

LOOP EQUATIONS AND A PROOF OF ZVONKINE'S qr -ELSV FORMULA

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ABSTRACT. We prove the 2006 Zvonkine conjecture that expresses Hurwitz numbers with completed cycles in terms of intersection numbers with the Chiodo classes via the so-called r -ELSV formula, as well as its orbifold generalization, the so-called qr -ELSV formula.

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1. INTRODUCTION

This paper is concerned with spin Hurwitz numbers, which have been conjectured by Zvonkine [Zvo06] to be expressible as integrals over the moduli space of curves, in a generalized ELSV formula, called Zvonkine's r -ELSV formula. In [KLPS19], the authors conjectured an orbifold generalization of this formula, called Zvonkine's qr -ELSV formula. In this paper, we prove the latter, and hence also the former, formula, via topological recursion and quadratic loop equations. We will introduce all of these concepts in this introduction.

1.1. q -orbifold r -spin Hurwitz numbers. In this section we introduce the q -orbifold r -spin Hurwitz numbers, following [OP06, Zvo06, SSZ12, SSZ15, KLPS19]. They are a very important and natural type of Hurwitz numbers; more precisely, they are a special case of completed Hurwitz numbers. Completed Hurwitz numbers were introduced by Okounkov and Pandharipande in [OP06] to establish a relation between Hurwitz numbers and relative Gromov-Witten invariants; in this section we recall their result specified for the q -orbifold r -spin case.

1.1.1. Completed cycles. A partition λ of an integer d is a non-increasing finite sequence $\lambda_1 \geq \dots \geq \lambda_l$ such that $\sum \lambda_i = d$.

It is known that the irreducible representations ρ_λ of the symmetric group S_d are in a natural one-to-one correspondence with the partitions λ of d . On the other hand, to a partition λ of d one can assign a central element $C_{p,\lambda}$ of the group algebra $\mathbb{C}S_p$ for any positive integer p . The coefficient of a given permutation $\sigma \in S_p$ in $C_{p,\lambda}$ is defined as the number of ways to choose and label l cycles of σ so that their lengths are $\lambda_1, \dots, \lambda_l$, and the remaining $p - d$ elements are fixed points of σ . Thus the coefficient of σ vanishes unless its cycle lengths are $\lambda_1, \dots, \lambda_l, 1, \dots, 1$. In particular, $C_{p,\lambda} = 0$ if $p < d$. Thus $C_{p,\lambda}$ is the sum of permutations with l numbered cycles of lengths $\lambda_1, \dots, \lambda_l$ and any number of non-numbered fixed points.

The collection of elements $C_{p,\lambda}$ for $p = 1, 2, \dots$ is called a *stable center element* C_λ . For example, the stable element $C_{(2)}$ is the sum of all transpositions in $\mathbb{C}S_p$, which is well-defined for each p , and in particular equals zero for $p = 1$.

Let λ be a partition of d and μ a partition of p . Consider the representation $\rho_\mu : S_p \rightarrow \text{End}(V)$, V being the representation space with the dimension given by the hook length formula. Since $C_{p,\lambda}$ lies in the center of $\mathbb{C}S_p$, its image under ρ_μ (extended to the group algebra) is an operator, corresponding to a multiplication by a constant. Denote this constant by $f_\lambda(\mu)$. Thus to a stable center element C_λ we have assigned a function f_λ defined on the set of all partitions, \mathcal{P} . We are interested in the vector space spanned by the functions f_λ .

To study this space, one defines some new functions on the set of partitions as follows:

$$(1.1) \quad \mathbf{p}_{r+1}(\mu) = \frac{1}{r+1} \sum_{i \geq 1} \left[\left(\mu_i - i + \frac{1}{2} \right)^{r+1} - \left(-i + \frac{1}{2} \right)^{r+1} \right] \quad (r \geq 0).$$

(The standard definition [OP06, p.11] involves certain additive constants that we have dropped to simplify the expression, since these constants play no role in this paper.)

Theorem 1.1 (Kerov, Olshansky [KO94]). *The vector space spanned by the functions f_λ coincides with the algebra generated by the functions $\mathbf{p}_1, \mathbf{p}_2, \dots$*

As a corollary, to each stable center element C_λ we can assign a polynomial in $\mathbf{p}_1, \mathbf{p}_2, \dots$ and, conversely, each \mathbf{p}_{r+1} corresponds to a linear combination of stable center elements C_λ .

Definition 1.2. The linear combination of stable center elements corresponding to \mathbf{p}_{r+1} is called the *completed $(r+1)$ -cycle* and denoted by $\bar{C}_{(r+1)}$.

The first completed cycles are:

$$(1.2) \quad \begin{aligned} \bar{C}_{(1)} &= C_{(1)}, \\ \bar{C}_{(2)} &= C_{(2)}, \\ \bar{C}_{(3)} &= C_{(3)} + C_{(1,1)} + \frac{1}{12}C_{(1)}, \\ \bar{C}_{(4)} &= C_{(4)} + 2C_{(2,1)} + \frac{5}{4}C_{(2)}, \\ \bar{C}_{(5)} &= C_{(5)} + 3C_{(3,1)} + 4C_{(2,2)} + \frac{11}{2}C_{(3)} + 4C_{(1,1,1)} + \frac{3}{2}C_{(1,1)} + \frac{1}{80}C_{(1)}. \end{aligned}$$

Definition 1.3. We say that a stable center element C_λ involved in the completed cycle $\bar{C}_{(r+1)}$ has *genus defect* $[r+2 - \sum(\lambda_i + 1)]/2$.

1.1.2. *r -spin Hurwitz numbers.* Let $g \in \mathbb{Z}_{\geq 0}$ and $r \in \mathbb{Z}_{\geq 1}$. Let $\vec{\mu} = (\mu_1, \dots, \mu_n)$ be an integer partition of length $n = \ell(\mu)$ such that $m := (\sum_{i=1}^n \mu_i + n + 2g - 2)/r$ is an integer, and let $d := |\mu| = \sum_{i=1}^n \mu_i$.

Recall that the completed $(r+1)$ -cycle can be considered as a central element of the group algebra $\mathbb{C}S_d$. An *r -factorization of type (μ_1, \dots, μ_n)* in the symmetric group S_d is a factorization

$$(1.3) \quad \sigma_1 \dots \sigma_m = \sigma$$

such that

- (i) the cycle lengths of σ equal μ_1, \dots, μ_n and
- (ii) for each permutation σ_i , the stable center element corresponding to its cyclic type enters the completed $(r+1)$ -cycle with a nonzero coefficient.

The product of these coefficients for i going from 1 to m is called the *weight* of the r -factorization.

Choose m points $y_1, \dots, y_m \in \mathbb{C}$ and a system of m loops $s_i \in \pi_1(\mathbb{C} \setminus \{y_1, \dots, y_m\})$, s_i going around y_i . Then to an r -factorization one can assign a family of stable maps from nodal curves to $\mathbb{C}P^1$. This is done in the following way.

- (i) Consider the covering of $\mathbb{C}P^1$ ramified over y_1, \dots, y_m , and ∞ with monodromies given by $\sigma_1, \dots, \sigma_m$ and σ^{-1} (relative to the chosen loops).

- (ii) If σ_i has l_i distinguished cycles and genus defect g_i , glue a curve of genus g_i with l_i marked points to the l_i preimages of the i -th ramification point that correspond to the distinguished cycles. The covering mapping is extended on this new component by saying that it is entirely projected to the i -th ramification point.

- (iii) Among the newly added components, contract those that are unstable.

One can easily check that the arithmetic genus of the curve C constructed in this way is equal to g . The complex structure on the newly added components of C can be chosen arbitrarily, which implies that in general we obtain not a unique stable map, but a family of stable maps.

An r -factorization is called *transitive* if the curve C assigned to the factorization is connected. To give a more formal description, consider for each σ_k , $k = 1, \dots, m$, the set of transpositions T_{σ_k} that consists of $(ij) \in S_d$ for all i, j belonging to possibly different distinguished cycles of σ_k . Then an r -factorization is called *transitive* if the subgroup of S_d generated by σ and $\cup_{k=1}^m T_{\sigma_k}$ acts transitively on $\{1, \dots, d\}$.

Definition 1.4. The *disconnected r -spin Hurwitz number* $h_{g;\vec{\mu}}^{\bullet,r}$ is the sum of weights of all r -factorizations of type (μ_1, \dots, μ_n) , divided by $|\mu|!m!$.

Definition 1.5. The *connected r -spin Hurwitz number* $h_{g;\vec{\mu}}^{\circ,r}$ is the sum of weights of transitive r -factorizations of type (μ_1, \dots, μ_n) , divided by $|\mu|!m!$.

Note that connected and disconnected r -spin Hurwitz numbers are related via the usual inclusion-exclusion formula.

1.1.3. *q -orbifold r -spin Hurwitz numbers.* The q -orbifold r -spin Hurwitz numbers arise as a generalization of the previous case, when one adds another ramification point with profile $[qq \dots q]$. In the language of the symmetric group this looks as follows.

Let $g \in \mathbb{Z}_{\geq 0}$, $r \in \mathbb{Z}_{\geq 1}$ and $q \in \mathbb{Z}_{\geq 1}$. Let $\vec{\mu} = (\mu_1, \dots, \mu_n)$ be an integer partition of length $n = \ell(\mu)$ such that $d := |\mu| = \sum_{i=1}^n \mu_i$ is divisible by q and $m := (d/q + n + 2g - 2)/r$ is an integer.

A q, r -factorization of type (μ_1, \dots, μ_n) in the symmetric group S_d is a factorization

$$(1.4) \quad \sigma_1 \dots \sigma_m \gamma = \sigma$$

such that

- (i) the cycle lengths of γ are all equal to q ,
- (ii) the cycle lengths of σ equal μ_1, \dots, μ_n and
- (iii) each permutation σ_i enters the completed $(r+1)$ -cycle with a nonzero coefficient.

The product of these coefficients for i going from 1 to m is called the *weight* of the r -factorization.

In a way completely analogous to the non-orbifold case we can define *transitive q, r -factorizations*. Then we can proceed to defining disconnected and connected q -orbifold r -spin Hurwitz numbers:

Definition 1.6. The *disconnected q -orbifold r -spin Hurwitz number* $h_{g;\vec{\mu}}^{\bullet,q,r}$ is the sum of weights of all q, r -factorizations of type (μ_1, \dots, μ_n) , divided by $|\mu|!m!$.

Definition 1.7. The *connected q -orbifold r -spin Hurwitz number* $h_{g;\vec{\mu}}^{\circ,q,r}$ is the sum of weights of transitive q, r -factorizations of type (μ_1, \dots, μ_n) , divided by $|\mu|!m!$.

Again, connected and disconnected q -orbifold r -spin Hurwitz numbers are related via the usual inclusion-exclusion formula.

Naturally, for $q = 1$ one recovers the r -spin Hurwitz numbers, for $r = 1$ one recovers the q -orbifold Hurwitz numbers, while for $q = r = 1$ one arrives at the classical simple Hurwitz numbers.

1.1.4. *Semi-infinite wedge formalism.* This subsection is devoted to writing q -orbifold r -spin Hurwitz numbers in terms of the *semi-infinite wedge formalism* (also known as *free-fermion formalism* to physicists).

First, we define the basic ingredients of this formalism. For a more complete introduction see e.g. [Joh15]. We will write $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ for the set of half-integers.

Definition 1.8. The Lie algebra \mathcal{A}_∞ is the \mathbb{C} -vector space of matrices $(A_{ij})_{i,j \in \mathbb{Z}'}$ with only finitely many non-zero diagonals, together with the commutator bracket.

In this algebra, we will consider the following elements:

- (1) The standard “matrix unit” elements $\{E_{i,j} \mid i, j \in \mathbb{Z}'\}$ such that $(E_{i,j})_{k,l} = \delta_{i,k}\delta_{j,l}$; here and below $\delta_{i,j}$ is the Kronecker delta;
- (2) The diagonal algebra elements (operators) $\mathcal{F}_n := \sum_{k \in \mathbb{Z}'} k^n E_{k,k}$. In particular, $C := \mathcal{F}_0$ is the *charge operator* and $E := \mathcal{F}_1$ is the *energy operator*. An algebra element A has energy $e \in \mathbb{Z}$ if $[A, E] = eA$;
- (3) For any non-zero integer n , the energy n element $\alpha_n := \sum_{k \in \mathbb{Z}'} E_{k-n,k}$.

The semi-infinite wedge space is a certain projective representation of this algebra, which we will construct now.

Definition 1.9. Let V be the vector space spanned by \mathbb{Z}' : $V = \bigoplus_{i \in \mathbb{Z}'} \mathbb{C}i$, where the i are basis elements. We define the *semi-infinite wedge space* $\mathcal{V} := \bigwedge^{\frac{\infty}{2}} V$ to be the span of all one-sided infinite wedge products

$$(1.5) \quad \underline{i}_1 \wedge \underline{i}_2 \wedge \cdots,$$

with $i_1 < i_2 < \cdots \in \mathbb{Z}'$, such that there exists a constant c with $i_k + k - \frac{1}{2} = c$ for large k . The constant c is called the *charge*.

Remark 1.10. Notice that \mathcal{A}_∞ has a natural representation on V , but this cannot be extended to \mathcal{V} easily, as one would have to deal with infinite sums.

Definition 1.11. For a partition λ , define

$$(1.6) \quad v_\lambda := \underbrace{\lambda_1 - \frac{1}{2}} \wedge \underbrace{\lambda_2 - \frac{3}{2}} \wedge \cdots.$$

In particular, define the vacuum $|0\rangle := v_\emptyset$ and let the covacuum $\langle 0|$ be its dual in \mathcal{V}^* .

Define \mathcal{V}_0 to be the charge-zero subspace of \mathcal{V} . Then $\mathcal{V}_0 = \bigoplus_{\lambda \in \mathcal{P}} \mathbb{C}v_\lambda$.

Definition 1.12. For an endomorphism O of \mathcal{V}_0 , define its *vacuum expectation value* or *disconnected correlator* to be

$$(1.7) \quad \langle O \rangle^\bullet := \langle 0|O|0\rangle.$$

Definition 1.13. Define a projective representation of \mathcal{A}_∞ on \mathcal{V}_0 as follows: for $i \neq j$ or $i = j > 0$, $E_{i,j}$ checks whether v_λ contains \underline{j} as a factor and replaces it by \underline{i} if it does. If $i = j < 0$, $E_{i,i}v_\lambda = -v_\lambda$ if v_λ does not contain \underline{i} . In all other cases it gives zero.

Equivalently, this gives a representation of the central extension $\tilde{\mathcal{A}}_\infty = \mathcal{A}_\infty \oplus \mathbb{C}1$, with commutation between “matrix unit” elements

$$(1.8) \quad [E_{a,b}, E_{c,d}] = \delta_{b,c}E_{a,d} - \delta_{a,d}E_{c,b} + \delta_{b,c}\delta_{a,d}(\delta_{b>0} - \delta_{d>0})1.$$

With these definitions, it is easy to see that C is identically zero on \mathcal{V}_0 and $Ev_\lambda = |\lambda|v_\lambda$. Therefore, any positive-energy operator annihilates the vacuum. Similarly, so do all \mathcal{F}_r .

The q -orbifold r -spin Hurwitz numbers can be represented as vacuum expectations of certain operators. We will write $\mu = a[\mu]_a + \langle \mu \rangle_a$ for the integral division of an integer μ by a natural number a . If $a = qr$, we may omit the subscript.

The q -orbifold r -spin Hurwitz numbers can be represented in terms of the semi-infinite wedge formalism as described in the following proposition.

Proposition 1.14. *The disconnected q -orbifold r -spin Hurwitz numbers can be expressed in terms of semi-infinite wedge formalism as*

$$(1.9) \quad h_{g;\bar{\mu}}^{\bullet,q,r} = \left\langle \left(\frac{\alpha_q}{q} \right)^{\frac{|\mu|}{q}} \frac{1}{\left(\frac{|\mu|}{q} \right)!} \frac{\mathcal{F}_{r+1}^m}{m!(r+1)^m} \prod_{i=1}^{l(\bar{\mu})} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle,$$

where the number of $(r+1)$ -completed cycles is

$$(1.10) \quad m = \frac{2g - 2 + l(\mu) + \frac{|\mu|}{q}}{r}.$$

This statement follows from the basic character formula for general Hurwitz numbers, see [OPo6].

Definition 1.15. The *generating series of q -orbifold r -spin Hurwitz numbers* is defined as

$$(1.11) \quad H^{\bullet,q,r}(\vec{\mu}, u) := \sum_{g=0}^{\infty} h_{g;\vec{\mu}}^{\bullet,q,r} u^{rm} = \left\langle e^{\frac{\alpha q}{r}} e^{u^r \frac{\tau_{r+1}}{r+1}} \prod_{i=1}^{l(\vec{\mu})} \frac{\alpha - \mu_i}{\mu_i} \right\rangle.$$

The *free energies* are defined as

$$(1.12) \quad H_{g,n}^{q,r}(X_1, \dots, X_n) := \sum_{\mu_1, \dots, \mu_n=1}^{\infty} h_{g;\vec{\mu}}^{\circ,q,r} e^{\sum_{i=1}^n \mu_i X_i}$$

With the help of semi-infinite wedge formalism, in [KLPS19] the following *quasi-polynomiality theorem* was proved in a purely combinatorial way:

Theorem 1.16 ([KLPS19]). *For $2g - 2 + \ell(\vec{\mu}) > 0$, the connected q -orbifold r -spin Hurwitz numbers can be expressed in the following way:*

$$(1.13) \quad h_{g;\vec{\mu}}^{\circ,q,r} = \prod_{i=1}^{l(\vec{\mu})} \frac{\mu_i^{[\mu_i]}}{[\mu_i]!} P_{\langle \vec{\mu} \rangle}(\mu_1, \dots, \mu_{l(\vec{\mu})}),$$

where P are symmetric polynomials in the variables $\mu_1, \dots, \mu_{l(\vec{\mu})}$ whose coefficients depend on the parameters $\langle \mu_1 \rangle, \dots, \langle \mu_{l(\vec{\mu})} \rangle$, and which has an upper bound on its degree in all variables that is independent of $\vec{\mu}$.

1.1.5. *Relative Gromov-Witten invariants and the Okounkov-Pandharipande formula.* The q -orbifold r -spin Hurwitz numbers were originally introduced in [OP06] because of their relation to relative Gromov-Witten invariants of $\mathbb{C}P^1$; this relation is a special case of the Okounkov-Pandharipande theorem from [OP06], which we would like to recall.

Let $\overline{\mathcal{M}}_{g,m;\mu_1, \dots, \mu_n;q}(\mathbb{C}P^1)$ be the space of stable genus g maps to $\mathbb{C}P^1$ relative to $\{\infty, 0\} \in \mathbb{C}P^1$ with profiles (μ_1, \dots, μ_n) and (q, q, \dots, q) respectively and with m marked points in the source curve, where $m = (|\mu|/q + n + 2g - 2)/r$. Let $[\overline{\mathcal{M}}_{g,m;\mu_1, \dots, \mu_n;q}(\mathbb{C}P^1)]^{\text{vir}}$ be its virtual fundamental class. See e.g. [Vako8] for the precise definition and main properties. Let $\omega \in H^2(\mathbb{C}P^1)$ be the Poincaré dual class of a point.

A special case of Okounkov-Pandharipande theorem from [OP06] states that

Theorem 1.17 (Okounkov-Pandharipande, [OP06]). *Connected q -orbifold, r -spin Hurwitz numbers are equal to certain relative Gromov-Witten invariants of $\mathbb{C}P^1$. Specifically, we have:*

$$(1.14) \quad h_{g;\vec{\mu}}^{\circ,q,r} = \frac{(r!)^m}{m!} \int_{[\overline{\mathcal{M}}_{g,m;\mu_1, \dots, \mu_n;q}(\mathbb{C}P^1)]^{\text{vir}}} \text{ev}_1^*(\omega) \psi_1^r \cdots \text{ev}_m^*(\omega) \psi_m^r$$

Here ev_i denotes the evaluation map $\overline{\mathcal{M}}_{g,m;\mu_1, \dots, \mu_n;q}(\mathbb{C}P^1) \rightarrow \mathbb{C}P^1$ at the i -th marked point, $i = 1, \dots, m$, and $\psi_i \in H^2(\overline{\mathcal{M}}_{g,m;\mu_1, \dots, \mu_n;q}(\mathbb{C}P^1))$ is the ψ -class corresponding to the i -th marked point.

1.2. **Chiodo classes and Zvonkine's conjecture.** The central objects in Zvonkine's conjecture are the so-called Chiodo classes, which are cohomology classes on the moduli spaces of stable curves $\overline{\mathcal{M}}_{g,n}$. In this section we briefly recall their definition, as well as properties relevant for our proof. More details can be found in [Chio8b, CR10, JPPZ17, SSZ15, KLPS19, CJ18].

1.2.1. *Geometric definition.* Let $r \geq 1$ be an integer and $g \geq 0$, $n \geq 1$, $1 \leq a_1, \dots, a_n \leq r$, and $s \geq 0$ be integers satisfying

$$(1.15) \quad (2g - 2 + n)s - \sum_{i=1}^n a_i \in r\mathbb{Z}$$

Let $[C, p_1, \dots, p_n] \in \mathcal{M}_{g,n}$ be a nonsingular curve with distinct marked points. Furthermore, let $\omega_{\log} = \omega_C(\sum p_i)$ be its log-canonical bundle. The condition (1.15) ensures that r th tensor roots L of the line bundle

$$(1.16) \quad \omega_{\log}^{\otimes s} \left(- \sum a_i p_i \right)$$

on C exist. There is a natural compactification of this moduli space of r th roots, denoted $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}^{r,s}$, which is an analog of the Deligne-Mumford compactification of $\mathcal{M}_{g,n}$ and was constructed in [Chio8a, Jaroo, AJ03, CCC07].

Let $\pi : C_{g;a_1,\dots,a_n}^{r,s} \rightarrow \overline{\mathcal{M}}_{g;a_1,\dots,a_n}^{r,s}$ be the universal curve and let $\mathcal{L} \rightarrow C_{g;a_1,\dots,a_n}^{r,s}$ be the universal r th root. The Chiodo class is the full Chern class of the derived push-forward $c(-R^\bullet \pi_* \mathcal{L})$.

In practice, we only need an expression for the pushforward of the Chiodo class to the compactified moduli space of curves $\overline{\mathcal{M}}_{g,n}$. There is an explicit formula for this pushforward in terms of tautological classes, which we recall below.

1.2.2. *Formula in terms of tautological classes.* The Chern characters of the derived push-forward $R^\bullet \pi_* \mathcal{L}$ are given by Chiodo's formula [Chio8b]. In order to give this formula, we first need to give some definitions. For any nodal curve in $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}^{r,s}$, the nodes must have automorphism group $\mathbb{Z}/r\mathbb{Z}$, inducing a primitive character on the cotangent line at each side of the branch (we pick one side). The line bundle L at this side is naturally a $\mathbb{Z}/r\mathbb{Z}$ -representation, because it is an r -th root. This representation is then an a -th power of the representation of the cotangent line at the point for some a . This a is locally constant on the boundary divisor, and hence we can split this divisor into components. We let j_a be the boundary map for the a -th component. We also write ψ', ψ'' for the ψ -classes at the two branches of the node (in general, we use standard notation for ψ and κ tautological classes, see e.g. [Vak03, Zvo12]). Then Chiodo's formula is

$$(1.17) \quad \text{ch}_m(R^\bullet \pi_* \mathcal{L}) = \frac{B_{m+1}(\frac{s}{r})}{(m+1)!} \kappa_m - \sum_{i=1}^n \frac{B_{m+1}(\frac{a_i}{r})}{(m+1)!} \psi_i^m + \frac{r}{2} \sum_{a=0}^{r-1} \frac{B_{m+1}(\frac{a}{r})}{(m+1)!} (j_a)_* \frac{(\psi')^m + (-1)^{m-1} (\psi'')^m}{\psi' + \psi''}.$$

The Bernoulli polynomials $B_l(x)$ used in this formula are generated by the function

$$(1.18) \quad \sum_{l=0}^{\infty} B_l(x) \frac{t^l}{l!} = \frac{te^{xt}}{e^t - 1}.$$

Let ϵ be the forgetful map

$$(1.19) \quad \epsilon : \overline{\mathcal{M}}_{g;a_1,\dots,a_n}^{r,s} \rightarrow \overline{\mathcal{M}}_{g,n}$$

We are interested in the pushforwards of the Chiodo classes

$$(1.20) \quad \begin{aligned} \mathcal{C}_{g,n}(r, s; a_1, \dots, a_n) &:= \epsilon_* c(-R^\bullet \pi_* \mathcal{L}) = \epsilon_* [c(R^1 \pi_* \mathcal{L}) / c(R^0 \pi_* \mathcal{L})] \\ &= \epsilon_* \exp \left(\sum_{m=1}^{\infty} (-1)^m (m-1)! \text{ch}_m(R^\bullet \pi_* \mathcal{L}) \right) \in H^{\text{even}}(\overline{\mathcal{M}}_{g,n}). \end{aligned}$$

The pushforwards of the Chiodo classes form a *cohomological field theory* in the sense of [KM94] (with non-flat unit if $s > r$), and can therefore be written explicitly in terms of the Givental graphs, see [LPSZ17].

1.2.3. *Zvonkine's qr -ELSV formula.* In [KLPS19] the authors proposed the following conjecture, which is a direct orbifold generalization of Zvonkine's conjecture.

Conjecture 1.18. [KLPS19, Conjecture 6.1] *q -orbifold r -spin Hurwitz numbers are given by the formula*

$$(1.21) \quad h_{g,\mu_1,\dots,\mu_n}^{o,q,r} = r^{2g-2+n} (qr)^{\frac{(2g-2+n)q + \sum_{j=1}^n \mu_j}{qr}} \prod_{j=1}^n \frac{\binom{\mu_j}{qr}}{[\mu_j]!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathcal{C}_{g,n}(qr, q; qr - \langle \mu_1 \rangle, \dots, qr - \langle \mu_n \rangle)}{\prod_{j=1}^n (1 - \frac{\mu_j}{qr} \psi_j)},$$

where $\mu = qr[\mu] + \langle \mu \rangle$ is the integral division of μ by qr .

This conjecture expresses the q -orbifold r -spin Hurwitz numbers as an explicit ELSV-like integral over the moduli space of curves, where the role of the Hodge class $1 - \lambda_1 + \dots \pm \lambda_g$ is played by the pushforward of the Chiodo class, $\mathcal{C}_{g,n}(r, s; a_1, \dots, a_n)$. We call this formula for the q -orbifold r -spin Hurwitz numbers Zvonkine's qr -ELSV formula.

This conjecture is already known for $q = r = 1$ (in this case it is the standard ELSV formula proved in [ELSV01], see also [GV03, DBKO⁺15]), $r = 1, q \geq 1$ (then it is the Johnson-Pandharipande-Tseng

formula proved in [JPT11], see also [DBLPS15]), and $r = 2, q \geq 1$ (proved in [BKL⁺21]). It is also known to hold for any $q, r \geq 1$ in genus $g = 0$ [BKL⁺21].

The main result of this paper is a proof of conjecture 1.18 in full generality:

Theorem 1.19. *Zvonkine's qr -ELSV formula holds.*

The proof of this theorem uses the formalism of CEO topological recursion explained below. Let us note one more fact before proceeding to that. Namely, our main result, theorem 1.19, together with Okounkov–Pandharipande's theorem (theorem 1.17) immediately imply the following purely intersection theory statement

Corollary 1.20.

$$(1.22) \quad \frac{(r!)^m}{m!} \int_{[\overline{\mathcal{M}}_{g,m;\mu_1,\dots,\mu_n,q}(\mathbb{C}P^1)]^{\text{vir}}} \text{ev}_1^*(\omega) \psi_1^r \cdots \text{ev}_m^*(\omega) \psi_m^r \\ = \int_{\overline{\mathcal{M}}_{g,n}} \frac{C_{g,n}(qr, q; qr - \langle \mu_1 \rangle, \dots, qr - \langle \mu_n \rangle)}{\prod_{j=1}^n (1 - \frac{\mu_j}{qr} \psi_j)} \cdot r^{2g-2+n} (qr)^{\frac{(2g-2+n)q + \sum_{j=1}^n \mu_j}{qr}} \prod_{j=1}^n \frac{\left(\frac{\mu_j}{qr}\right)^{[\mu_j]}}{[\mu_j]}.$$

1.3. Topological recursion.

1.3.1. *General setup.* The topological recursion of Chekhov, Eynard, and Orantin [CEo6, EOo7, Eyn14b] associates to a Riemann surface Σ (the so-called spectral curve) equipped with two functions $X, y: \Sigma \rightarrow \mathbb{C}$ and a symmetric bidifferential B on Σ^2 satisfying some extra conditions a family of meromorphic symmetric n -differentials (CEO-differentials) $\omega_{g,n}$ defined on $\Sigma^n, g \geq 0, n \geq 1$. We assume that dX is meromorphic and all critical points p_1, \dots, p_r of X are simple, y is holomorphic near p_i and $dy \neq 0$ at $p_i, i = 1, \dots, r$, and B has no singularities except for a double pole on the diagonal with biresidue 1. We set by definition $\omega_{0,1} = ydX, \omega_{0,2} = B$, and for $2g - 2 + n > 0$ we define:

$$(1.23) \quad \omega_{g,n}(z_{\{1,\dots,n\}}) = \frac{1}{2} \sum_{i=1}^r \text{Res}_{z \rightarrow p_i} \frac{\int_z^{\sigma_i(z)} \omega_{0,2}(\cdot, z_1)}{\omega_{0,1}(\sigma_i(z)) - \omega_{0,1}(z)} \left[\omega_{g-1,n+1}(z, \sigma_i(z), z_{\{2,\dots,n\}}) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{2,\dots,n\} \\ (g_i, |I_i|) \neq (0,0)}} \omega_{g_1,1+|I_1|}(z, z_{I_1}) \omega_{g_2,1+|I_2|}(\sigma_i(z), z_{I_2}) \right].$$

Here σ_i is the deck transformation for X near the point $p_i, i = 1, \dots, r$, and all $\omega_{-1,n}, n \geq 1$, are set to be equal to 0. Furthermore, for a set I , we write $z_I = \{z_i\}_{i \in I}$.

Eynard proved in [Eyn14a] that for $2g - 2 + n > 0$ the meromorphic differentials $\omega_{g,n}$ can be represented as linear combinations of the intersection numbers of some explicitly computed tautological classes on $\overline{\mathcal{M}}_{g,n}$ multiplied by some auxiliary differentials. Under some extra conditions, see [DBOSS14] and also [DNO⁺19, DNO⁺18], it is proved in [DBOSS14] that the meromorphic differentials $\omega_{g,n}$ can be represented in terms of the correlators of a semi-simple cohomological field theory of rank r , where the cohomological field theory is given explicitly in terms of Givental graphs [DSS13], and some other auxiliary differentials. More precisely, for $2g - 2 + n > 0$ the differentials $\omega_{g,n}$ are represented as

$$(1.24) \quad \omega_{g,n} = \sum_{\substack{i_1, \dots, i_n \\ a_1, \dots, a_n}} \int_{\overline{\mathcal{M}}_{g,n}} \alpha_{g,n}(e_{i_1}, \dots, e_{i_n}) \prod_{j=1}^n \psi_j^{a_j} d \left(\left(\frac{d}{dX} \right)^{a_j} \xi^{i_j}(z_j) \right),$$

where

$$(1.25) \quad \xi^i(z) := \int^z \frac{\omega_{0,2}(w_i, \cdot)}{dw_i} \Big|_{w_i=0}$$

for a local coordinate w_i near p_i , and $\alpha_{g,n}: V^{\otimes n} \rightarrow H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ form a cohomological field theory, where V is an r -dimensional vector space with basis $\langle e_1, \dots, e_r \rangle$.

1.3.2. *Particular spectral curves.* We consider the spectral curve data

$$(1.26) \quad \Sigma = \mathbb{CP}^1, \quad X(z) = -z^{qr} + \log z, \quad y(z) = z^q, \quad B(z_1, z_2) = dz_1 dz_2 / (z_1 - z_2)^2.$$

It is more convenient to work with this curve using the function $x = e^X = ze^{-z^{qr}}$. For this curve all the ingredients of the formula in equation (1.24) can be computed explicitly, and it is proved in [LPSZ17] that the expansions of $\omega_{g,n}$ in the variables x_1, \dots, x_n near $x_1 = \dots = x_n = 0$ are given by

$$(1.27) \quad \omega_{g,n} \sim d_1 \otimes \dots \otimes d_n \sum_{\mu_1, \dots, \mu_n=1}^{\infty} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathcal{C}_{g,n}(rq, q; qr - \langle \mu_1 \rangle, \dots, qr - \langle \mu_n \rangle)}{\prod_{j=1}^n (1 - \frac{\mu_j}{qr} \psi_j)} \cdot r^{2g-2+n} (qr)^{\frac{(2g-2+n)q + \sum_{j=1}^n \mu_j}{qr}} \prod_{j=1}^n \frac{\left(\frac{\mu_j}{qr}\right)^{[\mu_j]}}{[\mu_j]!} x_j^{\mu_j}.$$

Thus we have the following proposition.

Proposition 1.21 ([LPSZ17, SSZ15]). *Zvonkine's qr -ELSV formula holds if and only if the expansion of the CEO-differentials $\omega_{g,n}$ for the curve (1.26) in the variables x_1, \dots, x_n near $x_1 = \dots = x_n = 0$ is given by*

$$(1.28) \quad \omega_{g,n} - \delta_{g,0} \delta_{n,2} \frac{dx_1 dx_2}{(x_1 - x_2)^2} \sim d_1 \otimes \dots \otimes d_n \sum_{\mu_1, \dots, \mu_n=1}^{\infty} h_{g;\mu}^{\circ, q, r} \prod_{i=1}^n x_i^{\mu_i}.$$

Thus, an equivalent way to reformulate theorem 1.19 is

Theorem 1.22. *The expansion of the CEO-differentials $\omega_{g,n}$ for the curve (1.26) in the variables x_1, \dots, x_n near $x_1 = \dots = x_n = 0$ is given by equation (1.28).*

Remark 1.23. The spectral curve for the q -orbifold r -spin Hurwitz numbers in full generality was predicted in [MSS13] via the analysis of the so-called quantum curve.

Remark 1.24. Historically, this theorem was first formulated for $q = r = 1$ as the Bouchard-Mariño conjecture [BMno8], and this case was first proved in [EMS11] using the ELSV formula for Hurwitz numbers, see also [Eyn11]. In a similar way, this theorem was proved for any $q, r = 1$ in [BHSLM14, DLN16] using the Johnson-Pandharipande-Tseng formula. These proofs are not exactly what we want, since we want to use the inverse of their arguments, namely, we want to use this theorem in order to prove Zvonkine's qr -ELSV formula.

Remark 1.25. Proofs independent of Zvonkine's qr -ELSV formula are known in special cases. First of all, there are non-rigorous physics arguments in [BEMS11] for $q = r = 1$ and in [SSZ15] for $q = 1$, any r . Then there are rigorous proofs in [DBKO⁺15] for $q = r = 1$, in [DBLPS15] for any $q, r = 1$ (see also [KLS19] for an alternative argument for a part of that proof, and a discussion in [Lew18]), and in [BKL⁺21] for any $q, r = 2$. This theorem is also already known for any $q, r \geq 1$ in genus $g = 0$, see [KLPS19] for the unstable cases $n = 1, 2$ and [BKL⁺21] for $n \geq 3$.

1.3.3. *Loop equations.* We use a reformulation of the CEO topological recursion proved in [BEO15, BS17]. We say that a system of meromorphic differentials $\omega_{g,n}$ with possible poles at p_1, \dots, p_{qr} satisfies the *projection property* if $P_1 \cdots P_n \omega_{g,n} = \omega_{g,n}$ for $2g - 2 + n > 0$, where for any meromorphic differential λ we define

$$(1.29) \quad (P\lambda)(z) = \sum_{j=1}^{qr} \operatorname{Res}_{w \rightarrow p_i} \lambda(w) \int_{p_i}^w \omega_{0,2}(\cdot, z),$$

and by writing P_i we mean that we apply this operation to the i -th variable.

Denote

$$(1.30) \quad W_{g,n}(z_{\{1, \dots, n\}}) := \omega_{g,n}(z_{\{1, \dots, n\}}) / \prod_{j=1}^n dX(z_j).$$

We say that a system of meromorphic differentials $\omega_{g,n}$ with possible poles at p_1, \dots, p_{qr} satisfies the *linear loop equations* if for any $g \geq 0, n \geq 1$ the expression

$$(1.31) \quad W_{g,n}(z, z_{\{2, \dots, n\}}) + W_{g,n}(\sigma_i(z), z_{\{2, \dots, n\}})$$

is holomorphic in z for $z \rightarrow p_i$, for $i = 1, \dots, qr$.

We say that a system of meromorphic differentials $\omega_{g,n}$ with possible poles at p_1, \dots, p_{qr} satisfies the *quadratic loop equations* if for any $g \geq 0$, $n \geq 0$ the expression

$$(1.32) \quad W_{g-1, n+2}(z, \sigma_i(z), z_{\{1, \dots, n\}}) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1, \dots, n\}}} W_{g_1, 1+|I_1|}(z, z_{I_1}) W_{g_2, 1+|I_2|}(\sigma_i(z), z_{I_2})$$

is holomorphic in z for $z \rightarrow p_i$, for $i = 1, \dots, qr$.

Proposition 1.26 ([BEO15, BS17]). *A system of meromorphic differentials $\omega_{g,n}$ with $\omega_{0,1} = ydX$, $\omega_{0,2} = B$, satisfies the CEO topological recursion for the data (Σ, X, y, B) if and only if it satisfies the projection property, the linear loop equation, and the quadratic loop equation, where point p_i are the critical points of map X .*

1.3.4. *Quasi-polynomiality.* There is one property that is crucial for our proof scheme of the q -Zvonkine conjecture: the so-called *quasi-polynomiality*. For q -orbifold r -spin Hurwitz numbers this quasi-polynomiality is given in theorem 1.16, proved in [KLPS19]. Using [SSZ15, lemma 4.6], theorem 1.16 is equivalent to the following statement:

Proposition 1.27. *For $2g - 2 + n > 0$ the free energies of equation (1.12) are expansions of finite linear combinations of functions of the shape*

$$(1.33) \quad \prod_{j=1}^n \left(\frac{d}{dX} \right)^{a_j} \xi^{i_j}(z_j)$$

with the ξ^i defined by equation (1.25) for the spectral curve data given by equation (1.26).

Remark 1.28. Under the change $X \rightarrow x$, we get

$$(1.34) \quad H_{g,n}^{q,r}(x_1, \dots, x_n) = \sum_{\mu_1, \dots, \mu_n=1}^{\infty} h_{g; \vec{\mu}}^{\circ, q, r} \prod_{i=1}^n x_i^{\mu_i}.$$

We will often omit the superscripts q and r .

For more background on the importance of quasi-polynomiality, we refer the interested reader to [Lew18].

Relating this proposition to equations (1.24) and (1.28), we see that the free energies have the ‘right shape’ to satisfy topological recursion. In particular, proposition 1.27 implies the free energies can be interpreted as functions defined globally on the curve (1.26) rather than formal power series. We will use this viewpoint from now on.

The operator of the derivative $\frac{d}{dX} = x \frac{d}{dx}$ is denoted by D_x .

Note that the functions $d(D_x)^a \xi^i$, $i = 1, \dots, r$, $a = 0, 1, 2, \dots$, satisfy the projection property, that is, $Pd(D_x)^a \xi^i = d(D_x)^a \xi^i$, and the linear loop equation, that is, $d(D_x)^a \xi^i(z) + d(D_x)^a \xi^i(\sigma_j(z))$ is holomorphic for $z \rightarrow p_j$, for $j = 1, \dots, qr$. As a direct consequence of these two facts and proposition 1.27, we have:

Proposition 1.29. *The system of meromorphic differentials $d_1 \otimes \dots \otimes d_n H_{g,n}$ satisfies the projection property and the linear loop equations.*

Remark 1.30. Note that proposition 1.27 also implies that for $2g - 2 + n > 0$ the n -point functions $H_{g,n}$ themselves, once one puts them onto the spectral curve, satisfy a property similar to the linear loop equations. Namely, the sum $H_{g,n}(z, z_{\{2, \dots, n\}}) + H_{g,n}(\sigma_i(z), z_{\{2, \dots, n\}})$ is holomorphic at the i -th ramification point. This also follows from the fact that $(D_x)^a \xi^i(z) + (D_x)^a \xi^i(\sigma_j(z))$ is holomorphic for $z \rightarrow p_j$, $j = 1, \dots, qr$.

Thus theorem 1.22 is a corollary of proposition 1.26 and the following statement, whose proof is the technical core of this paper:

Theorem 1.31. *The system of meromorphic differentials $d_1 \otimes \dots \otimes d_n H_{g,n}$ on the curve (1.26) satisfies the quadratic loop equations.*

The rest of this paper is a proof of this theorem (reformulated as theorem 3.6 below), which is derived from the analysis of implications of the quadratic loop equations and their comparison with the so-called *cut-and-join equation* for the r -spin Hurwitz numbers. The cut-and-join equation for the r -spin Hurwitz numbers was proved in [SSZ12], see also [Roso8, Ale11], and converted in the form that we use in this paper in [BKL⁺21].

1.4. Further remarks. Though we tried to make this paper as self-contained as possible, the full proof of Zvonkine's conjecture from scratch includes several big steps performed in [SSZ12], [SSZ15], [LPSZ17], [KLPS19], and [BKL⁺21], and they are absolutely necessary for our proof. In particular, some familiarity with [BKL⁺21] may be very helpful to follow the technical steps of the proof below.

Our proof is definitely not of the kind that closes the whole area of research. For instance, neither the geometric interpretation of spin Hurwitz numbers in terms of relative Gromov-Witten invariants of $\mathbb{C}P^1$ (recalled in theorem 1.17 above), nor the geometric definition of the Chiodo classes and/or geometry of the moduli space of r -th roots (see section 1.2.1 above) played any role in the argument. We hope that a geometric proof of Zvonkine's conjecture (in the form of corollary 1.20) will be found (for instance, some ideas are discussed in a recent preprint [Lei18]).

Finally, we would like to mention that a quite general framework for topological recursion for Hurwitz numbers was recently proposed by Alexandrov, Chapuy, Eynard, and Harnad in [ACEH18]. The spectral curve data (1.26) is a special case of their proposal, while the r -spin Hurwitz numbers seem not to fit into their formalism (cf. the discussion of quantum curves in [ALS16]). It does not lead to any immediate contradiction, since the proof in [ACEH20] does not cover the cases we are interested here, but it would be extremely interesting to unify the point of view of [ACEH18] with the results of the present paper.

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2. THE CUT-AND-JOIN EQUATION AND QUADRATIC LOOP EQUATIONS

In studies of simple Hurwitz numbers the respective well-known cut-and-join equation plays a crucial role. For the r -spin Hurwitz numbers it is also possible to write an analog of the cut-and-join equation, which we now recall. For us this *spin* cut-and-join equation is instrumental, as the main technical result of the present paper is the derivation of the quadratic loop equation from it.

Let $\llbracket n \rrbracket := \{1, \dots, n\}$. The spin cut-and-join equation, [BKL⁺21, equation (17)], is

$$(2.1) \quad \frac{B_{g,n}}{r!} \tilde{H}_{g,n}(x_{\llbracket n \rrbracket}) = \sum_{\substack{m \geq 1, d \geq 0 \\ m+2d=r+1}} \frac{1}{m!} \sum_{l=1}^m \sum_{\substack{\{k\} \sqcup \sqcup_{j=1}^l K_j = \llbracket n \rrbracket \\ \sqcup_{j=1}^l M_j = \llbracket m \rrbracket \\ \forall j M_j \neq \emptyset}} \frac{1}{l!} \sum_{\substack{g_1, \dots, g_\ell \geq 0 \\ g = \sum_j g_j + m - \ell + d}} Q_{d,0,m}^{(k)} \left[\prod_{j=1}^{\ell} \tilde{H}_{g_j, |M_j| + |K_j|}(\xi_{M_j}, x_{K_j}) \right].$$

Here, $B_{g,n} := \frac{1}{r} (2g - 2 + n + \frac{1}{q} \sum_{i=1}^n D_{x_i})$ and

$$(2.2) \quad \sum_{d \geq 0} Q_{d,0,m}^{(k)} t^{2d} = \frac{t}{\zeta(t)} \frac{\zeta(tD_{x_k})}{tD_{x_k}} \circ \prod_{j=1}^m \frac{\zeta(tD_{\xi_j})}{t} \Big|_{\xi_j = x_k},$$

where $\zeta(t) := e^{t/2} - e^{-t/2}$. Furthermore,

$$(2.3) \quad \begin{aligned} \tilde{H}_{0,1} &:= H_{0,1} \\ \tilde{H}_{0,2}(\xi_1, \xi_2) &:= H_{0,2}(\xi_1, \xi_2) \\ \tilde{H}_{0,2}(\xi_1, x_2) &:= H_{0,2}(\xi_1, x_2) + H_{0,2}^{\text{sing}}(\xi_1, x_2) & H_{0,2}^{\text{sing}}(\xi_1, x_2) &:= \log\left(\frac{\xi_1 - x_2}{\xi_1 x_2}\right) \\ \tilde{H}_{g,n} &:= H_{g,n} - \delta_{2g-2+n,r} r! \frac{(2^{1-2g} - 1) B_{2g}}{2g!}, & & 2g - 2 + n > 0. \end{aligned}$$

Remark 2.1. Note that we have abused the notation above, defining $\tilde{H}_{0,2}(\xi_1, \xi_2)$ differently from $\tilde{H}_{0,2}(\xi_1, x_2)$, such that these two objects are different depending on whether they have two ξ -variables or one ξ - and one x -variable as arguments. This is a necessary evil, as otherwise the formulas would become very bulky.

This formula may seem rather daunting, so let us give some examples for small r . First, we calculate

$$(2.4) \quad \begin{aligned} Q_{0;0,m}^{(k)} &= \prod_{j=1}^m D_{\xi_j} \Big|_{\xi_j = x_k} ; \\ Q_{1;0,m}^{(k)} &= \frac{1}{24} \left((D_{x_k}^2 - 1) \circ \left(\prod_{j=1}^m D_{\xi_j} \Big|_{\xi_j = x_k} \right) + \sum_{l=1}^m \prod_{j=1}^m D_{\xi_j}^{1+2\delta_{j,l}} \Big|_{\xi_j = k} \right). \end{aligned}$$

In the 'non-spin' case, $r = 1$, the first sum only includes the summand for $m = 2$, $d = 0$, so the formula reduces to

$$(2.5) \quad B_{g,n} \tilde{H}_{g,n}(x_{\llbracket n \rrbracket}) = \frac{1}{2} \sum_{k=1}^n D_{\xi_1} D_{\xi_2} \left[\tilde{H}_{g-1,n+1}(\xi_1, \xi_2, x_{\llbracket n \rrbracket \setminus \{k\}}) + \sum_{\substack{g_1+g_2=g \\ K_1 \sqcup K_2 = \llbracket n \rrbracket \setminus \{k\}}} \tilde{H}_{g_1,|K_1|+1}(\xi_1, x_{K_1}) \tilde{H}_{g_2,|K_2|+1}(\xi_2, x_{K_2}) \right] \Big|_{\xi_1 = \xi_2 = x_k}.$$

A full derivation of this formula from the standard cut-and-join equation is available in [DBKO⁺15, section 3.3]. This equation should be interpreted as describing the removal of a transposition (completed 2-cycle) from a 2-factorization. Geometrically, this means removing a ramification point with simple ramification (partition $(2, 1^{d-2})$). After removing this, the two sheets which were glued together before either still belong to one connected curve, of genus one less (the linear term on the right-hand side) or now belong to two different curves (the quadratic term). Notice that in this equation the factor $\frac{1}{l!}$ cancels the overcounting coming from the decompositions of $\llbracket n \rrbracket$ and $\llbracket m \rrbracket$, which always give $l!$ identical terms.

In the case $r = 2$, we get either $m = 3$ and $d = 0$ or $m = 1$ and $d = 1$. Hence (cf. [BKL⁺21, equation (23)]),

$$(2.6) \quad \begin{aligned} B_{g,n} \tilde{H}_{g,n}(x_{\llbracket n \rrbracket}) &= \frac{1}{3} \sum_{k=1}^n \left(D_{\xi_1} D_{\xi_2} D_{\xi_3} \tilde{H}_{g-2,n+2}(\xi_1, \xi_2, \xi_3, x_{\llbracket n \rrbracket \setminus \{k\}}) \right) \Big|_{\xi_1 = \xi_2 = \xi_3 = x_k} \\ &+ \sum_{\substack{g_1+g_2=g-1 \\ \{k\} \sqcup K_1 \sqcup K_2 = \llbracket n \rrbracket}} \left(D_{x_k} \tilde{H}_{g_1,|K_1|+1}(x_k, x_{K_1}) \right) \left(D_{\xi_1} D_{\xi_2} \tilde{H}_{g_2,|K_2|+2}(\xi_1, \xi_2, x_{K_2}) \right) \Big|_{\xi_1 = \xi_2 = x_k} \\ &+ \frac{1}{3} \sum_{\substack{g_1+g_2+g_3=g \\ \{k\} \sqcup K_1 \sqcup K_2 \sqcup K_3 = \llbracket n \rrbracket}} \prod_{j=1}^3 D_{x_k} \tilde{H}_{g_j,|K_j|+1}(x_k, x_{K_j}) \\ &+ \frac{1}{12} (2D_{x_k}^3 - D_{x_k}) \tilde{H}_{g-1,n}(x_{\llbracket n \rrbracket}). \end{aligned}$$

As in the case before, the terms are related to removing a cycle from the 3-factorization, and considering the number of connected components of the resulting curve. A detailed exposition of the resulting

combinatorics is available in [SSZ12, section 5.2]. Because the completed 3-cycle is not equal to the non-completed 3-cycle, we get terms for each of the possible cycles to be removed, with extra coefficients. This is also what occurs for general r .

Our next goal is to express equation (2.1) in terms of z variables (global rational coordinates on the curve) applying the change of variables $x_i = x(z_i)$ and $\xi_i = x(w_i)$. This change of variables extends equation (2.1) originally written for formal power series to an equation for globally defined functions. With this extension in hand, we choose one of the critical points of $x(z)$, say p_i , and we take the sum of this extended equation and its local conjugate in z_1 . In other words, we symmetrize this equation near any of the ramification points of x . For notational simplicity, let us actually take the $(g, n+1)$ case of this equation, with added variable $x_0 = x(z_0)$, in which we symmetrize, and let us write $\bar{w} = \sigma_i(w)$. Let us also apply the operator $D_{x_1} \cdots D_{x_n}$ (converted to the variables z_1, \dots, z_n) to both sides of the equation.

Definition 2.2. The end result of these transformations is called the *symmetrized cut-and-join equation*.

The left hand side of the symmetrized cut-and-join equation is holomorphic by the linear loop equations and remark 1.30, and the right hand side becomes (up to terms, again holomorphic due to the linear loop equations and remark 1.30) equal to

$$(2.7) \quad \sum_{\substack{m \geq 1, d \geq 0 \\ m+2d=r+1}} \frac{1}{m!} \sum_{l=1}^m \sum_{\substack{\sqcup_{j=1}^l K_j = \llbracket n \rrbracket \\ \sqcup_{j=1}^l M_j = \llbracket m \rrbracket \\ \forall j M_j \neq \emptyset}} \frac{1}{l!} \sum_{\substack{g_1, \dots, g_\ell \geq 0 \\ g = \sum_j g_j + m - \ell + d}} \bar{Q}_{d,m}(z_0) \left[\prod_{j=1}^{\ell} \tilde{W}_{g_j, |M_j| + |K_j|}(w_{M_j}, z_{K_j}) \right] + \\ \sum_{\substack{m \geq 1, d \geq 0 \\ m+2d=r+1}} \frac{1}{m!} \sum_{l=1}^m \sum_{\substack{\sqcup_{j=1}^l K_j = \llbracket n \rrbracket \\ \sqcup_{j=1}^l M_j = \llbracket m \rrbracket \\ \forall j M_j \neq \emptyset}} \frac{1}{l!} \sum_{\substack{g_1, \dots, g_\ell \geq 0 \\ g = \sum_j g_j + m - \ell + d}} \bar{Q}_{d,m}(z_0) \left[\prod_{j=1}^{\ell} \tilde{W}_{g_j, |M_j| + |K_j|}(\bar{w}_{M_j}, z_{K_j}) \right].$$

Here, we use the notation

$$(2.8) \quad \tilde{W}_{g,m+n}(w_{\llbracket m \rrbracket}, z_{\llbracket n \rrbracket}) := \prod_{j=1}^m D_{\xi(w_j)} \prod_{i=1}^n D_{x(z_i)} \tilde{H}_{g,m+n}(\xi(w_{\llbracket m \rrbracket}), x(z_{\llbracket n \rrbracket})),$$

$$(2.9) \quad \sum_{d \geq 0} \bar{Q}_{d,m}(z_0) t^{2d} := \frac{t}{\zeta(t)} \frac{\zeta(t D_{x(z_0)})}{t D_{x(z_0)}} \circ \prod_{j=1}^m \left(\int_{w_j=z_0} \circ \frac{\zeta(t D_{x(w_j)})}{t D_{x(w_j)}} \right),$$

where

$$(2.10) \quad \int_{w=z} F(w) := \operatorname{Res}_{w=z} F(w) \frac{dX(w)}{X(w) - X(z)}.$$

We treat (2.10) as the definition of the operator \int of “setting two variables equal”. Although it was not stated explicitly in [BKL⁺21], this operator should be accurately defined by this residue formula, as it is the analytic continuation of the corresponding operator in coordinates x in the cut-and-join equation.

Note that it might be more natural to define the operator $\int_{w=z}$ as $\operatorname{Res}_{w=z} F(w) \frac{dx(w)}{x(w) - x(z)}$. The difference between this operator and (2.10) is not important when we apply it to a function that has no pole on the diagonal (which is the case in all statements in the rest of the paper), but in particular computations (2.10) appears to be more convenient, cf. the proof of proposition 3.3.

In order to simplify this a bit more, define the m -disconnected, n -connected correlators $\tilde{\mathcal{W}}_{g,m,n}(w_{\llbracket m \rrbracket} \mid z_{\llbracket n \rrbracket})$ (cf. [BE13]) by keeping only those terms in the inclusion-exclusion formula where each factor contains at least one w :

$$(2.11) \quad \tilde{\mathcal{W}}_{g,m,n}(w_{\llbracket m \rrbracket} \mid z_{\llbracket n \rrbracket}) := \sum_{l=1}^m \sum_{\substack{\sqcup_{j=1}^l K_j = \llbracket n \rrbracket \\ \sqcup_{j=1}^l M_j = \llbracket m \rrbracket \\ \forall j M_j \neq \emptyset}} \frac{1}{l!} \sum_{\substack{g_1, \dots, g_\ell \geq 0 \\ g = \sum_j g_j + m - \ell}} \prod_{j=1}^{\ell} \tilde{W}_{g_j, |K_j| + |M_j|}(z_{K_j}, w_{M_j}).$$

(The factor $\frac{1}{n!}$ is just a symmetry factor.) This is defined in such a way that $\tilde{\mathcal{W}}_{g,1,n}(z \mid z_{\llbracket n \rrbracket}) = \tilde{W}_{g,n+1}(z, z_{\llbracket n \rrbracket})$ and $\tilde{\mathcal{W}}_{g,n,0}(z_{\llbracket n \rrbracket} \mid \emptyset)$ is the disconnected correlator. The genus g here stands for the genus of all terms after all m w_j -points are glued to an $(m+1)$ -pointed sphere. Then we get for the right-hand side of the symmetrized cut-and-join equation

$$(2.12) \quad \sum_{\substack{m \geq 1, d \geq 0 \\ m+2d=r+1}} \frac{1}{m!} \bar{Q}_{d,m}(z_0) \left(\tilde{\mathcal{W}}_{g-d,m,n}(w_{\llbracket m \rrbracket} \mid z_{\llbracket n \rrbracket}) + \tilde{\mathcal{W}}_{g-d,m,n}(\bar{w}_{\llbracket m \rrbracket} \mid z_{\llbracket n \rrbracket}) \right).$$

To give an example, let us specify this expression in the cases of $r = 1$ and $r = 2$ discussed above. In the case $r = 1$ we necessarily have $m = 2, d = 0$. Note that $\bar{Q}_{0,2}(z_0) = \int_{w_1=z_0} \int_{w_2=z_0}$. Thus the symmetrized cut-and-join equation for $r = 1$ implies that the following expression is holomorphic in z_0 near the chosen ramification point of x :

$$(2.13) \quad \int_{w_1=z_0} \int_{w_2=z_0} \left(\tilde{\mathcal{W}}_{g,2,n}(w_1, w_2 \mid z_{\llbracket n \rrbracket}) + \tilde{\mathcal{W}}_{g,2,n}(\bar{w}_1, \bar{w}_2 \mid z_{\llbracket n \rrbracket}) \right).$$

For $r = 2$ we either have $m = 3, d = 0$, and $\bar{Q}_{0,3}(z_0) = \int_{w_1=z_0} \int_{w_2=z_0} \int_{w_3=z_0}$, or $m = 1, d = 1$, and

$$(2.14) \quad \bar{Q}_{1,1}(z_0) = \frac{1}{24} (D_{x(z_0)}^2 - 1) \int_{w_1=z_0} + \int_{w_1=z_0} \circ \frac{1}{24} D_{x(w_1)}^2.$$

Thus the symmetrized cut-and-join equation for $r = 2$ implies that the following expression is holomorphic in z_0 near the chosen ramification point of x :

$$(2.15) \quad \int_{w_1=z_0} \int_{w_2=z_0} \int_{w_3=z_0} \left(\tilde{\mathcal{W}}_{g,2,n}(w_1, w_2, w_3 \mid z_{\llbracket n \rrbracket}) + \tilde{\mathcal{W}}_{g,2,n}(\bar{w}_1, \bar{w}_2, \bar{w}_3 \mid z_{\llbracket n \rrbracket}) \right) \\ + \left(\frac{1}{24} (D_{x(z_0)}^2 - 1) \int_{w_1=z_0} + \int_{w_1=z_0} \circ \frac{1}{24} D_{x(w_1)}^2 \right) \left(\tilde{\mathcal{W}}_{g-1,1,n}(w_1 \mid z_{\llbracket n \rrbracket}) + \tilde{\mathcal{W}}_{g-1,1,n}(\bar{w}_1 \mid z_{\llbracket n \rrbracket}) \right).$$

3. PROOF OF THE QUADRATIC LOOP EQUATIONS VIA THE SYMMETRIZED CUT-AND-JOIN EQUATION

For the rest of the paper, we fix a ramification point p of x and let $z \mapsto \bar{z}$ be the local deck transformation.

Definition 3.1. Define the *symmetrizing operator* \mathcal{S}_z and the *anti-symmetrizing operator* Δ_z by

$$(3.1) \quad \begin{aligned} \mathcal{S}_z f(z) &:= f(z) + f(\bar{z}); \\ \Delta_z f(z) &:= f(z) - f(\bar{z}). \end{aligned}$$

We use the identity

$$(3.2) \quad \mathcal{S}_z f(\underbrace{z, \dots, z}_{r \text{ times}}) = 2^{1-r} \sum_{\substack{I \sqcup J = \llbracket r \rrbracket \\ |J| \text{ even}}} \left(\prod_{i \in I} \mathcal{S}_{z_i} \right) \left(\prod_{j \in J} \Delta_{z_j} \right) f(z_1, \dots, z_r) \Big|_{z_i=z},$$

which was also used in [BKL⁺21].

3.1. Symmetrization and anti-symmetrization of the regularized $W_{0,2}$. The main difficulty of the proof comes from the diagonal poles of $\tilde{W}_{0,2}$, so it is useful to give explicit formulae for it. In the global coordinate z we have [KLPS19, theorem 5.2]:

$$(3.3) \quad \begin{aligned} \tilde{W}_{0,2}(z, w) &= \frac{1}{X'(z) X'(w) (z-w)^2}; \\ \tilde{W}_{0,2}(w_1, w_2) &= \frac{1}{X'(w_1) X'(w_2) (w_1-w_2)^2} - \frac{x(w_1) x(w_2)}{(x(w_1) - x(w_2))^2}. \end{aligned}$$

Recall that in the cut-and-join equation, we need to use different formulas for $\tilde{W}_{0,2}$ if it has one w and one z as arguments (then it is the usual $W_{0,2}$) and if it has two w 's as arguments (in this case we use the regularized $W_{0,2}$). The latter is the one that can cause problems with diagonal poles. Hence, we should consider the action of \mathcal{S} and Δ on $\tilde{W}_{0,2}(w_1, w_2)$, to simplify many of the terms. As our spectral curve

(1.26) only has simple ramifications (ramification points are solutions of $dX = 0$), we can work in the local coordinate z defined by $X - X(p) = z^2/2$, so the involution is $\bar{z} = -z$.

$$(3.4) \quad \begin{aligned} \tilde{W}_{0,2}(w_1, w_2) &= \frac{1}{w_1 w_2 (w_1 + w_2)^2} + \text{holom}; \\ \mathcal{S}_{w_1} \mathcal{S}_{w_2} \tilde{W}_{0,2}(w_1, w_2) &= \frac{2}{w_1 w_2 (w_1 + w_2)^2} - \frac{2}{w_1 w_2 (w_1 - w_2)^2} + \text{holom} = -\frac{2}{(X(w_1) - X(w_2))^2} + \text{holom}; \\ \mathcal{S}_{w_1} \Delta_{w_2} \tilde{W}_{0,2}(w_1, w_2) &= \text{holom}; \\ \Delta_{w_1} \Delta_{w_2} \tilde{W}_{0,2}(w_1, w_2) &= \frac{2}{w_1 w_2 (w_1 + w_2)^2} + \frac{2}{w_1 w_2 (w_1 - w_2)^2} + \text{holom}. \end{aligned}$$

From this, it follows that any combination containing $\mathcal{S}_{w_1} \Delta_{w_2} \tilde{W}_{0,2}(w_1, w_2)$ is holomorphic. Note also that a simple residue argument implies that once $\Delta_{w_1} \Delta_{w_2} \tilde{W}_{0,2}(w_1, w_2)$ is used in an expression holomorphic in w_1 and w_2 near $w_1 = w_2 = 0$ and symmetric under the involution in both variables, the application of the operator to the whole expression $\int_{w_1=w_2}$ retains holomorphicity despite its poles on the diagonal $w_1 - w_2 = 0$ and on the antidiagonal $w_1 + w_2 = 0$.

In fact, in order to simplify the calculation a bit, we will redefine

$$(3.5) \quad \widetilde{\mathcal{S}_{w_1} \mathcal{S}_{w_2} \tilde{W}_{0,2}}(w_1, w_2) := \mathcal{S}_{w_1} \mathcal{S}_{w_2} \tilde{W}_{0,2}(w_1, w_2) + \frac{2}{(X(w_1) - X(w_2))^2};$$

$$(3.6) \quad \widetilde{\Delta_{w_1} \Delta_{w_2} \tilde{W}_{0,2}}(w_1, w_2) := \Delta_{w_1} \Delta_{w_2} \tilde{W}_{0,2}(w_1, w_2) - \frac{2}{(X(w_1) - X(w_2))^2},$$

i.e., during analysis of the RHS of (2.12), after we have written the expression in terms of \mathcal{S} and Δ symbolically, we do the said redefinition. It is clear that it does not change the expression — it just regroups some terms.

Then the $\widetilde{\mathcal{S}_{w_1} \mathcal{S}_{w_2} \tilde{W}_{0,2}}(w_1, w_2)$ is holomorphic, and we need only concern ourselves with $\tilde{W}_{0,2}(w_1, w_2)$ with two Δ 's acting on them. From now on, we will use these modified definitions of $\mathcal{S}\mathcal{S}$ and $\Delta\Delta$, and omit the tildes from notation.

3.2. Formal corollaries of the quadratic loop equations. From (1.32), the (g, n) quadratic loop equation states that

$$(3.7) \quad \int_{w=w_0} \tilde{\mathcal{W}}_{g,2,n}(w_0, \bar{w} \mid z_{\llbracket n \rrbracket}) \text{ is holomorphic in } w_0 \text{ near ramification points.}$$

Let us call $2g - 2 + n =: -\chi$ the *negative Euler characteristic* of a given quadratic loop equation.

Note that due to the symmetry of $\tilde{\mathcal{W}}_{g,2,n}$ in its first two arguments, the expression above can be rewritten as follows:

$$(3.8) \quad \int_{w=w_0} \tilde{\mathcal{W}}_{g,2,n}(w_0, \bar{w} \mid z_{\llbracket n \rrbracket}) = -\frac{1}{4} \int_{w=w_0} (\Delta_{w_0} \Delta_w - \mathcal{S}_{w_0} \mathcal{S}_w) \tilde{\mathcal{W}}_{g,2,n}(w_0, w \mid z_{\llbracket n \rrbracket}).$$

Note that

$$(3.9) \quad \int_{w=w_0} \mathcal{S}_{w_0} \mathcal{S}_w \tilde{\mathcal{W}}_{g,2,n}(w_0, w \mid z_{\llbracket n \rrbracket})$$

is holomorphic due to the linear loop equation, see (1.31), and thus the quadratic loop equation can be reformulated as the statement that

$$(3.10) \quad \int_{w=w_0} \Delta_{w_0} \Delta_w \tilde{\mathcal{W}}_{g,2,n}(w_0, w \mid z_{\llbracket n \rrbracket}) \text{ is holomorphic in } w_0 \text{ near ramification points.}$$

Now let us extend the quadratic loop equation onto $\tilde{\mathcal{W}}_{g,m,n}$ for $m > 2$. Namely, we have the following:

Proposition 3.2. *Suppose that a set of functions $(\tilde{\mathcal{W}}_{g,n})_{g,n}$ satisfies the quadratic loop equations up to negative Euler characteristic $-\chi$. Then, we get for any $s, g, n \geq 0$ such that $2g - 2 + n \leq -\chi$, that*

$$(3.11) \quad \int_{w=w_0} \Delta_{w_0} \Delta_w \tilde{\mathcal{W}}_{g,2+s,n}(w_0, w, w_{\llbracket s \rrbracket} \mid z_{\llbracket n \rrbracket}) \text{ is holomorphic in } w_0 \text{ near ramification points.}$$

Proof. With the help of the definition (2.11), it is easy to see that

$$(3.12) \quad \begin{aligned} \tilde{\mathcal{W}}_{g,2+s,n}(w_0, w, w_{\llbracket s \rrbracket} \mid z_{\llbracket n \rrbracket}) &= \tilde{\mathcal{W}}_{g-s,2,n+s}(w_0, w \mid w_{\llbracket s \rrbracket}, z_{\llbracket n \rrbracket}) \\ &+ \sum_{\substack{K_1 \sqcup K_2 = \llbracket n \rrbracket \\ M_1 \sqcup M_2 = \llbracket s \rrbracket \\ M_2 \neq \emptyset}} \sum_{\substack{g_1, g_2 \geq 0 \\ g_1 + g_2 = g - |M_1|}} \tilde{\mathcal{W}}_{g_1, 2, |M_1| + |K_1|}(w_0, w \mid w_{M_1}, z_{K_1}) \tilde{\mathcal{W}}_{g_2, |M_2|, |K_2|}(w_{M_2} \mid z_{K_2}). \end{aligned}$$

Note that after one applies $\big|_{w=w_0} \Delta_{w_0} \Delta_w$ to (3.12), the first term in the RHS, as well as the first factors in the terms in the sum in the second line of the equation, are holomorphic in w_0 , due to our assumption that quadratic loop equations are satisfied up to negative Euler characteristic $-\chi$. And the second factors in the terms in the sum in the second line are constant in w_0 . Thus, the whole expression is holomorphic in w_0 near ramification points. \square

Now we are ready to prove the following proposition, which is the main technical result of the present paper:

Proposition 3.3. *Suppose that a set of functions $(\tilde{\mathcal{W}}_{g,n})_{g,n}$ satisfies the quadratic loop equations up to negative Euler characteristic $-\chi$. Then, we get for any $N, g, n \geq 0$ such that $2g - 2 + n \leq -\chi$, that*

$$(3.13) \quad \sum_{k=0}^N \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \prod_{j=1}^{2k} \left(\big|_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right) \Delta_1 \cdots \Delta_{2k} \tilde{\mathcal{W}}_{g-\alpha_1-\dots-\alpha_{2k}, 2k, n}(w_1, \dots, w_{2k} \mid z_{\llbracket n \rrbracket}),$$

where $D_j := \frac{d}{dX(w_j)}$, is holomorphic in z near branch points of the spectral curve.

Proof. We use induction on N and g . First note that $k = 0$ can only occur if $N = 0$, and in this case, the statement is trivial, as the expression is constant in z .

For $N = 1$, the statement is just the quadratic loop equation, which holds by assumption, and furthermore, for $g = -1$ it is clearly zero.

Let us define, for $s \geq 0$,

$$(3.14) \quad \text{Hol}_{g,N,n,s}(z, \tilde{w}_{\llbracket s \rrbracket}) := \sum_{k=0}^N \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \prod_{j=1}^{2k} \left(\big|_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right) \Delta_1 \cdots \Delta_{2k} \tilde{\mathcal{W}}_{g-\alpha_1-\dots-\alpha_{2k}, 2k+s, n}(w_{\llbracket 2k \rrbracket}, \tilde{w}_{\llbracket s \rrbracket} \mid z_{\llbracket n \rrbracket}).$$

(we omit the dependence on $z_{\llbracket n \rrbracket}$ in the LHS for brevity).

Now let us fix some N_0 and g_0 and suppose that the statement of the proposition, which can now be rephrased as

$$(3.15) \quad \text{Hol}_{g,N,n,0}(z) \text{ is holomorphic in } z \text{ near ramification points,}$$

holds for all (g, N, n) such that

$$(3.16) \quad \begin{aligned} g &\leq g_0 + 1, \\ N &\leq N_0, \\ n &\leq -\chi + 2 - 2g. \end{aligned}$$

If we, under these assumptions, manage to prove the statement for $N = N_0 + 1$, $g = g_0 + 1$ (and for all $n \leq -\chi - 2g_0$), we will, by induction, achieve our goal (since, as explained above, the statement holds at the boundaries $N = 1$ and $g = -1$).

Note that under these assumptions we have the following statement:

$$(3.17) \quad \text{Hol}_{g,N,n,s}(z, \tilde{w}_{\llbracket s \rrbracket}) \text{ is holomorphic in } z \text{ near ramification points,}$$

for the same (g, N, n) as in (3.16) and all $s \geq 0$. The proof of this statement is completely analogous to the proof of proposition 3.2.

For brevity from now on we write (g, N) in place of (g_0, N_0) . We will express $\text{Hol}_{g+1, N+1, n, 0}(z)$ in terms of previous cases. First of all, we take

$$\begin{aligned}
 & D_{x(z)}^2 \text{Hol}_{g, N, n, 0}(z) \\
 &= \sum_{k=1}^N \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \sum_{i=1}^{2k} \prod_{j=1}^{2k} \left(\left| \int_{w_j=z} \frac{D_j^{2\alpha_j + 2\delta_{ij}}}{(2\alpha_j + 1)!} \right| \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}) \right) \\
 &+ \sum_{k=0}^N \frac{1}{(2k)!} \binom{2k}{2} \sum_{\substack{\alpha_1 + \dots + \alpha_{2k-2} \\ + \beta_1 + \beta_2 + k = N}} \prod_{j=1}^{2k-2} \left(\left| \int_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \right) \\
 &\quad \times \left| \int_{w_{2k-1}=z} \frac{D_{2k-1}^{2\beta_1+1}}{(2\beta_1 + 1)!} \right| \left| \int_{w_{2k}=z} \frac{D_{2k}^{2\beta_2+1}}{(2\beta_2 + 1)!} \right| \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}) \\
 &= \sum_{k=1}^N \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \sum_{i=1}^{2k} \prod_{j=1}^{2k} \left(\left| \int_{w_j=z} \frac{D_j^{2\alpha_j + 2\delta_{ij}}}{(2\alpha_j + 1)!} \right| \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}) \right) \\
 &+ \sum_{\substack{N_\alpha + N_\beta = N \\ N_\alpha \geq 0, N_\beta \geq 1}} \sum_{\beta_1 + \beta_2 + 1 = N_\beta} \left| \int_{w_{1'}=z} \frac{D_{1'}^{2\beta_1+1}}{(2\beta_1 + 1)!} \right| \left| \int_{w_{2'}=z} \frac{D_{2'}^{2\beta_2+1}}{(2\beta_2 + 1)!} \right| \sum_{k=0}^{N_\alpha} \frac{1}{(2k)!} \\
 &\quad \times \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N_\alpha} \prod_{j=1}^{2k} \left(\left| \int_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \right) \Delta_{w_{1'}} \Delta_{w_{2'}} \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k+1, 2k+2, n}(w_{1'}, w_{2'}, w_{\llbracket 2k \rrbracket}),
 \end{aligned} \tag{3.18}$$

where we have omitted the $z_{\llbracket n \rrbracket}$ arguments of $\tilde{\mathcal{W}}$ for brevity (and we will keep omitting them for the rest of this proof). D_j here and from now on stands for $D_{x(w_j)}$.

This whole expression is holomorphic in z , being the result of the application of $\frac{d^2}{dX^2(z)}$ to an expression holomorphic in z . We also see that the terms in the third-to-last line in this equation are already of the form which we see in $\text{Hol}_{g+1, N+1, n, 0}$. However, the terms corresponding to the second-to-last and the last lines contain odd derivatives in the second term, which are certainly absent from $\text{Hol}_{g+1, N+1, n, 0}$. To counteract these odd-derivative terms, we would like to subtract

$$\begin{aligned}
 & \sum_{\substack{N_\alpha + N_\beta = N \\ N_\alpha, N_\beta \geq 0}} \left| \int_{\tilde{z}=z} \frac{D_{x(\tilde{z})}^{2N_\beta}}{(2N_\beta)!} \right| \left| \int_{w_{1'}, w_{2'}=\tilde{z}} \Delta_{w_{1'}} \Delta_{w_{2'}} \text{Hol}_{g+1-N_\beta, N_\alpha, n, 2}(z, w_{1'}, w_{2'}) \right) \\
 &= \sum_{\substack{N_\alpha + N_\beta = N \\ N_\alpha, N_\beta \geq 0}} \left| \int_{\tilde{z}=z} \frac{D_{x(\tilde{z})}^{2N_\beta}}{(2N_\beta)!} \right| \left| \int_{w_{1'}, w_{2'}=\tilde{z}} \sum_{k=0}^{N_\alpha} \frac{1}{(2k)!} \right) \\
 &\quad \times \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N_\alpha} \prod_{j=1}^{2k} \left(\left| \int_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \right) \Delta_{w_{1'}} \Delta_{w_{2'}} \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g+1-N+k, 2k+2, n}(w_{1'}, w_{2'}, w_{\llbracket 2k \rrbracket}).
 \end{aligned} \tag{3.19}$$

Note that we include the $N_\beta = 0$ terms.

Proposition 3.2 and statement (3.17), under our induction assumption, imply that each expression

$$\left| \int_{w_{1'}, w_{2'}=\tilde{z}} \frac{D_{x(\tilde{z})}^{2N_\beta}}{(2N_\beta)!} \right| \Delta_{w_{1'}} \Delta_{w_{2'}} \text{Hol}_{g+1-N_\beta, N_\alpha, n, 2}(z, w_{1'}, w_{2'}) \tag{3.20}$$

is holomorphic in both z and \tilde{z} separately (at the ramification points), *once we do not apply the convention (3.6) to $\Delta_{w_{i'}} \Delta_{w_j} \tilde{\mathcal{W}}_{0,2}(w_{i'}, w_j)$, $i' = 1', 2'$, $j = 1, \dots, 2k$* . Note that in order to claim this, as per the conditions of proposition 3.2 and statement (3.17), we have to restrict n . Namely, for the holomorphicity in z we need the condition $n \leq -\chi + 2 - 2(g + 1 - N_\beta)$ to hold for all $0 \leq N_\beta \leq N$, and for the holomorphicity in \tilde{z} we need the condition $n \leq -\chi + 2 - 2(g + 1 - N + k)$ to hold for all $0 \leq k \leq N$. Both of these conditions are equivalent to $n \leq -\chi - 2g$, which is precisely what we want for our induction step.

Remarkably, after the application of $\int_{\tilde{z}=z}$ expression (3.20) remains holomorphic in z . In order to see this, let us prove that

$$(3.21) \quad \begin{aligned} & \operatorname{Res}_{\tilde{z} \rightarrow z} \frac{dX(\tilde{z})}{X(\tilde{z}) - X(z)} \frac{D_x^{2N\beta}}{(2N\beta)!} \Big|_{w_{1'}, w_{2'} = \tilde{z}} \Delta_{w_{1'}} \Delta_{w_{2'}} \operatorname{Hol}_{g+1-N\beta, N\alpha, n, 2}(z, w_{1'}, w_{2'}) \\ &= \frac{1}{2} \int_{|\tilde{z}|=\epsilon} \frac{dX(\tilde{z})}{X(\tilde{z}) - X(z)} \frac{D_x^{2N\beta}}{(2N\beta)!} \Big|_{w_{1'}, w_{2'} = \tilde{z}} \Delta_{w_{1'}} \Delta_{w_{2'}} \operatorname{Hol}_{g+1-N\beta, N\alpha, n, 2}(z, w_{1'}, w_{2'}), \end{aligned}$$

for $|z| < \epsilon$, where we assume that ϵ is a fixed number. Note two properties of the expression under the sign of the integral on the right hand side of equation (3.21):

- (1) its only poles in \tilde{z} are at $\tilde{z} = z$ and $\tilde{z} = -z$, and the residues at these two poles are equal to each other by the symmetry of this expression under the sign change;
- (2) it is holomorphic in z for $|\tilde{z}| = \epsilon$ and $|z| < \epsilon$.

The first property implies that equation (3.21) holds, the second property implies that the whole expression is holomorphic in z .

However, we want to use expression (3.19) assuming the convention (3.6) for the possible factors $\Delta_{w_{i'}} \Delta_{w_j} \tilde{W}_{0,2}(w_{i'}, w_j)$, $i' = 1', 2', j = 1, \dots, 2k$, for each k . In this way, it is not holomorphic, but by the previous paragraph it becomes holomorphic if we add the following terms:

$$(3.22) \quad \begin{aligned} & \sum_{j=1}^{2k} \frac{1}{(2N\beta)!} \Big|_{w_{1'}=w_j} D_{1'}^{2N\beta} \frac{D_j^{2\alpha_j}}{(2\alpha_j+1)!} \frac{2}{(X(w_{1'}) - X(w_j))^2} \Big|_{w_{2'}=w_{1'}} \Delta_{w_{2'}} \prod_{i \neq j} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{2'}, w_{\llbracket 2k \rrbracket \setminus \{j\}}) \\ &+ \sum_{j=1}^{2k} \frac{1}{(2N\beta)!} \Big|_{w_{2'}=w_j} D_{1'}^{2N\beta} \frac{D_j^{2\alpha_j}}{(2\alpha_j+1)!} \frac{2}{(X(w_{2'}) - X(w_j))^2} \Big|_{w_{1'}=w_{2'}} \Delta_{w_{2'}} \prod_{i \neq j} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{1'}, w_{\llbracket 2k \rrbracket \setminus \{j\}}) \\ &+ \sum_{l \neq j} \frac{1}{(2N\beta)!} \Big|_{w_{1'}=w_j} D_{1'}^{2N\beta} \Big|_{w_l=w_j} \frac{D_l^{2\alpha_l}}{(2\alpha_l+1)!} \frac{2}{(X(w_{1'}) - X(w_l))^2} \times \\ & \quad \frac{D_j^{2\alpha_j}}{(2\alpha_j+1)!} \frac{2}{(X(w_{1'}) - X(w_j))^2} \prod_{i \neq l, j} \Delta_{w_i} \tilde{W}_{g-N+k-1, 2k-2, n}(w_{\llbracket 2k \rrbracket \setminus \{l, j\}}) \end{aligned}$$

(we write these terms omitting the D -operators acting on the w 's which didn't appear in the $\tilde{W}_{0,2}(w_{i'}, w_j)$ factors and the sum over k). The sum of the first two summands in this expression is equal to

$$(3.23) \quad \begin{aligned} & \frac{2}{(2N\beta)!} \Big|_{w_{1'}=w_j} D_{1'}^{2N\beta} \frac{D_j^{2\alpha_j}}{(2\alpha_j+1)!} \frac{2}{(X(w_{1'}) - X(w_j))^2} \Delta_{w_{1'}} \prod_{i \neq j} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{1'}, w_{\llbracket 2k \rrbracket \setminus \{j\}}) \\ &= \frac{4}{(2N\beta)!(2\alpha_j+1)!} \operatorname{Res}_{w_{1'} \rightarrow w_j} \frac{dX(w_{1'})}{X(w_{1'}) - X(w_j)} \times \\ & \quad D_{1'}^{2N\beta} D_j^{2\alpha_j} \frac{1}{(X(w_{1'}) - X(w_j))^2} \Delta_{w_{1'}} \prod_{i \neq j} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{1'}, w_{\llbracket 2k \rrbracket \setminus \{j\}}) \\ &= \frac{4}{(2N\beta)!} \operatorname{Res}_{w_{1'} \rightarrow w_j} D_{1'}^{2N\beta} \left(\frac{1}{X(w_{1'}) - X(w_j)} \right) \frac{dX(w_{1'})}{(X(w_{1'}) - X(w_j))^{2+2\alpha_j}} \Delta_{w_{1'}} \prod_{i \neq j} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{1'}, w_{\llbracket 2k \rrbracket \setminus \{j\}}) \\ &= 4 \operatorname{Res}_{w_{1'} \rightarrow w_j} \frac{dX(w_{1'})}{(X(w_{1'}) - X(w_j))^{3+2N\beta+2\alpha_j}} \Delta_{w_{1'}} \prod_{i \neq j} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{1'}, w_{\llbracket 2k \rrbracket \setminus \{j\}}) \\ &= 4 \frac{D_j^{2\alpha_j+2N\beta+2}}{(2\alpha_j+2N\beta+2)!} \prod_{i=1}^{2k} \Delta_{w_i} \tilde{W}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}). \end{aligned}$$

By the same computation, the last summand in (3.22) is equal to zero.

Thus, if we add all the terms corresponding to (3.23) to (3.19), we get a holomorphic expression, which is then equal to

$$(3.24) \quad \sum_{\substack{N_\alpha + N_\beta = N \\ N_\alpha, N_\beta \geq 0}} \left(\left| \frac{D_x^{2N_\beta}}{(2N_\beta)!} \right|_{\tilde{w}_1, w_2 = \tilde{z}} \sum_{k=0}^{N_\alpha} \frac{1}{(2k)!} \right. \\ \times \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N_\alpha} \prod_{j=1}^{2k} \left(\left| \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \Delta_{w_1} \cdots \Delta_{w_{2k}} \tilde{\mathcal{W}}_{g+1-N+k, 2k+2, n}(w_1, w_2, w_{\llbracket 2k \rrbracket}) \right. \\ \left. \left. + 4 \sum_{k=1}^{N_\alpha} \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N_\alpha} \sum_{i=1}^{2k} \left| \frac{D_i^{2\alpha_i + 2N_\beta + 2}}{(2\alpha_i + 2N_\beta + 2)!} \right| \prod_{\substack{j=1 \\ j \neq i}}^{2k} \left(\left| \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}) \right) \right).$$

Subtracting expression (3.18) (which itself is holomorphic) from this, we get (note that the index N_β has been shifted here)

$$(3.25) \quad \sum_{\substack{N_\alpha + N_\beta = N+1 \\ N_\beta \geq 1}} \left(\sum_{\beta_1 + \beta_2 + 1 = N_\beta} \left| \frac{D_{1'}^{2\beta_1}}{(2\beta_1)!} \frac{D_{2'}^{2\beta_2}}{(2\beta_2)!} \right| \sum_{k=0}^{N_\alpha} \frac{1}{(2k)!} \sum_{\substack{\alpha_1 + \dots + \alpha_{2k} \\ + k = N_\alpha}} \prod_{j=1}^{2k} \left(\left| \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \right. \\ \left. \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g+1-N+k, 2k+2, n}(w_1, w_2, w_{\llbracket 2k \rrbracket}) \right) \\ + 4 \sum_{\substack{N_\alpha + N_\beta = N+1 \\ N_\beta \geq 1}} \left(\sum_{k=1}^{N_\alpha} \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N_\alpha} \sum_{i=1}^{2k} \left| \frac{D_i^{2\alpha_i + 2N_\beta}}{(2\alpha_i + 2N_\beta)!} \right| \prod_{\substack{j=1 \\ j \neq i}}^{2k} \left(\left| \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \right. \right. \\ \left. \left. \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}) \right) \right) \\ - \sum_{k=1}^N \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \sum_{i=1}^{2k} \prod_{j=1}^{2k} \left(\left| \frac{D_j^{2\alpha_j + 2\delta_{ij}}}{(2\alpha_j + 1)!} \right| \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g-N+k, 2k, n}(w_{\llbracket 2k \rrbracket}) \right),$$

which is holomorphic.

We claim that, up to a factor, this equals $\text{Hol}_{g+1, N+1, n, 0}(z)$. Indeed, let us extract the coefficient of a term

$$(3.26) \quad \frac{1}{(2k)!} \prod_{j=1}^{2k} \left| \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right| \prod_{i=1}^{2k} \Delta_{w_i} \tilde{\mathcal{W}}_{g+1-\sum \alpha_i, 2k, n}(w_{\llbracket 2k \rrbracket}),$$

where $\alpha_1 + \dots + \alpha_{2k} + k = N + 1$ (note that all terms in (3.25) are of this form and satisfy this condition). From the first, second, and third summands of expression (3.25) we get, respectively

$$(3.27) \quad 2 \sum_{1 \leq i < j \leq 2k} (2\alpha_i + 1)(2\alpha_j + 1); \\ 4 \sum_{i=1}^{2k} (2\alpha_i + 1) \cdot \alpha_i; \\ - \sum_{i=1}^{2k} (2\alpha_i)(2\alpha_i + 1);$$

where the α_i on the second line comes from the number of different ways of choosing N_β . Adding up these terms, we get

$$\begin{aligned}
 & 2 \sum_{1 \leq i < j \leq 2k} (2\alpha_i + 1)(2\alpha_j + 1) + 4 \sum_{i=1}^{2k} (2\alpha_i + 1) \cdot \alpha_i - \sum_{i=1}^{2k} (2\alpha_i)(2\alpha_i + 1) \\
 (3.28) \quad & = \sum_{1 \leq i \neq j \leq 2k} (2\alpha_i + 1)(2\alpha_j + 1) + \sum_{i=1}^{2k} 2\alpha_i(2\alpha_i + 1) \\
 & = \left(\sum_{i=1}^{2k} 2\alpha_i + 1 \right)^2 - \sum_{i=1}^{2k} (2\alpha_i + 1) \\
 & = (2N + 2)^2 - (2N + 2) = (2N + 2)(2N + 1).
 \end{aligned}$$

As this factor is independent of k and the α_j , this shows that expression (3.25) is equal to this factor times $\text{Hol}_{g+1, N+1, n, 0}$. Since expression (3.25) is holomorphic, and this whole reasoning works for any $n \leq -\chi - 2g$, this proves the induction step and thus the proposition. \square

Remark 3.4. In the induction step in the proof of proposition 3.3 for $\text{Hol}_{g+1, N+1, n, 0}$ we used $\text{Hol}_{g+1, i, n, 0}$, $i = 1, \dots, N$ for the same $g + 1$ case. It is easy to trace through the proof all instances where these terms occur: they always come from expression 3.19 for $N_\beta = 0$, $N_\alpha = k$. Applying the same induction argument, we obtain the following refinement of the statement of proposition 3.3: *if the quadratic loop equations are satisfied up to the negative Euler characteristic strictly less than $2g - 2 + n$, then for any $N \geq 1$ the following expression*

$$\begin{aligned}
 (3.29) \quad & \sum_{k=0}^N \frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \prod_{j=1}^{2k} \left(\int_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j + 1)!} \right) \Delta_1 \cdots \Delta_{2k} \tilde{\mathcal{W}}_{g-\alpha_1-\dots-\alpha_{2k}, 2k, n}(w_1, \dots, w_{2k} \mid z_{\llbracket n \rrbracket}) \\
 & - \frac{1}{(2N)!} \binom{N}{1} \int_{w_1=z} \int_{w_2=z} \Delta_1 \Delta_2 \tilde{\mathcal{W}}_{g, 2, n}(w_1, w_2 \mid z_{\llbracket n \rrbracket}) (\Delta_z W_{0,1}(z))^{2N-2}
 \end{aligned}$$

is holomorphic.

3.3. Quadratic loop equations from the cut-and-join equation. We prove the quadratic loop equations (in the form (3.10)) by induction from the cut-and-join equation (2.12). We distribute \mathcal{S} 's and Δ 's in cut-and-join equation according to equation (3.2) and express the result in terms of the form (3.13) with added \mathcal{S} 's. Then, we use inductive arguments both on the negative Euler characteristic $2g - 2 + n$ and on the number of Δ 's involved. In fact, we prove that any particular instance of the cut-and-join equation, so for any choice of r, g, n , is a combination of derivatives of linear and quadratic loop equations (for the same r), whose negative Euler characteristic is bounded from above by $2g - 2 + n$, and where the $2g - 2 + n$ quadratic loop equation occurs without derivatives and with a non-trivial coefficient. As the symmetrized cut-and-join equation is holomorphic and all the previous quadratic loop equations hold by induction, just as all linear loop equations, this will then prove the (g, n) quadratic loop equation holds.

By distributing the \mathcal{S} 's and Δ 's, we will always get an even number of Δ 's. Hence, up to diagonal poles, we can always write such a distribution as a product of linear and quadratic loop equations. By the discussion above, there are no possible diagonal poles between two \mathcal{S} 's or between an \mathcal{S} and a Δ , so we should focus our attention on the Δ factors.

Recall, from (2.12), that the symmetrized cut-and-join equation implies that

$$(3.30) \quad \mathcal{S}_{z_0} \sum_{\substack{m \geq 1, d \geq 0 \\ m+2d=r+1}} \frac{1}{m!} \bar{Q}_{d,m}(z_0) \tilde{\mathcal{W}}_{g-d,m,n}(w_{\llbracket m \rrbracket} \mid z_{\llbracket n \rrbracket})$$

is holomorphic. Here (we recall the definitions for the reader's convenience)

$$(3.31) \quad \sum_{d \geq 0} \bar{Q}_{d,m}(z_0) t^{2d} = \frac{t}{\zeta(t)} \frac{\zeta(tD_{x(z_0)})}{tD_{x(z_0)}} \circ \prod_{j=1}^m \left(\int_{w_j=z_0} \frac{\zeta(tD_{x(w_j)})}{tD_{x(w_j)}} \right);$$

$$\frac{\zeta(t)}{t} = \frac{e^{t/2} - e^{-t/2}}{t} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)! 2^{2k}} t^{2k};$$

$$\int_{w=z} F(w) := \operatorname{Res}_{w=z} F(w) \frac{dx(w)}{x(w) - x(z)}.$$

By our induction argument, we can omit any non-trivial contribution from $\frac{t dx(z_0)}{\zeta(t)} \frac{\zeta(tD_{x(z_0)})}{tD_{x(z_0)}}$, as it gives only a number of derivatives acting on symmetric terms that have inductively already been proved to be holomorphic.

Recall also proposition 3.3. In that proposition, the $2k$ and $2N$ are reminiscent of, respectively, m and $r+1$ in the cut-and-join equation, and they are written this way as we always have an even number of Δ 's ($2k$) and an even number of D 's ($2N - 2k$), the genus defect also being $N - k = \sum \alpha_i$. However, this proposition is only about the Δ part of any term, and it should still be multiplied with an \mathcal{S} part.

Furthermore, note that in proposition 3.3 we have omitted the factors $\frac{1}{2}$ coming from equations (3.2) and (3.31). As these give one factor for each Δ and D , respectively, and the sum of their exponents is constantly equal to N in equation (3.13), we may as well omit them.

Proposition 3.3 implies the following corollary.

Corollary 3.5. *Suppose that a set of functions $(W_{g,n})_{g,n}$ satisfies the quadratic loop equations up to negative Euler characteristic $-\chi$. Then, we get for any $r > 0$ and any $g, l, n \geq 0$ such that $r+1-l$ is even and $2g-2+n \leq -\chi$, that*

$$(3.32) \quad \sum_{\substack{m=l \\ m-1 \text{ even}}}^{r+1} \frac{1}{m!} \sum_{2\alpha_1 + \dots + 2\alpha_m + m = r+1} \prod_{j=1}^m \left(\int_{z_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j+1)!} \right) \sum_{\substack{I \subset \llbracket m \rrbracket \\ |I|=l}} \prod_{i \in \llbracket m \rrbracket \setminus I} \Delta_j \prod_{i \in I} \mathcal{S}_i \tilde{\mathcal{W}}_{g-\alpha_1-\dots-\alpha_m, m, n}(w_{\llbracket m \rrbracket} \mid z_{\llbracket n \rrbracket})$$

is holomorphic in z near branch points of the spectral curve.

Proof. For $l=0$, this is a reformulation of proposition 3.3, with $2k=m$ and $2N=r+1$.

In general we can rewrite it, by reshuffling, as

$$(3.33) \quad \sum_{\substack{N=0 \\ N \text{ even}}}^{r+1} \sum_{k=0}^N \left[\frac{1}{l!} \sum_{\substack{2\beta_1 + \dots + 2\beta_l + l \\ = r+1-N}} \prod_{j=1}^l \left(\int_{w'_j=z} \frac{D_{j'}^{2\beta_j}}{(2\beta_j+1)!} \right) \prod_{i'=1}^l \mathcal{S}_{i'} \right]$$

$$\left[\frac{1}{(2k)!} \sum_{\alpha_1 + \dots + \alpha_{2k} + k = N} \prod_{j=1}^{2k} \left(\int_{w_j=z} \frac{D_j^{2\alpha_j}}{(2\alpha_j+1)!} \right) \prod_{i=1}^{2k} \Delta_i \right] \tilde{\mathcal{W}}_{g-\beta_1-\dots-\beta_l-\alpha_1-\dots-\alpha_{2k}, 2k+l, n}(w'_{\llbracket l \rrbracket}, w_{\llbracket 2k \rrbracket} \mid z_{\llbracket n \rrbracket})$$

(in order to shorten the notation we use $D_{j'} := D_{x(w'_j)}$ and $D_j := D_{x(w_j)}$). In this formula, for a fixed choice of N , the k - and α -sums give something holomorphic by proposition 3.3, the extra \mathcal{S} 's do not change holomorphicity by the linear loop equations and the fact that $\mathcal{S}_{i'} \mathcal{S}_{j'} \tilde{W}_{0,2}(w'_i, w'_j)$ respectively $\mathcal{S}_{i'} \Delta_j \tilde{W}_{0,2}(w'_i, w_j)$ are holomorphic at the diagonal, and the operator $D_{i'}^{2\beta_i} D_{j'}^{2\beta_j}$ respectively $D_{i'}^{2\beta_i}$ do not change that. \square

Theorem 3.6. *The quadratic loop equations (3.10) hold for $(W_{g,n})_{g,n}$ in the case of r -spin Hurwitz numbers, i.e. for*

$$(3.34) \quad W_{g,n} - \delta_{g,0} \delta_{n,2} \frac{1}{(X_1 - X_2)^2} \sim \prod_{i=1}^n \left(\frac{d}{dX_i} \right) \sum_{\mu_1, \dots, \mu_n=1}^{\infty} h_{g;\mu}^{\circ, q, r} \prod_{i=1}^n e^{X_i \mu_i}.$$

Proof. As stated before, we use induction on the negative Euler characteristic.

So assume the quadratic loop equation has been proved up to $-\chi$, and consider the symmetrized cut-and-join equation for $2g - 2 + n = -\chi + 1$. All the sub-leading terms in the cut-and-join equation, i.e., those where $\bar{Q}_{d,m}(z_0)$ gives a non-trivial contribution from $\frac{t}{\zeta(t)} \frac{\zeta(tD_x(z_0))}{tD_x(z_0)}$, are already holomorphic by the induction hypothesis, equation (3.2), and corollary 3.5. In the leading term, by the same corollary (cf. also remark 3.4), everything is holomorphic, except possibly for the terms involving

$$(3.35) \quad \int_{w_1=z_0} \int_{w_2=z_0} \Delta_{w_1} \Delta_{w_2} \tilde{W}(w_1, w_2 \mid z_{\llbracket n \rrbracket}) \cdot \mathcal{S}_{z_0}(y(z_0))^{r-1}$$

(as $W_{0,1}(z_0) = y(z_0)$).

Hence, this term must be holomorphic as well, and because $y(z)$ (and hence $\mathcal{S}_z y(z)$) is non-zero at branchpoints of x , this shows

$$(3.36) \quad \int_{w_1=z_0} \int_{w_2=z_0} \Delta_{w_1} \Delta_{w_2} \tilde{W}(w_1, w_2 \mid z_{\llbracket n \rrbracket}) \text{ is holomorphic,}$$

which is exactly the quadratic loop equation. \square

Remark 3.7. Note that this proof generalizes the proofs of [BKL⁺21, theorems 14 & 15]. In particular, proposition 3.3 subsumes [BKL⁺21, lemma 16], although the proof is different.

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