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On the coupling of well posed differential models

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ABSTRACT

Consider the coupling of 2 evolution equations, each generating a global process. We prove that the resulting system generates a new global process. This statement can be applied to differential equations of various kinds. In particular, it also yields the well posedness of a predator–prey model, where the coupling is in the differential terms, and of an epidemiological model, which does not fit previous well posedness results.

1. Introduction

A variety of models describing the evolution in time of real situations is obtained coupling simpler models devoted to specific subsystems. In this paper we provide a framework where the well posedness of the *"big"* model follows from that of its parts.

Predictive models consisting of couplings of evolution equations, possibly of different types, are very common in the applications of mathematics. Here we only note that their use ranges, for instance, from epidemiology [1–3], to traffic modeling [4,5], to several specific engineering applications [6,7].

In this manuscript, the core result is set in a metric space, so that linearity plays no role whatsoever. This also allows the range of applicability of the general theorem to encompass, for instance, ordinary, partial and measure differential equations. In each of these cases, we obtain stability estimates tuned to the metric

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structure typical of the specific evolution equation considered, which can be, for example, the Euclidean norm in $\mathbb{R}^n$, the $\mathbf{L^1}$ norm in spaces of $\mathbf{BV}$ functions or some Wasserstein type distance between measures.

At the abstract level, the starting point is provided by the framework of evolution equations in metric spaces, see [8–13]. In this setting, an evolution equation is well posed as soon as it generates a *Global Process*, i.e., a Lipschitz continuous solution operator, see Definition 2. In other words, global processes substitute, in the time dependent case, semigroups that, in the autonomous case, have as trajectories the solutions to evolution equations.

Assume that two evolution equations are given, each depending on a parameter and each generating a global process, also depending on that parameter. We now let the parameter in an equation vary in time according to the other equation: a coupling between the two models is thus obtained. Theorem 2 ensures the well posedness of this coupled model, in the sense that it generates a new global process.

The assumptions required in this abstract construction are then verified in 5 sample situations: ordinary differential equations, initial and boundary value problems for renewal equations, measure valued balance laws and scalar conservation laws. Thus, we prove that any coupling of these equations results in a well posed model. Indeed, in each of these cases, we provide a full set of detailed stability estimates compatible with the abstract results. Note that assumptions ensuring global in time existence results are also provided.

Finally, we consider specific cases. First, we briefly show that Theorem 2 comprises the case of the traffic model introduced in [5], where a scalar conservation law is coupled to an ordinary differential equation.

Then, we detail the case of a predator–prey model inspired by [14], namely

$$\begin{cases} \partial_t \rho + \operatorname{div}_x \left(\rho\, V\left(t, x, p(t)\right)\right) = -\eta\left(\|p(t) - x\|\right)\, \rho(t,x) \\ \dot{p} = U\left(t, p, \rho(t)\right)\ . \end{cases} \tag{1}$$

While we refer to Section 4.1 for a detailed explanation of the terms in (1), here we remark that in (1) the coupling is not only in the source term of the partial differential equations, but also in the convective term, where no nonlocal term is involved (V is a function defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $P(t) \in \mathbb{R}^n$).

Then, we apply the general construction to a recent epidemiological model presented in [3] whose well posedness, to our knowledge, was not proved at the time of this writing. In this case, the coupling involves a boundary value problem for a renewal equation, see Section 4.2.

For all basic results on evolution equations in metric spaces, we refer to the extended treatises [8,9,12], whose wide bibliographies also give a detailed view on the whole field. Below, we follow the approach outlined in [10,11,13]. The different frameworks differ in their approaches but offer similar results. Related to Theorem 2 is, for instance, [12, Theorem 26]. However, here we follow a more quantitative approach to the various stability estimates.

We expect that also other equations fit in the framework introduced in Section 2. Natural candidates are, for instance, measure differential equations [15,16] and their coupling with ordinary differential equations as considered in [17]. A further class of couplings is that in [6], consisting of ordinary and partial differential equations similar to those comprised in Section 3.3. Very likely to comply with the present structure is also the general class of traffic models presented in [18].

This work is organized as follows. Section 2, once the basic notation is introduced, presents the general result. Each of the paragraphs in Section 3 is devoted to a particular evolution equation: its well posedness is proved obtaining those estimates that allow the application of Theorem 2. Specific models are then dealt with in Section 4. Finally, proofs are in the final Section 5. Further estimates and technical results of use in this paper are similar to known methods: we do not include them here but, for completeness, we make them publicly available in [19].

2. Definitions and abstract results

Below we rely on the framework established in [10,11,13], see [8,9,12] for an alternative, essentially equivalent, setting. Let (X, d) be a metric space and I be a real interval. First, a *local flow* on X provides a sort of tangent vector field to X.

Definition 1 (*[11, Definition 2.1]*)**.** Given $\delta > 0$ and a closed set $\mathcal{D} \subseteq X$, a *local flow* is a continuous map $F : [0, \delta] \times I \times \mathcal{D} \mapsto X$, such that $F(0, t_o)\, u = u$ for any $(t_o, u) \in I \times \mathcal{D}$ and which is Lipschitz in its first and third arguments uniformly in the second, i.e. there exists a $\mathbf{Lip}(F) > 0$ such that for all $\tau, \tau' \in [0, \delta]$ and $u, u' \in \mathcal{D}$

$$d\left(F(\tau, t_o)u, F(\tau', t_o)u'\right) \leq \mathbf{Lip}(F) \cdot \left(d(u, u') + |\tau - \tau'|\right) . \tag{2}$$

Given an evolution equation, a *global process* is a candidate for the solution operator, i.e., for the mapping assigning to initial datum u at time t_o and to time t the solution evaluated at time t.

Definition 2 (*[11, Definition 2.5]*)**.** Fix a family of sets $\mathcal{D}_{t_o} \subseteq \mathcal{D}$ for all $t_o \in I$, and a set

$$\mathcal{A} = \left\{(t, t_o, u) : t \geq t_o,\ t_o, t \in I \text{ and } u \in \mathcal{D}_{t_o}\right\} . \tag{3}$$

A *global process* on X is a map $P : \mathcal{A} \mapsto X$ such that, for all $u \in \mathcal{D}_{t_o}$ and $t_o, t_1, t_2 \in I$ with $t_2 \geq t_1 \geq t_o$,

$$P(t_o, t_o)u = u \tag{4}$$

$$P(t_1, t_o)u \in \mathcal{D}_{t_1} \tag{5}$$

$$P(t_2, t_1) \circ P(t_1, t_o)u = P(t_2, t_o)u. \tag{6}$$

In Theorem 1 below, a global process is constructed from a local flow by means of a suitable extension of *Euler Polygonals* to metric spaces.

Definition 3 (*[11, Definition 2.3]*)**.** Let F be a local flow. Fix $u \in \mathcal{D}$, $t_o \in I$, $\tau \in [0, \delta]$ with $t_o + \tau \in I$. For every $\varepsilon > 0$, let $k = \lfloor \tau/\varepsilon \rfloor$, where the symbol $\lfloor \cdot \rfloor$ denotes the integer part. An Euler ε-polygonal is

$$F^{\varepsilon}(\tau, t_o)\, u = F(\tau - k\varepsilon, t_o + k\varepsilon) \circ \bigcirc_{h=0}^{k-1} F(\varepsilon, t_o + h\varepsilon)\, u \tag{7}$$

whenever it is defined.

Above, we used the notation $\bigcirc_{h=0}^{k} f_h = f_k \circ f_{k-1} \circ \cdots \circ f_1 \circ f_0$.

For a local flow F, its corresponding Euler ε-polygonal F^{ε}, and any $t_o \in I$, introduce the notation:

$$\mathcal{D}^3_{t_o} = \left\{ \begin{array}{c} F^{\varepsilon_3}(\tau_3, t_o + \tau_1 + \tau_2) \circ F^{\varepsilon_2}(\tau_2, t_o + \tau_1) \circ F^{\varepsilon_1}(\tau_1, t_o)u \\ u \in \mathcal{D} : \text{is in } \mathcal{D} \text{ for all } \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \left]0, \delta\right] \text{ and all} \\ \tau_1, \tau_2, \tau_3 \geq 0 \text{ such that } t_o + \tau_1 + \tau_2 + \tau_3 \in I \end{array} \right\} . \tag{8}$$

The next result provides the basis for our construction of solutions to coupled problems.

Theorem 1 (*[11, Theorem 2.6]*)**.** *Let (X, d) be a complete metric space and $\mathcal{D}$ be a closed subset of X. Assume that for the local flow $F : [0, \delta] \times I \times \mathcal{D} \mapsto X$ there exist*

1. *a non decreasing map $\omega : [0, \delta] \to \mathbb{R}_+$ with $\int_0^{\delta} \frac{\omega(\tau)}{\tau}\, d\tau < +\infty$ such that*

$$d\left(F(k\tau, t_o + \tau) \circ F(\tau, t_o)u, F\left((k+1)\tau, t_o\right) u\right) \leq k\, \tau\, \omega(\tau) \tag{9}$$

whenever $\tau \in [0, \delta]$, $k \in \mathbb{N}$ and the left hand side above is well defined;

2. *a positive constant L such that*

$$d\left(F^{\varepsilon}(\tau,t_o)u_1, F^{\varepsilon}(\tau,t_o)u_2\right) \leq L\, d(u_1,u_2) \tag{10}$$

whenever $\varepsilon \in]0,\delta]$, $u_1,u_2 \in \mathcal{D}$, $\tau \geq 0$, $t_o, t_o+\tau \in I$ and the left hand side above is well defined.

Then, there exists a family of sets $\mathcal{D}_{t_o}$, for $t_o \in I$, and a unique global process (as in Definition 2) $P\colon \mathcal{A} \to X$ with the following properties:

1. *$\mathcal{D}^3_{t_o} \subseteq \mathcal{D}_{t_o}$ for any $t_o \in I$, with $\mathcal{D}^3_{t_o}$ as defined in (8);*
2. *P is Lipschitz continuous with respect to $(t,t_o,u) \in \mathcal{A}$;*
3. *P is tangent to F in the sense that for all $(t_o+\tau,t_o,u) \in \mathcal{A}$, with $\tau \in]0,\delta]$:*

$$\frac{1}{\tau}\, d\left(P(t_o+\tau,t_o)u, F(\tau,t_o)u\right) \leq \frac{2L}{\ln(2)} \int_0^{\tau} \frac{\omega(\xi)}{\xi}\, \mathrm{d}\xi\ . \tag{11}$$

A general condition to ensure that $\mathcal{A}$ is non empty is [11, Condition **(D)**]. Below, in the examples we consider, it explicitly stems out that $\mathcal{A} \neq \emptyset$.

We now head towards considering processes depending on parameters.

Definition 4. Let $(\mathcal{U}, d_{\mathcal{U}})$ and $(\mathcal{W}, d_{\mathcal{W}})$ be metric spaces. A *Lipschitz Process on $\mathcal{U}$ parametrized by $w \in \mathcal{W}$* is a family of maps $P^w\colon \mathcal{A}_{\mathcal{U}} \to \mathcal{U}$, with

$$\begin{aligned}
\mathcal{I} &= \left\{(t,t_o) \in I \times I : t \geq t_o\right\}\ ,\\
\mathcal{A}_{\mathcal{U}} &= \left\{(t,t_o,u) : (t,t_o) \in \mathcal{I}\,,\ u \in \mathcal{D}^{\mathcal{U}}_{t_o}\right\}\ ,\\
\mathcal{D}^{\mathcal{U}}_t &\subseteq \mathcal{U}\,,
\end{aligned}$$

such that for all $w \in \mathcal{W}$, P^w is a Global Process in the sense of Definition 2 and there exist positive constants C_u, C_t, C_w such that

$$d_{\mathcal{U}}\left(P^w(t,t_o)u_1, P^w(t,t_o)u_2\right) \leq e^{C_u(t-t_o)}\, d_{\mathcal{U}}(u_1,u_2)\,, \tag{12}$$

$$d_{\mathcal{U}}\left(P^w(t_1,t_o)u, P^w(t_2,t_o)u\right) \leq C_t\, |t_2-t_1|\,, \tag{13}$$

$$d_{\mathcal{U}}\left(P^{w_1}(t,t_o)u_o, P^{w_2}(t,t_o)u_o\right) \leq C_w\,(t-t_o)\, d_{\mathcal{W}}(w_1,w_2)\,. \tag{14}$$

We equip the product space $\mathcal{U} \times \mathcal{W}$ with the distance

$$d\left((u',w'),(u'',w'')\right) = d_{\mathcal{U}}(u',u'') + d_{\mathcal{W}}(w',w'').$$

Theorem 2. *Let $(\mathcal{U}, d_{\mathcal{U}})$ and $(\mathcal{W}, d_{\mathcal{W}})$ be complete. Let $P^w\colon \mathcal{A}_{\mathcal{U}} \to \mathcal{U}$ be a Lipschitz Process on $\mathcal{U}$ parametrized by $w \in \mathcal{W}$, and let $P^u\colon \mathcal{A}_{\mathcal{W}} \to \mathcal{W}$ be a Lipschitz Process on $\mathcal{W}$ parametrized by $\mathcal{U}$. Let C_u, C_w, and C_t be constants that satisfy (12)–(13)–(14) for both processes. Then,*

1. *Introducing $\mathcal{A}_F = \left\{(\tau,t_o,(u,w)) : \tau \geq 0,\ t_o, t_o+\tau \in I,\ (u,w) \in \mathcal{D}^{\mathcal{U}}_{t_o} \times \mathcal{D}^{\mathcal{W}}_{t_o}\right\}$, the map*

$$\begin{array}{rccc}
F & : & \mathcal{A}_F & \to & \mathcal{U} \times \mathcal{W}\\
 & & (\tau,t_o,(u,w)) & \mapsto & (P^w(t_o+\tau,t_o)u, P^u(t_o+\tau,t_o)w)
\end{array} \tag{15}$$

   is a local flow on $\mathcal{U} \times \mathcal{W}$.
2. *F satisfies the assumptions of Theorem 1 with*

$$L = e^{(C_u+C_w)T} \quad \text{and} \quad \omega(\tau) = C_t\, C_u\, \tau \tag{16}$$

   hence F generates a unique global process $P\colon \mathcal{A} \to \mathcal{U} \times \mathcal{W}$, for a suitable $\mathcal{A} \subseteq I \times I \times \mathcal{U} \times \mathcal{W}$, satisfying properties 1, 2 and 3 in Theorem 1.

3. For all $t_o \in I$ and $\tau \geq 0$ with $t_o + \tau \geq t_o$, we have

$$F(\tau, t_o)(\mathcal{D}^{\mathcal{U}}_{t_o} \times \mathcal{D}^{\mathcal{W}}_{t_o}) \subseteq (\mathcal{D}^{\mathcal{U}}_{t_o+\tau} \times \mathcal{D}^{\mathcal{W}}_{t_o+\tau}) \tag{17}$$

hence the process P is defined on $\mathcal{A}$ with

$$\mathcal{A} \supseteq \left\{(\tau, t_o, (u, w)) : \tau \geq 0,\ t_o, t_o + \tau \in I,\ (u, w) \in \mathcal{D}^{\mathcal{U}}_{t_o} \times \mathcal{D}^{\mathcal{W}}_{t_o}\right\}. \tag{18}$$

The proof is deferred to Section 5.1.

An analogous result can be proved defining the local flow F by means of local flows $F^{\mathcal{U}}_w$ and $F^{\mathcal{W}}_u$, provided these local flows satisfy the assumptions of Theorem 1 and have a Lipschitz continuous dependence on the parameter.

Theorem 3. *Consider two complete metric spaces $(\mathcal{U}, d_{\mathcal{U}})$ and $(\mathcal{W}, d_{\mathcal{W}})$. Let*

$$F^w : [0, \delta] \times I \times \mathcal{D}^{\mathcal{U}} \to \mathcal{U}, \qquad \text{and} \qquad F^u : [0, \delta] \times I \times \mathcal{D}^{\mathcal{W}} \to \mathcal{W},$$

be local flows parametrized by $w \in \mathcal{W}$ and $u \in \mathcal{U}$, respectively, so that there exists $\mathcal{L}$ such that for all $\tau \in [0, \delta]$ and $t \in I$,

$$\begin{array}{rcll} d_{\mathcal{U}}\left(F^{w_1}(\tau, t)u, F^{w_2}(\tau, t)u\right) & \leq & \mathcal{L}\, d_{\mathcal{W}}(w_1, w_2) & u \in \mathcal{D}^{\mathcal{U}} \qquad w_1, w_2 \in \mathcal{W} \\ d_{\mathcal{W}}\left(F^{u_1}(\tau, t)w, F^{u_2}(\tau, t)w\right) & \leq & \mathcal{L}\, d_{\mathcal{U}}(u_1, u_2) & u \in \mathcal{D}^{\mathcal{W}} \qquad u_1, u_2 \in \mathcal{U} \end{array}$$

Then, setting $\mathcal{D} = \mathcal{D}^{\mathcal{U}} \times \mathcal{D}^{\mathcal{W}}$, the coupling

$$\begin{array}{rcccc} \hat{F} & : & [0, \delta] \times I \times \mathcal{D} & \to & \mathcal{U} \times \mathcal{W} \\ & & (\tau, t, (u, w)) & \mapsto & (F^w(t, t_o)u, F^u(t, t_o)w) \end{array}$$

is a local flow in the sense of Definition 1. If moreover F^w and F^u satisfy assumptions 1 and 2 in Theorem 1, then $\hat{F}$ is tangent to the local flow F defined in (15) by means of the processes P^w and P^u defined through Theorem 1.

As a direct consequence of Theorem 3, by means of [20, Theorem 2.9], we have that whenever Theorem 2 applies, if $\hat{F}$ generates a global process $\hat{P}$, then $\hat{P}$ coincides with the process P constructed in Theorem 2.

3. General Cauchy problems

In the paragraphs below we consider differential equations depending on parameters that generate parametrized Lipschitz processes in the sense of Definition 4. Thus, any coupling of the processes below meets the requirements of Theorem 2 and generates a new Lipschitz process. Moreover, we verify that this new process eventually yields solutions to the coupled problem.

Throughout, $\hat{I}$ is a real interval containing 0. If $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm, while $\|x\|_V$ is the norm of x in the Banach space V. The open, respectively closed, ball centered at x with radius r is $B(x, r)$, respectively $\overline{B(x, r)}$.

3.1. Ordinary differential equations

This brief paragraph mainly serves as a paradigm for the subsequent ones. All proofs are deferred to [19]. Indeed, we begin by considering the classical Cauchy problem for an ordinary differential equation

$$\left\{\begin{array}{ll} \dot{u} = f(t, u, w) & t \in \hat{I} \\ u(t_o) = u_o \end{array}\right. \qquad \text{with} \qquad f : \hat{I} \times \mathbb{R}^n \times \mathcal{W} \to \mathbb{R}^n, \tag{19}$$

where $t_o \in \hat{I}$, $u_o \in \mathbb{R}^n$ and the parameter w is fixed in $\mathcal{W}$.

Definition 5. A map $u\colon I \to \mathbb{R}^n$ is a solution to (19) if $t_o \in I \subseteq \hat{I}$, $u(t_o) = u_o$, for a.e. $t \in I$, u is differentiable at t and $\dot{u}(t) = f(t, u(t), w)$.

The well posedness of (19) is an elementary result which we state below to allow subsequent couplings of (19) with other equations within the framework of Theorem 2.

Proposition 1. *Let $R > 0$. Define $\mathcal{D} = \overline{B(0,R)}$ in $\mathbb{R}^n$ and consider the Cauchy problem (19) under the assumptions*

(ODE1) *For all $u \in \mathcal{D}$ and all $w \in \mathcal{W}$, the map $t \mapsto f(t,u,w)$ is measurable.*
(ODE2) *There exist positive F_L, F_∞ such that for all $t \in \hat{I}$, $u_1, u_2 \in \mathcal{D}$ and $w_1, w_2 \in \mathcal{W}$*

$$\|f(t,u_1,w_1) - f(t,u_2,w_2)\| \leq F_L \left(\|u_1 - u_2\| + d_{\mathcal{W}}(w_1,w_2)\right) ; \tag{20}$$

$$\sup_{w \in \mathcal{W}} \|f(\cdot,\cdot,w)\|_{\mathbf{L}^\infty(\hat{I}\times\hat{\mathcal{D}};\mathbb{R}^n)} \leq F_\infty \,. \tag{21}$$

Then, there exists $T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz process on $\mathbb{R}^n$ parametrized by $\mathcal{W}$ in the sense of Definition 4, whose orbits solve (19) according to Definition 5, with

$$\begin{gathered} T \leq R/(2F_\infty) \;, \quad C_u = F_L \,, \quad C_t = F_\infty \,, \quad C_w = F_L\, e^{F_L T} \,, \\ \mathcal{D}_t = \overline{B\left(0, R - (T-t)\, \sup_{w\in\mathcal{W}} \|f(\cdot,\cdot,w)\|_{\mathbf{L}^\infty(\hat{I}\times\hat{\mathcal{D}};\mathbb{R}^n)}\right)} \,. \end{gathered} \tag{22}$$

Long time existence is also available.

Corollary 1. *Assume $\sup \hat{I} = +\infty$ and that, for every $R > 0$,* **(ODE1)** *and* **(ODE2)** *hold with $F_\infty = F_\infty(R)$ satisfying*

$$\limsup_{R\to+\infty} \frac{F_\infty(R)}{R\ln(R)} < +\infty.$$

Then, for all $t_o \in \hat{I}$, the solution to (19) exists for every $t \geq t_o$.

We now verify that Theorem 2 applies to the coupling of (19) with other Lipschitz Processes.

Proposition 2. *Set $\mathcal{U} = \mathbb{R}^n$. Assume that* **(ODE1)**–**(ODE2)** *hold. Let P^u be a Lipschitz Process on $\mathcal{W}$ parametrized by $u \in \mathcal{U}$. Call $P\colon \mathcal{A} \to \mathbb{R}^n \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the Process constructed in Theorem 2 coupling P^w, generated by (19), and P^u. If $([t_o,T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then*

$$\begin{array}{rcl} u : [t_o,T] & \to & \mathbb{R}^n \\ t & \mapsto & P_1(t,t_o)(u_o,w_o) \end{array} \quad \textit{solves} \quad \begin{cases} \dot{u} = \bar{f}(t,u) \\ u(t_o) = u_o \end{cases} \quad \textit{where } \bar{f}(t,u) = f\left(t,u,P_2(t,t_o)(u_o,w_o)\right)$$

in the sense of Definition 5.

3.2. The initial value problem for a renewal equation

We examine the following initial value problem for a first order partial differential equation

$$\begin{cases} \partial_t u + \operatorname{div}_x \left(v(t,x,w)\, u\right) = m(t,x,w)u + q(t,x,w) & (t,x) \in \hat{I} \times \mathbb{R}^n, \\ u(t_o,x) = u_o(x), & x \in \mathbb{R}^n \end{cases} \tag{23}$$

for $u_o \in \mathbf{L^1}(\mathbb{R}^n;\mathbb{R})$ and $t_o \in \hat{I}$. Proofs are deferred until Section 5.2.

Definition 6. For a fixed $w \in \mathcal{W}$, a function $u \in \mathbf{C^0}\left([t_o, T]; \mathbf{L^1}(\mathbb{R}^n; \mathbb{R})\right)$, where $[t_o, T] \subseteq \hat{I}$, is a solution to (23) if:

1. for any test function $\varphi \in \mathbf{C^\infty_c}(]t_o, T[\times \mathbb{R}^n; \mathbb{R})$,

$$\int_{t_o}^{T} \int_{\mathbb{R}^n} \Big(u(t,x)\, \partial_t \varphi(t,x) + u(t,x)\, v(t,x,w) \cdot \nabla_x \varphi(t,x) \\ + \left(m(t,x,w)\, u(t,x) + q(t,x,w)\right) \varphi(t,x) \Big) \, \mathrm{d}x \, \mathrm{d}t = 0;$$

2. $u(t_o, x) = u_o(x)$ for a.e. $x \in \mathbb{R}^n$.

Proposition 3. *Let $R > 0$ and set $\mathcal{U} = \mathbf{L^1}(\mathbb{R}^n; \mathbb{R})$. Define*

$$\mathcal{D} = \left\{ u \in \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}) \colon \max \left\{ \|u\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}, \|u\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})}, \mathrm{TV}(u) \right\} \le R \right\}.$$

Consider the Cauchy problem (23) under the assumptions

(IP1) *For all $w \in \mathcal{W}$, $v(\cdot,\cdot,w) \in \mathbf{C^0}(\hat{I} \times \mathbb{R}^n; \mathbb{R}^n)$, $v(t,\cdot,w) \in \mathbf{C^2}(\mathbb{R}^n; \mathbb{R}^n)$ for all $t \in \hat{I}$ and there exist positive constants V_1, V_L, V_∞ such that for all $t \in \hat{I}$*

$$\|v(t,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^n)} \le V_\infty \,; \qquad \|\nabla v(t,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^{n\times n})} \le V_L \,; \\ \|\nabla\nabla \cdot v(t,\cdot,w)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)} \le V_1 \,.$$

and, for all $w_1, w_2 \in \mathcal{W}$ and $t \in \hat{I}$,

$$\|v(t,\cdot,w_1) - v(t,\cdot,w_2)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R}^n)} \le V_L \, d_{\mathcal{W}}(w_1,w_2), \\ \|\nabla \cdot (v(t,\cdot,w_1) - v(t,\cdot,w_2))\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \le V_L \, d_{\mathcal{W}}(w_1,w_2).$$

(IP2) *For all $w \in \mathcal{W}$, $m(\cdot,\cdot,w) \in \mathbf{C^0}(\hat{I} \times \mathbb{R}^n; \mathbb{R})$ and there exist positive constants M_∞, M_L such that for all $t \in \hat{I}$ and for all $w, w_1, w_2 \in \mathcal{W}$*

$$\|m(t,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \mathrm{TV}\left(m(t,\cdot,w)\right) \le M_\infty \,; \\ \|m(t,\cdot,w_1) - m(t,\cdot,w_2)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \le M_L \, d_{\mathcal{W}}(w_1,w_2) \,.$$

(IP3) *For all $w \in \mathcal{W}$, $q(\cdot,\cdot,w) \in \mathbf{L^1}\left(\hat{I}; \mathbf{L^\infty}(\mathbb{R}^n; \mathbb{R})\right)$ and there exist positive constants Q_∞, Q_1, Q_L such that for all $t \in \hat{I}$ and for all $w, w_1, w_2 \in \mathcal{W}$,*

$$\|q(t,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \mathrm{TV}\left(q(t,\cdot,w)\right) \le Q_\infty \,; \\ \|q(t,\cdot,w)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \le Q_1, \\ \|q(t,\cdot,w) - q(t,\cdot,w_2)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \le Q_L \, d(w_1,w_2) \,.$$

Then, there exists $T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz process on $\mathcal{U}$ parametrized by $\mathcal{W}$ in the sense of Definition 4, whose orbits solve (23) in the sense of Definition 6, with

$$C_u = M_\infty \,, \quad C_t = V_\infty \, R \, e^{(M_\infty + 2V_L)T} + Q_1 \, e^{M_\infty T} + (M_\infty + V_L) \, R \, e^{(M_\infty + V_L)T} \,, \\ C_w = \left[V_L(2R + Q_\infty)(1 + (V_1 + M_\infty)T) + (Q_L + (M_L + V_L)(R + Q_\infty T))\right] e^{(M_\infty + V_L)T} \,, \\ \mathcal{D}_t = \left\{ u \in \mathcal{D} \colon \begin{array}{rcl} \|u\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} & \le & \alpha_1(t) \\ \|u\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} & \le & \alpha_\infty(t) \\ \mathrm{TV}(u) & \le & \alpha_{\mathrm{TV}}(t) \end{array} \right\} , \tag{24}$$

where

$$\begin{aligned}
\alpha_1(t) &= Re^{-M_\infty(T-t)} - Q_1(T-t)e^{M_\infty t}\,,\\
\alpha_\infty(t) &= Re^{-(M_\infty+V_L)(T-t)} - Q_\infty e^{(M_\infty+V_L)t}(T-t)\,,\\
\alpha_{\mathrm{TV}}(t) &= Re^{-(M_\infty+V_L)(T-t)}\left(1-(M_\infty+V_1)(T-t)\right)\\
&\quad -Q_\infty e^{(M_\infty+V_L)t}\left(1+(M_\infty+V_1)t\right)(T-t)\,.
\end{aligned} \tag{25}$$

Corollary 2. *Assume* $[0,+\infty)\subseteq \hat{I}$ *and that* **(IP1)**, **(IP2)**, *and* **(IP3)** *hold. Then the solution to* (23) *exists for every* $t\geq t_o$.

Continuing now to the act of coupling this Lipschitz process with another.

Proposition 4. *Set* $\mathcal{U}=\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})$. *Assume that* **(IP1)**–**(IP2)**–**(IP3)** *hold. Let* P^u *be a Lipschitz process on* $\mathcal{W}$, *parametrized by* $u\in\mathcal{U}$. *Call* $P\colon\mathcal{A}\to\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})\times\mathcal{W}$, *with* $P\equiv(P_1,P_2)$, *the process generated in* Theorem 2 *by the coupling of process* P^w, *found in* Proposition 3, *with* P^u. *If* $([t_o,T],t_o,u_o,w_o)\subseteq\mathcal{A}$, *then the map*

$$\begin{aligned}
u:[t_o,T] &\to (\mathbf{L^1}\cap\mathbf{BV})(\mathbb{R}^n;\mathbb{R})\\
t &\mapsto P_1(t,t_o)(u_o,w_o)
\end{aligned}$$

solves

$$\begin{cases}
\partial_t u+\mathrm{div}_x\left(\bar{v}(t,x)\,u\right)=\bar{m}(t,x)u+\bar{q}(t,x) & (t,x)\in[t_o,T]\times\mathbb{R}^n,\\
u(t_o,x)=u_o(x), & x\in\mathbb{R}^n
\end{cases}$$

in the sense of Definition (23), *where*

$$\bar{m}(t,x)=m\left(t,x,P_2(t,t_o)(u_o,w_o)\right),\qquad \bar{q}(t,x)=q\left(t,x,P_2(t,t_o)(u_o,w_o)\right)\,,$$
$$\bar{v}(t,x)=v\left(t,x,P_2(t,t_o)(u_o,w_o)\right)\,.$$

3.3. The boundary value problem for a linear balance law

Consider the model

$$\begin{cases}
\partial_t u+\partial_x\left(v(t,x)\,u\right)=m(t,x,w)\,u+q(t,x,w) & (t,x)\in\hat{I}\times\mathbb{R}_+\\
u(t,0)=b(t) & t\in\hat{I}\\
u(t_o,x)=u_o(x) & x\in\mathbb{R}_+\,.
\end{cases} \tag{26}$$

where $u_o\in\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})$, $t_o\in\hat{I}$ and $w\in\mathcal{W}$. Throughout, we choose left continuous representatives of **BV** functions. Proofs are deferred to Section 5.3.

Definition 7. For a fixed $w\in\mathcal{W}$, a function $u\in\mathbf{C^0}\left([t_o,T];\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})\right)$, with $[t_o,T]\subseteq\hat{I}$, such that $u(t)\in\mathbf{BV}(\mathbb{R}_+;\mathbb{R})$ for a.e. $t\in[t_o,T]$ is a solution to (26) if:

1. For all $\varphi\in\mathbf{C^\infty_c}(]t_o,T[\times\mathring{\mathbb{R}}_+;\mathbb{R})$

$$\begin{aligned}
\int_{t_o}^T\int_{\mathbb{R}_+}\Big(&u(t,x)\,\partial_t\varphi(t,x)+v(t,x)\,u(t,x)\,\partial_x\varphi(t,x)\\
&+\left(m(t,x,w)\,u(t,x)+q(t,x,w)\right)\,\varphi(t,x)\Big)\,\mathrm{d}x\,\mathrm{d}t \quad = \quad 0\,.
\end{aligned}$$

2. For a.e. $x\in\mathbb{R}_+$, $u(t_o,x)=u_o(x)$.
3. For a.e. $t\in[t_o,T]$, $\lim_{x\to0+}u(t,x)=b(t)$.

Proposition 5. *Let $\mathcal{U} = \mathbf{L^1}(\mathbb{R}_+;\mathbb{R})$ and fix $b \in \mathbf{BV}(\hat{I};\mathbb{R})$. For $R > 0$, define*

$$\mathcal{D} = \left\{ u \in \mathcal{U} \colon \max\left\{ \|u\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})}, \|u\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})}, \mathrm{TV}(u) + \left| b(\sup \hat{I}) - u(0) \right| \right\} \leq R \right\} . \tag{27}$$

Assume

(BP1) *There exist strictly positive constants $\check{v}, \hat{v}, V_1, V_\infty$ such that $v \in \mathbf{C^{0,1}}(\hat{I} \times \mathbb{R}_+; [\check{v}, \hat{v}])$ and for all $(t,x) \in \hat{I} \times \mathbb{R}_+$*

$$\begin{aligned} \mathrm{TV}\left(v(\cdot,x);\hat{I}\right) + \mathrm{TV}\left(v(t,\cdot)\right) &\leq V_\infty \,, \\ \mathrm{TV}\left(\partial_x v(t,\cdot)\right) + \|\partial_x v(t,\cdot)\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} &\leq V_L \,. \end{aligned}$$

(BP2) *For all $w \in \mathcal{W}$, $m(\cdot,\cdot,w) \in \mathbf{C^0}(\hat{I} \times \mathbb{R}_+;\mathbb{R})$ and there exist M_∞, M_L such that for all $t \in \hat{I}$, $w, w_1, w_2 \in \mathcal{W}$,*

$$\begin{aligned} \mathrm{TV}\left(m(t,\cdot,w)\right) + \|m(t,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} &\leq M_\infty \,, \\ \|m(t,\cdot,w_1) - m(t,\cdot,w_2)\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\leq M_L \, d_{\mathcal{W}}(w_1,w_2) \,. \end{aligned}$$

(BP3) *For all $w \in \mathcal{W}$, $q(\cdot,\cdot,w) \in \mathbf{C^0}\left(\hat{I};\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})\right)$ and there exist Q_1, Q_∞ such that for all $t \in \hat{I}$ and $w, w_1, w_2 \in \mathcal{W}$, and*

$$\begin{aligned} \|q(t,\cdot,w)\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\leq Q_1 \,, \\ \mathrm{TV}\left(q(t,\cdot,w)\right) + \|q(t,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} &\leq Q_\infty \,, \\ \|q(t,\cdot,w_1) - q(t,\cdot,w_2)\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} &\leq Q_L \, d_{\mathcal{W}}(w_1,w_2) \,. \end{aligned}$$

(BP4) *$b \in (\mathbf{L^1} \cap \mathbf{L^\infty} \cap \mathbf{BV})(\hat{I};\mathbb{R})$, is left continuous, and there exist positive constants B_1 and B_∞ such that*

$$\begin{aligned} \|b\|_{\mathbf{L^1}(\hat{I};\mathbb{R})} &\leq B_1 \,, \\ \mathrm{TV}(b) + \|b\|_{\mathbf{L^\infty}(\hat{I};\mathbb{R})} &\leq B_\infty \,. \end{aligned}$$

Then, there exists $R, T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz process on $\mathcal{U}$, parametrized by $\mathcal{W}$ in the sense of Definition 4, whose orbits solve (26) in the sense of Definition 7, with

$$\begin{aligned} C_u = M_\infty \,, \quad C_t &= [\hat{v}(B_1 + 2R + R(M_\infty + V_L)T) + M_\infty R + Q_1]e^{M_\infty T} \,, \\ C_w &= \left[B_\infty M_L + \hat{v}\, Q_L + \tfrac{1}{2}\hat{v}\, Q_\infty\, M_L\, T + M_L\, R + Q_L + \tfrac{1}{2}\, M_L\, Q_\infty\, T\right] e^{M_\infty T} \,, \\ \mathcal{D}_t &= \left\{ r \in \mathcal{U} \colon \begin{array}{c} \|u\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq \alpha_1(t) \,, \ \|u\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} \leq \alpha_\infty(t) \,, \\ \mathrm{TV}(u) + |b(t) - u(0)| \leq \alpha_{TV}(t) \end{array} \right\} \end{aligned} \tag{28}$$

where

$$\begin{aligned} \alpha_1(t) &= Re^{-M_\infty(T-t)} - (\hat{v}B_\infty + Q_1)(T-t)e^{M_\infty t} \\ \alpha_\infty(t) &= Re^{-M_\infty(T-t)} - Q_\infty(T-t) \\ \alpha_{\mathrm{TV}}(t) &= R\left(1 - (M_\infty + V_L)(T-t)\right)e^{(M_\infty+V_L)(T-t)} \\ &\quad -2Q_\infty(1 + (M_\infty + V_L)t)(T-t)e^{(M_\infty+V_L)t} \\ &\quad -B_\infty(M_\infty + V_L)(T-t)e^{(M_\infty+V_L)t} - \mathrm{TV}(b;[t,T])e^{(M_\infty+V_L)t} \,. \end{aligned}$$

A result entirely analogous to Corollary 2 can be proved also in the case of (26).

Proposition 6. *Set $\mathcal{U} = \mathbf{L^1}(\mathbb{R}_+;\mathbb{R})$. Assume* **(BP1)–(BP2)–(BP3)–(BP4)**. *Let P^u be a Lipschitz process on $\mathcal{W}$, parametrized by $u \in \mathcal{U}$. Set $P : \mathcal{A} \to \mathcal{U} \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, to be the process generated in Theorem 2 by the coupling of the process P^w, constructed in Proposition 5, with P^u. If $(t, t_o, (u_o, w_o)) \in \mathcal{A}$, then*

$$\begin{aligned} u : [t_o, T] \to & \quad \mathbf{L^1}(\mathbb{R}_+;\mathbb{R}) \\ t \quad & \mapsto P_1(t, t_o)\,(u_o, w_o) \end{aligned} \tag{29}$$

is a solution to

$$\begin{cases} \partial_t u + \partial_x\,(v(t,x)\,u) = \bar{m}(t,x)\,u + \bar{q}(t,x) & (t,x) \in [t_o, T] \times \mathbb{R}_+ \\ u(t,0) = b(t) & t \in [t_o, T] \\ u(t_o, x) = u_o(x) & x \in \mathbb{R}_+ \end{cases} \tag{30}$$

in the sense of Definition 7, where

$$\bar{m}(t,x) = m\,(t, x, P_2(t,t_o)\,(u_o, w_o))\ , \quad \bar{q}(t,x) = q\,(t, x, P_2(t,t_o)\,(u_o, w_o))\ . \tag{31}$$

3.4. Measure valued balance laws

Following [21], consider the following measure valued balance law

$$\begin{cases} \partial_t \mu + \partial_x\,(b(t,\mu,w)\,\mu) + c(t,\mu,w)\,\mu = \int_{\mathbb{R}_+} (\eta(t,\mu,w))\,(y)\ \mathrm{d}\mu(y) \ \ t \in \hat{I} \\ \mu(t_o) = \mu_o \end{cases} \tag{32}$$

for $\mu_o \in \mathcal{M}^+(\mathbb{R}_+)$, the set of bounded, positive Radon measures on $\mathbb{R}_+$ equipped with the following distance, induced by the dual norm of $\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})$, see [21, § 2]:

$$d_{\mathcal{M}}(\mu_1, \mu_2) = \sup \left\{ \int_{\mathbb{R}_+} \varphi\ \mathrm{d}(\,\mu_1 - \mu_2) : \varphi \in \mathbf{C^1}(\mathbb{R}_+;\mathbb{R}) \text{ and } \|\varphi\|_{\mathbf{W}^{1,\infty}} \le 1 \right\}\ . \tag{33}$$

We refer to [22] for basic measure theoretic results. Below, if X is a Banach space, then $\mathbf{BC}(\hat{I}; X)$ is the space of bounded continuous functions with the supremum norm. $\mathbf{BC^{\alpha,1}}(\hat{I} \times \mathcal{M}^+(\mathbb{R}_+); X)$ is the space of X valued functions which are bounded with respect to the $\|\cdot\|_X$ norm, Hölder continuous with exponent α with respect to time and Lipschitz continuous in the measure variable with respect to $d_{\mathcal{M}}$ in (33). These spaces are equipped with the norms

$$\begin{aligned} \|f\|_{\mathbf{BC}(\hat{I};X)} &= \sup_{t \in \hat{I}} \|f(t)\|_X\ , \\ \|f\|_{\mathbf{BC^{\alpha,1}}(\hat{I} \times \mathcal{M}^+(\mathbb{R}_+);X)} &= \sup_{t \in \hat{I}, \mu \in \mathcal{M}^+(\mathbb{R}_+)} \left(\|f(t,\mu)\|_X + \mathbf{Lip}\,(f(t,\cdot)) + \mathbf{H}\,(f(\cdot,\mu)) \right)\ , \\ \|f\|_{(\mathbf{BC} \cap \mathbf{W}^{1,\infty})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))} &= \sup_{x \in \mathbb{R}_+} \|f(x)\|_{\mathcal{M}(\mathbb{R}_+)} + \mathbf{Lip}(f)\,, \end{aligned}$$

where, with a slight abuse of notation,

$$\begin{aligned} \mathbf{Lip}\,(f(t,\cdot)) &= \sup_{\substack{\mu_1, \mu_2 \in \mathcal{M}^+(\mathbb{R}_+) \\ \mu_1 \neq \mu}} \left(\|f(t,\mu_1) - f(t,\mu_2)\|_X / d_{\mathcal{M}}(\mu_1, \mu_2) \right)\ , \\ \mathbf{H}\,(f(\cdot,\mu)) &= \sup_{s_1, s_2 \in \hat{I}} \left(\|f(s_1,\mu) - f(s_2,\mu)\|_X / |s_1 - s_2|^\alpha \right)\ , \\ \mathbf{Lip}(f) &= \sup_{\substack{x_1, x_2 \in \mathbb{R}_+ \\ x_1 \neq x_2}} \left(d_{\mathcal{M}}\,(f(x_1), f(x_2))\,/\|x_2 - x_1\| \right)\ . \end{aligned}$$

Definition 8. Given $T \in \hat{I}$ with $T > t_o$ and $w \in \mathcal{W}$, a function $\mu\colon [t_o, T] \to \mathcal{M}^+(\mathbb{R}_+)$ is a *weak solution* to (32) on the time interval $[t_o, T]$ if μ is narrowly continuous with respect to time (i.e., for every bounded function $\psi \in \mathbf{C^0}(\mathbb{R}_+;\mathbb{R})$, the map $t \mapsto \int_{\mathbb{R}_+} \psi(x)\,\mathrm{d}\mu(t,x)$ is continuous), and for all $\varphi \in (\mathbf{C^1} \cap \mathbf{W}^{1,\infty})([t_o, T] \times \mathbb{R}_+;\mathbb{R})$, the following equality holds:

$$\begin{aligned}
&\int_{t_o}^{T}\int_{\mathbb{R}_+} \left(\partial_t\varphi(t,x) + (b(t,\mu,w))(x)\,\partial_x\varphi(t,x) - (c(t,\mu,w))(x)\,\varphi(t,x)\right)\mathrm{d}\mu(t,x)\,\mathrm{d}t\\
&+\int_{t_o}^{T}\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t,x)\,\mathrm{d}[\eta(t,\mu,w)(y)](x)\right)\mathrm{d}\mu(t,y)\,\mathrm{d}t\\
=&\int_{\mathbb{R}_+}\varphi(T,x)\;\mathrm{d}\mu(T,x) - \int_{\mathbb{R}_+}\varphi(t_o,x)\;\mathrm{d}\mu_o(x)\,.
\end{aligned}$$

Proposition 7. *Let $R > 0$. Set $\mathcal{U} = \mathcal{M}^+(\mathbb{R})$ and let $\mathcal{D} = \{\mu \in \mathcal{M}^+(\mathbb{R}_+)\colon \mu(\mathbb{R}_+) \le R\}$. Consider the Cauchy problem (32) under the assumptions, for some positive constant $\hat{L}$,*

(MVBL1) *For every $w \in \mathcal{W}$, $b(\cdot,\cdot,w) \in \mathbf{BC}^{\alpha,1}(\hat{I} \times \mathcal{D}; \mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R}))$. Further, for every $w, w_1, w_2 \in \mathcal{W}$, $t \in \hat{I}$, and $\mu \in \mathcal{D}$, $b(t,\mu,w)(0) \ge 0$, and, for some $B > 0$,*

$$\begin{aligned}
\|b(t,\mu,w)\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})} &\le B\,,\\
\|b(\cdot,\mu,w_1) - b(\cdot,\mu,w_2)\|_{\mathbf{BC}(\hat{I};\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R}))} &\le \hat{L}\; d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

(MVBL2) *For every $w \in \mathcal{W}$, $c(\cdot,\cdot,w) \in \mathbf{BC}^{\alpha,1}(\hat{I} \times \mathcal{D}; \mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R}))$. Further, there exists a positive constant $C \ge 0$ such that, for all $w, w_1, w_2 \in \mathcal{W}$, $\mu \in \mathcal{D}$ and $t \in \hat{I}$,*

$$\begin{aligned}
\|c(t,\mu,w)\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})} &\le C\,,\\
\|c(\cdot,\mu,w_1) - c(\cdot,\mu,w_2)\|_{\mathbf{BC}(\hat{I};\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R}))} &\le \hat{L}\; d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

(MVBL3) *For all $w \in \mathcal{W}$, $\eta(\cdot,\cdot,w) \in \mathbf{BC}^{\boldsymbol{\alpha},\mathbf{1}}\left(\hat{I} \times \mathcal{D};\ (\mathbf{BC} \cap \mathbf{W}^{1,\infty})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))\right)$. Further, there exists an $E > 0$ such that, for all $w, w_1, w_2 \in \mathcal{W}$, $t \in \hat{I}$, and $\mu \in \mathcal{D}$,*

$$\begin{aligned}
\|\eta(t,\mu,w)\|_{(\mathbf{BC}\cap\mathbf{W}^{1,\infty})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))} &\le E\,,\\
\|\eta(\cdot,\mu,w_1) - h(\cdot,\mu,w_2)\|_{\mathbf{BC}(\hat{I};\mathbf{BC}\cap\mathbf{W}^{1,\infty})(\mathbb{R}_+;\mathcal{M}^+(\mathbb{R}_+))} &\le \hat{L}\; d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

Then, there exist $T > 0$, such that $[0,T] \subseteq \hat{I}$, and a Lipschitz Process on $\mathcal{M}^+(\mathbb{R}^n)$, parametrized by $\mathcal{W}$ in the sense of Definition 4 whose orbits solve (32) in the sense of Definition 8, with

$$\begin{aligned}
C_u = 3(B+C+E)\,,\quad C_t = (B+C+E)\,e^{2(B+C+E)T}R\,,\\
C_w = C^*(T,B,C,E)\;R\;L\;e^{5(B+C+E)T}\,,\\
\mathcal{D}_t = \left\{\mu \in \mathcal{D}\colon \mu(\mathbb{R}_+) \le Re^{-3(B+C+E)(T-t)}\right\}\,.
\end{aligned} \tag{34}$$

The proof is a direct consequence of [21, Theorem 2.10] and, hence, it is omitted. In particular, C^* in (34) is the constant defined in [21, Item (iv), Theorem 2.10].

Proposition 8. *Set $\mathcal{U} = \mathcal{M}^+(\mathbb{R}^n)$. Fix $T > 0$ and assume that* **(MVBL1)**–**(MVBL2)**–**(MVBL3)** *hold. Let P^u be a Lipschitz process on $\mathcal{W}$, parametrized by $u \in \mathcal{U}$. Call $P\colon \mathcal{A} \to \mathbb{R}^n \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the Process constructed in Theorem 2 coupling P^w, found in Proposition 7, and P^u. If $([t_o,T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then the map*

$$\begin{array}{rcccl}
\mu & : & [t_o,T] & \to & \mathcal{M}^+(\mathbb{R}^n)\\
 & & t & \mapsto & P_1(t,t_o)(\mu,w)
\end{array} \tag{35}$$

solves the measure valued balance law

$$\begin{cases} \partial_t \mu + \partial_x \left(\bar{b}(t,\mu)\,\mu\right) + \bar{c}(t,\mu)\,\mu = \int_{\mathbb{R}_+} \left(\bar{\eta}(t,\mu)\right)(y)\ \mathrm{d}\mu(y) \ \ t \in \hat{I} \\ \mu(t_o) = \mu_o \end{cases}$$

in the sense of Definition 8, where

$$\bar{b}(t,\mu) = b\left(t,\mu,P_2(t,t_o)(\mu_o,w_o)\right)\,, \qquad \bar{c}(t,\mu) = c\left(t,\mu,P_2(t,t_o)(\mu_o,w_o)\right)\,,$$
$$\bar{\eta}(t,\mu) = \eta\left(t,\mu,P_2(t,t_o)(\mu_o,w_o)\right)\,.$$

The proof is deferred to Section 5.4.

3.5. Scalar nonlinear conservation laws

We now consider the following scalar nonlinear conservation law in one space dimension:

$$\begin{cases} \partial_t u + \partial_x f(t,u,w) = 0 & (t,x) \in \hat{I} \times \mathbb{R}, \\ u(t_o,x) = u_o(x) & x \in \mathbb{R} \end{cases} \tag{36}$$

for $t_o \in \hat{I}$, $u_o \in \mathbf{L^1}(\mathbb{R};\mathbb{R})$, $w \in \mathcal{W}$, with $f\colon \hat{I} \times \mathbb{R} \times \mathcal{W} \to \mathbb{R}$ a given function.

Definition 9. Fix $w \in \mathcal{W}$ and $[t_o,T] \subseteq \hat{I}$. We say that a map $u \in \mathbf{C^0}\left([t_o,T];\mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ is a solution to problem (36) if it is a Kružkov–Entropy solution, i.e.

$$\begin{aligned} \int_{t_o}^{T} \int_{\mathbb{R}} \left[|u-k|\,\partial_t\varphi + \mathrm{sign}(u-k)\ \left(f(t,u,w) - f(t,k,w)\right)\,\partial_x\varphi\right] \mathrm{d}x\,\mathrm{d}t \\ \geq \int_{\mathbb{R}} |u(T,x)-k|\,\varphi(T,x)\,\mathrm{d}x - \int_{\mathbb{R}} |u_o(x)-k|\,\varphi(t_o,x)\,\mathrm{d}x\,, \end{aligned} \tag{37}$$

for all non-negative test functions $\varphi \in \mathbf{C^\infty_c}(\hat{I} \times \mathbb{R};\mathbb{R}_+)$, and for all $k \in \mathbb{R}$.

Proposition 9. *Let $R > 0$ and t_o, T be such that $[t_o,T] \subseteq \hat{I}$. Choose $\mathcal{U} = \mathbf{L^1}(\mathbb{R};\mathbb{R})$ and define $\mathcal{D} = \{u \in \mathcal{U} \colon \mathrm{TV}(u) \leq R\}$. Consider the Cauchy problem*

$$\begin{cases} \partial_t u + \partial_x f(u,w) = 0 & (t,x) \in [t_o,T] \times \mathbb{R}, \\ u(t_o,x) = u_o(x) & x \in \mathbb{R} \end{cases} \tag{38}$$

under the assumptions

(CL1) *For all $w \in \mathcal{W}$, the map $u \mapsto f(u,w)$ is piecewise twice continuously differentiable.*
(CL2) *There exists a positive F_L such that for all $u_1, u_2 \in \mathbb{R}$ and all $w, w_1, w_2 \in \mathcal{W}$*

$$|f(u_1,w) - f(u_2,w)| \leq F_L\,|u_1 - u_2|$$
$$\mathbf{Lip}\left(f(\cdot,w_1) - f(\cdot,w_2)\right) \leq F_L\,d_{\mathcal{W}}(w_1,w_2)$$

Then, there exists a Lipschitz Process on $\mathbf{L^1}(\mathbb{R};\mathbb{R})$, parametrized by $\mathcal{W}$, whose orbits are solutions to (36) in the sense of Definition 9, with constants in (12)–(13)–(14)

$$C_u = 0\,, \quad C_t = F_L\,R\,, \quad C_w = F_L\,R\,, \quad \mathcal{D}_t = \mathcal{D}.$$

The proof is classical and follows, for instance, from [23, Theorem 2.14 and Theorem 2.15].

Remark 1. The present treatment is limited to *homogeneous*, i.e., with a flux independent of x, conservation laws. Note that general 2×2 systems of conservation laws can *not* be approached by means of Theorem 2 while, for instance, we do comprehend a nonlocal coupling of the form

$$\begin{cases} \partial_t u + \partial_x f\left(u, \int_{\mathbb{R}} w \, \mathrm{d}x\right) = 0 \\ u(0,x) = u_o(x) \end{cases} \qquad \begin{cases} \partial_t w + \partial_x g\left(w, \int_{\mathbb{R}} u \, \mathrm{d}x\right) = 0 \\ w(0,x) = w_o(x)\,. \end{cases}$$

Proposition 10. *Set $\mathcal{U} = \mathbf{L^1}(\mathbb{R};\mathbb{R})$. Assume that* **(CL1)**–**(CL2)** *hold. Let P^u be a Lipschitz process on $\mathcal{W}$, parametrized by $u \in \mathcal{U}$. Call $P\colon \mathcal{A} \to \mathbb{R}^n \times \mathcal{W}$, with $P \equiv (P_1, P_2)$, the Process constructed in Theorem 2 coupling P^w, generated by (38), to P^u. If $([t_o, T], t_o, u_o, w_o) \subseteq \mathcal{A}$, then*

$$\begin{array}{rl} u : [t_o, T] \to & \mathbf{L^1}(\mathbb{R};\mathbb{R}) \\ t \quad \mapsto & P_1(t, t_o)(u_o, w_o) \end{array} \qquad \textit{solves} \qquad \begin{cases} \partial_t u + \partial_x \bar{f}(t,u) = 0 \\ u(t_o) = u_o, \end{cases}$$

in the sense of Definition 9, where $\bar{f}(t,u) = f\left(u, P_2(t,t_o)(u_o,w_o)\right)$.

The proof is left until Section 5.5.

4. Specific coupled problems

The abstract framework developed in Section 2, thanks to the proofs in the subsequent paragraphs, allows to prove the Lipschitz well posedness of several models.

As a first example, consider the model introduced in [5], where a large and slow vehicle positioned at $y = y(t)$ affects the overall traffic density $\rho = \rho(t,x)$. The resulting model [5, Formula (2.1)] consists in the coupling of the Lighthill–Whitham [24] and Richards [25] macroscopic model describing the evolution of ρ coupled with an ordinary differential equation for y, that is

$$\begin{cases} \partial_t \rho + \partial_x f\left(x, y(t), \rho\right) = 0 \\ \dot{y} = w\left(\rho(t,y)\right) \end{cases} \tag{39}$$

Clearly, this coupled problem fits in Theorem 2 thanks to Proposition 10 and Proposition 2, once the functions f and w meet reasonable requirements.

In the next paragraphs, we consider in particular the case of a predator–prey system (Section 4.1) and that of an epidemiological model (Section 4.2). To our knowledge, this latter well posedness is first proved here.

4.1. Predators and prey

On the basis of the games introduced in [14] we consider the following predator–prey model:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x\left(\rho\, V\left(t,x,p(t)\right)\right) = -\eta\left(\|p(t) - x\|\right)\, \rho(t,x) \\ \rho(0,x) = \bar{\rho}(x) \end{cases} \quad \text{where} \quad \begin{cases} \dot{p} = U\left(t,p,\rho(t)\right) \\ p(0) = \bar{p} \end{cases} \tag{40}$$

We consider a specific example, letting $\rho = \rho(t,x)$ be the density of some prey species moving in $\mathbb{R}^N$ and $p = p(t)$ be the position in $\mathbb{R}^N$ of a predator hunting it. To escape the predator, prey adopt a strategy defined by the speed

$$V(t,x,p) = -\frac{p - x}{\alpha + \|p - x\|^2}\, \psi\left(\|p - x\|^2\right) \tag{41}$$

where the term $\dfrac{p-x}{\alpha+\|p-x\|^2}$ stands for the escape direction of the prey. The positive term α in the denominator smooths the normalization. The function ψ describes the relevance of the predator p to the prey at x as a function of the distance $\|p-x\|$. The function $\eta=\eta\left(\|p-x\|\right)$ describes the effect of the feeding of the predator at p on the prey at x. On the other hand, the predator hunts moving towards the region of highest (mean) prey density, i.e., with speed

$$U(t,p,\rho)=\left(\nabla\varphi*\rho\right)(p)\,, \tag{42}$$

where φ is an averaging kernel.

Here, we show that (40) fits in the general framework presented in Section 2. Indeed, with reference to Section 3.2, set

$$\begin{array}{rclrclrcl} \mathcal{U} &=& \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\,, & u &=& \rho\,, & v(t,x,w) &=& V(t,x,w)\,,\\ \mathcal{W} &=& \mathbb{R}^N\,, & w &=& p\,, & m(t,x,w) &=& -\eta\left(\|w-x\|\right)\,,\\ & & & & & & q(t,x,w) &=& 0\,, \end{array} \tag{43}$$

while with reference to Section 3.1, set

$$\begin{array}{rclrcl} \mathcal{U} &=& \mathbb{R}^N\,, & u &=& p\,, \\ \mathcal{W} &=& \mathbf{L^1}(\mathbb{R}^N;\mathbb{R})\,, & w &=& \rho\,, \end{array} \qquad f(t,u,w)=U(t,u,w)\,. \tag{44}$$

Proposition 11. *Fix positive $\alpha, r_\rho, r_p, r_\eta$ and mollifiers*

(V) *Let V be as in (41) with $\psi\in\mathbf{C^\infty_c}(\mathbb{R}^N;\mathbb{R}_+)$, with $\operatorname{spt}\psi\subseteq B(0,r_\rho)$ and $\int_{B(0,r_\rho)}\psi\,\mathrm{d}\xi=1$.*

(U) *Let U be defined in (42) with $\varphi\in\mathbf{C^\infty_c}(\mathbb{R};\mathbb{R})$, positive, with $\operatorname{spt}\varphi\subseteq[-r_p,r_p]$ in (42).*

($\boldsymbol{\eta}$) *$\eta\in\mathbf{C^\infty_c}(\mathbb{R}^N;\mathbb{R})$, positive, with $\operatorname{spt}\eta\subseteq B(0,r_\eta)$.*

Then, conditions **(IP1)**–**(IP2)**–**(IP3)** *and* **(ODE1)**–**(ODE2)** *are all satisfied. Therefore, model (40) defines a unique global process in the sense of Definition 2.*

Proof. Consider first **(IP1)**. By (41), V is a smooth function and the exponential factor ensures all the required boundedness conditions. We also have that $\|\nabla_p V\|_{\mathbf{L}^\infty(\mathbb{R}_+\times\mathbb{R}^N\times\mathbb{R}^N;\mathbb{R}^{N\times N})}$ is bounded, proving the first Lipschitz requirement in **(IP1)**. Prove now the latter inequality:

$$\begin{aligned} &\int_{\mathbb{R}^N}\left|\nabla_x\cdot\left(V(t,x,p_1)-V(t,x,p_2)\right)\right|\mathrm{d}x\\ \le&\int_{B(p_1,r_p)\cup B(p_2,r_p)}\sup_{p\in\mathbb{R}^N}\|\nabla_p\nabla_x\cdot V(t,x,p)\|\,\mathrm{d}x\;\|p_2-p_1\| \end{aligned}$$

proving also the latter requirement in **(IP1)**.

To prove **(IP2)**, compute:

$$\begin{aligned} \|m(t,\cdot,w)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}+\mathrm{TV}\left(m(t,\cdot,w)\right) &= \max_{B(0,r_\eta)}|\eta|+\|\eta'\|_{\mathbf{L^1}(B(0,r_\eta);\mathbb{R})}\,;\\ \|m(t,\cdot,w_1)-m(t,\cdot,w_2)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} &\le \int_{B(w_1,r_\eta)\cup B(w_2,r_\eta)}\sup_{B(0,r_\eta)}|\eta'|\|w_2-w_1\|\,\mathrm{d}x\\ &\le \mathcal{O}(1)\,\|\eta'\|_{\mathbf{L}^\infty(B(0,r_\eta);\mathbb{R})}\,\|w_2-w_1\|\,. \end{aligned}$$

Clearly, due to (43), **(IP3)** is immediate.

The regularity required in **(ODE1)** is immediate. Pass to the Lipschitz estimate:

$$\begin{aligned}
&\|U(t,p_1,\rho_1) - U(t,p_2,\rho_2)\| \\
\leq\ & \|U(t,p_1,\rho_1) - U(t,p_1,\rho_2)\| + \|U(t,p_1,\rho_2) - U(t,p_2,\rho_2)\| \\
\leq\ & \|\nabla\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \|\rho_1 - \rho_2\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})} + \left\|\nabla^2\varphi * \rho_2\right\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^{N\times N})} \|p_1 - p_2\| \,.
\end{aligned}$$

Finally, the latter boundedness in **(ODE2)** is proved as follows:

$$\sup_{\rho\in\mathcal{D}_\rho} \|U(\cdot,\cdot,\rho)\| \quad \leq \quad \sup_{\rho\in\mathcal{D}_\rho} \|\nabla\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^N;\mathbb{R}^N)} \|\rho\|_{\mathbf{L}^1(\mathbb{R}^N;\mathbb{R})}$$

completing the proof by the definition of $\mathcal{D}_\rho$.

By Proposition 3, the balance law in (40) defines a global process P_1. Similarly, Proposition 1 ensures that the ordinary differential equation in (40) generates a global process P_2. Now, Propositions 2 and 4 ensure that the global process P obtained from P_1 and P_2 through Theorem 2 yields a solution to the coupled problem (40). □

4.2. Modeling vaccination strategies

Consider the model presented in [3, § 2]:

$$\left\{\begin{aligned}
\dot{S} &= -\rho_S\, I\, S - p(t) \\
\partial_t V + \partial_\tau V &= -\rho_V\, I\, V \\
\dot{I} &= (\rho_S\, S + \textstyle\int_0^{T_*} \rho_V\, V) I - \vartheta\, I - \mu\, I \\
\dot{R} &= \vartheta\, I + V(t,T_*) \\
V(t,0) &= p(t)\,.
\end{aligned}\right. \tag{45}$$

It describes a population consisting of susceptibles, $S = S(t)$, of infected that are also infective, $I = I(t)$, and recovered individuals, $R = R(t)$. The vaccination rate is $p = p(t)$ and vaccinated individuals need a time T_* to get immunized. More precisely, $V = V(t,\tau)$ is the number of individuals at time t vaccinated at time $t - \tau$, for $\tau \in [0, T_*]$. Thus, at time T_*, vaccinated individual enter the R population.

The positive constants ρ_S, ϑ and μ quantify the infectivity rate, the recovery rate and the mortality rate, respectively. The function $\rho_V = \rho_V(\tau)$ describes the infectivity rate of individuals vaccinated after time τ from being dosed.

Note that model (45) is triangular, in the sense that the evolution of the R population results from that of the other ones, without affecting them.

Model (45), once the R population is omitted, fits in the abstract framework presented in Section 2. Indeed, with reference to the notation used in Section 3.1, we pose

$$\begin{gathered}
\mathcal{U} = \mathbb{R}^2\,, \qquad \mathcal{W} = \mathbf{L}^1([0,T_*];\mathbb{R})\,, \qquad u = \begin{bmatrix} S \\ I \end{bmatrix}, \qquad w = V\,, \\
f(t,u,w) = \begin{bmatrix} -\rho_S\, u_1\, u_2 - p(t) \\ \left(\rho_S\, u_1 + \int_0^{T_*} \rho_V(\tau)\, w(\tau)\, \mathrm{d}\tau - \vartheta - \mu\right) u_2 \end{bmatrix},
\end{gathered} \tag{46}$$

while with reference to Section 3.3, we set

$$\begin{aligned}
&\mathcal{U} = \mathbf{L}^1([0,T_*];\mathbb{R}) \\
&\mathcal{W} = \mathbb{R}^2
\end{aligned} \qquad x = \tau\,, \quad u = V\,, \quad w = \begin{bmatrix} S \\ I \end{bmatrix}, \qquad \begin{aligned}
v(t,x) &= 1 \\
m(t,x,w) &= -\rho_V(x)\, w_2 \\
q(t,x,w) &= 0 \\
b(t) &= p(t)\,.
\end{aligned} \tag{47}$$

The well posedness of (45) now follows once we verify that Proposition 2 and Proposition 6 can be applied.

Proposition 12. *Fix positive r, T_*, ρ_S and choose $p \in \mathbf{BV}(\mathbb{R}_+;\mathbb{R})$, $\rho_V \in \mathbf{BV}([0,T_*];\mathbb{R})$. Then, problem (45) defines a unique global process P, in the sense of Definition 2, defined on all initial data*

$$S_o, I_o, R_o \in [0,r] \quad \text{and} \quad V_o \in \mathbf{L^1}([0,T_*];\mathbb{R}_+) \text{ with } \mathrm{TV}(V_o) + \|V_o\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq r\,. \tag{48}$$

P is Lipschitz continuous as a function of time and of the initial data, with respect to the Euclidean norm in (S_o, I_o, R_o) and to the $\mathbf{L^1}$ norm in V.

Proof. Verifying **(ODE1)** is immediate. The Lipschitz continuity required in **(ODE2)** follows from the boundedness $u \in \mathcal{D}_\mathcal{U}$, which is a closed ball in $\mathcal{U} = \mathbb{R}^2$ and from the choice of ρ_V, see Section 3.1. Hence, Proposition 1 applies.

Conditions **(BP1)** and **(BP3)** are immediate. The first requirement in **(BP2)** follows from the choice of ρ_V and the boundedness of $\mathcal{D}_\mathcal{U}$. The second is ensured by the linearity of m and the boundedness of ρ_V. Since p has bounded variation, **(BP4)** is satisfied on any bounded time interval. Hence, also Proposition 5 can be applied.

Then, Proposition 2 and Proposition 6, through Theorem 2, ensure the well posedness of the coupled system (46)–(47).

We now verify the well posedness of the R component. From (45), using (77), we have

$$V(t,\tau) = \begin{cases} V_o(\tau + t_o - t)\, \exp\left(-\int_{t_o}^t \rho_V(s)\, I(s)\, \mathrm{d}s\right) & \text{if } t \leq \tau + t_o\,, \\ p(t-\tau)\, \exp\left(-\int_{t-\tau}^t \rho_V(s)\, I(s)\, \mathrm{d}s\right) & \text{if } t > \tau + t_o\,. \end{cases}$$

This shows that the map $t \mapsto V(t,T_*)$ is sufficiently regular for the equation for R, namely $\dot{R} = \vartheta\, I(t) + V(t,T_*)$, to be explicitly solved: $R(t) = R_o + \int_0^t (I(s) + V(s,T_*))\, \mathrm{d}s$. Thus, the full model (45) is well posed. □

5. Technical details

5.1. Proofs for Section 2

Proof of Theorem 2. We begin by showing F is a local flow in the sense of Definition 1. F is continuous as it is a pairing of two continuous functions. Further

$$F(0,t_o)(u,w) = (P^w(t_o,t_o)u, P^u(t_o,t_o)w) = (u,w)\,.$$

We prove the Lipschitz continuity in time and with respect to initial conditions of F:

$$\begin{aligned}
& d\left(F(\tau_1,t_o)(u_1,w_1), F(\tau_2,t_o)(u_2,w_2)\right) \\
\leq\ & d_\mathcal{U}\left(P^{w_1}(t_o+\tau_1,t_o)u_1, P^{w_1}(t_o+\tau_1,t_o)u_2\right) + d_\mathcal{U}\left(P^{w_1}(t_o+\tau_1,t_o)u_2, P^{w_2}(t_o+\tau_1,t_o)u_2\right) \\
& + d_\mathcal{U}\left(P^{w_2}(t_o+\tau_1,t_o)u_2, P^{w_2}(t_o+\tau_2,t_o)u_2\right) \\
& + d_\mathcal{W}\left(P^{u_1}(t_o+\tau_1,t_o)w_1, P^{u_1}(t_o+\tau_1,t_o)w_2\right) + d_\mathcal{W}\left(P^{u_1}(t_o+\tau_1,t_o)w_2, P^{u_2}(t_o+\tau_1,t_o)w_2\right) \\
& + d_\mathcal{W}\left(P^{u_2}(t_o+\tau_1,t_o)w_2, P^{u_2}(t_o+\tau_2,t_o)w_2\right) \\
\leq\ & (e^{C_u\delta} + C_w\,\delta)\; d\left((u_1,w_1),(u_2,w_2)\right) + 2\,C_t\,|\tau_1 - \tau_2|\,.
\end{aligned}$$

Thus F is indeed a local flow in the sense of Definition 1, with $\mathbf{Lip}(F) = e^{C_u\delta} + C_w\,\delta + 2C_t$.

We now show that F satisfies the assumptions of Theorem 1. Consider (9):

$$\begin{aligned} & d\left(F(k\tau, t_o+\tau)\circ F(\tau,t_o)(u,w), F((k+1)\tau,t_o)(u,w)\right) \\ = \; & d_{\mathcal{U}}\left(P^{P^u(t_o+\tau,t_o)w}(t_o+(k+1)\tau, t_o+\tau)P^w(\tau,t_o)u, P^w\left(t_o+(k+1)\tau,t_o\right)u\right) \qquad (49) \\ + \; & d_{\mathcal{W}}\left(P^{P^w(t_o+\tau,t_o)u}(t_o+(k+1)\tau, t_o+\tau)P^u(t_o+\tau,t_o)w, P^u\left(t_o+(k+1)\tau,t_o\right)w\right). \qquad (50) \end{aligned}$$

We consider only the term (49), since the latter is entirely similar. By (6), we have

$$P^w\left(t_o+(k+1)\tau,t_o\right)u = P^w(t_o+(k+1)\tau, t_o+\tau)\; P^w(t_o+\tau,t_o)u,$$

hence, via (13) and (14),

$$\begin{aligned} & d_{\mathcal{U}}\left(P^{P^u(t_o+\tau,t_o)w}(t_o+(k+1)t, t_o+\tau)P^w(t_o+\tau,t_o)u, P^w\left(t_o+(k+1)\tau,t_o\right)u\right) \\ \leq \; & k\,\tau\, C_t\, C_w \tau\,. \end{aligned} \qquad (51)$$

Combining (51) with the analogous estimate bounding (50), we end up with

$$d\left(F(k\tau,t_o+\tau)\circ F(\tau,t_o)(u,w), F\left((k+1)\tau,t_o\right)(u,w)\right) \leq k\,\tau\,\omega(\tau)$$

where ω is as in (16). Thus (9) is satisfied.

We consider the second condition in Theorem 1, namely (10). Note that Euler polygonals for the local flow F, see Definition 3, can be written recursively, as

$$F^\varepsilon(\tau,t_o)(u,w) = F(\tau-k\varepsilon, t_o+k\varepsilon)\circ F^\varepsilon(k\varepsilon,t_o)(u,w).$$

For any $\tau\in[0,\delta]$ and for any (u,w), $(\bar u,\bar w)$ in $\mathcal{U}\times\mathcal{W}$, we have

$$\begin{aligned} d\left(F(\tau,t_o)(u,w), F(\tau,t_o)(\bar u,\bar w)\right) = \; & d_{\mathcal{U}}(P^w(t_o+\tau,t_o)u, P^{\bar w}(t_o+\tau,t_o)\bar u) \\ & + d_{\mathcal{W}}(P^u(t_o+\tau,t_o)w, P^{\bar u}(t_o+\tau,t_o)\bar w)\,. \end{aligned}$$

For the first of these summands, by the triangle inequality, we have

$$d_{\mathcal{U}}\left(P^w(t_o+\tau,t_o)u, P^{\bar w}(t_o+\tau,t_o)\bar u\right) \leq e^{C_u\tau}\, d_{\mathcal{U}}(u,\bar u) + C_w\,\tau\, d_{\mathcal{W}}(w,\bar w).$$

The second term is estimated analogously, leading to

$$d\left(F(\tau,t_o)(u,w), F(\tau,t_o)(\bar u,\bar w)\right) \leq \left(e^{C_u\tau}+C_w\,\tau\right)\, d\left((u,w),(\bar u,\bar w)\right)\,. \qquad (52)$$

Estimate (52) is of use in the following:

$$\begin{aligned} & d\left(F^\varepsilon(\tau,t_o)(u,w), F^\varepsilon(\tau,t_o)(\bar u,\bar w)\right) \\ \leq \; & \left(e^{C_u(\tau-k\varepsilon)}+C_w\,(\tau-k\varepsilon)\right)\, d\left(F^\varepsilon(k\varepsilon,t_o)(u,w), F^\varepsilon(k\varepsilon,t_o)(\bar u,\bar w)\right)\,. \end{aligned}$$

It remains to estimate the distance in the latter right hand side. We have for any $k\in\mathbb{N}\setminus\{0\}$,

$$F^\varepsilon(k\varepsilon,t_o)(u,w) = F(\varepsilon,t_o)\; F^\varepsilon\left((k-1)\varepsilon,t_o\right)(u,w),$$

and thus using iteratively (52),

$$d\left(F^\varepsilon(k\varepsilon,t_o)(u,w), F^\varepsilon(k\varepsilon,t_o)(\bar u,\bar w)\right) \leq \left(e^{C_u\varepsilon}+C_w\,\varepsilon\right)^k d\left((u,w),(\bar u,\bar w)\right).$$

Therefore,

$$\begin{aligned} & d\left(F^{\varepsilon}(\tau,t_o)(u,w),F^{\varepsilon}(\tau,t_o)(\bar{u},\bar{w})\right) \\ \leq & \left(e^{C_u(\tau-k\varepsilon)}+C_w\left(\tau-k\varepsilon\right)\right)\left(e^{C_u\varepsilon}+C_w\,\varepsilon\right)^k\ d\left((u,w),(\bar{u},\bar{w})\right). \end{aligned}$$

Hence, (10) is satisfied provided there exists a positive L such that for all $\varepsilon>0$ and $t\in[0,T]$

$$\left(e^{C_u(\tau-k\varepsilon)}+C_w\left(\tau-k\varepsilon\right)\right)\left(e^{C_u\varepsilon}+C_w\,\varepsilon\right)^k\leq L,$$

where $k=\lfloor\frac{\tau}{\varepsilon}\rfloor$. Indeed, since $e^a+b\leq e^{a+b}$ for all $a,b\in\mathbb{R}_+$, we have

$$\left(e^{C_u(\tau-k\varepsilon)}+C_w(\tau-k\varepsilon)\right)\left(e^{C_u\varepsilon}+C_w\,\varepsilon\right)^k\leq e^{(C_u+C_w)(\tau-k\varepsilon)}\left(e^{(C_u+C_w)\varepsilon}\right)^k=e^{(C_u+C_w)\tau}$$

so that $L=e^{(C_u+C_w)\delta}$.

Finally, note that (17) directly follows from the definition (15) of F, together with the properties $P^w(t_o+\tau,t_o)\mathcal{D}^{\mathcal{U}}_{t_o}\subseteq\mathcal{D}^{\mathcal{U}}_{t_o+\tau}$, which holds for all $w\in\mathcal{W}$, and $P^u(t_o+\tau,t_o)\mathcal{D}^{\mathcal{W}}_{t_o}\subseteq\mathcal{D}^{\mathcal{W}}_{t_o+\tau}$, which holds for all $u\in\mathcal{U}$. Therefore, with reference to (8), we have $\mathcal{D}^3_{t_o}\supseteq(\mathcal{D}^{\mathcal{U}}_{t_o}\times\mathcal{D}^{\mathcal{W}}_{t_o})$ and Condition 1. in Theorem 1 completes the proof of (18).

Proof of Theorem 3. The continuity of $\hat{F}$ is immediate. The Lipschitz continuity follows from the triangle inequality and a Lipschitz constant is $\mathbf{Lip}(\hat{F})=\mathcal{L}+\max\left\{\mathbf{Lip}(F^w)\,,\ \mathbf{Lip}(F^u)\right\}$. Hence, $\hat{F}$ is a local flow according to Definition 1.

Concerning the tangency condition, compute

$$\begin{aligned} \frac{1}{\tau}\,d\left(\hat{F}(\tau,t_o)(u,w),F(\tau,t_o)(u,w)\right)=&\ \frac{1}{\tau}\,d_{\mathcal{U}}\left(F^w(\tau,t_o)u,P^w(t_o+\tau,t_o)u\right)\\ &+\frac{1}{\tau}\,d_{\mathcal{W}}\left(F^u(\tau,t_o)w,P^u(t_o+\tau,t_o)w\right) \end{aligned}$$

and the first order tangency condition (11) allows to complete the proof. □

5.2. Proofs for Section 3.2

With reference to (23) and (26), introduce for $\bar{t},t\in\hat{I}$ and $\bar{x},x\in\mathbb{R}_+$ the characteristics

$$t\mapsto\mathcal{X}(t;\bar{t},\bar{x})\text{ solves }\begin{cases}\dot{x}=v(t,x,w)\\ x(\bar{t})=\bar{x},\end{cases}\quad\text{and }t\mapsto\mathcal{T}(x;\bar{x},\bar{t})\text{ solves }\begin{cases}t'=1/v(t,x,w)\\ t(\bar{x})=\bar{t}\,,\end{cases}\tag{53}$$

and in the sequel we omit the dependence on w. The $\mathcal{T}$ characteristics are introduced now for completeness, but used only in Section 5.3. As is well known, see for instance [26, Lemma 5] and the references therein, the unique solution to (23) is

$$u(t,x)=u_o\left(\mathcal{X}(t_o;t,x)\right)\,\mathcal{E}_w(t_o,t,x)+\int_{t_o}^{t}q\left(s,\mathcal{X}(s;t,x),w\right)\,\mathcal{E}_w(s,t,x)\,\mathrm{d}s\tag{54}$$

where the characteristics $\mathcal{X}$ are defined by (53) and

$$\mathcal{E}_w(\tau,t,x)=\exp\int_{\tau}^{t}\left(m\left(s,\mathcal{X}(s;t,x),w\right)-\mathrm{div}v\left(s,\mathcal{X}(s;t,x)\right)\right)\mathrm{d}s\,.$$

Below, we often use the substitution $y\leftrightarrow x$, where

$$y=\mathcal{X}(t;t_o,x)\quad\text{ with Jacobian }\quad J(t,y)=\exp\left(\int_t^{t_o}\nabla\cdot v\left(s,\mathcal{X}(s;\tau,y)\right)\mathrm{d}s\right),\tag{55}$$

for more details see for instance [26, Proof of Proposition 3].

Lemma 1. *Assume* **(IP1)** *holds and use the notation* (53). *Let* $u \in (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R}^n;\mathbb{R})$. *Then, for all* $t_o, t \in \hat{I}$

$$\int_{\mathbb{R}^n} |u(\mathcal{X}(t;t_o,x)) - u(x)| \, \mathrm{d}x \leq \frac{V_\infty}{V_L} \left(e^{V_L|t-t_o|} - 1\right) \mathrm{TV}(u) \,. \tag{56}$$

This Lemma is an extension of [20, Lemma 2.3] to $\mathbb{R}^n$. For the proof, refer to [19].

Define the parametrized mapping P^w by

$$P^w : \begin{array}{ccc} \mathcal{A} & \to & \mathcal{U} \\ (t,t_o,u_o) & \mapsto & u(t) \end{array} \quad \text{where} \quad u(t) \text{ is given by (54);} \tag{57}$$

Below, by **(IP1)** and **(IP2)**, for all $t, \tau \in \hat{I}$, $x \in \mathbb{R}^n$ and $w \in \mathcal{W}$, we use the uniform estimate

$$0 \leq \mathcal{E}_w(\tau,t,x) \leq e^{(M_\infty + V_L)|t-\tau|} \,. \tag{58}$$

Lemma 2. *For all* $w \in \mathcal{W}$, P^w *in* (57) *is a global process according to Definition 2.*

Proof of Lemma 2. That P^w satisfies (4) is an immediate consequence of its definition (54). The uniqueness of the solution ensures that (6) is satisfied.

Fix $t_o, t \in I$, with $t_o \leq t$, and $r_o \in \mathcal{D}_{t_o}$. It remains to show (5), that is, $u(t) = P^w(t,t_o)u_o \in \mathcal{D}_t$ for each $w \in \mathcal{W}$.

1. We begin by showing that, if $\|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_1(t_o)$, then $\|u(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_1(t)$. Making use of **(IP2)**–**(IP3)**–(25) –(54)–(55), see also [26, Proposition 3, **(H3)**],

$$\begin{aligned} & \|u(t)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \\ \leq & \left(\|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} + \|q(\cdot,\cdot,w)\|_{\mathbf{L^1}([t_o,t]\times\mathbb{R}^n;\mathbb{R})}\right) \exp\left(\int_{t_o}^t \|m(\tau,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \, \mathrm{d}\tau\right) \\ \leq & \left(\alpha_1(t_o) + Q_1(t-t_o)\right) e^{M_\infty(t-t_o)} \\ \leq & \, \alpha_1(t) \,, \end{aligned} \tag{59}$$

   as required.
2. Assuming now that $\|u_o\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_\infty(t_o)$, we show that $\|u(t)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \leq \alpha_\infty(t)$, We use (25)–(54), see also [26, Proposition 3, **(H4)**], together with **(IP1)**, **(IP2)**, **(IP3)** and (58). Then,

$$\begin{aligned} \|u(t)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} & \leq \left(\|u_o\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + Q_\infty(t-t_o)\right) e^{(M_\infty+V_L)(t-t_o)} \\ & \leq \left(\alpha_\infty(t_o) + Q_\infty(t-t_o)\right) e^{(M_\infty+V_L)(t-t_o)} \\ & \leq \alpha_\infty(t) \,, \end{aligned}$$

   as required.
3. Finally, we show that, if $u_o \in \mathcal{D}_{t_o}$, then $\mathrm{TV}\left(u(t)\right) \leq \alpha_{TV}(t)$. We use **(IP1)**–**(IP2)**–**(IP3)**–(25) –(54)–(55)–(58), see also [26, Formula (31)]:

$$\begin{aligned} \mathrm{TV}\left(u(t)\right) \leq \Bigg[& \mathrm{TV}(u_o) + \int_{t_o}^t \mathrm{TV}\left(q(s,\cdot,w)\right) \mathrm{d}s \\ & + \left(\|u_o\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} + \int_{t_o}^t \|q(s,\cdot,w)\|_{\mathbf{L^\infty}(\mathbb{R}^n;\mathbb{R})} \, \mathrm{d}s\right) \\ & \times \int_{t_o}^t \left(\mathrm{TV}\left(m(s,\cdot,w)\right) + \|\nabla\nabla\cdot v(s)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}^n)}\right) \mathrm{d}s \Bigg] \, e^{(M_\infty+V_L)|t-\tau|} \end{aligned} \tag{60}$$

Since $u_o \in \mathcal{D}_{t_o}$, by (24), $\mathrm{TV}(u_o) \leq \alpha_{\mathrm{TV}}(t_o)$ and we have that (60) becomes

$$\begin{aligned}
&\mathrm{TV}\left(u(t)\right)\\
\leq &\Bigg[\alpha_{\mathrm{TV}}(t_o) + Q_\infty(t-t_o)\\
&+\left(Re^{-(M_\infty+V_L)(T-t_o)} - Q_\infty e^{(M_\infty+V_L)t_o}(T-t_o) + Q_\infty(t-t_o)\right)(M_\infty+V_1)(t-t_o)\Bigg]\\
&\quad\times e^{(M_\infty+V_L)(t-t_o)}\\
\leq &\,\alpha_{\mathrm{TV}}(t)\,,
\end{aligned}$$

completing the proof of (5). □

Proof of Proposition 3. We define the mapping P^w by (57). That this defines a process is a consequence of Lemma 2.

It remains to show the three Lipschitz continuity estimates (12), (13), and (14).

1. Lipschitz continuity w.r.t initial data. By the linear structure of (23), from (59) we immediately have

$$\|P^w(t,t_o)(u_o-\bar{u}_o)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})} \leq e^{M_\infty(t-t_o)}\,\|u_o-\bar{u}_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}$$

which is compatible with the choice of C_u in (24).

2. Lipschitz continuity in time. By direct computations based on (54), for $t \geq t_o$:

$$\begin{aligned}
&\|P^w(t,t_o)u_o - u_o\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})}\\
\leq &\int_{\mathbb{R}^n} |u_o\left(\mathcal{X}(t_o;t,x)\right) - u_o(x)|\,\mathcal{E}_w(t_o,t,x)\,\mathrm{d}x\\
&+\int_{\mathbb{R}^n}\int_{t_o}^t |q\left(\tau,\mathcal{X}(\tau;t,x),w\right)|\,\mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x + \int_{\mathbb{R}^n}|u_o(x)|\,|\mathcal{E}_w(t_o,t,x)-1|\,\mathrm{d}x
\end{aligned}$$

and we consider the latter three terms separately. First, use (58) and Lemma 1, for $t \geq t_o$,

$$\begin{aligned}
\int_{\mathbb{R}^n} |u_o\left(\mathcal{X}(t_o;t,x)\right) - u_o(x)|\mathcal{E}_w(t_o,t,x)\,\mathrm{d}x &\leq \int_{\mathbb{R}^n} |u_o\left(\mathcal{X}(t_o;t,x)\right) - u_o(x)|\,\mathrm{d}x\; e^{(M_\infty+V_L)(t-t_o)}\\
&\leq V_\infty\,\mathrm{TV}(u_o)\,e^{(M_\infty+2V_L)(t-t_o)}\,(t-t_o)\,.
\end{aligned}$$

To deal with the second term, after using the coordinates (55) and **(IP2)**–**(IP3)**, one finds

$$\int_{\mathbb{R}^n}\int_{t_o}^t |q\left(\tau,\mathcal{X}(\tau;t,x),w\right)|\,\mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \leq Q_1\,e^{M_\infty(t-t_o)}\,(t-t_o).$$

Finally, the third term follows by (58),

$$\int_{\mathbb{R}^n}|u_o(x)|\,|\mathcal{E}_w(t_o,t,x)-1|\,\mathrm{d}x \leq (M_\infty+V_L)\,\|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\,e^{(M_\infty+V_L)(t-t_o)}(t-t_o).$$

Adding up, we have

$$\begin{aligned}
\|P^w(t,t_o)u_o - u_o\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq\; & V_\infty\,\mathrm{TV}(u_o)\,e^{(M_\infty+2V_L)(t-t_o)}\,(t-t_o)\\
&+Q_1\,e^{M_\infty(t-t_o)}\,(t-t_o)\\
&+(M_\infty+V_L)\,\|u_o\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\,e^{(M_\infty+V_L)(t-t_o)}(t-t_o)
\end{aligned}$$

which agrees with the choice of C_t in (24).

3. Lipschitz continuity w.r.t parameters. Thanks to **(IP1)**, **(IP2)**, and **(IP3)**, the necessary computations are a consequence of [26, (H5)].
4. Choice of T. The time T has to be chosen so that $\alpha_1(0) > 0$, $\alpha_\infty(0) > 0$ and $\alpha_{\mathrm{TV}}(0) > 0$. Clearly, by (25), for T sufficiently small, these requirements are all met.

Proof of Corollary 2. Note that the constants defined in **(IP1)**, **(IP2)**, and **(IP3)** do not depend on R. Moreover T has to be chosen such that $\alpha_1(0) > 0$, $\alpha_\infty(0) > 0$ and $\alpha_{\mathrm{TV}}(0) > 0$, which are equivalent to

$$\begin{cases} Re^{-M_\infty T} - Q_1 T > 0 \\ Re^{-(M_\infty + V_L)T} - Q_\infty T > 0 \\ Re^{-(M_\infty + V_L)T}\left(1 - (M_\infty + V_1)\,T\right) - Q_\infty T > 0. \end{cases}$$

The proof ends setting $T = \min\left\{\frac{1}{2(M_\infty+V_1)}, \frac{\ln(2)}{M_\infty+V_L}\right\}$, provided R is sufficiently big. □

Proof of Proposition 4. The Lipschitz continuity of P ensured by Theorem 2 shows that P_1 is $\mathbf{L^1}$–Lipschitz continuous, and hence in $\mathbf{C^0}([t_o, T]; \mathbf{L^1}(\mathbb{R}^n; \mathbb{R}))$ as required.

We focus our attention now on the first item in Definition 6, the second being immediate. To ease reading, for any test function $\varphi \in \mathbf{C^\infty_c}(]t_o, T[\times \mathbb{R}^n; \mathbb{R})$ we introduce the notation

$$\mathcal{I}_\varphi(u, w) = u\,\partial_t\varphi + u\,v \cdot \nabla_x\varphi + \left(m(\cdot,\cdot,w)\,u + q(\cdot,\cdot,w)\right)\varphi. \tag{61}$$

We want to prove that, for any $\varphi \in \mathbf{C^\infty_c}(]t_o, T[\times \mathbb{R}^n; \mathbb{R})$,

$$\int_{\mathbb{R}^n}\int_{t_o}^{T} \mathcal{I}_\varphi\left(P(t, t_o)(u_o, w_o)\right) \mathrm{d}t\ \mathrm{d}x = 0.$$

We begin by discretizing the time domain. For a given $k \in \mathbb{N} \setminus \{0\}$ and $i = 0, \ldots, k$, introduce $t_i = t_o + i(T - t_o)/k$ and $(\tilde{u}_i, \tilde{w}_i) = P(t_{i-1}, t_o)(u_o, w_o)$. Splitting the integral then gives

$$\begin{aligned} &\int_{t_o}^{T}\int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(P(t, t_o)(u_o, w_o)\right) \mathrm{d}x\ \mathrm{d}t \\ &= \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \left(\mathcal{I}_\varphi\left(P(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) - \mathcal{I}_\varphi\left(F(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right)\right) \mathrm{d}x\ \mathrm{d}t \\ &\quad + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n} \mathcal{I}_\varphi\left(F(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) \mathrm{d}x\ \mathrm{d}t\ . \end{aligned} \tag{62}$$

We compute the terms on the last two lines separately, our goal is to show that they both converge to zero as $k \to \infty$.

For the first,

$$\begin{aligned} &\mathcal{I}_\varphi\left(P(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) - \mathcal{I}_\varphi\left(F(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) \\ &= \partial_t\varphi\left(P_1(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i) - F_1(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) && (63) \\ &\quad + \Big(\, P_1(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i) v(t, x, P_2(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i)) \\ &\qquad - F_1(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i) v(t, x, F_2(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i))\,\Big) \cdot \nabla_x\varphi && (64) \\ &\quad + \Big(\, m\left(t, x, P_2(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) P_1(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i) && (65) \\ &\qquad - m\left(t, x, F_2(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) F_1(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\,\Big)\,\varphi && (66) \\ &\quad + \left(q\left(t, x, P_2(t, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right) - q\left(t, x, F_2(t - t_{i-1}, t_{i-1})(\tilde{u}_i, \tilde{w}_i)\right)\right)\varphi. && (67) \end{aligned}$$

Recall that the tangency condition (11) ensures

$$\frac{1}{t-t_{i-1}}\left\|P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\leq\frac{2L}{\ln(2)}\int_0^{t-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi$$
$$\frac{1}{t-t_{i-1}}\,d_{\mathcal{W}}\left(P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i),F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\leq\frac{2L}{\ln(2)}\int_0^{t-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi$$

with L and ω defined as in (16), so that, considering (63),

$$\begin{aligned}&\left|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}(\partial_t\varphi\ )\left(P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\mathrm{d}x\ \mathrm{d}t\right|\\&\leq\frac{L}{\ln(2)}\left\|\partial_t\varphi\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R})}(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,.\end{aligned}\tag{68}$$

Considering the next term (64),

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\big[P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)v(t,x,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i))\\&\qquad-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)v(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i))\big]\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\\&=\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\big[P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\big]\qquad(69)\\&\qquad\times v(t,x,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i))\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\\&\quad+\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\\&\qquad\big[v(t,x,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i))-v(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i))\big]\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\,.\qquad(70)\end{aligned}$$

For (69), using **(IP1)** and the same approach as for (68), we get

$$\begin{aligned}&\left|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\big[P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\big]v(t,x,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i))\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\right|\\&\leq\frac{L}{\ln(2)}V_\infty\left\|\nabla_x\varphi\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R}^n)}(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,.\end{aligned}\tag{71}$$

For the second term (70), using **(IP1)** again, we have,

$$\begin{aligned}&\Big|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\\&\qquad\times\big[v(t,x,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i))-v(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i))\big]\cdot\nabla_x\varphi\,\mathrm{d}t\,\mathrm{d}x\Big|\\&\leq\frac{L}{\ln(2)}R\|\nabla_x\varphi\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}^n;\mathbb{R}^n)}V_L(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,.\end{aligned}\tag{72}$$

Pass to (65)–(66) and using again (63):

$$\begin{aligned}&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\Big|\ \big(\ m\left(t,x,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)\\&\qquad-m\left(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\ \big)\ \varphi\Big|\,\mathrm{d}x\,\mathrm{d}t\\&\leq\int_{t_{i-1}}^{t_i}\left\|m\left(t,\cdot,P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)-m\left(t,\cdot,F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\right\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\\&\qquad\times\|P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&+\int_{t_{i-1}}^{t_i} \|m\left(t,\cdot,F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \\
&\qquad \times \|P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}t \\
&\le M_L\,R\,\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\int_{t_{i-1}}^{t_i} d_{\mathcal{W}}\left(P_2(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i),F_2(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\mathrm{d}t \\
&\quad +M_\infty\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\int_{t_{i-1}}^{t_i}\|P_1(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)-F_1(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\|_{\mathbf{L^1}(\mathbb{R}^n;\mathbb{R})}\,\mathrm{d}t \\
&\le \frac{L}{\ln(2)}\left(M_L\,R+M_\infty\right)\|\varphi\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})}\,(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,. \qquad (73)
\end{aligned}
$$

Concerning (67), the tangency condition (11) implies

$$
\begin{aligned}
&\left|\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}[q(t,x,P_2(t,t_{i-1})(\tilde{u},\tilde{w}))-q(t,x,F_2(t-t_{i-1},t_{i-1})(\tilde{u},\tilde{w}))]\varphi(t)\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\le \frac{L}{\ln(2)}\,Q_L\,\|\varphi\|_{L^\infty([t_o,T]\times\mathbb{R}^n)}(t_i-t_{i-1})^2\int_0^{t_i-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,. \qquad (74)
\end{aligned}
$$

Computing the sum over all time intervals, we get:

$$
\begin{aligned}
&\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\left(\mathcal{I}_\varphi\left(P(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)-\mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\right)\mathrm{d}x\;\mathrm{d}t \\
\le\;& \sum_{i=1}^k[(68)]+[(71)]+[(72)]+[(73)]+[(74)] \\
\le\;& \frac{L}{\ln(2)}\,\mathcal{C}\int_0^{(T-t_o)/k}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\;\frac{(T-t_o)^2}{k} \\
\underset{k\to+\infty}{\longrightarrow}\;& 0\,,
\end{aligned}
$$

where $\mathcal{C}$ depends on the test function φ and the constants from **(IP1)**–**(IP2)**–**(IP3)**.

Pass now to estimate (62). Temporarily, for $i=0,\dots,k$, define $(u_i(t),w_i(t))=F(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)$. Then $u_i(t)=P^{\tilde{w}_i}(t,t_{i-1})\tilde{u}_i$, and thus it satisfies

$$
\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\mathcal{I}_\psi(u_i(t),\tilde{w}_i)\,\mathrm{d}x\,\mathrm{d}t=0 \qquad \forall\,\psi\in\mathbf{C}^\infty_{\mathbf{c}}(]t_{i-1},t_i[\times\mathbb{R}^n;\mathbb{R})\,. \qquad (75)
$$

Then, each summand in (62) can be estimated as follows:

$$
\begin{aligned}
&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde{u}_i,\tilde{w}_i)\right)\mathrm{d}x\,\mathrm{d}t \\
=&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\mathcal{I}_\varphi\left(u_i(t),\tilde{w}_i\right)\mathrm{d}x\,\mathrm{d}t \\
+&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\left[\left(m(t,x,\tilde{w}_i)-m(t,x,w_i(t))\right)u_i(t)+\left(q(t,x,\tilde{w}_i)-q(t,x,w_i(t))\right)\right]\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
+&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}u_i(t)\left(v(t,x,w_i(t))-v(t,x,\tilde{w}_i)\right)\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t \\
\le&\int_{t_{i-1}}^{t_i}\int_{\mathbb{R}^n}\mathcal{I}_\varphi\left(u_i(t),\tilde{w}_i\right)\mathrm{d}x\,\mathrm{d}t+\|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n;\mathbb{R}^n)}\frac{1}{2}(M_L\,R+Q_L)\,\mathcal{C}\,(t_i-t_{i-1})^2 \\
+&\|\nabla_x\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R}^n;\mathbb{R}^n)}\,\frac{1}{2}\,V_L\,R\,\mathcal{C}\,(t_i-t_{i-1})^2\,, \qquad (76)
\end{aligned}
$$

where $\mathcal{C}$ is the Lipschitz constant of $t \mapsto w(t)$ and we used the equality $w(t_{i-1}) = \tilde{w}_i$. The latter two summands in (76) are treated as the terms above.

It can be shown that

$$\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\varphi \left(u_i(t), \tilde{w}_i\right) \mathrm{d}x \, \mathrm{d}t = \int_{\mathbb{R}^n} \left(u_i(t_i, x)\ \varphi(t_i, x) - u_i(t_{i-1}, x)\ \varphi(t_{i-1}, x)\right) \mathrm{d}x \, \mathrm{d}t \, .$$

refer to [19] for more details.

Passing to the sum (62), and remembering that $u_i(t_{i-1}, x) = \tilde{u}_i = P_1(t_{i-1}, t_o)(u_o, w_o)$,

$$\begin{aligned}
\sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}^n} \mathcal{I}_\varphi \left(u_i(t), \tilde{w}_i\right) \mathrm{d}x \, \mathrm{d}t \quad &\leq \quad \sum_{i=1}^{k-1} (t_i - t_{i-1}) \frac{2L}{\ln(2)} \int_0^{t_i - t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \ \left\| \varphi(t_i) \right\|_{\mathbf{L}^\infty(\mathbb{R}^n;\mathbb{R})} \\
&\leq \quad \frac{2L}{\ln(2)} \left\| \varphi \right\|_{\mathbf{L}^\infty([t_o,T];\mathbb{R}^n;\mathbb{R})} (T - t_o) \int_0^{(T-t_o)/k} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \\
&\underset{k \to +\infty}{\longrightarrow} \ 0,
\end{aligned}$$

as required. □

5.3. Proofs for Section 3.3

Similar to the previous sections, for each $w \in \mathcal{W}$ the unique solution to (26) in the sense of Definition 7 is

$$u(t,x) = \begin{cases} u_o\left(\mathcal{X}(t_o; t, x)\right) \mathcal{E}_w(t_o, t, x) & \\ \qquad + \displaystyle\int_{t_o}^{t} q\left(\tau, \mathcal{X}(\tau; t, x), w\right) \mathcal{E}_w(\tau, t, x) \, \mathrm{d}\tau & x \geq \mathcal{X}(t; t_o, 0) \\ b\left(\mathcal{T}(0; t, x)\right) \mathcal{E}_w\left(\mathcal{T}(0; t, x), t, x\right) & \\ \qquad + \displaystyle\int_{\mathcal{T}(0;t,x)}^{t} q\left(\tau, \mathcal{X}(\tau; t, x), w\right) \mathcal{E}_w(\tau, t, x) \, \mathrm{d}\tau & x < \mathcal{X}(t; t_o, 0) \end{cases} \tag{77}$$

where now

$$\mathcal{E}_w(\tau, t, x) = \exp \int_\tau^t \left(m\left(s, \mathcal{X}(s; t, x), w\right) - \partial_x v\left(s, \mathcal{X}(s; t, x)\right)\right) \mathrm{d}s \ . \tag{78}$$

Working under the assumptions of Proposition 5, we define the parametrized mapping P^w, which we propose is a process, by

$$P^w : \begin{array}{ccc} \mathcal{A} & \to & \mathcal{U} \\ (t, t_o, u_o) & \mapsto & u(t) \end{array} \quad \text{where} \quad u(t) \text{ is given by (77);} \tag{79}$$

where $\mathcal{A}$ is generated by the sets $\mathcal{D}_t$ as given by (28).

Lemma 3. *The mapping P^w as defined in (79) is a process in the sense of Definition 2.*

The proof is in [19].

Proof of Proposition 5. The mapping P^w, as given by (79), is a process for any $w \in \mathcal{W}$ by Lemma 3. It remains to show that P^w is a Lipschitz process on $\mathcal{U}$ parametrized by $w \in \mathcal{W}$, i.e., it satisfies (12), (13), and (14), with C_u, C_t and C_w given by (28).

1. Lipschitz Continuity w.r.t. Initial Data. Consider two initial data $u_1, u_2 \in \mathcal{D}$, $t_o, t \in I$ with $t_o < t$, and $w \in \mathcal{W}$.
   To begin, assume that $x \in [0, \sigma(t)[$. Then, it is easy to see from (77) that

$$|P^w(t, t_o)u_1 - P^w(t, t_o)u_2|(x) = 0,$$

as b, q and m are independent of the choice of initial data u_o. Similarly, for $x \in [\sigma(t), +\infty[$,

$$|P^w(t,t_o)u_1 - P^w(t,t_o)u_2|(x) = |u_1(\mathcal{X}(t_o;t,x)) - u_2(\mathcal{X}(t_o;t,x))|\, \mathcal{E}_w(t_o,t,x).$$

Thus, using the substitution $y = \mathcal{X}(t_o;t,x)$,

$$\begin{aligned} d_{\mathcal{U}}\left(P^w(t,t_o)u_1, P^w(t,t_o)u_2\right) &= \int_{\sigma(t)}^{+\infty} |u_1(\mathcal{X}(t_o;t,x)) - u_2(\mathcal{X}(t_o;t,x))|\mathcal{E}_w(t_o,t,x)\,\mathrm{d}x \\ &\leq e^{M_\infty(t-t_o)}\|u_1(0) - u_2(0)\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})}\,. \end{aligned}$$

2. Lipschitz Continuity w.r.t. Time. Consider $u_o \in \mathcal{D}$, $t_o, t \in I$, and $w \in \mathcal{W}$.
We have

$$\begin{aligned} d_{\mathcal{U}}(P^w(t,t_o)u_o, u_o) \quad \leq \quad & \|P^w(t,t_o)u_o - u_o\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)} \\ & + \|P^w(t,t_o)u_o - u_o\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R}_+)}\,. \end{aligned} \tag{80}$$

Focusing on the first term of (80), using (77), **(BP1)**, **(BP2)**, **(BP3)**, **(BP4)**, and that $u_o \in \mathcal{D}$,

$$\begin{aligned} & \|P^w(t,t_o)u_o - u_o\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)} \\ \leq & \int_0^{\sigma(t)} |b(\mathcal{T}(0;t,x))\mathcal{E}_w(\mathcal{T}(0;t,x),t,x) - u_o(x)|\,\mathrm{d}x \\ & \quad + \int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t |q(\tau,\mathcal{X}(\tau;t,x),w)\mathcal{E}_w(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x \\ \leq & \ \hat{v}(B_1 + \|u_o\|_{\mathbf{L^\infty}(\mathbb{R}_+;\mathbb{R})} + Q_1)e^{M_\infty(t-t_o)}(t-t_o) \\ & \quad + \int_{t_o}^t v(y,0)|u_o(\mathcal{X}(t;0,y))||e^{\int_y^t m(s,\mathcal{X}(s;y,0),w)\mathrm{d}s} - e^{\int_y^t \partial_x v(s,\mathcal{X}(s;y,0))\mathrm{d}s}|\,\mathrm{d}y \\ \leq & \ \hat{v}(B_1 + R + Q_1)e^{M_\infty T}(t-t_o) + \hat{v}R(M_\infty + V_L)(t-t_o)^2 e^{(M_\infty+V_L)(t-t_o)}\,. \end{aligned}$$

For the second term of (80), once again from (77),

$$\begin{aligned} & \|P^w(t,t_o)u_o - u_o\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R}_+)} \\ \leq & \int_{\sigma(t)}^{+\infty} |u_o(\mathcal{X}(t_o;t,x))\mathcal{E}_w(t_o,t,x) - u_o(x)|\,\mathrm{d}x + \int_{\sigma(t)}^{+\infty}\int_{t_o}^t |q(\tau,\mathcal{X}(\tau;t,x),w)|\mathcal{E}_w(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \\ \leq & \left[\hat{v}\mathrm{TV}(u_o;\mathbb{R}_+) + M_\infty\|u_o\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} + Q_1\right] e^{M_\infty(t-t_o)}(t-t_o) \\ \leq & \left[\hat{v}R + M_\infty R + Q_1\right] e^{M_\infty(t-t_o)}(t-t_o)\,, \end{aligned}$$

where we used the **BV** estimates in [19].
Concluding, we thus have

$$d_{\mathcal{U}}(P^w(t,t_o)u_o, u_o) \leq \left[\hat{v}(B_1 + 2R + R(M_\infty + V_L)T) + M_\infty R + Q_1\right] e^{M_\infty T}(t-t_o).$$

3. Lipschitz Continuity w.r.t. Parameters. Consider $u_o \in \mathcal{D}$, $t_o, t \in I$ and $w_1, w_2 \in \mathcal{W}$.
We have

$$\begin{aligned} d_{\mathcal{U}}(P^{w_1}(t,t_o)u_o, P^{w_2}(t,t_o)u_o) \leq & \|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)} \\ & + \|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R}_+)} \end{aligned} \tag{81}$$

For the first term of (81),

$$
\begin{aligned}
&\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\|_{\mathbf{L^1}([0,\sigma(t)[;\mathbb{R}_+)} \\
&\leq \int_0^{\sigma(t)} |b(\mathcal{T}(0;t,x))|\,|\mathcal{E}_{w_1}(\mathcal{T}(0;t,x),t,x) - \mathcal{E}_{w_2}(\mathcal{T}(0;t,x),t,x)|\,\mathrm{d}x && (82)\\
&\quad + \int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t |q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)|\,\mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}x && (83)\\
&\quad + \int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t |q(\tau,\mathcal{X}(\tau;t,x),w_2)|\,|\mathcal{E}_{w_2}(\tau,t,x) - \mathcal{E}_{w_1}(\tau,t,x)|\,\mathrm{d}x\,. && (84)
\end{aligned}
$$

Focussing first on (82), we use **(BP2)**, and get

$$
\begin{aligned}
&\int_0^{\sigma(t)} |b\left(\mathcal{T}(0;t,x)\right)|\,|\mathcal{E}_{w_1}(\mathcal{T}(0;t,x),t,x) - \mathcal{E}_{w_2}(\mathcal{T}(0;t,x),t,x)|\,\mathrm{d}x \\
&\leq B_\infty e^{M_\infty(t-t_o)} \int_{t_o}^t \int_y^t v(y,0)|m(s,\mathcal{X}(s;y,0),w_1) - m(s,\mathcal{X}(s;y,0),w_2)|\,\mathrm{d}s\,\mathrm{d}y \\
&\leq B_\infty M_L e^{M_\infty(t-t_o)}(t-t_o) d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

For (83), using **(BP3)**,

$$
\begin{aligned}
&\int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t |q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)|\,\mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x \\
&\leq Q_L\,\hat{v}\,e^{M_\infty(t-t_o)}\,d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

Finally, for (84), we have

$$
\begin{aligned}
&\int_0^{\sigma(t)} \int_{\mathcal{T}(0;t,x)}^t |q(\tau,\mathcal{X}(\tau;t,x),w_2)|\,|\mathcal{E}_{w_2}(\tau,t,x) - \mathcal{E}_{w_1}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x \\
&\leq Q_\infty e^{M_\infty(t-t_o)} \int_{t_o}^t \int_0^{\sigma(\tau)} \int_\tau^t |m(s,\mathcal{X}(s;\tau,\xi),w_1) - m(s,\mathcal{X}(s;\tau,\xi),w_2)|\,\mathrm{d}s\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\leq Q_\infty e^{M_\infty(t-t_o)} \int_{t_o}^t \int_\tau^t \int_{\mathcal{X}(s;t_o,0)}^{\mathcal{X}(s;\tau,0)} |m(s,y,w_1) - m(s,y,w_2)|\,\mathrm{d}s\,\mathrm{d}y\,\mathrm{d}\tau \\
&\leq Q_\infty M_L e^{M_\infty(t-t_o)} \frac{1}{2}(t-t_o)^2 d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\|_{\mathbf{L^1}(J_1;\mathbb{R}_+)} \\
&\leq \left[B_\infty M_L + \hat{v}Q_L + \frac{1}{2}Q_\infty M_L(t-t_o)\right] e^{M_\infty(t-t_o)}(t-t_o) d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned} \tag{85}
$$

Focusing now on the second term of (81), we have

$$
\begin{aligned}
&\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\|_{\mathbf{L^1}([\sigma(t),+\infty[;\mathbb{R})} \\
&\leq \int_{\sigma(t)}^{+\infty} |u_o\left(\mathcal{X}(t_o;t,x)\right)||\mathcal{E}_{w_1}(t_o,t,x) - \mathcal{E}_{w_2}(t_o,t,x)|\,\mathrm{d}x && (86)\\
&\quad + \int_{\sigma(t)}^{+\infty} \int_{t_o}^t |q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)|\mathcal{E}_{w_1}(\tau,t,x)\,\mathrm{d}\tau\,\mathrm{d}x && (87)\\
&\quad + \int_{\sigma(t)}^{+\infty} \int_{t_o}^t |q(\tau,\mathcal{X}(\tau;t,x),w_2)||\mathcal{E}_{w_1}(\tau,t,x) - \mathcal{E}_{w_2}(\tau,t,x)|\,\mathrm{d}\tau\,\mathrm{d}x\,. && (88)
\end{aligned}
$$

Looking at term (86),

$$\begin{aligned}
&\int_{\sigma(t)}^{+\infty} |u_o(\mathcal{X}(t_o;t,x))|\, |\mathcal{E}_{w_1}(t_o,t,x) - \mathcal{E}_{w_2}(t_o,t,x)|\, \mathrm{d}x \\
&\leq \|u_o\|_{\mathbf{L}^\infty(\mathbb{R}_+;\mathbb{R})} e^{M_\infty(t-t_o)} \int_0^{+\infty} \int_{t_o}^t |m(s,\mathcal{X}(s;t_o,y),w_1) - m(s,\mathcal{X}(s;t_o,y),w_2)|\, \mathrm{d}s\, \mathrm{d}x \\
&\leq M_L R e^{M_\infty(t-t_o)} (t-t_o) d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

Next, for the term (87),

$$\begin{aligned}
&\int_{\sigma(t)}^{+\infty} \int_{t_o}^t |q(\tau,\mathcal{X}(\tau;t,x),w_1) - q(\tau,\mathcal{X}(\tau;t,x),w_2)| \mathcal{E}_{w_1}(\tau,t,x)\, \mathrm{d}\tau\, \mathrm{d}x \\
&\leq e^{M_\infty(t-t_o)} \int_{t_o}^t \int_{\sigma(\tau)}^{+\infty} |q(\tau,y,w_1) - q(\tau,y,w_2)|\, \mathrm{d}y\, \mathrm{d}\tau \\
&\leq Q_L e^{M_\infty(t-t_o)} (t-t_o) d_{\mathcal{W}}(w_1,w_2).
\end{aligned}$$

Finally, for term (88),

$$\begin{aligned}
&\int_{\sigma(t)}^{+\infty} \int_{t_o}^t |q(\tau,\mathcal{X}(\tau;t,x),w_2)| |\mathcal{E}_{w_1}(\tau,t,x) - \mathcal{E}_{w_2}(\tau,t,x)|\, \mathrm{d}\tau\, \mathrm{d}x \\
&\leq Q_\infty e^{M_\infty(t-t_o)} \int_{t_o}^t \int_{\sigma(t)}^{+\infty} \int_\tau^t |m(s,\mathcal{X}(s;\tau,\xi),w_1) - m(s,\mathcal{X}(s;\tau,\xi),w_2)|\, \mathrm{d}s\, \mathrm{d}\xi\, \mathrm{d}\tau \\
&\leq \frac{1}{2} M_L Q_\infty e^{M_\infty(t-t_o)} (t-t_o)^2 d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned}$$

Thus, combining these estimates together we have

$$\begin{aligned}
&\|P^{w_1}(t,t_o)u_o - P^{w_2}(t,t_o)u_o\|_{\mathbf{L^1}(J_1;\mathbb{R}_+)} \\
\leq\ & \left[M_L R + Q_L + \tfrac{1}{2} M_L Q_\infty (t-t_o)\right] e^{M_\infty(t-t_o)} d_{\mathcal{W}}(w_1,w_2)\,.
\end{aligned} \tag{89}$$

Due to the assumption $u_o \in \mathcal{D}$, we have $\|u_o\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq R$. Hence, substituting (85) and (89) into (81), and as $(t-t_o) < T$, we get

$$d_{\mathcal{U}}(P^{w_1}(t,t_o)u_o, P^{w_2}(t,t_o)u_o) \leq C_w (t-t_o) d_{\mathcal{W}}(w_1,w_2) \tag{90}$$

where C_w is as in (28), as required. □

Proof of Proposition 6. For fixed $t_o \in I$, $u_o \in \mathcal{U}$, and $w \in \mathcal{W}$, define by $\Pi_{(t_o,u_o,w_o)} : \{(s,s_o) \in [t_o,T]^2 : s \geq s_o\} \times \mathcal{U} \to \mathcal{U}$ to be the process with $s \mapsto \Pi_{(t_o,u_o,w_o)}(s,s_o)\rho_o$ being the solution of

$$\begin{cases}
\partial_t \rho + \partial_x \left(v(t,x)\,\rho\right) = \bar{m}(t,x)\,\rho + \bar{q}(t,x) & (t,x) \in [s_o,T] \times \mathbb{R}_+ \\
\rho(t,0) = b_o(t) & t \in [s_o,T] \\
\rho(s_o,x) = \rho_o(x) & x \in \mathbb{R}_+
\end{cases} \tag{91}$$

with $\bar{m}$ and $\bar{q}$ the given by (31). For notational simplicity, we write $\Pi_{(t_o,u_o,w_o)} = \Pi$ when the (t_o,u_o,w_o) when no confusion arises.

The mapping Π is Lipschitz continuous with respect to time and initial data, for some constant $\mathcal{L} > 0$, as $\bar{m}$ and $\bar{q}$ satisfy correspondingly **(BP2)** and **(BP3)**, which do not explicitly depend on w.

By this construction, $t \mapsto \Pi_{(t_o,u_o,w_o)}(t,t_o)u_o$ is the solution of (30).

From [20, Theorem 2.9], we have

$$\begin{aligned}
&\left\|u(t) - \Pi_{(t_o,u_o,w_o)}(t,t_o)u_o\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \\
\leq\ & \mathcal{L}\int_{t_o}^{t} \liminf_{h\to 0+} \tfrac{1}{h}\left\|u(\tau+h) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \mathrm{d}\tau \\
=\ & \mathcal{L}\int_{t_o}^{t} \liminf_{h\to 0+} \tfrac{1}{h}\left\|P_1(\tau+h,\tau)P(\tau,t_o)(u_o,w_o) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \mathrm{d}\tau\,.
\end{aligned}$$

Thus it suffices to show, for any $0 \leq t_o \leq \tau \in [0,T]$, that

$$\liminf_{h\to 0+} \frac{1}{h}\left\|P_1(\tau+h,\tau)P(\tau,t_o)(u_o,w_o) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} = 0.$$

The tangency condition (11) ensures that

$$\frac{1}{h}\left\|P_1(\tau+h,\tau)u(\tau) - P^{P_2(\tau,t_o)(u_o,w_o)}(\tau+h)u(\tau)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq \mathcal{O}(1)\int_0^h \frac{\omega(\xi)}{\xi}\, d\xi \to 0$$

as $h \to 0$.

Further, it can be shown, using formula (77), that

$$\left\|P^{P_2(\tau,t_o)(u_o,w_o)}(\tau+h,\tau)u(\tau) - \Pi_{(t_o,u_o,w_o)}(\tau+h,\tau)u(\tau)\right\|_{\mathbf{L^1}(\mathbb{R}_+;\mathbb{R})} \leq \mathcal{O}(1)\, h^2,$$

with the constant $\mathcal{O}(1)$ depending on the constants laid out in **(BP1)**–**(BP4)**, R and T. Thus this also converges to zero as $h \to 0$, completing our proof. □

5.4. Proofs for Section 3.4

Lemma 4. *The mapping μ defined by (35) in Proposition 8 is narrowly continuous.*

The proof is in [19].

Proof of Proposition 8. *The Narrow Continuity:* This is a consequence of Lemma 4.

Distributional Solution: To simplify calculations we define, for a test function $\varphi \in (\mathbf{C^1} \cap \mathbf{W}^{1,\infty})([t_o,T]\times\mathbb{R};\mathbb{R})$,

$$\begin{aligned}
\mathcal{I}_\varphi(\mu,w) = &\int_{\mathbb{R}_+} \left(\partial_t\varphi(\cdot,x) + b(\cdot,\mu,w)(x)\partial_x\varphi(\cdot,x) - c(\cdot,\mu,w)(x)\varphi(\cdot,x)\right) d\mu(\cdot,x) \\
&+ \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+} \varphi(\cdot,x) d[\eta(\cdot,\mu,w)(y)](x)\right) d\mu(\cdot,y).
\end{aligned}$$

By a density argument, it suffices to check the integral equality in Definition 8 for $\varphi \in \mathbf{C^1_c}([t_o,T]\times\mathbb{R}_+;\mathbb{R})$. We discretize the time domain. For a spacing $k \in \mathbb{N}$, and $i = 0,\ldots,k$, we introduce the grid points $t_i = t_o + \frac{i(T-t_o)}{k}$, and the associated $(\tilde{\mu}_i,\tilde{w}_i) = P(t_{i-1},t_o)(u_o,w_o)$. We then split the integral,

$$\begin{aligned}
&\int_{t_o}^{T} \mathcal{I}_\varphi\left(P(t,t_o)(\mu_o,w_o)\right) \mathrm{d}t \\
&= \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \underbrace{\left[\mathcal{I}_\varphi\left(P(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\right) - \mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\right)\right]}_{A_{1,i}(t)} \mathrm{d}t \qquad (92)\\
&\quad + \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} \underbrace{\mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\right)}_{A_{2,i}(t)} \mathrm{d}t\,. \qquad (93)
\end{aligned}$$

Our first goal is to demonstrate that (92) vanishes in the limit $k \to \infty$. Focusing on $A_{1,i}$, we split the integral to get

$$
\begin{aligned}
& A_{1,i}(t) \\
= & \int_{\mathbb{R}_+} \partial_t \varphi(t,x) \, \mathrm{d}(P_1(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i) - F_1(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))\,(x) && (94)\\
& + \int_{\mathbb{R}_+} b\left(t, P(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\right)(x) \partial_x \varphi(t,x) \, \mathrm{d}P_1(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,(x) \\
& - \int_{\mathbb{R}_+} b(t, F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(x) \partial_x \varphi(t,x) \, \mathrm{d}F_1(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,(x) && (95)\\
& + \int_{\mathbb{R}_+} c(t, F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(x) \varphi(t,x) \, \mathrm{d}F_1(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,(x) \\
& - \int_{\mathbb{R}_+} c(t, P(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(x) \varphi(t,x) \, \mathrm{d}P_1(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,(x) && (96)\\
& + \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x) \quad \mathrm{d}[\eta(t, P(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(y)]\,(x) \right) \mathrm{d}P_1(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,(y) \\
& - \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}_+} \varphi(t,x) \, \mathrm{d}[\eta(t, F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(y)]\,(x) \right) && (97)\\
& \qquad \mathrm{d}F_1(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,(y)\,.
\end{aligned}
$$

We now deal with each of these terms separately. To simplify the notation we will set

$$
\begin{array}{rclcl}
P_i(t) & \equiv & (\mu_{i,P}(t), w_{i,P}(t)) & = & P(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i), \\
F_i(t) & \equiv & (\mu_{i,F}(t), w_{i,F}(t)) & = & F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\,.
\end{array} \tag{98}
$$

We will make extensive use of the relation (11), which gives

$$
d(P_i(t), F_i(t)) \le \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{w(\xi)}{\xi} \, \mathrm{d}\xi \tag{99}
$$

for L as in (16). For (94),

$$
\begin{aligned}
\left| \int_{\mathbb{R}_+} \partial_t \varphi(t,x) \; d\left(\mu_{i,P}(t) - \mu_{i,F}(t)\right)(x) \right| & \le \|\partial_t \varphi\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})} d_{\mathcal{M}}\left(\mu_{i,P}(t), \mu_{i,F}(t)\right) \\
& \le \|\partial_t \varphi\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})} \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \;.
\end{aligned}
$$

Next, for (95), calling $L_b = \sup_{t\in[0,T], w\in\mathcal{W}} \mathbf{Lip}(b(t,\cdot,w))$,

$$
\begin{aligned}
& \left| \int_{\mathbb{R}_+} b(t,P_i(t))(x) \partial_x \varphi(t,x) \; d\mu_{i,P}(t)(x) - \int_{\mathbb{R}_+} b(t,F_i(t))(x) \partial_x \varphi(t,x) \; d\mu_{i,F}(t)(x) \right| \\
= & \left| \int_{\mathbb{R}_+} \left[b(t,P_i(t))(x) - b(t, F(t,t_{i-1})(\tilde{\mu}_i,\tilde{w}_i))(x) \right] \partial_x \varphi(t,x) \; d\mu_{i,P}(t)(x) \right. \\
& \left. + \int_{\mathbb{R}_+} b(t,F_i(t))(x) \partial_x \varphi(t,x) \; d\left(\mu_{i,P}(t) - \mu_{i,F}(t)\right)(x) \right| \\
\le & \|\partial_x \varphi\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})} (RL_b + R\hat{L} + B) \frac{2L}{\ln 2}(t-t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \;.
\end{aligned}
$$

Repeat the same calculations for (96) and set $L_c = \sup_{t\in[0,T],w\in\mathcal{W}} \mathbf{Lip}\,(c(t,\cdot,w))$,

$$\begin{aligned}&\left|\int_{\mathbb{R}_+} c(t,F_i(t))(x)\varphi(t,x)\ dw_{i,F}(t)(x) - \int_{\mathbb{R}_+} c(t,P_i(t))(x)\varphi(t,x)\ d\mu_{i,P}(t)(x)\right|\\ &\le \|\varphi\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})}(RL_c + R\hat{L} + C)\frac{2L}{\ln 2}(t-t_{i-1})\int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\ .\end{aligned}$$

Finally, for the term (97), we find

$$\begin{aligned}&\Bigg|\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t,x)\ d[\eta(t,P_i(t))(y)](x)\right)\ d\mu_{i,P}(t)(y)\\ &\quad - \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t,x)\ d[\eta(t,F_i(t))(y)](x)\right)\ dw_{i,F}(t)(y)\Bigg|\\ &\le \|\varphi\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})}R\left(\sup_{\substack{t\in[0,T]\\ w\in\mathcal{W}}}\mathbf{Lip}(\eta(t,\cdot,w)) + \hat{L} + E\right)\frac{2L}{\ln 2}(t-t_{i-1})\int_0^{t-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,.\end{aligned}$$

Combining these four estimates together, we have for a constant $\mathcal{C}$, independent of k,

$$\left|\sum_{i=1}^k\int_{t_{i-1}}^{t_i} A_{1,i}(t)\,\mathrm{d}t\right| \le \mathcal{C}\sum_{i=1}^k\frac{(t_i-t_{i-1})^2}{2}\int_0^{\frac{T-t_o}{k}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi \to 0 \text{ as } k\to+\infty.$$

Now,

$$\begin{aligned}A_{2,i}(t) &= \mathcal{I}_\varphi\left(F(t-t_{i-1},t_{i-1})(\tilde{\mu}_i,\tilde{w}_i)\right)\\ &= \mathcal{I}_\varphi\left(\mu_{i,F}(t),\tilde{w}_i\right)\\ &\quad + \int_{\mathbb{R}_+}(b(t,\mu_{i,F}(t),w_{i,F}(t))(x) - b(t,\mu_{i,F}(t),\tilde{w}_i)(x))\partial_x\varphi(t,x)\ d\mu_{i,F}(t)(x)\\ &\quad + \int_{\mathbb{R}_+}(c(t,\mu_{i,F}(t),\tilde{w}_i)(x) - c(t,\mu_{i,F}(t),w_{i,F}(t))(x))\varphi(t,x)\ d\mu_{i,F}(t)(x)\\ &\quad + \int_{\mathbb{R}_+}\left(\int_{\mathbb{R}_+}\varphi(t,x)\,d[\eta(t,\mu_{i,F}(t),w_{i,F}(t))(y) - \eta(t,\mu_{i,F}(t),\tilde{w}_i)(y)](x)\right)d\mu_{i,F}(t)(x)\end{aligned}$$

and hence

$$\begin{aligned}A_{2,i}(t) &\le \mathcal{I}_\varphi\left(\mu_{i,F}(t),\tilde{w}_i\right)\\ &\quad + \hat{L}R\left(2\|\varphi\|_{\mathbf{W}^{1,\infty,(\mathbb{R}_+;\mathbb{R})}} + \|\partial_x\varphi\|_{\mathbf{W}^{1,\infty,(\mathbb{R}_+;\mathbb{R})}}\right)\frac{2L}{\ln 2}(t-t_{i-1})\int_0^{t-t_{i-1}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\,.\end{aligned}\tag{100}$$

The second term will thus converge to zero in the summation. Hence we concentrate on the summation of the first term.

In the next calculation, we will use the fact

$$\begin{aligned}&\int_{\mathbb{R}_+}\varphi(T,x)\,\mathrm{d}(\mu_{k,F}(T) - P_1(T,t_o)(u_o,w_o))\,(x)\\ &= \int_{\mathbb{R}_+}\varphi(T,x)\,\mathrm{d}(F_1(T-t_{k-1},t_{k-1})P(t_{k-1},t_o)(u_o,w_o) - P_1(T,t_{k-1})P(t_{k-1},t_o)(u_o,w_o))\,(x)\\ &\le \|\varphi(T)\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})}\frac{2L}{\ln 2}\frac{T-t_o}{k}\int_0^{\frac{T-t_o}{k}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi \to 0,\ \text{as } k\to\infty.\end{aligned}$$

Focusing on the summation of the first term in (100)

$$\begin{aligned}\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\mathcal{I}_\varphi\left(\mu_{i,F}(t),\tilde{w}_i\right)\mathrm{d}t &= \sum_{i=1}^{k}\left(\int_{\mathbb{R}_+}\varphi(t_i,x)\ d\mu_{i,F}(t_i)(x)-\int_{\mathbb{R}_+}\varphi(t_{i-1},x)\ d\tilde{\mu}_i(x)\right)\\ &= \int_{\mathbb{R}_+}\varphi(T,x)\ d\mu_{T,F}(T)(x)-\int_{\mathbb{R}_+}\varphi(t_o,x)\ d\mu_o(x)\\ &\quad+\sum_{i=1}^{k}\left(\int_{\mathbb{R}_+}\varphi(t_i,x)\ d(\mu_{i,F}(t_i)-\tilde{\mu}_{i+1})(x)\right)\\ &\underset{k\to+\infty}{\longrightarrow}\int_{\mathbb{R}_+}\varphi(T,x)\,d(P_1(T,t_o)(u_o,w_o))(x)-\int_{\mathbb{R}_+}\varphi(t_o,x)\ d\mu_o(x),\end{aligned}$$

where we use that

$$\sum_{i=1}^{k}\left(\int_{\mathbb{R}_+}\varphi(t_i,x)\ d(\mu_{i,F}(t_i)-\tilde{\mu}_{i+1})(x)\right)\le\|\varphi\|_{\mathbf{W}^{1,\infty}(\mathbb{R}_+;\mathbb{R})}\frac{2L}{\ln 2}T\int_0^{\frac{T-t_o}{k}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\underset{k\to+\infty}{\longrightarrow}0,$$

completing the proof. □

5.5. Proofs for Section 3.5

Proof of Proposition 10. We assume for simplicity that both processes P^u and P^w share the same constants C_u, C_w, C_t in (12)–(13)–(14).

The properties of P ensured by Theorem 2 show that $P_1 \in \mathbf{C^0}([t_o,T];\mathbf{L^1}(\mathbb{R}^n;\mathbb{R}))$ as required by Definition 9.

Introduce the following notation. For any $k\in\mathbb{R}$ and $\varphi\in\mathbf{C^\infty_c}(\hat{I}\times\mathbb{R};\mathbb{R}_+)$, denote

$$\begin{aligned}\mathcal{I}_{\varphi,k}(u,w) &= \int_{\mathbb{R}}\left[|u-k|\,\partial_t\varphi+q_k(u,w)\,\partial_x\varphi\right]\mathrm{d}x\ ,\\ q_k(u,w) &= \mathrm{sign}(u-k)\ \left(f(u,w)-f(k,w)\right)\ .\end{aligned}$$

Fix $N\in\mathbb{N}\setminus\{0\}$ and, for every $i\in\{0,\dots,N\}$, define $t_i=t_o+i\frac{T-t_o}{N}$ and, for $t\in[t_{i-1},T]$,

$$\begin{array}{rcrcl}&&(\tilde{u}_i,\tilde{w}_i) &=& P(t_{i-1},t_o)(u_o,w_o)\,,\\ \bar{P}_i(t,x) &\equiv& (u_{i,P}(t,x),w_{i,P}(t)) &=& P(t,t_{i-1})(\tilde{u}_i,\tilde{w}_i)(x)\,,\\ \bar{F}_i(t,x) &\equiv& (u_{i,F}(t,x),w_{i,F}(t)) &=& \left(P^{\tilde{w}_i}\left(t,t_{i-1}\right)\tilde{u}_i(x),P^{\tilde{u}_i}\left(t,t_{i-1}\right)\tilde{w}_i\right)\ .\end{array}\tag{101}$$

We now prove in 2 steps that

$$\begin{aligned}\int_{t_o}^{T}\mathcal{I}_{\varphi,k}(P\left(t,t_o\right)(u_o,w_o))\,\mathrm{d}t \ge &\int_{\mathbb{R}}\left|P_1\left(T,t_o\right)(u_o,w_o)-k\right|\varphi(T,x)\,\mathrm{d}x\\ &-\int_{\mathbb{R}}\left|u_o(x)-k\right|\varphi(0,x)\,\mathrm{d}x\ .\end{aligned}\tag{102}$$

Step 1: We prove the inequality

$$\int_{t_o}^{T}\mathcal{I}_{\varphi,k}(P\left(t,t_o\right)(u_o,w_o))\,\mathrm{d}t\ge\limsup_{N\to+\infty}\sum_{i=1}^{N}\int_{t_{i-1}}^{t_i}\mathcal{I}_{\varphi,k}(u_{i.F}(t),\tilde{w}_i)\,\mathrm{d}t\ .\tag{103}$$

To this aim, write

$$\int_{t_o}^{T}\mathcal{I}_{\varphi,k}(P\left(t,t_o\right)(u_o,w_o))\,\mathrm{d}t = \int_{t_o}^{T}\int_{\mathbb{R}}\left|P_1\left(t,t_o\right)(u_o,w_o)(x)-k\right|\partial_t\varphi(t,x)\,\mathrm{d}x\ \mathrm{d}t\tag{104}$$

$$+\int_{t_o}^{T}\int_{\mathbb{R}}q_k\left(P\left(t,t_o\right)(u_o,w_o)(x)\right)\partial_x\varphi(t,x)\,\mathrm{d}x\ \mathrm{d}t\tag{105}$$

We proceed towards the estimate of (104). For every $i \in \{1, \dots, N\}$ and $k \in \mathbb{R}$, using (11) with L and ω given by (16), we have

$$\begin{aligned}
&\left| \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} \left[|u_{i,P}(t,x) - k| \partial_t \varphi(t,x) - |u_{i,F}(t,x) - k| \partial_t \varphi(t,x) \right] \mathrm{d}x \ \mathrm{d}t \right| \\
&\leq \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |u_{i,P}(t,x) - u_{i,F}(t,x)| \partial_t \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \\
&\leq \frac{L}{\ln(2)} \frac{(T-t_o)^2}{N^2} \|\partial_t \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \, .
\end{aligned}$$

Therefore, the term (104) is estimated as:

$$\begin{aligned}
&\int_{t_o}^{T} \int_{\mathbb{R}} |P_1(t,t_o)(u_o,w_o)(x) - k| \, \partial_t \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \\
&= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |u_{i,P}(t,x) - k| \, \partial_t \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \\
&\geq \sum_{i=1}^{N} \left[\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |u_{i,F}(t,x) - k| \, \partial_t \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \right] \\
&\qquad - \frac{L}{\ln(2)} \frac{(T-t_o)^2}{N} \|\partial_t \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi
\end{aligned}$$

and the last term converges to 0 as $N \to +\infty$. Thus, the term (104) is estimated as follows:

$$[(104)] \geq \limsup_{N\to+\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |u_{i,F}(t,x) - k| \, \partial_t \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \ . \tag{106}$$

We pass now to the term (105). For every $i \in \{1, \dots, N\}$ and $k \in \mathbb{R}$, since q_k is Lipschitz continuous [27, Lemma 3] and using (11), L_f from **(CL2)**, L and ω from (16),

$$\begin{aligned}
&\int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k\left(\bar{P}_i(t,x)\right) \partial_x \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t - \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i) \, \partial_x \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \\
&\leq L_f \frac{2L}{\ln(2)} \|\partial_x \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \int_{t_{i-1}}^{t_i} (t - t_{i-1}) \int_0^{t-t_{i-1}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \ \mathrm{d}t \\
&\quad + L_f \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |u_{i,F}(t,x) - k| \cdot d_{\mathcal{W}}(w_{i,P}(t), \tilde{w}_i) \cdot |\partial_x \varphi(t,x)| \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \frac{L_f}{2} \|\partial_x \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \left(L_f \, C_t(R+k) + \frac{2L}{\ln(2)} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \right) \frac{(T-t_o)^2}{N^2} \, .
\end{aligned}$$

Therefore, (105) is estimated as

$$\begin{aligned}
&\int_{t_o}^{T} \int_{\mathbb{R}} q_k\left(P(t,t_o)(u_o,w_o)(x)\right) \partial_x \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \\
&\geq \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i) \, \partial_x \varphi(t,x) \, \mathrm{d}x \ \mathrm{d}t \\
&\quad - \frac{L_f}{2} \|\partial_x \varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})} \left(L_f \, C_t(R+k) + \frac{2L}{\ln(2)} \int_0^{\frac{T-t_o}{N}} \frac{\omega(\xi)}{\xi} \, \mathrm{d}\xi \right) \frac{(T-t_o)^2}{N^2}
\end{aligned}$$

and the last term converges to 0 as $N \to +\infty$. Thus,

$$[(105)] \geq \limsup_{N\to+\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i)\, \partial_x \varphi(t,x)\, \mathrm{d}x\ \mathrm{d}t\ . \tag{107}$$

Combining (106) and (107), the proof of Step 1, namely (103), is completed.

Step 2: Now we prove that

$$\begin{aligned} &\liminf_{N\to+\infty} \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i)\, \mathrm{d}t \\ &\geq \int_{\mathbb{R}} |P_1(T,t_o)(u_o,w_o)(x) - k|\, \varphi(T,x)\, \mathrm{d}x - \int_{\mathbb{R}} |u_o(x) - k|\, \varphi(t_o,x)\, \mathrm{d}x \end{aligned} \tag{108}$$

Fix $i \in \{1,\dots,N\}$. For $\varepsilon > 0$ sufficiently small, consider $\chi_\varepsilon \in \mathbf{C}_{\mathbf{c}}^{\infty}\left(]t_{i-1}, t_i[; [0,1]\right)$ such that $\chi_\varepsilon(t) = 1$ for $t \in [t_{i-1}+\varepsilon, t_i - \varepsilon]$ and define $\varphi_\varepsilon = \varphi \cdot \chi_\varepsilon$. Then, by Definition 9 and the choice of χ_ε, we have that for every $\varepsilon > 0$ sufficiently small,

$$\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi_\varepsilon,k}(u_{i,F}(t,x), \tilde{w}_i)\, \mathrm{d}t \geq 0.$$

This implies that

$$\begin{aligned} \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i)\, \mathrm{d}t &\geq \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi-\varphi_\varepsilon,k}(u_{i.F}(t), \tilde{w}_i)\, \mathrm{d}t \\ &= \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} |u_{i,F}(t,x) - k| \partial_t \left(\varphi - \varphi_\varepsilon\right)(t,x)\, \mathrm{d}x\ \mathrm{d}t \qquad (109) \\ &\quad + \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i)\, \partial_x(\varphi - \varphi_\varepsilon)(t,x)\ \mathrm{d}x\ \mathrm{d}t \qquad (110) \end{aligned}$$

for every $\varepsilon > 0$ sufficiently small. Moreover the continuity in time of $u_{i,F}$ implies that

$$\lim_{\varepsilon\to 0^+} [(109)] = \int_{\mathbb{R}} |u_{i,F}(t_i,x) - k|\, \varphi(t_i,x)\ \mathrm{d}x - \int_{\mathbb{R}} |u_{i,F}(t_{i-1},x) - k|\, \varphi(t_{i-1},x)\ \mathrm{d}x\,,$$

while, by the Dominated Convergence Theorem, we deduce that

$$\lim_{\varepsilon\to 0^+} [(110)] = \lim_{\varepsilon\to 0^+} \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}} q_k(u_{i,F}(t,x), \tilde{w}_i)\, \partial_x(\varphi - \varphi_\varepsilon)(t,x)\, \mathrm{d}x\ \mathrm{d}t = 0.$$

Therefore, we get

$$\begin{aligned} &\int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i)\ \mathrm{d}t \\ &\geq \int_{\mathbb{R}} |u_{i,F}(t_i,x) - k|\, \varphi(t_i,x)\, \mathrm{d}x - \int_{\mathbb{R}} |u_{i,F}(t_{i-1},x) - k|\, \varphi(t_{i-1},x)\ \mathrm{d}x\ . \end{aligned}$$

Summing over i, we obtain that

$$\begin{aligned} &\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \mathcal{I}_{\varphi,k}(u_{i.F}(t), \tilde{w}_i)\, \mathrm{d}t \\ &\geq \sum_{i=1}^{N} \int_{\mathbb{R}} |u_{i,F}(t_i,x) - k|\, \varphi(t_i,x)\, \mathrm{d}x - \sum_{i=1}^{N} \int_{\mathbb{R}} |u_{i,F}(t_{i-1},x) - k|\, \varphi(t_{i-1},x)\, \mathrm{d}x \\ &= \int_{\mathbb{R}} |u_{N,F}(T,x) - k|\, \varphi(T,x)\, \mathrm{d}x - \int_{\mathbb{R}} |u_o(x) - k|\, \varphi(t_o,x)\, \mathrm{d}x \qquad (111) \\ &\quad + \sum_{i=1}^{N-1} \int_{\mathbb{R}} \left(|u_{i,F}(t_i,x) - k| - |u_{i+1,F}(t_i,x) - k|\right)\, \varphi(t_i,x)\, \mathrm{d}x\ . \qquad (112) \end{aligned}$$

We now estimate the first term in (111):

$$\begin{aligned}
&\int_{\mathbb{R}} |u_{N,F}(T,x)-k|\,\varphi(T,x)\,\mathrm{d}x - \int_{\mathbb{R}} |P_1(T,t_o)(u_o,w_o)(x)-k|\,\varphi(T,x)\,\mathrm{d}x\\
=&\int_{\mathbb{R}} \left(|F_1(T-t_{N-1},t_{N-1})(\tilde{u}_{N-1},\tilde{w}_{N-1})(x)-k| - |P_1(T,t_o)(u_o,w_o)(x)-k|\right)\varphi(T,x)\,\mathrm{d}x
\end{aligned}$$

and, using L and ω as in (16), we get

$$\begin{aligned}
&\left|\int_{\mathbb{R}} \left(|F_1(T-t_{N-1},t_{N-1})(\tilde{u}_{N-1},\tilde{w}_{n-1})(x)-k| - |P_1(T,t_o)(u_o,w_o)(x)-k|\right)\varphi(T,x)\,\mathrm{d}x\right|\\
\leq&\int_{\mathbb{R}} \big|F_1(T-t_{N-1},t_{N-1})P(t_{N-1},t_o)(u_o,w_o)(x)\\
&\qquad -P_1(T,t_{N-1})P(t_{N-1},t_o)(u_o,w_o)(x)\big|\varphi(T,x)\,\mathrm{d}x\\
\leq&\frac{2L}{\ln(2)}\,\frac{T-t_o}{N}\int_0^{\frac{T-t_o}{N}}\frac{\omega(\xi)}{\xi}\,\mathrm{d}\xi\\
\to&\,0 \quad \text{as } N\to+\infty\,.
\end{aligned}$$

We now estimate (112) using (101) and (11)

$$\begin{aligned}
&\sum_{i=1}^{N-1}\int_{\mathbb{R}} \big||u_{i,F}(t_i,x)-k| - |u_{i+1,F}(t_i,x)-k|\big|\,\varphi(t_i,x)\,\mathrm{d}x\\
\leq&\sum_{i=1}^{N-1}\int_{\mathbb{R}} |u_{i,F}(t_i,x)-u_{i+1,F}(t_i,x)|\,\varphi(t_i,x)\,\mathrm{d}x\\
\leq&\,\|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})}\sum_{i=1}^{N-1}\left\|P^{\tilde{w}_i}(t_i,t_{i-1})\tilde{u}_i - P_1(t_i,t_{i-1})\tilde{u}_i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\leq&\,\frac{2\,L}{\ln 2}\|\varphi\|_{\mathbf{L}^\infty([t_o,T]\times\mathbb{R};\mathbb{R})}(T-t_o)\int_0^{(T-t_o)/N}\frac{\omega(\tau)}{\tau}\,\mathrm{d}\tau\\
\to&\,0 \quad \text{as } N\to+\infty\,.
\end{aligned}$$

The obtained estimates for (111) and (112), as $N\to+\infty$, proved Step 2, namely (108). □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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