nu}^\perp}$ on $\mathbb{P}^{2r-1}$ corresponds to the circle action given by

$$\gamma_{e^{\imath\vartheta}} : [S_1 : S_0] \mapsto \left[e^{-\imath\,\nu_1\,\vartheta}\, S_1 : e^{\imath\,\nu_2\,\vartheta}\, S_0\right]. \tag{68}$$

Hence if we set $\boldsymbol{\nu}_2 := \begin{pmatrix} \nu_2 & \cdots & \nu_2 \end{pmatrix}$, $\boldsymbol{\nu}_1 := \begin{pmatrix} \nu_1 & \cdots & \nu_1 \end{pmatrix} \in \mathbb{Z}^r$ then $\gamma = \gamma^{\boldsymbol{\nu}_1, -\boldsymbol{\nu}_2}$, where notation is as in (65).

We can use $f_{\boldsymbol{\nu}_1,-\boldsymbol{\nu}_2}$ in (62) to identify $M'^T_{\boldsymbol{\nu}} \cong M^T_{\boldsymbol{\nu}}$ with the unit circle bundle $X_{\boldsymbol{\nu}_1,-\boldsymbol{\nu}_2}$ over $\mathbb{P}^{r-1} \times \overline{\mathbb{P}^{r-1}}$, with projection $\pi_{\boldsymbol{\nu}_1,-\boldsymbol{\nu}_2} : [S_1 : S_0] \mapsto ([S_1], [S_0])$. Since γ covers the trivial action on $\mathbb{P}^{r-1} \times \overline{\mathbb{P}^{r-1}}$, $\mathbb{P}(\boldsymbol{\nu}_1, -\boldsymbol{\nu}_2) = \mathbb{P}^{r-1} \times \overline{\mathbb{P}^{r-1}}$.

The connection form $\alpha_{\boldsymbol{\nu}_1,-\boldsymbol{\nu}_2}$ on $M'^T_{\boldsymbol{\nu}} \cong X_{\boldsymbol{\nu}_1,-\boldsymbol{\nu}_2}$, as unit circle bundle in $\mathcal{O}_{\mathbb{P}^{r-1}}(-1) \boxtimes \mathcal{O}_{\overline{\mathbb{P}^{r-1}}}(-1)$, is as follows. Let

$$\Xi : (Z, W) \in S^{2r-1}_{\mathbf{v}_1} \times S^{2r-1}_{\mathbf{v}_2} \mapsto [Z : W] \in M'^{T}_{\mathbf{v}}$$

and let $\jmath : S^{2r-1}_{\mathbf{v}_1} \times S^{2r-1}_{\mathbf{v}_2} \hookrightarrow \mathbb{C}^r \times \mathbb{C}^r$ be the inclusion; clearly, $S^{2r-1}_{\mathbf{v}_1} = S^{2r-1}(1/\sqrt{v_1})$ and $S^{2r-1}_{\mathbf{v}_2} = S^{2r-1}(1/\sqrt{v_2})$ where $S^{2r-1}(r)$ is the sphere centered at the origin of radius $r > 0$. Then $\Xi^*(\alpha_{\mathbf{v}_1,-\mathbf{v}_2}) = \jmath^*(\tilde{\alpha}_{\mathbf{v}_1,-\mathbf{v}_2})$, where

$$\tilde{\alpha}_{\mathbf{v}_1,-\mathbf{v}_2} := \frac{\imath}{2}\left[v_1 \sum_{j=1}^{r}\left(z_{j1}\, \mathrm{d}\overline{z}_{j1} - \overline{z}_{j1}\, \mathrm{d}z_{j1}\right) - v_2 \sum_{j=1}^{r}\left(z_{j0}\, \mathrm{d}\overline{z}_{j0} - \overline{z}_{j0}\, \mathrm{d}z_{j0}\right)\right].$$

The corresponding Kähler structure ω on $\mathbb{P}^{r-1} \times \overline{\mathbb{P}^{r-1}}$ is then uniquely determined by the condition that

$$2\,\Xi^*\left(\pi^*_{\mathbf{v}_1,-\mathbf{v}_2}(\omega)\right) = 2\,\jmath^*\left(\mathrm{d}\tilde{\alpha}_{\mathbf{v}_1,-\mathbf{v}_2}\right).$$

Either by direct inspection, or by appealing to Corollary 3.2, one can verify that $\omega = \pi_1^*(\omega_{FS}) - \pi_2^*(\omega_{FS})$ (π_j is the projection of $\mathbb{P}^{r-1} \times \overline{\mathbb{P}^{r-1}}$ onto the j-th factor). Furthermore, by (53) we have $\Phi_{\mathbf{v}_1,\mathbf{v}_2} = v_1 + v_2$ (constant) and so by (60) we conclude that $\eta_{\mathbf{v}_1,-\mathbf{v}_2} = (v_1 + v_2)^{-1}\,\omega$.

It is evident from (67) that σ_Υ (see Theorem 3.1) is the section of $\mathcal{O}_{\mathbb{P}^r}(1)\boxtimes\mathcal{O}_{\overline{\mathbb{P}^r}}(1)$ given by the bi-homogeneous polynomial $S_1 \cdot \overline{S}_0$. Hence $\overline{M}^G_{\mathbf{v}} \subset \mathbb{P}^r \times \overline{\mathbb{P}^r}$ is a (holomorphic) (1, 1)-divisor.

Example 3.2 Let us consider the representation $\mu_2^{\oplus r}$ on $W_2^{\oplus r}$; thus $\mathbf{K} = \mathbf{2} := (2 \quad \cdots \quad 2)$, $\mathbf{L} = \mathbf{0}$ in (27). The functions $F_{2,j} : \mathbb{C}^3 \to \mathbb{C}^2$ in (7) and (8) are given by

$$F_{2,1} : (z_0 \quad z_1 \quad z_2) \mapsto \left(\sqrt{2}\, z_0 \quad z_1\right), \quad F_{2,2} : (z_0 \quad z_1 \quad z_2) \mapsto \left(z_1 \quad \sqrt{2}\, z_2\right).$$

For $j = 0, 1, 2$ let us define $S_j : \left(\mathbb{C}^3\right)^r \to \mathbb{C}^r$ by setting

$$S_j(Z_1, \ldots, Z_r) := (z_{1,j} \quad \cdots \quad z_{r,j});$$

then by (28)

$$\Phi_{\mathbf{0},\mathbf{2}}([Z]) = \frac{\imath}{\|Z\|^2}\begin{pmatrix} 2\,\|S_0(Z)\|^2 + \|S_1(Z)\|^2 & \sqrt{2}\left(S_1(Z)^t\,\overline{S_0(Z)} + S_2(Z)^t\,\overline{S_1(Z)}\right) \\ \sqrt{2}\left(S_0(Z)^t\,\overline{S_1(Z)} + S_1(Z)^t\,\overline{S_2(Z)}\right) & \|S_1(Z)\|^2 + 2\,\|S_2(Z)\|^2 \end{pmatrix}. \tag{69}$$

Assume $v_1 > v_2 > 0$. With the unitary change of coordinates

$$Z \in \left(\mathbb{C}^3\right)^r \mapsto (S_1(Z) \quad S_2(Z) \quad S_0(Z)) \in (\mathbb{C}^r)^3,$$

$M^T_{\mathbf{v}}$ may be identified with

$$M'^{T}_{\mathbf{v}} := \Big\{[S_1 : S_2 : S_0] \in \mathbb{P}^{3r-1} = \mathbb{P}(\mathbb{C}^r \oplus \mathbb{C}^r \oplus \mathbb{C}^r) \\ : (v_1 - v_2)\,\|S_1\|^2 + 2\,v_1\,\|S_2\|^2 = 2\,v_2\,\|S_0\|^2\Big\}.$$

Furthermore, if we identify $T^1_{\boldsymbol{v}^\perp}$ with S^1 as in (32), its action on $M'^T_{\boldsymbol{v}}$ corresponds to

$$\gamma_{e^{\imath\vartheta}}\big([S_0 : S_1 : S_2]\big) := \Big[e^{-\imath(v_1-v_2)\vartheta}\, S_1 : e^{-2\imath v_1\vartheta}\, S_2 : e^{2\imath v_2\vartheta}\, S_0\Big]. \tag{70}$$

Let us define $\mathbf{a}_{\boldsymbol{v}} \in \mathbb{N}^{2r}$ and $\mathbf{b}_{\boldsymbol{v}} \in \mathbb{N}^r$ by setting

$$\mathbf{a}_{\boldsymbol{v}} := (\, v_1 - v_2 \quad \cdots \quad v_1 - v_2 \quad 2\,v_1 \quad \cdots \quad 2\,v_1\,), \qquad \mathbf{b}_{\boldsymbol{v}} := (\,2\,v_2 \quad \cdots 2\,v_2\,),$$

where $v_1 - v_2$ and $2\,v_1$ are repeated r times. Then by (70) we have $\gamma = \gamma^{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}}$ (see (65)). By means of $f_{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}}$, we can identify $M'^T_{\boldsymbol{v}}$ with the unit circle bundle

$$X_{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}} \subset \mathcal{O}_{\mathbb{P}^{2r-1}}(-1)\boxtimes\mathcal{O}_{\overline{\mathbb{P}^{r-1}}}(-1),$$

with respect to the Hermitian metric induced by $h_{\mathbf{a}_{\boldsymbol{v}}}$ and $h_{\mathbf{b}_{\boldsymbol{v}}}$, with projection $\pi_{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}} : [S_1 : S_2 : S_0] \mapsto ([S_1 : S_2], [S_0])$. The structure S^1-action given by clockwise fibre rotation is

$$r_{e^{-\imath\vartheta}} : [S_1 : S_2 : S_0] \mapsto \Big[e^{-\imath\vartheta/2}\, S_1 : e^{-\imath\vartheta/2}\, S_2 : e^{\imath\vartheta/2}\, S_0\Big].$$

Thus γ may be identified with the contact lift $\tilde{\mu}^{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}}$ to $X_{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}}$ of the Hamiltonian S^1-action $\mu^{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}}$ on $(\mathbb{P}^{2r-1} \times \overline{\mathbb{P}^{r-1}}, 2\,\omega_{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}})$ having moment map $\Phi_{\mathbf{a}_{\boldsymbol{v}},\mathbf{b}_{\boldsymbol{v}}}$ (see the discussion following (59)). Hence (N_0, J'_0, Ω'_0) in §3.2.2 with $N = M$ and $S^1 \cong T^1_{\boldsymbol{v}_\perp}$ is in this case $\big(\mathbb{P}(\mathbf{a}_{\boldsymbol{v}}, -\mathbf{b}_{\boldsymbol{v}}), \eta_{\mathbf{a}_{\boldsymbol{v}},-\mathbf{b}_{\boldsymbol{v}}}\big)$.

We can rewrite (70) as

$$\gamma_{e^{\imath\vartheta}}\big([S_0 : S_1 : S_2]\big) := \Big[e^{-\imath(v_1+v_2)\vartheta}\, S_1 : e^{-2\imath(v_1+v_2)\vartheta}\, S_2 : S_0\Big]. \tag{71}$$

Passing to the quotient group $T^1(\boldsymbol{v})$ in (39), this is the action $\overline{\gamma}_{e^{\imath\vartheta}} : [S_1 : S_2 : S_0] \in M'^T_{\boldsymbol{v}} \mapsto \big[e^{-\imath\vartheta}\, S_1 : e^{-2\imath\vartheta}\, S_2 : S_0\big] \in M'^T_{\boldsymbol{v}}$. The latter is functionally independent of $\boldsymbol{v}_\perp$, and it follows that the quotients $\mathbb{P}(\mathbf{a}_{\boldsymbol{v}}, -\mathbf{b}_{\boldsymbol{v}})$ are all isomorphic as complex orbifolds when $v_1 > v_2 > 0$.

Let us come to a general representation $W_{\mathbf{L},\mathbf{K}}$. Let us introduce some terminology.

Definition 3.3 If $W_{\mathbf{L},\mathbf{K}}$ is a representation fullfilling the equivalent conditions of Proposition 2.5, let

$$\mathcal{I}(\mathbf{L},\mathbf{K}) := \big\{(a,j) \,:\, a \in \{1,\ldots,r\}, j \in \{0,\ldots,k_a\}\big\}.$$

Given $\boldsymbol{v} = (\, v_1 \quad v_2\,) \in \mathbb{Z}^2$, let us define $n_{\boldsymbol{v}} : \mathcal{I}(\mathbf{L},\mathbf{K}) \to \mathbb{Z}$ by setting

$$n_{\boldsymbol{v}}(a,j) := -v_2\,(k_a - j + l_a) + v_1\,(l_a + j). \tag{72}$$

Let us assume that $\Phi_{\mathbf{L},\mathbf{K}}\big(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\big) \cap \mathbb{R}_+ \cdot \imath\,\boldsymbol{v} \neq \emptyset$, and that $\Phi_{\mathbf{L},\mathbf{K}}$ is transverse to $\mathbb{R}_+ \cdot \imath\,\boldsymbol{v}$. Then, by Proposition 2.3 and Theorem 2.5, $\boldsymbol{v}$ lies in the interior of one of the wedges cut out by the rays through the integral vectors $\boldsymbol{v}_{k_a,j_a,l_a}$ defined in (31). It follows that:

1. $n_{\boldsymbol{v}}(a,j) \neq 0$ for every $(a,j) \in \mathcal{I}(\mathbf{L},\mathbf{K})$;
2. there exist (a,j), $(b,h) \in \mathcal{I}(\mathbf{L},\mathbf{K})$ such that $n_{\boldsymbol{v}}(a,j) \cdot n_{\boldsymbol{v}}(b,h) < 0$.

Definition 3.4 Under the previous assumptions, let us define

$$\mathcal{P}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}) := \left\{(a,j) \in \mathcal{I}(\mathbf{L},\mathbf{K}) \,:\, n_{\boldsymbol{v}}(a,j) > 0\right\}, \tag{73}$$

$$\mathcal{N}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}) := \left\{(a,j) \in \mathcal{I}(\mathbf{L},\mathbf{K}) \,:\, n_{\boldsymbol{v}}(a,j) < 0\right\}. \tag{74}$$

Then $\mathcal{I}(\mathbf{L},\mathbf{K})$ is the disjoint union of $\mathcal{P}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})$ and $\mathcal{N}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})$, both of which are non-empty. Furthermore, let us define

$$\mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}) := \left(|n_{\boldsymbol{v}}(a,j)|\right)_{(a,j)\in\mathcal{P}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})} \in \mathbb{N}^{|\mathcal{P}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})|},$$
$$\mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}) := \left(|n_{\boldsymbol{v}}(a,j)|\right)_{(a,j)\in\mathcal{N}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})} \in \mathbb{N}^{|\mathcal{N}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})|}.$$

Theorem 3.2 *Let $W_{\mathbf{L},\mathbf{K}}$ be a representation fullfilling the equivalent conditions of Proposition 2.5. Suppose that $\boldsymbol{v} = (\,v_1 \quad v_2\,)$, $v_1 \neq v_2$, and that*

1. $\Phi_{\mathbf{L},\mathbf{K}}\left(\mathbb{P}(W_{\mathbf{L},\mathbf{K}})\right) \cap \mathbb{R}_+ \cdot \imath\,\boldsymbol{v} \neq \emptyset$;
2. $\Phi_{\mathbf{L},\mathbf{K}}$ is transverse to $\mathbb{R}_+ \cdot \imath\,\boldsymbol{v}$.

Then the $\boldsymbol{v}$-th DH-reduction of $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$ is

$$\left(\mathbb{P}(\mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}), -\mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})), \eta_{\mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})}\right). \tag{75}$$

Furthermore, if $W_{\mathbf{L},\mathbf{K}}$ is a uniform representation (Definition 2.3) then the complex orbifold $\mathbb{P}(\mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}), -\mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}))$ remains constant as $\boldsymbol{v}$ ranges in the interior of one of the wedges cut out by the rays through the $\boldsymbol{v}_{k_a,j_a,l_a}$'s.

Remark 3.3 As discussed in §3.2.1, (75) is a weighted projective subvariety and a Kähler suborbifold of the weighted projective space

$$\left(\mathbb{P}(\mathbf{c}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})), \eta^{\mathbf{c}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})}_{\mathbf{d}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})}\right),$$

where

$$\mathbf{c}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})_{ij} := \mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})_i + \mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})_j, \quad \mathbf{d}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})_{ij} := \mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})_i \cdot \mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})_j.$$

Proof of Theorem 3.2 By (28) we have with $M = \mathbb{P}(W_{\mathbf{L},\mathbf{K}})$

$$\begin{aligned} M^T_{\boldsymbol{v}} = \Bigg\{ [Z] : v_2 & \sum_{a=1}^{r}\left(\|F_{k_a,1}(Z_a)\|^2 + l_a\,\|Z_a\|^2\right) \\ & = v_1 \sum_{a=1}^{r}\left(\|F_{k_a,2}(Z_a)\|^2 + l_a\,\|Z_a\|^2\right)\Bigg\}. \end{aligned} \tag{76}$$

In view of (7) and (8), the relation in (76) may be rewritten

$$
\begin{aligned}
0 &= \sum_{(a,j)\in\mathcal{I}(\mathbf{L},\mathbf{K})} n_{\mathbf{v}}(a,j)\,|z_{a,j_a}|^2 \\
&= \sum_{(a,j)\in\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})} |n_{\mathbf{v}}(a,j)|\,|z_{a,j_a}|^2 - \sum_{(a,j)\in\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})} |n_{\mathbf{v}}(a,j)|\,|z_{a,j_a}|^2 .
\end{aligned}
\tag{77}
$$

This can be reformulated as follows. Let us consider $\mathbb{C}^{|\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|}$ and $\mathbb{C}^{|\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|}$, with coordinates $Z=(z_{a,j})_{(a,j)\in\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$, $W=(w_{a,j})_{(a,j)\in\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$, respectively. On $\mathbb{C}^{|\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|}$ and $\mathbb{C}^{|\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|}$ we have the positive definite Hermitian products given by

$$
\begin{aligned}
h_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}(Z,Z') &= \sum_{(a,j)\in\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})} |n_{\mathbf{v}}(a,j)|\, z_{a,j}\,\overline{z'_{a,j}},\\
h_{\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}(W,W') &= \sum_{(a,j)\in\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})} |n_{\mathbf{v}}(a,j)|\, w_{a,j}\,\overline{w'_{a,j}},
\end{aligned}
$$

and so by (76)

$$
\begin{aligned}
M^T_{\mathbf{v}} \cong M'^T_{\mathbf{v}} := \Big\{ [Z:W] \in \mathbb{P}\Big(\mathbb{C}^{|\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|} \oplus \mathbb{C}^{|\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|}\Big) :\\
h_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}(Z,Z) = h_{\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}(W,W)\Big\}.
\end{aligned}
\tag{78}
$$

Therefore $M^T_{\mathbf{v}}$ may be identified by $f_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$ in (62) with the unit circle bundle in

$$
X_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})} \subset \mathcal{O}_{\mathbb{P}^{|\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|-1}}(-1) \boxtimes \mathcal{O}_{\overline{\mathbb{P}^{|\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|-1}}}(-1),
$$

relative to the Hermitian metric induced by $h_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$ and $h_{\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$; the bundle projection is $\pi : [Z:W] \mapsto ([Z],[W])$.

In the notation (65), the action of $T^1_{\mathbf{v}^\perp}$ on $M'^T_{\mathbf{v}}$ given by restriction of $\phi_{\mathbf{L},\mathbf{K}}$ is

$$
\begin{aligned}
&\gamma^{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}_{e^{\imath\vartheta}}\big(\big[(z_{a,j}):(w_{a,j})\big]\big)\\
&\quad = \Big[\Big(e^{-\imath\, n_{\mathbf{v}}(a,j)\,\vartheta}\, z_{a,j}\Big) : \Big(e^{-\imath\, n_{\mathbf{v}}(a,j)\,\vartheta}\, w_{a,j}\Big)\Big]\\
&\quad = \Big[\Big(e^{-\imath\, |n_{\mathbf{v}}(a,j)|\,\vartheta}\, z_{a,j}\Big) : \Big(e^{\imath\, |n_{\mathbf{v}}(a,j)|\,\vartheta}\, w_{a,j}\Big)\Big].
\end{aligned}
\tag{79}
$$

$\gamma^{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$ corresponds, under the previous identification, to the contact lift $\tilde{\mu}^{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$ of the Hamiltonian action $\mu^{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$ (see (59)) on

$$
\Big(\mathbb{P}^{|\mathcal{P}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|-1} \times \overline{\mathbb{P}^{|\mathcal{N}_{\mathbf{v}}(\mathbf{L},\mathbf{K})|-1}},\, 2\,\omega_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}\Big),
$$

with moment map $\Phi_{\mathbf{a}_{\mathbf{v}}(\mathbf{L},\mathbf{K}),\mathbf{b}_{\mathbf{v}}(\mathbf{L},\mathbf{K})}$ (recall (53)). Thte first statement of the Theorem follows from this.

Let us assume that $W_{\mathbf{L},\mathbf{K}}$ is a uniform representation. By definition, there is $s\in\mathbb{Z}$ (independent of a) such that $k_a + 2\,l_a = s$ for $a=1,\ldots,r$. Then (72) may be rewritten

$$
n_{\mathbf{v}}(a,j) = -v_2\, s + (v_1+v_2)\,(l_a + j).
\tag{80}
$$

Therefore, (79) may be rewritten

$$
\begin{aligned}
&\gamma_{e^{\imath\vartheta}}^{\mathbf{a}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K}),-\mathbf{b}_{\boldsymbol{v}}(\mathbf{L},\mathbf{K})}\left(\left[(z_{a,j}) : (w_{a,j})\right]\right) \\
&\quad = \left[\left(e^{\imath\,[v_2\,s-(l_a+j)\,(v_1+v_2)\,\vartheta]}\,z_{a,j}\right) : \left(e^{\imath\,[v_2\,s-(v_1+v_2)\,(l_a+j)\,\vartheta]}\,w_{a,j}\right)\right] \\
&\quad = \left[\left(e^{-\imath\,(v_1+v_2)\,(l_a+j)\,\vartheta]}\,z_{a,j}\right) : \left(e^{-\imath\,(v_1+v_2)\,(l_a+j)\,\vartheta}\,w_{a,j}\right)\right].
\end{aligned}
\tag{81}
$$

After passing to the quotient group $T^1(\boldsymbol{v})$ in (39), we obtain the action $\left[(z_{a,j}) : (w_{a,j})\right] \mapsto \left[\left(e^{-\imath\,(l_a+j)\,\vartheta]}\,z_{a,j}\right) : \left(e^{-\imath\,(l_a+j)\,\vartheta]}\,w_{a,j}\right)\right]$, which is functionally independent of $\boldsymbol{v}$, and the claim can be readily deduced from this. □

3.2.4 The case of μ_k and $v_1 \gg v_2$

Let us focus on the special case of μ_k, for $k \geq 2$ and $\boldsymbol{v}$ in the in the range $v_1 \gg v_2$. For any positive sequence $\boldsymbol{a} = (\,a_1 \quad \cdots \quad a_k\,)$, the quotient of the sphere $S^{2k-1}_{\boldsymbol{a}} \subset \mathbb{C}^k$ by the circle action with weights $(\,1 \quad 2 \quad \cdots \quad k\,)$ is $\mathbb{P}(1,2,\ldots,k)$; the image in $\mathbb{P}(1,2,\ldots,k)$ of $S^{2k-1}_{\boldsymbol{a}} \cap (z_1 = 0)$ is a canonically embedded copy of $\mathbb{P}(2,\ldots,k)$, independent of $\mathbf{a}$. We shall denote by $\jmath : \mathbb{P}(2,\ldots,k) \hookrightarrow \mathbb{P}(1,2,\ldots,k)$ the inclusion, which is a holomorphic orbifold embedding.

Theorem 3.3 *Under the previous assumptions, suppose $v_1 \gg v_2$. Then $\overline{M}^T_{\boldsymbol{v}} \cong \mathbb{P}(1,2,\ldots,k)$. Furthermore, there is a smooth isotopy of orbifold embeddings*

$$J : [0,1] \times \mathbb{P}(2,\ldots,k) \to \mathbb{P}(1,2,\ldots,k)$$

such that:

1. $J_0 = \jmath$;
2. $J_1\big(\mathbb{P}(2,\ldots,k)\big) = \overline{M}^G_{\boldsymbol{v}}$;
3. $J_t\big(\mathbb{P}(2,\ldots,k)\big)$ *is a symplectically embedded orbifold in* $(\overline{M}^T_{\boldsymbol{v}}, \Omega_0)$ *for every* $t \in [0,1]$;

In particular, $\overline{M}^G_{\boldsymbol{v}}$ is diffeomorphic to $\mathbb{P}(2,\ldots,k)$.

The following argument will produce $J_t\big(\mathbb{P}(2,\ldots,k)\big)$ as the zero locus of a smoothly varying family of transverse sections of the orbifold line bundle in Theorem 3.1.

Proof of Theorem 3.3 We have $M = \mathbb{P}^k = \mathbb{P}(W_k)$. By (7), (8) and (9), $M^T_{\boldsymbol{v}}$ is contained in the affine open set $\mathbb{A}^k_0 = (z_0 \neq 0)$. More explicitly, let us define $\mathbf{a}_{\boldsymbol{v}}(k) \in \mathbb{N}^k$ by setting

$$\mathbf{a}_{\boldsymbol{v}}(k)_j := v_1\,j - v_2\,(k-j);$$

thus $\mathbf{a}_{\boldsymbol{v}}(k)_j > 0$ for $j = 1,\ldots,k$ if $v_1 > (k-1)\,v_2$. Then

$$M^T_{\boldsymbol{v}} = \left\{\left[\frac{1}{\sqrt{k\,v_2}} : v_1 : \cdots : v_k\right] \in \mathbb{P}^k \,:\, \sum_{j=1}^k \mathbf{a}_{\boldsymbol{v}}(k)_j\,|v_j|^2 = 1\right\} \cong S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)}. \tag{82}$$

Being irreducible, μ_k is uniform, hence $T^1(\boldsymbol{v}) = S^1(\boldsymbol{v})$ in (39). Under the isomorphism $\kappa_{\boldsymbol{v}} : S^1 \cong T^1_{\boldsymbol{v}_\perp}$ in (32), $T^1_{\boldsymbol{v}_\perp} \cap Z(G)$ corresponds to the subgroup of S^1 of $(v_1 + v_2)$-th roots of unity; thus the quotient map $T^1_{\boldsymbol{v}_\perp} \to T^1(\boldsymbol{v})$ corresponds to the Lie group epimorphism $e^{\imath\vartheta} \in S^1 \mapsto e^{\imath(v_1+v_2)\vartheta} \in S^1$.

Identified with S^1 as in (32), $T^1_{\boldsymbol{v}^\perp}$ acts on $M^T_{\boldsymbol{v}}$ as

$$\begin{aligned}\gamma_{e^{\imath\vartheta}}&\left(\left[\frac{1}{\sqrt{k\,v_2}} : v_1 : \cdots : v_k\right]\right)\\ &= \left[\frac{1}{\sqrt{k\,v_2}} : e^{-\imath\vartheta(v_1+v_2)}\,v_1 : \cdots : e^{-\imath j\vartheta(v_1+v_2)}\,v_j : \cdots : e^{-\imath k\vartheta(v_1+v_2)}\,v_k\right].\end{aligned} \tag{83}$$

Passing to the action $\overline{\gamma}$ of the quotient group $T^1(\boldsymbol{v}) \cong S^1$, we conclude that $J_0 = J_0'$, and $\overline{M}^T_{\boldsymbol{v}} \cong \mathbb{P}(1,2,\ldots,k)$. Furthermore, the intersection $S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)} \cap (v_1 = 0)$ is clearly $\overline{\gamma}$-invariant, and it projects down to $\mathbb{P}(2,\ldots,k) \subset \mathbb{P}(1,2,\ldots,k)$.

As $\overline{\gamma}$ is effective, any character χ of $T^1(\boldsymbol{v})$ defines an orbifold line bundle L_χ on $\overline{M}^T_{\boldsymbol{v}}$. We shall write $L = L_1$ if $\chi = \chi_1$ corresponds to the identity of S^1. Any function $f : S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)} \to \mathbb{C}$ which is the restriction of a $\mathcal{C}^\infty$ (respectively, holomorphic) function on $\mathbb{C}^k$ and satisfies $f \circ \overline{\gamma}_{e^{-\imath\vartheta}} = e^{\imath\vartheta} f$ determines a $\mathcal{C}^\infty$ (respectively, holomorphic) section σ_f of $L_{\mathfrak{a}}$.

With abuse of notation, in view of (82) let us regard Φ_{12} as defined on $S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)}$; by (6),

$$\Phi_{12}(V) = \frac{\imath}{(k\,v_2)^{-1} + \|V\|^2}\left[\frac{1}{\sqrt{v_2}}\,v_1 + \sum_{j=1}^{k-1}\sqrt{(k-j)\,(j+1)}\,v_{j+1}\,\overline{v_j}\right]. \tag{84}$$

Let us consider the continuous function $\Lambda : [0,1] \times S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)} \to \mathbb{C}$ given by

$$\begin{aligned}\Lambda :\ &(t, V)\\ &\mapsto \frac{\imath}{(k\,v_2)^{-1} + t\,\|V\|^2}\left[\frac{1}{\sqrt{v_2}}\,v_1 + t\sum_{j=1}^{k-1}\sqrt{(k-j)\,(j+1)}\,v_{j+1}\,\overline{v_j}\right];\end{aligned} \tag{85}$$

we shall write $\Lambda_t(V) := \Lambda(t,V)$.

Let $(e_1,\ldots,e_k)$ denote the standard basis of $\mathbb{C}^k$, and let $(e_1^*,\ldots,e_k^*)$ be the dual basis. Then

$$\Lambda_0 = \imath\,k\,\sqrt{v_2}\,e_1^*\big|_{S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)}},\quad \Lambda_1 = \Phi_{12},\quad \Lambda_t \circ \overline{\gamma}_{e^{-\imath\vartheta}} = e^{\imath\vartheta}\,\Lambda_t,\ \forall\, t \in [0,1]; \tag{86}$$

in particular, Λ_t corresponds to a $\mathcal{C}^\infty$ section σ_{Λ_t} of L_1.

The following is left to the reader:

Lemma 3.8 *Let $\|\cdot\| : \mathbb{C}^k \to \mathbb{R}$ be the standard Euclidean norm. If $v_1 \geq 2\,(k-1)\,v_2$, then $\|V\| \leq \sqrt{2/v_1}$ for all $V \in S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)}$.*

Using (85) and Lemma 3.8, one can also prove the following two Lemmas.

Lemma 3.9 *Let us set $\tilde{\Lambda}_t := -\imath\,(k\,\sqrt{v_2})^{-1}\,\Lambda_t$, and let us view $\tilde{\Lambda}_t$ as defined on $\mathbb{C}^k$ (by the same functional equation). Then, uniformly in $V \in S^{2k-1}_{\mathbf{a}_{\boldsymbol{v}}(k)}$ we have*

$$\mathrm{d}_V\tilde{\Lambda}_t = e_1^* + O\left(\sqrt{\frac{v_2}{v_1}}\right).$$

Lemma 3.10 *There exists $C > 0$ (independent of k, t and $\mathbf{v}$) such that if $V \in S^{2k-1}_{\mathbf{a}_v(k)}$ and $\Lambda_t(V) = 0$ for some $t \in [0,1]$, then $|v_1| \leq C\,k\,(\sqrt{v_2}/v_1)$.*

The general $V \in S^{2k-1}_{\mathbf{a}_v(k)}$ has the form

$$V = \sum_{j=1}^{k} \frac{r_j}{\sqrt{\mathbf{a}_v(k)_j}}\, e_j, \qquad \text{where} \qquad r_j \in \mathbb{C}, \quad \sum_{j=1}^{k} |r_j|^2 = 1. \tag{87}$$

Lemma 3.10 and (87) imply that if $V \in S^{2k-1}_{\mathbf{a}_v(k)}$ and $\Lambda_t(V) = 0$ for some $t \in [0,1]$, then $v_1 = r_1/\sqrt{\mathbf{a}_v(k)_1}$ where $r_1 \in \mathbb{C}$ satisfies

$$|r_1| \leq C\,k\,\frac{\sqrt{v_2}}{v_1}\,\sqrt{\mathbf{a}_v(k)_1} \leq C\,k\,\sqrt{\frac{v_2}{v_1}}. \tag{88}$$

Hence, if $R' = R'(V) := \sum_{j=2}^{k} r_j\, e_j$ then

$$v_1/v_2 > 2\,C^2\,k^2 \quad \Rightarrow \quad \|R'\|^2 = 1 - |r_1|^2 \geq 1 - C^2\,k^2\,(v_2/v_1) \geq 1/2.$$

Hence there exists $j \in \{2,\ldots,k\}$ such that $|r_j| \geq 1/\sqrt{2\,k}$. Perhaps after renumbering, we may assume that $j = 2$.

Therefore, we can draw the following conclusion.

Lemma 3.11 *Suppose $v_1/v_2 \gg 0$. If $V \in S^{2k-1}_{\mathbf{a}_v(k)}$ and $\Lambda_t(V) = 0$ for some $t \in [0,1]$ then, perhaps after a renumbering of $(2,\ldots,k)$ we have*

$$V = \frac{r_1}{\sqrt{\mathbf{a}_v(k)_1}}\, e_1 + \frac{r_2}{\sqrt{\mathbf{a}_v(k)_2}}\, e_2 + S(V), \tag{89}$$

where $S(V) \in \mathrm{span}_{\mathbb{C}}(e_3,\ldots,e_k)$, r_1 satisfies (88) and $|r_2| \geq 1/\sqrt{2\,k}$.

Let us set

$$\begin{aligned} N_V :=& \sqrt{v_1}\left[-\frac{1}{\sqrt{\mathbf{a}_v(k)_1}}\,\overline{r}_2\, e_1 + \frac{1}{\sqrt{\mathbf{a}_v(k)_2}}\,\overline{r}_1\, e_2\right] \\ =& -\frac{\overline{r_2}}{\sqrt{1-(k-1)\frac{v_2}{v_1}}}\, e_1 + \frac{\overline{r_1}}{\sqrt{2-(k-2)\frac{v_2}{v_1}}}\, e_2. \end{aligned} \tag{90}$$

Then $\mathrm{span}_{\mathbb{C}}(N_V) \subseteq T_V S^{2k-1}_{\mathbf{a}_v(k)}$ and $\|N_V\| > 1/(2k)$ by Lemma 3.11. In view of Lemma 3.9, we obtain for every $e^{\imath\theta} \in S^1$

$$\mathrm{d}_V\tilde{\Lambda}_t\big(e^{\imath\theta}N_V\big) = -\frac{e^{\imath\theta}\,\overline{r_2}}{\sqrt{1-(k-1)\frac{v_2}{v_1}}} + O\left(\sqrt{\frac{v_2}{v_1}}\right). \tag{91}$$

It follows that $\mathrm{d}_V\tilde{\Lambda}_t$ restricts to a surjective $\mathbb{R}$-linear map $\mathrm{span}_{\mathbb{C}}(N_V)\to\mathbb{C}$; therefore the same is true *a fortiori* of the restriction of $\mathrm{d}_V\Lambda_t$ to $T_V S^{2k-1}_{\mathbf{a}_\nu(k)}$.

Thus we conclude the following:

Lemma 3.12 *Suppose* $v_1/v_2 \gg 0$, $V\in S^{2k-1}_{\mathbf{a}_\nu(k)}$, $t\in[0,1]$, *and* $\Lambda_t(V)=0$. *Then* $\mathrm{d}_V\Lambda_t|_{T_V S^{2k-1}_{\mathbf{a}_\nu(k)}}\to\mathbb{C}$ *is a surjective* $\mathbb{R}$*-linear map.*

Lemma 3.12 has the following consequences:

Corollary 3.3 *In the situation of Lemma 3.12,* $Z_t := \Lambda_t^{-1}(0)\subset S^{2k-1}_{\mathbf{a}_\nu(k)}$ *is a smooth* γ*-invariant submanifold of* $S^{2k-1}_{\mathbf{a}_\nu(k)}$*, of (real) codimension* 2.

Corollary 3.4 $\overline{Z}_t := Z_t/\gamma \subset \overline{M}^T_\nu$ *is a smoothly embedded orbifold of real codimension* 2.

Corollary 3.5 *Let* $\mathcal{Z} := \Lambda^{-1}(0)\subset[0,1]\times S^{2k-1}_{\mathbf{a}_\nu(k)}$. *Then*:

1. $\mathcal{Z}$ is a submanifold (with boundary) of codimension 2 of $[0,1]\times S^{2k-1}_{\mathbf{a}_\nu(k)}$;
2. the projection $p:\mathcal{Z}\to[0,1]$ is a submersion;
3. $Z_t = p^{-1}(t)$ for every t.

$T^1(\boldsymbol{\nu})$ acts on $[0,1]\times S^{2k-1}_{\mathbf{a}_\nu(k)}$ trivially on the first factor and via $\overline{\gamma}$ on the second, and this action preserves $\mathcal{Z}$ in view of (86). The product metric on $[0,1]\times S^{2k-1}_{\mathbf{a}_\nu(k)}$ restricts to an invariant Riemannian metric $g_{\mathcal{Z}}$ on $\mathcal{Z}$. By $g_{\mathcal{Z}}$, we can define an invariant horizontal distribution for p, whence an invariant horizontal vector field, whose integral curves are the horizontal lifts of [0, 1] for $g_{\mathcal{Z}}$. These horizontal lifts define an invariant family ψ_p of paths, one for each $p\in Z_0$; for each t, the assignment $\psi^t : p\in Z_0\mapsto\psi_p(t)\in Z_t$ is a $\overline{\gamma}$-equivariant diffeomorphism. Therefore, ψ^t descends to a smoothly varying family of orbifold diffeomorphisms $\overline{\psi}^t:\overline{Z}_0\to\overline{Z}_t$. In particular, $\overline{Z}_0$ is diffeomorphic to $\overline{Z}_1$.

Let $\mathbf{a}_\nu(k)' := \big(\mathbf{a}_\nu(k)_2,\ldots,\mathbf{a}_\nu(k)_k\big)$. Then in view of (86)

$$Z_0 = \{v_1=0\}\cap S^{2k-1}_{\mathbf{a}_\nu(k)'} = \{0\}\times S^{2k-3}_{\mathbf{a}_\nu(k)'}; \tag{92}$$

by (83), $\overline{Z}_0\cong\mathbb{P}(2,3,\ldots,k)$. Thus every $\overline{Z}_t\subset\overline{M}^T_\nu$ is diffeomorphic to $\mathbb{P}(2,3,\ldots,k)$.

Let us show that every $\overline{Z}_t$ is symplectically embedded in $(\overline{M}^T_\nu,\Omega_0)$. By construction, $S^{2k-1}_{\mathbf{a}_\nu(k)}\cong M^T_\nu = \Psi^{-1}_{\nu_\perp}(0)$ ($\Psi_{\nu_\perp}$ is as in Lemma 3.1). Let $q: S^{2k-1}_{\mathbf{a}_\nu(k)}\to\overline{M}^T_\nu$ be the projection, and let $\imath: S^{2k-1}_{\mathbf{a}_\nu(k)}\hookrightarrow\mathbb{C}^k\cong\mathbb{A}^k_0\subset\mathbb{P}^k$ be the inclusion; then $q^*(\Omega_0)=\imath^*(\omega_{FS})$.

Let $\omega_0 := (\imath/2)\sum_{j=1}^k \mathrm{d}v_j\wedge\mathrm{d}\overline{v}_j$ be the standard symplectic structure on $\mathbb{C}^k$. Expressing ω_{FS} in affine coordinates, by a standard computation we obtain on $\mathbb{A}^k_0$

$$\omega_{FS} = \omega_0 + R_2(V), \tag{93}$$

where R_2 is a differential form vanishing to second order at the origin. By Lemma 3.8, along $S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}$ we have $\|V\|^2 \leq 2/v_1 \leq 2\, v_2/v_1$; hence (93) implies that $\omega_{FS} = \omega_0 + O(v_2/v_1)$ on $S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}$. Therefore,

$$q^*(\Omega_0) = \imath^*(\omega_{FS}) = \imath^*(\omega_0) + O\left(\frac{v_2}{v_1}\right). \tag{94}$$

With $\tilde{\Lambda}_t : \mathbb{A}^n_0 \cong \mathbb{C}^k \to \mathbb{C}$ as in Lemma 3.9, let us set $\tilde{Z}_t := \tilde{\Lambda}_t^{-1}(0)$; thus $Z_t = \tilde{Z}_t \cap S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}$.

Let $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{2k-1}, \epsilon_{2k})$ be the real basis $(e_1, \imath\, e_1, \ldots, e_k, \imath\, e_k)$ of $\mathbb{C}^k \cong \mathbb{R}^{2k}$. Then by Lemma 3.9

$$\mathrm{d}_V\tilde{\Lambda}_t = \epsilon_1^* + \imath\, \epsilon_2^* + O\left(\sqrt{\frac{v_2}{v_1}}\right) \qquad (V \in S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}), \tag{95}$$

and this implies that if $v_1/v_2 \gg 0$ then $\ker(\mathrm{d}_V\tilde{\Lambda}_t)$ is a (real) symplectic vector subspace of $(\mathbb{C}^k, \omega_0)$ whenever $V \in S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}$ and $t \in [0, 1]$. Given this and (94), we conclude the following:

Lemma 3.13 *If $v_1/v_2 \gg 0$, then the following holds. For every $t \in [0, 1]$ and $V \in S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}$ such that $\Lambda_t(V) = 0$, the tangent space $T_V\tilde{Z}_t$ is a symplectic vector subspace of $(\mathbb{C}^k, \omega_{FS})$.*

Corollary 3.6 *If $v_1/v_2 \gg 0$, there exists a $\overline{\gamma}$-invariant open neighborhood $U \subseteq \mathbb{C}^k$ of $S^{2k-1}_{\mathbf{a}_\mathbf{v}(k)}$, such that $\tilde{Z}'_t := \tilde{Z}_t \cap U$ is a symplectic submanifold of real codimension 2 of $(\mathbb{C}^k, \omega_{FS})$, for every $t \in [0, 1]$.*

Let $\jmath_t : \tilde{Z}'_t \hookrightarrow \mathbb{C}^k$ be the inclusion, and set $\omega_t := \jmath_t^*(\omega_{FS})$. The restriction $\psi_t := \Psi_{\mathbf{v}_\perp} \circ \jmath_t$ is the moment map for the action of $T^1_{\mathbf{v}^\perp}$ on $(\tilde{Z}'_t, \omega_t)$, and $Z_t = \psi_t^{-1}(0)$. Hence $\overline{Z}_t := \tilde{Z}'_t/\gamma$, with the reduced symplectic structure $\overline{\omega}_t$, is the symplectic reduction of $(\tilde{Z}'_t, \omega_t)$, and as such it is a symplectic suborbifold of $(\overline{M}^T_\mathbf{v}, \Omega_0)$.

4 $M^G_{\mathcal{O}}$

We shall assume throughout that $\mathbf{0} \notin \Phi(M)$ and that Φ is transverse to $\mathcal{C}(\mathcal{O})$, and focus on $\overline{M}^G_{\mathcal{O}}$ and its relation to $\overline{M}^G_\mathbf{v}$. We do not assume that M be projective.

Given that Φ is transverse to $\mathcal{C}(\mathcal{O})$, ϕ has rank ≥ 3 along $M^G_{\mathcal{O}}$, meaning that for every $m \in M^G_{\mathcal{O}}$ the evaluation map $\mathrm{val}_m : \xi \in \mathfrak{g} \mapsto \xi_M(m) \in T_m M^G_{\mathcal{O}}$ has rank ≥ 3 [4, 15]. Let us give a direct proof for the reader's convenience.

Proposition 4.1 *Given that Φ is transverse to $\mathcal{C}(\mathcal{O})$, for any $m \in M^G_{\mathcal{O}}$ the evaluation map $\mathrm{val}_m : \mathfrak{g} \to T_m M$ is injective on* $\ker\big(\Phi(m)\big)$.

Proof If $m \in M^G_{\mathcal{O}}$, then by equivariance Φ is transverse to $\mathcal{C}(\mathcal{O})$ at m if and only if it is transverse to the ray $\mathbb{R}_+ \Phi(m)$ at m. Hence, $\mathrm{d}_m\Phi(T_mM) + \mathbb{R}\,\Phi(m) = \mathfrak{g}^\vee$. Suppose that $\xi \in \ker\big(\Phi(m)\big)$, and that $\xi_M(m) = 0$. Pick $\alpha \in \mathfrak{g}^\vee$. Then there exists $v \in T_mM$ and $\lambda \in \mathbb{R}$ such that $\alpha = \mathrm{d}_m\Phi(v) + \lambda\,\Phi(m)$. Thus

$$\begin{aligned}\alpha(\xi) &= \mathrm{d}_m\Phi(v)(\xi) + \lambda\,\Phi(m)(\xi)\\ &= \mathrm{d}_m\Phi(v)(\xi) = \mathrm{d}_m\Phi^\xi(v) = 2\,\omega\big(\xi_M(m), v\big) = 0.\end{aligned}$$

Thus $\alpha(\xi) = 0\ \forall\,\alpha \in \mathfrak{g}^\vee$, whence $\xi = 0$.

For example, when $\phi = \phi_{\mathbf{L},\mathbf{K}}$ for a uniform representation (Definition 2.3), ϕ is bound to have constant rank 3 along $M^G_{\mathcal{O}}$.

We shall accordingly distinguish two cases: 1): ϕ has constant rank 3 along $M^G_{\mathcal{O}}$; 2): ϕ is generically locally free along $M^G_{\mathcal{O}}$. Before, however, it is in order to sum up some general facts.

If $m \in M^G_{\mathcal{O}}$, then by definition there exist unique $\lambda_{\mathbf{v}}(m) > 0$ and $h_m T \in G/T$ such that

$$\Phi(m) = \imath\,\lambda_{\mathbf{v}}(m)\,h_m \begin{pmatrix} v_1 & 0\\ 0 & v_2\end{pmatrix} h_m^{-1}. \tag{96}$$

The applications $\lambda_{\mathbf{v}}$ and $m \mapsto h_m T$ are $\mathcal{C}^\infty$. Furthermore, $h_{\mu_g(m)}\,T = g\,h_m\,T$ and $\lambda_{\mathbf{v}} = \lambda_{\mathbf{v}} \circ \mu_g$ by the equivariance of Φ.

Let us define

$$T^1_{\mathbf{v}_\perp,m} := h_m\,T^1_{\mathbf{v}_\perp}\,h_m^{-1}, \quad T_m := h_m\,T\,h_m^{-1} \qquad (m \in M^G_{\mathcal{O}}). \tag{97}$$

Then $T^1_{\mathbf{v}_\perp,m} \leqslant T_m \leqslant G$ are well-defined, and

$$T^1_{\mathbf{v}_\perp,\mu_g(m)} = g\,T_{\mathbf{v}_\perp,m}\,g^{-1} \leqslant T_{\mu_g(m)} = g\,T_m\,g^{-1} \qquad (g \in G,\ m \in M^G_{\mathcal{O}}). \tag{98}$$

In particular, if $g \in T_m$ then $T_{\mu_g(m)} = T_m$; hence $T_{m'} = T_m$ for every $m' \in T_m \cdot m$; similarly for $T_{\mathbf{v}_\perp,m}$.

Definition 4.1 Let us define the action $\rho : S^1 \times M^G_{\mathcal{O}} \to M^G_{\mathcal{O}}$ by setting

$$\rho_{e^{\imath\vartheta}}(m) := \phi_{h_m\,\kappa_{\mathbf{v}}(e^{\imath\vartheta})\,h_m^{-1}}(m),$$

where $\kappa_{\mathbf{v}} : S^1 \to T^1_{\mathbf{v}^\perp}$ is as in (32).

Thus the ρ-orbit of $m \in M^G_{\mathcal{O}}$ is $T_m \cdot m$. The following facts are more or less well-known, and are either discussed in [4], or can be deduced using arguments in [4, 6]:

Lemma 4.1 *$M^G_{\mathcal{O}} \subset M$ is a compact and connected G-invariant hypersurface, and ρ is locally free. The isotropic leaves of $M^G_{\mathcal{O}}$ are the ρ-orbits. Hence, the quotient $\overline{M}^G_{\mathcal{O}}$ is an orbifold of real dimension $2d-2$, with a reduced symplectic structure $\omega_{\overline{M}^G_{\mathcal{O}}}$.*

Let $p : M^G_{\mathcal{O}} \to \overline{M}^G_{\mathcal{O}}$ be the projection. Then $p(M^G_{\mathbf{v}})$ is diffeomorphic to $\overline{M}^G_{\mathbf{v}}$ in (33); with abuse of notation, we shall write $\overline{M}^G_{\mathbf{v}} = p(M^G_{\mathbf{v}})$. We have seen that $\overline{M}^G_{\mathbf{v}}$ has an intrinsic symplectic structure $\omega_{\overline{M}_{\nu}^G}$, and that $(\overline{M}_{\nu}^G, \omega_{\overline{M}_{\nu}^G})$ is a symplectic suborbifold of $(\overline{M}_{\nu}^T, \omega_{\overline{M}_{\nu}^T})$. Arguing as in §3.1 one obtains the following.

Lemma 4.2 *Under the previous identification, $(\overline{M}_{\nu}^G, \omega_{\overline{M}_{\nu}^G})$ is a symplectic suborbifold of $(\overline{M}_{\mathcal{O}}^G, \omega_{\overline{M}_{\mathcal{O}}^G})$.*

Furthermore, we have:

Lemma 4.3 *For every $e^{\imath\vartheta} \in S^1$, $g \in G$, $m \in M_{\mathcal{O}}^G$ we have*

$$\rho_{e^{\imath\vartheta}} \circ \phi_g(m) = \phi_g \circ \rho_{e^{\imath\vartheta}}(m).$$

Corollary 4.1 *ϕ (restricted to $M_{\mathcal{O}}^G$) descends to a smooth action*

$$\overline{\phi} : G \times \overline{M}_{\mathcal{O}}^G \to \overline{M}_{\mathcal{O}}^G.$$

Furthermore, $\overline{\phi}$ is symplectic for $\omega_{\overline{M}_{\mathcal{O}}^G}$.

In view of (96) and Definition 4.1, $\Phi|_{M_{\mathcal{O}}^G}$ is ρ-invariant, and therefore it descends to a smooth function $\overline{\Phi} : \overline{M}_{\mathcal{O}}^G \to \mathfrak{g}$.

Corollary 4.2 *$\overline{\phi}$ is Hamiltonian for $2\,\omega_{\overline{M}_{\mathcal{O}}^G}$, with moment map $\overline{\Phi}$.*

4.1 Case 1)

In this case, we shall establish in Theorem 4.1 that $\overline{M}_{\mathcal{O}}^G$ factors symplectically as the product of $\overline{M}_{\nu}^G$ and a coadjoint orbit.

Proposition 4.2 *If the rank of ϕ along $M_{\mathcal{O}}^G$ is generically 3, then it is 3 everywhere on $M_{\mathcal{O}}^G$. Furthermore, the stabilizer $F_m \leqslant G$ of any $m \in M_{\mathcal{O}}^G$ is 1-dimensional subgroup $F_m \leqslant T_m$, transverse to $T^1_{\nu_\perp,m}$ in T_m.*

This will be the case, for instance, if μ is associated to a uniform representation, in which case the connected component of F_m is $Z(G)$.

Proof of Proposition 4.2 Let us first assume that $m \in M_{\nu}^G$, so that $T_m = T$. Then any $g \in F_m$ commutes with $\Phi(m)$, therefore $g \in T$ since $\nu_1 \neq \nu_2$. Thus $F_m \leqslant T$. Since the action of $T^1_{\nu_\perp}$ is locally free at m, F_m has to be transverse to $T^1_{\nu_\perp}$ in T. The general case follows from this and (98).

For $\overline{m} \in \overline{M}_{\mathcal{O}}^G$, let $\overline{F}_{\overline{m}}$ denote the stabilizer of $\overline{m}$ for $\overline{\mu}$.

Corollary 4.3 *Under the hypothesis of Proposition 4.2, $\overline{F}_{\overline{m}} = T_m$, for any $\overline{m} \in \overline{M}^G_{\mathcal{O}}$ and $m \in p^{-1}(\overline{m})$. In particular, $\overline{F}_{\overline{m}} = T$, for any $\overline{m} \in \overline{M}^G_{\boldsymbol{\nu}}$.*

Corollary 4.4 *Under the hypothesis of Proposition 4.2, $\overline{\phi}$ is trivial on $Z(G)$. If, in addition, $\nu_1 + \nu_2 \neq 0$, then $\lambda_{\boldsymbol{\nu}}$ is constant.*

Proof of Corollary 4.4 For any $\overline{m} \in \overline{M}^G_{\mathcal{O}}$, $\overline{F}_{\overline{m}}$ is a maximal torus, hence contains $Z(G)$. This proves the first statement. As to the second, $\lambda_{\boldsymbol{\nu}}$ descends to a well-defined smooth function on $\overline{M}^G_{\mathcal{O}}$, which we shall denote by the same symbol. Furthermore, the Hamiltonian function for the (trivial) action of $Z(G)$ on $(\overline{M}^G_{\mathcal{O}}, 2\omega_{\overline{M}^G_{\mathcal{O}}})$ is $\langle \overline{\Phi}, \imath I_2 \rangle = \lambda_{\boldsymbol{\nu}} (\nu_1 + \nu_2)$. Since $\nu_1 + \nu_2 \neq 0$, $\lambda_{\boldsymbol{\nu}}$ needs to be contant.

By (96), if $\overline{m} \in \overline{M}^G_{\mathcal{O}}$ and $m \in p^{-1}(\overline{m})$ we have

$$\phi_{h_m^{-1}}(m) \in M^G_{\boldsymbol{\nu}}, \qquad \overline{\phi}_{h_m^{-1}}(\overline{m}) \in \overline{M}^G_{\boldsymbol{\nu}}.$$

Thus we obtain well-defined and $\mathcal{C}^\infty$ orbifold maps

$$\Delta : \overline{m} \in \overline{M}^G_{\mathcal{O}} \mapsto \left(\overline{\phi}_{h_m^{-1}}(\overline{m}), h_{\overline{m}} T \right) \in \overline{M}^G_{\boldsymbol{\nu}} \times (G/T), \tag{99}$$

and

$$\Theta : (\overline{m}, hT) \in \overline{M}^G_{\boldsymbol{\nu}} \times (G/T) \mapsto \overline{\phi}_h(\overline{m}) \in \overline{M}^G_{\mathcal{O}}. \tag{100}$$

Notice that Δ and Θ are well-defined by Corollary 4.3, and $\Theta = \Delta^{-1}$. Hence Δ and Θ are diffeomorphism. Furthermore, G acts on $\overline{M}^G_{\boldsymbol{\nu}} \times (G/T)$ by

$$\alpha_g(\overline{m}, hT) := (\overline{m}, ghT).$$

It is clear from (100) that Θ intertwines α and $\overline{\phi}$, that is, $\Theta \circ \alpha_g = \overline{\phi}_g \circ \Theta$ for all $g \in G$.

Let us identify G/T with $\mathbb{P}^1$ by the equivariant diffeomorphism

$$\sigma : hT \in G/T \mapsto [h\, e_2] \in \mathbb{P}^1,$$

where (e_1, e_2) is the standard basis of $\mathbb{C}^2$. We have proved the following:

Proposition 4.3 *Under the hypothesis of Proposition 4.2, $\overline{M}^G_{\mathcal{O}}$ is equivariantly diffeomorphic to $\overline{M}^G_{\boldsymbol{\nu}} \times \mathbb{P}^1$.*

By the Künneth formula, we obtain:

Corollary 4.5 *Under the hypothesis of Proposition 4.2, there is a ring isomorphism $H^*(\overline{M}^G_{\mathcal{O}}) \cong H^*(\overline{M}^G_{\boldsymbol{\nu}}) \otimes H^*(\mathbb{P}^1)$.*

Let us set $\omega_{G/T} := \sigma^*(\omega_{FS})$, where ω_{FS} is the Fubini-Study form. On $\overline{M}^G_{\boldsymbol{\nu}} \times (G/T)$ consider the product symplectic structure $\omega_{\overline{M}^G_{\boldsymbol{\nu}}} \oplus \omega_{G/T}$. Let us assume that $\nu_1 + \nu_2 \neq 0$; then $\lambda_{\boldsymbol{\nu}} > 0$ is a constant (Corollary 4.4), and we may consider the symplectic form

$$\omega'_{G/T} := 2\,(\nu_1 + \nu_2)\,\lambda_{\boldsymbol{\nu}}\,\omega_{G/T}.$$

We can strengthen Proposition 4.3 in the following manner:

Theorem 4.1 *Under the assumptions on Proposition 4.2, assume in addition that* $\nu_1 + \nu_2 \neq 0$. *Then*

$$\Delta : (\overline{M}^G_{\mathcal{O}}, \omega_{\overline{M}^G_{\mathcal{O}}}) \to \left(\overline{M}^G_{\boldsymbol{\nu}} \times (G/T), \omega_{\overline{M}^G_{\boldsymbol{\nu}}} \oplus \omega'_{G/T}\right)$$

is a symplectomorphism.

Remark 4.1 The assumption that $\nu_1 + \nu_2 \neq 0$ is guaranteed in the case of $\mathbb{P}(W_{\mathbf{L},\mathbf{K}})$, by Corollary 2.10.

Proof of Theorem 4.1 $\overline{M}^G_{\mathcal{O}}$ is the $\overline{\mu}$-saturation of $\overline{M}^G_{\boldsymbol{\nu}}$; furthermore, $\overline{M}^G_{\boldsymbol{\nu}}$ maps diffeomorphically under Δ onto $\overline{M}^G_{\boldsymbol{\nu}} \times \{I_2\,T\}$. Since $\overline{\phi}$ is symplectic on $(\overline{M}^G_{\mathcal{O}}, \omega_{\overline{M}^G_{\mathcal{O}}})$, α is symplectic on $\left(\overline{M}^G_{\boldsymbol{\nu}} \times (G/T), \omega_{\overline{M}^G_{\boldsymbol{\nu}}} \oplus \omega_{G/T}\right)$, and Δ intertwines the two symplectic actions, it suffices to prove the statement along $\overline{M}^G_{\boldsymbol{\nu}}$. Explicitly, suppose $\overline{m}_0 \in \overline{M}^G_{\boldsymbol{\nu}}$ and $\overline{m} = \overline{\phi}_g(\overline{m}_0)$ for some $g \in G$; then $\Delta \circ \overline{\phi}_g = \alpha_g \circ \Delta$ implies $\mathrm{d}_{\overline{m}}\Delta \circ \mathrm{d}_{\overline{m}_0}\overline{\phi}_g = \mathrm{d}_{\Delta(\overline{m}_0)}\alpha_g \circ \mathrm{d}_{\overline{m}_0}\Delta$. Hence if $\mathrm{d}_{\overline{m}}\Delta$ is a linear symplectomorphism for every $\overline{m} \in \overline{M}^G_{\boldsymbol{\nu}}$, then it is so also for every $\overline{m} \in \overline{M}^G_{\mathcal{O}}$.

For every $\upsilon \in \mathfrak{g}$, let $\upsilon_{\overline{M}^G_{\mathcal{O}}}$ denote the corresponding orbifold vector field on $\overline{M}^G_{\mathcal{O}}$ (see [11]). If ξ, η, $\mathfrak{a}$ are as in (16), Lemma 3.3 and Corollary 3.1 imply that there is a symplectic direct sum of orbifold (uniformized) tangent bundles

$$\jmath^*(T\overline{M}^G_{\mathcal{O}}) = T\overline{M}^G_{\boldsymbol{\nu}} \oplus \jmath^*(\mathfrak{a}_{\overline{M}^G_{\mathcal{O}}}),$$

where $\jmath : \overline{M}^G_{\boldsymbol{\nu}} \hookrightarrow \overline{M}^G_{\mathcal{O}}$ is the inclusion.

Let us fix $\overline{m} \in \overline{M}^G_{\boldsymbol{\nu}}$, so that $\Delta(\overline{m}) = (\overline{m}, I_2\,T)$. We have

$$T_{(\overline{m}, I_2\,T)}\left(\overline{M}^G_{\boldsymbol{\nu}} \times (G/T)\right) \cong T_{\overline{m}}(\overline{M}^G_{\boldsymbol{\nu}}) \oplus T_{I_2\,T}(G/T) \cong T_{\overline{m}}(\overline{M}^G_{\boldsymbol{\nu}}) \times \mathfrak{a};$$

in both cases, the two summands are symplectically orthogonal. Furthermore, it is apparent from our definition of Δ that, in terms of the previous isomorphisms $T_{\overline{m}}\overline{M}^G_{\mathcal{O}} \cong T_{\overline{m}}(\overline{M}^G_{\boldsymbol{\nu}}) \times \mathfrak{a} \cong T_{(\overline{m}, I_2\,T)}\left(\overline{M}^G_{\boldsymbol{\nu}} \times (G/T)\right)$, $\mathrm{d}_{\overline{m}}\Delta$ corresponds to the identity map $T_{\overline{m}}(\overline{M}^G_{\boldsymbol{\nu}}) \times \mathfrak{a} \to T_{\overline{m}}(\overline{M}^G_{\boldsymbol{\nu}}) \times \mathfrak{a}$. Therefore, we are reduced to comparing the symplectic structures on $\mathfrak{a}$ coming from $\omega_{G/T}$ and from $\overline{M}^G_{\mathcal{O}}$.

On the one hand, with ω_0 the standard symplectic structure on $\mathbb{C}^2$,

$$\omega_{G/T, I_2\,T}(\xi, \eta) = \omega_0(\xi\,e_1, \eta_1) = \frac{\imath}{2}\left(\sum_{j=1}^{2} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_j\right)\left(\begin{pmatrix}\imath\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}\right) = -1.$$

On the other,

$$\begin{aligned}\omega_{\overline{M}_{\mathcal{O}}^G,\overline{m}}\big(\xi_{\overline{M}_{\mathcal{O}}^G}(\overline{m}),\eta_{\overline{M}_{\mathcal{O}}^G}(\overline{m})\big) =&\mathrm{d}_{\overline{m}}\overline{\Phi}^{\xi}\left(\eta_{\overline{M}_{\mathcal{O}}^G}(\overline{m})\right)\\ =&\big\langle\big[\eta,\overline{\Phi}(\overline{m})\big],\xi\big\rangle=-2\,(v_1+v_2)\,\lambda_{\mathbf{v}}.\end{aligned}$$

4.2 Case 2)

Let us relax the assumption that the rank of ϕ is everywhere 3 on $M_{\mathbf{v}}^G$. On $\overline{M}_{\mathbf{v}}^G\times\overline{B(\mathbf{0},\pi/2)}$ let us define a relation $\sim$ as follows: $(\overline{m}_1,z_1)\sim(\overline{m}_2,z_2)$ if and only if either $(\overline{m}_1,z_1)=(\overline{m}_2,z_2)$, or else $z_j=(\pi/2)\,e^{\imath\,\theta_j}$, $j=1,2$, and $\overline{m_2}=\overline{\phi}_{D(\theta_1,\theta_2)}(\overline{m}_1)$, where

$$D(\theta_1,\theta_2):=\begin{pmatrix}e^{\imath\,(\theta_2-\theta_1)} & 0\\ 0 & e^{\imath\,(\theta_1-\theta_2)}\end{pmatrix}.$$

Let $\hat{M}_{\mathcal{O}}^G:=\overline{M}_{\mathbf{v}}^G\times\overline{B(\mathbf{0},\pi/2)}/\sim$ denote the corresponding identification space. If the rank of ϕ along $M_{\mathcal{O}}^G$ is constant and equal to three, as in Proposition 4.3, then T acts trivially on $\overline{M}_{\mathbf{v}}^G$; hence there is a homeomorphism $\hat{M}_{\mathcal{O}}^G=\overline{M}_{\mathbf{v}}^G\times S^2$.

Theorem 4.2 *Suppose that* $0\notin\Phi(M)$, *and that* Φ *is transverse to* $\mathcal{C}(\mathcal{O})$. *Then*:

1. $\overline{M}_{\mathcal{O}}^G$ is homeomorphic to $\hat{M}_{\mathcal{O}}^G$.
2. For every q we have an isomorphism

$$H^q\big(\overline{M}_{\mathcal{O}}^G\big)\cong H^{q-2}\big(\overline{M}_{\mathbf{v}}^G\big)\oplus H^q\big(\overline{M}_{\mathbf{v}}^G\big).$$

Proof of Theorem 4.2 Let us consider the $\mathbb{R}$-linear isomorphism

$$B:z\in\mathbb{C}\mapsto B_z:=\imath\begin{pmatrix}0 & z\\ \overline{z} & 0\end{pmatrix}\in\mathfrak{a}\subset\mathfrak{g}. \tag{101}$$

Lemma 4.4 *For any* $z\in\mathbb{C}$, *we have*

$$e^{B_z}=\begin{pmatrix}\cos(|z|) & \imath\,\dfrac{\sin(|z|)}{|z|}\,z\\ \imath\,\dfrac{\sin(|z|)}{|z|}\,\overline{z} & \cos(|z|)\end{pmatrix}=\cos(|z|)\,I_2+B_{\sin(|z|)\,z/|z|}.$$

The previous expression is well-defined only for $z\neq0$, but $\sin(w)/w$ extends to an even analytic function $F(w^2)$ on $\mathbb{C}$; therefore $\sin(|z|)\,z/|z|=F(|z|^2)\,z$ extends to a real-analytic function of z. We shall regard e^{B_z} as a real-analytic function $\mathbb{C}\to G$.

Proof of Lemma 4.4 The statement follows from a computation based on the identities

$$B_z^{2k} = (-1)^k \, |z|^{2k} \, I_2 = (\imath \, |z|)^{2k} \, I_2, \quad B_z^{2k+1} = (-1)^k \, |z|^{2k} \, B_z = (\imath \, |z|)^{2k} \, B_z.$$

Let $D_{\boldsymbol{v}}$ be the diagonal matrix with diagonal entries $(\, v_1 \quad v_2 \,)$. Then by Lemma 4.4 we have

$$e^{B_z} \, D_{\boldsymbol{v}} \, e^{-B_z} = \begin{pmatrix} v_1 \, \cos(|z|)^2 + v_2 \, \sin(|z|)^2 & \imath \, (v_2 - v_1) \, \cos(|z|) \, \sin(|z|) \, \dfrac{z}{|z|} \\ \imath \, (v_1 - v_2) \, \cos(|z|) \, \sin(|z|) \, \dfrac{\overline{z}}{|z|} & v_2 \, \cos(|z|)^2 + v_1 \, \sin(|z|)^2 \end{pmatrix}. \tag{102}$$

The function $\lambda_{\boldsymbol{v}} : M_{\mathcal{O}}^G \to \mathbb{R}$, being G-invariant, descends to a smooth function on $\overline{M}_{\mathcal{O}}^G$, that will be denoted by the same symbol.

Corollary 4.6 *Let $\overline{\Phi}_{T^1_{\boldsymbol{v}_\perp}} : \overline{M}_{\mathcal{O}}^G \to \imath \, \mathbb{R}$ be the moment map for the Hamiltonian action of $T^1_{\boldsymbol{v}_\perp}$ on the symplectic orbifold $(\overline{M}_{\mathcal{O}}^G, \omega_{\overline{M}_{\mathcal{O}}^G})$. Let us identify $T^1_{\boldsymbol{v}_\perp}$ with S^1 by the isomorphism $\kappa_{\boldsymbol{v}}$ in* (32). *Then for every $\overline{m} \in \overline{M}_{\boldsymbol{v}}^G$ and $z \in \mathbb{C}$ we have*

$$\overline{\Phi}_{T^1_{\boldsymbol{v}_\perp}} \left(\overline{\phi}_{e^{B_z}}(\overline{m}) \right) = \imath \left(v_1^2 - v_2^2 \right) \lambda_{\boldsymbol{v}}(\overline{m}) \, \sin(|z|)^2.$$

Let us set $\boldsymbol{v}' := (\, v_2 \quad v_1 \,)$, $M_{\boldsymbol{v}'}^G := \Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{v}')$. Hence,

$$\overline{M}_{\boldsymbol{v}'}^G := \overline{\Phi}^{-1}(\mathbb{R}_+ \cdot \boldsymbol{v}') = p\big(M_{\boldsymbol{v}'}^G\big).$$

Furthermore,

$$M_{\boldsymbol{v}'}^G = \phi_\gamma\big(M_{\boldsymbol{v}}^G\big), \quad \overline{M}_{\boldsymbol{v}'}^G = \overline{\phi}_\gamma\left(\overline{M}_{\boldsymbol{v}}^G\right), \quad \gamma := \begin{pmatrix} 0 & \imath \\ \imath & 0 \end{pmatrix} = e^{B_{\pi/2}}. \tag{103}$$

Proposition 4.4 *The map*

$$F : (\overline{m}, z) \in \overline{M}_{\boldsymbol{v}}^G \times \overline{B(\mathbf{0}, \pi/2)} \mapsto \overline{\phi}_{e^{B_z}}(\overline{m}) \in \overline{M}_{\mathcal{O}}^G.$$

satisfies the following properties:

1. F is surjective;
2. F restricts to a diffeomorphism $\overline{M}_{\boldsymbol{v}}^G \times B(\mathbf{0}, \pi/2) \to \overline{M}_{\mathcal{O}}^G \setminus \overline{M}_{\boldsymbol{v}}^G$;
3. F induces a homeomorphism between $\hat{M}_{\mathcal{O}}^G \cong \overline{M}_{\mathcal{O}}^G$.

Proof of Proposition 4.4 Let us prove that F is surjective. First note that $\overline{M}_{\boldsymbol{v}}^G = F\big(\overline{M}_{\boldsymbol{v}}^G \times \{0\}\big)$ and that $\overline{M}_{\boldsymbol{v}'}^G = F\big(\overline{M}_{\boldsymbol{v}}^G \times \{\pi/2\}\big)$ by (103). Pick $\overline{m} \in \overline{M}_{\mathcal{O}}^G \setminus \big(\overline{M}_{\boldsymbol{v}}^G \cup \overline{M}_{\boldsymbol{v}'}^G\big)$. Then there exists $g \in G$ such that $\overline{m} \in \overline{\phi}_g(\overline{M}_{\boldsymbol{v}}^G)$, and we need to show that g may be chosen of the form e^{B_z}, for some $z \in B(\mathbf{0}, \pi/2)$. We know that g is neither diagonal nor antidiagonal. Furthermore, since $\overline{M}_{\boldsymbol{\nu}}^G$ is T-invariant, we are free to replace g by any element in $g\,T$. In particular, we may assume $g \in SU(2)$ and then, muliplying by a suitable diagonal matrix in $SU(2)$, that it has the form

$$g = \begin{pmatrix} \cos(x) & -\sin(x)\,e^{-\imath\gamma} \\ \sin(x)\,e^{\imath\gamma} & \cos(x) \end{pmatrix}.$$

Perhaps multiplying by $-I_2$, we may further assume that $\cos(x) > 0$, and since g is not diagonal we may assume $x \in (-\pi/2, 0) \cup (0, \pi/2)$. If $x \in (0, \pi/2)$, set $z = \imath\, x\, e^{-\imath\gamma}$; we conclude from Lemma 4.4 that $g = e^{B_z}$. If $x \in (-\pi/2, 0)$, replace it by $x' = -x \in (0, \pi/2)$ to reach the same conclusion.

Let us prove that F is injective on $\overline{M}_{\boldsymbol{\nu}}^G \times B(\mathbf{0}, \pi/2)$. Suppose $(\overline{m}_j, z_j) \in \overline{M}_{\boldsymbol{\nu}}^G \times B(\mathbf{0}, \pi/2)$ and $F(\overline{m}_1, z_1) = F(\overline{m}_2, z_2)$. We may assume that $|z_j| > 0$ for $j = 1, 2$. We have, by definition of F,

$$\overline{\phi}_{e^{B_{z_1}}}(\overline{m}_1) = \overline{\phi}_{e^{B_{z_2}}}(\overline{m}_2) \quad \Rightarrow \quad \overline{m}_2 = \overline{\phi}_{e^{-B_{z_2}}\,e^{B_{z_1}}}(\overline{m}_1).$$

Since $\nu_1 \neq \nu_2$, this forces

$$e^{B_{-z_2}}\,e^{B_{z_1}} = e^{-B_{z_2}}\,e^{B_{z_1}} \in T.$$

Computing the (1, 2) entry of the latter product by Lemma 4.4, we obtain

$$\imath\left[\cos(|z_2|)\,\sin(|z_1|)\,\frac{z_1}{|z_1|} - \sin(|z_2|)\,\cos(|z_1|)\,\frac{z_2}{|z_2|}\right] = 0.$$

Given that $|z_j| \in (0, \pi/2)$, this implies $z_1 = z_2$; it also follows therefore that $\overline{m}_1 = \overline{m}_2$.

Let us prove that F is an orbifold embedding on $\overline{M}_{\boldsymbol{\nu}}^G \times B(\mathbf{0}, \pi/2)$. We can lift (the restriction of) F to a map

$$\tilde{F} : (m, z) \in M_{\boldsymbol{\nu}}^G \times B(\mathbf{0}, \pi/2) \mapsto \phi_{e^{B_z}}(m) \in M_{\mathcal{O}}^G \setminus M_{\boldsymbol{\nu}'}^G.$$

Let S^1 act on $M_{\boldsymbol{\nu}}^G \times B(\mathbf{0}, \pi/2)$ by the product of the action of $T^1_{\boldsymbol{\nu}_\perp} \cong S^1$ on $M_{\boldsymbol{\nu}}^G$ and the trivial action on $B(\mathbf{0}, \pi/2)$. If ρ is as in Definition 4.1, it follows from Lemma 4.3 that $\tilde{F}$ is S^1-equivariant, and F is the map induced by $\tilde{F}$ on the quotient spaces. To prove the claim, it thus suffices to show that $\tilde{F}$ is a (local) diffeomorphism. We know that $\tilde{F}$ is a local diffeorphism along $M_{\boldsymbol{\nu}}^G \times \{0\}$. If $m \in M_{\boldsymbol{\nu}}^G$ and $v \in T_m M_{\boldsymbol{\nu}}^G$, then for any $z \in B(\mathbf{0}, \pi/2)$ we have

$$\mathrm{d}_{(m,z)}F\big((v, 0)\big) = \mathrm{d}_m\phi_{e^{B_z}}(v), \tag{104}$$

which is tangent to $\phi_{e^{B_z}}(M_{\boldsymbol{\nu}}^G)$ at $\phi_{e^{B_z}}(m)$. On the other hand, for $\delta \sim 0 \in \mathbb{C}$ we have

$$e^{B_{z+\delta}} = e^{B_z}\,e^{B_\delta} = e^{B_z}\,e^{B_\delta - \frac{1}{2}[B_z, B_\delta] + R_3(\delta)}.$$

Hence

$$\mathrm{d}_{(m,z)}F\big((0, \delta)\big) = \mathrm{d}_m\phi_{e^{B_z}}\left((B_\delta)_M(m) - \frac{1}{2}\,[B_z, B_\delta]_M(m)\right). \tag{105}$$

Since $[B_z, B_\delta]$ is diagonal and $T_m M_{\boldsymbol{\nu}}^G$ is T-invariant, $[B_z, B_\delta]_M(m) \in T_m M_{\boldsymbol{\nu}}^G$. On the other hand, $(B_\delta)_M(m) \neq 0$ for $\delta \neq 0$, and is normal to $M_{\boldsymbol{\nu}}^G$. Hence it follows (104) and (105) that

$$\mathrm{d}_{(m,0)}F : T_{(m,z)}\left(M_{\boldsymbol{v}}^G \times B(\mathbf{0},\pi/2)\right) \cong T_m M_{\boldsymbol{v}}^G \times \mathbb{C} \to T_{\phi_{e^{B_z}}(m)} M_{\mathcal{O}}^G$$

is an isomorphism of real vector spaces.

Finally, let us show that the topology of $\overline{M}_{\mathcal{O}}^G$ is indeed the quotient topology of F. Clearly F is continuous, hence $F^{-1}(U)$ is open for every $U \subset \overline{M}_{\mathcal{O}}^G$. Suppose by contradiction that $F^{-1}(U)$ is open for some $U \subset \overline{M}_{\mathcal{O}}^G$ which is not open. Let $\overline{m} \in U$ be such that there exists a sequence $\overline{m}_j \in \overline{M}_{\mathcal{O}}^G$, $j = 1, 2, \ldots$, such that $\overline{m}_j \to \overline{m}$ and $\overline{m}_j \notin U$ for every j. The subset $R := \{\overline{m}\}_j \cup \{\overline{m}\} \subset \overline{M}_{\mathcal{O}}^G$ is compact, and since F is proper so is $F^{-1}(R)$. Consider $(\overline{n}_j, z_j) \in M_{\boldsymbol{v}}^G \times \overline{B(\mathbf{0},\pi/2)}$ such that $F(\overline{n}_j, z_j) = \overline{m}_j$ for every j. Perhaps passing to a subsequence, we may assume $\overline{n}_j \to \overline{n} \in M_{\boldsymbol{v}}^G$ and $z_j \to z \in \overline{B(\mathbf{0},\pi/2)}$, and therefore by continuity and uniqueness of the limit $F(\overline{n}, z) = \overline{m} \in U$. Hence $(\overline{n}, z) \in F^{-1}(U)$, and since the latter is open by assumption we need to have $(\overline{n}_j, z_j) \in F^{-1}(U)$ for all $j \gg 0$. But then $\overline{m}_j = F(\overline{n}_j, z_j) \in U$, a contradiction.

These considerations may be repeated inverting the roles of $\boldsymbol{v}$ and $\boldsymbol{v}'$. Thus, we can replace F in the statement of Proposition 4.4 by a similarly defined map

$$F' : (\overline{m}, \eta) \in \overline{M}_{\boldsymbol{v}'}^G \times \overline{B(\mathbf{0},\pi/2)} \mapsto \overline{\phi}_{e^{B_\eta}}(\overline{m}) \in \overline{M}_{\mathcal{O}}^G,$$

and prove an analogue of Proposition 4.4. In particular, we obtain two diffeomorphisms

$$\overline{M}_{\boldsymbol{v}}^G \times B^*(\mathbf{0},\pi/2) \xrightarrow{F} \overline{M}_{\mathcal{O}}^G \setminus \left(\overline{M}_{\boldsymbol{v}}^G \cup \overline{M}_{\boldsymbol{v}'}^G\right) \xleftarrow{F'} \overline{M}_{\boldsymbol{v}'}^G \times B^*(\mathbf{0},\pi/2),$$

where $B^*(\mathbf{0},\pi/2) := B(\mathbf{0},\pi/2) \setminus \{\mathbf{0}\}$.

Lemma 4.5 *Suppose* $(\overline{m}_1, z_1) \in \overline{M}_{\boldsymbol{v}}^G \times B^*(\mathbf{0},\pi/2)$, $(\overline{m}_2, z_2) \in \overline{M}_{\boldsymbol{v}'}^G \times B^*(\mathbf{0},\pi/2)$, *and* $F(\overline{m}_1, z_1) = F'(\overline{m}_2, z_2)$. *Then* $|z_1| + |z_2| = \pi/2$.

Proof of Lemma 4.5 Let $\overline{m} := F(\overline{m}_1, z_1)$. Then $\overline{m}$, $\overline{m}_1$, $\overline{m}_2$ are all in the same G-orbit. Therefore, $\lambda_{\boldsymbol{v}}(\overline{m}_1) = \lambda_{\boldsymbol{v}}(\overline{m}) = \lambda_{\boldsymbol{v}}(\overline{m}_2)$. By (102) and Corollary 4.6 and their analogues with $\boldsymbol{v}$ and $\boldsymbol{v}'$ interchanged, we have

$$\overline{\Phi}_{T^1_{v_\perp}}(\overline{m}) = \imath\left(v_1^2 - v_2^2\right)\lambda_{\boldsymbol{v}}(\overline{m})\,\sin(|z_1|)^2 = \imath\left(v_1^2 - v_2^2\right)\lambda_{\boldsymbol{v}}(\overline{m})\,\cos(|z_2|)^2.$$

Since $|z_1|, |z_2| \in (0, \pi/2)$, this forces $|z_1| + |z_2| = \pi/2$.

Let us set

$$U := F\left(\overline{M}_{\boldsymbol{v}}^G \times B(\mathbf{0}, 3\,\pi/8)\right), \quad U' := F'\left(\overline{M}_{\boldsymbol{v}'}^G \times B(\mathbf{0}, 3\,\pi/8)\right). \tag{106}$$

Then U, $U' \subset \overline{M}_{\mathcal{O}}^G$ are open and diffeomorphic to $\overline{M}_{\boldsymbol{v}}^G \times B(\mathbf{0}, 3\,\pi/8)$ by Proposition 4.4 and its analogue for F'. Furthermore, by Lemma 4.5,

$$
\begin{aligned}
U'^c :=& F'\left(\left\{(\overline{m}, z) \in \overline{M}^G_{\mathbf{v}'} \times \overline{B(\mathbf{0}, \pi/2)} : |z| \geq \frac{3}{8}\,\pi\right\}\right)\\
=& F\left(\left\{(\overline{m}, z) \in \overline{M}^G_{\mathbf{v}} \times \overline{B(\mathbf{0}, \pi/2)} : |z| \leq \frac{1}{8}\,\pi\right\}\right) \subset U.
\end{aligned}
\tag{107}
$$

Hence $\{U, U'\}$ is an open cover of $M^G_{\mathcal{O}}$. By (106) and (107) we have

$$
U \cap U' = F\left(\overline{M}^G_{\mathbf{v}} \times A\left(\mathbf{0}, \frac{1}{8}\,\pi, \frac{3}{8}\,\pi\right)\right),
\tag{108}
$$

where for $a<b<0$ we set $A(\mathbf{0}, a, b) = \{z \in \mathbb{C} : a<|z|<b\}$. Also, F induces a diffeomorphism $\overline{M}^G_{\mathbf{v}} \times A(\mathbf{0}, \pi/8, 3\,\pi/8)$ and $U \cap U'$. Therefore, in view of (108) and the Künneth formula, the Mayer-Vietoris sequence for the open cover $\{U, U'\}$ of $\overline{M}^G_{\mathcal{O}}$ has the form

$$
\begin{aligned}
\ldots \to H^q\left(\overline{M}^G_{\mathcal{O}}\right) &\to H^q\left(\overline{M}^G_{\mathbf{v}}\right) \oplus H^q\left(\overline{M}^G_{\mathbf{v}}\right)\\
&\to H^q\left(\overline{M}^G_{\mathbf{v}}\right) \oplus H^{q-1}\left(\overline{M}^G_{\mathbf{v}}\right) \to H^{q+1}\left(\overline{M}^G_{\mathcal{O}}\right) \to \cdots,
\end{aligned}
\tag{109}
$$

which splits in short exact sequences

$$
0 \to H^{q-1}\left(\overline{M}^G_{\mathbf{v}}\right) \to H^{q+1}\left(\overline{M}^G_{\mathcal{O}}\right) \to H^{q+1}\left(\overline{M}^G_{\mathbf{v}}\right) \to 0.
$$

Acknowledgements I am endebted to the referee for various interesting comments and for suggesting several improvements in presentation.

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

References

1. Atiyah, M.F.: Convexity and commuting Hamiltonians. Bull. Lond. Math. Soc. **14**, 1–15 (1982)
2. Bröcker, T., Tom Dieck, T.: Representations of compact Lie groups. Corrected reprint of the 1985 translation. Graduate Texts in Mathematics, **98**. Springer-Verlag, New York (1995)
3. Duistermaat, J.J., Heckman, G.J.: On the variation in the cohomology of the symplectic form of the reduced phase space. Invent. Math. **69**(2), 259–268 (1982)
4. Galasso, A., Paoletti, R.: Equivariant asymptotics of Szegö kernels under Hamiltonian U(2)-actions. Ann. Mat. Pura Appl. (4) **198**(2), 639–683 (2019)

5. Guillemin, V., Sternberg, S.: Geometric quantization and multiplicities of group representations. Invent. Math. **67**(3), 515–538 (1982)
6. Guillemin, V., Sternberg, S.: Homogeneous quantization and multiplicities of group representations. J. Funct. Anal. **47**(3), 344–380 (1982)
7. Guillemin, V., Sternberg, S.: Convexity properties of the moment mapping. Invent. Math. **67**(3), 491–513 (1982)
8. Guillemin, V., Sternberg, S.: Symplectic Techniques in Physics, 2nd edn. Cambridge University Press, Cambridge (1990)
9. Kirwan, F.C.: Cohomology of Quotients in Symplectic and Algebraic Geometry, Mathematical Notes, 31. Princeton University Press, Princeton (1984)
10. Kostant, B.: Quantization and unitary representations. I. Prequantization, Lectures in modern analysis and applications, III, pp. 87–208. Lecture Notes in Math., Vol. **170**, Springer, Berlin (1970)
11. Lerman, E., Tolman, S.: Hamiltonian torus actions on symplectic orbifolds and toric varieties. Trans. Am. Math. Soc. **349**(10), 4201–4230 (1997)
12. Martin, S.: Symplectic quotients by a nonabelian group and by its maximal torus. arXiv:0001002
13. Mumford, D., Fogarty, J., Kirwan, F.C.: Geometric invariant theory, Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], **34**. Springer-Verlag, Berlin (1994)
14. Ness, L.: A stratification of the null cone via the moment map, With an appendix by David Mumford. Am. J. Math. **106**(6), 1281–1329 (1984)
15. Paoletti, R.: Asymptotics of Szegö kernels under Hamiltonian torus actions. Israel J. Math. **191**(1), 363–403 (2012). https://doi.org/10.1007/s11856-011-0212-4
16. Paoletti, R.: Lower-order asymptotics for Szegö and Toeplitz kernels under Hamiltonian circle actions, Recent advances in algebraic geometry, pp. 321–369, London Math. Soc. Lecture Note Ser., **417**, Cambridge University Press, Cambridge (2015)
17. Paoletti, R.: Polarized orbifolds associated to quantized Hamiltonian torus actions. J. Geom. Phys. **170**, 104363, 20 pp (2021)
18. Varadarajan, V.S.: An introduction to harmonic analysis on semisimple Lie groups, Corrected reprint of the 1989 original. Cambridge Studies in Advanced Mathematics, **16**, Cambridge University Press, Cambridge (1999)
19. Weinstein, A.: Symplectic V-manifolds, periodic orbits of Hamiltonian systems, and the volume of certain Riemannian manifolds. Commun. Pure Appl. Math. **30**(2), 265–271 (1977)