Université Paris VI - Pierre et Marie Curie

École Doctorale Paris Centre

Thèse de doctorat

Discipline : Mathématiques

présentée par

Olivier Haution

Steenrod operations and quadratic forms

dirigée par Nikita Karpenko

Soutenue le 9 décembre 2009 devant le jury composé de :

M. Antoine Ducros	Université Pierre et Marie Curie	
M. Nikita Karpenko	Université Pierre et Marie Curie	Directeur
M. Ahmed Laghribi	Université d'Artois	
M. Burt Totaro	Cambridge University	Rapporteur

Rapporteurs :

M. Fabien Morel	Ludwig-Maximilians-Universität München
M. Burt Totaro	Cambridge University

Institut de Mathématiques de Jussieu
175, rue du chevaleret
75 013 Paris

École doctorale Paris centre
Case 188
4 place Jussieu
75 252 Paris cedex 05

Résumé : Aucune des opérations de Steenrod sur les groupes de Chow modulo un nombre premier p n'est disponible lorsque la caractéristique du corps de base est égale à p. Nous construisons des opérations sur les groupes de Chow de la restriction à un corps de déploiement des variétés projectives homogènes sous l'action d'un groupe algébrique semi-simple. Ces opérations envoient un cycle rationnel sur un cycle rationnel sous l'hypothèse que le corps de base admette une forme de résolution des singularités, satisfaite lorsque sa caractéristique diffère du nombre p, d'après un résultat récent de Gabber. On retrouve ainsi une forme faible des opérations de Steenrod déjà construites, et ce par une méthode assez différente. Nous prouvons que l'hypothèse de résolution des singularités n'est pas nécessaire pour construire le premier carré de Steenrod ($p = 2$). Nous déduisons de cette construction un théorème sur le premier indice de Witt des formes quadratiques.

Une autre partie de ce travail consiste a fournir une preuve directe du fait que les motifs de Chow des quadriques projectives lisses se décomposent de la même manière, que les coefficients soient dans $\mathbb{Z}$ ou $\mathbb{Z}/2$.

Mots-Clefs : Groupes de Chow, groupes de Grothendieck, variétés projectives homogènes, opérations d'Adams, opérations de Steenrod, théorèmes de Riemann-Roch, indices de Witt, décompositions motiviques.

Abstract : Steenrod operations on the Chow groups modulo a prime number p are not available when the characteristic of the base field is equal to p. We build operations on the restriction to a splitting field of the Chow group of a smooth projective homogeneous variety under a semi-simple linear algebraic group. These operations respect rationality of cycles provided that the base field admits a form of resolution of singularities, which is given by a result of Gabber when the base field has a characteristic different from p. Therefore we recover a weak form of Steenrod operations, in the cases when they are already constructed, using a very different approach. We show that the first Steenrod square ($p = 2$) can be constructed without using resolution of singularities. As a consequence we prove a theorem on the parity of the Witt index of a quadratic form.

Another part of this work consists of proving directly that Chow motives of smooth projective quadrics decompose in the same way when the coefficients are either $\mathbb{Z}$ or $\mathbb{Z}/2$.

Keywords : Chow groups, Grothendieck groups, projective homogeneous varieties, Steenrod operations, Adams operations, Riemann-Roch theorems, Witt indices, motivic decompositions.

2000 Mathematics Subject Classification : 11E04, 14C15, 14C40, 19E08, 19E20.

Remerciements

Ma reconnaissance va bien sûr d'abord a mon directeur de thèse, Nikita Karpenko. Ses idées et encouragements ont sans aucun doute eu une influence décisive sur la réalisation de ce travail. Ça a été pour moi une expérience extrêmement stimulante d'être initié à la théorie géométrique des formes quadratiques par lui. Il m'a été très profitable de pouvoir étudier l'excellent livre [EKM08] bien avant sa publication.

L'idée, centrale dans ce travail, d'utiliser les méthodes de l'article [Ati66], a été suggérée par Fabien Morel. Je l'en remercie, ainsi que Bruno Kahn, avec qui j'ai eu l'opportunité de pouvoir discuter de la question de l'existence d'opérations de Steenrod modulo p en caractéristique p.

Le séjour que j'ai effectué à l'université de Bielefeld avant de débuter ma thèse m'a certainement été très bénéfique. Je remercie l'université de Bielefeld pour m'avoir permis d'effectuer ce séjour dans de si bonnes conditions. A cette occasion j'ai pu faire connaissance des jeunes mathématiciens Mathieu Florence, Nikita Semenov et Kirill Zainoulline. Je les remercie pour le temps qu'il m'ont consacré et les nombreuses choses qu'ils m'ont apprises.

Je voudrais aussi remercier Fabien Morel et Burt Totaro pour l'intérêt qu'ils ont montré à mon travail en acceptant d'être rapporteurs de cette thèse. Je remercie Antoine Ducros et Ahmed Laghribi d'avoir accepté de faire partie du jury.

Je remercie Maksim Zhykhovich pour les enrichissantes discussions que nous avons eues.

Je tiens aussi à remercier le rapporteur qui a examiné l'article intitulé "Lifting of coefficients for Chow motives of quadrics", ainsi que Skip Garibaldi. Leurs remarques m'ont permis d'améliorer le texte qui constitue le cinquième chapitre de cette thèse.

Je souhaite remercier tous les professeurs qui ont su susciter mon intérêt pour les mathématiques au cours de ma formation, et m'ont progressivement orienté vers le domaine des mathématiques auquel cette thèse prétend contribuer. Je pense en particulier à Bruno Arsac, Jean Barge, Jean Lannes et Vasudevan Srinivas.

Finalement, je remercie ma famille, Hélène et mes amis pour le soutien et les encouragements qu'ils m'ont apportés.

Paris, le 16 décembre 2009,
Olivier Haution.

Contents

Introduction

Dans cette thèse nous nous intéressons à deux questions n'entretenant pas de rapport logique, mais ayant pour point commun d'être liées à la théorie des formes quadratiques sur un corps arbitraire.

Le premier des deux problèmes, et le principal par le travail qu'il a requis et l'intérêt des résultats obtenus, est la question de l'existence d'opérations de Steenrod pour le groupes de Chow sur un corps de caractéristique arbitraire.

Lorsque la caractéristique du corps de base est différente d'un nombre premier p, les opérations de Steenrod dans les groupes de Chow à coefficients dans l'anneau $\mathbb{Z}/p$ ont été construites par Voevodsky dans le cadre plus général de la cohomologie motivique à coefficients dans $\mathbb{Z}/p$ dans [Voe03]. Brosnan a fourni une construction similaire dans [Bro03], mais en restant au niveau des groupes de Chow, ce qui la rend plus élémentaire. Ces constructions suivent le même schéma que la construction originale de Steenrod. Karpenko-Elman-Merkurjev puis Boisvert ont fourni ensuite une construction un peu différente, évitant le recours aux groupes de Chow équivariants. Ces constructions ne fonctionnent pas lorsque la caractéristique du corps de base est égale à p, par exemple parce que le groupe de Chow équivariant sous l'action du groupe $\mathbb{Z}/p$ du point est trivial ([MV99, §4.3, Proposition 3.3]).

Une approche de nature très différente, qui a été une source d'inspiration importante pour cette thèse, est celle fournie par Levine dans [Lev07]. Il s'agit d'utiliser le fait que les groupes de Chow constituent la théorie cohomologique orientée additive universelle lorsque le corps de base a pour caractéristique zéro. Une approche similaire consiste à remarquer que certaines opérations de Landweber-Novikov dans le cobordisme algébrique descendent en des opérations au niveau des groupes de Chow (à coefficients finis), toujours lorsque le corps de base est de caractéristique zéro. L'hypothèse de caractéristique zéro provient du fait que l'on utilise la résolution des singularités d'Hironaka ([Hir64]) et le théorème de factorisation faible de [AKMW02], comme signalé dans [LM07, Remark 1.2.20] ou [LP09, §2.7]. A la lumière de ces considérations, il devient nettement plus raisonnable de conjecturer l'existence d'opérations de Steenrod pour les groupes de Chow lorsque le corps de base est de caractéristique arbitraire.

Un des objectifs de ce texte est de prouver qu'il suffit d'utiliser la résolution des singularités, et non le théorème de factorisation faible, pour construire une forme faible, mais suffisante pour certaines applications, des opérations de Steenrod.

Lorsque l'on s'intéresse au nombre premier deux, nous construisons une forme faible de la première opération de Steenrod, sans aucune restriction sur le corps de base, ni à propos de sa caractéristique, ni à propos de la possibilité de résoudre les singularités des variétés définies sur ce corps. L'existence de cette opération a des conséquences intéressantes concernant le schéma de déploiement des formes quadratiques, et nous obtenons des restrictions sur le premier indice de Witt, nouvelles lorsque la caractéristique du corps de base est égale à deux.

La seconde question étudiée est celle du relèvement des coefficients pour les motifs de Chow des quadriques projectives lisses. On peut extraire du travail de Vishik ([Vis04]) le fait que les motifs à coefficients entiers de telles quadriques se décomposent de la même facon que les motifs à coefficients modulo deux, ces derniers étant bien sûr à priori plus commodes à étudier. Nous fournissons une preuve directe et élémentaire de ce résultat. Le cinquième chapitre de cette thèse, qui traite cette question, a été accepté pour publication dans le livre à paraître “Quadratic forms, linear algebraic groups, and cohomology”, dans la collection “Developments in Mathematics”.

Le plan de la thèse est le suivant. Tout d'abord nous rappelons les résultats généraux concernant les groupes de Grothendieck des schémas, et précisons le vocabulaire et les notations que allons utiliser. En fait l'exposé que constitue le premier chapitre pourrait se suffire logiquement à lui-même, dans une certaine mesure . Nous avons choisi de citer sans preuves les théorèmes les plus classiques (invariance homotopique, suite de localisation, théorèmes du fibré projectif, théorème de dévissage, ...) dont les preuves sont discutées en de nombreux autres endroits. Par contre nous avons inclus des preuves des résultats pour lesquels il est un peu moins commun de trouver des détails dans la littérature existante, comme par exemple la multiplicativité de la filtration topologique, les liens entre anneau de Chow et anneau gradué associé à la filtration topologique, ou la notion de classe de Chern à valeurs dans K^0. Le premier chapitre, malgré sa longueur, ne contient certainement aucun résultat, ni argument, qui soit ignoré des spécialistes.

Le second chapitre est consacré au théorème d'Adams-Riemann-Roch. Celui-ci s'exprime à l'aide de la classe de Bott θ^p que nous commençons par étudier. Nous fournissons une décomposition particulière de la classe de Bott $\theta^p(E)$ d'un fibré vectoriel quelconque E sur une variété X. Si $\mathrm{F}_\gamma^n K^0(X)$ dsigne le

n-ième terme de la gamma filtration, nous prouvons que

$$\theta^p(E) \in \sum_{k=0}^{\mathrm{rank}(E)} p^{\mathrm{rank}(E)-k} \cdot \mathrm{F}_\gamma^{k(p-1)}\, K^0(X).$$

Cette décomposition joue un rôle essentiel dans la construction des opérations de Steenrod que nous proposons dans le chapitre suivant. Elle est tout à fait analogue à une décomposition fournie par Atiyah dans le cadre topologique ([Ati66]). Afin de pouvoir définir le premier carré de Steenrod homologique sur les variétés singulières, nous avons besoin de la notion d'opération d'Adams homologique introduite par Soulé dans [Sou85]. Leur construction repose sur le théorème de Riemann-Roch pour les opérations d'Adams dans la K-théorie avec support. Même si on se limite au groupes de K-théorie d'ordre 0, K^0, la construction de telles opérations ([GS87]) n'est pas tout à fait élémentaire. Nous exposons en détail une construction peut-être un peu plus élémentaire de telles opérations, inspirée par [Ati66], et prouvons le théorème de Riemann-Roch associé à ces opérations, en suivant le processus classique de déformation vers le fibré normal.

Dans le troisième chapitre, nous commençons par énoncer un résultat non-publié de Gabber, affirmant l'existence d'altérations de degré premier à p, l'entier p étant un nombre premier distinct de la caractéristique du corps de base. Ce résultat remplace complètement la résolution des singularités d'Hironaka pour ce qui est de la construction des opérations de Steenrod. Désignons par Ch le groupe de Chow à coefficients modulo un nombre premier p. Nous construisons des opérations

$$\mathrm{S}\colon \mathrm{Ch}(\overline{X}) \to \mathrm{Ch}(\overline{X})$$

lorsque X est par exemple une variété lisse projective homogène sous l'action d'un groupe algébrique linéaire semi-simple sur un corps F (dont la caractéristique est arbitraire), avec $\overline{F}/F$ une extension de corps déployant X, et $\overline{X} = X \times_F \overline{F}$. Ces opérations vérifient des propriétés similaires aux opérations de Steenrod, en particulier il s'agit d'homomorphismes d'anneaux commutant avec le pull-back par rapport à un morphisme arbitraire de variétés satisfaisant les conditions ci-dessus. De plus si x est un cycle de codimension un, on a la formule

$$\mathrm{S}(x) = x + x^p \in \mathrm{Ch}(\overline{X}).$$

Un cycle de $\mathrm{Ch}(\overline{X})$ est dit rationel s'il est la restriction d'un cycle de $\mathrm{Ch}(X)$. Le résultat de Gabber permet d'affirmer que ces opérations envoient cycles rationnels sur cycles rationnels, lorsque la caractéristique du corps de base est différente de p.

Lorsque $p = 2$, et X est une variété éventuellement singulière sur un corps arbitraire, nous construisons un homomorphisme

$$\mathrm{Sq}_1\colon \mathrm{Ch}_\bullet(X) \to \widetilde{\mathrm{Ch}}_{\bullet+1}(X)$$

où $\widetilde{\mathrm{Ch}}(X)$ désigne le groupe Chow à coefficients entiers $\mathrm{CH}(X)$ modulo son sous-groupe de torsion, tensorisé par $\mathbb{Z}/2$. Cet homomorphisme est compatible au push-forward par rapport aux morphismes projectifs, et peut être considéré comme un version faible du *premier carré de Steenrod homologique*. Il permet de montrer que la première opération construite précédemment

$$\mathrm{S}^1\colon \mathrm{Ch}^\bullet(\overline{X}) \to \mathrm{Ch}^{\bullet+1}(\overline{X})$$

respecte la rationalité des cycles, en évitant le recours au théorème de Gabber. Ce dernier résultat est donc valable indépendamment de la caractéristique de corps de base.

Nous concluons ce chapitre par une preuve de la relation

$$\mathrm{S}^1 \circ \mathrm{S}^1 = 0 \mod 2,$$

qui peut être considerée comme la plus simple des relations d'Adem.

C'est concernant le nombre premier deux que nous obtenons les résultats les plus intéressants (non soumis à l'hypothèse d'existence de résolution des singularités). Les applications que nous avons à l'esprit concernent les formes quadratiques, le choix de ce nombre premier semble donc assez raisonnable. Cependant, la véritable raison qui nous a poussé à nous restreindre à ce nombre premier (dans les paragraphes III.2 et III.3) est que nous sommes incapables de construire une quelconque opération de Steenrod non triviale modulo un nombre premier $p \neq 2$, lorsque le corps de base a pour caractéristique p. Moralement, nous utilisons la procédure de normalisation d'une variété algébrique, qui permet de "résoudre les singularités en codimension 1", et le premier carré de Steenrod ($p = 2$) est la seule opération de Steenrod qui réduise la dimension des cycles de 1. En général, la k-ième opération de Steenrod modulo p envoie un cycle de dimension n sur un cycle de dimension $n - k(p-1)$.

Le quatrième chapitre est dédié à quelques applications des résultats obtenus dans les chapitres précédents.

Nous commençons par établir une propriété de divisibilité pour certains nombres caractéristiques des variétés projectives lisses sur un corps arbitraire, par exemple nous prouvons que le *nombre de Segre* $c_{\dim X}(-T_X)$ d'une telle variété (sans composante connexe de dimension zéro) est divisible par deux. Notons que par exemple lorsque X est une quadrique projective anisotrope, c'est aussi une conséquence du théorème de Springer. Il s'agit d'observations géométriques classiques (on peut tout-à-fait étendre le corps de base pour formuler et prouver ces résultats), cependant une hypothèse sur la caractéristique du corps de base est habituellement associée à de telles observations. Une méthode pour obtenir ces divisibilités est d'ailleurs d'utiliser les opérations de Steenrod. Ce résultat apparaît ici comme une conséquence immédiate du théorème d'Adams-Riemann-Roch, ce qui fournit une indication que l'utilisation des opérations d'Adams peut parfois se substituer à celle des opérations de

Steenrod. Cette remarque a constitué une motivation importante pour développer les techniques présentées ici. Ces divisibilités peuvent être considérées comme des préliminaires nécessaires à la formulation de certains théorèmes nettement plus subtils à propos des nombres caractéristiques, de nature plus arithmétique, tels que [Ros06, Theorem 9.1] à propos des correspondances de Rost, ou alors la formule du degré. De tels résultats ne semblent pas pour le moment pouvoir être obtenus à l'aide des opérations que nous construisons ici (même si l'on admet qu'il est possible de résoudre les singularités), et la validité de ces théorèmes est d'ailleurs une question ouverte lorsque la caractéristique du corps de base est arbitraire.

Ensuite nous utilisons le fait que le premier carré de Steenrod préserve la rationalité des cycles pour prouver le théorème suivant

Théorème 1. Soit q une forme quadratique non-dégénérée et anisotrope. Si q est de dimension paire, son premier indice de Witt est soit pair, soit égal à un. Si q est de dimension impaire, alors son premier indice de Witt est impair.

Ce théorème a été prouvé par Karpenko, sous l'hypothèse supplémentaire que le corps de base a une caractéristique différente de deux. D'autre part, il apparaît clairement dans le livre [EKM08] que la seule raison de faire cette hypothèse est l'absence de construction connue des opérations de Steenrod modulo deux, lorsque le corps de base a pour caractéristique deux. Le théorème de Karpenko est en fait plus précis, mais puisque nous ne disposons que de la première opération de Steenrod, nous ne pouvons prouver qu'une version partielle du théorème. Nous mentionnons aussi que la version complète du théorème de Karpenko découle de la validité du théorème de Gabber.

Finalement nous discutons quelques propriétés d'intégralité du caractère de Chern. Il est écrit dans l'introduction de [Ati66] que cet article a d'abord été une tentative de reprouver, par une méthode différente, un théorème d'Adams concernant l'intégralité du caractère de Chern, sous l'hypothèse supplémentaire que l'espace considéré est *sans torsion*. Nous prouvons l'analogue algébrique du théorème d'Adams, lorsque le théorème de Gabber est disponible. Nous obtenons aussi un résultat partiel qui ne repose pas sur le théorème de Gabber (et donc est valide sans hypothèse sur le corps de base), correspondant au fait que le premier carré de Steenrod existe inconditionnellement. L'approche utilisée ressemble à celle d'Atiyah, mais il est intéressant de noter que nous n'avons pas besoin d'hypothèse analogue à celle que l'espace soit sans torsion.

Bien que les résultats nouveaux obtenus ici concernent les corps de caractéristique positive, nous évitons autant que possible de nous restreindre à cette situation (une exception notable étant la construction des opérations d'Adams "avec supports" présentée dans le paragraphe II.3). Nos arguments fournissent donc une approche unifiée, fonctionnant indépendamment de la caractéristique du corps de base ; en particulier nous présentons de nouvelles preuves de certains résultats déjà connus.

Le dernier chapitre peut être lu de manière complétement indépendante, et se suffit logiquement à lui-même à l'intérieur de ce texte. Nous prouvons le théorème suivant.

Théorème 2. Le foncteur évident de la catégorie des motifs de Chow à coefficients entiers des quadriques projectives lisses vers la catégorie des motifs de Chow à coefficients modulo deux de ces mêmes quadriques induit une bijection sur les classes d'isomorphismes d'objets.

La difficulté de cet énoncé concerne surtout les quadriques de dimension paire, pour lesquelles l'action du groupe de Galois sur le groupe de Chow peut être non triviale. D'une manière générale, les motifs des variétés projectives homogènes sous un groupe algébrique pour lequel l'action du groupe de Galois sur le diagramme du Dynkin est non-triviale (« outer type ») sont habituellement plus difficiles à étudier. Le théorème ci-dessus se place dans ce cadre pour le cas particulier des quadriques.

L'intérêt de ce théorème provient du fait qu'il est considérablement plus aisé d'étudier les motifs de quadriques à coefficients modulo deux, que ces motifs à coefficients entiers.

Finalement, nous avons inclus deux appendices. Dans le premier nous fournissons une définition alternative de l'homomorphisme de déformation, en suivant [Gil05]. Cette définition est très similaire à celle donnée par Rost dans [Ros96] pour les modules de cycles, et les arguments utilisés sont très proches de ceux de [EKM08].

Le deuxième appendice traite de la notion classique de fibré tangent relatif. Il permet de définir le *fibré tangent virtuel* d'une variété régulière, un élément du groupe de Grothendieck de la variété, qui généralise la notion de fibré tangent d'une variété lisse au cas légèrement plus général d'une variété régulière.

Basic notations

We shall use the following general notations (a list of more specific notations is given page 158):

$\mathcal{O}_X$	— structure sheaf of the scheme X.
$\mathcal{O}_{X,x}$	— local ring at the point x of the scheme X.
$F(X)$	— function field of the integral scheme X over the field F.
$\operatorname{Sym}\mathcal{E}$	— symmetric sheaf of algebras of the locally free sheaf $\mathcal{E}$.
$\mathbb{A}^n$	— affine space of space of dimension n over the base field.
$\mathbb{G}_m$	— multiplicative group $\mathbb{A}^1 - \{0\}$.
$\mathbb{P}^n$	— projective space of dimension n over the base field.
$E^\vee$	— dual of the vector bundle E.
$\mathbb{P}(E), \mathbb{P}(\mathcal{E})$	— the projective bundle $\operatorname{Proj}_X(\operatorname{Sym}\mathcal{E}^\vee)$ over the scheme X associated with the vector bundle $E = \operatorname{Spec}_X(\operatorname{Sym}\mathcal{E}^\vee)$, with sheaf of sections $\mathcal{E}$.
$\mathbb{P}^n_X$	— $\mathbb{P}(E)$ with E the trivial vector bundle of rank n over the scheme X.
$\mathcal{O}(1)$	— canonical line bundle over $\mathbb{P}^n$ or $\mathbb{P}(E)$.
$E \oplus 1$	— direct sum of the vector bundle E with the trivial line bundle.
T_X	— tangent bundle of the smooth scheme X.
$X \hookrightarrow Y$	— a closed embedding.
$Y - X$	— open complement of the closed subset X in Y.
$\mathbb{Z}/n$	— cyclic group of order n.
$\mathbb{Z}[1/n]$	— ring of fractions of $\mathbb{Z}$ with respect to the powers of n.
$A[t], A[[t]]$	— polynomial, power series algebra in one variable over the ring A.
$A^\times$	— multiplicative group of invertible elements in the ring A.
$\operatorname{im} f$	— image of the morphism f.

A *variety* is a separated quasi-projective finite type scheme over a field. A *regular variety* is a variety whose local rings are all regular local ring. A *smooth variety* is a variety whose diagonal embedding is a regular closed embedding.

Chapter I

Grothendieck groups, Chow groups and characteristic classes

In this chapter we fix the notations, and prove some results that we will need later on. Most notably, we prove that the topological filtration for smooth varieties is multiplicative in Proposition I.3.10, following the lines of [Gil05, § 2.5.11], where a proof of this fact for higher K-groups is sketched. We try to stay as elementary as possible, and as we are interested in K_0 we avoid any reference to higher K-groups.

One can define Gysin pull-backs along regular closed embeddings in the Grothendieck groups of coherent sheaves both by means of the natural Tor formula, and by deformation to the normal bundle. We show that these two constructions coincide in Proposition I.2.28,

One advantage of defining pull-backs by means of the deformation homomorphism is that it allows too see that is respects the topological filtration (Proposition I.2.29). Another advantage is that this definition is completely analogous to the definition of Gysin pull-backs for Chow groups. This is used in the proof of Proposition I.3.17, when the Chow group is compared with the graded group associated with the topological filtration on K_0. However defining pull-backs with the Tor formula can also be convenient. For example such a pull-back along the diagonal induces the ring structure on $K(X)$ for X smooth (together with the external product). Also pull-backs along flat morphisms and regular closed embeddings are treated in a uniform way, and a statement such as Lemma I.2.8 (the pull-back along a composite is the composite of pull-backs) is easier to prove (compare with [EKM08, Proposition 55.1, Lemma 51.10, § 104.F] where a double deformation scheme is used).

We also study some relations between the graded group associated with the topological filtration and the Chow group. These relations will allow us, in the third chapter, to construct operations on the Chow group by studying the interaction of the operations in K_0 with the topological filtration.

Finally we recall the construction of Chern classes for various cohomological theories, including the less classical notion of Chern classes with values in K^0 defined by means of the gamma operations. These theories with their Chern classes are particular examples of what we will call *presheaves with characteristic class.* This notion provides a convenient framework that will allow us in the next chapter to prove important decompositions of some Todd homomorphisms, with respect to powers of a prime number .

No original statement is contained in this chapter. We included many proofs and basic definitions since we gathered the informations from different sources. The general treatments of the material of this chapter that we used are [FL85] and [sga71], but we used [Ful98] and other sources for specific statements.

Here by a *variety*, we mean a separated quasi-projective finite type scheme over a field. A variety need not be irreducible nor reduced. We chose to restrict ourselves to the consideration of varieties in most statements, although part of the theory of Grothendieck works with much more general schemes. For example, it will not appear clearly when the presence of an ample invertible sheaf is needed. The only situation that we shall meet, which is outside the framework of varieties and their morphisms, will be the generic point $F(X) \to X$ of a variety, and this is why we consider the more general notion of a noetherian scheme for some statements in the beginning of this chapter.

On the other hand, the hypothesis that a variety is *regular* (*i.e* every local ring of X is a regular local ring) will usually not be made without a particular reason. Such an hypothesis can be avoided in some places, but this would in general bring in additional complexity without having noticeable consequences for us. For example we shall define push-forwards in the Grothendieck groups with supports only when our varieties will be regular. Another illustration of this phenomenon is the fact that it is more convenient to speak of Chern classes when our theory has a product.

I.1 NOTATIONS AND BASIC FACTS

Grothendieck groups

For a noetherian scheme X, we write $\mathsf{VB}(X)$ for the category of locally free sheaves on X. We denote the Grothendieck group of $\mathsf{VB}(X)$ by $K^0(X)$. A vector bundle E on X has class $[E]$ in $K^0(X)$.

The tensor product of $\mathcal{O}_X$-sheaves induces a biexact functor

$$\mathsf{VB}(X) \times \mathsf{VB}(X) \to \mathsf{VB}(X), \tag{I.1.a}$$

which, in turn, induces a bilinear map

$$K^0(X) \times K^0(X) \to K^0(X).$$

This gives $K^0(X)$ a ring structure.

An arbitrary morphism of noetherian schemes $f\colon X \to Y$ induces an exact functor

$$\mathcal{E} \mapsto f^*\mathcal{E} = \mathcal{E} \times_Y X, \;\; \mathsf{VB}(Y) \to \mathsf{VB}(X)$$

hence a group homomorphism

$$f^*\colon K^0(Y) \to K^0(X).$$

The associativity of the tensor product ensures that the two constructions just described are compatible, therefore we get for K^0 the structure of a presheaf of rings on the category of noetherian schemes and arbitrary morphisms.

Let $\mathsf{M}(X)$ be the category of coherent sheaves over a noetherian scheme X and $K_0(X)$ the Grothendieck group of $\mathsf{M}(X)$. A coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ has class $[\mathcal{F}]$ in $K_0(X)$.

A flat morphism of noetherian schemes $f\colon X \to Y$ induces an exact functor

$$\mathcal{F} \mapsto f^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X, \;\; \mathsf{M}(Y) \to \mathsf{M}(X)$$

hence a group homomorphism

$$f^*\colon K^0(Y) \to K^0(X).$$

Let $f\colon X \to Y$ be a projective morphism of noetherian schemes, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-sheaf. The $\mathcal{O}_Y$-modules $\mathsf{R}^i f_*\mathcal{F}$ (i-th higher direct image) are coherent because f is proper and Y is noetherian ([Gro61, Proposition 3.2.1]). If $X \xrightarrow{j} \mathbb{P}^r_Y \xrightarrow{p} Y$ is a factorization of f with j a closed embedding and p the standard projection, we have $\mathsf{R}^i f_* = j_* \circ \mathsf{R}^i p_*$ because j_* is an exact functor. The functors $\mathsf{R}^i p_*$ vanish for $i > r$ ([FL85, Property R2 p.105]).

We define a push-forward

$$f_*\colon K_0(X) \to K_0(Y)$$

by setting

$$f_*[\mathcal{F}] = \sum_i (-1)^i [\mathsf{R}^i f_* \mathcal{F}].$$

This is well-defined because an exact sequence of coherent $\mathcal{O}_X$-sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

gives an exact sequence of coherent $\mathcal{O}_Y$-sheaves

$$\cdots \to \mathsf{R}^i f_*(\mathcal{F}_1) \to \mathsf{R}^i f_*(\mathcal{F}_2) \to \mathsf{R}^i f_*(\mathcal{F}_3) \to \mathsf{R}^{i+1} f_*(\mathcal{F}_1) \to \cdots .$$

Remark I.1.1. If $\mathcal{E}_\bullet$ is a bounded complex of $\mathcal{O}_X$-sheaves with differential $d_i \colon \mathcal{E}_i \to \mathcal{E}_{i-1}$, and $\mathsf{H}_i(\mathcal{E}_\bullet)$ its i-th homology sheaf, then we have short exact sequences of $\mathcal{O}_X$-sheaves

$$0 \to \ker(d_i) \to \mathcal{E}_i \to \operatorname{im}(d_i) \to 0$$

and

$$0 \to \operatorname{im}(d_{i+1}) \to \ker(d_i) \to \mathsf{H}_i(\mathcal{E}_\bullet) \to 0.$$

It follows that we have in $K_0(X)$ the equality

$$\sum_i (-1)^i [\mathcal{E}_i] = \sum_i (-1)^i [\mathsf{H}_i(\mathcal{E}_\bullet)].$$

In particular, when

$$E^\bullet_{p,q} \Rightarrow A_{p+q}$$

is a convergent bounded spectral sequence then for every integer n we have in $K_0(X)$ the equality

$$\sum_{p,q} (-1)^{p+q} [E^n_{p,q}] = \sum_{p,q} (-1)^{p+q} [E^\infty_{p,q}].$$

Since A_i admits a filtration with successive quotients $E^\infty_{p,q}$ for $p+q=i$, we have in $K_0(X)$

$$[A_i] = \sum_{p+q=i} [E^\infty_{p,q}].$$

Finally we get in $K_0(X)$, for all integer n

$$\sum_{p,q} (-1)^{p+q} [E^n_{p,q}] = \sum_i (-1)^i [A_i].$$

Let $f \colon X \to Y$ and $g \colon Y \to Z$ be two projective morphisms. Given a coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$, the Leray spectral sequence has E^2 term

$$E^2_{p,q} = \mathsf{R}^p g_* (\mathsf{R}^q f_* \mathcal{F}) \Rightarrow \mathsf{R}^{p+q} (g \circ f)_* \mathcal{F}.$$

It follows from Remark I.1.1 that we have $g_* \circ f_* = (g \circ f)_*$.

When A is a commutative ring, we will write $K^0(A)$ and $K_0(A)$ for the groups $K^0(\operatorname{Spec}(A))$ and $K_0(\operatorname{Spec}(A))$.

Localization sequence, homotopy invariance

Theorem I.1.2 (localization sequence)**.** *Let $i\colon Y \hookrightarrow X$ be a closed embedding of varieties, with open complement $u\colon U \to X$. Then we have an exact sequence*

$$K_0(Y) \xrightarrow{i_*} K_0(X) \xrightarrow{u^*} K_0(U) \to 0.$$

A proof can be found in [FL85, Chapter VI, Proposition 3.2] or [sga71, IX, Proposition 1.1].

Theorem I.1.3 (homotopy invariance, [sga71, IX, Proposition 1.6])**.** *Let $p\colon \mathcal{E} \to X$ be a vector bundle. Then the pull-back $p^*\colon K_0(X) \to K_0(\mathcal{E})$ is an isomorphism.*

Products

Let X and Y be varieties over a common field F. If $\mathcal{E}$ (*resp.* $\mathcal{F}$) is a coherent sheaf over X (*resp.* Y), we shall write $\mathcal{E} \times \mathcal{F}$ for the fiber product $\mathcal{E} \times_F \mathcal{F}$. This is a coherent sheaf over $X \times Y = X \times_F Y$. Let $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ be the two projections. The association

$$(\mathcal{E},\mathcal{F}) \mapsto \mathcal{E} \times \mathcal{F} \simeq {p_X}^*\mathcal{E} \oplus {p_Y}^*\mathcal{F} \simeq {p_X}^*\mathcal{E} \underset{X\times Y}{\times} {p_Y}^*\mathcal{F}$$

induces an exact functor

$$\mathsf{M}(X) \times \mathsf{M}(Y) \to \mathsf{M}(X \times Y).$$

In turn, this induces a group homomorphism

$$K_0(X) \times K_0(Y) \to K_0(X \times Y)$$

which we shall denote as $(x, y) \mapsto x \times y$.

The association

$$(\mathcal{E},\mathcal{F}) \mapsto \mathcal{E} \boxtimes \mathcal{F} = {p_X}^*\mathcal{E} \otimes {p_Y}^*\mathcal{F}$$

defines biexact functors

$$\mathsf{M}(X) \times \mathsf{M}(Y) \to \mathsf{M}(X \times Y) \quad , \quad \mathsf{VB}(X) \times \mathsf{VB}(Y) \to \mathsf{VB}(X \times Y),$$

which induces bilinear maps

$$K_0(X) \times K_0(Y) \to K_0(X \times Y) \quad , \quad K^0(X) \times K^0(Y) \to K^0(X \times Y)$$

which we shall denote as $(x, y) \mapsto x \boxtimes y$. .

Note that if f, g are flat (*resp.* proper) then we have $(f \times g)^* = f^* \boxtimes g^*$ (*resp.* $(f \times g)_* = f_* \boxtimes g_*$).

Let $\Delta \colon X \to X \times X$ be the diagonal embedding for a variety X. Then the composite

$$\mathsf{VB}(X) \times \mathsf{VB}(X) \xrightarrow{-\boxtimes-} \mathsf{VB}(X \times X) \xrightarrow{\Delta^*} \mathsf{VB}(X)$$

coincides with the biexact functor (I.1.a) defining the ring structure on $K^0(X)$. It follows that we have

$$\Delta^*(x \boxtimes y) = x \cdot y. \tag{I.1.b}$$

Lemma I.1.4. *Let X and Y be two varieties over a common field. Let $X' \hookrightarrow X$ and $Y' \hookrightarrow Y$ be closed subvarieties. Then we have an isomorphism of $\mathcal{O}_{X \times Y}$-sheaves*

$$\mathcal{O}_{X'} \boxtimes \mathcal{O}_{Y'} \simeq \mathcal{O}_{X' \times Y'}.$$

Proof. Let F be the common base field. We have isomorphisms

$$\begin{aligned}
\mathcal{O}_{X'} \boxtimes \mathcal{O}_{Y'} &\simeq (\mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times Y}) \otimes_{\mathcal{O}_{X \times Y}} (\mathcal{O}_{Y'} \otimes_{\mathcal{O}_Y} \mathcal{O}_{X \times Y}) \\
&\simeq \mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times Y} \otimes_{\mathcal{O}_{X \times Y}} \mathcal{O}_{X \times Y} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \\
&\simeq \mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times Y} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \\
&\simeq \mathcal{O}_{X'} \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_F \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \\
&\simeq \mathcal{O}_{X'} \otimes_F \mathcal{O}_{Y'} \\
&\simeq \mathcal{O}_{X' \times Y'}.
\end{aligned}$$

□

Poincare homomorphism

For a variety X, there is a canonical homomorphism:

$$\delta \colon K^0(X) \to K_0(X)$$

induced by the obvious exact embedding of categories.

More generally, the tensor product of sheaves gives rise to a biexact functor

$$\mathsf{VB}(X) \times \mathsf{M}(X) \to \mathsf{M}(X),$$

which gives $K_0(X)$ the structure of a module over $K^0(X)$, which we shall denote as

$$K^0(X) \otimes K_0(X) \to K_0(X) \quad , \quad \lambda \otimes x \mapsto \lambda \cdot x.$$

Moreover, if $f \colon X \to Y$ is a proper morphism, and if we make $K^0(X)$ a module over $K^0(Y)$ *via* f^*, then f_* is a $K^0(Y)$-module homomorphism (*projection formula*). In other words if $\mu \in K^0(Y)$ and $x \in K_0(X)$, then we have in $K_0(Y)$ the equality

$$f_*\Big(f^*(\mu) \cdot x\Big) = \mu \cdot f_*(x) \tag{I.1.c}$$

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-sheaf equipped with a finite resolution

$$0 \to \mathcal{E}_n \to \cdots \to \mathcal{E}_0 \to \mathcal{F} \to 0,$$

where each $\mathcal{E}_i$ is a locally free sheaf, the element

$$r(\mathcal{F}) = \sum_{i=0}^{n} (-1)^i [\mathcal{E}_i] \in K^0(X) \tag{I.1.d}$$

does not depend on the choice of the resolution of $\mathcal{F}$ by locally free sheaves. We have

$$\delta \circ r(\mathcal{F}) = [\mathcal{F}]. \tag{I.1.e}$$

If we have an exact sequence of coherent $\mathcal{O}_X$-sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

such that each $\mathcal{F}_i$ admits a finite resolution by locally free $\mathcal{O}_X$-sheaves then we have in $K^0(X)$ the equality

$$r(\mathcal{F}_2) = r(\mathcal{F}_1) + r(\mathcal{F}_3). \tag{I.1.f}$$

It follows the $r(\mathcal{F})$ depends only on $[\mathcal{F}] \in K_0(X)$.

Rank homomorphisms

If $\pi_0(X)$ is the set of connected components of a scheme X, the ring $K^0(X)$ has an augmentation

$$\mathrm{rank}\colon K^0(X) \to \mathbb{Z}^{\pi_0(X)}$$

which is a ring epimorphism taking the class of vector bundle to the rank of its restriction to each connected component of X. A splitting of this epimorphism is induced by the pull-back along the natural faithfully flat morphism

$$X \to \mathrm{Spec}(F^{\pi_0(X)}).$$

Lemma I.1.5. *Let X be a zero-dimensional variety. Then the homomorphism*

$$\mathrm{rank}\colon K^0(X) \to \mathbb{Z}^{\pi_0(X)}$$

is an isomorphism

Proof. We can assume that X is connected. Then it is the spectrum of a local (Artin) ring. Therefore any locally free sheaf on X is free. □

Let $\rho(X)$ be the (finite) set of irreducible components of a scheme X. Then there is a split epimorphism of abelian groups

$$\operatorname{rank}\colon K_0(X) \to \mathbb{Z}^{\rho(X)}$$

defined as follows. Let $\eta_i\colon \operatorname{Spec}(F(x_i)) \to X$ be the generic points of X, associated with the irreducible components X_i of X. Then the morphism

$$\eta = \coprod_i \eta_i\colon \coprod_i \operatorname{Spec}(F(x_i)) \to X$$

is flat. The homomorphism rank is the pull-back

$$\eta^*\colon K_0(X) \to K_0\left(\coprod_i \operatorname{Spec}(F(x_i))\right) = \mathbb{Z}^{\rho(X)}.$$

The homomorphism

$$\mathbb{Z}^{\rho(X)} \to K_0(X) \quad , \quad (a_1, \cdots, a_{\rho(X)}) \mapsto \sum_i a_i[\mathcal{O}_{X_i}]$$

gives a splitting of η^*.

Note that there is a canonical (faithfully) flat morphism $x\colon X \to \operatorname{Spec}(F^{\pi_0(X)})$. This gives a flat morphism $x \circ \eta = d\colon \coprod_i \operatorname{Spec}(F(x_i)) \to \operatorname{Spec}(F^{\pi_0(X)})$. The pull-back along d fits in a commutative diagram

$$\begin{array}{ccc} K^0(X) & \xrightarrow{\delta} & K_0(X) \\ \downarrow{\scriptstyle \operatorname{rank}} & & \downarrow{\scriptstyle \operatorname{rank}} \\ \mathbb{Z}^{\pi_0(X)} & \xrightarrow[d^*]{} & \mathbb{Z}^{\rho(X)} \end{array} \tag{I.1.g}$$

The gamma filtration

Definition I.1.6. If X is a variety, its Grothendieck group of locally free sheaves $K^0(X)$ is endowed with the *gamma filtration*

$$\mathrm{F}_\gamma^n K^0(X) \subset \mathrm{F}_\gamma^{n-1} K^0(X) \subset \cdots \subset \mathrm{F}_\gamma^0 K^0(X) = K^0(X).$$

It can be defined as the ring filtration generated by the conditions

a) For any projective bundle $p\colon X \to Y$ and any integer n, we have

$$(p^*)^{-1}(\mathrm{F}_\gamma^n K^0(X)) \subset \mathrm{F}_\gamma^n K^0(Y).$$

b) For any variety X, we have $\ker(\operatorname{rank}) \subset \mathrm{F}_\gamma^1 K^0(X)$.

We record some immediate consequences of this definition.

Lemma I.1.7. *We have* $\ker(\mathrm{rank}) = \mathrm{F}^1_\gamma K^0(X)$.

Proof. Any element $x \in \mathrm{F}^1_\gamma K^0(X)$ is so that there is a composite of projective bundles $p\colon Y \to X$ such that $p^*(x)$ belongs to the ideal $\ker(\mathrm{rank})$ of $K^0(Y)$. Since $\mathrm{rank} \circ p^*$ coincide with the rank function of $K^0(X)$, we see that $\mathrm{rank}(x) = 0$, as required. ☐

Lemma I.1.8. *Let* $f\colon X \to Y$ *be a morphism. For every integer* n*, we have*

$$f^*(\mathrm{F}^n_\gamma K^0(Y)) \subset \mathrm{F}^n_\gamma K^0(X).$$

Proof. Take $y \in \mathrm{F}^n_\gamma K^0(Y)$. There is a composite of projective bundles $p\colon Y' \to Y$ such that $p^*(y)$ decomposes as a $\mathbb{Z}$-linear combination of products of n elements of rank zero. Form the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{\phi} & Y' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

Then $\phi^* \circ p^*(y)$ is a $\mathbb{Z}$-linear combination of products of n elements of rank zero, hence belongs to $\mathrm{F}^n_\gamma K^0(X')$. But π is also a composite of projective bundles, hence

$$(\pi^*)^{-1}\big(\phi^* \circ p^*(y)\big) \subset \mathrm{F}^n_\gamma K^0(X).$$

Since $\pi^* \circ f^* = \phi^* \circ p^*$, we see that $f^*(y) \in \mathrm{F}^n_\gamma K^0(X)$, which proves the lemma. ☐

Another equivalent definition of the gamma filtration will be given in Proposition I.4.8.

The topological filtration

Definition I.1.9. The Grothendieck group of coherent sheaves $K_0(X)$ is endowed with the *topological filtration*

$$0 = \mathrm{F}^{\mathsf{top}}_{-1} K_0(X) \subset \mathrm{F}^{\mathsf{top}}_0 K_0(X) \subset \cdots \subset \mathrm{F}^{\mathsf{top}}_{\dim X} K_0(X) = K_0(X).$$

The subgroup $\mathrm{F}^{\mathsf{top}}_n K_0(X)$ is generated by the classes of coherent sheaves $[\mathcal{O}_Z]$, as Z runs over all closed subvarieties of dimension $\leq n$.

We will write $\mathrm{gr}^{\mathsf{top}} K_0(X) = \bigoplus_i \mathrm{gr}^{\mathsf{top}}_i K_0(X)$ for the graded group associated with the topological filtration . The push-forward along a proper morphism $f\colon X \to Y$ induces a graded group homomorphism $\mathrm{gr}^{\mathsf{top}} K_0(X) \to \mathrm{gr}^{\mathsf{top}} K_0(Y)$, so that we can view $\mathrm{gr}^{\mathsf{top}} K_0$ as a covariant functor from the category of varieties and proper morphisms to the category of graded groups.

Lemma I.1.10. *Let U be an open subvariety of a variety X. Then for all integer i, the restriction map*

$$\mathrm{F}_i^{\mathsf{top}} K_0(X) \to \mathrm{F}_i^{\mathsf{top}} K_0(U)$$

is surjective.

Proof. By definition, the group $\mathrm{F}_i^{\mathsf{top}} K_0(U)$ is generated by classes $[\mathcal{O}_Y]$ for Y a closed integral subvariety of U of dimension $\leq i$. Given such a variety Y, the closure $\overline{Y}$ of Y in X has the same dimension as Y. Endow $\overline{Y}$ with the reduced scheme structure. Then $[\mathcal{O}_{\overline{Y}}] \in \mathrm{F}_i^{\mathsf{top}} K_0(X)$ restricts to $[\mathcal{O}_Y] \in \mathrm{F}_i^{\mathsf{top}} K_0(U)$. □

Lemma I.1.11. *Let X be a variety of pure dimension n over a field F. Then we have a split exact sequence*

$$0 \to \mathrm{F}_{n-1}^{\mathsf{top}} K_0(X) \to K_0(X) \xrightarrow{\text{rank}} \mathbb{Z}^{\rho(X)} \to 0,$$

where $\rho(X)$ is the set of irreducible components of X.

Proof. The only thing to prove is exactness at $K_0(X)$. It is clear that

$$\text{rank}\left(\mathrm{F}_{n-1}^{\mathsf{top}} K_0(X) \right) = 0.$$

Now an arbitrary element $x \in K_0(X)$ can be written $[\mathcal{F}] - [\mathcal{G}]$, for some coherent $\mathcal{O}_X$-sheaves $\mathcal{F}$ and $\mathcal{G}$. For every generic point x_i of X, take an open variety U_i containing x_i and no x_j for $j \neq i$, and such that both $\mathcal{F}|_{U_i}$ and $\mathcal{G}|_{U_i}$ are free.

For $i \neq j$, let $C_{i,j}$ be the closure of $U_i \cap U_j$ in U_i. The set $U_i \cap U_j$ contains every generic point of $C_{i,j}$, but does not contain the point x_i. It follows that the point x_i does not belong to $C_{i,j}$ (otherwise it would necessarily be a generic point of $C_{i,j}$). We replace each U_i with

$$U_i - \bigcup_{j \neq i} C_{i,j},$$

which is still a non-empty open subvariety of X containing x_i, on which both $\mathcal{F}$ and $\mathcal{G}$ are free.

Therefore we can assume that $U_i \cap U_j = \emptyset$ when $i \neq j$. Let U be the reunion of the various open subschemes U_i. The sheaves $\mathcal{F}$ and $\mathcal{G}$ need not be free over U, but they are free on each connected component of U. Hence if $\text{rank}(x) = 0$ then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic on U, and therefore $u^*(x) = 0$. Since U is dense and X has pure dimension n, the reduced closed complement Z of U has dimension $\leq n-1$. Using the localization sequence Theorem I.1.2

$$K_0(Z) \to K_0(X) \xrightarrow{u^*} K_0(U) \to 0,$$

we see that x belongs to the image of the map $K_0(Z) \to K_0(X)$, which is contained in $\mathrm{F}_{n-1}^{\mathsf{top}} K_0(X)$. □

Remark I.1.12. One can also consider the full subcategory $\mathsf{M}^1(X)$ of $\mathsf{M}(X)$ consisting of those $\mathcal{O}_X$-sheaves supported in codimension at least one in X. This is a Serre subcategory of the abelian category $\mathsf{M}(X)$, and the quotient abelian category is naturally isomorphic to the category of modules of finite length over $\prod_i F(x_i)$ the x_i being the generic points of X. The Grothendieck group of this category is isomorphic to $\mathbb{Z}^{\rho(X)}$, and Quillen's localization gives a long exact sequence

$$\cdots \to K_0(\mathsf{M}^1(X)) \to K_0(X) \to \mathbb{Z}^{\rho(X)} \to 0.$$

The group $\mathrm{F}^{\mathsf{top}}_{n-1} K_0(X)$ coincides with the image of $K_0(\mathsf{M}^1(X)) \to K_0(\mathsf{M}(X)) = K_0(X)$ and we get the short exact of the proposition.

The behaviour of this decomposition of $K_0(X)$ with respect to push-forwards is studied in the following lemma.

Lemma I.1.13. *Let $f\colon X \to Y$ be a proper surjective morphism of integral varieties of the same dimension. Then we have a commutative diagram with split exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{F}^{\mathsf{top}}_{n-1} K_0(X) & \longrightarrow & K_0(X) & \xrightarrow{\text{rank}} & \mathbb{Z} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle \cdot \deg(F(X)/F(Y))} & & \\ 0 & \longrightarrow & \mathrm{F}^{\mathsf{top}}_{n-1} K_0(Y) & \longrightarrow & K_0(Y) & \xrightarrow[\text{rank}]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Proof. We need to verify the commutativity of the square on the right. The generic fiber of f coincides with the generic point of X by our assumptions. Hence we have a cartesian square

$$\begin{array}{ccc} \mathrm{Spec}(F(X)) & \xrightarrow{x} & X \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f} \\ \mathrm{Spec}(F(Y)) & \xrightarrow[y]{} & Y \end{array}$$

It follows from Proposition I.2.26 below that the square on the left in the diagram

$$\begin{array}{ccccc} K_0(X) & \xrightarrow{x^*} & K_0(F(X)) & \xrightarrow{\text{rank}} & \mathbb{Z} \\ \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle g_*} & & \Big\downarrow{\scriptstyle \cdot \deg(F(X)/F(Y))} \\ K_0(Y) & \xrightarrow{y^*} & K_0(F(Y)) & \xrightarrow{\text{rank}} & \mathbb{Z} \end{array}$$

is commutative (this is a quite degenerate case of Proposition I.2.26 which can, of course, be checked directly).

Let E be an $F(X)$-vector space. Then $g_*[E] = [g_*E]$ as g is affine, and g_*E is E considered as an $F(Y)$-vector space, its dimension is therefore

$$\dim_{F(Y)} g_*E = (\dim_{F(X)} E) \cdot \deg(F(X)/F(Y)).$$

This proves the commutativity of the square on the right hand side. Hence the exterior square is commutative, and this is precisely the square appearing in the statement of the lemma. □

Proposition I.1.14. *Let $f\colon X \to Y$ be a proper morphism of varieties over a field F, X' a closed n-dimensional integral subvariety of X with image Y' in Y. Then we have*

$$f_*[\mathcal{O}_{X'}] = \deg(F(X')/F(Y')) \cdot [\mathcal{O}_{Y'}] \mod \mathrm{F}^{\mathsf{top}}_{n-1} K_0(X),$$

where by $\deg(F(X')/F(Y'))$ we mean the degree of the extension of function fields in case $\dim Y' = \dim X'$, and 0 otherwise.

Proof. Replacing X by X' and Y by Y', we can assume that $f\colon X \to Y$ is a surjective proper morphism of integral varieties. We have $\dim X \geq \dim Y$, and in the case where $\dim X > \dim Y$, there is nothing to prove. We can therefore assume that $\dim X = \dim Y$. The statement now appears as a consequence of Lemma I.1.13. □

Corollary I.1.15. *Let $f\colon X \to Y$ be a proper morphism of varieties, and $\mathcal{F}$ a coherent $\mathcal{O}_X$-sheaf supported in dimension n. Then*

$$f_*[\mathcal{F}] = [f_*\mathcal{F}] \mod \mathrm{F}^{\mathsf{top}}_{n-1} K_0(X).$$

Proof. We can assume that $\mathcal{F} = \mathcal{O}_X$, that f is surjective, and that X and Y are integral varieties of the same dimension. Then we have an isomorphism of $F(Y)$-vector spaces

$$F(Y) \otimes_{\mathcal{O}_Y} \mathcal{O}_X \simeq F(X),$$

hence $\operatorname{rank}[f_*\mathcal{O}_X] = \deg(F(X)/F(Y))$. This is also the rank of $f_*[\mathcal{O}_X]$ by Proposition I.1.14, and we conclude using the exact sequence of Lemma I.1.11. □

Corollary I.1.16 (Birational invariance for $\mathrm{gr}^{\mathsf{top}} K_0$)**.** *For any birational proper morphism $f\colon X \to Y$, we have $f_*[\mathcal{O}_X] = [\mathcal{O}_Y] \in \mathrm{gr}^{\mathsf{top}} K_0(Y)$.*

Lemma I.1.17. *For a variety X, we have*

$$\delta^{-1}(\mathrm{F}^{\mathsf{top}}_1 K_0(X)) = \mathrm{F}^1_\gamma K^0(X),$$

where $\delta\colon K^0(X) \to K_0(X)$ is the Poincare homomorphism.

Proof. By definition of the gamma filtration and Lemma I.1.11, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{F}^1_\gamma K^0(X) & \longrightarrow & K^0(X) & \longrightarrow & \mathbb{Z}^{\pi_0(X)} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle d^*} & & \\ 0 & \longrightarrow & \mathrm{F}^{\mathsf{top}}_{n-1} K_0(X) & \longrightarrow & K_0(X) & \longrightarrow & \mathbb{Z}^{\rho(X)} & \longrightarrow & 0. \end{array}$$

The square on the right is (I.1.g), it induces the vertical arrow on the left. The map

$$d\colon \operatorname{Spec}\left(\prod_{x\in\rho(X)} F(x)\right) \to \operatorname{Spec}\left(F^{\pi_0(X)}\right)$$

induces the morphism d^*. In order to check that d^* is injective, one can assume that X is connected, as d splits as the coproduct of the corresponding maps for each irreducible component of X. In this case, we just need to check that it is non-zero, and this is obvious.

The statement is a consequence of the injectivity of d^*. □

Regular varieties

If X is regular, any coherent $\mathcal{O}_X$-sheaf admits a finite resolution by locally free sheaves. It follows that δ is an isomorphism, the inverse being induced by the mapping r of (I.1.d). We will therefore identify the groups $K^0(X)$ and $K_0(X)$, writing $K(X)$ for both. We will also write $K(X)_{(n)}$ for $\mathrm{F}^{\mathsf{top}}_n K_0(X)$.

If X_i are irreducible components (*i.e.* connected components) of X then

$$K(X) = \coprod_i K(X_i).$$

We define a filtration on $K(X)$ with

$$K(X)^{(n)} := \coprod_i \mathrm{F}^{\mathsf{top}}_{(\dim X_i - n)} K_0(X_i) = \coprod_i K(X_i)_{(\dim X_i - n)}$$

and put $\operatorname{gr} K(X)$ for the associated graded group.

I.2 Gysin maps

Gysin homomorphisms

Definition I.2.1. (See [FL85, Chapter V, § 4] and [Ful98, Example 15.1.8].)

A closed embedding $i\colon X \hookrightarrow Y$ such that the coherent $\mathcal{O}_Y$-sheaf $i_*\mathcal{O}_X$ admits a finite resolution by locally free $\mathcal{O}_Y$-sheaves will be called *perfect*.

Example I.2.2. A closed embedding $i\colon X \hookrightarrow Y$ is a perfect closed embedding in the following particular cases:

— The variety X is regular.

— The embedding i is a regular closed embedding, the resolution of $i_*\mathcal{O}_X$ being locally given by the Koszul complex.

If $i\colon X \hookrightarrow Y$ is a perfect morphism, then for any locally free $\mathcal{O}_X$-sheaf $\mathcal{F}$, the coherent $\mathcal{O}_Y$-sheaf $i_*\mathcal{F}$ admits a finite resolution by locally free $\mathcal{O}_Y$-sheaves.

We define the *Gysin push-forward*

$$i_*\colon K^0(X) \to K^0(Y)$$

by setting (the map r was defined in (I.1.d))

$$i_*[\mathcal{F}] = r(i_*(\mathcal{F})).$$

This is well-defined since an exact sequence of locally-free $\mathcal{O}_X$-sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

gives an exact sequence of coherent $\mathcal{O}_Y$-sheaves

$$0 \to i_*(\mathcal{F}_1) \to i_*(\mathcal{F}_2) \to i_*(\mathcal{F}_3) \to 0,$$

hence we have $r \circ i_*(\mathcal{F}_2) = r \circ i_*(\mathcal{F}_1) + r \circ i_*(\mathcal{F}_3)$ by (I.1.f).

Lemma I.2.3. *Let $f\colon X \hookrightarrow Y$ be a perfect closed embedding. Then we have a commutative diagram*

$$\begin{array}{ccc} K^0(X) & \xrightarrow{\delta} & K_0(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ K^0(Y) & \xrightarrow{\delta} & K_0(Y). \end{array}$$

Proof. For a locally free $\mathcal{O}_X$-sheaf $\mathcal{E}$, we have, by (I.1.e)

$$\delta \circ f_*[\mathcal{E}] = \delta \circ r\Big(f_*(\mathcal{E})\Big) = [f_*(\mathcal{E})] = f_* \circ \delta[\mathcal{E}]. \qquad \square$$

Lemma I.2.4 (projection formula)**.** *Let $f\colon X \hookrightarrow Y$ be a perfect closed embedding. Then we have, for all $a \in K^0(X)$ and $b \in K^0(Y)$, the formula in $K^0(Y)$*

$$f_*\Big(a \cdot f^*(b)\Big) = f_*(a) \cdot b.$$

Proof. We can assume that $a = [\mathcal{E}]$ and $b = [\mathcal{F}]$ for some vector bundles $\mathcal{E}$ over X and $\mathcal{F}$ over Y. Let $\mathcal{E}_\bullet \to \mathcal{E} \to 0$ be a resolution of the coherent

$\mathcal{O}_Y$-sheaf $\mathcal{E}$ by locally-free $\mathcal{O}_Y$-sheaves. As $\mathcal{F}$ is a flat $\mathcal{O}_Y$-sheaf, the complex $\mathcal{E}_\bullet \otimes_{\mathcal{O}_Y} \mathcal{F} \to \mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F} \to 0$ is exact, hence

$$f_*(a) \cdot b = f_*[\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}].$$

But we have an isomorphism of $\mathcal{O}_X$-sheaves

$$\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F} \simeq \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{F},$$

completing the proof. □

Definition I.2.5. Let $f\colon X \to Y$ be a morphism of varieties. We say that f has *finite Tor-dimension* if for every coherent $\mathcal{O}_Y$-sheaf $\mathcal{F}$ there is an integer n such that

$$\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}) = 0 \text{ for all } k \geq n.$$

Example I.2.6. Examples of morphisms of finite Tor-dimension include perfect closed embeddings (in particular regular closed embeddings) and flat morphisms.

Let $f\colon X \to Y$ be a morphism of finite Tor-dimension. The coherent $\mathcal{O}_Y$-sheaves $\mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})$ have a natural structure of coherent $\mathcal{O}_X$-sheaves. For a short exact sequence of coherent $\mathcal{O}_Y$-sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

we have a long exact sequence

$$\cdots \to \mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}_1) \to \mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}_2) \to \mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}_3) \to \mathsf{Tor}_{i-1}^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F}_1) \to \cdots,$$

hence the association

$$\mathcal{F} \mapsto \sum_i (-1)^i \left[\mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{F})\right]$$

induces a *Gysin pull-back*

$$f^*\colon K_0(Y) \to K_0(X).$$

Lemma I.2.7. *Let $f\colon X \to Y$ be a morphism of finite Tor-dimension. Then we have a commutative diagram*

$$\begin{array}{ccc} K^0(X) & \xrightarrow{\delta} & K_0(X) \\ {\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle f^*} \\ K^0(Y) & \xrightarrow{\delta} & K_0(Y). \end{array}$$

Proof. Take a locally free $\mathcal{O}_Y$-sheaf $\mathcal{E}$. Then, since $\mathcal{E}$ is flat $\mathcal{O}_Y$-sheaf

$$i^* \circ \delta[\mathcal{E}] = \sum_k (-1)^k [\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_X)] = [\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{O}_X] = \delta \circ i^*[\mathcal{E}]. \qquad \square$$

Lemma I.2.8. *Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms of finite Tor-dimension. Then $g \circ f$ is of finite Tor-dimension and*

$$f^* \circ g^* = (g \circ f)^* \colon K_0(Z) \to K_0(X).$$

Proof. This follows from the spectral sequence associated with any coherent $\mathcal{O}_Z$-sheaf $\mathcal{F}$

$$E^2_{p,q} = \mathsf{Tor}_p^{\mathcal{O}_Y}\Big(\mathcal{O}_X, \mathsf{Tor}_q^{\mathcal{O}_Z}(\mathcal{O}_Y, \mathcal{F})\Big) \Rightarrow \mathsf{Tor}_{p+q}^{\mathcal{O}_Z}(\mathcal{O}_X, \mathcal{F}),$$

and from Remark I.1.1. $\square$

Proposition I.2.9 (projection formula)**.** *Let $f\colon X \hookrightarrow Y$ be a perfect closed embedding, $a \in K^0(X)$ and $b \in K^0(Y)$. Then we have in $K^0(X)$*

$$f_*\Big(f^*(b) \cdot a\Big) = b \cdot f_*(a).$$

Proof. We can assume that a and b are classes of vector bundles $\mathcal{A}$ over X and $\mathcal{B}$ over Y. Let $\mathcal{A}_\bullet \to \mathcal{A} \to 0$ be a resolution of the $\mathcal{O}_Y$-sheaf $f_*\mathcal{A}$ by locally free $\mathcal{O}_Y$-sheaves. Then, since $\mathcal{B}$ is a flat $\mathcal{O}_Y$-sheaf,

$$\mathcal{B} \otimes_{\mathcal{O}_Y} \mathcal{A}_\bullet \to \mathcal{B} \otimes_{\mathcal{O}_Y} f_*\mathcal{A} \to 0$$

is a resolution of $\mathcal{B} \otimes_{\mathcal{O}_Y} f_*\mathcal{A}$, and the latter is isomorphic to $f_*(f^*\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{A})$, proving the lemma. $\square$

Proposition I.2.10 (projection formula)**.** *Let $f\colon X \hookrightarrow Y$ be a perfect closed embedding, $a \in K^0(X)$ and $b \in K_0(Y)$. Then we have in $K_0(X)$*

$$f_*\Big(a \cdot f^*(b)\Big) = f_*(a) \cdot b,$$

where K^0 acts on K_0 on the left.

Proof. Take a vector bundle $\mathcal{E}$ over X and a coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$. Then

$$\begin{aligned} f_*\Big([\mathcal{E}] \cdot f^*[\mathcal{F}]\Big) &= \sum_i (-1)^i f_*\Big[\mathcal{E} \otimes_{\mathcal{O}_X} \mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)\Big] \\ &= \sum_i (-1)^i \Big[\mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{E})\Big] \end{aligned}$$

Let $\mathcal{E}_\bullet \to \mathcal{E} \to 0$ be a resolution of the $\mathcal{O}_Y$-sheaf $\mathcal{E}$ by locally free $\mathcal{O}_Y$-sheaves. Then $\mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{E})$ is the i-th homology of the complex $\mathcal{E}_\bullet \otimes_{\mathcal{O}_Y} \mathcal{F} \to 0$. Hence we have in $K_0(Y)$

$$f_*[\mathcal{E}] \cdot [\mathcal{F}] = \sum_i (-1)^i [\mathcal{E}_i \otimes_{\mathcal{O}_Y} \mathcal{F}] = \sum_i (-1)^i \Big[\mathsf{Tor}_i^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{E})\Big] = f_*\Big([\mathcal{E}] \cdot f^*[\mathcal{F}]\Big),$$

as required. $\square$

Definition I.2.11. We say that two arbitrary morphisms $i\colon X \to Y$ and $y\colon Y' \to Y$ are *Tor-independent* when

$$\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0$$

for every integer $k > 0$. We also say that the square

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is *Tor-independent.*

We consider a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ \downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[i]{} & Y. \end{array} \tag{I.2.a}$$

Assume that i (hence i') is a closed embedding. Let $\mathcal{I}$ (*resp.* $\mathcal{I}'$) be the sheaf of ideals corresponding to i (*resp.* i'). Then the normal cone C of i (*resp.* C' of i') is the scheme over X (*resp.* X') associated with the sheaf of algebras

$$\bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1} \quad (\textit{resp.} \quad \bigoplus_n (\mathcal{I}')^n/(\mathcal{I}')^{n+1}).$$

There is a natural surjective homomorphism of $\mathcal{O}_{Y'}$-sheaves $\mathcal{I}^n \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \to (\mathcal{I}')^n$ for every integer n. These homomorphisms induce a surjective homomorphism of $\mathcal{O}_{X'}$-algebras

$$\bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \to \bigoplus_n (\mathcal{I}')^n/(\mathcal{I}')^{n+1},$$

giving a closed embedding

$$k\colon C \hookrightarrow C \times_X X'.$$

Definition I.2.12. We say that the morphisms i and y are *transverse*, or that the commutative square (I.2.a) is *transverse* when the morphism i is a closed embedding and the closed embedding k is an isomorphism [1].

Remark I.2.13. If i is a Cartier divisor, then the square is transverse if and only if i' is a Cartier divisor.

1. This includes the case when X' is empty.

Remark I.2.14. When i is a regular closed embedding of constant relative dimension, the cone C is a vector bundle of rank $\dim Y - \dim X$ over X. Assume that X' is not empty. Then $C \times_X X'$ is a vector bundle of the same rank over X'. Since $\dim C' = \dim Y'$ and $\dim C \times_X X' = \dim Y - \dim X + \dim X'$, we see that for i and j to be transverse, it is necessary that

$$\dim Y - \dim X = \dim Y' - \dim X'.$$

Conversely if X' is reduced and equidimensional of dimension $\dim Y' - \dim Y + \dim X$, then i and j are transverse. This is the case, for instance, when j is a closed embedding and Y' meets X regularly in Y.

Proposition I.2.15. *Let $i\colon X \hookrightarrow Y$ be the embedding of a Cartier divisor, and $y\colon Y' \to Y$ an arbitrary morphism. Then for $k \geq 2$ we have*

$$\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0.$$

Moreover i and y are Tor-independent if and only if they are transverse.

Proof. We still consider the fiber square (I.2.a), and let $\mathcal{I}$ be the sheaf of $\mathcal{O}_Y$-ideals corresponding to the closed embedding i. Consider the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0.$$

The corresponding long exact sequence of Tor associated with tensorisation with $\mathcal{O}_{Y'}$ over $\mathcal{O}_Y$ reads

$$\cdots \to \mathsf{Tor}_{k+1}^{\mathcal{O}_Y}(\mathcal{O}_Y, \mathcal{O}_{Y'}) \to \mathsf{Tor}_{k+1}^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) \to \mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_{Y'}) \to \cdots.$$

Since $\mathcal{O}_Y$ and $\mathcal{I}$ are locally free $\mathcal{O}_Y$-modules, it follows that

$$\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0 \quad (k \geq 2).$$

Let $\mathcal{I}'$ be the sheaf of $\mathcal{O}_{Y'}$-ideals defining i'. We have the exact sequence

$$0 \to \mathsf{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) \to \mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \to \mathcal{I}' \to 0.$$

By Remark I.2.13, i and y are transverse if and only if $\mathcal{I}'$ is locally free of rank one, which occurs if and only if $\mathsf{Tor}_1^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'}) = 0$. □

Remark I.2.16. The case of i and y are arbitrary regular closed embedding is considered in [sga71, VII, Proposition 2.5] or [Tho93, Lemme 3.2]. The restriction to the case of i a Cartier divisor in Proposition I.2.15 allows us to take y to be an arbitrary morphism. This has also the advantage of allowing us to avoid the local arguments given in *loc.cit.*.

Note that while the hypothesis of Proposition I.2.15 does depend on the orientation of the square (I.2.a), the conclusion does not.

Refined Gysin maps

Theorem I.2.17 (Devissage Theorem)**.** *Let $i\colon X \hookrightarrow X'$ be a closed embedding which is a homeomorphism. Then the push-forward*

$$i_*\colon K_0(X) \to K_0(X')$$

is an isomorphism. Its inverse is given by

$$[\mathcal{F}] \mapsto \sum_n \Big[(\mathcal{J}^n \cdot \mathcal{F})/(\mathcal{J}^{n+1} \cdot \mathcal{F})\Big] .$$

where $\mathcal{J}$ is the nilpotent ideal sheaf of X in X'.

Proof. See for example [Wei09, Chapter II, Theorem 6.3]. □

From this theorem, we see that if Y is a closed subvariety of X, then the Grothendieck group of coherent $\mathcal{O}_X$-sheaves supported on Y is isomorphic to $K_0(Y)$. Then any coherent $\mathcal{O}_X$-sheaf supported on Y has a well-defined class

$$[\mathcal{F}]_Y = \sum_k \Big[(\mathcal{I}^k \cdot \mathcal{F})/(\mathcal{I}^{k+1} \cdot \mathcal{F})\Big] \in K_0(Y),$$

where $\mathcal{I}$ is the ideal sheaf of Y in X.

Indeed if n is an integer such that $\mathcal{I}^n$ annihilates $\mathcal{F}$, then the sheaf of ideals $\mathcal{I}^n$ defines an infinitesimal neighbourhood $\widetilde{Y}$ of Y in X. We can apply Theorem I.2.17 to the closed embedding $Y \hookrightarrow \widetilde{Y}$ defined by the sheaf of ideals $\mathcal{J} = \mathcal{I}/\mathcal{I}^n \subset \mathcal{O}_X/\mathcal{I}^n$. Then we have for every integer k an isomorphism

$$(\mathcal{J}^k\mathcal{F})/(\mathcal{J}^{k+1}\mathcal{F}) \simeq (\mathcal{I}^k\mathcal{F})/(\mathcal{I}^{k+1}\mathcal{F}),$$

hence $[\mathcal{F}] \mapsto [\mathcal{F}]_Y$ is the inverse of the push-forward $K_0(Y) \to K_0(\widetilde{Y})$ given by Theorem I.2.17.

Lemma I.2.18. *Let $i\colon Z \hookrightarrow Y$ be a closed embedding of varieties, with Y a closed subvariety of a variety X. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-sheaf supported on Z. Then we have in $K_0(Y)$*

$$i_*([\mathcal{F}]_Z) = [\mathcal{F}]_Y.$$

Proof. Let $\mathcal{I}_Y$ (*resp.* $\mathcal{I}_Z$) be the ideal sheaf of Y (*resp.* Z) in X. Let n be an integer such that ${\mathcal{I}_Z}^n$ annihilates $\mathcal{F}$. Then ${\mathcal{I}_Y}^n \subset {\mathcal{I}_Z}^n$ also annihilates $\mathcal{F}$, and we can consider $\mathcal{F}$ both as an $\mathcal{O}_{\widetilde{Y}}$-sheaf and as an $\mathcal{O}_{\widetilde{Z}}$-sheaf. Let $\widetilde{Y}$ (*resp.* $\widetilde{Z}$) be the closed subvariety of X defined by the sheaf of ideals ${\mathcal{I}_Y}^n$ (*resp.* ${\mathcal{I}_Z}^n$). We have a commutative square of closed embeddings

$$\begin{array}{ccc} Z & \xrightarrow{i} & Y \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle y} \\ \widetilde{Z} & \xrightarrow[j]{} & \widetilde{Y} \end{array}$$

We have in $K_0(\widetilde{Y})$

$$y_* \circ i_*([\mathcal{F}]_Z) = j_* \circ z_*([\mathcal{F}]_Z) = j_*[\mathcal{F}] = [\mathcal{F}] = y_*([\mathcal{F}]_Y).$$

We conclude using Theorem I.2.17 asserting that $y_* \colon K_0(Y) \to K_0(\widetilde{Y})$ is an isomorphism. □

Definition I.2.19. Consider a fiber square

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with y, x closed embeddings, and f a morphism of finite Tor-dimension. Let $\mathcal{F}$ be a coherent $\mathcal{O}_{Y'}$-sheaf. Then the $\mathcal{O}_X$-sheaves

$$\mathsf{Tor}_k^{\mathcal{O}_Y}(y_*(\mathcal{F}), \mathcal{O}_X)$$

are supported on $f^{-1}(Y') = X'$. We define in $K_0(X')$ an element

$$f^!(\mathcal{F}) = \sum_k (-1)^k \left[\mathsf{Tor}_k^{\mathcal{O}_Y}(y_*(\mathcal{F}), \mathcal{O}_X)\right]_{X'}. \tag{I.2.b}$$

If we have an exact sequence of coherent $\mathcal{O}_{Y'}$-sheaves,

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

then we have a long exact sequence

$$\cdots \to \mathsf{Tor}_{k+1}^{\mathcal{O}_Y}(y_*(\mathcal{F}_3), \mathcal{O}_X) \to \mathsf{Tor}_k^{\mathcal{O}_Y}(y_*(\mathcal{F}_1), \mathcal{O}_X) \to \mathsf{Tor}_k^{\mathcal{O}_Y}(y_*(\mathcal{F}_2), \mathcal{O}_X) \to \cdots$$

of coherent $\mathcal{O}_X$-sheaves supported on X'. If $\mathcal{I}$ is the ideal sheaf of X' in X, let $\widetilde{X}$ be the infinitesimal neighborhood of X' defined in X by the ideal sheaf $\mathcal{I}^n$, where n is an integer such that

$$\mathcal{I}^n \cdot \mathsf{Tor}_k^{\mathcal{O}_Y}(y_*(\mathcal{F}_i), \mathcal{O}_X) = 0$$

for all $i = 1, 2, 3$ and $k \geq 0$. Then the long exact sequence above is a sequence of $\mathcal{O}_{\widetilde{X}}$-sheaves, and we have in $K_0(\widetilde{X})$ the equality

$$\sum_{k \geq 0, i=1,2,3} (-1)^{k+i} \left[\mathsf{Tor}_k^{\mathcal{O}_Y}(y_*(\mathcal{F}_i), \mathcal{O}_X)\right] = 0.$$

Applying the isomorphism $K_0(\widetilde{X}) \to K_0(X')$ to the formula above, we see that the formula (I.2.b) gives a well defined map

$$f^! \colon K_0(Y') \to K_0(X').$$

Remark I.2.20. In particular when $x = \mathrm{id}_X$ and $y = \mathrm{id}_Y$ we have $f^! = f^*$.

Lemma I.2.21. *Consider a diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

where each square is cartesian, f is of finite Tor-dimension, vertical arrows are closed embeddings. Then the morphisms $f^!\colon K_0(Y'') \to K_0(X'')$ and $f^!\colon K_0(Y') \to K_0(X')$ satisfy

$$x_* \circ f^! = f^! \circ y_*\colon K_0(Y'') \to K_0(X').$$

Proof. Let $y'\colon Y' \hookrightarrow Y$ be the closed embedding, and $\mathcal{F}$ a coherent $\mathcal{O}_{Y''}$-sheaf. Then by Lemma I.2.18, we have in $K_0(X')$

$$\begin{aligned} x_* \circ f^![\mathcal{F}] &= x_*\Big(\sum_k (-1)^k \left[\mathsf{Tor}_k^{\mathcal{O}_Y}\left({y'}_* \circ y_*(\mathcal{F}), \mathcal{O}_X\right)\right]_{X''} \Big) \\ &= \sum_k (-1)^k \left[\mathsf{Tor}_k^{\mathcal{O}_Y}\left({y'}_* \circ y_*(\mathcal{F}), \mathcal{O}_X\right)\right]_{X'} \\ &= f^! \circ y_*[\mathcal{F}], \end{aligned}$$

as required. ☐

Lemma I.2.22. *Let $f\colon X \to Y$ be a morphism of finite Tor-dimension, and Z a closed subvariety of X. Let $V = Z \times_Y X$, and $\mathcal{F}$ be a coherent $\mathcal{O}_Y$-sheaf supported on Z. Then we have in $K_0(V)$*

$$f^!([\mathcal{F}]_Z) = \sum_k (-1)^k \left[\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)\right]_V.$$

Proof. Let $\mathcal{I}$ be the sheaf of ideals defining Z in X. Choose an integer n such that $\mathcal{I}^n$ annihilates $\mathcal{F}$, and $(\mathcal{I} \cdot \mathcal{O}_X)^n$ annihilates $\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)$ for every integer k. Let $z\colon Z \hookrightarrow \widetilde{Z}$ and $v\colon V \hookrightarrow \widetilde{V}$ be the corresponding infinitesimal neighbourhoods. Since

$$\mathcal{I}^n \cdot \mathcal{O}_X = (\mathcal{I} \cdot \mathcal{O}_X)^n,$$

we have $\widetilde{Z} \times_Y X = \widetilde{V}$. We can apply Lemma I.2.21, and get that

$$v_* \circ f^!([\mathcal{F}]_Z) = f^! \circ z_*([\mathcal{F}]_Z) = f^![\mathcal{F}] = \sum_k (-1)^k \left[\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)\right]_{\widetilde{V}}.$$

By Lemma I.2.18 the latter is equal to

$$v_*\Big(\sum_k (-1)^k \left[\mathsf{Tor}_k^{\mathcal{O}_Y}(\mathcal{F}, \mathcal{O}_X)\right]_V \Big).$$

We conclude using Theorem I.2.17 asserting that $v_*\colon K_0(V) \to K_0(\widetilde{V})$ is an isomorphism. ☐

Lemma I.2.23. *Consider a cartesian square*

$$\begin{array}{ccc} X' & \longrightarrow & Y' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

with f a morphism of finite Tor-dimension, and x, y closed embeddings. Then we have

$$x_* \circ f^! = f^* \circ y_* \colon K_0(Y') \to K_0(X)$$

Proof. This follows at once from Lemma I.2.21, in view of Remark I.2.20. □

Lemma I.2.24. *Consider a diagram*

$$\begin{array}{ccc} X'' & \longrightarrow & Y'' \\ \downarrow & & \downarrow{\scriptstyle y_1} \\ X' & \xrightarrow[g]{} & Y' \\ \downarrow & & \downarrow{\scriptstyle y_2} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where each square is cartesian, f and g are of finite Tor-dimension, vertical arrows are closed embeddings, and the bottom square is Tor-independent. Then we have

$$f^! = g^! \colon K_0(Y'') \to K_0(X'').$$

Proof. Let $\mathcal{F}$ be a coherent $\mathcal{O}_{Y''}$-sheaf. We have a spectral sequence

$$E^2_{p,q} = \mathsf{Tor}_p^{\mathcal{O}_{Y'}}\Big((y_1)_*(\mathcal{F}), \mathsf{Tor}_q^{\mathcal{O}_Y}(\mathcal{O}_X, \mathcal{O}_{Y'})\Big) \Rightarrow \mathsf{Tor}_{p+q}^{\mathcal{O}_Y}\Big((y_2 \circ y_1)_*(\mathcal{F}), \mathcal{O}_X\Big).$$

Because of the assumption of Tor-independence, it collapses at E^2 and we get

$$\mathsf{Tor}_p^{\mathcal{O}_{Y'}}\Big((y_1)_*(\mathcal{F}), \mathcal{O}_{X'}\Big) \simeq \mathsf{Tor}_p^{\mathcal{O}_Y}\Big((y_2 \circ y_1)_*(\mathcal{F}), \mathcal{O}_X\Big).$$

This proves the claim. □

Corollary I.2.25. *If the cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow[f]{} & Y \end{array}$$

is Tor-independent, f and g are morphisms of finite Tor-dimension, and vertical arrows are closed embeddings, then

$$f^! = g^* \colon K_0(Y') \to K_0(X').$$

Proof. We already noticed that $g^! = g^*\colon K_0(Y') \to K_0(X')$. Now we apply Lemma I.2.24 with $Y'' = Y'$, and get the statement. □

Combining Lemma I.2.23 and Corollary I.2.25, we obtain

Proposition I.2.26. *If the cartesian square*

$$\begin{array}{ccc} X' & \xrightarrow{g} & Y' \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is Tor-independent, x, y are closed embeddings, and f, g are of finite Tor-dimension, then

$$f^* \circ y_* = x_* \circ g^*\colon K_0(Y') \to K_0(X).$$

Note that the hypothesis of Proposition I.2.26 are automatically satisfied when the square is cartesian, f is flat, and x, y are closed embeddings.

Euler class of line bundles

For a line bundle

$$L = \operatorname{Spec}(\operatorname{Sym}\mathcal{L}^\vee)$$

over a variety X, with sheaf of sections $\mathcal{L}$, its Euler class $e(L) = e(\mathcal{L})$ is the endomorphism of $K^0(X)$ (*resp.* $K_0(X)$) given by multiplication by (*resp.* action of) the element $1 - [\mathcal{L}^\vee]$.

Lemma I.2.27. *If $s\colon X \hookrightarrow L$ is the zero section of the line bundle L over X, then*

$$e(L) = s^* \circ s_*,$$

*the homomorphism $s_*\colon K^0(X) \to K^0(L)$ (*resp. *$s^*\colon K_0(L) \to K_0(X)$) being the Gysin push-forward (*resp. *pull-back) associated with the regular closed embedding s.*

Proof. Let $\mathcal{L}$ be the sheaf of sections of line bundle L, and $p\colon L \to X$ the line bundle map. Let $x_0 \in K_0(X)$. Then using Lemma I.2.8 and the projection formula Proposition I.2.10 we have in $K_0(X)$

$$s^* \circ s_*(x_0) = s^* \circ s_* \circ s^* \circ p^*(x_0) = s^*\Big(s_*[\mathcal{O}_X] \cdot p^*(x_0)\Big) = (s^* \circ s_*[\mathcal{O}_X]) \cdot x_0.$$

Similarly, using the projection formula Proposition I.2.9 instead of Proposition I.2.10, we see that for $x^0 \in K^0(X)$, we have in $K^0(X)$

$$s^* \circ s_*(x^0) = (s^* \circ s_*[\mathcal{O}_X]) \cdot x^0.$$

Therefore, it will be enough to prove that

$$s^* \circ s_*[\mathcal{O}_X] = 1 - [\mathcal{L}^\vee] \in K^0(X).$$

We have an exact sequence of $\mathcal{O}_L$-sheaves

$$0 \to p^*\mathcal{L}^\vee \to \mathcal{O}_L \to s_*\mathcal{O}_X \to 0.$$

We can express $s_*[\mathcal{O}_X] \in K^0(L)$ with the help of this resolution by locally free $\mathcal{O}_L$-sheaves. We get $s_*[\mathcal{O}_X] = [\mathcal{O}_L] - p^*[\mathcal{L}^\vee]$. Applying the pull-back s^*, we obtain the expected formula. □

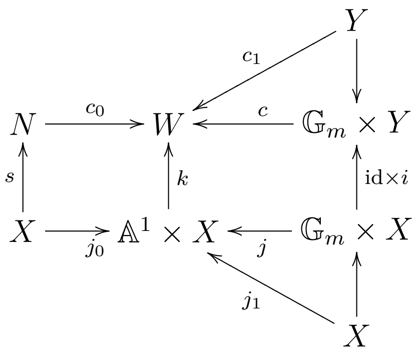
Deformation to the normal cone

Let $i\colon X \hookrightarrow Y$ be a closed embedding, with normal cone N over X. We define the *deformation homomorphism*

$$\sigma\colon K_0(Y) \to K_0(N)$$

as follows.

Let W be the deformation scheme of i, *i.e.* the open complement of the strict transform of $\{0\} \times Y$ in the blow-up of $\mathbb{A}^1 \times Y$ along $\{0\} \times X$. There is a closed embedding $\mathbb{P}(N \oplus 1) - \mathbb{P}(N) = N \hookrightarrow W$ with open complement $(\mathbb{A}^1 \times Y) - (\{0\} \times Y) = \mathbb{G}_m \times Y$. We have a commutative square



(I.2.c)

The morphisms c_0 and j_0 are principal Cartier divisors, with open complements $\mathbb{G}_m \times Y$ and $\mathbb{G}_m \times X$. The closed embedding s is the zero section of the normal cone N.

Take an element $x \in K_0(Y)$, and an antecedent $y \in K_0(W)$ of $[\mathcal{O}_{\mathbb{G}_m}] \boxtimes x$ under the epimorphism $c^*\colon K_0(W) \to K_0(\mathbb{G}_m \times Y)$. Then define

$$\sigma(x) = s^* \circ {c_0}^*(y).$$

To check that this element does not depend on the choice of y, notice that two choices differ by $(c_0)_*(z)$, for some $z \in K_0(N)$ (by localization Theorem I.1.2), and, using Lemma I.2.27, we have

$${c_0}^* \circ (c_0)_*(z) = e(L)(z),$$

where L is normal bundle of c_0. This bundle is trivial, as c_0 is a principal Cartier divisor, hence its Euler class given is given by the action of $[L]-[\mathcal{O}_N] = 0$, hence $\sigma(x)$ does not depend on the choice of y.

Proposition I.2.28. *Let $i\colon X \hookrightarrow Y$ be a regular closed embedding, and $p\colon N \to X$ its normal bundle. Then we have*

$$\sigma = p^* \circ i^*.$$

Proof. We use the notations of the diagram (I.2.c). By homotopy invariance Theorem I.3.4 the pull-back p^* is an isomorphism, and by Lemma I.2.8 its inverse coincides with s^*. For the same reasons, the homomorphisms ${j_0}^*$ and ${j_1}^*$ both coincide with $(q^*)^{-1}$, where $q\colon \mathbb{A}^1 \times X \to X$ is the second projection.

Let $x \in K_0(Y)$, and $y \in K_0(W)$ be such that $c^*(y) = [\mathcal{O}_{\mathbb{G}_m}] \boxtimes x$. We have $\sigma(x) = {c_0}^*(y)$ and $x = {c_1}^*(y)$, hence using Lemma I.2.8

$$\begin{aligned}(p^*)^{-1} \circ \sigma(x) &= s^* \circ \sigma(x) = s^* \circ {c_0}^*(y)\\ &= {j_0}^* \circ k^*(y) = {j_1}^* \circ k^*(y)\\ &= i^* \circ {c_1}^*(y) = i^*(x).\end{aligned}$$

□

We can define the Gysin pullback $i^\star\colon K_0(Y) \to K_0(X)$ along a regular closed embedding $i\colon X \hookrightarrow Y$ by the formula

$$i^\star = (p^*)^{-1} \circ \sigma$$

where $p\colon N \to X$ is the normal bundle of i. We see with Proposition I.2.28 that indeed $i^\star = i^*$.

Proposition I.2.29. *Let $i\colon X \hookrightarrow Y$ be a regular closed embedding, and $y\colon Y' \hookrightarrow Y$ a closed embedding. Let C be the normal bundle of the closed embedding $X \cap Y' \hookrightarrow Y'$, N the normal bundle of i, $f\colon C \hookrightarrow N$ the natural closed embedding, and $\sigma\colon K_0(Y) \to K_0(N)$ the deformation homomorphism. Then*

$$\sigma[\mathcal{O}_{Y'}] = f_*[\mathcal{O}_C].$$

Proof. Let W' be the deformation scheme associated with i'. Then we have cartesian squares

$$\begin{array}{ccccc} C & \xrightarrow{c_0'} & W' & \xleftarrow{j'} & \mathbb{G}_m \times Y' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle \mathrm{id}\times y} \\ N & \xrightarrow[c_0]{} & W & \xleftarrow[j]{} & \mathbb{G}_m \times Y \end{array}$$

The square on the right hand side is Tor-independent because j is a flat morphism. The square on the left hand side is transverse by Remark I.2.13, hence Tor-independent by Proposition I.2.15.

As an antecedent of $[\mathcal{O}_{\mathbb{G}_m}] \boxtimes [\mathcal{O}_{Y'}]$, one can take $w_*[\mathcal{O}_{W'}]$, by Proposition I.2.26 applied to the square on the right hand side. Then, using again Proposition I.2.26 for w, f, c_0 and c_0', we see that

$$c_0{}^* \circ w_*[\mathcal{O}_W] = f_* \circ c_0'^*[\mathcal{O}_W] = f_*[\mathcal{O}_C]. \qquad \square$$

I.3 The graded ring associated with the topological filtration

Comparison with Chow groups

In this paragraph, we describe a natural transformation between the Chow group CH and the graded group associated with the topological filtration $\mathrm{gr}^{\mathsf{top}} K_0$.

Let $Z(X)$ be the free abelian group with generators the integral subvarieties of a variety X over a field F. To a closed integral subvariety Z of X corresponds a *prime cycle* $[Z]$ in $Z(X)$. Letting $Z_k(X)$ be the subgroup generated by the k-dimensional integral subvarieties of X, we obtain a grading on $Z(X)$. If $f\colon X \to Y$ is a proper morphism of F-varieties, and Z a closed subvariety of X, we define an element of $Z(Y)$ as

$$f_*[Z] = \begin{cases} \deg\Big(F(Z)/F(f(Z))\Big) \cdot [f(Z)] \text{ if } \dim Z = \dim f(Z), \\ 0 \text{ otherwise.} \end{cases}$$

The push forward $f_*\colon Z(X) \to Z(Y)$ respects the gradings and passes to rational equivalence.

Definition I.3.1. The Chow group $\mathrm{CH}(X)$ of a variety X is the quotient of $Z(X)$ by rational equivalence. We shall write $\mathrm{CH}_k(X)$ for the k-th (homologically) graded piece.

When X is smooth with connected components X_i, the group $\mathrm{CH}(X)$ has the structure of a graded ring, where

$$\mathrm{CH}^q(X) = \coprod_i \mathrm{CH}_{\dim X_i - q}(X_i).$$

Proposition I.3.2 ([Ful98, Example 15.1.5])**.** *The group homomorphism*

$$Z_k(X) \to \mathrm{gr}_k^{\mathsf{top}} K_0(X) \quad , \quad [Z] \mapsto [\mathcal{O}_Z] \quad \mathrm{mod} \;\; \mathrm{F}_{k-1}^{\mathsf{top}} K_0(X)$$

passes to rational equivalence and induces a surjective graded group homomorphism,

$$\varphi = \varphi_X\colon \mathrm{CH}(X) \to \mathrm{gr}^{\mathsf{top}} K_0(X),$$

which commutes with proper push-forwards.

Proof. The association above gives a homomorphism

$$\varphi_X \colon Z(X) \to \operatorname{gr}^{\mathsf{top}} K_0(X)$$

which is surjective and respects the gradings. If $f\colon X \to Y$ is a proper morphism of varieties, then it follows from Proposition I.1.14 that we have

$$f_* \circ \varphi_X = \varphi_Y \circ f_*. \tag{I.3.a}$$

If $W \hookrightarrow X \times \mathbb{P}^1$ is a subvariety such that the morphism $f\colon W \to \mathbb{P}^1$ is dominant, then the fiber over any rational point of $\mathbb{P}^1$ is a Cartier divisor in W associated with the line bundle $f^*\mathcal{O}(1)$. It follows that the element of $\operatorname{gr}^{\mathsf{top}} K_0(W)$ associated to such a fiber is

$$[\mathcal{O}_W] - [f^*\mathcal{O}(1)]$$

and therefore does not depend on the choice of the rational point of $\mathbb{P}^1$. Let $p\colon X \times \mathbb{P}^1 \to X$ be the projection. We have shown that the element $p_* \circ \varphi_{X\times\mathbb{P}^1}[f^{-1}(a)]$ of $K_0(X)$ does not depend on the choice of the rational point a of $\mathbb{P}^1$. By (I.3.a) above, this element coincides with $\varphi_X \circ p_*[f^{-1}(a)]$.

By [EKM08, Proposition 57.6], rational equivalence on X is generated by relations

$$[f^{-1}(a)] = [f^{-1}(b)]$$

where a and b runs over the rational points of $\mathbb{P}^1$, and W over the closed subvarieties of $X \times \mathbb{P}^1$ dominant over $\mathbb{P}^1$.

It follows that φ_X passes to rational equivalence. □

We provide an alternate description of the map φ in the next section.

Proposition I.3.3. *The map φ commutes with pull-backs along flat morphisms.*

Proof. Let $f\colon X \to Y$ be a flat morphism of varieties. Take an integral closed subvariety Z of Y and form the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & Y \end{array}$$

It will be enough to prove that

$$\varphi \circ f^* \circ i_*[Z] = f^* \circ \varphi \circ i_*[Z] = f^*[\mathcal{O}_Z].$$

We have, both in $\operatorname{gr}^{\mathsf{top}} K_0$ (Proposition I.2.26) and in CH ([EKM08, Proposition 49.20]), the formula

$$j_* \circ g^* = f^* \circ i_*.$$

Applying $\varphi = \varphi_X$, we get

$$\begin{aligned}\varphi \circ f^* \circ i_*[Z] &= j_* \circ \varphi \circ g^*[Z] = j_* \circ \varphi[W] \\ &= j_*[\mathcal{O}_W] = j_* \circ g^*[\mathcal{O}_Z] \\ &= f^* \circ i_*[\mathcal{O}_Z] = f^*[\mathcal{O}_Z],\end{aligned}$$

which concludes the proof □

Brown-Gersten-Quillen spectral sequence

Here we quickly describe a classical way of dealing with the topological filtration, especially useful when considering higher K-groups. This allows to give an alternate definition of the transformation φ. We will not really need the point of view developed in this paragraph in the sequel.

Let X be a variety over a field F. For a point $x \in X$, we let $F(x)$ be its residue field, $\overline{\{x\}}$ the closed subvariety of X with generic point x, and we define

$$\dim x = \dim \overline{\{x\}} = \operatorname{tr.deg}(F(x)/F).$$

For a variety X, we denote by $K_n(X)$ Quillen higher K-groups of the category $\mathsf{M}(X)$ of coherent sheaves on X. The abelian category $\mathsf{M}(X)$ admits a filtration by exact subcategories

$$\mathsf{M}_{-1}(X) = 0 \subset \mathsf{M}_0(X) \subset \cdots \subset \mathsf{M}_{\dim X - 1}(X) \subset \mathsf{M}_{\dim X}(X) = \mathsf{M}(X).$$

Taking Quillen n-th K-groups, we obtain a *topological filtration* on $K_n(X)$ by setting

$$K_n(X)_{(m)} = \operatorname{im}\Big(K_n(\mathsf{M}_m(X)) \to K_n(\mathsf{M}(X)) = K_n(X)\Big),$$

This of course coincides with our previous definition of the topological filtration when $n = 0$. When R is a commutative ring, we use the notation $K_n(R)$ for $K_n(\operatorname{Spec}(R))$.

A combination of localization and devissage yields the *BGQ spectral sequence*

$$E^1_{p,q} = \bigoplus_{\dim x = p} K_{p+q}(F(x)) \Rightarrow K_{p+q}(X)$$

the filtration induced on $K_{p+q}(X)$ being the topological filtration. By this we mean that

$$E^\infty_{p,q} \simeq \operatorname{gr}^{\mathsf{top}}_p K_{p+q}(X).$$

The differential

$$E^1_{p+1,-p} = \bigoplus_{\dim x = p+1} F(x)^\times \to E^1_{p,-p} = Z_p(X)$$

is given by, for $\dim x = p+1$

$$\operatorname{div}\colon F(x)^\times \to Z_p\left(\overline{\{x\}}\right), \quad f \mapsto \sum_{\dim y = p} \operatorname{ord}_y(f)\cdot\left[\overline{\{y\}}\right].$$

Therefore we have an isomorphism

$$E^2_{p,-p} \simeq \mathrm{CH}_p(X)$$

Note that $E^1_{p,q} = 0$ for $p+q<0$, hence there is a surjective edge homomorphism

$$\varphi\colon E^2_{p,-p} \simeq \mathrm{CH}_p(X) \to E^\infty_{p,-p} \simeq \mathrm{gr}^{\mathsf{top}}_p K_0(X),$$

which is precisely the morphism φ_X considered above.

Multiplicativity of the topological filtration

Proposition I.3.4 (Homotopy invariance of the topological filtration)**.** *Let $p\colon E \to X$ be a vector bundle of rank d. Then the induced pull-back homomorphisms*

$$p^*\colon\ \mathrm{F}^{\mathsf{top}}_\bullet K_0(X) \to \mathrm{F}^{\mathsf{top}}_{\bullet+d} K_0(E) \quad \textit{and} \quad \mathrm{gr}^{\mathsf{top}}_\bullet K_0(X) \to \mathrm{gr}^{\mathsf{top}}_{\bullet+d} K_0(E)$$

are isomorphisms.

Proof. By induction on n, using the isomorphisms $\mathrm{F}^{\mathsf{top}}_n K_0(-) \simeq 0$ for $n<0$ and the commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{F}^{\mathsf{top}}_{d+n-1} K_0(E) & \to & \mathrm{F}^{\mathsf{top}}_{d+n} K_0(E) & \to & \mathrm{gr}^{\mathsf{top}}_{d+n} K_0(E) & \to & 0 \\ & & \uparrow {\scriptstyle \mathrm{F}^{\mathsf{top}}_{n-1}(p^*)} & & \uparrow {\scriptstyle \mathrm{F}^{\mathsf{top}}_{n}(p^*)} & & \uparrow {\scriptstyle \mathrm{gr}^{\mathsf{top}}_{n}(p^*)} & & \\ 0 & \to & \mathrm{F}^{\mathsf{top}}_{n-1} K_0(X) & \to & \mathrm{F}^{\mathsf{top}}_{n} K_0(X) & \to & \mathrm{gr}^{\mathsf{top}}_{n} K_0(X) & \to & 0 \end{array}$$

we see by the 5-lemma that

- $\mathrm{F}^{\mathsf{top}}_n(p^*)$ is an isomorphism for all $n \iff \mathrm{gr}^{\mathsf{top}}_n(p^*)$ is an isomorphism for all n.
- $\mathrm{gr}^{\mathsf{top}}_n(p^*)$ is an epimorphism for all $n \Longrightarrow \mathrm{F}^{\mathsf{top}}_n(p^*)$ is an epimorphism for all n.

First note that the map $\mathrm{F}^{\mathsf{top}}_\bullet(p^*)\colon\ \mathrm{F}^{\mathsf{top}}_\bullet K_0(X) \to \mathrm{F}^{\mathsf{top}}_{\bullet+d} K_0(E)$ is injective, as the restriction of the monomorphism $p^*\colon K_0(X) \to K_0(E)$ (which is split by the pull-back s^* along the zero section $s\colon X \hookrightarrow E$).

By Proposition I.3.3, we have a commutative diagram

$$\begin{array}{ccc} \mathrm{CH}_{\bullet+d}(E) & \xrightarrow{\varphi_E} & \mathrm{gr}^{\mathsf{top}}_{\bullet+d} K_0(X) \\ \uparrow {\scriptstyle p^*} & & \uparrow {\scriptstyle \mathrm{gr}^{\mathsf{top}}_\bullet(p^*)} \\ \mathrm{CH}_\bullet(X) & \xrightarrow[\varphi_X]{} & \mathrm{gr}^{\mathsf{top}}_\bullet K_0(X) \end{array}$$

It follows from the homotopy invariance of Chow groups ([EKM08, Theorem 57.13]) that the map on the left hand side is surjective. Since φ_E is also surjective, the map $\mathrm{gr}^{\mathsf{top}}_{\bullet}(p^*)$ on the right is surjective.

By the preliminary remarks in the beginning of the proof, we get that $\mathrm{F}^{\mathsf{top}}_{\bullet}(p^*)$ is an epimorphism, hence an isomorphism. Then we get that $\mathrm{gr}^{\mathsf{top}}_{\bullet}(p^*)$ is an isomorphism. □

Remark I.3.5 ([Gil05, Lemma 85])**.** One can also prove directly that the kernel of natural transformation $\varphi\colon \mathrm{CH} \to \mathrm{gr}^{\mathsf{top}} K_0$ is homotopy invariant. This kernel is controlled by differentials in the BGQ spectral sequence. It is proven in [Gil81, Theorem 8.3] that all E^2 terms of this spectral sequence are homotopy invariant, therefore that the whole spectral sequence is homotopy invariant from E^2 on, which yields the homotopy invariance not only of $E^{\infty}_{p,-p} = \mathrm{gr}^{\mathsf{top}}_p K_0$ but also of $E^{\infty}_{p,q} = \mathrm{gr}^{\mathsf{top}}_p K_{p+q}$.

Corollary I.3.6 ([Gil05, Theorem 83])**.** *Let $i\colon X \hookrightarrow Y$ be a regular closed embedding of constant codimension c. Then for any integer k*

$$i^*(\mathrm{F}^{\mathsf{top}}_k K_0(Y)) \subset \mathrm{F}^{\mathsf{top}}_{k-c} K_0(X).$$

Proof. Let $p\colon N \to X$ be the normal bundle of i. Let Z be a k-dimensional integral closed subvariety of Y. Then the normal cone C of the induced closed embedding $Z \cap X \hookrightarrow Z$ has pure dimension k by [Ful98, B.6.6, p.436], hence if $j\colon C \hookrightarrow N$ is the natural closed embedding, we have by Propositions I.2.28 and I.2.29

$$p^* \circ i^*[\mathcal{O}_Z] = \sigma[\mathcal{O}_Z] = j_*[\mathcal{O}_C] \in \mathrm{F}^{\mathsf{top}}_k K_0(N).$$

The statement now follows from Proposition I.3.4. □

It follows that a regular closed embedding $i\colon X \hookrightarrow Y$ of constant codimension c induces a homomorphism

$$i^*\colon \mathrm{gr}^{\mathsf{top}}_n K_0(Y) \to \mathrm{gr}^{\mathsf{top}}_{n-c} K_0(X).$$

Combining Corollary I.3.6 and Lemma I.2.27, we get

Corollary I.3.7. *Let L be a line bundle over a variety X. Then the Euler class of L sends $\mathrm{F}^{\mathsf{top}}_n K_0(X)$ to $\mathrm{F}^{\mathsf{top}}_{n-1} K_0(X)$.*

Remark I.3.8. This corollary can be proved directly, as in the course of the proof of [sga71, X, Théorème 1.3.2].

We can now proceed with the proof of the multiplicativity of the topological filtration.

Lemma I.3.9 ([Gil05, Lemma 82]). *Let X and Y be varieties over a common field, $x \in \mathrm{F}_n^{\mathsf{top}} K_0(X)$ and $y \in \mathrm{F}_m^{\mathsf{top}} K_0(Y)$. Then*

$$x \boxtimes y \in \mathrm{F}_{n+m}^{\mathsf{top}} K_0(X \times Y).$$

Proof. Let $U \subset X$ be a n-dimensional subvariety and $V \subset Y$ a m-dimensional subvariety. Then $[\mathcal{O}_U] \boxtimes [\mathcal{O}_V] = [\mathcal{O}_{U\times V}]$ by Lemma I.1.4, and $U \times V$ has dimension $n + m$ □

Therefore this external product induces an external product at the level of the associated graded groups, which will denoted by the same symbol ⊠.

Proposition I.3.10 ([Gil05, § 2.5.11]). *Let X be a smooth variety. The topological filtration is a ring filtration,* i.e. *for every elements $x \in K(X)^{(m)}$ and $y \in K(X)^{(n)}$, the product $x \cdot y$ belongs to $K(X)^{(n+m)}$.*

Proof. The element $x \boxtimes y$ belongs to $K(X \times X)^{(n+m)}$ by Lemma I.3.9. From (I.1.b), we know that $x \cdot y = \Delta^*(x \boxtimes y)$. By Lemma I.2.7, we see that we can compute $x \cdot y$ as $\Delta^*(x \boxtimes y)$, where $\Delta^* \colon K_0(X \times X) \to K_0(X)$ is the Gysin pull-back. Hence by Corollary I.3.6, the element $x \cdot y$ belongs to $K(X)^{(n+m)}$. □

This gives $\operatorname{gr} K(X)$ a ring structure, for any smooth variety X.

Projective bundles

Lemma I.3.11 ([FL85, Chapter V, Lemma 3.8]). *Let E be a vector bundle of rank $r + 1$ on a variety X. Let $p \colon \mathbb{P}(E) \to X$ be the associated projective bundle. Then the pull-back*

$$p^* \colon K_0(X) \to K_0(\mathbb{P}(E))$$

is injective.

For all integer n and all $x \in K_0(X)$ we have

$$p^*(x) \in \mathrm{F}_{n+r}^{\mathsf{top}} K_0(\mathbb{P}(E)) \Longrightarrow x \in \mathrm{F}_n^{\mathsf{top}} K_0(X).$$

In other words the map

$$p^* \colon \operatorname{gr}_\bullet^{\mathsf{top}} K_0(X) \to \operatorname{gr}_{\bullet+r}^{\mathsf{top}} K_0(\mathbb{P}(E))$$

is also injective.

Proof. We consider the Euler class of the canonical line bundle

$$h = e(\mathcal{O}(1)) \colon K_0(\mathbb{P}(E)) \to K_0(\mathbb{P}(E)),$$

which coincides with the action of $1 - [\mathcal{O}(-1)] \in K^0(\mathbb{P}(E))$. For a positive integer m, we denote by $h^m = h \circ \cdots \circ h$ the m-th iterate of h.

We have for all integers $0 > k \geq -r$, $i \geq 0$, $j > 0$, and all coherent $\mathcal{O}_X$-sheaves $\mathcal{F}$ ([FL85, p.106])

$$\mathsf{R}^i p_*\Big(p^*\mathcal{F} \otimes \mathcal{O}(k)\Big) = 0 \ \ , \ \ p_*(p^*\mathcal{F}) \simeq \mathcal{F} \ \ , \ \ \mathsf{R}^j p_*(p^*\mathcal{F}) = 0.$$

It follows that for a coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$, we have in $K_0(X)$

$$p_* \circ h^r \circ p^*[\mathcal{F}] = \sum_i \sum_{k=0}^{r} (-1)^{i+k} \binom{r}{k} \Big[\mathsf{R}^i p_*(p^*\mathcal{F} \otimes \mathcal{O}(-k))\Big] = [\mathcal{F}].$$

This shows that the map p^* is injective.

Now if x is an element of $K_0(X)$ such that $p^*(x) \in \mathrm{F}^{\mathsf{top}}_{n+r} K_0(\mathbb{P}(\mathcal{E}))$, then by Corollary I.3.7 we have

$$h^r \circ p^*(x) \in \mathrm{F}^{\mathsf{top}}_n K_0(\mathbb{P}(E)),$$

hence $x = p_*\Big(h^r \circ p^*(x)\Big) \in p_*(\mathrm{F}^{\mathsf{top}}_n K_0(\mathbb{P}(E))) \subset \mathrm{F}^{\mathsf{top}}_n K_0(X)$, which concludes the proof. □

Proposition I.3.12 ([sga71, IX, Théorème 1.3.2])**.** *Let X be a variety, and k, n integers. We have*

$$\mathrm{F}^k_\gamma K^0(X) \cdot \mathrm{F}^{\mathsf{top}}_n K_0(X) \subset \mathrm{F}^{\mathsf{top}}_{n-k} K_0(X).$$

Proof. By Lemma I.3.11, it is enough to check the statement after pulling-back along an arbitrary projective bundle. By definition of the gamma filtration, we see that it is then enough to show that for a vector bundle E over X we have

$$\Big([E] - \mathrm{rank}(E)\Big) \cdot \mathrm{F}^{\mathsf{top}}_n K_0(X) \subset \mathrm{F}^{\mathsf{top}}_{n-1} K_0(X).$$

Using the splitting principle, we reduce to the case where E is a line bundle. Then multiplication by $([E] - \mathrm{rank}(E))$ is the opposite of the Euler class of the line bundle $E^\vee$, and the proposition follows from Corollary I.3.7. □

It follows from Proposition I.3.12 that for a variety X, the Poincare homomorphism δ satisfies $\delta(\mathrm{F}^k_\gamma K^0(X)) \subset \mathrm{F}^{\mathsf{top}}_k K_0(X)$ and therefore induces a homomorphism of graded groups:

$$\delta_G \colon \ \mathrm{gr}_\gamma K^0(X) \to \mathrm{gr}^{\mathsf{top}} K_0(X). \tag{I.3.b}$$

Corollary I.3.13. *Let X be a regular variety. We have* $\mathrm{F}^{\dim X+1}_\gamma K^0(X) = 0$.

Remark I.3.14. The hypothesis of regularity is not needed for this statement. For a proof of this when X is an arbitrary noetherian scheme possessing an ample invertible sheaf, see [FL85, Chapter V, Corollary 3.10], where a topological filtration on K^0 is introduced.

The following corollary will be useful.

Corollary I.3.15. *Let $f\colon X \to Y$ be a projective morphism, with X an equidimensional variety and Y a variety. Then*

$$f_* \circ \delta\Big(\mathrm{F}_\gamma^k K^0(X)\Big) \subset \mathrm{F}^{\mathsf{top}}_{\dim X - k} K_0(Y).$$

We conclude with the statement of the projective bundle theorem.

Theorem I.3.16 (Projective bundle Theorem, [sga71, VI, Théorème 1.1 and IX, Corollaire 3.2])**.** *Let E be a vector bundle of constant rank d over a variety X. Let $p\colon \mathbb{P}(E) \to X$ be the associated projective bundle. Then the pull-back homomorphism p^* makes $K^0(\mathbb{P}(E))$ a free $K^0(X)$-module, and $K_0(\mathbb{P}(E))$ a free $K_0(X)$-module with basis*

$$\Big(e\Big(\mathcal{O}(1)\Big)\Big)^i, \quad 0 \le i < d.$$

Comparison with the Chow ring

Proposition I.3.17. *The map $\varphi\colon \mathrm{CH} \to \mathrm{gr}^{\mathsf{top}} K_0$ commutes with pull-backs along regular closed embeddings.*

Proof. Let $f\colon X \hookrightarrow Y$ be a regular closed embedding of varieties. As in the proof of Proposition I.3.3, we take an integral subvariety Z of Y, form the cartesian square

$$\begin{array}{ccc} W & \xrightarrow{g} & Z \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{f} & Y \end{array}$$

and prove that

$$\varphi \circ f^* \circ i_*[Z] = f^* \circ \varphi \circ i_*[Z] = f^*[\mathcal{O}_Z].$$

Let $p\colon N \to X$ be the normal bundle of f. We consider the deformation homomorphism $\sigma = p^* \circ f^*$ in both theories. Let C be the normal cone of the closed embedding $Z \cap X \hookrightarrow Z$, and $j\colon C \hookrightarrow N$ the natural closed embedding.

By [EKM08, Proposition 52.7], we have in $\mathrm{CH}(N)$

$$\sigma \circ i_*[Z] = j_*[C].$$

By Proposition I.2.29, we have in $K_0(N)$

$$\sigma \circ i_*[\mathcal{O}_Z] = j_*[\mathcal{O}_C],$$

hence

$$\sigma \circ \varphi[Z] = \varphi \circ \sigma[Z],$$

and φ commutes with deformation homomorphisms. Then φ commutes with $f^* = (p^*)^{-1} \circ \sigma$, by Proposition I.3.3 applied to p. □

Corollary I.3.18. *The morphism φ commutes with pull-backs along arbitrary morphisms of smooth varieties.*

Proof. If $f\colon X \to Y$ is an arbitrary morphism of smooth varieties, then it factors as $X \hookrightarrow X \times Y \to Y$, a regular closed embedding (the graph of f) followed by a flat morphism (the projection to the second factor). The claim now follows from Proposition I.3.3 and Proposition I.3.17. □

Lemma I.3.19. *Let X and Y be varieties over a common field, $x \in \mathrm{CH}(X)$ and $y \in \mathrm{CH}(Y)$. Then we have in $\mathrm{gr}^{\mathsf{top}} K_0(X \times Y)$ the equality*

$$\varphi_{X\times Y}(x \times y) = \varphi_X(x) \boxtimes \varphi_Y(y).$$

Proof. Both sides of the formula are bilinear in x an y, so that it will be enough to consider the case when $x = [A]$ and $y = [B]$ for $A \hookrightarrow X$ and $B \hookrightarrow Y$ closed subvarieties. Using Lemma I.1.4, we have

$$\begin{aligned}\varphi_{X\times Y}([A] \times [B]) &= \varphi_{X\times Y}([A \times B]) = [\mathcal{O}_{A\times B}]\\ &= [\mathcal{O}_A \boxtimes \mathcal{O}_B] = [\mathcal{O}_A] \boxtimes [\mathcal{O}_B]\\ &= \varphi_X[A] \boxtimes \varphi_Y[B],\end{aligned}$$

as required. □

The combination of Proposition I.3.17 and Lemma I.3.19 yields

Corollary I.3.20. *If X is smooth, the map φ_X is a ring homomorphism.*

I.4 Presheaves with characteristic class

The results and arguments exposed in this section are classical (mostly from [Gro58] and [Hir66] to the best of our knowledge), but we want to state them in a way suited to our purposes.

Let F be a field. The category of varieties (*resp.* smooth varieties) over F, with arbitrary morphisms of varieties, will be denoted as Sch/F (*resp.* Sm/F).

At some point we shall also consider the category Reg/F of regular varieties over F, and flat morphisms.

In the sequel the category C will be either Sch/F, Sm/F or Reg/F. These categories share the following property:

If $X \in \mathsf{C}$ and $p\colon Y \to X$ is a projective bundle, then $Y \in \mathsf{C}$ and p is a morphism in C.

For a presheaf A on C and a morphism $f\colon X \to Y$ in C we shall write f^* for the morphism $A(f)\colon A(Y) \to A(X)$.

Definition I.4.1. A presheaf A of commutative rings on C is a *presheaf with characteristic class* if the following conditions hold.

a) If $f\colon X \to Y$ is a projective bundle and $X, Y \in \mathsf{C}$, then $f^*\colon A(Y) \to A(X)$ is injective.

b) There is a morphism of presheaves of abelian groups on C, the *characteristic class*
$$c_t = \sum_i c_i \cdot t^i\colon K^0 \to (A(-)[[t]])^\times.$$

c) For every variety $X \in \mathsf{C}$, and every line bundle L on X, the element $c_1(L)$ is a nilpotent element of the ring $A(X)$, $c_0(L) = 1$, and $c_k(L) = 0$ for $k > 1$.

The associated *total characteristic class* is the composite $c = c_t|_{t=1} = \rho_1 \circ c_t$ where $\rho_1\colon A(-)[t] \to A$ is evaluation at $t = 1$.

A morphism of presheaves with characteristic class $f\colon (A, c_t^A) \to (B, c_t^B)$ is a morphism $f\colon A \to B$ of presheaves of rings such that $f \circ c_t^A = c_t^B$.

Note that a characteristic class is allowed to be trivial, that is $c_i = 0$ for $i > 0$ is possible. However one can easily check that the characteristic classes that will be introduced here are indeed not trivial.

We now record some direct consequences of the axioms in Definition I.4.1.

Lemma I.4.2. *Let A be a presheaf with characteristic class c, and E a vector bundle on a variety X. Then for every integer $k > 0$ the element $c_k(E)$ is nilpotent in the ring $K^0(X)$. Moreover we have*
$$c_k(E) = 0 \quad \big(k > \operatorname{rank}(E)\big).$$

Proof. We can easily reduce to the case when X is connected.

Let r be the rank of E. Take a composite of projective bundle $p\colon Y \to X$ such that $[p^*E]$ splits in $K^0(Y)$ as $\sum_{i=1}^r [L_i]$ with $L_i, i = 1, \cdots, r$ line bundles on Y. Then by (c), we have $c_t(L_i) = 1 + c_1(L_i) \cdot t$, and by (b)
$$c_t(p^*E) = \prod_{i=1}^r c_t(L_i) = \sum_k p^* \circ c_k(E) \cdot t^k,$$
hence $p^* \circ c_k(E)$ is the k-th symmetric polynomial of the family $c_1(L_i), i = 1, \cdots, r$. Since the elements $c_1(L_i)$ are nilpotent by (c), it follows that the elements $p^* \circ c_k(E)$ are also nilpotent for $k > 0$. When $k \geq r + 1$, we see that $p^* \circ c_k(E) = 0$.

We conclude using (a). $\square$

Chern classes

We now describe canonical structures of a presheaves with characteristic class on some examples of cohomological theories, that will be useful for us.

If $\mathcal{E}$ is a locally free sheaf, we write $\Lambda^i\mathcal{E}$ for its i-th external power. This operation induces, for every variety X, a functor

$$\Lambda^i\colon \mathsf{VB}(X) \to \mathsf{VB}(X).$$

Given an exact sequence

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0$$

the locally free sheaf $\Lambda^i(\mathcal{E}_2)$ admits a filtration whose successive quotients are isomorphic to

$$\Lambda^k\mathcal{E}_3 \otimes \Lambda^{i-k}\mathcal{E}_1$$

hence the association

$$\mathcal{E} \mapsto \lambda_t(\mathcal{E}) = \sum_i [\Lambda_i(\mathcal{E})] \cdot t^i \in (K^0(X)[[t]])^\times$$

induces a group homomorphism

$$\lambda_t\colon K^0 \to (K^0(X)[[t]])^\times,$$

and individual *lambda operations*

$$\lambda^i\colon K^0 \to K^0(X)$$

such that

$$\lambda_t = \sum_i \lambda^i \cdot t^i.$$

Now we define the *gamma operations* by

$$\gamma_t = \lambda_{\frac{t}{1-t}} = \sum_i \gamma^i(x) \cdot t^i \in K^0(X)[[t]].$$

For variety X over a field F, we consider the idempotent endomorphism

$$\mathrm{rank}\colon K^0(X) \to K^0(X)$$

defined as the composition $K^0(X) \xrightarrow{\mathrm{rank}} K^0(F^{\pi_0(F)}) \xrightarrow{p^*} K^0(X)$ where $p\colon X \to \mathrm{Spec}(F^{\pi_0(F)})$ is natural surjective morphism.

This rank homomorphism can be used to define the i-th *Chern class with values in* K^0 as

$$x \mapsto c_i(x) = (-1)^i\gamma^i\Big(x^\vee - \mathrm{rank}(x)\Big),$$

where $x \mapsto x^\vee$ is the involution of $K^0(X)$ induced by the duality on $\mathsf{VB}(X)$.

We also define the total Chern class

$$x \mapsto c_t(x) = \sum_i c_i(x)t^i = \gamma_{-t}\Big(x^\vee - \mathrm{rank}(x)\Big). \tag{I.4.a}$$

Remark I.4.3. For a line bundle

$$L = \mathrm{Spec}(\mathrm{Sym}\,\mathcal{L}^\vee)$$

over a variety X, with sheaf of sections $\mathcal{L}$, we have

$$\begin{aligned} c_t(L) &= \lambda_{\frac{-t}{1+t}}(L^\vee)\cdot\lambda_{\frac{-t}{1+t}}(1)^{-1} \\ &= \Big(1-\frac{t}{1+t}[L^\vee]\Big)\cdot(1+t)^{-1} \\ &= 1+t\cdot(1-[L^\vee]), \end{aligned}$$

hence

$$c_0(L) = 1 \quad \text{and} \quad c_k(L) = 0 \quad (k > 1). \tag{I.4.b}$$

The first Chern class with values in K^0 is given by the formula

$$c_1(L) = c_1(\mathcal{L}) = 1 - [L^\vee] \tag{I.4.c}$$

and coincides with the Euler class of L.

This convention agrees with [Pan04] and [LM07], but is dual to the choice made in [FL85] and [Sou85].

Example I.4.4 (Chern classes with values in K^0)**.** We have seen in Lemma I.3.11 that when $p\colon Y \to X$ is a projective bundle, the pull-back

$$p^*\colon K^0(X) \to K^0(Y)$$

is injective. This is Condition (a) of Definition I.4.1 for the presheaf K^0 on Sch/F.

The Chern class of (I.4.a) is a morphism

$$c_t\colon K^0 \to (K^0(-)[[t]])^\times \quad , \quad x \mapsto \gamma_{-t}\Big(x^\vee - \mathrm{rank}(x)\Big)$$

as required for Condition (b).

Condition (c) is (I.4.b), and nilpotency of $c_1(L) = 1-[L^\vee]$, when L is a line bundle, can be checked using the presence of an ample line bundle on every object of our category Sch/F, by a formal computation as in [FL85, Chapter 3, Lemma 1.4].

Therefore we have shown that (I.4.a) gives K^0 the structure of a presheaf with characteristic class on Sch/F.

Example I.4.5 (Chern classes with values in Chow groups)**.** We will always endow the Chow group functor with its usual Chern class (defined using the projective bundle theorem). This makes CH a presheaf with characteristic class on Sm/F. For a line bundle L, with zero section $s\colon X \hookrightarrow L$, its first Chern class is given by

$$c_1(L) = s^* \circ s_*(1),$$

and coincides once again with the Euler class.

Finally, we notice that the Chern classes with values in K^0 could also have been defined by means of the projective bundle theorem.

Proposition I.4.6. *Let E be a vector bundle of rank r on a variety X, and $p\colon \mathbb{P}(E) \to X$ the associated projective bundle. Let*

$$h = e(\mathcal{O}(1)) = 1 - [\mathcal{O}(-1)] \in K^0(\mathbb{P}(E)).$$

Then we have

$$\sum_{i=0}^{r+1} (-1)^{r+1-i} h^{r+1-i} \cdot p^* \circ c_i(E) = 0.$$

Proof. We have $c_t(\mathcal{O}(1)) = 1 + t \cdot h$ hence

$$c_t(-\mathcal{O}(1)) = \frac{1}{1 + t \cdot h} = \sum_{k=0}^{\infty} (-1)^k t^k h^k.$$

There is a surjective vector bundle morphism $p^*E \to \mathcal{O}(1)$ over $\mathbb{P}(E)$, hence the class $p^*[E] - [\mathcal{O}(1)] \in K^0(\mathbb{P}(E))$ is represented by a vector bundle of rank r over $\mathbb{P}(E)$. Therefore, by Lemma I.4.2, we have $c_{r+1}(p^*[E] - [\mathcal{O}(1)]) = 0$, hence

$$0 = \sum_{i=0}^{r+1} c_{r+1-i}(-\mathcal{O}(1)) \cdot p^* \circ c_i(E) = \sum_{i=0}^{r+1} (-1)^{r+1-i} h^{r+1-i} \cdot p^* \circ c_i(E),$$

as required. □

The gamma filtration

The gamma filtration was defined in Definition I.1.6. The next couple of propositions provide an explanation for the name of this filtration.

Proposition I.4.7. *Let X be a variety, $x \in K^0(X)$ and i an integer. Then we have*

$$c_i(x) = (-1)^i \gamma^i\Big(x^\vee - \operatorname{rank}(x)\Big) \in \mathrm{F}^i_\gamma K^0(X).$$

Moreover the ideal $\mathrm{F}^1_\gamma K^0(X)$ coincides with the set of first Chern classes $c_1(x)$ for $x \in K^0(X)$.

Proof. Take a composite of projective bundles $p\colon Y \to X$ such that

$$p^*\Big(x - \operatorname{rank}(x)\Big) = \sum_i([L_i]-1) - \sum_j([M_j]-1)$$

for some line bundles L_i and M_j on $\mathbb{P}(\mathcal{E})$. Then

$$\begin{aligned}
p^* \circ c_t(x) &= \gamma_{-t}\Big(p^*(x^\vee) - \operatorname{rank} \circ p^*(x)\Big)\\
&= \gamma_{-t}\Big(\sum_i([L_i^\vee]-1) - \sum_j([M_j^\vee]-1)\Big)\\
&= \prod_i \gamma_{-t}([L_i^\vee]-1)\cdot \prod_j \gamma_{-t}([M_j^\vee]-1)^{-1}\\
&= \prod_i \Big(1+t(1-[L_i^\vee])\Big)\prod_j\Big(1+t(1-[M_j^\vee])\Big)^{-1}\\
&\subset \prod_i \Big(1+t\ker(\operatorname{rank})\Big)\prod_j\Big(1+t\ker(\operatorname{rank})\Big)^{-1}\\
&\subset \sum_k t^k\Big(\ker(\operatorname{rank})\Big)^k,
\end{aligned}$$

The first statement now follows from Condition (a) of the definition I.1.6 of the gamma filtration.

Note that γ^1 is the identity, hence $c_1\colon x \mapsto x^\vee - \operatorname{rank}(x)$. Since rank is a split idempotent endomorphism of $K^0(X)$, we have $\operatorname{im}(\operatorname{id}-\operatorname{rank}) = \ker(\operatorname{rank})$. The latter is equal to $\mathrm{F}^1_\gamma K^0(X)$ by Lemma I.1.7, and the second statement follows. □

Using arguments that shall not be fully reproduced here, one can prove a stronger statement, that will not be used in the sequel.

Proposition I.4.8. *The gamma filtration is the ring filtration generated by the conditions*

$$c_i(x) = (-1)^i\gamma^i\Big(x^\vee - \operatorname{rank}(x)\Big) \in \mathrm{F}^i_\gamma K^0(X)$$

for all varieties X, and every element $x \in K^0(X)$.

Proof. Let $\mathrm{G}^k_\gamma K^0(X)$ be the k-th term of the ring filtration defined by the conditions $c_i(x) \in \mathrm{G}^i_\gamma K^0(X)$ for every integer i and every $x \in K^0(X)$. From Proposition I.4.7 we know that

$$\mathrm{G}^1_\gamma K^0(X) = \mathrm{F}^1_\gamma K^0(X) \quad \text{and} \quad \mathrm{G}^k_\gamma K^0(X) \subset \mathrm{F}^k_\gamma K^0(X)$$

Using [FL85, Chapter V, Corollary 2.5], we see that for any projective bundle $p\colon Y \to X$ we have

$$(p^*)^{-1}\,\mathrm{G}^k_\gamma K^0(Y) \subset \mathrm{G}^k_\gamma K^0(X)$$

hence the filtration $\mathrm{G}_\gamma^\bullet K^0(X)$ satisfies the generating conditions of the filtration $\mathrm{F}_\gamma^\bullet K^0(X)$ and

$$\mathrm{F}_\gamma^k K^0(X) \subset \mathrm{G}_\gamma^k K^0(X),$$

for every integer k. □

Example I.4.9 (Chern classes with values in $\mathrm{gr}_\gamma K^0$)**.** The presheaf $\mathrm{gr}_\gamma K^0$ on Sch/F satisfies Condition (a) of Definition I.4.1. This is an immediate consequence of Condition (a) of Definition I.1.6.

Let c_k be the Chern classes with values in K^0 of Example I.4.4 above. Using Proposition I.4.7 above, we define a characteristic class c^γ by the formula

$$c_i^\gamma(E) = c_i(E) \mod \mathrm{F}_\gamma^{i+1} K^0(X) \in \mathrm{gr}_\gamma^i K^0(X)$$

for a vector bundle E on a variety X, and an integer i.

Condition (c) of Definition I.4.1 follows from the corresponding statement concerning the Chern class with values in K^0 of Example I.4.4.

This gives $\mathrm{gr}_\gamma K^0$ the structure of a presheaf with characteristic class on Sch/F. We shall sometimes call c_i^γ the *i-th Chern classes with values in* $\mathrm{gr}_\gamma K^0$.

Todd homomorphisms

We now introduce some notations in order to state the next proposition. For any variety $X \in \mathsf{C}$ ($= \mathsf{Sch}/F$, Sm/F or Reg/F), we write $K_+^0(X)$ for the submonoid of $(K^0(X), +)$ generated by the classes of vector bundles, this give a presheaf of monoids K_+^0 on C.

Given a presheaf A of commutative rings on C, we write $A_{\mathsf{mult}}(X)$ for the monoid associated with the multiplicative law of the ring $A(X)$. This defines another presheaf of monoids A_{mult} on C.

Proposition I.4.10 ([Pan04, Proposition 2.2.3])**.** *Let A be a presheaf with characteristic class c, and $u \in A(\mathrm{Spec}(F))[[t]]$ be a power series. Then there exists a unique morphism of presheaves of monoids on* C

$$\tau_u \colon K_+^0 \to A_{\mathsf{mult}}$$

such that for a line bundle $\mathcal{L}$ on a variety $X \in \mathsf{C}$, we have in $A(X)$

$$\tau_u(\mathcal{L}) = u\Big(c_1(\mathcal{L})\Big).$$

Proof. Let $\mathcal{E}$ be a vector bundle of rank r over a variety X in C.

We first prove unicity. Take a composite of projective bundles $p\colon Y \to X$ such that $p^*[\mathcal{E}] = \sum_i [\mathcal{L}_i]$ in $K^0(Y)$, with $\mathcal{L}_i$ line bundles on Y. Then any morphism τ_u satisfying the conditions of the proposition satisfies the formula

$$p^* \circ \tau_u(\mathcal{E}) = \tau_u(p^*\mathcal{E}) = \prod_i u\Big(c_1(\mathcal{L}_i)\Big). \tag{I.4.d}$$

By Condition (a) of Definition I.4.1, the map p^* is injective, and the unicity of τ_u follows.

We now construct the morphism τ_u. Write the symmetric power series

$$\prod_{i=1}^{r} u(t_i)$$

as a power series $\mathfrak{u}(\sigma_1, \cdots, \sigma_r)$ in the symmetric functions $\sigma_1, \cdots, \sigma_r$ in the variables t_i, for $i = 1, \cdots, r$. The power series $\mathfrak{u}$ has coefficients in the ring $A(\mathrm{Spec}(F))$. Consider the ring $A(X)$ as an $A(\mathrm{Spec}(F))$-algebra *via* the pull-back along the structural morphism $X \to \mathrm{Spec}(F)$, and define

$$\tau_u(\mathcal{E}) = \mathfrak{u}\Big(c_1(\mathcal{E}), \cdots, c_r(\mathcal{E})\Big) \in A(X).$$

This is well-defined because the elements $c_i(\mathcal{E})$ are nilpotent for $i > 0$ by Lemma I.4.2. This gives a functorial construction satisfying the required formula on line bundles.

If $[\mathcal{E}_1] = [\mathcal{E}_2] + [\mathcal{E}_3]$ in $K^0(X)$ we take a composite of projective bundles $p\colon Y \to X$ such that the vector bundles $p^*\mathcal{E}_i$, for $i = 2, 3$ split as sums of line bundles (at least as elements of $K^0(X)$). Equation (I.4.d) shows that $\tau_u(\mathcal{E}_1) = \tau_u(\mathcal{E}_2) \cdot \tau_u(\mathcal{E}_3)$. □

We shall call τ_u the *Todd homomorphism* associated with u with values in the presheaf with characteristic class A on C.

Lemma I.4.11. *Let A be a presheaf with characteristic class, and $u \in A(\mathrm{Spec}(F))[[t]]$ be a power series. Assume that the constant term of u is invertible in the ring $A(\mathrm{Spec}(F))$. Then the Todd homomorphism τ_u induces a natural transformation of presheaves of abelian groups*

$$\tau_u\colon K^0 \to A^\times.$$

Proof. Let v be the power series multiplicatively inverse to u. If $\mathcal{L}$ is a line bundle, then $\tau_u(\mathcal{L})$ has multiplicative inverse $\tau_v(\mathcal{L})$. Therefore $x \mapsto \tau_u(x) \cdot \tau_v(x)$ is the Todd homomorphism associated by Proposition I.4.10 with the constant power series 1. By unicity, it coincides with the constant morphism

$$x \mapsto 1, \quad K^0_+ \to A_{\mathsf{mult}},$$

hence τ_u has values in presheaf $A^\times$ of invertible elements in A_{mult}. Then τ_u naturally extends to a morphism of presheaves of abelian groups $K^0 \to A^\times$. □

One can perform a similar construction in an additive fashion.

Proposition I.4.12. *Let A be a presheaf with characteristic class c, and $u \in A(\mathrm{Spec}(F))[[t]]$ be a power series. Then there exists a unique morphism of presheaves of abelian groups*

$$\alpha_u \colon K^0 \to A$$

such that for a line bundle $\mathcal{L}$ we have

$$\alpha_u(\mathcal{L}) = u\Big(c_1(\mathcal{L})\Big).$$

We shall call α_u the *additive Todd homomorphism* associated with u with values in the presheaf with characteristic class A on C.

The examples of additive Todd homomorphisms that we shall consider will additionally satisfy the condition of the next lemma.

Lemma I.4.13. *Let A be a presheaf with characteristic class c, and $u \in A(\mathrm{Spec}(F))[[t]]$ be a power series. Assume that for every pair of line bundles $(\mathcal{L}, \mathcal{L}')$ on any variety X, we have in $A(X)$*

$$\Big(u \circ c_1(\mathcal{L})\Big) \cdot \Big(u \circ c_1(\mathcal{L}')\Big) = u \circ c_1(\mathcal{L} \otimes \mathcal{L}').$$

Then the additive Todd homomorphism associated with u and with values in A is a morphism of presheaves of rings.

Proof. Let $\mathcal{E}$ and $\mathcal{F}$ be vector bundles on a variety $X \in \mathsf{C}$, and $p\colon$ a composite of projective bundles such that both $p^*[\mathcal{E}] = \sum_i [\mathcal{L}_i]$ and $p^*[\mathcal{F}] = \sum_j [\mathcal{M}_j]$ in $K^0(Y)$, with $\mathcal{L}_i$, $\mathcal{M}_j$ line bundles on Y. Then we have in $A(Y)$

$$\begin{aligned}
\tau_u \circ p^*([\mathcal{E}] \cdot [\mathcal{F}]) &= \tau_u \left(\sum_{i,j} [\mathcal{L}_i \otimes \mathcal{M}_j] \right) \\
&= \sum_{i,j} u([\mathcal{L}_i \otimes \mathcal{M}_j]) \\
&= \sum_{i,j} \Big(u \circ c_1(\mathcal{L}_i)\Big) \cdot \Big(u \circ c_1(\mathcal{M}_j)\Big) \\
&= \tau_u \circ p^*(\mathcal{E}) \cdot \tau_u \circ p^*(\mathcal{F}).
\end{aligned}$$

We conclude using Condition (a) of Definition I.4.1. □

Remark I.4.14. When A is an oriented cohomology theory in the sense of [LM07], the condition of the lemma can be formulated purely in terms of u and of the formal group law $\mathrm{FGL}(x, y)$ of A as follows

$$u(x) \cdot u(y) = u\Big(\,\mathrm{FGL}(x, y)\Big) \in A(\mathrm{Spec}(F))[[x, y]].$$

Any presheaf of rings on C is endowed with an external product $\boxtimes$ as follows. Let X and Y be varieties in C such that $X \times Y \in \mathsf{C}$. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the two projections. Then the external product is given by

$$(x, y) \mapsto p_X^*(x) \cdot p_Y^*(y) = x \boxtimes y.$$

Lemma I.4.15. *Let A be a presheaf with characteristic class on C. Let X and Y be varieties in C such that $X \times Y \in \mathsf{C}$, and $\mathcal{E}$ (*resp. *$\mathcal{F}$) be a vector bundle over the variety X (*resp. *Y). Take a power series u with coefficients in $A(\mathrm{Spec}(F))$, and let τ_u be the associated Todd homomorphism with values in A. Then we have*

$$\tau_u(\mathcal{E} \times \mathcal{F}) = \tau_u(\mathcal{E}) \boxtimes \tau_u(\mathcal{F})$$

If u satisfies the condition of Lemma I.4.13, i.e. *if the corresponding additive Todd homomorphism α_u is ring morphism then*

$$\alpha_u(\mathcal{E} \boxtimes \mathcal{F}) = \alpha_u(\mathcal{E}) \boxtimes \alpha_u(\mathcal{F})$$

Proof. Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ be the two projections. Then $\mathcal{E} \times \mathcal{F} \simeq p_X^*\mathcal{E} \oplus p_Y^*\mathcal{F}$, so that

$$\begin{aligned}
\tau_u(\mathcal{E} \times \mathcal{F}) &= \tau_u(p_X^*\mathcal{E}) \cdot \tau_u(p_Y^*\mathcal{F}) \\
&= \Big(p_X^* \circ \tau_u(\mathcal{E})\Big) \cdot \Big(p_Y^* \circ \tau_u(\mathcal{F})\Big) \\
&= \tau_u(\mathcal{E}) \boxtimes \tau_u(\mathcal{F}).
\end{aligned}$$

In case α_u is a ring morphism, we have

$$\begin{aligned}
\alpha_u(\mathcal{E} \boxtimes \mathcal{F}) &= \alpha_u(p_X^*\mathcal{E} \cdot p_Y^*\mathcal{F}) \\
&= \alpha_u(p_X^*\mathcal{E}) \cdot \alpha_u(p_Y^*\mathcal{F}) \\
&= \alpha_u(\mathcal{E}) \boxtimes \alpha_u(\mathcal{F}).
\end{aligned}$$

□

Adams operations, Chern character, top Chern class

In this paragraph we apply the theory of presheaves with characteristic class developped above to define some classical natural transformations, and state some of their basic properties.

Example I.4.16 (Adams operations)**.** Choose an integer $n \in \mathbb{Z}$. We define the *n-th Adams operation*

$$\psi^n : K^0 \to K^0$$

as the additive Todd homomorphism associated with the power series

$$u_n(t) = \frac{1}{(1-t)^n}$$

with values in the presheaf K^0 on Sch/F.

If $\mathcal{L}$ is a line bundle on a variety X, we have

$$\psi^n([\mathcal{L}]) = \frac{1}{(1 - c_1(\mathcal{L}))^n} = [\mathcal{L}]^n.$$

It follows that the power series u_n satisfies the condition of Lemma I.4.13, and that ψ^n is ring homomorphism. Note that $\psi^0 = \mathrm{rank}$, $\psi^1 = \mathrm{id}$ and $\psi^{-1}(x) = [x^\vee]$ for any variety X, and $x \in K^0(X)$.

Also $\psi^n \cdot \psi^m$ and ψ^{nm} are both additive Todd homomorphism associated with u_{nm} and therefore coincide as morphism of presheaves of rings $K^0 \to K^0$.

It follows from Lemma I.4.15 that we have

$$\psi^n(x \boxtimes y) = \psi^n(x) \boxtimes \psi^n(y)$$

when X and Y are varieties over a common field, and $x \in K^0(X)$, $y \in K^0(Y)$.

Because of the properties of the Adams operations described above, we shall often restrict ourselves to the case of n a prime number.

The behaviour of the Adams operation with respect to the topological filtration will be studied in some details in Chapter III. For the moment we state a couple of easy facts concerning the Adams operation.

Lemma I.4.17. *Let X be a variety, and $x \in K^0(X)$ be such that*

$$\delta(x) \in \mathrm{F}_k^{\mathsf{top}} K_0(X),$$

where $\delta\colon K^0(X) \to K_0(X)$ is the Poincare homomorphism. Then

$$\delta \circ \psi^n(x) \in \mathrm{F}_k^{\mathsf{top}} K_0(X).$$

Proof. The hypothesis $\delta(x) \in \mathrm{F}_k^{\mathsf{top}} K_0(X)$ implies that there is a closed subvariety $i\colon Y \hookrightarrow X$, with $\dim Y \leq k$, such that $\delta(x)$ belongs to the image of the map $i_*\colon K_0(Y) \to K_0(X)$. Let $u\colon U \to X$ be the open complement of i. Then we have

$$u^* \circ \delta \circ \psi^n(x) = \delta \circ u^* \circ \psi^n(x) = \delta \circ \psi^n \circ u^*(x)$$

Since $u^*(x) = 0$, the localization theorem I.1.2 implies that $\delta \circ \psi^n(x)$ also belongs to the image of i_*. In particular it belongs to $\mathrm{F}_k^{\mathsf{top}} K_0(X)$. □

Lemma I.4.18. *Let X be a variety, $x \in K^0(X)$, and p a prime number. Then we have*

$$\psi^p(x) = x^p \mod pK^0(X).$$

Proof. The association $x \mapsto x^p$ induces a morphism of presheaves of abelian groups $K^0 \to \mathbb{Z}/p \otimes K^0$, because p is prime. If $\mathcal{L}$ is a line bundle then

$$[\mathcal{L}]^p = \frac{1}{(1 - c_1(\mathcal{L}))^p},$$

hence the morphism induced by $x \mapsto x^p$ is the additive Todd homomorphism associated with the power series

$$\frac{1}{(1-t)^p}$$

with values in the presheaf with characteristic class $\mathbb{Z}/p \otimes K^0$. Therefore it coincides with the composite

$$K^0 \xrightarrow{\psi^p} K^0 \to \mathbb{Z}/p \otimes K^0. \qquad \square$$

Example I.4.19 (Chern character)**.** The *Chern character* ch is the additive Todd homomorphism associated with the power series

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

with values in the presheaf $\mathbb{Q} \otimes \mathrm{CH}$ on Sm/F. If $\mathcal{L}$ and $\mathcal{L}'$ are line bundles over a variety X, we have

$$e^{c_1(\mathcal{L}\otimes\mathcal{L}')} = e^{c_1(\mathcal{L})+c_1(\mathcal{L}')} = e^{c_1(\mathcal{L})} \cdot e^{c_1(\mathcal{L}')}$$

because the Chow group functor has additive formal group law ([EKM08, Proposition 57.26]). Therefore by Lemma I.4.13, we have constructed a morphism of presheaves of rings on Sm/F

$$\mathrm{ch}\colon K \to \mathbb{Q} \otimes \mathrm{CH}\,.$$

For every integer n, we shall denote by ch_n the composite $K \xrightarrow{\mathrm{ch}} \mathbb{Q} \otimes \mathrm{CH} \to \mathbb{Q} \otimes \mathrm{CH}^n$.

The following lemma explains the interaction between the Chern character and the Adams operations.

Lemma I.4.20. *Let X be a smooth variety, and $x \in K(X)$. For every integers k and n, we have*

$$\mathrm{ch}_n \circ \psi^k(x) = k^n \cdot \mathrm{ch}_n(x).$$

Proof. The morphisms

$$\mathrm{ch} \circ \psi^k \quad \text{and} \quad \sum_n k^n \cdot \mathrm{ch}_n$$

are both additive Todd homomorphisms associated with the power series

$$e^{k \cdot t} = \sum_{n=0}^{\infty} \frac{k^n}{n!} t^n$$

in the variable t with values in the presheaf CH on Sm/F. Therefore they coincide by the unicity statement in Proposition I.4.12. □

We now describe a Todd homomorphism that will be used in the proof of Riemann-Roch theorems.

Example I.4.21 (Top Chern class)**.** Evaluating the polynomial

$$\lambda_x(E^\vee) = \sum_i \lambda^i(E^\vee) \cdot x^i$$

at $x = -1$ for every vector bundle E gives a morphism of presheaves of monoids

$$x \mapsto \lambda_{-1}(x^\vee) \colon K^0_+ \to K^0_{\mathsf{mult}}$$

which coincides with the Todd homomorphism associated with the power series t with values in the presheaf with characteristic classes K^0.

Lemma I.4.22. *Let N be a vector bundle of constant rank over a variety X, and $s \colon X \hookrightarrow \mathbb{P}(N \oplus 1)$ the zero section of its projective completion. Then there is a vector bundle Q on $\mathbb{P}(N \oplus 1)$ satisfying $s^*Q = N$, and such that for any $x \in K^0(\mathbb{P}(N \oplus 1))$, we have*

$$s_* \circ s^*(x) = \lambda_{-1}(Q^\vee) \cdot x.$$

Proof. Let Q be the vector bundle over $\mathbb{P}(N \oplus 1)$ fitting in the exact sequence

$$0 \to \mathcal{O}(-1) \to p^*(N \oplus 1) \to Q \to 0.$$

Then $s^*Q = N$, and the map $\mathcal{O}_{\mathbb{P}(N \oplus 1)} = p^*(0 \oplus 1) \to p^*(N \oplus 1) \to Q$ gives a section of Q whose scheme of zeroes is X ([FL85, Chapter IV, Proposition 2.7]), hence its dual fits in the Koszul resolution (see [Ful98, p.282])

$$0 \to \Lambda^d Q^\vee \to \cdots \to \Lambda^2 Q^\vee \to Q^\vee \to \mathcal{O}_{\mathbb{P}(N \oplus 1)} \to \mathcal{O}_X \to 0,$$

where d is the rank of N. It follows that

$$s_*[\mathcal{O}_X] = \lambda_{-1}(Q^\vee).$$

We conclude using the projection formula Lemma I.2.4. □

Comparison with the Chow ring, characteristic classes

Recall that we have defined the functor $\operatorname{gr} K$ as the restriction of the presheaf of abelian groups $\operatorname{gr}^{\mathsf{top}} K_0$ to the category Sm/F. The product in K indeed induces the structure of a presheaf of rings on $\operatorname{gr} K$ by Proposition I.3.10.

Proposition I.4.23. *The morphism of presheaves of rings on* Sm/F

$$\delta_G\colon \operatorname{gr}_\gamma K^0|_{\mathsf{Sm}/F} \to \operatorname{gr} K$$

(see (I.3.b)*) and the Chern class with values* $\operatorname{gr}_\gamma K^0$ *of Example I.4.9 gives* $\operatorname{gr} K$ *the structure of a presheaf with characteristic class on* Sm/F*.*

Proof. We need to check the conditions of Definition I.4.1.

Condition (a) is Lemma I.3.11.

Condition (c) formally follows from the corresponding properties of the Chern class with values in $\operatorname{gr}_\gamma K^0$ (which in turn followed from the property of the Chern class with values in K^0). □

We shall call this characteristic class *the Chern class with values in* $\operatorname{gr} K$.

Proposition I.4.24. *The natural transformation* $\varphi\colon \mathrm{CH} \to \operatorname{gr} K$ *is a morphism of presheaves with characteristic class on* Sm/F*.*

Proof. Let L be line bundle on a smooth variety X, and $s\colon X \hookrightarrow L$ its zero section. We check that $\varphi_X \circ c_1(L) = c_1(L)$, which will be sufficient, in view of the splitting principle. We have the equality

$$s^* \circ s_*(1) = c_1(L)$$

both in the Chow group of X and in $K_0(X)$ (by Lemma I.2.27), and φ commutes with both the push-forward s_* (Proposition I.3.2) and the pull-back s^* (Proposition I.3.17), and sends 1 to 1. This concludes the proof. □

Our next aim is to prove that the morphism φ_X has torsion kernel for every variety X. This can be reduced to a Riemann-Roch-type theorem, and we begin by considering the case of a smooth X, because in this case the statement is easier to prove, and we can provide an explicit map in the reverse direction, which will be useful later on. This is based on [Ful98, Example 15.3].

In the course of the proof of the next proposition, when the base field will not be perfect, we shall need the notion of Chern classes for Chow groups of regular, not necessarily smooth, varieties. The projective bundle theorem for

Chow groups allows to construct a *total Chern class homomorphism* ([EKM08, p.250]), for an arbitrary variety X

$$c\colon K^0(X) \to \mathrm{Aut}(\mathrm{CH}_\bullet(X)).$$

If $f\colon X \to Y$ is a flat morphism, and $x \in K^0(Y)$, we have ([EKM08, Proposition 54.5])

$$f^* \circ c(x) = c\Big(f^*(x)\Big) \circ f^*.$$

When k is an integer, X an equidimensional variety, and $x \in K^0(X)$, we shall write $c_k(x)$ for the k-codimensional homogeneous component of the element $c(x)([X])$. Using [EKM08, Proposition 58.15], one can check that this agrees with our previous definition of $c_k(x)$ when X is smooth.

This generalization does not exactly fit in our setting of presheaves with characteristic classes, because we are not dealing with a presheaf of rings anymore. However the splitting principle is still available, and reasonings similar to Proposition I.4.10 are still possible, only less pleasant to formulate.

Proposition I.4.25. *Let X be a smooth connected variety, and Y an integral subvariety of codimension n. Then we have for all integers* $0 < k < n$

$$c_k[\mathcal{O}_Y] = 0 \in \mathrm{CH}(X).$$

Moreover

$$c_n[\mathcal{O}_Y] = (-1)^{n-1}(n-1)! \cdot [Y] \in \mathrm{CH}(X).$$

Proof. Let $u\colon U \to X$ be the open complement of Y in X. Then for $0 < k < n$, we have

$$u^* \circ c_k[\mathcal{O}_Y] = c_k \circ u^*[\mathcal{O}_Y] = c_k(0) = 0.$$

Using the localization sequence for Chow groups, we see that $c_k[\mathcal{O}_Y] \in i_* \mathrm{CH}(Y) \subset \mathrm{CH}^{\geq n}(X)$, where $i\colon Y \hookrightarrow X$ is the closed embedding. Since for $0 < k < n$, we have $c_k[\mathcal{O}_Y] \in \mathrm{CH}^{<n}(X)$, and we get

$$c_k[\mathcal{O}_Y] = 0 \in \mathrm{CH}(X).$$

Note that this argument also shows that $c_n[\mathcal{O}_Y] = m \cdot [Y]$ for some integer $m \in \mathbb{Z}$, and what remains to be proven is that $m = (-1)^{n-1}(n-1)!$.

We now prove the second assertion. It will be more convenient to prove the slightly stronger statement in which the hypothesis of smoothness of X is replaced by the assumption that X is a regular variety. We use the remarks preceding the statement of the proposition to define $c_n[\mathcal{O}_Y]$ in this case.

The set of points $y \in Y$ such that the local ring $\mathcal{O}_{Y,y}$ is not regular is closed in Y ([Gro65, Corollaire 6.12.5]), hence in X, and does not contain the generic point of Y, therefore its open complement U in X meets Y. The variety $Y \cap U$ is a regular connected variety of codimension n in the smooth variety U. Using the localization sequence for Chow groups and the fact that U has closed complement of codimension at least $n+1$ in X, we see that, while proving the second statement of the proposition that we can further assume that Y is regular variety.

Under this additional assumption, the closed embedding is regular, as is any closed embedding of regular varieties by [Bou07, Proposition 2, §5, N°3, p.65].

We first consider the case when i is the zero section $Y \hookrightarrow \mathbb{P}(N \oplus 1) = X$ for a vector bundle N on a regular variety connected Y. Then by Lemma I.4.22, there is a vector bundle Q on $\mathbb{P}(N \oplus 1)$ such that we have $\mathrm{CH}^n(\mathbb{P}(N \oplus 1))$

$$c_n \circ i_*[\mathcal{O}_Y] = c_n \circ \lambda_{-1}[Q^\vee].$$

A computation ([Ful98, §15.3]) shows that we have in $\mathrm{CH}^n(\mathbb{P}(N \oplus 1))$ the equality

$$c_n \circ \lambda_{-1}[Q^\vee] = (-1)^{n-1}(n-1)! \cdot c_n(Q).$$

As already noticed in the course of the proof of Lemma I.4.22, the vector bundle Q has a regular section σ whose scheme of zeroes is Y. In other words, there is a regular closed embedding σ fitting in a transverse square

$$\begin{array}{ccc} Y & \xrightarrow{\ i\ } & \mathbb{P}(N \oplus 1) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \sigma} \\ \mathbb{P}(N \oplus 1) & \xrightarrow[s]{} & Q \end{array}$$

where s is the zero section of Q. By [EKM08, Corollary 55.4], we have $i_*[Y] = \sigma^* \circ s_*[\mathbb{P}(N \oplus 1)]$ in $\mathrm{CH}(\mathbb{P}(N \oplus 1))$. Note that $\sigma^* = (p^*)^{-1} = s^*$, where $p \colon Q \to \mathbb{P}(N \oplus 1)$ is the vector bundle. Therefore $i_*[Y] = s^* \circ s_*[\mathbb{P}(N \oplus 1)] = c_n(Q)$ by the self-intersection formula [Ful98, Corollary 6.3]. Hence we have proven the statement in this particular case.

Then one reduces to this case by deformation to the normal bundle, as follows. Consider the commutative diagram of varieties (see [Ful98, Chapter 5])

$$\begin{array}{ccccc} \mathbb{P}(N \oplus 1) & \xrightarrow{\ k_0\ } & W & \xleftarrow{\ k_\infty\ } & X \\ {\scriptstyle s}\uparrow & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle i} \\ Y & \xrightarrow[i_0]{} & Y \times \mathbb{P}^1 & \xleftarrow[i_\infty]{} & Y \end{array}$$

where W is the blow-up of $X \times \mathbb{P}^1$ along $Y \times \{\infty\}$, whose exceptional divisor is $\mathbb{P}(N \oplus 1)$, the vector bundle N being the normal bundle of i. Horizontal arrows are Cartier divisors, hence every square is transverse by Remark I.2.13, and Tor-independent by Proposition I.2.15. Using Proposition I.2.26 and the case of the zero section s already considered, we have

$$\begin{aligned} {k_0}^* \circ c_n[\mathcal{O}_{Y\times\mathbb{P}^1}] &= c_n \circ {k_0}^* \circ j_*[\mathcal{O}_{Y\times\mathbb{P}^1}] \\ &= c_n \circ s_* \circ {i_0}^*[\mathcal{O}_{Y\times\mathbb{P}^1}] \\ &= c_n \circ s_*[\mathcal{O}_Y] \\ &= (-1)^{n-1}(n-1)! \cdot [Y] \\ &= {k_0}^*\Big((-1)^{n-1}(n-1)! \cdot [Y \times \mathbb{P}^1]\Big). \end{aligned}$$

Letting $\alpha = c_n[\mathcal{O}_{Y\times\mathbb{P}^1}] - (-1)^{n-1}(n-1)! \cdot [Y \times \mathbb{P}^1]$ we have ${k_0}^*(\alpha) = 0$. Using the same argument as in the beginning of the proof, we see that there is a cycle $y \in \mathrm{CH}(Y \times \mathbb{P}^1)$ such that $\alpha = j_*(y)$. Then

$$0 = {k_0}^*(\alpha) = {k_0}^* \circ j_*(y) = s_* \circ {i_0}^*(y).$$

The push forward s_* is injective, a left inverse being given by p_* where $p\colon \mathbb{P}(E\oplus 1) \to Y$ is the projective bundle, hence $i_0^*(y) = 0$. Note that the pull-backs ${i_0}^*$ and ${i_\infty}^*$ coincide: if $v\colon Y\times\mathbb{A}^1 \to Y\times\mathbb{P}^1$ is the open complement of $Y\times\{1\}$, and $q\colon Y \times \mathbb{A}^1 \to Y$ the second projection, then q is an isomorphism by homotopy invariance I.1.3, and we have by Lemma I.2.8

$${i_0}^* = (q^*)^{-1} \circ v^* = {i_\infty}^*.$$

We get $i_\infty^*(y) = 0$ hence

$$\begin{aligned} 0 &= i_* \circ {i_\infty}^*(y) = {k_\infty}^* \circ j_*(y) = {k_\infty}^*(\alpha) \\ &= c_n \circ {k_\infty}^*[\mathcal{O}_{Y\times\mathbb{P}^1}] - (-1)^{n-1}(n-1)! \cdot {k_\infty}^*[Y \times \mathbb{P}^1] \\ &= c_n[\mathcal{O}_Y] - (-1)^{n-1}(n-1)! \cdot [Y], \end{aligned}$$

as required. □

Remark I.4.26. We will prove in Corollary IV.3.4 that for X smooth, and $x \in K(X)^{(n)}$, we have

$$2c_{n+1}(x) \in n! \cdot \mathrm{CH}^{n+1}(X) + (\text{torsion subgroup of } \mathrm{CH}^{n+1}(X)).$$

When X is smooth and its Chow group is generated by birational projective images of regular varieties (for example when the base field admits resolution of singularities, or when X is a split projective homogeneous variety under a semi-simple algebraic group), , we will prove in Corollary IV.3.7 that we have

$$\prod_{p \text{ prime}} p^{[t/(p-1)]} \cdot c_{n+t}(x) \in (n+t-1)! \cdot \mathrm{CH}^{n+t}(X) + (\text{torsion subgroup}),$$

where t is an integer satisfying $0 \le t < n$ and $[t/(p-1)]$ is the greatest integer j such that $j(p-1) \le t$.

Proposition I.4.27 ([Ful98, Example 15.3.6])**.** *If X is smooth, there is a homomorphism of abelian groups $\eta_X\colon \operatorname{gr} K(X) \to \mathrm{CH}(X)$ such that $\varphi_X \circ \eta_X$ and $\eta_X \circ \varphi_X$ are multiplication by $(-1)^{n-1}(n-1)!$ in degree n.*

Proof. We define
$$\eta_X^n\colon \operatorname{gr}^n K(X) \to \mathrm{CH}^n(X)$$
by setting $\eta_X^n(x) = c_n(x)$. Proposition I.4.25 above shows that $c_n(x) \in \mathrm{CH}^n(X)$, that η_X is well-defined (*i.e.* passes to the associated graded group), and that $\varphi_X \circ \eta_X$ and $\eta_X \circ \varphi_X$ are multiplication by $(-1)^{n-1}(n-1)!$ in degree n. □

Corollary I.4.28. *For every smooth variety X, the morphism φ_X is surjective with torsion kernel.*

Indeed the assumption of smoothness in the statement of Corollary I.4.28 is not necessary.

This can be proven by several methods. One can show that the differentials landing in $E^2_{p,-p} = \mathrm{CH}_p$ in the BGQ spectral sequence have finite order. This is done in [GS87, Theorem 8.2] by considering the action of the Adams operations on the BGQ spectral sequence "with supports" (see also [Mer09, Theorem 3.4]).

Here we provide a proof using Baum-Fulton-Mac Pherson's Riemann-Roch Theorem.

Proposition I.4.29. *Let X be a variety. The morphism*
$$\varphi_X\colon \mathrm{CH}(X) \to \operatorname{gr}^{\mathsf{top}} K_0(X)$$
is surjective with torsion kernel.

Proof. We already know that it is surjective, and it will be enough to provide a left inverse of $\varphi_X \otimes \mathbb{Q}$.

In [Ful98, Theorem 18.3], a natural transformation
$$\tau\colon K_0(-) \to \mathrm{CH}(-) \otimes \mathbb{Q}$$
of functors from the category of varieties and proper morphisms to the category of abelian groups is constructed. Let X be a variety. Consider both on $K_0(X)$ and on $\mathrm{CH}(X)$ the filtration whose k-term is generated by $i_*K_0(Z)$ and $i_* \mathrm{CH}(Z) \otimes \mathbb{Q}$ for all closed embeddings $i\colon Z \hookrightarrow X$ with $\dim Z \leq k$. The associated graded groups are $\operatorname{gr}^{\mathsf{top}}_\bullet K_0(X)$ and $\mathrm{CH}_\bullet(X) \otimes \mathbb{Q}$, and since τ commutes with push-forwards, it induces a morphism of graded abelian groups
$$\operatorname{gr}(\tau_X)\colon \operatorname{gr}^{\mathsf{top}}_\bullet K_0(X) \to \mathrm{CH}_\bullet(X) \otimes \mathbb{Q}.$$
It is also proven in [Ful98, Theorem 18.3] that
$$\tau_X[\mathcal{O}_Z] = [Z] \mod \mathrm{CH}_{<\dim Z}(X) \otimes \mathbb{Q},$$

hence

$$\mathrm{gr}(\tau_X) \circ (\varphi_X \otimes \mathbb{Q}) = \mathrm{id}_{\mathrm{CH}(X)\otimes\mathbb{Q}}.$$

as required. □

Remark I.4.30. When X is smooth, one sees using the explicit map η_X of Proposition I.4.27 that the map φ_X is indeed an isomorphism in low codimension (0,1 and 2). Of course this is not true when X is singular, already in codimension one. For $\mathrm{gr}^{\mathsf{top}}_{\dim X-1} K_0(X)$ is a quotient of the Picard group of X (say X integral), and there are examples of varieties with zero Picard group and $\mathrm{CH}^1(X)$ a non-zero torsion group, see for instance [Ful98, Example 2.1.3].

Torsion free varieties

Two conditions on a variety X will play an important role in the sequel, when we will construct the Steenrod operations on Chow groups modulo a prime p. [2]

The first one is the fact that the morphism $\varphi_X\colon \mathrm{CH}(X) \to \mathrm{gr}^{\mathsf{top}} K_0(X)$ is an isomorphism. This will allows us to work with $\mathrm{gr}^{\mathsf{top}} K_0(X)$ instead of $\mathrm{CH}(X)$. The second condition is that the group $\mathrm{gr}^{\mathsf{top}} K_0(X)$ is torsion free. This will be needed for the construction of operations at the level of $\mathbb{Z}/p \otimes \mathrm{gr}^{\mathsf{top}} K_0(X)$.

These two conditions can be reformulated as a single condition, introduced in the following definition.

Definition I.4.31. We say that a variety is *torsion free* if its Chow group is a torsion free group.

Example I.4.32. Any variety whose Chow motive splits as sum of twisted Tate motives is torsion free. In particular, cellular varieties are torsion free.

The next proposition follows immediately from Proposition I.4.29.

Proposition I.4.33. *If X is a torsion free variety then the graded rings $\mathrm{gr}^{\mathsf{top}} K_0(X)$ and $\mathrm{CH}(X)$ are isomorphic.*

The following statements will be for us a convenient way to use the fact that a variety is torsion free, when constructing operations on $\mathbb{Z}/p \otimes \mathrm{gr}^{\mathsf{top}} K_0$.

Lemma I.4.34. *Let X be a torsion free variety, $x \in \mathrm{F}^{\mathsf{top}}_k K_0(X)$ and $y \in K_0(X)$ satisfying, for some integers n and m, the congruence*

$$mx = mny \mod \mathrm{F}^{\mathsf{top}}_{k-1} K_0(X).$$

Then the image of x in $\mathbb{Z}/n \otimes \mathrm{gr}^{\mathsf{top}}_k K_0(X)$ is zero.

2. When constructing the first Steenrod square in Section III.2, we will be able to avoid these conditions though.

Proof. Since $\mathrm{CH}(X)$ is torsion free, it is isomorphic to $\mathrm{gr}^{\mathsf{top}} K_0(X)$, which in turn, will be torsion free. It follows that if $y = 0$, then $x \in \mathrm{F}^{\mathsf{top}}_{k-1} K_0(X)$, and its image in the k-th graded group is zero.

Hence we may assume $y \neq 0$. Let d be the greatest integer such that $y \in \mathrm{F}^{\mathsf{top}}_d K_0(X)$ and $y \notin \mathrm{F}^{\mathsf{top}}_{d-1} K_0(X)$ (such an integer certainly exists because $y \notin \bigcap_i \mathrm{F}^{\mathsf{top}}_i K_0(X) = 0$). Assume $d > k$. Then $mny \in \mathrm{F}^{\mathsf{top}}_k K_0(X) \subset \mathrm{F}^{\mathsf{top}}_{d-1} K_0(X)$, hence mny is zero in the torsion-free group $\mathrm{gr}^{\mathsf{top}}_d K_0(X)$, which is incompatible with the choice of the integer d. Hence $d \leq k$ and $y \in \mathrm{F}^{\mathsf{top}}_k K_0(X)$. Using the fact that $\mathrm{gr}^{\mathsf{top}}_k K_0(X)$ is torsion free, the assertion follows. □

Corollary I.4.35. *Let X be a torsion free variety, and n an integer. Let $u_k \in \mathrm{F}^{\mathsf{top}}_k K_0(X)$ for $k = d, \cdots, \dim X$ and assume that*

$$\sum_{k=d}^{\dim X} n^{k-d} u_k \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(X).$$

Then for all $k = d, \cdots, \dim X$, the image of u_k in $\mathbb{Z}/n \otimes \mathrm{gr}^{\mathsf{top}}_k K_0(X)$ is zero.

Proof. We see that for all $k = d, \cdots, \dim X$ we have a congruence

$$n^{k-d} u_k = n^{k-d} \left(n \cdot \sum_{i=k+1}^{\dim X} n^{i-d-1} u_i \right) \quad \mathrm{mod}\ \mathrm{F}^{\mathsf{top}}_{k-1} K_0(X),$$

so that we may apply Lemma I.4.34 above, with $m = n^{k-d}$ and $x = u_k$. □

Chapter II

Adams operations

In this chapter we discuss the Adams-Riemann-Roch theorem. The statement of this theorem involves Bott's class, the *inverse Todd genus* of the n-th Adams operation. We provide a special decomposition of this class when $n = p$ is prime, with respect to powers of the prime number p on one hand, and to the gamma filtration on the other hand. We use the convenient framework of *presheaves with characteristic class* introduced in the previous section.

In order the construct homological operations for Chow groups in the next chapter, we will need *homological Adams operations*. These operations are constructed using a Riemann-Roch theorem for Adams operations on the K-theory with supports. We recall some basic facts about K^0-groups with supports, and provide a construction of Adams operations with supports, in some sense more elementary than the one provided in [GS87], but which requires to distinguish cases, depending on the characteristic of the base field.

Next we prove the Adams-Riemann-Roch theorem for these operations with supports, deduce from it the usual form of Adams-Riemann-Roch Theorem, and construct homological Adams operations.

II.1 Bott's class

In this section n will be an integer ≥ 2. When n will be required to be prime we shall rather denote it by p.

We now define several Todd homomorphisms that will be related to the n-th Adams operation. We consider the Todd homomorphisms

$$w^{\gamma,n} \colon K^0 \to (\operatorname{gr}_\gamma K^0)^\times, \quad w^{\mathrm{CH},n} \colon K^0 \to (\mathrm{CH})^\times \quad \text{and} \quad w^{K,n} \colon K^0 \to (\operatorname{gr} K)^\times$$

associated with the power series in the variable x

$$x^{n-1} + 1.$$

and respectively with values in the presheaf with characteristic class $\operatorname{gr}_\gamma K^0$ on Sch/F, CH on Sm/F and $\operatorname{gr} K$ on Sm/F. Individual components $w_k^{\gamma,n}$, $w_k^{\mathrm{CH},n}$ and $w_k^{K,n}$ are defined by composition with the projections $\operatorname{gr}_\gamma K^0 \to \operatorname{gr}_\gamma^k K^0$, $\mathrm{CH} \to \mathrm{CH}^k$ and $\operatorname{gr} K \to \operatorname{gr}^k K$.

Note that if $n = 2$, these Todd homomorphisms are associated with the power series $1 + x$, and therefore by Proposition I.4.10 they respectively coincide with the total Chern class of Example I.4.9, the usual Chern class for Chow groups, and the class of Proposition I.4.23.

If $\mathcal{L}_i, i = 1, \cdots, m$ are line bundles and σ_k the k-th elementary symmetric polynomial in m variables then

$$w_k^{\gamma,n}\Big(\sum_{i=1}^{m}[\mathcal{L}_i]\Big) = \sigma_k\Big(c_1(\mathcal{L}_\bullet)^{n-1}\Big), \tag{II.1.a}$$

where c_1 is the first Chern class with values. Similar formulae of course hold $w^{\mathrm{CH},n}$ and $w^{K,n}$.

The following lemma is similar to [Kar98, Lemma 2.16].

Lemma II.1.1. *We have a commutative diagram of presheaves of abelian groups on* Sm/F

$$\begin{array}{ccc} K & \xrightarrow{w^{\mathrm{CH},n}} & (\mathrm{CH})^\times \\ {\scriptstyle w^{\gamma,n}}\downarrow & \searrow^{w^{K,n}} & \downarrow{\scriptstyle \varphi} \\ (\operatorname{gr}_\gamma K^0|_{\mathsf{Sm}/F})^\times & \xrightarrow[\delta_G]{} & (\operatorname{gr} K)^\times \end{array}$$

Proof. This is a consequence of the unicity in Proposition I.4.10, and of Proposition I.4.24. □

We also consider the Todd homomorphism

$$\theta^n \colon K_+^0 \to (K^0)^\times$$

associated with the power series in the variable t

$$\frac{1-(1-t)^n}{t}$$

with values in the presheaf K^0 on Sch/F. In the terminology of [Pan04], θ^n is the *inverse Todd genus* of the n-th Adams operation. This morphism of presheaves of monoids is uniquely determined be the equation

$$\theta^n([\mathcal{L}]) = 1 + [\mathcal{L}]^{-1} + \cdots + [\mathcal{L}]^{1-n}$$

for all line bundles $\mathcal{L}$.

From now on, we choose a prime integer p. The following lemma explicits a relation between the classes θ^p and $w^{\gamma,p}$.

Proposition II.1.2. *Let X be a connected variety and $\mathcal{E}$ a vector bundle over X. Then there exists elements $e_k \in \mathrm{F}_\gamma^{k(p-1)} K^0(X)$ satisfying in $K^0(X)$*

$$\theta^p(\mathcal{E}) = \sum_{k=0}^{\mathrm{rank}(\mathcal{E})} p^{\mathrm{rank}(\mathcal{E})-k} e_k.$$

Moreover, we have in $\mathrm{gr}_\gamma^{k(p-1)} K^0(X)$

$$e_k = (-1)^{k(p-1)} w_k^{\gamma,p}(\mathcal{E}) \mod \mathrm{F}_\gamma^{k(p-1)+1} K^0(X).$$

Proof. It is easy to see that there exists a polynomial T_p, with integral coefficients, such that we have in the polynomial ring $\mathbb{Z}[x]$: [1]

$$\frac{1-(1-x)^p}{x} = (-1)^{p-1}x^{p-1} + p\Big(1 + x \cdot T_p(x)\Big)$$

We consider the presheaf of rings

$$K_\gamma^0 = \bigoplus_k \mathrm{F}_\gamma^k K^0(-) \cdot t^k \subset \mathbb{Z}[t] \otimes K^0.$$

Let $c = \sum_i c_i$ be the Chern class with values in K^0 of Example I.4.4. We have $c_i(\mathcal{E}) \in \mathrm{F}_\gamma^i K^0(X)$ for every variety X and all vector bundles $\mathcal{E}$ over X by Proposition I.4.7, and the association

$$\mathcal{E} \mapsto \sum_i c_i(\mathcal{E}) t^i x^i \quad K^0 \to (K_\gamma^0(-)[[x]])^\times$$

gives K_γ^0 the structure of a presheaf with characteristic class. We define Θ^p as the Todd homomorphism associated with the power series in the variable x

$$x^{p-1}t + p\Big(1 + x \cdot T_p(x)\Big),$$

with values in K_γ^0. Since

$$\Big(x^{p-1}t + p(1 + x \cdot T_p(x))\Big)\Big|_{t=(-1)^{p-1}} = \frac{1-(1-x)^p}{x}$$

is the power series defining θ^p as a Todd homomorphism, we have, by unicity in Proposition I.4.10, a commutative diagram of presheaves of monoids on Sch/F

$$\begin{array}{ccc} & & (K_\gamma^0)_{\mathsf{mult}} \\ & \overset{\Theta^p}{\nearrow} & \big\downarrow {\scriptstyle t=(-1)^{p-1}} \\ K_+^0 & \xrightarrow[\theta^p]{} & (K^0)_{\mathsf{mult}} \end{array}$$

1. This is precisely the place where we need p to be prime number, and not an arbitrary integer n. However when n is a prime power, we can still obtain a decomposition of $\theta^n(\mathcal{E})$, an example of such a decomposition is given in the proof of Proposition III.3.4

Consider the ring filtration

$$\cdots \subset F_n(K^0_\gamma) \subset \cdots \subset F_0(K^0_\gamma) = K^0_\gamma.$$

generated by the conditions

$$\mathrm{F}^n_\gamma\, K^0(X) \cdot t^n \subset F_n(K^0_\gamma) \text{ and } p \in F_1(K^0_\gamma).$$

It follows from the splitting principle and the obvious case of line bundles that

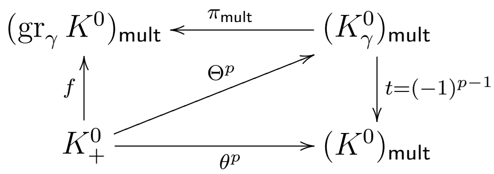
$$\Theta^p(\mathcal{E}) \in F_{\mathrm{rank}(\mathcal{E})}(K^0_\gamma(X))$$

for every connected variety X and every vector bundle $\mathcal{E}$ on X. The first assertion follows.

There is a morphism of presheaves of graded rings $K^0_\gamma \to \mathrm{gr}_\gamma K^0$

$$\pi\colon a_k \cdot t^k \mapsto (a_k \mod \mathrm{F}^{k+1}_\gamma\, K^0(-)),$$

which allows us to enlarge the commutative diagram above as



The transformation f is easily seen, by the splitting principle, to be the Todd homomorphism associated with the power series in the variable x

$$x^{p-1} + p,$$

with values in the presheaf $\mathrm{gr}_\gamma K^0$.

In order to prove the second assertion, we may assume that $[\mathcal{E}]$ splits as $\sum_i [\mathcal{L}_i]$, for some line bundles $\mathcal{L}_i$ with $i = 1, \cdots, m$. Then writing σ_k for the k-th elementary symmetric polynomial in m variables, we have using (II.1.a),

$$\begin{aligned} f(\mathcal{E}) &= \prod_i \Big(c_1(\mathcal{L}_i)^{p-1} + p\Big) \\ &= \sum_{k+j=\mathrm{rank}(\mathcal{E})} p^j \cdot \sigma_k\Big(c_1(\mathcal{L}_\bullet)^{p-1}\Big) \\ &= \sum_{k+j=\mathrm{rank}(\mathcal{E})} p^j \cdot w_k^{\gamma,p}(\mathcal{E}). \end{aligned}$$

This proves the second assertion. □

It will sometimes be convenient to be able extend the morphism θ^n to the whole K^0 instead of the submonoid of positive elements K^0_+. Consider the presheaf of rings $K^0 \otimes \mathbb{Z}[1/n]$ on Sch/F. It inherits the structure of a presheaf with characteristic class in an obvious fashion. The power series

$$\frac{1-(1-x)^n}{x}$$

has constant coefficient n, hence is invertible in the ring

$$\Big(K^0(\text{point}) \otimes \mathbb{Z}[1/n]\Big)[[x]] = \mathbb{Z}[1/n][[x]].$$

By Lemma I.4.11, the associated Todd homomorphism extends to a morphism of presheaves of abelian groups on Sch/F

$$\theta^n \colon K^0 \to (K^0 \otimes \mathbb{Z}[1/n])^\times.$$

Corollary II.1.3. *Let X be a connected variety, and $e \in K^0(X)$. Then there exists elements $e_k \in \mathrm{F}_\gamma^{k(p-1)} K^0(X)$ satisfying in $K^0(X) \otimes \mathbb{Z}[1/p]$*

$$\theta^p(e) = \sum_k e_k \otimes p^{\operatorname{rank}(e)-k}.$$

Moreover, we have in $\operatorname{gr}_\gamma^{k(p-1)} K^0(X)$

$$e_k = (-1)^{k(p-1)} w_k^{\gamma,p}(e) \mod \mathrm{F}_\gamma^{k(p-1)+1} K^0(X).$$

Proof. Write e as $[\mathcal{E}_+] - [\mathcal{E}_-]$, with $\mathcal{E}_+, \mathcal{E}_-$ vector bundles on X. Then $\theta^p(e) = \theta^p(\mathcal{E}_+) \cdot \theta^p(\mathcal{E}_-)^{-1}$. The set of elements that can be written $1+u$ with

$$u \in \mathrm{F}_\gamma^1 K^0(X) \otimes 1 + \sum_{k>0} \mathrm{F}_\gamma^{k(p-1)} K^0(X) \otimes p^{-k}$$

form a subgroup $G(X)$ of $(K^0(X) \otimes \mathbb{Z}[1/p])^\times$. Then by Proposition II.1.2 above, $\theta^p(\mathcal{E}_-)$ can be written $p^{\operatorname{rank}(\mathcal{E}_-)}(1+x)$, with $x \in G(X)$. Let

$$\frac{1}{1+t} = F(t) = 1 - t + \cdots$$

be the power series in the variable t multiplicatively inverse of $1+t$. Then

$$\theta^p(\mathcal{E}_-)^{-1} = p^{-\operatorname{rank}(\mathcal{E}_-)} \cdot F(x),$$

and $F(x)$ is a well-defined element of $K^0(X)$ because x is nilpotent, and indeed belongs to $G(X)$ because $G(X)$ is stable under addition and multiplication. The first assertion now easily follows.

An argument completely similar to the proof of the second assertion of Proposition II.1.2, replacing K^0_+ with K^0, $(K^0)_{\mathsf{mult}}$ with $(K^0 \otimes \mathbb{Z}[1/p])^\times$, ..., gives a proof of the second assertion. □

The following lemma will be used in the course of the proof of the Riemann-Roch theorem. The class λ_{-1} was defined in Example I.4.21.

Lemma II.1.4. *Let $\mathcal{E}$ be a vector bundle on a variety X. Then we have in $K^0(X)$ the equality*

$$\psi^n \circ \lambda_{-1}(\mathcal{E}^\vee) = \lambda_{-1}(\mathcal{E}^\vee) \cdot \theta^n(\mathcal{E})$$

Proof. The maps

$$x \mapsto \psi^n \circ \lambda_{-1}(x^\vee) \quad \text{and} \quad x \mapsto \lambda_{-1}(x^\vee) \cdot \theta^n(x)$$

are both Todd homomorphisms

$$K^0_{\mathsf{mult}} \to (K^0)^\times$$

associated with the power series in the variable t

$$1 - (1-t)^n.$$

The claim is a consequence of the unicity in Proposition I.4.10. □

II.2 Grothendieck group with supports

Here we follow the presentation of [GS87] (see also [Sou92, Chapter 1]).

Definition II.2.1. Let Y be a closed subset of a scheme X. Let $\mathsf{C}_Y(X)$ be the category of bounded complexes of locally free $\mathcal{O}_X$-sheaves acyclic off Y. We define $K^0_Y(X)$ as the Grothendieck group of $\mathsf{C}_Y(X)$ modulo the subgroup generated by acyclic complexes. This coincides with the free abelian group on objects of $\mathsf{C}_Y(X)$ modulo the identification of quasi-isomorphic complexes, and the relations $[\mathcal{E}_2] = [\mathcal{E}_1] + [\mathcal{E}_3]$ for exact sequences of complexes

$$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0.$$

Given two closed subsets Y and Z of X there is a *cup-product*

$$\cup \colon K^0_Y(X) \otimes K^0_Z(X) \to K^0_{Y\cap Z}(X)$$

induced by the tensor product of complexes.

Let $f \colon X' \to X$ be a morphism of varieties, Y a closed subset of X, and $Y' \supset f^{-1}(Y)$ a closed subset of Y'. There is a pull-back $f^* \colon K^0_Y(X) \to K^0_{Y'}(X')$.

A particular case is when $Y \subset Y'$ are two closed subset of a variety X, and $f = \mathrm{id}_X$. Then we have *change of support map* $K^0_Y(X) \to K^0_Z(X)$.

When the subset Y' will not be specified, we will denote by f^* the pull-back $K^0_Y(X) \to K^0_{f^{-1}(Y)}(X')$.

Let $i\colon Y \hookrightarrow X$ be a closed embedding. Then any $\mathcal{O}_X$-module $\mathcal{F}$ supported on Y is annihilated by some power of $\mathcal{I}$, the ideal sheaf of Y on X. Such a $\mathcal{F}$ has a class

$$[\mathcal{F}]_Y = \sum_k \left[(\mathcal{I}^k \cdot \mathcal{F})/(\mathcal{I}^{k+1} \cdot \mathcal{F})\right] \in K_0(Y).$$

Then given closed subvarieties Y and Z of a variety X, we define a *cap-product*

$$\begin{array}{rrcl} \cap\colon & K^0_Y(X) \otimes K_0(Z) & \to & K_0(Y \cap Z) \\ & [\mathcal{E}_\bullet] \otimes [M] & \mapsto & x \cap y = \sum_i (-1)^i \left[\mathsf{H}_i(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} M)\right]_{Y \cap Z}, \end{array}$$

where for a complex of vector bundles $\mathcal{F}_\bullet$, we denote by $\mathsf{H}_i(\mathcal{F}_\bullet)$ its i-th homology sheaf.

We will make repeated use of the following technical lemma.

Lemma II.2.2. *Let $E^\bullet_{p,q}$ be a convergent bounded spectral sequence of coherent $\mathcal{O}_X$-sheaves, with abutment A_{p+q}. Assume that there is a closed subset Y of X such that every term $E^n_{p,q}$ is supported on Y. Then for every integer i, the sheaf A_i is supported on Y, and for every integer n we have in $K_0(Y)$ the equality*

$$\sum_{p,q} (-1)^{p+q} \left[E^n_{p,q}\right]_Y = \sum_i (-1)^i \left[A_i\right]_Y.$$

Proof. Let $\mathcal{I}$ be the sheaf of ideals of Y in X. There is an integer m, and an infinitesimal neighborhood V of Y defined by the sheaf of ideals $\mathcal{I}^m$ in X, such $\mathcal{I}^m$ annihilates every term $E^n_{p,q}$, making these sheaves $\mathcal{O}_V$-modules. Then $\mathcal{I}^m$ annihilates each A_i; we also view them as $\mathcal{O}_V$-modules.

Let $j\colon Y \hookrightarrow V$ be the closed embedding. Using Remark I.1.1, we see that have, for every integer n, the chain of equalities in $K_0(V)$

$$\begin{aligned} j_*\left(\sum_{p,q} (-1)^{p+q} \left[E^n_{p,q}\right]_Y\right) &= \sum_{p,q} (-1)^{p+q} \left[E^n_{p,q}\right] \\ &= \sum_i (-1)^i \left[A_i\right] \\ &= j_*\left(\sum_i (-1)^i \left[A_i\right]_Y\right). \end{aligned}$$

By Theorem I.2.17, the push-forward $j_*\colon K_0(Y) \to K_0(V)$ is an isomorphism, whence the result. ☐

Lemma II.2.3 (projection formula)**.** *If $f\colon X \to X'$ is a projective morphism, Z a closed subvariety of X', Y closed subvariety of X, and $\varphi\colon Y \to f(Y)$, $\phi\colon Y \cap f^{-1}(Z) \to f(Y) \cap Z$ the induced morphisms, we have*

$$\phi_*\Big(f^*(a) \cap b\Big) = a \cap \varphi_*(b)$$

in $K_0(f(Y) \cap Z)$, *for all* $a \in K^0_Z(X')$ *and* $b \in K_0(Y)$.

Proof. Again we assume that $a = [\mathcal{A}_\bullet]$ and $b = [B]$, for some complex $\mathcal{A}_\bullet$ of coherent $\mathcal{O}_{X'}$-sheaves supported on Z, and some coherent $\mathcal{O}_Y$-sheaf B. The projection formula says that for every integer p we have an isomorphism of complexes

$$\mathcal{A}_\bullet \otimes_{\mathcal{O}_{X'}} \mathsf{R}^p\varphi_*(B) \simeq \mathsf{R}^p f_*\Big((f^*\mathcal{A}_\bullet) \otimes_{\mathcal{O}_X} B\Big). \tag{II.2.a}$$

Let $\mathcal{E}_\bullet = (f^*\mathcal{A}_\bullet) \otimes_{\mathcal{O}_X} B$. Then the spectral sequences

$$^{I}E_2^{p,q} = \mathsf{H}_q\Big(\mathsf{R}^p f_*(\mathcal{E}_\bullet)\Big) \quad \text{and} \quad ^{II}E_2^{p,q} = \mathsf{R}^p f_*\Big(\mathsf{H}_q(\mathcal{E}_\bullet)\Big)$$

have isomorphic abutments, namely the image of $\mathcal{E}_\bullet$ under the $(p+q)$-th right hyper-derived functor of f_*.

The complex $\mathsf{R}^p f_*(\mathcal{E}_\bullet)$ has homology supported on $f(Y) \cap Z$ by (II.2.a). It follows that every term $^{I}E^n_{p,q}$ for $n \geq 2$ is supported on $f(Y) \cap Z$. On the other hand, the homology of the complex $\mathcal{E}_\bullet$ is supported on $Y \cap f^{-1}(Z)$, hence the same holds for every term $^{II}E_2^{p,q}$ with $n \geq 2$. We can now apply Lemma II.2.2 to conclude. □

Lemma II.2.4. *Let* Y, Y', Z *be closed subsets of a variety* X. *Let* $a \in K^0_Y(X)$, $b \in K^0_{Y'}(X)$ *and* $c \in K_0(Z)$. *Then we have the formula*

$$(a \cup b) \cap c = a \cap (b \cap c).$$

Proof. We can assume that we have taken elementary elements $a = [\mathcal{A}_\bullet], b = [\mathcal{B}_\bullet]$ and $c = [C]$, with $\mathcal{A}_\bullet \in \mathsf{C}_Y(X)$, $\mathcal{B}_\bullet \in \mathsf{C}_{Y'}(X)$ and C a coherent sheaf of $\mathcal{O}_Z$-modules. Then there is a double complex spectral sequence

$$E^2_{p,q} = \mathsf{H}_p\Big(\mathcal{A}_\bullet \otimes_{\mathcal{O}_X} \mathsf{H}_q(\mathcal{B}_\bullet \otimes_{\mathcal{O}_X} C)\Big) \Rightarrow \mathsf{H}_{p+q}(\mathcal{A}_\bullet \otimes_{\mathcal{O}_X} \mathcal{B}_\bullet \otimes_{\mathcal{O}_X} C),$$

where every term is supported in $Y \cap Y' \cap Z$. We conclude again using Lemma II.2.2. □

Lemma II.2.5. *Let* $g \colon X' \to X$ *be a morphism of finite Tor-dimension, and* Y *a closed subset of* X. *Then for every* $a \in K^0_Y(X)$ *and* $\lambda \in K_0(X)$, *we have in* $K_0(g^{-1}(Y))$ *the equality*

$$g^*(a) \cap g^*(\lambda) = g^!(a \cap \lambda),$$

where $g^! \colon K_0(Y) \to K_0(g^{-1}(Y))$ *is the refined Gysin map.*

Proof. For a complex $\mathcal{E}_\bullet \in \mathsf{C}_Y(X)$, we have the Künneth spectral sequence

$$E^2_{p,q} = \mathsf{Tor}_p^{\mathcal{O}_X}\Big(\mathsf{H}_q(\mathcal{E}_\bullet), \mathcal{O}_{X'}\Big) \Rightarrow \mathsf{H}_{p+q}(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}).$$

In this spectral sequence, every term $E^n_{p,q}$ is supported on $g^{-1}(Y)$.

We can assume that $a = [\mathcal{A}_\bullet]$ and $\lambda = [M]$, for some coherent $\mathcal{O}_X$-sheaf M and some $\mathcal{A}_\bullet \in \mathsf{C}_Y(X)$, and set $\mathcal{E}_\bullet = \mathcal{A}_\bullet \otimes_{\mathcal{O}_X} M$. Then we have in $K_0(g^{-1}(Y))$,

$$\begin{aligned} g^!(a \cap \lambda) &= g^!\Big(\sum_q (-1)^q \,[\mathsf{H}_q(\mathcal{E}_\bullet)]_Y\Big) \\ &= \sum_{p,q} (-1)^{p+q} \left[E^2_{p,q}\right]_{g^{-1}(Y)} \qquad \text{by Lemma I.2.22} \\ &= \sum_i (-1)^i \,[\mathsf{H}_i(\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})]_{g^{-1}(Y)} \qquad \text{by Lemma II.2.2} \\ &= \sum_i (-1)^i \left[\mathsf{H}_i\left((\mathcal{A}_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}) \otimes_{\mathcal{O}_{X'}} (M \otimes_{\mathcal{O}_X} \mathcal{O}_{X'})\right)\right]_{g^{-1}(Y)} \\ &= g^*(a) \cap g^*(\lambda), \end{aligned}$$

as required. □

Proposition II.2.6 ([GS87, Lemma 1.9]). *Let Y be a closed subscheme of a regular scheme X. Then the natural morphism*

$$\begin{array}{rccl} - \cap [\mathcal{O}_Y]\colon & K^0_Y(X) & \to & K_0(Y) \\ & [\mathcal{E}_\bullet] & \mapsto & \sum_i (-1)^i \,[\mathsf{H}_i(\mathcal{E}_\bullet)]_Y \end{array}$$

is an isomorphism.

The inverse map is given by associating to a coherent $\mathcal{O}_Y$-sheaf $\mathcal{F}$ the class of a resolution by locally free $\mathcal{O}_X$-sheaves of the induced $\mathcal{O}_X$-sheaf $\mathcal{F}$.

If we are given a commutative square

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{g} & X' \end{array}$$

with i and j closed embeddings, f a projective morphism, and X' a regular variety we define a push-forward

$$(g, f)_*\colon K^0_Y(X) \to K^0_{Y'}(X')$$

to be the unique morphism fitting in the commutative diagram

$$\begin{array}{ccc} K^0_Y(X) & \xrightarrow{g_*} & K^0_{Y'}(X') \\ {\scriptstyle -\cap[\mathcal{O}_Y]}\downarrow & & \downarrow{\scriptstyle -\cap[\mathcal{O}_{Y'}]} \\ K_0(Y) & \xrightarrow[f_*]{} & K_0(Y') \end{array} \tag{II.2.b}$$

In particular, the map $(g, f)_*$ depends only on f and not on the map g fitting in the square above (g need not even be projective), but this notation has the advantage of helping keep track of the varieties X, Y, X', Y', and we shall use sometimes it for this reason.

In order to lighten the notations, when $Y \subset X$ is a closed subset, $g\colon X \to X'$ a morphism which is projective when restricted to Y, with X' a regular variety, and $f\colon Y \to g(Y)$ the induced morphism we shall write

$$g_* = (g, f)_*\colon K^0_Y(X) \to K^0_{g(Y)}(X').$$

When X is regular, and $Y \subset Y'$ are closed subsets of X, the *change of support map* can be viewed as the push-forward

$$(\mathrm{id}_X, i)_*\colon K^0_Y(X) \to K^0_{Y'}(X).$$

The push-forward map $(g, f)_*\colon K^0_Y(X) \to K^0_{Y'}(X')$ factors through the change of support map above.

Remark II.2.7. One can more generally define Gysin push-forwards directly at the level of K^0 with supports, when X' is not assumed to be regular, at least when g is a closed embedding of finite Tor-dimension, see [GS87, Lemma 1.10]. We shall however need to consider the case of an arbitrary projective morphism of regular varieties for the statement of the Adams-Riemann-Roch with denominators, and for the definition of homological Adams operations.

Lemma II.2.8 (projection formula)**.** *Consider a fiber square*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{g} & X' \end{array}$$

with f a surjective morphism, i, j closed embeddings, g projective, and X' regular. We have

$$g_*\Big(g^*(a) \cup b\Big) = a \cup g_*(b).$$

Proof. We have

$$\begin{aligned} g_*(g^*(a) \cup b) \cap [\mathcal{O}_{X'}] &= f_*\Big((g^*(a) \cup b) \cap [\mathcal{O}_X]\Big) && \text{by (II.2.b)} \\ &= f_*\Big(g^*(a) \cap (b \cap [\mathcal{O}_X])\Big) && \text{by Lemma II.2.4} \\ &= a \cap f_*(b \cap [\mathcal{O}_X]) && \text{by Lemma II.2.3} \\ &= a \cap \Big(g_*(b) \cap [\mathcal{O}_{X'}]\Big) && \text{by (II.2.b)} \\ &= \Big(a \cup g_*(b)\Big) \cap [\mathcal{O}_{X'}] && \text{by Lemma II.2.4.} \end{aligned}$$

We conclude using the fact that $- \cap [\mathcal{O}_{X'}]$ is an isomorphism by Proposition II.2.6. □

Lemma II.2.9 ([GS87, Lemma 1.11])**.** *Consider a cartesian square*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow{g} & X' \end{array}$$

where i, j are closed embeddings, and j, g of finite Tor-dimension. Assume that the square is Tor-independent. Let Z be a closed subset of Y'. Then we have

$$g^* \circ j_* = i_* \circ f^* \colon K^0_Z(Y') \to K^0_W(X).$$

Proof. Let $x \in K^0_Z(Y')$. Then

$$\begin{aligned} \Big(i_* \circ f^*(x)\Big) \cap [\mathcal{O}_X] &= f^*(x) \cap [\mathcal{O}_Y] && \text{by (II.2.b)} \\ &= f^!(x \cap [\mathcal{O}_{Y'}]) && \text{by Lemma II.2.5} \\ &= g^!(x \cap [\mathcal{O}_{Y'}]) && \text{by Lemma I.2.23} \\ &= g^!(j_*(x) \cap [\mathcal{O}_{X'}]) && \text{by (II.2.b)} \\ &= \Big(g^* \circ j_*(x)\Big) \cap [\mathcal{O}_X] && \text{by Lemma II.2.5.} \end{aligned}$$

Again, we conclude by Proposition II.2.6. □

Let Y be a closed subset of a variety X. We define an action of $K^0(X)$ on $K^0_Y(X)$ by the composition

$$K^0(X) \otimes K^0_Y(X) \to K^0_X(X) \otimes K^0_Y(X) \xrightarrow{\cup} K^0_Y(X) \quad , \quad \lambda \otimes x \mapsto \lambda \cdot x.$$

When $i \colon Y \hookrightarrow X$ is a closed embedding and X a regular variety, this action is induced by the action of $K^0(Y)$ on $K_0(Y)$, and we have the formula

$$(\lambda \cdot x) \cap [\mathcal{O}_Y] = i^*(\lambda) \cdot (x \cap [\mathcal{O}_Y]), \tag{II.2.c}$$

for $\lambda \in K^0(X)$ and $x \in K^0_Y(X)$.

If $a \in K^0_Y(X)$, $b \in K^0_{Y'}(X)$ and $\lambda \in K^0(X)$ we have

$$a \cup (\lambda \cdot b) = \lambda \cdot (a \cup b), \tag{II.2.d}$$

by associativity of the cup product. Similarly, we also have the formula, for $a \in K^0_Y(X)$, $\lambda \in K^0(X)$ and $f \colon X' \to X$

$$f^*(\lambda \cdot a) = f^*(\lambda) \cdot f^*(a). \tag{II.2.e}$$

Given varieties X and X' over a common field, and $Y \subset X$, $Y' \subset X'$ closed subsets, there is an external product

$$\begin{aligned} \boxtimes \colon \ K^0_Y(X) \otimes K^0_{Y'}(X') &\to K^0_{Y \times Y'}(X \times X') \\ x \otimes y &\mapsto x \boxtimes y = p^*(x) \cup (p')^*(y) \end{aligned}$$

where $p\colon X \times X' \to X$ and $p'\colon X \times X' \to X'$ are the two projections.

It follows from (II.2.d) and (II.2.e) that we have for $a \in K^0_Y(X)$, $b \in K^0_{Y'}(X')$ and $\lambda \in K^0(X)$, $\mu \in K^0(X')$

$$(\lambda \boxtimes \mu) \cdot (a \boxtimes b) = (\lambda \cdot a) \boxtimes (\mu \cdot b). \tag{II.2.f}$$

The cap-product is compatible with external products in the sense made precise by the following lemma.

Lemma II.2.10. *Let X and X' be varieties over a common field. Let $a \in K^0_Y(X), b \in K^0_{Y'}(X')$ and $c \in K_0(Y), d \in K_0(Y')$, we have in $K_0(Y \times Y')$ the equality*

$$(a \boxtimes b) \cap (c \boxtimes d) = (a \cap c) \boxtimes (b \cap d).$$

Proof. We can assume that $a = [\mathcal{A}_\bullet], b = [\mathcal{B}_\bullet], c = [C]$ and $d = [D]$, where $\mathcal{A}_\bullet \in \mathsf{C}_Y(X)$, $\mathcal{B}_\bullet \in \mathsf{C}_{Y'}(X')$, C a coherent $\mathcal{O}_Y$-sheaf and D a coherent $\mathcal{O}_{Y'}$-sheaf. We have an isomorphism of complexes of $\mathcal{O}_{X\times X'}$-sheaves

$$(\mathcal{A}_\bullet \boxtimes \mathcal{B}_\bullet) \otimes_{\mathcal{O}_{X\times X'}} (C \boxtimes D) \simeq (\mathcal{A}_\bullet \otimes_{\mathcal{O}_X} C) \boxtimes (\mathcal{B}_\bullet \otimes_{\mathcal{O}_{X'}} D).$$

Let $\mathcal{E}_\bullet = \mathcal{A}_\bullet \otimes_{\mathcal{O}_X} C$ and $\mathcal{F}_\bullet = \mathcal{B}_\bullet \otimes_{\mathcal{O}_{X'}} D$. There is a spectral sequence

$$E^2_{p,q} = \mathsf{H}_p\Big(\mathcal{E}_\bullet \boxtimes \mathsf{H}_q(\mathcal{F}_\bullet)\Big) \Rightarrow \mathsf{H}_{p+q}(\mathcal{E}_\bullet \boxtimes \mathcal{F}_\bullet),$$

where every term is supported on $Y \times Y'$.

Since the external product is a biexact functor we have

$$E^2_{p,q} = \mathsf{H}_p(\mathcal{E}_\bullet) \boxtimes \mathsf{H}_q(\mathcal{F}_\bullet),$$

and the result follows, in view of Lemma II.2.2. □

Corollary II.2.11. *Let X_1 and X_2 be two regular varieties over the same field such that $X_1 \times X_2$ is regular. Let $f\colon X_1' \to X_1$ and $g\colon X_2 \to X_2'$ be projective morphisms, and $Y_1' \subset X_1'$, $Y_2' \subset X_2'$ closed subsets. Then for all $a \in K^0_{Y_1'}(X_1')$ and $b \in K^0_{Y_2'}(X_2')$ we have in $K^0_{f(Y_1'\times Y_2')}(X_1 \times X_2)$ the equality*

$$(f \times g)_*(a \boxtimes b) = f_*(a) \boxtimes g_*(b).$$

II.3 Adams operations on the K-theory with supports

The Adams operations

$$\psi^n\colon K^0_Y(X) \to K^0_Y(X)$$

are constructed in [GS87, § 4] (see also [Sou92, Chapter I, § 5]), in a fashion working independently of the characteristic of the base field. In this section we describe another construction of the Adams operations.

The properties of Adams operations on the K-theory with supports that we will need can be summarized as follows.

Proposition II.3.1. *Let F be a field. The operations ψ^n satisfies the following properties.*

(a) If $\mathcal{P}_F$ is the category of pairs (X,Y) where X is a variety over F and Y a closed subset of X, and morphisms $(X,Y) \to (X',Y')$ are morphisms of F-varieties $X \to X'$ satisfying $f^{-1}(Y') \subset Y$, then $(X,Y) \mapsto K^0_Y(X)$ is a presheaf of abelian groups on $\mathcal{P}_F$ and ψ^n is an endomorphism of the presheaf of abelian groups $(Y,X) \mapsto K^0_Y(X)$.

(b) If $Y \subset X$ and $Y' \subset X'$ are closed subsets of varieties X and X' over F, $x \in K^0_Y(X)$ and $x' \in K^0_{Y'}(X')$ then we have in $K^0_{Y\times Y'}(X \times X')$

$$\psi^n(x \boxtimes x') = \psi^n(x) \boxtimes \psi^n(x').$$

(c) If Y and Z are closed subsets of a variety X over F, $y \in K^0_Y(X)$ and $z \in K^0_Z(X)$ then we have in $K^0_{Y\cap Z}(X)$

$$\psi^n(y) \cup \psi^n(z) = \psi^n(y \cup z).$$

(d) If X is a regular variety over F, and $x \in K^0_X(X)$, then we have in $K(X)$

$$\psi^n(x \cap [\mathcal{O}_X]) = \psi^n(x) \cap [\mathcal{O}_X],$$

where $\psi^n \colon K(X) \to K(X)$ is the Adams operation defined in Example I.4.16.

Note that if Condition (a) holds then (b) and (c) are equivalent.

The aim of this paragraph is to describe a more elementary (than the one given in [GS87]) construction of the Adams operations with supports satisfying the conditions of Proposition II.3.1. We restrict our attention to the p-th Adams operation (p a prime number), as the other Adams operations can be constructed for these primary operations.

We first notice that, when the characteristic of the base field is p, the p-th Adams operation on K-theory with supports can be easily constructed.

Theorem II.3.2. *Let F be field of characteristic p, and Ψ be the absolute Frobenius morphism of X. Then the pull-back along Ψ induces a morphism*

$$\psi^p \colon K^0_Y(X) \to K^0_Y(X)$$

for all closed subsets Y of X, which satisfies the conditions of Proposition II.3.1.

Proof. If Y is a closed subset of X, the set $\Psi^{-1}(Y)$ coincides with Y as Ψ is the identity as a map of topological spaces. We get therefore a pull-back

$$\Psi^*\colon K^0_Y(X) \to K^0_Y(X),$$

and Conditions (a) and (c) are satisfied.

If $\mathcal{L}$ is a line bundle over X then there is natural morphism of vector bundles over X

$$\mathcal{L}^{\otimes p} \to \Psi^*\mathcal{L}$$

which is an isomorphism, as can be checked locally. It follows that

$$\psi^p = \Psi^*\colon K^0(X) \to K^0(X).$$

When X is regular, the Frobenius map Ψ is flat [2], and Condition (d) is a consequence of Proposition II.2.5 applied with $X = X' = Y = Y'$ and $g = \Psi$. □

Therefore from now on, in this paragraph, we assume that p is a prime number different from the characteristic of the base field, and construct the p-th Adams operation following method inspired by [Ati66] and [Köc00]. We additionally assume that the base field contains primitive p-th roots of unity (we are indeed interested in the case $p = 2$ in the sequel, and see also Remark II.3.12 below).

For a closed subset Y of a variety X, and a group G, we consider the category $\mathsf{C}_Y(X, G)$ of complexes of G-vector bundles on X acyclic off Y, the variety X being endowed with the trivial G-action.

We will also write $\mathsf{VB}(X, G)$ for the category of G-vector bundles on a variety X, with trivial G-action.

In what follows $\mathbb{Z}/p$ will denote the cyclic group of order p, and $\mathfrak{S}_n$ the n-th symmetric group .

Definition II.3.3. Let G be a group, and $H \subset G$ a subgroup of finite index, X be a variety with trivial G(and H)-action, and Y a subset of X. Then the forgetful functor

$$\mathrm{Res}^G_H\colon \mathsf{VB}(X, G) \to \mathsf{VB}(X, H)$$

admits a left and right adjoint

$$\mathrm{Ind}^G_H\colon \mathsf{VB}(X, H) \to \mathsf{VB}(X, G).$$

2. Reduce to the case of the spectrum of a regular local complete ring, and use Cohen's structure theorem. Indeed flatness of the Frobenius morphism is equivalent to regularity of the variety by [Kun69].

If $\mathcal{O}_X[G]$ (*resp.* $\mathcal{O}_X[H]$) is the sheaf of group algebras over X of G (*resp.* H), we have for all $\mathcal{E} \in \mathsf{VB}(X, H)$

$$\mathrm{Ind}_H^G(\mathcal{E}) \simeq \mathcal{E} \otimes_{\mathcal{O}_X[H]} \mathcal{O}_X[G],$$

and this coincides with the coinduced module $\mathrm{Hom}_{\mathcal{O}_X[H]}(\mathcal{O}_X[G], \mathcal{E})$, because H has finite index in G.

The exact functors above induce a pair of adjoint functors

$$\mathrm{Res}_H^G \colon \mathsf{C}_Y(X, G) \to \mathsf{C}_Y(X, H) \quad \text{and} \quad \mathrm{Ind}_H^G \colon \mathsf{C}_Y(X, H) \to \mathsf{C}_Y(X, G).$$

If $\mathcal{E}_\bullet$ is a bounded complex of vector bundles, and n an integer, the total tensor power $(\mathcal{E}_\bullet)^{\otimes n}$ is endowed with a $\mathfrak{S}_n$-action. If $\sigma \in \mathfrak{S}_n$ is the transposition (kl) and $e_1 \otimes \cdots \otimes e_n \in \mathcal{E}_{i_1} \otimes \cdots \otimes \mathcal{E}_{i_n} \subset (\mathcal{E}_\bullet)^{\otimes n}_m$ then

$$\sigma \cdot (e_1 \otimes \cdots \otimes e_n) = (-1)^{i_k \cdot i_l} e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(n)} \in \mathcal{E}_{i_{\sigma(1)}} \otimes \cdots \otimes \mathcal{E}_{i_{\sigma(n)}} \subset (\mathcal{E}_\bullet)^{\otimes n}_m.$$

If $0 \leq k \leq n$, we view $\mathfrak{S}_k \times \mathfrak{S}_{n-k}$ as the subgroup of $\mathfrak{S}_n$ consisting of those permutations fixing both subsets $\{1, \cdots, k\}$ and $\{k, \cdots, n\}$ of the set $\{1, \cdots, n\}$. In particular we make the convention that $\mathfrak{S}_0 \times \mathfrak{S}_n = \mathfrak{S}_n \times \mathfrak{S}_0 = \mathfrak{S}_n$.

The next lemma is similar to the binomial formula

$$(e + f)^n = \sum_k \binom{n}{k} e^k f^{n-k}.$$

Lemma II.3.4. *Let $n \geq 2$ be an integer, and $\mathcal{E}_\bullet, \mathcal{F}_\bullet$ be complexes of vector bundles on a variety X with trivial $\mathfrak{S}_n$-action. Then we have an isomorphism of complexes of $\mathfrak{S}_n$-vector bundles over X*

$$(\mathcal{E}_\bullet \oplus \mathcal{F}_\bullet)^{\otimes n} \simeq \bigoplus_{k=0}^{n} \mathrm{Ind}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{\mathfrak{S}_n} \Big((\mathcal{E}_\bullet)^{\otimes k} \otimes (\mathcal{F}_\bullet)^{\otimes n-k} \Big).$$

Proof. We start with the case $n = 2$. Then forgetting the $\mathfrak{S}_2$-action we have the direct sum decomposition

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)^{\otimes 2} = \mathcal{E}_\bullet^{\otimes 2} \oplus (\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet) \oplus (\mathcal{F}_\bullet \otimes \mathcal{E}_\bullet) \oplus \mathcal{F}_\bullet^{\otimes 2}.$$

The action of $\mathfrak{S}_2$ fixes each of the three summands $\mathcal{E}_\bullet^{\otimes 2}$, $\mathcal{F}_\bullet^{\otimes 2}$ and $(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet) \oplus (\mathcal{F}_\bullet \otimes \mathcal{E}_\bullet)$. The action of $\mathfrak{S}_2$ on $\mathcal{E}_\bullet^{\otimes 2}$ and $\mathcal{F}_\bullet^{\otimes 2}$ is the natural one, and $\mathfrak{S}_2$ acts on the other summand as on $\mathrm{Ind}_{\mathfrak{S}_1 \times \mathfrak{S}_1}^{\mathfrak{S}_2}(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)$. This proves the formula for the case $n = 2$.

When $n \geq 3$, the group $\mathfrak{S}_n$ is generated by the subgroups $\mathfrak{S}_{n-1} \times \mathfrak{S}_1$ and $\mathfrak{S}_1 \times \mathfrak{S}_{n-1}$. Therefore it will be enough to prove that the formula holds after applying to both sides the functor $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}}$ for each of the two embeddings $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ mentioned above. Choose such an embedding. Then for every integer $0 < k < n$, the set

$$\mathfrak{S}_{n-1} \backslash \mathfrak{S}_n / (\mathfrak{S}_k \times \mathfrak{S}_{n-k})$$

consists of two elements: the class of the identity and the class of the transposition $(1n)$. It follows that the functor $\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \circ \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}$ coincides with [3]

$$\mathrm{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}} \circ \mathrm{Res}^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}_{\mathfrak{S}_1 \times \mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}} \oplus \mathrm{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k-1}} \circ \mathrm{Res}^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k-1} \times \mathfrak{S}_1}, \quad \text{(II.3.a)}$$

see for example [Ser78, Proposition 22, § 7.3, p.75].

Now, assuming that the embedding $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ is $\mathfrak{S}_1 \times \mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ (the other case being completely similar), and reasoning by induction on n, we have

$$\begin{aligned}
\mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} (\mathcal{E}_\bullet \oplus \mathcal{F}_\bullet)^{\otimes n} &\simeq (\mathcal{E}_\bullet \oplus \mathcal{F}_\bullet) \otimes \bigoplus_{k=0}^{n-1} \mathrm{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_k \times \mathfrak{S}_{n-1-k}} \Big((\mathcal{E}_\bullet)^{\otimes k} \otimes (\mathcal{F}_\bullet)^{\otimes n-1-k} \Big) \\
&\simeq \mathcal{F}_\bullet \otimes (\mathcal{F}_\bullet)^{\otimes (n-1)} \oplus \bigoplus_{k=1}^{n-1} \bigg(\mathrm{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_{k-1} \times \mathfrak{S}_{n-k}} \Big(\mathcal{E}_\bullet \otimes (\mathcal{E}_\bullet)^{\otimes k-1} \otimes (\mathcal{F}_\bullet)^{\otimes n-k} \Big) \\
&\oplus \mathrm{Ind}^{\mathfrak{S}_{n-1}}_{\mathfrak{S}_k \times \mathfrak{S}_{n-1-k}} \Big(\mathcal{F}_\bullet \otimes (\mathcal{E}_\bullet)^{\otimes k} \otimes (\mathcal{F}_\bullet)^{\otimes n-k-1} \Big) \bigg) \oplus \mathcal{E}_\bullet \otimes (\mathcal{E}_\bullet)^{\otimes (n-1)} \\
&\simeq \mathrm{Res}^{\mathfrak{S}_n}_{\mathfrak{S}_{n-1}} \bigg(\bigoplus_{k=0}^{n} \mathrm{Ind}^{\mathfrak{S}_n}_{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} \Big((\mathcal{E}_\bullet)^{\otimes k} \otimes (\mathcal{F}_\bullet)^{\otimes n-k} \Big) \bigg)
\end{aligned}$$

where we have used the decomposition (II.3.a) for the last isomorphism. □

Definition II.3.5. Let X be a variety, and assume that the base field contains the primitive p-th roots of unity. The ring representation ring of $\mathbb{Z}/p$ is naturally isomorphic to the group ring $\mathbb{Z}[\mu_p]$ where μ_p is the group of p-th roots of unity, and we associate with a p-th root of unity ρ a $\mathbb{Z}/p$-line bundle $\mathcal{L}_\rho$ on the variety X, the latter being endowed with the trivial $\mathbb{Z}/p$-action. We consider the functor

$$\mathsf{VB}(X, \mathbb{Z}/p) \to \mathsf{VB}(X) \quad , \quad \mathcal{E} \mapsto \mathcal{E}(\rho) = \mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\rho, \mathcal{E}).$$

It is an exact functor because $\mathcal{L}_\rho$ is a projective object in $\mathsf{VB}(X, \mathbb{Z}/p)$, being a direct summand of the free $\mathbb{Z}/p$-vector bundle $\mathcal{O}_X[\mathbb{Z}/p]$ corresponding to the regular representation of $\mathbb{Z}/p$.

3. This is similar to Pascal's rule

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Then for any complex $\mathcal{E}_\bullet$ of $\mathbb{Z}/p$-vector bundles on X, acyclic off a closed subset Y, and any $\rho \in \mu_p$, we consider the complex

$$\mathcal{E}_\bullet(\rho) = \mathcal{E}(\rho)_\bullet \in \mathsf{C}_Y(X)$$

of vector bundles over X acyclic off Y.

Lemma II.3.6. *Let p be a prime number and X a variety with trivial $\mathfrak{S}_p$-action. Let ρ and ν be two primitive p-th roots of unity, and $\mathcal{F}_\bullet$ a complex of $\mathfrak{S}_p$-vector bundles on X, which is acyclic off a closed subset Y. For any embedding $\mathbb{Z}/p \subset \mathfrak{S}_p$ there is an isomorphism in $\mathsf{C}_Y(X)$*

$$(\mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{F}_\bullet)(\rho) \simeq (\mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{F}_\bullet)(\nu).$$

Proof. Let $k \in \mathbb{Z}/p$ be the element such that $\rho^k = \nu$, and let $\tau \in \mathfrak{S}_p$ be the cyclic permutation which is the image of $1 \in \mathbb{Z}/p$ under the embedding $\mathbb{Z}/p \subset \mathfrak{S}_p$. As any two cyclic permutations of order p are conjugate in $\mathfrak{S}_p$, we can find $\sigma \in \mathfrak{S}_p$ such that $\tau^k\sigma = \sigma\tau$.

Composition with the automorphism of $\mathcal{O}_X[\mathfrak{S}_p]$ induced by σ gives an isomorphism

$$\mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{O}_X[\mathfrak{S}_p], \mathcal{L}_\rho) \to \mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{O}_X[\mathfrak{S}_p], \mathcal{L}_\nu),$$

and therefore an isomorphism of the coinduced $\mathfrak{S}_p$-vector bundles

$$\mathrm{Ind}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{L}_\rho \simeq \mathrm{Ind}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{L}_\nu.$$

Then we have a chain of isomorphisms, natural in $\mathcal{E} \in \mathsf{VB}(X, \mathfrak{S}_p)$

$$\begin{aligned}
(\mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{E})(\rho) &\simeq \mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\rho, \mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{E}) \\
&\simeq \mathrm{Hom}_{\mathsf{VB}(X,\mathfrak{S}_p)}(\mathrm{Ind}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{L}_\rho, \mathcal{E}) \\
&\simeq \mathrm{Hom}_{\mathsf{VB}(X,\mathfrak{S}_p)}(\mathrm{Ind}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{L}_\nu, \mathcal{E}) \\
&\simeq \mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\nu, \mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{E}) \\
&\simeq (\mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{E})(\nu).
\end{aligned}$$

The statement follows. □

Lemma II.3.7. *Let p be a prime number and X a variety with trivial $\mathfrak{S}_p$-action. Let ρ be a primitive p-th root of unity, and $\mathcal{F}_\bullet$ a complex of $\mathfrak{S}_k \times \mathfrak{S}_{p-k}$-vector bundles on X which is acyclic outside of a closed subset Y, with $0 < k < p$. Consider the induced complex of $\mathfrak{S}_p$-vector bundles*

$$\mathcal{I}_\bullet = \mathrm{Ind}^{\mathfrak{S}_p}_{\mathfrak{S}_k \times \mathfrak{S}_{p-k}} \mathcal{F}_\bullet.$$

Choose an embedding $\mathbb{Z}/p \subset \mathfrak{S}_p$. There is an isomorphism in $\mathsf{C}_Y(X)$

$$(\mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{I}_\bullet)(\rho) \simeq (\mathrm{Res}^{\mathfrak{S}_p}_{\mathbb{Z}/p} \mathcal{I}_\bullet)(1).$$

Proof. The cyclic group $\mathbb{Z}/p$ acts freely on the set $\mathfrak{S}_p/(\mathfrak{S}_k \times \mathfrak{S}_{p-k})$ because $\mathfrak{S}_k \times \mathfrak{S}_{p-k}$ contains no non-trivial element of order dividing p. Let S be the set of orbits for this action. Define

$$\mathcal{E}_\bullet = \bigoplus_{\sigma \in S} \operatorname{Res}_{\{1\}}^{\mathfrak{S}_k \times \mathfrak{S}_{p-k}} \mathcal{F}_\bullet \in \mathsf{C}_Y(X).$$

Then we have an isomorphism in $\mathsf{C}_Y(X, \mathbb{Z}/p)$

$$\operatorname{Res}_{\mathbb{Z}/p}^{\mathfrak{S}_p} \mathcal{I}_\bullet \simeq \operatorname{Ind}_{\{1\}}^{\mathbb{Z}/p} \mathcal{E}_\bullet$$

and therefore

$$\begin{aligned}
\operatorname{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\rho, \operatorname{Res}_{\mathbb{Z}/p}^{\mathfrak{S}_p} \mathcal{I}_\bullet) &\simeq \operatorname{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\rho, \operatorname{Ind}_{\{1\}}^{\mathbb{Z}/p} \mathcal{E}_\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{VB}(X)}(\operatorname{Res}_{\{1\}}^{\mathbb{Z}/p} \mathcal{L}_\rho, \mathcal{E}_\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{VB}(X)}(\mathbb{A}^1_X, \mathcal{E}_\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{VB}(X)}(\operatorname{Res}_{\{1\}}^{\mathbb{Z}/p} \mathcal{L}_1, \mathcal{E}_\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_1, \operatorname{Ind}_{\{1\}}^{\mathbb{Z}/p} \mathcal{E}_\bullet) \\
&\simeq \operatorname{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_1, \operatorname{Res}_{\mathbb{Z}/p}^{\mathfrak{S}_p} \mathcal{I}_\bullet). \qquad \square
\end{aligned}$$

The definition of the Adams operation given in the next theorem is similar to the "geometrical description" of the Adams operation given in [Ati66, Equation (2.7)].

Theorem II.3.8. *Assume that the base field contains a primitive p-th root of unity ρ. The map*

$$\psi^p \colon \mathsf{C}_Y(X) \to K_Y^0(X) \quad , \quad \mathcal{E}_\bullet \mapsto [(\mathcal{E}_\bullet)^{\otimes p}(1)] - [(\mathcal{E}_\bullet)^{\otimes p}(\rho)],$$

where $(\mathcal{E}_\bullet)^{\otimes p}$ is the p-th total tensor power of the complex $\mathcal{E}_\bullet$, endowed with the action of $\mathbb{Z}/p$, induces a group homomorphism

$$\psi^p \colon K_Y^0(X) \to K_Y^0(X)$$

which does not depend on the choice of ρ.

Proof. It follows from Lemma II.3.6 that ψ^p does not depend on the choice of ρ, since the action of $\mathbb{Z}/p$ on $(\mathcal{E}_\bullet)^{\otimes p}$ is the restriction of the action of $\mathfrak{S}_p$.

If $\mathcal{E}_\bullet$ is acyclic then $(\mathcal{E}_\bullet)^{\otimes p}$ is also acyclic. Since $\mathcal{F} \mapsto \mathcal{F}(1)$ and $\mathcal{F} \mapsto \mathcal{F}(\rho)$ are exact functors, it follows that $(\mathcal{E}_\bullet)^{\otimes p}(1)$ and $(\mathcal{E}_\bullet)^{\otimes p}(\rho)$ are both acyclic.

The same argument shows that if $\mathcal{E}_\bullet$ is supported outside of Y, then so are both $(\mathcal{E}_\bullet)^{\otimes p}(1)$ and $(\mathcal{E}_\bullet)^{\otimes p}(\rho)$.

Now let

$$0 \to \mathcal{E}'_\bullet \to \mathcal{E}_\bullet \to \mathcal{E}''_\bullet \to 0 \tag{II.3.b}$$

be an exact sequence in $\mathsf{C}_Y(X)$. We use a construction from [BGS88, § f] (see also [Sou92, Chapter IV, § 3, p.80]). Let $p\colon X \times \mathbb{P}^1 \to X$ be the projection to the first factor, and σ be the section of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ vanishing only at ∞. Then σ induces an embedding $p^*\mathcal{E}'_\bullet \to \mathcal{E}'_\bullet \boxtimes \mathcal{O}(1)$, and $p^*\mathcal{E}'_\bullet$ embeds diagonally in $(\mathcal{E}'_\bullet \boxtimes \mathcal{O}(1)) \oplus \mathcal{E}_\bullet$. Let $\widetilde{\mathcal{E}_\bullet}$ be the complex of coherent $\mathcal{O}_{X\times\mathbb{P}^1}$-sheaves fitting in the exact sequence

$$0 \to p^*\mathcal{E}'_\bullet \to (\mathcal{E}'_\bullet \boxtimes \mathcal{O}(1)) \oplus p^*\mathcal{E}_\bullet \to \widetilde{\mathcal{E}_\bullet} \to 0.$$

Indeed $\widetilde{\mathcal{E}_\bullet}$ is a complex of locally free sheaves supported outside of $Y \times \mathbb{P}^1$. Letting $i_0, i_\infty\colon X \hookrightarrow X \times \mathbb{P}^1$ be the two closed embeddings corresponding to the rational points 0 and ∞ of $\mathbb{P}^1$, we have

$$i_0{}^*(\widetilde{\mathcal{E}_\bullet}) = \mathcal{E}_\bullet \quad \text{and} \quad i_\infty{}^*(\widetilde{\mathcal{E}_\bullet}) = \mathcal{E}'_\bullet \oplus \mathcal{E}''_\bullet.$$

Note that it is clear from the construction that ψ^p commutes with pull-backs. Since the maps $i_0{}^*, i_\infty{}^*\colon K^0_Y(X \times \mathbb{P}^1) \to K^0_Y(X)$ coincide, we have

$$\psi^p(\mathcal{E}_\bullet) = \psi^p \circ i_0{}^*(\widetilde{\mathcal{E}_\bullet}) = i_0{}^* \circ \psi^p(\widetilde{\mathcal{E}_\bullet}) = i_\infty{}^* \circ \psi^p(\widetilde{\mathcal{E}_\bullet}) = \psi^p \circ i_\infty{}^*(\widetilde{\mathcal{E}_\bullet}) = \psi^p(\mathcal{E}'_\bullet \oplus \mathcal{E}''_\bullet).$$

It follows that we can assume that (II.3.b) splits as a sequence of complexes.

Then by Lemma II.3.4, we have for any p-th root of unity ν

$$[(\mathcal{E}_\bullet)^{\otimes p}(\nu)] \simeq \sum_{k=0}^{p} \Big[\Big(\operatorname{Ind}^{\mathfrak{S}_p}_{\mathfrak{S}_k \times \mathfrak{S}_{p-k}} (\mathcal{E}'_\bullet)^{\otimes k} \otimes (\mathcal{E}''_\bullet)^{\otimes p-k}\Big)(\nu)\Big].$$

The terms in the sum for $0 < k < p$ do not depend on the choice of $\nu \in \mu_p$ by Lemma II.3.7, in particular they coincide for $\nu = \rho$ and $\nu = 1$. We get

$$\psi^p(\mathcal{E}_\bullet) = \psi^p(\mathcal{E}'_\bullet) + \psi^p(\mathcal{E}''_\bullet),$$

as required. □

Lemma II.3.9. *Assume that the base field contains a primitive p-th root of unity. Let ρ be a p-th root of unity, and $\mathcal{E}_\bullet, \mathcal{F}_\bullet$ two elements of $\mathsf{C}_Y(X, \mathbb{Z}/p)$. Then we have an isomorphism in $\mathsf{C}_Y(X)$*

$$(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)(\rho) \simeq \bigoplus_{\nu \in \mu_p} \mathcal{E}_\bullet(\nu) \otimes \mathcal{F}_\bullet(\nu^{-1}\rho).$$

Proof. We use the direct sum decomposition in $\mathsf{VB}(X, \mathbb{Z}/p)$

$$\mathcal{O}_X[\mathbb{Z}/p] = \bigoplus_{\mu \in \mu_p} \mathcal{L}_\mu.$$

Then for any $\mathcal{A} \in \mathsf{VB}(X, \mathbb{Z}/p)$ we have a natural decomposition

$$\mathcal{A} \simeq \mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{O}_X[\mathbb{Z}/p], \mathcal{A}) \simeq \bigoplus_{\mu \in \mu_p} \mathcal{L}_\mu \otimes \mathcal{A}(\mu),$$

and therefore for any $\mathcal{A}_\bullet \in \mathsf{C}_Y(X, \mathbb{Z}/p)$

$$\mathcal{A}_\bullet \simeq \bigoplus_{\mu \in \mu_p} \mathcal{L}_\mu \otimes \mathcal{A}_\bullet(\mu).$$

This gives isomorphisms

$$\begin{aligned}\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet &\simeq \Big(\bigoplus_{\mu \in \mu_p} \mathcal{L}_\mu \otimes \mathcal{E}_\bullet(\mu)\Big) \otimes \Big(\bigoplus_{\nu \in \mu_p} \mathcal{L}_\nu \otimes \mathcal{F}_\bullet(\nu)\Big) \\ &\simeq \bigoplus_{\mu \in \mu_p} \mathcal{L}_\mu \otimes \Big(\bigoplus_{\nu \in \mu_p} \mathcal{E}_\bullet(\nu) \otimes \mathcal{F}_\bullet(\nu^{-1}\mu)\Big)\end{aligned}$$

Now we apply the functor $\mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\rho, -)$ and use the fact that if $\mathcal{A}$ has trivial $\mathbb{Z}/p$ action then there are natural isomorphisms

$$\mathrm{Hom}_{\mathsf{VB}(X,\mathbb{Z}/p)}(\mathcal{L}_\rho, \mathcal{L}_\mu \otimes \mathcal{A}) \simeq \begin{cases} 0 & \text{if } \rho \neq \mu \\ \mathcal{A} & \text{if } \rho = \mu, \end{cases}$$

to conclude the proof. □

Proposition II.3.10. *The map ψ^p of Theorem II.3.8 satisfies the conditions of Proposition II.3.1.*

Proof. Condition (a) is clear from the construction.

Let ρ be a primitive p-th root of unity. If $\mathcal{E}_\bullet, \mathcal{F}_\bullet \in \mathsf{C}_Y(X)$ then applying Lemma II.3.9 to $(\mathcal{E}_\bullet)^{\otimes p}$ and $(\mathcal{F}_\bullet)^{\otimes p}$ we get

$$\begin{aligned}[(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)(\rho)] &= \sum_{\nu \in \mu_p} [(\mathcal{E}_\bullet)^{\otimes p}(\nu) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\nu^{-1}\rho)] \\ &= [(\mathcal{E}_\bullet)^{\otimes p}(1) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\rho)] + [(\mathcal{E}_\bullet)^{\otimes p}(\rho) \otimes (\mathcal{F}_\bullet)^{\otimes p}(1)] \\ &\quad + (p-2)[(\mathcal{E}_\bullet)^{\otimes p}(\rho) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\rho)],\end{aligned}$$

where we have used Lemma II.3.6 for the last equality. Similarly, applying Lemma II.3.9 with the root 1 we get

$$\begin{aligned}[(\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet)(1)] &= \sum_{\nu \in \mu_p} [(\mathcal{E}_\bullet)^{\otimes p}(\nu) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\nu^{-1})] \\ &= [(\mathcal{E}_\bullet)^{\otimes p}(1) \otimes (\mathcal{F}_\bullet)^{\otimes p}(1)] + (p-1)[(\mathcal{E}_\bullet)^{\otimes p}(\rho) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\rho)].\end{aligned}$$

This yields

$$\begin{aligned}\psi^p[\mathcal{E}_\bullet \otimes \mathcal{F}_\bullet] &= [(\mathcal{E}_\bullet)^{\otimes p}(1) \otimes (\mathcal{F}_\bullet)^{\otimes p}(1)] + [(\mathcal{E}_\bullet)^{\otimes p}(\rho) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\rho)] \\ &\quad - [(\mathcal{E}_\bullet)^{\otimes p}(1) \otimes (\mathcal{F}_\bullet)^{\otimes p}(\rho)] - [(\mathcal{E}_\bullet)^{\otimes p}(\rho) \otimes (\mathcal{F}_\bullet)^{\otimes p}(1)] \\ &= \psi^p[\mathcal{E}_\bullet] \otimes \psi^p[\mathcal{F}_\bullet],\end{aligned}$$

proving Condition (c).

Now if $\mathcal{L}$ is a line bundle on a variety X with trivial $\mathfrak{S}_p$-action, let $L_\bullet = \cdots \to 0 \to \mathcal{L} \to 0 \to \cdots$ be the complex with $\mathcal{L}$ in degree zero. Then $(L_\bullet)^{\otimes p}$ is the complex $\cdots \to 0 \to \mathcal{L}^{\otimes p} \to 0 \to \cdots$, on which $\mathbb{Z}/p$ acts trivially. It follows that in $K_X^0(X)$

$$\psi^p[L_\bullet] = [(L_\bullet)^{\otimes p}(1)] - [(L_\bullet)^{\otimes p}(\rho)] = [(L_\bullet)^{\otimes p}(1)] = [L_\bullet]^p. \tag{II.3.c}$$

Let $\tau_X \colon K(X) \to K_X^0(X)$ be the inverse of the isomorphism $-\cap[\mathcal{O}_X] \colon K_X^0(X) \to K(X)$. If $f \colon X \to Y$ a flat morphism of regular varieties, we know by Condition (a) and Proposition II.2.5 that

$$f^* \circ \tau_X^{-1} \circ \psi^n \circ \tau_X = \tau_Y^{-1} \circ \psi^n \circ \tau_Y \circ f^*,$$

hence $X \mapsto \tau_X^{-1} \circ \psi^n \circ \tau_X$ defines an endomorphism φ of the presheaf of abelian groups K on Reg/F. Using (II.3.c), we see that φ is the additive Todd homomorphism associated with the power series

$$\frac{1}{(1-t)^n}$$

in the variable t and with values in the presheaf K on Reg/F. Therefore $\varphi = \psi^p$ as an endomorphism of K, as required for Condition (d). □

Remark II.3.11. As pointed out by Grothendieck in [sga71, XIV, § 1], a similar technique can be used to construct a lambda-ring structure on $K_Y^0(X)$ when the base field has characteristic zero. This involves considering the action of the group $\mathfrak{S}_k$ on the k-th tensor power of a complex, and then defining its k-th exterior power as the class of the subcomplex corresponding to the sign representation of $\mathfrak{S}_k$.

For the construction of the p-th Adams operation it is enough to consider representations of $\mathbb{Z}/p$, and one can work with arbitrary fields of characteristic not p (containing primitive p-th primitive roots of unity).

Remark II.3.12. For an arbitrary prime number p and a variety X over a field F of characteristic not p, let L/F be a finite Galois extension of degree prime to p such that L contains primitive p-roots of unity, and $j \colon X_L = X \times_F L \to X$ the induced morphism. If Y is a closed subset of X, we define the operation

$$\widetilde{\psi}^p = j_* \circ \psi_L^p \circ j^* \colon K_Y^0(X) \to K_Y^0(X),$$

where $\psi_L^p \colon K^0_{j^{-1}(Y)}(X_L) \to K^0_{j^{-1}(Y)}(X_L)$ is the Adams operation with supports defined by the procedure described above, over the field L.

This operation $\widetilde{\psi}^p$ can provide a substitute for the p-th Adams operation with supports when we are dealing with p-primary properties, in particular for the construction of homological Steenrod operations for the modulo p Chow groups.

II.4 Adams Riemann-Roch Theorem and homological Adams operations

Most arguments used here can be found in [Sou85], where higher K-groups are considered.

Adams-Riemann-Roch Theorem

We first notice that Adams operations commute with change of supports maps.

Lemma II.4.1. *Let $i\colon Y \hookrightarrow Y'$ be a closed embedding of subvarieties of a regular variety X. Let $(\mathrm{id}_X, i)_*\colon K^0_Y(X) \to K^0_{Y'}(X)$ be the change of support map. Then*

$$\psi^n \circ (\mathrm{id}_X, i)_* = (\mathrm{id}_X, i)_* \circ \psi^n.$$

Proof. This follows from Condition (a) of Proposition II.3.1 since the change of support map is the pull-back along $(X, Y') \to (X, Y)$. □

Before proceeding with the proof of the Riemann-Roch theorem, we prove the following general lemma.

Lemma II.4.2. *Let $Y \hookrightarrow X$ be a closed embedding of regular varieties. Then the blow-up of X along Y is a regular variety, as is its exceptional divisor.*

Proof. The closed embedding is a regular closed embedding by [Bou07, Proposition 2, § 5, N°3, p.65]. Let N its normal bundle, and B the blow-up of X along Y. Using [Bou07, Proposition 7, §6, N°4, p.76], we see that the exceptional divisor $\mathbb{P}(N)$ is regular because it is locally isomorphic to the product of a projective space with an open subvariety of the regular variety X.

Now the open complement of $\mathbb{P}(N)$ in B is isomorphic to the open complement of Y in X, hence is regular. Hence if y is point of $B - \mathbb{P}(N)$, the local ring $\mathcal{O}_{B,y}$ is regular. If y belongs to the closed complement $\mathbb{P}(N)$, then the ring $\mathcal{O}_{\mathbb{P}(N),y}$ is regular. The kernel of the epimorphism $\mathcal{O}_{B,y} \to \mathcal{O}_{\mathbb{P}(N),y}$ is generated by a non-invertible element (because $y \in \mathbb{P}(N)$), which is not a zero divisor (because $\mathbb{P}(N)$ is a Cartier divisor in B). Then by [Bou06b, Corollaire 1, Chapter VIII, §5, N°3, p.55], the ring $\mathcal{O}_{B,y}$ is regular. □

We now state a lemma which will be used in the course of the proof of the next proposition.

Lemma II.4.3. *Let $s\colon X \hookrightarrow X'$ be a closed embedding, and $p\colon X' \to X$ a retraction of s (*i.e.* a morphism such that $p \circ s = \mathrm{id}_X$). Assume that X' is a regular variety. Let Y be closed subvariety of X, $a \in K^0_Y(X)$ and $b \in K^0_X(X)$. Then we have in $K^0_Y(X')$*

$$p^*(a) \cup (s, \mathrm{id}_X)_*(b) = (s, \mathrm{id}_Y)_*(a \cup b).$$

Proof. We have in $K_0(Y)$, by Lemma II.2.4 and the definition of push-forwards with supports (II.2.b)

$$\Big((s,\mathrm{id}_Y)_*(a\cup b)\Big)\cap[\mathcal{O}_Y]=(a\cup b)\cap[\mathcal{O}_Y]=a\cap(b\cap[\mathcal{O}_Y]).$$

On the other hand, using again Lemma II.2.4 and (II.2.b)

$$\Big(p^*(a)\cup(s,\mathrm{id}_X)_*(b)\Big)\cap[\mathcal{O}_Y]=p^*(a)\cap\Big((s,\mathrm{id}_X)_*(b)\cap[\mathcal{O}_Y]\Big)=p^*(a)\cap(b\cap[\mathcal{O}_Y]).$$

Applying Lemma II.2.3 we get that

$$s^*\circ p^*(a)\cap(b\cap[\mathcal{O}_Y])=p^*(a)\cap(b\cap[\mathcal{O}_Y]).$$

We have $s^*\circ p^*=\mathrm{id}_{K^0_Y(X)}$ by hypothesis. Since X' is regular this proves the statement, in view of Proposition II.2.6. □

Next we prove the Riemann-Roch theorem for the prototype of a regular closed embedding with normal bundle N.

Proposition II.4.4. *Let N be a vector bundle over a regular variety X, and $s\colon X\hookrightarrow\mathbb{P}(N\oplus 1)$ the closed embedding corresponding to the zero section of N. Let Y be a closed subvariety of X and $x\in K^0_Y(X)$. Then we have in $K^0_Y(\mathbb{P}(N\oplus 1))$*

$$\psi^n\circ s_*(x)=s_*\Big(\theta^n(N)\cdot\psi^n(x)\Big).$$

Proof. Note that the variety $\mathbb{P}(N\oplus 1)$ is regular by Lemma II.4.2. Let $p\colon\mathbb{P}(N\oplus 1)\to X$ be the projective bundle. Lemma II.4.3 above gives

$$s_*(x)=(s,\mathrm{id}_Y)_*(x)=p^*(x)\cup(s,\mathrm{id}_X)_*(1_X),\tag{II.4.a}$$

where $1_X\in K^0_X(X)$ is the image of $[\mathcal{O}_X]$ under the natural morphism $K^0(X)\to K^0_X(X)$, *i.e.* the class of the complex $\cdots\to 0\to\mathcal{O}_X\to 0\to\cdots$ with $\mathcal{O}_X$ in degree 0. This element is also characterized by the formula $1_X\cap[\mathcal{O}_X]=[\mathcal{O}_X]$.

Recall that for any endomorphism g of $\mathbb{P}(N\oplus 1)$ over X, the map $(g,p)_*$ depends only on p, hence

$$(s\circ p,p)_*=(\mathrm{id}_{\mathbb{P}(N\oplus 1)},p)_*\colon K^0_{\mathbb{P}(N\oplus 1)}(\mathbb{P}(N\oplus 1))\to K^0_X(\mathbb{P}(N\oplus 1)).$$

We have in $K^0_X(\mathbb{P}(N\oplus 1))$, using Lemma II.4.1,

$$\begin{aligned}\psi^n\circ(s,\mathrm{id}_X)_*(1_X)&=(\mathrm{id}_{\mathbb{P}(N\oplus 1)},p\circ s)_*\circ\psi^n\circ(s,\mathrm{id}_X)_*(1_X)\\&=(s\circ p,p)_*\circ(\mathrm{id}_{\mathbb{P}(N\oplus 1)},s)_*\circ\psi^n\circ(s,\mathrm{id}_X)_*(1_X)\\&=(s\circ p,p)_*\circ\psi^n\circ(\mathrm{id}_{\mathbb{P}(N\oplus 1)},s)_*\circ(s,\mathrm{id}_X)_*(1_X)\\&=(s\circ p,p)_*\circ\psi^n\circ(s,s)_*(1_X).\end{aligned}$$

By Condition (c) of Proposition II.3.1, we have in $K(\mathbb{P}(N \oplus 1))$

$$\begin{aligned}\Big(\psi^n \circ (s,s)_*(1_X)\Big) \cap [\mathcal{O}_{\mathbb{P}(N\oplus 1)}] &= \psi^n\Big(\Big((s,s)_*(1_X)\Big) \cap [\mathcal{O}_{\mathbb{P}(N\oplus 1)}]\Big)\\ &= \psi^n \circ s_*[\mathcal{O}_X]\end{aligned}$$

By Lemma I.4.22 and Lemma II.1.4, there is a vector bundle Q on $\mathbb{P}(N\oplus 1)$ such that $s^*Q = N$ and

$$\psi^n \circ s_*[\mathcal{O}_X] = \psi^n \circ \lambda_{-1}(Q^\vee) = \lambda_{-1}(Q^\vee) \circ \theta^n(Q) = s_* \circ \theta^n(N).$$

This can be rewritten as

$$\Big(\psi^n \circ (s,s)_*(1_X)\Big) \cap [\mathcal{O}_{\mathbb{P}(N\oplus 1)}] = \Big((s,s)_*\Big(\theta^n(N)\cdot 1_X\Big)\Big) \cap [\mathcal{O}_{\mathbb{P}(N\oplus 1)}].$$

Since $- \cap [\mathcal{O}_{\mathbb{P}(N\oplus 1)}]$ is an isomorphism, we get in $K^0_{\mathbb{P}(N\oplus 1)}(\mathbb{P}(N\oplus 1))$

$$\psi^n \circ (s,s)_*(1_X) = (s,s)_*\Big(\theta^n(N)\cdot 1_X\Big).$$

It follows that we have in $K^0_X(\mathbb{P}(N\oplus 1))$

$$\begin{aligned}\psi^n \circ (s,\mathrm{id}_X)_*(1_X) &= (s\circ p, p)_* \circ \psi^n \circ (s,s)_*(1_X)\\ &= (s\circ p,p)_* \circ (s,s)_*\Big(\theta^n(N)\cdot 1_X\Big)\\ &= (s,\mathrm{id}_X)_*\Big(\theta^n(N)\cdot 1_X\Big).\end{aligned}$$

Finally, we compute in $K^0_Y(\mathbb{P}(N\oplus 1))$

$$\begin{aligned}\psi^n \circ (s,\mathrm{id}_Y)_*(x) &= \psi^n\Big(p^*(x) \cup (s,\mathrm{id}_X)_*(1_X)\Big) && \text{by (II.4.a)}\\ &= \Big(p^* \circ \psi^n(x)\Big) \cup \psi^n \circ (s,\mathrm{id}_X)_*(1_X) && \text{by II.3.1, (c), (a)}\\ &= \Big(p^* \circ \psi^n(x)\Big) \cup (s,\mathrm{id}_X)_*\Big(\theta^n(N)\cdot 1_X\Big)\\ &= (s,\mathrm{id}_Y)_*\Big(\psi^n(x) \cup \Big(\theta^n(N)\cdot 1_X\Big)\Big) && \text{by Lemma II.4.3}\\ &= (s,\mathrm{id}_Y)_*\Big(\theta^n(N)\cdot\psi^n(x)\Big) && \text{by (II.2.d),}\end{aligned}$$

which proves the formula. □

We now prove the Riemann-Roch theorem for an arbitrary (regular) closed embedding of regular varieties, by deforming the situation to the situation of Proposition II.4.4 above.

Theorem II.4.5 (Riemann-Roch Theorem without denominators)**.** *Let $i\colon X \hookrightarrow X'$ be a closed embedding of regular varieties with normal bundle N, Y a closed subvariety of X, and $x \in K^0_Y(X)$. Then we have in $K^0_Y(X')$ the equality*

$$\psi^n \circ i_*(x) = i_*\Big(\theta^n(N)\cdot\psi^n(x)\Big).$$

Proof. The arguments below are analog to those already used in the proof of Proposition I.4.25.

We have the deformation diagram (see [Ful98, Chapter 5])

$$\begin{array}{ccccc} \mathbb{P}(N \oplus 1) & \xrightarrow{k_0} & W & \xleftarrow{k_\infty} & X' \\ \uparrow{\scriptstyle s} & & \uparrow{\scriptstyle j} & & \uparrow{\scriptstyle i} \\ X & \xrightarrow[i_0]{} & X \times \mathbb{P}^1 & \xleftarrow[i_\infty]{} & X \end{array}$$

where W is the blow-up of $X' \times \mathbb{P}^1$ along $X \times \{\infty\}$, with exceptional divisor $\mathbb{P}(N \oplus 1)$. The squares are cartesian, and transverse by Remark I.2.13, hence Tor-independent by Proposition I.2.15. Note that by Lemma II.4.2, every variety in the diagram is a regular variety.

Let $z \in K^0_{Y\times\mathbb{P}^1}(X \times \mathbb{P}^1)$. We have in $K^0_Y(\mathbb{P}(N \oplus 1))$ the chain of equalities

$$\begin{aligned} {k_0}^* \circ \psi^n \circ j_*(z) &= \psi^n \circ {k_0}^* \circ j_*(z) && \text{by II.3.1, (a)} \\ &= \psi^n \circ s_* \circ {i_0}^*(z) && \text{by Lemma II.2.9} \\ &= s_*\Big(\theta^n(N) \cdot \psi^n \circ {i_0}^*(z)\Big) && \text{by Proposition II.4.4} \\ &= s_* \circ {i_0}^*\Big(\theta^n(N \times \mathbb{P}^1) \cdot \psi^n(z)\Big) && \text{by II.3.1, (a)} \\ &= {k_0}^* \circ j_*\Big(\theta^n(N \times \mathbb{P}^1) \cdot \psi^n(z)\Big) && \text{by Lemma II.2.9.} \end{aligned}$$

We consider the element

$$\alpha = \psi^n \circ j_*(z) - j_*\Big(\theta^n(N \times \mathbb{P}^1) \cdot \psi^n(x)\Big) \in K^0_{Y\times\mathbb{P}^1}(W).$$

Note that $\psi^n \circ j_*(z)$ belongs to the image of the push-forward

$$j_* \colon K^0_{Y\times\mathbb{P}^1}(X \times \mathbb{P}^1) \to K^0_{Y\times\mathbb{P}^1}(W).$$

Indeed $\psi^n \circ j_*(z) = j_*(u)$ where $u \in K^0_{Y\times\mathbb{P}^1}(X \times \mathbb{P}^1)$ is the unique element such that

$$u \cap [\mathcal{O}_{Y\times\mathbb{P}^1}] = \Big(\psi^n \circ j_*(z)\Big) \cap [\mathcal{O}_{Y\times\mathbb{P}^1}].$$

Therefore we can find $y \in K^0_{Y\times\mathbb{P}^1}(X \times \mathbb{P}^1)$ such that $\alpha = j_*(y)$. Then we have

$$0 = {k_0}^*(\alpha) = {k_0}^* \circ j_*(y) = s_* \circ {i_0}^*(y).$$

Since s_* is injective (a left inverse is given by p_*), we get ${i_0}^*(y) = 0$, hence ${i_\infty}^*(y) = 0$.

Finally, we compute

$$\begin{aligned} 0 = i_* \circ {i_\infty}^*(y) = {k_\infty}^* \circ j_*(y) = {k_\infty}^*(\alpha) && \text{by Lemma II.2.9}\\ = \psi^n \circ {k_\infty}^* \circ j_*(z) - {k_\infty}^* \circ j_*\Big(\theta^n(N\times \mathbb{P}^1)\cdot \psi^n(z)\Big) && \text{by II.3.1, (a)}\\ = \psi^n \circ i_* \circ {i_\infty}^*(z) - i_* \circ {i_\infty}^*\Big(\theta^n(N\times \mathbb{P}^1)\cdot \psi^n(z)\Big) && \text{by Lemma II.2.9}\\ = \psi^n \circ i_* \circ {i_\infty}^*(z) - i_*\Big(\theta^n(N)\cdot \psi^n \circ {i_\infty}^*(z)\Big) && \text{by II.3.1, (a).} \end{aligned}$$

Letting $z = x \boxtimes [\mathcal{O}_{\mathbb{P}^1}]$, we get the desired formula. □

Theorem II.4.6 (Riemman-Roch Theorem with denominators)**.** *Consider a commutative square*

$$\begin{array}{ccc} Y & \xrightarrow{f} & Y' \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ X & \xrightarrow[g]{} & X' \end{array}$$

with f, g projective morphisms, i, j closed embeddings, and X, X' regular varieties. Then for all $x \in K^0_Y(X)$, we have in $K^0_{Y'}(X') \otimes \mathbb{Z}[1/n]$ the equality

$$(g,f)_*\Big(\theta^n(\mathcal{N}_g)\cdot\psi^n(x)\Big) = \psi^n \circ (g,f)_*(x),$$

where $\mathcal{N}_g = -\mathcal{T}_g = [g^*\mathcal{T}_{X'}] - [\mathcal{T}_X]$ is the virtual normal bundle of g (Definition B.3).

Proof. Using Lemma II.4.1 we can assume that f is surjective.

Note that we have a closed embedding $Y \hookrightarrow X \times_{X'} Y'$, and the map $g_* \colon K^0_Y(X) \to K^0_{Y'}(X')$ factors through the change of support morphism $K^0_Y(X) \to K^0_{X\times_{X'}Y'}(X)$. Using again Lemma II.4.1, we see that it is enough to prove the theorem under the additional assumption that the square is cartesian (f being assumed to be surjective).

Decomposing g as $X \hookrightarrow \mathbb{P}^r \times X' \to X'$ we see that it will be enough to treat separately the cases of g a regular closed embedding, and g a trivial projective bundle.

The case of g a regular closed embedding has been settled in Theorem II.4.5, hence we can assume that $X = \mathbb{P}^r \times X'$ and that g is the second projection. Then the projective bundle theorem I.3.16 implies that the external product map

$$K_0(\mathbb{P}^r) \otimes K_0(Y') \to K_0(\mathbb{P}^r \times Y')$$

is an isomorphism, hence so is the external product map, by Lemma II.2.10,

$$K^0_{\mathbb{P}^r}(\mathbb{P}^r) \otimes K^0_{Y'}(X') \to K^0_{\mathbb{P}^r\times Y'}(\mathbb{P}^r \times X').$$

It follows that we can assume that $x = a \boxtimes y$ with $a \in K^0_{\mathbb{P}^r}(\mathbb{P}^r)$ and $y \in K^0_{Y'}(X')$. Then letting $p \colon \mathbb{P}^r \to \operatorname{Spec}(F)$ be the structure morphism, and $T_{\mathbb{P}^r}$ the tangent bundle of the variety $\mathbb{P}^r$, we have

$$\begin{aligned} g_*\Big(\theta^n(\mathcal{N}_g)\cdot\psi^n(x)\Big) &= g_*\Big(\theta^n(-T_{\mathbb{P}^r}\times X')\cdot\psi^n(x)\Big) \\ &= g_*\Big(\theta^n(-T_{\mathbb{P}^r}\times X')\cdot\Big(\psi^n(a)\boxtimes\psi^n(y)\Big)\Big) && \text{by II.3.1, (b)} \\ &= g_*\Big(\Big(\theta^n(-T_{\mathbb{P}^r})\cdot\psi^n(a)\Big)\boxtimes\psi^n(y)\Big) && \text{by (II.2.f)} \\ &= p_*\Big(\theta^n(-T_{\mathbb{P}^r})\cdot\psi^n(a)\Big)\boxtimes\psi^n(y) && \text{by Corollary II.2.11.} \end{aligned}$$

On the other hand we have

$$\begin{aligned} \psi^n\circ g_*(x) &= \psi^n\circ g_*(a\boxtimes y) \\ &= \psi^n\Big(p_*(a)\boxtimes y\Big) && \text{by Corollary II.2.11} \\ &= \Big(\psi^n\circ p_*(a)\Big)\boxtimes\psi^n(y) && \text{by II.3.1, (b).} \end{aligned}$$

Therefore it will be enough to prove that for all $a \in K^0_{\mathbb{P}^r}(\mathbb{P}^r) = K^0(\mathbb{P}^r)$, we have in $\mathbb{Z}[1/n] = K^0(\operatorname{Spec}(F)) \otimes \mathbb{Z}[1/n]$ the formula

$$\psi^n\circ p_*(a) = p_*\Big(\theta^n(-T_{\mathbb{P}^r})\cdot\psi^n(a)\Big).$$

By the projective bundle theorem and Theorem II.4.5, we can further assume that $a = [\mathcal{O}_{\mathbb{P}^r}]$. The formula to be proven becomes

$$1 = p_*\circ\theta^n(-T_{\mathbb{P}^r}),$$

since the n-th Adams operation on $K^0(\operatorname{Spec}(F))$ is the identity, and $p_*[\mathcal{O}_{\mathbb{P}^r}] = 1$. This is proved in Lemma II.4.7 below. □

Lemma II.4.7. *Let $p \colon \mathbb{P}^r \to \operatorname{Spec}(F)$ be the projective space of dimension r, with tangent bundle $T_{\mathbb{P}^r}$. Then we have in $K_0(\operatorname{Spec}(F)) \otimes \mathbb{Z}[1/n] = \mathbb{Z}[1/n]$ the equality*

$$1 = p_*\circ\theta^n(-T_{\mathbb{P}^r}).$$

Proof. We follow the arguments of the proof of [Pan04, Lemma 1.10.2], and also use the notation

$$\{\mathbb{P}^k\} = (p_k)_*\circ\theta^n(-T_{\mathbb{P}^k}),$$

for all integer k, where $p_k \colon \mathbb{P}^k \to \operatorname{Spec}(F)$ is the projection.

Let $\xi = c_1(\mathcal{O}(1)) = 1 - [\mathcal{O}(-1)]$. We see by induction on r that for every linear subspace $\mathbb{P}^k \hookrightarrow \mathbb{P}^r$ with $0 \le k \le r$ we have

$$[\mathcal{O}_{\mathbb{P}^k}] = \xi^{r-k}.$$

Indeed any linear hypersurface $\mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^r$ is a Cartier divisor associated with the line bundle $\mathcal{O}(1)$, hence $[\mathcal{O}_{\mathbb{P}^{r-1}}] = \xi$. If $k \leq r-1$, and $i \colon \mathbb{P}^{r-1} \hookrightarrow \mathbb{P}^r$ is a linear embedding with $\mathbb{P}^{r-1}$ containing $\mathbb{P}^k$, then using the induction hypothesis, and the projection formula (I.1.c), we get in $K(\mathbb{P}^r)$

$$[\mathcal{O}_{\mathbb{P}^k}] = i_*[\mathcal{O}_{\mathbb{P}^k}] = i_*(\xi^{r-1-k}) = i_* \circ i^*(\xi^{r-1-k}) = \xi^{r-1-k} \cdot [\mathcal{O}_{\mathbb{P}^{r-1}}] = \xi^{r-k}.$$

Using Theorem II.4.5, we get

$$\{\mathbb{P}^k\} = p_*\Big(\theta^n(-T_{\mathbb{P}^r}) \cdot \psi^n(\xi^{r-k})\Big). \tag{II.4.b}$$

We now show by induction on the integer r that $\{\mathbb{P}^r\} = 1$, the case $r = 0$ being obvious.

By the projective bundle Theorem I.3.16 the group $K(\mathbb{P}^r \times \mathbb{P}^r)$ is free on the basis $\xi^i \boxtimes \xi^j$ for $0 \leq i,j \leq r$. Let $\Delta \colon \mathbb{P}^r \hookrightarrow \mathbb{P}^r \times \mathbb{P}^r$ be the diagonal embedding. Then there are integers $a_{i,j}$ for $0 \leq i,j \leq r$ such that in $K(\mathbb{P}^r \times \mathbb{P}^r)$

$$\Delta_*[\mathcal{O}_{\mathbb{P}^r}] = \sum_{0 \leq i,j \leq r} a_{i,j} \cdot \xi^i \boxtimes \xi^j. \tag{II.4.c}$$

By the projection formula and the formula $\xi^{r+1} = 0$, we have

$$\Delta_*(\xi^r) = \sum_{0 \leq j \leq r} a_{0,j} \cdot \xi^r \boxtimes \xi^j = \sum_{0 \leq i \leq r} a_{i,0} \cdot \xi^i \boxtimes \xi^r.$$

Applying the push-forward along each projection $\mathbb{P}^r \times \mathbb{P}^r \to \mathbb{P}^r$, and using the formula $p_*(\xi^i) = 1$ for $0 \leq i \leq r$ and compatibility of push-forwards with the external product following from Corollary II.2.11, we get

$$a_{r,0} = a_{0,r} = 1 \quad , \quad a_{i,0} = a_{0,j} = 0 \text{ for } 0 \leq i,j < r. \tag{II.4.d}$$

Let $\pi \colon \mathbb{P}^r \times \mathbb{P}^r \to \mathbb{P}^r$ be the second projection. The same argument yields in $K(\mathbb{P}^r)$

$$[\mathcal{O}_{\mathbb{P}^r}] = \pi_* \circ \Delta_*[\mathcal{O}_{\mathbb{P}^r}] = \sum_{0 \leq i,j \leq r} a_{i,j} \cdot \xi^j$$

hence

$$0 = \psi^n(\xi^r) + \sum_{i,j>0} a_{i,j} \cdot \psi^n(\xi^j). \tag{II.4.e}$$

Then we compute

$$\begin{aligned}
[\mathcal{O}_{\mathbb{P}^r}] &= \psi^n \circ \pi_* \circ \Delta_*[\mathcal{O}_{\mathbb{P}^r}] \\
&= \pi_*\Big(\big(\theta^n(-T_{\mathbb{P}^r}) \boxtimes [\mathcal{O}_{\mathbb{P}^r}]\big) \cdot \psi^n \circ \Delta_*[\mathcal{O}_{\mathbb{P}^r}]\Big) && \text{by Theorem II.4.5} \\
&= \sum_{0 \le i,j \le r} a_{i,j} \cdot \pi_*\Big(\big(\theta^n(-T_{\mathbb{P}^r}) \cdot \psi^n(\xi^i)\big) \boxtimes \psi^n(\xi^j)\Big) && \text{by (II.4.c)} \\
&= \sum_{0 \le i,j \le r} a_{i,j} \cdot p_*\big(\theta^n(-T_{\mathbb{P}^r})\psi^n(\xi^i)\big) \cdot \psi^n(\xi^j) && \text{by Corollary II.2.11} \\
&= \sum_{0 \le i,j \le r} a_{i,j} \cdot \{\mathbb{P}^{r-i}\} \cdot \psi^n(\xi^j) && \text{by (II.4.b)} \\
&= \{\mathbb{P}^0\} \cdot [\mathcal{O}_{\mathbb{P}^r}] + \{\mathbb{P}^r\} \cdot \psi^n(\xi^r) + \sum_{i,j>0} a_{i,j} \cdot \{\mathbb{P}^{r-i}\} \cdot \psi^n(\xi^j) && \text{by (II.4.d).}
\end{aligned}$$

By induction hypothesis, we get

$$0 = \{\mathbb{P}^r\} \cdot \psi^n(\xi^r) + \sum_{i,j>0} a_{i,j} \cdot \psi^n(\xi^j).$$

Comparing this formula with (II.4.e), we obtain

$$(\{\mathbb{P}^r\} - 1) \cdot \psi^n(\xi^r) = 0.$$

But we can compute, using again the formula $\xi^{r+1} = 0$,

$$\psi^n(\xi^r) = \psi^n(\xi)^r = \big(\xi(2-\xi)\big)^r = 2^r\xi^r,$$

which is a non-zero element of the torsion-free group $K(\mathbb{P}^r)$. We get $\{\mathbb{P}^r\} = 1$, as required. □

It will be sometimes more convenient to use the following equivalent formulation of Theorem II.4.6

Corollary II.4.8. *In the situation of Theorem II.4.6, we have*

$$\theta^n(-\mathcal{T}_{X'}) \cdot \psi^n \circ (g,f)_*(x) = (g,f)_*\big(\theta^n(-\mathcal{T}_X) \cdot \psi^n(x)\big),$$

where $\mathcal{T}_X$ and $\mathcal{T}_{X'}$ are the virtual tangent bundles of the regular varieties X and X' (Definition B.3).

Proof. This follows from Theorem II.4.6 and Lemma II.2.8. □

As a consequence we get the Adams-Riemann-Roch theorems without supports, which can be, of course, proven directly (without using K^0-groups with supports) by the same method.

Also proceeding as in [FL85], we can indeed replace the assumption of regularity on the varieties by an assumption of regularity for the morphism f in the theorems below. The functor K must then be replaced by K^0, and $f_*\colon K(Y) \to K(X)$ by an appropriate notion of push-forward along a perfect morphism, such as the one described in [Ful98, Example 15.1.8].

Theorem II.4.9 (Adams-Riemann-Roch for closed embeddings of regular varieties)**.** *Let $f: Y \hookrightarrow X$ be a closed embedding of regular varieties, with normal bundle N. Then we have*

$$\psi^n \circ f_* = f_* \circ \theta^n(N) \circ \psi^n : K(Y) \to K(X).$$

This statement can be found in [FL85, Chapter V, Theorem 6.3] or [Pan04, Theorem 2.5.3] when Y and X are smooth.

Theorem II.4.10 (Adams-Riemann-Roch with denominators)**.** *Let $f: Y \to X$ be a projective morphism of regular varieties, with virtual normal bundle $\mathcal{N}_f$. Then we have*

$$\psi^n \circ f_* = f_* \circ \theta^n(\mathcal{N}_f) \circ \psi^n : K(Y) \otimes \mathbb{Z}[1/n] \to K(X) \otimes \mathbb{Z}[1/n].$$

This theorem can be found in [FL85, Chapter V, Theorem 7.4] or [Pan04, Theorem 2.5.4] when Y and X are smooth.

When p is a prime integer, it follows from Theorem II.4.9, Proposition II.1.2, and Corollary I.3.15 that if Y is a smooth connected closed subvariety of a smooth variety X, with $[\mathcal{O}_Y] \in K(X)^{(q)}$ for some integer q, we can find elements $x_k \in K(X)^{(q+k(p-1))}$ for $k = 0, \cdots, q$ such that:

$$\psi^p[\mathcal{O}_Y] = \sum_{k=0}^{q} p^{q-k} x_k.$$

Lemma II.4.11. *Let X be a regular variety and $x \in K(X)^{(q)}$. If p is a prime number, we have the following congruence:*

$$\psi^p(x) = p^q x \mod K(X)^{(q+1)}.$$

Proof. We can assume that $x = [\mathcal{O}_Z]$ for Z a closed subvariety of X, of codimension q. Proceeding as in the proof of [sga71, VII, 4.11], one may find an open subscheme U of X such that the reduced closed complement Y of U in X is contained in Z, with $Z - Y \neq \emptyset$, and $i: Z \cap U \hookrightarrow U$ is a regular closed embedding. Writing $j: Y \hookrightarrow X$ and $u: U \to X$ for the closed and open embeddings, we have the exact localization sequence of Theorem I.1.2:

$$K_0(Y) \xrightarrow{j_*} K(X) \xrightarrow{u^*} K(U) \to 0.$$

Using Theorem II.4.9, Proposition II.1.2 and Corollary I.3.15, we see that

$$\psi^p \circ i_*[\mathcal{O}_{Z\cap U}] = p^q \cdot i_*[\mathcal{O}_{Z\cap U}] \mod K(U)^{(q+1)}.$$

By Proposition I.2.26, we have $u^*[\mathcal{O}_Z] = i_*[\mathcal{O}_{Z\cap U}]$, hence

$$u^*(\psi^n[\mathcal{O}_Z] - p^q \cdot [\mathcal{O}_Z]) \in K(U)^{(q+1)}.$$

By Lemma I.1.10 we know that the pull-back $u^*: K(X)^{(q+1)} \to K(U)^{(q+1)}$ is surjective, and $j_*(K_0(Y)) \subset K(X)^{(q+1)}$. This proves the congruence. $\square$

Homological Adams operations

Now after [Sou85] we make the following definition.

Definition II.4.12. Let X be a variety. Choose a closed embedding $g\colon X \hookrightarrow \Gamma$ in a smooth variety Γ with tangent bundle T_Γ. Then the unique morphism

$$\psi_n\colon K_0(X) \otimes \mathbb{Z}[1/n] \to K_0(X) \otimes \mathbb{Z}[1/n],$$

satisfying, for all $x \in K^0_X(\Gamma)$, the formula

$$\psi_n(x \cap [\mathcal{O}_\Gamma]) = \Big(\theta^n(-T_\Gamma) \cdot \psi^n(x)\Big) \cap [\mathcal{O}_\Gamma],$$

will be called the *n-th homological Adams operation.* Note that it follows from Lemma II.2.8 that an equivalent formula is

$$\psi_n(x \cap [\mathcal{O}_\Gamma]) = \theta^n(-g^*T_\Gamma) \cdot (\psi^n(x) \cap [\mathcal{O}_\Gamma]). \qquad \text{(II.4.f)}$$

We prove in the next proposition that the morphism ψ_n does not depend on the choice of g and Γ.

Proposition II.4.13. *Let $X \hookrightarrow \Gamma$ and $X \hookrightarrow \Gamma'$ be two closed embeddings in smooth varieties Γ and Γ'. Then if $x \in K^0_X(\Gamma)$ and $x' \in K^0_X(\Gamma')$ are elements satisfying $x \cap [\mathcal{O}_X] = x' \cap [\mathcal{O}_X]$, we have in $K_0(X) \otimes \mathbb{Z}[1/n]$ the equality*

$$\Big(\theta^n(-T_\Gamma) \cdot \psi^n(x)\Big) \cap [\mathcal{O}_\Gamma] = \Big(\theta^n(-T_{\Gamma'}) \cdot \psi^n(x')\Big) \cap [\mathcal{O}_{\Gamma'}].$$

Proof. We apply Corollary II.4.8 to the square

$$\begin{array}{ccc} X & \xrightarrow{=} & X \\ \downarrow & & \downarrow \\ \Gamma \times \Gamma' & \xrightarrow[g]{} & \Gamma \end{array}$$

where vertical arrows are the closed embeddings, and g is the projection to the first factor. Let $x'' \in K^0_X(\Gamma \times \Gamma')$ be the element such that $x'' \cap [\mathcal{O}_{\Gamma\times\Gamma'}] = x \cap [\mathcal{O}_\Gamma]$. This gives in $K^0_X(\Gamma) \otimes \mathbb{Z}[1/n]$ the formula

$$\theta^n(-T_\Gamma) \cdot \psi^n(x) = g_*\Big(\theta^n(-T_{\Gamma\times\Gamma'}) \cdot \psi^n(x'')\Big). \qquad \text{(II.4.g)}$$

In a similar fashion, letting $g'\colon \Gamma \times \Gamma' \to \Gamma'$ be the second projection, we get in $K^0_X(\Gamma') \otimes \mathbb{Z}[1/n]$ the equality

$$\theta^n(-T_{\Gamma'}) \cdot \psi^n(x') = g'_*\Big(\theta^n(-T_{\Gamma\times\Gamma'}) \cdot \psi^n(x'')\Big). \qquad \text{(II.4.h)}$$

Now we apply $- \cap [\mathcal{O}_\Gamma]$ to (II.4.g) and $- \cap [\mathcal{O}_{\Gamma'}]$ to (II.4.h), and we get the formula in the statement, in view of (II.2.b) with $f = \mathrm{id}_X$. □

Proposition II.4.14. *Let $f\colon X \to Y$ be a projective morphism. Then*

$$f_* \circ \psi_n = \psi_n \circ f_*.$$

Proof. We choose embeddings $g_X\colon X \hookrightarrow \Gamma$ and $g_Y\colon Y \hookrightarrow \Omega$ and apply Corollary II.4.8 to the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle (g_X, f\circ g_Y)}\downarrow & & \downarrow{\scriptstyle g_Y} \\ \Gamma \times \Omega & \xrightarrow[p]{} & \Omega \end{array}$$

This gives, for $x \in K^0_X(\Gamma \times \Omega)$,

$$\theta^n(-T_\Omega) \cdot \Big(\psi^n \circ p_*(x)\Big) = p_*\Big(\theta^n(-T_{\Gamma\times\Omega}) \cdot \psi^n(x)\Big).$$

Now we apply $- \cap [\mathcal{O}_\Omega]$ and use (II.2.b) to conclude the proof. □

Proposition II.4.15. *Let $u\colon U \to X$ be an open immersion. Then*

$$u^* \circ \psi_n = \psi_n \circ u^*.$$

Proof. Choose a smooth variety Γ with a closed embedding $X \hookrightarrow \Gamma$. Let Y be the reduced closed complement of U in X. Then we have a fiber square with vertical arrows close embeddings

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ \downarrow & & \downarrow \\ \Gamma - Y & \xrightarrow[g]{} & \Gamma. \end{array}$$

We have for $x \in K^0_X(\Gamma)$ using Condition (a) of Proposition II.3.1, and (II.2.e), the equality in $K^0_U(\Gamma - Y) \otimes \mathbb{Z}[1/n]$

$$g^*(\theta^n(-T_\Gamma) \cdot \psi^n(x)) = \theta^n(-T_{\Gamma-Y}) \cdot \Big(\psi^n \circ g^*(x)\Big).$$

We apply $- \cap [\mathcal{O}_U]$ to both sides, and use Lemma II.2.5 to conclude. □

Lemma II.4.16. *Let X be a regular variety, and $\mathcal{T}_X$ its virtual tangent bundle (Definition B.3). Then for every element $x \in K(X) \otimes \mathbb{Z}[1/n]$, we have in $K(X) \otimes \mathbb{Z}[1/n]$*

$$\psi_n(x) = \theta^n(-\mathcal{T}_X) \cdot \psi^n(x).$$

Proof. Let $g\colon X \hookrightarrow \Gamma$ be a (regular) closed embedding in a smooth variety Γ, and N its normal bundle. There is a unique element $y \in K^0_X(X)$ such that $y \cap [\mathcal{O}_X] = x$. Then $g_*(y) \cap [\mathcal{O}_\Gamma] = x$ by (II.2.b).

Then we have in $K(X) \otimes \mathbb{Z}[1/n]$

$$\begin{aligned}
\psi_n(x) &= \theta^n(-g^*T_\Gamma) \cdot \Big(\psi^n \circ g_*(y) \cap [\mathcal{O}_\Gamma]\Big) && \text{by (II.4.f)}\\
&= \theta^n(-g^*T_\Gamma) \cdot \Big(g_*\Big(\theta^n(N) \cdot \psi^n(y)\Big) \cap [\mathcal{O}_\Gamma]\Big) && \text{by Theorem II.4.5}\\
&= \theta^n(-g^*T_\Gamma) \cdot \Big(\Big(\theta^n(N) \cdot \psi^n(y)\Big) \cap [\mathcal{O}_X]\Big) && \text{by (II.2.b)}\\
&= \theta^n(N - g^*T_\Gamma) \cdot \Big(\psi^n(y) \cap [\mathcal{O}_X]\Big) && \text{by (II.2.c)}\\
&= \theta^n(-\mathcal{T}_X) \cdot \psi^n(x) && \text{by Proposition II.3.1, (d),}
\end{aligned}$$

where we have additionally used Lemma B.5 for the last equality. □

Lemma II.4.17. *Let X and Y be two varieties over a common field. Then we have for all $x \in K_0(X)$ and $y \in K_0(Y)$ the equality in $K_0(X \times Y) \otimes \mathbb{Z}[1/n]$*

$$\psi_n(x) \boxtimes \psi_n(y) = \psi_n(x \boxtimes y).$$

Proof. Choose closed embeddings $X \hookrightarrow \Gamma$ and $Y \hookrightarrow \Omega$, with Γ and Ω smooth varieties over the common base field, and elements $x' \in K^0_X(\Gamma), y' \in K^0_Y(\Omega)$ such that $x' \cap [\mathcal{O}_X] = x$ and $y' \cap [\mathcal{O}_Y] = y$. Then using successively Lemma II.2.10, Formula (II.2.f), Condition (b) of Proposition II.3.1 and [EKM08, Corollary 104.8], we have in $K_0(X \times Y)$

$$\begin{aligned}
\psi_n(x) \boxtimes \psi_n(y) &= \Big(\Big(\theta^n(-T_\Gamma) \cdot \psi^n(x')\Big) \cap [\mathcal{O}_\Gamma]\Big) \boxtimes \Big(\Big(\theta^n(-T_\Omega) \cdot \psi^n(y')\Big) \cap [\mathcal{O}_\Omega]\Big)\\
&= \Big(\Big(\theta^n(-T_\Gamma) \cdot \psi^n(x')\Big) \boxtimes \Big(\theta^n(-T_\Omega) \cdot \psi^n(y')\Big)\Big) \cap [\mathcal{O}_{\Gamma\times\Omega}]\\
&= \Big(\theta^n(-T_\Gamma) \boxtimes \theta^n(-T_\Omega)\Big) \cdot \Big(\psi^n(x') \boxtimes \psi^n(y')\Big) \cap [\mathcal{O}_{\Gamma\times\Omega}]\\
&= \theta^n(-T_{\Gamma\times\Omega}) \cdot \psi^n(x' \boxtimes y') \cap [\mathcal{O}_{\Gamma\times\Omega}].
\end{aligned}$$

We conclude using the fact that

$$(x' \boxtimes y') \cap [\mathcal{O}_{\Gamma\times\Omega}] = x \boxtimes y,$$

which is consequence of Lemma II.2.10. □

Chapter III

Steenrod operations

In this chapter, we first construct Steenrod operations on the modulo p Chow groups (p a prime number),

$$\mathrm{Ch}(\overline{X}) \to \mathrm{Ch}(\overline{X}),$$

where X is a smooth variety, $\overline{X} = X \times_F \overline{F}$, with $\overline{F}/F$ a field extension such that $\operatorname{gr} K(\overline{X})$ is *generated by p-regular classes* (Definition III.1.3) and $\overline{X}$ is *torsion-free* (Definition I.4.31). An example of such a variety X is a smooth projective homogeneous variety under a semi-simple linear algebraic group, with $\overline{F}/F$ a splitting extension, for example an algebraic closure.

Then we state a theorem of Gabber, and explain how it combines with Adams-Riemann-Roch Theorem to give rise to a proof of the fact that the Steenrod operations above preserve rationality of cycles.

Next, we define $\widetilde{\mathrm{Ch}}(X)$ as $\mathrm{CH}(X)/\left(2\,\mathrm{CH}(X) + \text{torsion subgroup}\right)$, and view it as a (covariant) functor on the category of varieties with projective morphisms to the category of abelian groups. The purpose of the second section is to construct the *first Steenrod square of homological type*, which is a natural transformation

$$\mathrm{Sq}_1\colon\ \mathrm{Ch}_\bullet \to \widetilde{\mathrm{Ch}}_{\bullet-1}\,.$$

The associated cohomological operation will induce an operation on the reduced Chow group (modulo two) of geometrically torsion free varieties which will coincide with the operation S^1 already constructed in the first section, when it is defined.

We made the choice to restrict to cohomological operations in the first section, where an arbitrary prime number p is considered, and to define the homological first Steenrod square in the second section. The reasons for this choice are twofold. First a condition such as being geometrically torsion free seemed to us quite hard to verify for a given singular variety, whence the restriction to cohomological operations in the first section. On the other hand, as the second section contains some new unconditional results, we tried to state them in the strongest form we could, even at the cost of using the more complicated notion of Adams operations with supports. Note that no condition such as being a torsion free variety appears for the first Steenrod square, but we still need to work with Chow groups modulo torsion.

III.1 Resolution of singularities

Definition III.1.1. An *alteration* of an integral variety X is an integral variety X' with a projective generically finite morphism $X' \to X$. The *degree* of the alteration X' is the degree of the extension of function fields $F(X')/F(X)$.

Given a prime integer p, we shall say that a field F *admits p-resolution of singularities* if every integral variety X over F admits a regular alteration of degree prime to p.

It seems that Ofer Gabber has recently proven the following

Theorem III.1.2 (Ofer Gabber). *Any field of characteristic not p admits p-resolution of singularities.*

The only reference that we have found concerning this result is [Gab].

Definition III.1.3. Let X be a smooth variety and p a prime integer. We will say that $\operatorname{gr} K(X)$ *is generated by p-regular classes*, if $\operatorname{gr} K(X)$ is generated, as a ring, by classes $[\mathcal{O}_Z]$, where Z is an integral variety admitting a regular alteration of degree prime to p.

Theorem III.1.4. *Let X be a smooth variety and $x \in K(X)^{(q)}$. Assume that* $\operatorname{gr} K(X)$ *is generated by p-regular classes. Then we may find elements $x_k \in K(X)^{(q+k(p-1))}$ such that the element*

$$\psi^p(x) - \sum_{k=0}^{q} p^{q-k} x_k$$

is of p-th power torsion in $K(X)$.

In addition, we can choose the element x_0 in order to have

$$x_0 = x \mod K(X)^{(q+1)}.$$

Finally, there is an element $y \in K(X)$ such that

$$x_q - x^p - py$$

is of p-th power torsion in $K(X)$.

Proof. The last statement follows at once from the first and Lemma I.4.18.

The group $K(X)^{(q)}$ is additively generated by the subgroup $K(X)^{(q+1)}$ and products $[\mathcal{O}_{Z_1}]\cdots[\mathcal{O}_{Z_n}]$, where Z_i for $i=1,\cdots,n$ are integral varieties admitting a regular alteration of degree prime to p and satisfying $\sum_i(\dim X-\dim Z_i)=q$. It follows by descending induction on q that the group $K(X)^{(q)}$ is additively generated by products satisfying $\sum_i(\dim X-\dim Z_i)\geq q$.

The claims that we want to prove are compatible with multiplication, hence it is enough to prove them for $x=[\mathcal{O}_Z]$, with Z an integral variety admitting a regular alteration $f\colon A\to Z$, of degree d prime to p. Let $j\colon Z\hookrightarrow X$ be the closed embedding. By Proposition I.1.14, we have in $K(X) \mod K(X)^{(q+1)}$ the congruence

$$j_*\circ f_*[\mathcal{O}_A]=d\cdot j_*[\mathcal{O}_Z] \mod K(X)^{(q+1)}.$$

The morphism f is projective, hence $g=j\circ f$ factors as $A\xrightarrow{i}\mathbb{P}^r_X\xrightarrow{p}X$, with i a closed embedding, and p the standard projection. The closed embedding $i\colon A\hookrightarrow\mathbb{P}^r_X$ is a regular closed embedding, as is any closed embedding of regular varieties ([Bou07, Proposition 2, § 5, N°3, p.65]). Let N be its normal bundle, and T_p the tangent bundle of p. Then the virtual normal bundle of g is, by Definition B.3

$$\mathcal{N}_g=[N]-i^*[T_p]\in K(A).$$

We have $\operatorname{rank}(\mathcal{N}_g)=q$ by Proposition B.6. The Adams-Riemann-Roch theorem with denominators II.4.10, and Corollary II.1.3 yield in the ring $\mathbb{Z}[1/p]\otimes K(X)$

$$\begin{aligned} d\otimes\psi^p[\mathcal{O}_Z] &= 1\otimes(\psi^p\circ g_*[\mathcal{O}_A])\\ &= g_*\Big(\theta^p(\mathcal{N}_g)\cdot(1\otimes\psi^p[\mathcal{O}_A])\Big)\\ &= g_*\circ\theta^p(\mathcal{N}_g)\\ &= \sum_{k=0}^{q}p^{q-k}\otimes g_*(z_k), \end{aligned}$$

the elements z_k being liftings of $w_k^{K,p}(\mathcal{N}_g)$ to $K(A)$. Then the elements $g_*(z_k)$ belong to $K(X)^{(q+k(p-1))}$ by Corollary I.3.15.

Pick integers u and v such that $du=1-p^{q+1}v$. Then we have in the ring $\mathbb{Z}[1/p]\otimes K(X)$

$$1\otimes\psi^p[\mathcal{O}_Z]=p^q\otimes\Big(u\cdot g_*(z_0)+pv\cdot\psi^p[\mathcal{O}_Z]\Big)+\sum_{k=1}^{q}u\cdot p^{q-k}\otimes g_*(z_k)$$

We take

$$x_0=u\cdot g_*(z_0)+pv\cdot\psi^p[\mathcal{O}_Z],$$

which belongs to $K(X)^{(q)}$ by Lemma I.4.17, and

$$x_k = u \cdot g_*(z_k) \in K(X)^{(q+k(p-1))} \qquad (k > 0).$$

The first formula follows modulo p-th power torsion in the ring $K(X)$.

As to the choice of x_0, we have $w_0^{K,p}(\mathcal{N}_f) = [\mathcal{O}_A]$, hence

$$g_*(z_0) = g_*[\mathcal{O}_A] \mod K(X)^{(q+1)} = d[\mathcal{O}_Z] \mod K(X)^{(q+1)}.$$

Now we compute, using Lemma II.4.11

$$\begin{aligned} x_0 &= u \cdot g_*(z_0) + pv \cdot \psi^p[\mathcal{O}_Z] \\ &= du \cdot [\mathcal{O}_Z] + p^{q+1}v \cdot [\mathcal{O}_Z] \mod K(X)^{(q+1)} \\ &= [\mathcal{O}_Z] \mod K(X)^{(q+1)}, \end{aligned}$$

as claimed. □

We consider the following condition on a smooth variety X over a field F and a prime integer p:

Condition III.1.5. The ring $\operatorname{gr} K(X)$ is generated by p-regular classes, and there is an extension $\overline{F}$ of F such that the variety $\overline{X} := X \times_{\operatorname{Spec}(F)} \operatorname{Spec}(\overline{F})$ is torsion free (Definition I.4.31). We require additionally that $\operatorname{gr} K(\overline{X})$ is generated by p-regular classes (Definition III.1.3).

We define the functor $\operatorname{Ch} := \mathbb{Z}/p \otimes \operatorname{CH}$ and, following [Ati66], we will build natural operations, for X satisfying Condition III.1.5 above,

$$\mathrm{S}^k_{\overline{X}} \colon \operatorname{Ch}^\bullet(\overline{X}) \to \operatorname{Ch}^{\bullet+k(p-1)}(\overline{X})$$

which preserve rationality of cycles.

Before explaining the construction, we give two examples of varieties satisfying Condition III.1.5 above.

Example III.1.6. Let X be a split projective homogeneous variety under a semi-simple algebraic group over an arbitrary field F. Set $\overline{F} = F$. Then the Chow motive of the variety $X = \overline{X}$ splits as a sum of Tate motives by the results of [Köc91], hence the variety X is torsion free. The ring $\operatorname{CH}(X)$ and $\operatorname{gr} K(X)$ are isomorphic by Proposition I.4.33. Using the results of [Dem74], we get that the ring $\operatorname{gr} K(X)$ is generated by p-regular classes, for any prime integer p.

Example III.1.7. Let X be a projective homogeneous variety under a semi-simple algebraic group over a field F admitting p-resolution of singularities. Let $\overline{F}$ be an algebraic closure of F. Then the ring $\operatorname{gr} K(X)$ is generated by p-regular classes. By Example III.1.6, the split projective homogeneous variety $\overline{X}$ is torsion free, and the ring $\operatorname{gr} K(\overline{X})$ is generated by p-regular classes.

Let $\overline{F}/F$ be a field extension. If G is a presheaf on Sm/F with values in the category of abelian groups, like $\operatorname{gr} K$ or the Chow group functors CH or Ch, we will write $\overline{\mathsf{G}}(X)$ for the image of the map $\mathsf{G}(X) \to \mathsf{G}(\overline{X})$. We also write $\overline{\alpha}$ for the image of an element α under the map $\mathsf{G}(X) \to \overline{\mathsf{G}}(X)$.

Lemma III.1.8. *Let X be a smooth variety over a field F, and $\overline{F}/F$ an extension such that the variety $\overline{X}$ is torsion free. Then the rings $\overline{\mathrm{Ch}}(X)$ and $\mathbb{Z}/p \otimes \overline{\operatorname{gr} K}(X)$ are isomorphic.*

Proof. The ring isomorphism $\varphi_{\overline{X}} \colon \mathrm{CH}(\overline{X}) \to \operatorname{gr} K(\overline{X})$ of Proposition I.4.33 induces an injective ring homomorphism $\overline{\mathrm{CH}}(X) \to \overline{\operatorname{gr} K}(X)$. The latter is surjective because φ_X is so. The lemma follows after tensorisation with $\mathbb{Z}/p$. □

Cohomological Steenrod operations

We take a smooth variety X satisfying Condition III.1.5, and a prime number p. Note that the group $K(\overline{X})$ is then torsion-free. Let $x \in K(\overline{X})^{(q)}$. Choosing x_k as in Theorem III.1.4, we call y_k the image of x_k under the natural epimorphism:

$$K(\overline{X})^{(q+k(p-1))} \to \mathbb{Z}/p \otimes \operatorname{gr}^{q+k(p-1)} K(\overline{X})$$

and put $y_k = 0$ for $k > q$. It follows from Lemma I.4.35 that the elements y_k do not depend on the choice of the elements x_k.

Furthermore, if $x \in K(\overline{X})^{(q+1)}$ or if x is divisible by p in $K(\overline{X})^{(q)}$, then for all k, x_k may be chosen to be divisible by p or in $K(\overline{X})^{(q+k(p-1)+1)}$, hence $y_k = 0$. This gives group homomorphisms:

$$\mathbb{Z}/p \otimes \operatorname{gr}^q K(\overline{X}) \to \mathbb{Z}/p \otimes \operatorname{gr}^{q+k(p-1)} K(\overline{X}).$$

Since the Adams operation commutes with the pull-back along the flat morphism $\overline{X} \to X$, if x belongs to the image of the pull-back $K(X)^{(q)} \to K(\overline{X})^{(q)}$, then the element x_k can be chosen to belong to the image of $K(X)^{(q+k(p-1))} \to K(\overline{X})^{(q+k(p-1))}$, for every integer k. Therefore we get group homomorphisms:

$$\mathbb{Z}/p \otimes \overline{\operatorname{gr}^q K}(X) \to \mathbb{Z}/p \otimes \overline{\operatorname{gr}^{q+k(p-1)} K}(X).$$

Using Lemma III.1.8, we obtain, for every positive integer k, a group homomorphism S^k_X fitting in the diagram:

$$\begin{array}{ccc} \overline{\mathrm{Ch}}^{\bullet}(X) & \xrightarrow{\mathrm{S}^k_X} & \overline{\mathrm{Ch}}^{\bullet+k(p-1)}(X) \\ \downarrow & & \downarrow \\ \mathrm{Ch}^{\bullet}(\overline{X}) & \xrightarrow{\mathrm{S}^k_{\overline{X}}} & \mathrm{Ch}^{\bullet+k(p-1)}(\overline{X}) \end{array}$$

Proposition III.1.9. *The operations* $\mathrm{S}^\bullet$ *have the following properties, for all smooth varieties* X *and* Y *over a common field and satisfying Condition III.1.5:*

(i) $\mathrm{S}^0_X = \mathrm{id}$.

(ii) If $x \in \overline{\mathrm{Ch}}^q(X)$ *and* $k \notin [0,q]$ *then* $\mathrm{S}^k_X(x) = 0$.

(iii) For $x \in \overline{\mathrm{Ch}}^q(X)$, $\mathrm{S}^q_X(x) = x^p$.

(iv) If $f\colon X \to Y$ *is any morphism, then* $\mathrm{S}^\bullet_X \circ f^* = f^* \circ \mathrm{S}^\bullet_Y$.

(v) If $x, y \in \overline{\mathrm{Ch}}(X)$, *then* $\mathrm{S}^\bullet_X(x \cdot y) = \mathrm{S}^\bullet_X(x) \cdot \mathrm{S}^\bullet_X(y)$.

(vi) $\mathrm{S}^\bullet_{X\times Y} = \mathrm{S}^\bullet_X \times \mathrm{S}^\bullet_Y$.

Proof. Formulas (i) and (iii) from Theorem III.1.4 and Corollary I.3.20.

Statement (ii) holds by definition of the elements y_k.

Statement (iv) follows from Proposition I.3.3 and the fact that Adams operations commute with pull-backs.

Compatibility with product (v) follows from the multiplicativity of the Adams operations and Corollary I.3.20.

Finally (vi) follows from (iv) and (v). □

Proposition III.1.10 (Wu formula). *Let* $i\colon Z \hookrightarrow X$ *be a closed embedding of smooth varieties, with normal bundle* N. *Assume that* X *satisfies Condition III.1.5. Then we have*

$$\mathrm{S}^\bullet_X[\overline{Z}] = i_* \circ w^{\mathrm{CH},p}_\bullet(\overline{N}).$$

Proof. By proposition II.4.9, Proposition II.1.2 and Lemma II.1.1, we have

$$\begin{aligned}\varphi_{\overline{X}} \circ \mathrm{S}^\bullet_X[\overline{Z}] &= i_* \circ w^{K,p}(\overline{N}) \mod p\\ &= i_* \circ \varphi_{\overline{Z}} \circ w^{\mathrm{CH},p}(\overline{N}) \mod p\\ &= \varphi_{\overline{X}} \circ i_* \circ w^{\mathrm{CH},p}(\overline{N}) \mod p.\end{aligned}$$

The result follows from the fact that $\varphi_{\overline{X}}$ is an injective map (Proposition I.4.33). □

Remark III.1.11. We need to resolve singularities on the varieties corresponding to the prime cycles to which we want to apply the Steenrod operations. It follows that, when the integer p is small enough, say $p = 2$ or 3, and the base field is differentially finite over a perfect subfield, but does not necessarily admits p-resolution of singularities, if X is a torsion-free variety of arbitrary dimension, we get operations on low dimensional cycles modulo p, say $\overline{\mathrm{Ch}}_2(X) \to \overline{\mathrm{Ch}}_0(X)$ or $\overline{\mathrm{Ch}}_3(X) \to \overline{\mathrm{Ch}}_1(X)$ for Chow groups modulo 2, and $\overline{\mathrm{Ch}}_3(X) \to \overline{\mathrm{Ch}}_0(X)$ for Chow groups modulo 3, see [CP08] and [CP09].

III.2 The first Steenrod square

In this section, we restrict ourselves to the case $p = 2$.

The first homological square

Lemma III.2.1. *Let X be a normal and connected variety of dimension d. Then there are elements $y \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(X)$ and $z \in \mathrm{F}^{\mathsf{top}}_{d-2} K_0(X)$ satisfying in $K_0(X) \otimes \mathbb{Z}[1/2]$ the equation*

$$\psi_2[\mathcal{O}_X] = 2^{-d}[\mathcal{O}_X] + 2^{-d-1}y + 2^{-s}z,$$

for some integer s.

Proof. The set S of points $x \in X$ such that $\mathcal{O}_{X,x}$ is not a regular local ring is closed in X ([Gro65, Corollaire 6.12.5]). We consider S as a closed subscheme of X, by endowing it with the reduced scheme structure.

Since X is normal, the subscheme S has codimension at least two in X. Indeed X is assumed to be integral, and if x is a point of codimension one in X, the local ring $\mathcal{O}_{X,x}$ is integrally closed of dimension one, hence a discrete valuation ring by [Bou06a, Proposition 11, Chapter VII, §1, N°7, p.209]. Such a ring is regular, a regular system of parameters being given by any uniformizing parameter.

Let U be the open complement of S in X. It is a regular variety. Using Lemma II.4.16, Proposition II.1.2, and the fact that $\psi^2[\mathcal{O}_X] = [\mathcal{O}_X]$, we find elements $y_U \in \mathrm{F}^1_\gamma K^0(U)$ and $z_U \in \mathrm{F}^2_\gamma K^0(U)$ such that we have in $K_0(U) \otimes \mathbb{Z}[1/2]$

$$\psi_2[\mathcal{O}_U] = 2^{-d}[\mathcal{O}_U] + 2^{-d-1}y_U + 2^{-2d}z_U$$

We have the localization exact sequence from Theorem I.1.2

$$K_0(S) \to K_0(X) \xrightarrow{u^*} K(U) \to 0.$$

Choose, using Lemma I.1.10, elements $y \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(X)$ and $z \in \mathrm{F}^{\mathsf{top}}_{d-2} K_0(X)$ such that $u^*(y) = y_U$ and $u^*(z) = z_U$. Then by Proposition II.4.15, we have in $K_0(U) \otimes \mathbb{Z}[1/2]$

$$\begin{aligned} u^* \circ \psi_2[\mathcal{O}_X] &= \psi_2 \circ u^*[\mathcal{O}_X] \\ &= \psi_2[\mathcal{O}_U] \\ &= 2^{-d}[\mathcal{O}_U] + 2^{-d-1}y_U + 2^{-2d}z_U \\ &= u^*(2^{-d}[\mathcal{O}_Y] + 2^{-d-1}y + 2^{-2d}z). \end{aligned}$$

Since S has codimension at least two in X, it follows that

$$\ker(u^* \otimes \mathrm{id}) \subset \mathrm{F}^{\mathsf{top}}_{d-2} K_0(X) \otimes \mathbb{Z}[1/2].$$

We complete the proof of the lemma by setting $s = 2d$. □

Proposition III.2.2. *Let Z be an integral variety of dimension d. Then there exist an integer s and elements $y \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(Z)$, and $z \in \mathrm{F}^{\mathsf{top}}_{d-2} K_0(Z)$ whose images in $\mathbb{Z}[1/2] \otimes K_0(Z)$ satisfy*

$$\psi_2[\mathcal{O}_Z] = 2^{-d}[\mathcal{O}_Z] + 2^{-d-1}y + 2^{-s}z.$$

Proof. We proceed by induction on the dimension d of Z. The statement is clear when $d = 0$ because ψ_2 is the identity in this case.

When d is arbitrary, let $p\colon \widetilde{Z} \to Z$ be the normalization of Z. It is a finite birational morphism ([Bou06a, Théorème 2, Chapter V, § 3, N°2, p.59]), hence by Corollary I.1.16, there is an element $\delta \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(Z)$ such that

$$p_*[\mathcal{O}_{\widetilde{Z}}] + \delta = [\mathcal{O}_Z]. \tag{III.2.a}$$

By Proposition II.4.14 and Lemma III.2.1 applied to the normal and connected variety $\widetilde{Z}$, we find elements $\widetilde{y} \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(\widetilde{Z})$ and $\widetilde{z} \in \mathrm{F}^{\mathsf{top}}_{d-2} K_0(\widetilde{Z})$ such that we have in $\mathbb{Z}[1/2] \otimes K_0(Z)$

$$\psi_2 \circ p_*[\mathcal{O}_{\widetilde{Z}}] = p_* \circ \psi_2[\mathcal{O}_{\widetilde{Z}}] = 2^{-d}p_*[\mathcal{O}_{\widetilde{Z}}] + 2^{-d-1}p_*(\widetilde{y}) + 2^{-r}p_*(\widetilde{z}).$$

By (III.2.a), we have

$$\begin{aligned}\psi_2[\mathcal{O}_Z] &= 2^{-d}p_*[\mathcal{O}_{\widetilde{Z}}] + 2^{-d-1}p_*(\widetilde{y}) + 2^{-r}p_*(\widetilde{z}) + \psi_2(\delta)\\ &= 2^{-d}[\mathcal{O}_Z] + 2^{-d-1}\Big(p_*(\widetilde{y}) - 2\delta\Big) + 2^{-r}p_*(\widetilde{z}) + \psi_2(\delta).\end{aligned}$$

Using the induction hypothesis, we find an element $\eta \in \mathrm{F}^{\mathsf{top}}_{d-2} K_0(Z)$ and an integer t such that

$$\psi_2(\delta) = 2^{-d-1}\delta + 2^{-r-t}\eta.$$

Then we can set $y = p_*(\widetilde{y}) - \delta \in \mathrm{F}^{\mathsf{top}}_{d-1} K_0(Z)$, $z = 2^t p_*(\widetilde{z}) + \eta \in \mathrm{F}^{\mathsf{top}}_{d-2} K_0(Z)$ and $s = r + t$, and we get the desired formula. □

Corollary III.2.3. *Let X be a variety, and $x \in \mathrm{F}^{\mathsf{top}}_{k} K_0(X)$. Then there exists an element $y(x) \in \mathrm{F}^{\mathsf{top}}_{k-1} K_0(X)$ whose image satisfies the congruence in $(K_0(X) \otimes \mathbb{Z}[1/2]) \mod (\mathrm{F}^{\mathsf{top}}_{k-2} K_0(X) \otimes \mathbb{Z}[1/2])$*

$$\psi_2(x) = 2^{-k}x + 2^{-k-1}y(x).$$

All elements $y(x) \in \mathrm{F}^{\mathsf{top}}_{k-1} K_0(X)$ satisfying the congruence above have the same image in $\mathrm{gr}^{\mathsf{top}}_{k-1} K_0(X)/(2\text{-torsion})$, i.e. *the difference of two such elements belongs to $\mathrm{F}^{\mathsf{top}}_{k-2} K_0(X)$ when multiplied by an appropriate power of 2.*

Proof. By linearity, it is enough to prove existence of $y(x)$ for elements x of the form $i_*[\mathcal{O}_Z]$ where $i\colon Z \hookrightarrow X$ is the closed embedding of an integral variety Z. The existence of $y(x)$ follows from the compatibility of the topological filtration with push-forward, Proposition II.4.14, and Proposition III.2.2.

We now prove the second statement. Let $x \in \mathrm{F}_k^{\mathsf{top}} K_0(X)$ and y, y' elements of $\mathrm{F}_{k-1}^{\mathsf{top}} K_0(X)$ satisfying congruences in $(K_0(X) \otimes \mathbb{Z}[1/2]) \mod (\mathrm{F}_{k-2}^{\mathsf{top}} K_0(X) \otimes \mathbb{Z}[1/2])$

$$\psi_2(x) = 2^{-k}x + 2^{-k-1}y = 2^{-k}x + 2^{-k-1}y'.$$

Then $y - y' \in \mathrm{F}_{k-2}^{\mathsf{top}} K_0(X) \otimes \mathbb{Z}[1/2]$, hence $y - y'$ is of 2-power torsion in $\mathrm{gr}_{k-1}^{\mathsf{top}} K_0(X)$. □

In order to state the next proposition, we introduce the following notation. For a variety X and an integer k we define

$$^2G_k(X) = \Big(\mathrm{gr}_k^{\mathsf{top}} K_0(X)/(2\text{-torsion})\Big) \otimes \mathbb{Z}/2.$$

In other words $^2G_k(X)$ is the quotient of $\mathrm{F}_k^{\mathsf{top}} K_0(X)$ by the subgroup generated by

— elements of $2 \cdot \mathrm{F}_k^{\mathsf{top}} K_0(X)$,

— elements $x \in \mathrm{F}_k^{\mathsf{top}} K_0(X)$ such that $2^j \cdot x \in \mathrm{F}_{k-1}^{\mathsf{top}} K_0(X)$ for some integer j.

This gives a functor $^2G_\bullet$ from the category of varieties and projective morphisms to the category of graded abelian groups.

Proposition III.2.4. *The association $x \mapsto y(x)$ of Corollary III.2.3 induces a natural transformation of functors with values in the category of graded abelian groups*

$$\mathbb{Z}/2 \otimes \mathrm{gr}_\bullet^{\mathsf{top}} K_0 \to {}^2G_{\bullet-1}.$$

Proof. The association $x \mapsto y(x)$ gives a homomorphism of abelian groups

$$\mathrm{F}_k^{\mathsf{top}} K_0(X) \to {}^2G_{k-1}(X) \quad , \quad x \mapsto y_k(x).$$

Now take $x \in \mathrm{F}_{k-1}^{\mathsf{top}} K_0(X) \subset \mathrm{F}_k^{\mathsf{top}} K_0(X)$. Corollary III.2.3 gives an element $y_{k-1}(x) \in \mathrm{F}_{k-2}^{\mathsf{top}} K_0(X)$ satisfying in $(K_0(X) \otimes \mathbb{Z}[1/2]) \mod (\mathrm{F}_{k-3}^{\mathsf{top}} K_0(X) \otimes \mathbb{Z}[1/2])$ the congruence

$$\psi_2(x) = 2^{1-k}x + 2^{-k}y_{k-1}(x) = 2^{-k}x + 2^{-k-1}(2x + 2y_{k-1}(x)).$$

It follows that $y_k(x) = 2(x + y_{k-1}(x))$ which is zero in $^2G_{k-1}(X)$, since $x + y_{k-1}(x) \in \mathrm{F}_{k-1}^{\mathsf{top}} K_0(X)$.

We get a homomorphism of abelian groups

$$\mathbb{Z}/2 \otimes \mathrm{gr}_\bullet^{\mathsf{top}} K_0 \to {}^2G_{\bullet-1}.$$

The fact that it commutes with push-forwards along projective morphisms follows from Proposition II.4.14. □

One can compose the transformation of Proposition III.2.4 on the left with the natural transformation $\varphi\colon \operatorname{Ch} \to \mathbb{Z}/2 \otimes \operatorname{gr}^{\mathsf{top}}_{\bullet} K_0$, and we can replace ${}^2G$ with a functor built out of the Chow group, as in the following definition.

Definition III.2.5. We define $\widetilde{\operatorname{Ch}}(X)$ as $\operatorname{CH}(X)/(2\operatorname{CH}(X)+\text{torsion subgroup})$. This gives a functor from the category of varieties and projective morphisms to the category of graded abelian groups.

We have a natural transformation

$$\rho\colon {}^2G_{\bullet} \to \widetilde{\operatorname{Ch}}_{\bullet},$$

induced for every variety X by

$$\operatorname{gr}^{\mathsf{top}}_{\bullet} K_0(X)/(2\text{-torsion}) \to \operatorname{gr}^{\mathsf{top}}_{\bullet} K_0(X)/(\text{torsion}) \xleftarrow{\widetilde{\varphi_X}} \operatorname{CH}_{\bullet}(X)/(\text{torsion}),$$

where $\widetilde{\varphi_X}$ is the map induced by the morphism φ of Proposition I.3.2. The map $\widetilde{\varphi_X}$ is an isomorphism by Proposition I.4.29 (Corollary I.4.28 when X is smooth). For every variety X, the morphism ρ_X is surjective, and its kernel consists of images modulo two of torsion elements.

Theorem III.2.6. *The association $x \mapsto y(x)$ of Corollary III.2.3 induces a natural transformation*

$$\operatorname{Sq}_1\colon \operatorname{Ch}_{\bullet} \to \widetilde{\operatorname{Ch}}_{\bullet-1}\,.$$

of functors from the category of arbitrary varieties and projective maps to the category of graded abelian groups.

Proof. We define Sq_1 as the composite

$$\operatorname{Ch}_{\bullet} \xrightarrow{\varphi} \mathbb{Z}/2 \otimes \operatorname{gr}^{\mathsf{top}}_{\bullet} K_0 \to {}^2G_{\bullet-1} \xrightarrow{\rho} \widetilde{\operatorname{Ch}}_{\bullet-1}$$

where the map in the middle is the transformation of Proposition III.2.4. □

Definition III.2.7. The natural transformation Sq^1 of Theorem III.2.6 above will be called the *first homological Steenrod square.*

Remark III.2.8. One can expect the map Sq_1 to lift to a map

$$\operatorname{Ch}_{\bullet} \to \operatorname{Ch}_{\bullet-1}\,.$$

However one can not expect that Sq_1 descends to a map

$$\widetilde{\operatorname{Ch}}_{\bullet} \to \widetilde{\operatorname{Ch}}_{\bullet-1},$$

as suggested by Example III.2.9 below.

Example III.2.9. Let X be an anisotropic projective 3-dimensional quadric over a field F of characteristic not two, defined by a quadratic form of type $\langle\langle a,b\rangle\rangle \perp c$. In this case, by [Kar90, Theorem 5.3] there is an element $l_0 \in \mathrm{F}_1^{\mathsf{top}} K_0(X)$ such that $l_0 \notin \mathrm{F}_0^{\mathsf{top}} K_0(X)$ and $2l_0 \in \mathrm{F}_0^{\mathsf{top}} K_0(X)$. Note that $\varphi_X\colon \mathrm{CH}(X) \to \mathrm{gr}^{\mathsf{top}} K_0(X)$ is an isomorphism : this follows from Proposition I.4.27 in codimension $0,1$ and 2, and from the fact that $\mathrm{CH}_0(X) \simeq \mathbb{Z}$, see for example [EKM08, Corollary 71.4]. Let $x \in \mathrm{Ch}_1(X)$ be the antecedent of $l_0 \mod \mathrm{F}_0^{\mathsf{top}} K_0(X)$ under φ_X, and $y \in \mathrm{Ch}_0(X)$ the class of a point of degree two. We have $\varphi_X(y) = 2l_0$,

$$\psi_2(l_0) = l_0 = 2^{-1}l_0 + 2^{-2}(2l_0),$$

hence $\mathrm{Sq}_1(x) = y$, which is non-zero in $\widetilde{\mathrm{Ch}}_0(X)$. On the other hand x is zero in $\widetilde{\mathrm{Ch}}_1(X)$.

If $\overline{F}/F$ is a splitting field extension for X (Definition IV.2.1), note that $\overline{x}$ vanishes as an element of $\mathrm{Ch}(\overline{X})$ because this group is torsion free, and x is a torsion element. The cycle $\overline{y}$ belongs to $2\,\mathrm{CH}(\overline{X})$, hence also vanishes in $\mathrm{Ch}(\overline{X})$. Therefore this example does not provide an obstruction to the statement that Sq_1 induces a map

$$\overline{\mathrm{Ch}}_\bullet(X) \to \overline{\mathrm{Ch}}_{\bullet-1}(X),$$

and it does indeed (when the variety $\overline{X}$ is torsion free), as explained in the beginning of Section III.3.

For a variety X, we denote by

$$\mathrm{Sq}_1^X\colon \mathrm{Ch}_\bullet(X) \to \widetilde{\mathrm{Ch}}_{\bullet-1}(X)$$

the morphism of graded abelian groups induced on X by the transformation of Theorem III.2.6. We also denote by $\widetilde{x}$ the image in $\widetilde{\mathrm{Ch}}_\bullet(X)$ of an element $x \in \mathrm{Ch}_\bullet(X)$.

Proposition III.2.10. *Let X and Y be two varieties over the same field. We have*

$$\mathrm{Sq}_1^{X\times Y}(x \times y) = \mathrm{Sq}_1^X(x) \times \widetilde{y} + \widetilde{x} \times \mathrm{Sq}_1^Y(y).$$

Proof. Let $a \in \mathrm{F}_m^{\mathsf{top}} K_0(X)$ and $b \in \mathrm{F}_n^{\mathsf{top}} K_0(Y)$. We have in $(K_0(X \times Y) \otimes \mathbb{Z}[1/2]) \mod (\mathrm{F}_{m+n-2}^{\mathsf{top}} K_0(X \times Y) \otimes \mathbb{Z}[1/2])$ the congruence

$$\begin{aligned}\psi_2(a \boxtimes b) &= \psi_2(a) \boxtimes \psi_2(b)\\ &= \Big(2^{-m}a + 2^{-m-1}y_m(a)\Big) \boxtimes \Big(2^{-n}a + 2^{-n-1}y_n(b)\Big)\\ &= 2^{-n-m}a \boxtimes b + 2^{-m-n-1}\Big(y_n(a) \boxtimes b + a \boxtimes y_m(b)\Big),\end{aligned}$$

by Lemma II.4.17, Corollary III.2.3 and Lemma I.3.9. The statement follows in view of the unicity in Corollary III.2.3, and Lemma I.3.19 asserting that the map φ is compatible with external products. □

The first cohomological square

When X is a regular connected variety and q an integer, we set

$$^2G^q(x) = {}^2G_{\dim X - q}(X).$$

Proposition III.2.11. *Let X be a regular connected variety, and $x \in K(X)^{(q)}$. Then there exists an element $y(x) \in K(X)^{(q+1)}$ such that we have in $(K(X) \otimes \mathbb{Z}[1/2]) \mod (K(X)^{(q+2)} \otimes \mathbb{Z}[1/2])$ the congruence*

$$\psi^2(x) = 2^q x + 2^{q-1} y(x).$$

Moreover the image in $^2G^{q+1}(X)$ of such an element $y(x)$ is uniquely determined by x.

Proof. This follows from Corollary III.2.3 and Lemma II.4.16. □

Remark III.2.12. When one is interested only in operations on smooth varieties, the use of Adams operations with supports is not required, and one can of course prove Proposition III.2.11 directly using (cohomological) Adams operations.

Theorem III.2.13. *The association $x \mapsto y(x)$ of Proposition III.2.11 induces, for every integer q, a natural transformation*

$$\mathrm{Ch}^q \to \widetilde{\mathrm{Ch}}^{q+1}$$

of presheaves of graded abelian groups on the category of regular equidimensional varieties and arbitrary morphisms.

When $q = 0$, this is the zero functor.

When $q = 1$, and if we restrict to the category Sm/F, this is the transformation induced by $x \mapsto \widetilde{x}^2$.

Proof. The existence of the morphism

$$\mathrm{Ch}^q(X) \to \widetilde{\mathrm{Ch}}^{q+1}(X)$$

for all regular varieties X and integers q follows from Proposition III.2.11 the same way Proposition III.2.4 followed from Corollary III.2.3.

The fact that these morphisms commute with pull-back follows from the fact that Adams operations commute with pull-back, and unicity in Proposition III.2.11.

For the case $q = 0$, note that any element $x \in K(X)$ is congruent to $\mathrm{rank}(x) \cdot [\mathcal{O}_X]$ modulo $K(X)^{(1)}$ by Lemma I.1.11, and $\psi^2[\mathcal{O}_X] = [\mathcal{O}_X]$. Again we use unicity in Proposition III.2.11.

For the case $q = 1$, the first Chow group $\mathrm{CH}^1(X)$ of a regular variety X is generated by first Chern classes of line bundles. But if $\mathcal{L}$ is a line bundle, we have

$$\psi^2 \circ c_1(\mathcal{L}) = \psi^2(1-[\mathcal{L}^\vee]) = 1-[\mathcal{L}^\vee]^2 = 2(1-[\mathcal{L}^\vee])-(1-[\mathcal{L}^\vee])^2 = 2c_1(\mathcal{L})-c_1(\mathcal{L})^2,$$

and we conclude using unicity in Proposition III.2.11. □

Definition III.2.14. The operation Sq^1 of Theorem III.2.13 above will be called the *first cohomological Steenrod square*. We shall write

$$\mathrm{Sq}^1_X \colon \mathrm{Ch}^q(X) \to \widetilde{\mathrm{Ch}}^{q+1}(X)$$

for the homomorphism of abelian groups, when X is a regular variety.

Lemma III.2.15. *Let X be regular variety, and $\mathcal{T}_X$ its virtual tangent bundle (Definition B.3). We have*

$$\mathrm{Sq}^1_X = c_1(\mathcal{T}_X) + \mathrm{Sq}_1^X .$$

Proof. This follows from Lemma II.4.16, and Corollary II.1.3 combined with Lemma II.1.1. □

Proposition III.2.16. *Let X and Y be two regular varieties over the same field, and such that $X \times Y$ is a regular variety. Then for all $a \in \mathrm{Ch}(X)$ and $b \in \mathrm{Ch}(Y)$, we have in $\widetilde{\mathrm{Ch}}(X \times Y)$ the equality*

$$\mathrm{Sq}^1_{X\times Y}(a \times b) = \mathrm{Sq}^1_X(a) \times \widetilde{b} + \widetilde{a} \times \mathrm{Sq}^1_Y(b).$$

Proof. We have, using Lemma III.2.15, Lemma B.4 and Proposition III.2.10

$$\begin{aligned}
\mathrm{Sq}^1(a \times b) &= c_1(\mathcal{T}_{X\times Y})(\widetilde{a \times b}) + \mathrm{Sq}_1(a \times b)\\
&= c_1(\mathcal{T}_X)(\widetilde{a}) \times \widetilde{b} + \widetilde{a} \times c_1(\mathcal{T}_Y)(\widetilde{b}) + \mathrm{Sq}_1(a) \times \widetilde{b} + \widetilde{a} \times \mathrm{Sq}_1(b)\\
&= \mathrm{Sq}^1(a) \times \widetilde{b} + \widetilde{a} \times \mathrm{Sq}^1(b),
\end{aligned}$$

as required. □

Proposition III.2.17. *Let X be a smooth variety. Then the group homomorphism*

$$\mathrm{Sq}^1_X \colon \mathrm{Ch}(X) \to \widetilde{\mathrm{Ch}}(X)$$

is a derivation, i.e *satisfies*

$$\mathrm{Sq}^1_X(a \cdot b) = \mathrm{Sq}^1_X(a) \cdot \widetilde{b} + \widetilde{a} \cdot \mathrm{Sq}^1_X(b)$$

for all $a, b \in \mathrm{Ch}(X)$.

Proof. Let $\Delta\colon X \hookrightarrow X \times X$ be the diagonal embedding. Then, using Proposition III.2.16 and compatibility of Sq^1 with pull-backs, we have in $\widetilde{\mathrm{Ch}}(X)$

$$\begin{aligned}\mathrm{Sq}^1_X(a \cdot b) &= \mathrm{Sq}^1_X \circ \Delta^*(a \times b) = \Delta^* \circ \mathrm{Sq}^1_{X\times X}(a \times b) = \Delta^*\Big(\mathrm{Sq}^1_X(a) \times \widetilde{b} + \widetilde{a} \times \mathrm{Sq}^1_X(b)\Big)\\ &= \mathrm{Sq}^1_X(a) \cdot \widetilde{b} + \widetilde{a} \cdot \mathrm{Sq}^1_X(b),\end{aligned}$$

as required. □

We conclude this section with a formula for smooth cycles.

Proposition III.2.18. *Let X be a smooth variety and $i\colon Y \hookrightarrow X$ be a closed embedding, with Y a smooth variety. If N is the normal bundle of i, then we have in* $\widetilde{\mathrm{Ch}}(X)$

$$\mathrm{Sq}^1_X[Y] = i_* \circ \widetilde{c_1(N)}.$$

Proof. We can assume that X and Y are connected. Let q be the codimension of Y in X. We apply Adams-Riemann-Roch Theorem II.4.9 to the closed embedding i, and Proposition II.1.2,

$$\begin{aligned}\psi^2[\mathcal{O}_Y] &= i_* \circ \theta^2(N)\\ &= 2^q[\mathcal{O}_Y] + 2^{q-1} i_* \circ c_1(N) \mod i_*(\mathrm{F}^2_\gamma\, K^0(Y)).\end{aligned}$$

By Corollary I.3.15, we have $i_* \circ c_1(N) \in i_*(\mathrm{F}^1_\gamma\, K^0(Y)) \subset K(X)^{(q+1)}$ and $i_*(\mathrm{F}^2_\gamma\, K^0(Y)) \subset K(X)^{(q+2)}$. This concludes the proof, in view of Lemma II.1.1. □

III.3 ADEM RELATION

As in the previous section, we restrict our attention to the prime $p = 2$.

We have seen in Proposition III.2.17 that Sq^1_X is a *derivation* when X is smooth. On can ask whether Sq^1_X is a *differential*, *i.e* if $\mathrm{Sq}^1_X \circ \mathrm{Sq}^1_X = 0$; but we need first to be able to regard Sq^1_X as an endomorphism of some abelian group. Note that Remark III.2.8 suggests that some informations will be lost in the process.

Let X be a smooth variety over a field F, and $\overline{F}/F$ a field extension such that the variety $\overline{X}$ is torsion free. Then, since Sq^1 commutes with pull-backs, we have a commutative diagram

$$\begin{array}{ccc}\mathrm{Ch}(X) & \xrightarrow{\mathrm{Sq}^1_X} & \widetilde{\mathrm{Ch}}(X)\\ \downarrow & & \downarrow\\ \mathrm{Ch}(\overline{X}) & \xrightarrow[\mathrm{Sq}^1_{\overline{X}}]{} & \widetilde{\mathrm{Ch}}(\overline{X}).\end{array}$$

Since the variety $\overline{X}$ is assumed torsion free, we have an isomorphism

$$\mathrm{Ch}(\overline{X}) \simeq \widetilde{\mathrm{Ch}}(\overline{X}),$$

and moreover the pull-back $\mathrm{Ch}(X) \to \mathrm{Ch}(\overline{X})$ factors through the map $\widetilde{\mathrm{Ch}}(X) \to \widetilde{\mathrm{Ch}}(\overline{X})$, hence $\mathrm{Sq}^1_{\overline{X}}$ induces a morphism

$$\overline{\mathrm{Sq}}^1 = \overline{\mathrm{Sq}}^1_X \colon \overline{\mathrm{Ch}}^{\bullet}(X) \to \overline{\mathrm{Ch}}^{\bullet+1}(X). \tag{III.3.a}$$

The next proposition states that the morphism $\overline{\mathrm{Sq}}^1$ coincides with the construction of Section III.1, when these morphisms can be compared. Its hypothesis are fulfilled when X is a smooth projective homogeneous variety under the action of a semi-simple algebraic group, over an arbitrary field.

Proposition III.3.1. *Assume that X is a smooth variety over a field F, and that $\overline{F}/F$ is a field extension such that $\overline{X} = X \times_F \overline{F}$ is a torsion free variety, and $\mathrm{gr}\, K(\overline{X})$ is generated by 2-regular classes. The operation $\mathrm{S}^1_{\overline{X}}$ of the previous section fits in the commutative square*

$$\begin{array}{ccc} \overline{\mathrm{Ch}}(X) & \xrightarrow{\overline{\mathrm{Sq}}^1} & \overline{\mathrm{Ch}}(X) \\ \downarrow & & \downarrow \\ \mathrm{Ch}(\overline{X}) & \xrightarrow[\mathrm{S}^1_{\overline{X}}]{} & \mathrm{Ch}(\overline{X}) \end{array}$$

In particular the operation $\mathrm{S}^1_{\overline{X}}$ preserves rationality of cycles.

Proof. This is clear from our constructions. □

When additionally $\mathrm{gr}\, K(X)$ is generated by 2-regular classes, then the operations S^1_X and $\overline{\mathrm{Sq}}^1$ coincide, since the vertical arrows in the square above are injective.

Now that we are able to view the first Steenrod square as an endomorphism of the presheaf of abelian groups $\overline{\mathrm{Ch}}$, it makes sense to ask the question of the validity of Adem relations. We will now prove the Adem relation involving only the first Steenrod square, namely $\overline{\mathrm{Sq}}^1 \circ \overline{\mathrm{Sq}}^1 = 0$.

The idea is to use the formula

$$\psi^2 \circ \psi^2 = \psi^4$$

together with explicit decompositions of the Todd homomorphisms θ^2 and θ^4.

We begin with a lemma, strengthening Corollary II.1.3, in the case $p = 2$.

Lemma III.3.2. *Let c_k be the Chern classes with value in K^0 introduced in Example I.4.4. Let X be a variety, and x an element of $K^0(X)$. We have in* $\mathbb{Z}[1/2] \otimes K^0(X)$

$$\theta^2(x) = \sum_{k=0}^{\infty} (-1)^k 2^{\mathrm{rank}(x)-k} \cdot c_k(x).$$

Proof. The association

$$x \mapsto \sum_{k=0}^{\infty} (-1)^k 2^{\mathrm{rank}(x)-k} \cdot c_k(x)$$

induces a morphism of presheaves of abelian groups $\tau \colon K^0 \to (\mathbb{Z}[1/2] \otimes K^0)^{\times}$. If $\mathcal{L}$ is a line bundle, we have

$$\tau(\mathcal{L}) = 2 - c_1(\mathcal{L}),$$

hence τ is the Todd homomorphism associated with the power series

$$\frac{1 - (1 - t^2)}{t} = -t + 2.$$

By the unicity in Proposition I.4.10, τ and θ^2 coincide on K^0_+, hence on the whole K^0. □

We now turn to the Todd homomorphism θ^4. We define

$$c^3 = \sum_k c^3_k \cdot t^k$$

as the Todd homomorphism associated with the power series

$$1 + t^2$$

with value in the presheaf K^0 on Sm/F.

Lemma III.3.3. *Let X be a variety, and x an element of $K^0(X)$ of rank q. We have in $\mathbb{Z}[1/2] \otimes K^0(X) \mod \mathrm{F}^3_\gamma K^0(X)$ the congruence*

$$\theta^4(x) = 2^{2q} + 2^{2q-1}\Big(-3c_1(x)\Big) + 2^{2q-2}\Big(9c_2(x) + 4c^3_1(x)\Big).$$

Proof. This follows from a careful inspection of the formula

$$\frac{1 - (1 - t)^4}{t} = -t^3 + 4 + 2t(-3 + 2t).$$ □

In order to prove the relation $\overline{\mathrm{Sq}}^1 \circ \overline{\mathrm{Sq}}^1 = 0$, we will need some form of resolution of singularities. But we can work over an arbitrary extension of the base field when proving the relation above, which makes it easier to find generators of the Chow group admitting desingularization. Indeed the hypothesis of the following proposition are satisfied when X is a projective homogeneous variety under the action of a semi-simple linear algebraic group over an arbitrary field F, by Example III.1.6 applied to $\overline{X} = X \times_F \overline{F}$, where $\overline{F}$ is a splitting field for X.

Proposition III.3.4. *Let $\overline{F}/F$ be a field extension. Let X be a smooth variety over the field F such that $\overline{X} = X \times_F \overline{F}$ is a torsion-free variety. Assume that $\operatorname{gr} K(\overline{X})$ is generated by 2-regular classes, in other words $\overline{X}$ satisfies Condition III.1.5 for the prime 2 and the trivial field extension $\overline{F}/\overline{F}$.*

Then the group homomorphism $\overline{\mathrm{Sq}}^1$ of (III.3.a) satisfies

$$\overline{\mathrm{Sq}}^1 \circ \overline{\mathrm{Sq}}^1 = 0.$$

Proof. There is no loss of generality in making the additional assumption that $F = \overline{F}$, because the map $\overline{\mathrm{Ch}}(X) \to \mathrm{Ch}(\overline{X})$ is injective.

Using the fact that $\mathrm{S}^1 = \mathrm{Sq}^1 = \overline{\mathrm{Sq}}^1$ is a derivation (Proposition III.2.17), we can assume that we are given an element $x \in K(X)^{(q)}$ satisfying $d \cdot x = f_*[\mathcal{O}_Y]$ where Y is a regular connected variety with $\dim X - \dim Y = q$, the morphism $f\colon Y \to X$ is projective and d is an odd integer.

The projective morphism $f\colon Y \to X$ can be decomposed as $Y \hookrightarrow \mathbb{P}^r_X \to X$, a regular closed embedding followed by a (trivial) projective bundle. Let $\mathcal{N}_f = f^*[T_X] - \mathcal{T}_Y$ be the virtual normal bundle of f (Definition B.3). It is an element of $K(Y)$ of rank q by Proposition B.6. The Riemann-Roch theorem with denominators II.4.10 for the second Adams operation and the morphism f gives in $\mathbb{Z}[1/2] \otimes K(X)$

$$f_* \circ \theta^2(\mathcal{N}_f) = d \cdot \psi^2(x).$$

By Lemma III.3.2, we have

$$\sum_k 2^{q-k} f_* \circ c_k(\mathcal{N}_f) = d \cdot \psi^2(x).$$

It follows that

$$d \cdot \psi^2(x) = 2^q d \cdot x + 2^{q-1} x_1 + 2^{q-2} x_2 \mod K(X)^{(q+3)}, \qquad \text{(III.3.b)}$$

with $x_1 = f_* \circ c_1(\mathcal{N}_f)$ and $x_2 = f_* \circ c_2(\mathcal{N}_f)$. Note that

$$\psi^2(x) = d \cdot \psi^2(x) + (1-d) \cdot \psi^2(x)$$

hence if $y(x) \in K(X)^{(q+1)}$ is the element produced by Proposition III.2.11, we have the congruence in $K(X) \mod K(X)^{(q+2)}$

$$2^q x + 2^{q-1} y(x) = 2^q x + 2^{q-1} x_1 + (1-d)\Big(2^q x + 2^{q-1} y(x)\Big).$$

Since d is odd, and X torsion free, it follows that $\varphi_X(x_1) = \mathrm{Sq}^1(x)$.

Applying Theorem III.1.4 with $p = 2$ (or Proposition III.2.11) to $x_1 \in K(X)^{(q+1)}$, we find an element $x_1' \in K(X)^{(q+2)}$ satisfying

$$\psi^2(x_1) = 2^{q+1} x_1 + 2^q x_1' \mod K(X)^{(q+3)}.$$

Our aim is to prove that x'_1 belongs to $2K(X)^{(q+2)} + K(X)^{(q+3)}$. We have

$$\psi^2(x_2) = 2^{q+2}x_2 \mod K(X)^{(q+3)}$$

by Lemma II.4.11, hence applying the operation ψ^2 to (III.3.b) we obtain the congruence in $K(X) \mod K(X)^{(q+3)}$

$$d \cdot \psi^2 \circ \psi^2(x) = 2^{2q}d \cdot x + 2^{2q-1}(3x_1) + 2^{2q-2}(5x_2 + 2x'_1). \tag{III.3.c}$$

From Lemma III.3.3, we get in $K(Y) \mod \mathrm{F}^3_\gamma K^0(Y)$ the congruence

$$\theta^4(\mathcal{N}_f) = 2^{2q} + 2^{2q-1}\Big(- 3c_1(\mathcal{N}_f)\Big) + 2^{2q-2}\Big(9c_2(\mathcal{N}_f) + 4c_1^3(\mathcal{N}_f)\Big) \tag{III.3.d}$$

We now apply the Riemann-Roch theorem with denominators II.4.10 for the 4-th Adams operation to the morphism f. This yields in $\mathbb{Z}[1/2] \otimes K(X)$ the equality

$$f_* \circ \theta^4(\mathcal{N}_f) = \psi^4 \circ f_*[\mathcal{O}_Y] = d \cdot \psi^2 \circ \psi^2(x)$$

In other words we have, so to speak, f_*(III.3.d) – (III.3.c) $\in K(X)^{(q+3)}$, that is, the element

$$\begin{aligned} &- 2^{2q-1}3\Big(x_1 + x_1\Big) + 2^{q-2}\Big(9x_2 - 5x_2 + 4f_* \circ c_1^3(\mathcal{N}_f) - 2x'_1\Big) \\ &= 2^{2q}\Big(- 3x_1 + x_2 + f_* \circ c_1^3(\mathcal{N}_f)\Big) - 2^{2q-1}x'_1 \end{aligned}$$

belongs to $K(X)^{(q+3)}$. Therefore, we see that $2^{2q-1}x'_1$ belongs $2^{2q}K(X) + K(X)^{(q+3)}$, and since the variety X is torsion free, we can use Lemma I.4.34 to see that x'_1 belongs to $2K(X)^{(q+2)} + K(X)^{(q+3)}$, as required. □

Chapter IV

Applications

In this chapter, we provide some applications of the constructions made in the previous chapters.

We prove some divisibilities of characteristic numbers of smooth projective varieties. These relations are well-known, but we give here a proof working independently of the characteristic of the base field, and without any assumption of resolution of singularities.

Using the first Steenrod square, we prove a theorem about the parity of the first Witt index of quadratic form, which is valid over any field. The statement in the case of a field of characteristic 2 is new, to the best of our knowledge.

The last section is devoted to the study of integrality properties of the Chern character. Indeed, it is mentioned in the introduction of [Ati66] that the article of Atiyah first grew as an attempt to prove the integrality theorem of [Ada61] on the Chern character by more elementary methods. We prove an algebraic analog of Adams's Theorem in Proposition IV.3.5, under the assumption of resolution of singularities, and give one unconditional statement for the prime 2 in Proposition IV.3.3. It should be noticed that in contrast with [Ati66], where only *spaces without torsion* are considered, we do not require our variety to be *torsion free* (see Definition I.4.31) for this statement.

IV.1 Divisibility of some characteristic numbers

Proposition IV.1.1. *Let X be a smooth connected projective variety of positive dimension, with tangent bundle T_X, and p a prime number. Then for all integers k we have*

$$\deg w_k^{\mathrm{CH},p}(-T_X) = 0 \mod p.$$

Proof. We also write deg for the K-theoretical degree induced by the push-forward under the structural morphism, and the identification $K(\text{point}) = \mathbb{Z}$. A closed embedding of X in a projective space is a regular closed embedding, hence the structural map of X factors as a regular closed embedding followed by a projective bundle. By the Adams Riemann-Roch theorem with denominators II.4.10, the following diagram is commutative

$$\begin{array}{ccc} K(X) \otimes \mathbb{Z}[1/p] & \xrightarrow{\theta^p(-T_X)\circ\psi^p} & K(X) \otimes \mathbb{Z}[1/p] \\ \Big\downarrow \deg & & \Big\downarrow \deg \\ \mathbb{Z}[1/p] & \xrightarrow[\psi^p=\mathrm{id}]{} & \mathbb{Z}[1/p]. \end{array}$$

In particular

$$\deg \circ \psi_p[\mathcal{O}_X] = \deg \circ \theta^p(-T_X) = \deg[\mathcal{O}_X] \in \mathbb{Z} \subset \mathbb{Z}[1/p]. \tag{IV.1.a}$$

The lemma being obvious when $k(p-1) \neq \dim X$ for dimensional reasons, we may assume that $\dim X = k(p-1)$.

Let $\widetilde{K}(X)$ be the quotient of $K(X)$ by its p-power torsion subgroup. Note that the degree map factors through $\widetilde{K}(X)$. We view $\widetilde{K}(X)$ as a subgroup of $K(X) \otimes \mathbb{Z}[1/p]$. Then by Corollary II.1.3, the element $p^{2k} \cdot \theta^p(-T_X)$ belongs to $\widetilde{K}(X)$, and is congruent modulo $p\widetilde{K}(X)$ to $w_k^{K,p}(-T_X)$. By (IV.1.a), its degree has to be divisible by p. In view of Lemma II.1.1, this proves the claim. □

Remark IV.1.2. Under some assumptions about the characteristic of the base field, this lemma is well-known. More general forms have been proven using various methods : Steenrod operations in [Mer03], algebraic cobordism in [Lev07], and reduced power operations in [Pan04, § 2.6.7].

When the characteristic of the base field is p, the case $p = 2$ has been considered in [Ros08], using the Frobenius map.

Setting $d_X = [\frac{\dim X}{p-1}]$ and

$$s_p(X) = \frac{w_{d_X}^{\mathrm{CH},p}(-T_X)}{p} \mod p \in \mathbb{Z}/p,$$

we have the primitivity property, as in [Lev07, Lemma 11]:

Lemma IV.1.3. *Let X and Y be two smooth connected projective varieties of positive dimensions over a common field, and p a prime integer. Then*

$$s_p(X \times Y) = 0 \mod p.$$

Proof. Again we may assume that $\dim(X \times Y) = d(p-1)$. By [EKM08, Corollary 104.8] and Lemma I.4.15, we have

$$\begin{aligned} w_d^{\mathrm{CH},p}(-T_{X\times Y}) &= w_d^{\mathrm{CH},p}(-T_X \times T_Y) \\ &= \sum_{i+j=d} w_i^{\mathrm{CH},p}(-T_X) \times w_j^{\mathrm{CH},p}(-T_Y) \end{aligned}$$

This element is zero unless $(p-1)|\dim X$. But in this case, $\dim Y = k(p-1)$ for some integer k, and

$$\begin{aligned}\deg w_d^{\mathrm{CH},p}(-T_{X\times Y}) &= \deg\Big(w_{d-k}^{\mathrm{CH},p}(-T_X)\times w_k^{\mathrm{CH},p}(-T_Y)\Big)\\ &= \deg w_{d-k}^{\mathrm{CH},p}(-T_X)\cdot \deg w_k^{\mathrm{CH},p}(-T_Y)\\ &= 0 \mod p^2.\end{aligned}$$

□

IV.2 Application to quadratic forms

Here we state some consequences of the existence of the operations constructed in Sections III.2 and III.1.

Definition IV.2.1. Let X be a smooth projective quadric over a field F. A *splitting field extension* for X is a field extension $\overline{F}/F$ such that $\overline{X} = X\times_{\mathrm{Spec}(F)}\mathrm{Spec}(\overline{F})$ contains a projective space of the maximal possible dimension, $\dim X/2$ or $(\dim X-1)/2$ depending on the parity of $\dim X$. For example $\overline{F}/F$ could be an algebraic closure.

Theorem IV.2.2. *Let φ be an anisotropic non-degenerate quadratic form over an arbitrary field. Let $\mathfrak{i}_1$ be the first Witt index of φ. If $\dim\varphi-\mathfrak{i}_1$ is odd then $\mathfrak{i}_1=1$.*

Proof. Let X be the smooth projective quadric of φ, and $\overline{F}/F$ a splitting field extension for X. We shall use the standard notation h^i, l_i for the basis of $\mathrm{Ch}(\overline{X})$, found in [EKM08], and also quickly recalled in Section V.2. We also use the notion of a cycle contained in another ([EKM08, p.313]). Let $\pi\in\overline{\mathrm{Ch}}(X)$ be the 1-primordial cycle of X ([EKM08, p.323]). It is the "minimal" ([EKM08, Definition 73.5]) rational cycle containing $h^0\times l_{\mathfrak{i}_1-1}$. Write

$$\pi = h^0\times l_{\mathfrak{i}_1-1} + l_{\mathfrak{i}_1-1}\times h^0 + v.$$

Then $v\in\mathrm{Ch}(\overline{X})$ does not contain $h^0\times l_{\mathfrak{i}_1-1}$ nor $l_{\mathfrak{i}_1-1}\times h^0$ by [EKM08, Lemma 73.15].

We apply the homomorphism $\overline{\mathrm{Sq}}^1$ of (III.3.a), and we have, by Proposition III.2.16 and Lemma IV.2.3 below,

$$\overline{\mathrm{Sq}}^1(h^0\times l_{\mathfrak{i}_1-1}) = h^0\times\overline{\mathrm{Sq}}^1(l_{\mathfrak{i}_1-1}) = (\dim\varphi-\mathfrak{i}_1)\cdot(h^0\times l_{\mathfrak{i}_1-2}).$$

(Here we have written $l_{-1}=0$.) Assume that $\mathfrak{i}_1\neq 1$, so that the element above is non-zero. Then

$$\overline{\mathrm{Sq}}^1(\pi) = h^0\times l_{\mathfrak{i}_1-2} + l_{\mathfrak{i}_1-2}\times h^0 + \mathrm{Sq}^1(v).$$

By [EKM08, Corollary 73.21] the cycle $\mathrm{Sq}^1(v)$ contains $h^1\times l_{\mathfrak{i}_1-1}+l_{\mathfrak{i}_1-1}\times h^1$. It is then necessary that the cycle v, hence the rational cycle π, contains $h^1\times l_{\mathfrak{i}_1}+l_{\mathfrak{i}_1}\times h^1$. But it is outside of the "shell triangles", *i.e.* is forbidden by [EKM08, Lemma 73.12]. □

Lemma IV.2.3. *Let X be a smooth projective quadric, and $\overline{X}$ the restriction of X to a splitting field. For every integer i such that $2i \leq \dim X$, let $l_i \in \mathrm{Ch}_i(\overline{X})$ be the class of the projectivisation of a totally isotropic space of dimension $i+1$ (and $l_{-1} = 0$). Then we have in $\mathrm{Ch}_{i-1}(\overline{X})$*

$$\overline{\mathrm{Sq}}^1(l_i) = (\dim X - i + 1) \cdot l_{i-1}.$$

Proof. Note that [EKM08, Corollary 78.2] is valid over a field of arbitrary characteristic. Hence if $j \colon \mathbb{P}^i \hookrightarrow \overline{X}$ is the closed embedding of a totally isotropic subspace, with normal bundle N, we have

$$c_1(N) = (\dim X - i + 1) \cdot H,$$

where H is the hyperplane class in $\mathrm{Ch}^1(\mathbb{P}^i)$. The cycle $j_*(H) \in \mathrm{Ch}_{i-1}(\overline{X})$ is equal to l_{i-1}, and the formula follows by Proposition III.2.18. □

As a corollary of Theorem IV.2.2, we get a generalization of [EKM08, Corollary 79.6] that includes the case of fields of characteristic 2.

Corollary IV.2.4. *The relative higher Witt indices of a non-degenerate quadratic form of odd dimension are odd.*

The relative higher Witt indices of a non-degenerate quadratic form of even dimension are equal to 1 or even.

Proof. Let φ be the non-degenerate quadratic form. Then we have the Witt decomposition ([EKM08, Theorem 8.5])

$$\varphi = \varphi_{an} \perp h,$$

with φ_{an} an anisotropic form, and h an hyperbolic form. The bilinear form associated with h is non-degenerate, hence, by [EKM08, Proposition 7.22], the quadratic form φ_{an} is non-degenerate. Therefore we can apply Theorem IV.2.2 to φ_{an}, which proves that the statement holds for the first Witt index of φ. Moreover the argument above shows that the anisotropic part of the non-degenerate quadratic form $\varphi_{F(\varphi)}$ is non-degenerate, so that we can proceed by induction. □

In case the base field admits 2-resolution of singularities we can prove a more precise statement.

Theorem IV.2.5. *Let φ be an anisotropic non-degenerate quadratic form over a field admitting 2-resolution of singularities. Let $\mathfrak{i}_1$ be the first Witt index of φ. Then*

$$\mathfrak{i}_1 \leq 2^{v_2((\dim \varphi) - \mathfrak{i}_1)}.$$

Proof. Since φ is non-degenerate, its projective quadric X is smooth. We are in the situation of Example III.1.7 for the quadric X.

The operations $\mathrm{S}^\bullet$ constructed here satisfy the Wu formula (Proposition III.1.10), so that the analog of [EKM08, Corollary 78.3] holds. This gives, in the Chow group of a split smooth quadric, a computation of $\mathrm{S}^\bullet(l_i)$, where l_i is the class of totally isotropic subspace of projective dimension i.

Concerning the hyperplane class h in the Chow group of a smooth quadric, the computation of $\mathrm{S}^\bullet(h^i)$, as given in [EKM08, Lemma 78.4], follows from Proposition III.1.9, (iii) and (v).

In addition to these computations, the proof of [EKM08, Proposition 79.4] only deals with the reduced Chow group $\overline{\mathrm{Ch}}$, uses analog of Proposition III.1.9, (i), (ii) and (vi), and relies on [EKM08, Corollary 73.12], which holds over an arbitrary field. □

Combined with Gabber's theorem III.1.2 implying that any field of characteristic not 2 admits 2-resolution of singularities, this gives a new proof of Hoffmann's conjecture, originally proved in [Kar03].

IV.3 Integrality of the Chern character

The purpose of this section is to study integrality properties of the Chern character ch defined in Example I.4.19, and to prove analogs of [Ada61, Theorem 1].

Lemma IV.3.1. *Let X be a smooth variety. Let $x \in K(X)$ and n an integer such that for every integer $0 < k < n$ we have $c_k(x) = 0$. Then for every integer j such that $0 \leq j < 2n$ we have in $\mathbb{Q} \otimes \mathrm{CH}^j(X)$ the relation*

$$\mathrm{ch}_j(x) = \frac{(-1)^{j-1}}{(j-1)!} \otimes c_j(x).$$

Proof. We use the formula

$$\sum_{k=0}^{j-1} (-1)^k (j-k)! \cdot \mathrm{ch}_{j-k} \cdot c_k = (-1)^{j-1} \cdot j \cdot c_j \qquad \text{(IV.3.a)}$$

of [Ful98, Example 3.2.3].

If $j < n$ we see that

$$(j-k)! \cdot \mathrm{ch}_j(x) = (-1)^{j-1} \cdot j \cdot c_j(x) = 0.$$

Now for $n \leq j < 2n$, using again (IV.3.a) and the relation above, we get

$$j! \cdot \mathrm{ch}_j(x) = (-1)^{j-1} \cdot j \cdot c_j(x),$$

as required. □

For a variety X, and an element $x \in \mathbb{Q} \otimes \mathrm{CH}(X)$, we define its *p-adic valuation* $v_p(x)$ as the supremum of the (usual) p-adic valuation of a rational number a such that the cycle $a^{-1} \cdot x$ is the image of an integral cycle in $\mathrm{CH}(X)$. We have $v_p(0) = -\infty$.

Lemma IV.3.2. *Let X be a smooth variety and $x \in K(X)^{(q)}$. We have $\mathrm{ch}_i(x) = 0$ for every integer $i \leq q-1$ and $v_p(\mathrm{ch}_q(x)) \geq 0$.*

Proof. This follows from the combination of Lemma IV.3.1 and Proposition I.4.25.

This can also be proved without referring to Lemma IV.3.1, using directly the Grothendieck-Riemann-Roch theorem in place of Lemma I.4.25. Indeed, assuming for simplicity that the base field is perfect, given a closed embedding $i\colon Y \hookrightarrow X$ of codimension q, with X and Y smooth, and with normal bundle N, we have

$$\mathrm{ch}[\mathcal{O}_Y] = i_* \circ \mathrm{Td}(N) = [Y] \mod \mathbb{Q} \otimes \mathrm{CH}^{\geq q+1}(X),$$

so that the claim holds for $x = [\mathcal{O}_Y]$. Here $\mathrm{Td}(N)$ is the Todd class of N, *i.e* the Todd homomorphism associated with the power series

$$\frac{t}{1 - e^{-t}}$$

with values in CH, applied to $[N] \in K(Y)$.

When Y is possibly not smooth, we reduce to the situation of a regular closed embedding by taking an open smooth subvariety of U of X, and proceed as in the proof of Proposition I.4.25, using localization for Chow groups. □

Proposition IV.3.3. *Let X be a smooth variety, and $x \in K(X)^{(q)}$. Then we have*

$$2\,\mathrm{ch}_{q+1}(x) \in \mathrm{im}\Big(\mathrm{CH}^{q+1}(X) \to \mathbb{Q} \otimes \mathrm{CH}^{q+1}(X)\Big).$$

Proof. By Proposition III.2.11, we can write in $\mathbb{Q} \otimes K(X)$

$$1 \otimes \psi^2(x) = 2^q \otimes x + 2^{q-1} \otimes x_1 + 2^{-s} \otimes z,$$

with $x_1 \in K(X)^{(q+1)}$ and $z \in K(X)^{(q+2)}$. Then we apply the homomorphism ch_{q+1} and, since by Lemma IV.3.2 we have $\mathrm{ch}_{q+1}(z) = 0$, we get, using Lemma I.4.20

$$2^{q+1}\,\mathrm{ch}_{q+1}(x) = 2^q\,\mathrm{ch}_{q+1}(x) + 2^{q-1}\,\mathrm{ch}_{q+1}(x_1).$$

Hence $2\,\mathrm{ch}_{q+1}(x) = \mathrm{ch}_{q+1}(x_1)$. Again by Lemma IV.3.2, the latter belongs to the image of the map $\mathrm{CH}^{q+1}(X) \to \mathbb{Q} \otimes \mathrm{CH}^{q+1}(X)$. □

Combining Proposition IV.3.3 with Lemma IV.3.1, we obtain

Corollary IV.3.4. *Let X be a smooth variety, and $x \in K(X)^{(q)}$. Then we have*

$$2c_{q+1}(x) \in q! \cdot \mathrm{CH}^{q+1}(X) + (\textit{torsion subgroup}).$$

For a rational number $a \in \mathbb{Q}$, we use the notation $[a]$ for its integral part, *i.e.* the greatest integer which is smaller than a.

Proposition IV.3.5. *Let X be a smooth variety such that* $\operatorname{gr} K(X)$ *is generated by p-regular classes, and $x \in K(X)^{(q)}$. Then we have for every integer $n \geq 0$*

$$v_p\Big(\operatorname{ch}_{q+n}(x)\Big) \geq -\left[\frac{n}{p-1}\right]$$

Proof. We proceed by descending induction on q, the case $q = \dim X + 1$ being empty, for all integers n.

By Proposition III.1.4 , we can write in $\mathbb{Q} \otimes K(X)$

$$1 \otimes \psi^p(x) = \sum_k p^{q-k} \otimes x_k,$$

with $x_k \in K(X)^{(q+k(p-1))}$ and $x_0 = x \mod K(X)^{(q+1)}$. Let $i = [n/(p-1)]$. Then we apply the homomorphism ch_{q+n} and we get in $\mathbb{Q} \otimes \mathrm{CH}^{q+n}(X)$, in view of Lemma I.4.20

$$p^{q+n} \operatorname{ch}_{q+n}(x) = \sum_{k=0}^{i} p^{q-k} \operatorname{ch}_{q+n}(x_k).$$

It follows that

$$(p^n - 1) \operatorname{ch}_{q+n}(x) = \operatorname{ch}_{q+n}(x_0 - x) + \sum_{k=1}^{i} p^{-k} \operatorname{ch}_{q+n}(x_k).$$

We apply the induction hypothesis to x_k, for $1 \leq k \leq i$, and we get

$$v_p\Big(\operatorname{ch}_{q+n}(x_k)\Big) \geq -\left[\frac{n-k(p-1)}{p-1}\right] = k - \left[\frac{n}{p-1}\right]$$

Applying the induction hypothesis to $x_0 - x \in K(X)^{(q+1)}$, we have

$$v_p\Big(\operatorname{ch}_{q+n}(x_0 - x)\Big) \geq -\left[\frac{n-1}{p-1}\right] \geq -\left[\frac{n}{p-1}\right].$$

The claim follows. □

Remark IV.3.6. The proof of Proposition IV.3.5 is modeled upon the proof of [Ati66, Theorem 7.1] concerning *spaces without torsion*, but here we do not assume that X is torsion free. Instead we use the fact that $x = x_0 \mod K(X)^{(q+1)}$.

Given an integer t, we define the t-th *Todd number* as

$$\tau_t = \prod_{p \text{ prime}} p^{[t/(p-1)]}.$$

Combining Proposition IV.3.5 with Lemma IV.3.1, we obtain

Corollary IV.3.7. *Let X be a smooth variety, $x \in K(X)^{(q)}$ and t an integer $< q$. Assume that X is generated by p-regular classes for every prime integer p. Then we have*

$$\tau_t \cdot c_{q+t}(x) \in (q+t-1)! \cdot \mathrm{CH}^{q+t}(X) + (\textit{torsion subgroup}).$$

Chapter V

Lifting of coefficients for Chow motives of quadrics [1]

V.1 Introduction

Alexander Vishik has given a description of the Chow motives of quadrics with integral coefficients in [Vis04]. It uses much subtler methods than the ones used to give a similar description with coefficients in $\mathbb{Z}/2$, found for example in [EKM08], but the description obtained is the same ([EKM08], Theorems 93.1 and 94.1). The result presented here allows to recover Vishik's results from the modulo 2 description.

In order to state the main result, we first define the categories involved. Let Λ be a commutative ring, and $\mathrm{CH}(-,\Lambda)$ the Chow group with coefficients in Λ functor . We write $\mathcal{Q}_F$ for the class of smooth projective quadrics over a field F. We consider the additive category $\mathcal{C}(\mathcal{Q}_F,\Lambda)$, where objects are (coproducts of) quadrics in $\mathcal{Q}_F$ and if X, Y are two such quadrics, $\mathrm{Hom}(X,Y)$ is the group of correspondences of degree 0, namely $\mathrm{CH}_{\dim X}(X\times Y,\Lambda)$. We write $\mathcal{CM}(\mathcal{Q}_F,\Lambda)$ for the idempotent completion of $\mathcal{C}(\mathcal{Q}_F,\Lambda)$. This is the category of graded Chow motives of smooth projective quadrics with coefficients in Λ. If $(X,\rho),(Y,\sigma)$ are two such motives then we have :

$$\mathrm{Hom}\Big((X,\rho),(Y,\sigma)\Big) = \sigma\circ \mathrm{CH}_{\dim X}(X\times Y,\Lambda)\circ\rho.$$

We will prove the following :

Theorem V.1.1. *The functor $\mathcal{CM}(\mathcal{Q}_F,\mathbb{Z})\to\mathcal{CM}(\mathcal{Q}_F,\mathbb{Z}/2)$ induces a bijection on the isomorphism classes of objects.*

The proof mostly relies on the low rank of the homogeneous components of the Chow groups of quadrics when passing to a splitting field. These components are almost always indecomposable if we take into account the Galois

1. This chapter will appear in the volume "Quadratic forms, linear algebraic groups, and cohomology" in the series "Developments in Mathematics".

action. The only exception is the component of rank 2 when the discriminant is trivial but in this case the Galois action on the Chow group is trivial which allows the proof to go through.

It seems that Theorem V.1.1 may be deduced from [Vis04] (see Theorem E.11.2 p.254 in [Kah09]). Here we try to give a more direct and self-contained proof.

V.2 Chow groups of quadrics

We first recall some facts and fix the notations that we will use.

If L/F is a field extension, and S a scheme over F, we write S_L for the scheme $S \times_{\mathrm{Spec}(F)} \mathrm{Spec}(L)$. Similarly, for a F-vector space U, we write U_L for $U \otimes_F L$, and for a cycle $x \in \mathrm{CH}(S)$, the element $x_L \in \mathrm{CH}(S_L)$ is the pull-back of x along the flat morphism $S_L \to S$.

We say that a cycle in $\mathrm{CH}(S_L)$ is *F-rational* (or simply *rational* when no confusion seems possible) if it can be written as x_L for some cycle $x \in \mathrm{CH}(S)$, *i.e.* if it belongs to the image of the pull-back homomorphism $\mathrm{CH}(S) \to \mathrm{CH}(S_L)$.

Let F be a field and φ be a non-degenerate quadratic form on an F-vector space V of dimension $D+2$. The associated projective quadric X is smooth of dimension $D = 2d$ or $2d+1$. Let L/F be a splitting extension for X, *i.e.* a field extension such that V_L has a totally isotropic subspace of dimension $d+1$. We write h^i, l_i for the usual basis of $\mathrm{CH}(X_L)$, where $0 \leq i \leq d$. The class h is the pull-back of the hyperplane class of the projective space of V_L, the class l_i is the class of the projectivisation of a totally isotropic subspace of V_L of dimension $i+1$.

If D is even, then $\mathrm{CH}_d(X_L)$ is freely generated by h^d and l_d. In this case, there are exactly two classes of maximal totally isotropic spaces, l_d and ${l_d}'$. They correspond to spaces exchanged by a reflection and verify the relation $l_d + {l_d}' = h^d$.

The group $\mathrm{Aut}(L/F)$ acts on $\mathrm{CH}(X_L)$. It acts trivially on the i-th homogeneous component of $\mathrm{CH}(X_L)$, as long as $2i \neq D$.

See [EKM08] for proofs of all these facts.

In the next proposition, X is a smooth projective quadric of dimension $D = 2d$ associated with a quadratic space (V, φ) over the field F, L/F is a splitting extension for X, and $\operatorname{disc} X$ is the discriminant algebra of φ.

Proposition V.2.1. *Under the natural* $\mathrm{Aut}(L/F)$*-actions, we can identify the pair* $\{l_d, {l_d}'\}$ *and the connected components of* $\mathrm{Spec}(\operatorname{disc} X \otimes L)$.

Proof. We consider the scheme $G(\varphi)$ of maximal totally isotropic subspaces of V, *i.e* the grassmannian variety of isotropic $(d+1)$-dimensional subspaces of V. The scheme $G(\varphi)_L$ has two connected components exchanged by any reflection of the quadratic space (V, φ). There is a faithfully flat morphism $G(\varphi) \to \mathrm{Spec}(\operatorname{disc} X)$ (see [EKM08], §85, p.357), hence the connected components of $G(\varphi)_L$ are in correspondence with those of $\mathrm{Spec}(\operatorname{disc} X \otimes L)$, in a way respecting the natural $\mathrm{Aut}(L/F)$-actions.

Now two maximal totally isotropic subspaces lie in the same connected component of $G(\varphi)_L$ if and only if the corresponding d-dimensional closed subvarieties of the quadric X_L are rationally equivalent (see [EKM08], §86, p.358). Therefore the pair $\{l_d, {l_d}'\}$ is $\mathrm{Aut}(L/F)$-isomorphic to the pair of connected components of $G(\varphi)_L$. The statement follows. □

V.3 Lifting of coefficients

We now give a useful characterization of rational cycles (Proposition V.3.5). The proof will rely on the following theorem ([Ros98], Proposition 9):

Theorem V.3.1 (Rost's nilpotence for quadrics)**.** *Let X be a smooth projective quadric over z field F, and let $\alpha \in \mathrm{End}_{\mathcal{CM}(\mathcal{Q}_F,\Lambda)}(X)$. If $\alpha_L \in \mathrm{CH}(X_L^2)$ vanishes for some field extension L/F, then α is nilpotent.*

We will use the following classical corollaries :

Corollary V.3.2 ([EKM08, Corollary 92.5])**.** *Let X be a smooth projective quadric over a field F and L/F a field extension. Let π a projector in $\mathrm{End}_{\mathcal{CM}(\mathcal{Q}_L,\Lambda)}(X_L)$ that is the restriction of some element in $\mathrm{End}_{\mathcal{CM}(\mathcal{Q}_F,\Lambda)}(X)$. Then there exist a projector φ in $\mathrm{End}_{\mathcal{CM}(\mathcal{Q}_F,\Lambda)}(X)$ such that $\varphi_L = \pi$.*

Corollary V.3.3 ([EKM08, Corollary 92.7])**.** *Let $f\colon (X,\rho) \to (Y,\sigma)$ be a morphism in $\mathcal{CM}(\mathcal{Q}_F,\Lambda)$. If f_L is an isomorphism for some field extension L/F then f is an isomorphism.*

For the following proposition we use ideas from [VY07] and [PSZ08].

Proposition V.3.4. *For any $n \geq 1$, the functor $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n) \to \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ is bijective on the isomorphism classes of objects.*

Proof. We are clearly in the situation $(\star)$ of [VY07], § 2, p.587 for the functor $\mathcal{C}(\mathcal{Q}_F, \mathbb{Z}/2^n) \to \mathcal{C}(\mathcal{Q}_F, \mathbb{Z}/2)$. The statement then follows from Propositions 2.5 and 2.2 of [VY07] (see also Corollary 2.7 of [PSZ08]). □

Any smooth projective quadric admits a (non-canonical) finite Galois splitting extension, of degree a power of 2. This will be used together with the following proposition when we will need to prove that a given cycle with integral coefficients (and defined over some extension of the base field) is rational.

Proposition V.3.5. *Let $X, Y \in \mathcal{Q}_F$ and L/F be a splitting Galois extension of degree m for X and Y. A correspondence in $\mathrm{CH}((X \times Y)_L, \mathbb{Z})$ is rational if and only if it is invariant under the group $\mathrm{Gal}(L/F)$ and its image in $\mathrm{CH}\Big((X \times Y)_L, \mathbb{Z}/m\Big)$ is rational.*

Proof. We write Z for $X \times Y$. We first prove that if x is a $\mathrm{Gal}(L/F)$-invariant cycle in $\mathrm{CH}(Z_L)$ then $[L : F] \cdot x$ is rational.

Let $\tau \colon L \to \overline{F}$ be a separable closure so that we have an $\overline{F}$-isomorphism $L \otimes \overline{F} \to \overline{F} \times ... \times \overline{F}$ given by $u \otimes 1 \mapsto \Big(\tau \circ \gamma(u)\Big)_{\gamma \in \mathrm{Gal}(L/F)}$.

We have a cartesian square :

$$\begin{array}{ccc} Z_L & \longleftarrow & Z_{L \otimes \overline{F}} \\ \downarrow & & \downarrow \\ Z & \longleftarrow & Z_{\overline{F}} \end{array}$$

It follows that we have a commutative diagram of pull-backs and push-forwards:

$$\begin{array}{ccc} \mathrm{CH}(Z_L) & \longrightarrow & \mathrm{CH}(Z_{L \otimes \overline{F}}) \\ \downarrow & & \downarrow \\ \mathrm{CH}(Z) & \longrightarrow & \mathrm{CH}(Z_{\overline{F}}) \end{array}$$

The top map followed by the map on the right is:

$$x \mapsto \sum_{\gamma \in \mathrm{Gal}(L/F)} t^*({}^{\gamma}x)$$

where $t \colon Z_{\overline{F}} \to Z_L$ is the map induced by τ. Using the commutativity of the diagram and the injectivity of t^*, we see that the composite $\mathrm{CH}(Z_L) \to \mathrm{CH}(Z) \to \mathrm{CH}(Z_L)$ maps x to $\sum {}^{\gamma}x$, where γ runs in $\mathrm{Gal}(L/F)$. The claim follows.

Now suppose that u is a cycle in $\mathrm{CH}(Z_L, \mathbb{Z})$ invariant under $\mathrm{Gal}(L/F)$, and that its image in $\mathrm{CH}(Z_L, \mathbb{Z}/m)$ is rational. We can find a rational cycle v in $\mathrm{CH}(Z_L, \mathbb{Z})$ and a cycle δ in $\mathrm{CH}(Z_L, \mathbb{Z})$ such that $m\delta = v - u$. Since $\mathrm{CH}(Z_L, \mathbb{Z})$ is torsion-free, δ is invariant under $\mathrm{Gal}(L/F)$. The first claim ensures that $v - u$ is rational, hence u is rational. □

Let us remark that if $X \in \mathcal{Q}_F$, L/F is a splitting extension, and $2i < \dim X$ then $2l_i = h^{\dim X - i} \in \mathrm{CH}(X_L)$ is always rational. It follows that $2\,\mathrm{CH}_i(X_L)$ consists of rational cycles when $2i \neq \dim X$.

V.4 Surjectivity in main Theorem

Proposition V.4.1. *The functor $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}) \to \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ is surjective on the isomorphism classes of objects.*

Proof. Let $(X, \pi) \in \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$ and L/F a finite splitting Galois extension for X of degree 2^n. By Proposition V.3.4, we can lift the isomorphism class of (X, π) to the isomorphism class of some $(X, \tau) \in \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n)$.

Assume that we have found a $\mathrm{Gal}(L/F)$-invariant projector ρ in $\mathrm{CH}_{\dim X}(X \times X)_L$ which gives modulo 2^n the projector τ_L. By Proposition V.3.5, ρ is a rational cycle, and Corollary V.3.2 provides a projector p such that $\rho = p_L$. Write $\tilde{p} \in \mathrm{CH}(X \times X, \mathbb{Z}/2^n)$ for the image of p. Consider the morphism $(X, \tau) \to (X, \tilde{p})$ given by $\tilde{p} \circ \tau$. Since $\tilde{p}_L = \rho \mod 2^n = \tau_L$, this morphism becomes an isomorphism (the identity) after extending scalars to L hence is an isomorphism by Corollary V.3.3. It follows that the isomorphism class of $(X, p) \in \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z})$ is a lifting of the class of $(X, \tau) \in \mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n)$.

We now build the projector ρ. For any commutative ring Λ, projectors in $\mathrm{CH}((X \times X)_L, \Lambda)$ are in bijective correspondence with ordered pairs of subgroups of $\mathrm{CH}(X_L, \Lambda)$ which form a direct sum decomposition. This bijection is compatible with the natural $\mathrm{Gal}(L/F)$-actions. A projector is of degree 0 if and only if the two summands in the associated decomposition are graded subgroups of $\mathrm{CH}(X_L, \Lambda)$.

When $\dim X$ is odd or when $\mathrm{disc}\, X$ is a field, each homogeneous component of $\mathrm{CH}(X_L, \Lambda)$ is $\mathrm{Gal}(L/F)$-indecomposable, hence $\mathrm{Gal}(L/F)$-invariant projectors of degree 0 of $\mathrm{CH}(X_L, \Lambda)$ are in one-to-one correspondence with the subsets of $\{0, \cdots, \dim X\}$. It follows that we can lift any $\mathrm{Gal}(L/F)$-invariant projector of degree 0 with coefficients in $\mathbb{Z}/2^n$ to an integral $\mathrm{Gal}(L/F)$-invariant projector of degree 0.

When $\dim X$ is even and $\mathrm{disc}\, X$ is trivial, $\mathrm{CH}_i(X_L, \Lambda)$ is indecomposable if $2i \neq 2d_X = \dim X$. The group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}_{d_X}(X_L, \Lambda)$. If the rank of the restriction of $(\tau_L)_*$ to $\mathrm{CH}_{d_X}(X_L, \mathbb{Z}/2^n)$ is 0 or 2, the projector τ_L clearly lifts to a $\mathrm{Gal}(L/F)$-invariant projector in $\mathrm{CH}_{\dim X}(X \times X)_L$.

The last case is when the rank is 1. We fix a decomposition of the group $\mathrm{CH}_{d_X}(X_L, \mathbb{Z})$ into the direct sum of rank 1 summands. Any such decomposition of $\mathrm{CH}_{d_X}(X_L, \Lambda)$ is then given by some element of $\mathrm{SL}_2(\Lambda)$. The next lemma ensures that we can lift any element of $\mathrm{SL}_2(\mathbb{Z}/2^n)$ to $\mathrm{SL}_2(\mathbb{Z})$, thus that τ_L lifts to a projector with integral coefficients. It remains to notice that $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}_{\dim X}(X \times X)_L$ since $\mathrm{disc}\, X$ is trivial, to conclude the proof. □

Lemma V.4.2 ([PSZ08], Lemma 2.14)**.** *For any positive integers k and p, the reduction homomorphism $\mathrm{SL}_k(\mathbb{Z}) \to \mathrm{SL}_k(\mathbb{Z}/p)$ is surjective.*

Proof. Since $\mathbb{Z}/p$ is a semi-local commutative ring, it follows from Corollary 9.3, Chapter V, p.267 of [Bas68], applied with $A = \mathbb{Z}$ and $\underline{q} = p\mathbb{Z}$, that any matrix in $\mathrm{SL}_k(\mathbb{Z}/p)$ is the image modulo p of a product of elementary matrices with integral coefficients. Such a product in particular belongs to $\mathrm{SL}_k(\mathbb{Z})$, as required. □

V.5 Injectivity in main Theorem

In order to prove injectivity in Theorem V.1.1, we may assume that we are given two motives $(X, \rho), (Y, \sigma)$ in $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z})$ and an isomorphism between their images in $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2)$. We will build an isomorphism with integral coefficients between the two motives (which will not, in general, be an integral lifting of the original isomorphism with finite coefficients).

We fix a finite Galois splitting extension L/F for X and Y of degree 2^n. Using Proposition V.3.4 we may assume that there exists an isomorphism α between (X, ρ) and (Y, σ) in $\mathcal{CM}(\mathcal{Q}_F, \mathbb{Z}/2^n)$. By Proposition V.3.5 and Corollary V.3.3, it is enough to build an isomorphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$ which reduces to a rational correspondence modulo 2^n and which is equivariant under $\mathrm{Gal}(L/F)$.

Let d_X be such that $\dim X = 2d_X$ or $2d_X + 1$ and d_Y defined similarly for Y. Let $r(X, \rho)$ be the rank of $\mathrm{CH}_{d_X}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ if $\dim X$ is even and $r(X, \rho) = 0$ if $\dim X$ is odd. We define $r(Y, \sigma)$ in a similar fashion. We will distinguish cases using these integers.

A basis of $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ gives an isomorphism of (X_L, ρ_L) with twists of Tate motives, thus choosing bases for the groups $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ and $\mathrm{CH}(Y_L) \cap \mathrm{im}\,(\sigma_L)_*$, we can see morphisms between the two motives as matrices.

We fix a basis (e_i) of $\mathrm{CH}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ as follows : we choose $e_i \in \mathrm{CH}_i(X_L)$ among the cycles $h^{\dim X - i}, l_i$ for $2i \neq \dim X$. We are done when $r(X, \rho) = 0$.

If $r(X, \rho) = 2$ we complete the basis with $e_{d_X} = l_{d_X}, e'_{d_X} = l'_{d_X} \in \mathrm{CH}_{d_X}(X_L)$.

If $r(X, \rho) = 1$ we choose a generator e_{d_X} of $\mathrm{CH}_{d_X}(X_L) \cap \mathrm{im}\,(\rho_L)_*$ to complete the basis.

We choose a basis (f_i) for $\mathrm{CH}(Y_L) \cap \mathrm{im}\,(\sigma_L)_*$ in a similar way.

If we write $\tilde{\rho}$ and $\tilde{\sigma}$ for the reduction modulo 2^n of ρ and σ, these bases reduce to bases $(\tilde{e}_i)$ of $\mathrm{CH}(X_L, \mathbb{Z}/2^n) \cap \mathrm{im}(\tilde{\rho}_L)_*$ and $(\tilde{f}_i)$ of $\mathrm{CH}(Y_L, \mathbb{Z}/2^n) \cap \mathrm{im}(\tilde{\sigma}_L)_*$. In these homogeneous bases the matrix of a correspondence of degree 0 is diagonal by blocks. The sizes of the blocks are the ranks of the homogeneous components of $\mathrm{im}\,(\rho_L)_*$.

Lemma V.5.1. *If* $r(X, \rho) = 1$ *then* $\mathrm{disc}\, X$ *is trivial.*

Proof. Assume $\mathrm{disc}\, X$ is not trivial. The correspondence ρ induces a projection of $\mathrm{CH}_{d_X}(X_L)$ which is equivariant under the action of $\mathrm{Gal}(L/F)$. But $\mathrm{CH}_{d_X}(X_L)$ is indecomposable as a $\mathrm{Gal}(L/F)$-module. It follows that $(\rho_L)_*$ is either the identity or 0 when restricted to $\mathrm{CH}_{d_X}(X_L)$, hence $r(X, \rho) \neq 1$. □

Corollary V.5.2. *If* $r(X, \rho) \neq 2$ *then* $\mathrm{Gal}(L/F)$ *acts trivially on* $\mathrm{im}\,(\rho_L)_*$.

Lemma V.5.3. *If* $r(X, \rho) = 2$ *then* $r(Y, \rho) = 2$, $\dim Y = \dim X$ *and* $\mathrm{disc}\, Y = \mathrm{disc}\, X$.

Proof. Since the isomorphism $(\alpha_L)_*$ is graded, the d_X-th homogeneous component of $\mathrm{im}(\alpha_L)_*$ has rank 2. This image is a subgroup of the Chow group with coefficients in $\mathbb{Z}/2^n$ of a split quadric, thus the only possibility is that $\dim Y$ is even, $d_X = d_Y$ and $r(Y,\sigma) = 2$.

The isomorphism $(\alpha_L)_*$ is equivariant under the action of $\mathrm{Gal}(L/F)$. It follows that an element of the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}(X_L, \mathbb{Z}/2^n)$ if and only if it acts trivially on $\mathrm{CH}(Y_L, \mathbb{Z}/2^n)$. Every element of $\mathrm{Gal}(L/F)$ acting non-trivially on $\mathrm{CH}(X_L, \mathbb{Z}/2^n)$ (*resp.* $\mathrm{CH}(Y_L, \mathbb{Z}/2^n)$) necessarily exchanges the integral cycles l_{d_X} and $l_{d_X}{}'$ (*resp.* l_{d_Y} and $l_{d_Y}{}'$). Therefore the pair of integral cycles $\{l_{d_X}, l_{d_X}{}'\} \subset \mathrm{CH}(X_L)$ is $\mathrm{Gal}(L/F)$-isomorphic to $\{l_{d_Y}, l_{d_Y}{}'\} \subset \mathrm{CH}(Y_L)$. By proposition V.2.1, we have a $\mathrm{Gal}(L/F)$-isomorphism between the split étale algebras $\mathrm{disc}\, X \otimes L$ and $\mathrm{disc}\, Y \otimes L$. Hence $\mathrm{disc}\, X$ and $\mathrm{disc}\, Y$ correspond to the same cocycle in $\mathrm{H}^1(\mathrm{Gal}(L/F), \mathbb{Z}/2)$, thus are isomorphic. □

We now proceed with the proof of the injectivity.

Let us first assume that $r(X,\rho) \neq 2$. Then $r(Y,\sigma) \neq 2$ by the preceding lemma. By Corollary V.5.2 the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{im}\,(\rho_L)_*$ and on $\mathrm{im}\,(\sigma_L)_*$, therefore any morphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$ is defined by a cycle invariant under $\mathrm{Gal}(L/F)$.

Since the isomorphism α_L is of degree 0, its matrix in our graded bases of the modulo 2^n Chow groups is diagonal. Let $\lambda_i \in (\mathbb{Z}/2^n)^\times$ be the coefficients in the diagonal so that we have $(\alpha_L)_*(\tilde{e}_i) = \lambda_i \tilde{f}_i$ for all i such that $\mathrm{CH}_i(X_L) \cap \mathrm{im}\,(\rho_L)_* \neq \emptyset$.

If $r(X,\rho) = 1$ then λ_{d_X} is defined and we consider the cycle $\beta = (\lambda_{d_X})^{-1}\alpha_L$. If $r(X,\rho) = 0$, we just put $\beta = \alpha_L$.

Now we take $k_i \in \mathbb{Z}/2^n$ such that $\lambda_i^{-1} = 2k_i + 1$. Let $\Delta \in \mathrm{End}(X_L, \tilde{\rho}_L)$ be the identity morphism. We consider the rational cycle

$$\gamma = \Delta + 2\sum k_i \tilde{e}_i \times \tilde{e}_{\dim X - i},$$

where the sum is taken over all i such that $\mathrm{CH}_i(X_L) \subset \mathrm{im}\,(\rho_L)_*$ (which implies in case $r(X,\rho) = 1$ that we do not take $i = d_X$). The composite $\beta \circ \gamma$ is rational, and its matrix in our bases is the identity matrix. This correspondence lifts to an isomorphism with integral coefficients $(X_L, \rho_L) \to (Y_L, \sigma_L)$.

Next assume that $r(X,\rho) = 2$. Then we have $\dim X = \dim Y$, $r(Y,\sigma) = 2$ and $\mathrm{disc}\, X = \mathrm{disc}\, Y$ by Lemma V.5.3. The matrix of $(\alpha_L)_*$ is diagonal by blocks :

$$\begin{pmatrix} \nu_{i_1} & & & & & & \\ & \ddots & & & & & \\ & & \nu_{i_r} & & & & \\ & & & B & & & \\ & & & & \nu_{i_{r+1}} & & \\ & & & & & \ddots & \\ & & & & & & \nu_{i_p} \end{pmatrix}$$

where $\nu_i \in (\mathbb{Z}/2^n)^\times$ and $B \in \mathrm{GL}_2(\mathbb{Z}/2^n)$.

Now if $\operatorname{disc} X = \operatorname{disc} Y$ is a field, there is an element in $\mathrm{Gal}(L/F)$ that simultaneously exchanges the cycles in the bases $\{l_{d_X}, {l_{d_X}}'\}$ of $\mathrm{CH}_{d_X}(X_L, \mathbb{Z}/2^n)$ and $\{l_{d_Y}, {l_{d_Y}}'\}$ of $\mathrm{CH}_{d_Y}(Y_L, \mathbb{Z}/2^n)$. It follows that we may write B as :

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

for some a and b in $\mathbb{Z}/2^n$. The determinant of B is $(a+b)(a-b) \in (\mathbb{Z}/2^n)^\times$, hence $(a-b) \in (\mathbb{Z}/2^n)^\times$. Thus we may replace α_L by $(a-b)^{-1}\alpha_L$ and assume that $a = b+1$.

As before we may write $\nu_i^{-1} = 2k_i + 1$ and replace α_L by the rational cycle :

$$\alpha_L \circ (\Delta + 2\sum k_i \tilde{e}_i \times \tilde{e}_{\dim X - i})$$

the sum being taken over all i such that $\mathrm{CH}_i(X_L) \cap \operatorname{im}(\rho_L)_* \neq \emptyset$ and $i \neq d_X$. Therefore we may assume that $\nu_i = 1$ for all i and that we have a matrix of the shape (I_r being the identity block of size r):

$$\begin{pmatrix} \mathrm{I}_s & & & 0 \\ & a{+}1 & a & \\ & a & a{+}1 & \\ 0 & & & \mathrm{I}_t \end{pmatrix}$$

The matrix of the rational cycle $h^{d_X} \times h^{d_Y} \in \mathrm{CH}((X \times Y)_L, \mathbb{Z}/2^n)$ is :

$$\begin{pmatrix} 0 & & & 0 \\ & 1 & 1 & \\ & 1 & 1 & \\ 0 & & & 0 \end{pmatrix}$$

Now $\alpha_L - a(h^{d_X} \times h^{d_Y})$ is rational and its matrix is the identity. This cycle is invariant under $\mathrm{Gal}(L/F)$ and lifts to an isomorphism $(X_L, \rho_L) \to (Y_L, \sigma_L)$.

It remains to treat the case when $\operatorname{disc} X$ is trivial. In this case the group $\mathrm{Gal}(L/F)$ acts trivially on $\mathrm{CH}(X_L)$ (and on $\mathrm{CH}(Y_L)$ since $\operatorname{disc} Y$ is also trivial). As before, composing with a rational cycle, we may assume that $\nu_i = 1$ for all i. We write $\det B^{-1} = 2k + 1$.

The cycle $\Delta + k(h^{d_X} \times h^{d_X}) \in \mathrm{End}(X_L, \tilde{\rho}_L)$ is rational and its matrix in our basis is :

$$\begin{pmatrix} \mathrm{I}_p & & & 0 \\ & 1{+}k & k & \\ & k & 1{+}k & \\ 0 & & & \mathrm{I}_r \end{pmatrix}$$

We see that the determinant of this matrix is $1+2k$. Therefore the composite $\alpha_L \circ (\Delta + k(h^{d_X} \times h^{d_X}))$ has determinant 1. We use Lemma V.4.2 to conclude, which completes the proof of Theorem V.1.1.

Appendix A

Deformation homomorphism

The aim of this section is to provide an alternate definition of the deformation homomorphism, which has the advantage of working with higher K-groups, and to prove that it coincides with our previous definition (p.42) in the case of K_0. It should be noticed that even in this case the definition involves K_1 groups.

For a variety X, we shall write $K_n(X)$ for Quillen higher K-groups of the category $\mathsf{M}(X)$ of coherent sheaves on a variety X, and $K^n(X)$ for the K-groups of the category $\mathsf{VB}(X)$ of locally free sheaves on X.

If $i\colon X \hookrightarrow Y$ is a closed embedding and $f\colon X \to Y$ a flat map, we have exact functors

$$i_*\colon \mathsf{M}(X) \to \mathsf{M}(Y) \text{ and } f^*\colon \mathsf{M}(Y) \to \mathsf{M}(X)$$

hence homomorphisms $i_*\colon K_n(X) \to K_n(Y)$ and $f^*\colon K_n(Y) \to K_n(X)$ for every integer n.

Theorem A.4 (localization)**.** *Let $i\colon X \hookrightarrow Y$ be a closed embedding, and $u\colon U \to X$ the open complement. Then there is a long exact sequence*

$$\cdots \to K_1(U) \xrightarrow{\partial} K_0(X) \xrightarrow{i_*} K_0(Y) \xrightarrow{u^*} K_0(U) \to 0.$$

Moreover this sequence is natural in the following sense. Consider a commutative diagram of cartesian squares

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y & \xleftarrow{u} & U \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ X' & \xrightarrow{i'} & Y' & \xleftarrow{u'} & U' \end{array}$$

*with U the open complement of X in Y (*resp. *U' of X' in Y').*

If g is flat, then we have a commutative diagram

$$\begin{CD}
\cdots @>>> K_1(U) @>{\partial}>> K_0(X) @>{i_*}>> K_0(Y) @>{u^*}>> K_0(U) @>>> 0\\
@. @A{h^*}AA @A{f^*}AA @A{g^*}AA @A{h^*}AA @.\\
\cdots @>>> K_1(U') @>{\partial}>> K_0(X') @>{i'_*}>> K_0(Y') @>{(u')^*}>> K_0(U') @>>> 0
\end{CD}$$

If g is a closed embedding, then we have a commutative diagram

$$\begin{CD}
\cdots @>>> K_1(U) @>{\partial}>> K_0(X) @>{i_*}>> K_0(Y) @>{u^*}>> K_0(U) @>>> 0\\
@. @V{h_*}VV @V{f_*}VV @V{g_*}VV @V{h_*}VV @.\\
\cdots @>>> K_1(U') @>{\partial}>> K_0(X') @>{i'_*}>> K_0(Y') @>{(u')^*}>> K_0(U') @>>> 0
\end{CD}$$

Proof. See [Qui73, Proposition 3.2, Remark 3.4]. □

Given a closed embedding $X \hookrightarrow Y$ with normal cone N, we define the deformation homomorphism

$$\sigma\colon K_0(Y) \to K_0(N)$$

as follows.

Let $\partial\colon K_1(\mathbb{G}_m \times Y) \to K_0(N)$ be the connecting homomorphism in the long exact localization sequence associated with the closed embedding $N \hookrightarrow W$, where W is the deformation scheme considered in (I.2.c). Given $y \in K_0(Y)$, we set

$$\sigma(y) = \partial(\{t\} \boxtimes y) \in K_0(N)$$

where the element $\{t\} \in K^1(\mathbb{G}_m) = K^1(F[t, t^{-1}])$ is defined as the image of the element $t \in (F[t, t^{-1}])^\times$.

There is a strong analogy between this deformation homomorphism and the deformation homomorphism for cycles modules of [Ros96]. Indeed we will follow closely the exposition of [EKM08] for most arguments below.

Lemma A.5. *Let $X \hookrightarrow Y$ be a closed embedding of varieties, U its open complement, and*

$$\partial\colon K_1(U) \to K_0(X)$$

be the associated connecting homomorphism. Let Z be a variety over the same field as Y, and

$$\partial^Z\colon K_1(U \times Z) \to K_0(X \times Z)$$

be the connecting homomorphism associated with the closed embedding $X \times Z \hookrightarrow Y \times Z$. Then for all $u \in K_1(U)$ and $z \in K_0(Z)$, we have

$$\partial^Z(u \boxtimes z) = \partial(u) \boxtimes z.$$

Proof. Let $\mathsf{M}_Y(X)$ (*resp.* $\mathsf{M}_{Y\times Z}(X \times Z)$) be the Serre subcategory of $\mathsf{M}(X)$ (*resp.* $\mathsf{M}(X \times Z)$) consisting of sheaves supported on Y (*resp.* $Y \times Z$). We can assume that $z = [\mathcal{F}]$ for some coherent $\mathcal{O}_Z$-sheaf. Then external product with $\mathcal{F}$ induces exact functors

$$\mathsf{M}(X) \to \mathsf{M}(X \times Z) \quad \text{and} \quad \mathsf{M}_Y(X) \to \mathsf{M}_{Y\times Z}(X \times Z)$$

commuting with the inclusions $\mathsf{M}_Y(X) \subset \mathsf{M}(X)$ and $\mathsf{M}_{Y\times Z}(X \times Z) \subset \mathsf{M}(X \times Z)$. The statement is now a consequence of the functoriality statement in Quillen localization Theorem [Qui73, Theorem 5], (see also [Sri96, end of Chapter 6, p.124-125] for a detailed proof of this functoriality). □

Lemma A.6 ([EKM08, Example 49.32])**.** *Let F be a field, and $\{t\} \in K^1(\mathbb{G}_m) = K^1(F[t,t^{-1}])$ be the image of the element $t \in (F[t,t^{-1}])^\times$. Let X be a variety over F, and $\partial\colon K_1(\mathbb{G}_m \times X) \to K_0(X)$ be the connecting homomorphism in the long exact localization sequence associated with the closed embedding $X = \{0\} \times X \hookrightarrow \mathbb{A}^1 \times X$. Then for all $x \in K_0(X)$, we have*

$$\partial(\{t\} \boxtimes x) = x.$$

Proof. By Lemma A.5, we can assume that $X = \mathrm{Spec}(F)$. By linearity it will be enough to show that $\partial\{t\} = 1$.

Using the explicit formula found for instance in [Bas68, Chapter IX, Proposition 6.1, p.492], we see that the homomorphism ∂ sends $\{t\}$ to the class of the F-vector space

$$\mathrm{coker}\left(F[t] \xrightarrow{\cdot t} F[t]\right) \simeq F[t]/(t \cdot F[t]) \simeq F,$$

which is precisely $1 \in K_0(\mathrm{Spec}(F))$. □

Lemma A.7 ([EKM08, Lemma 51.9])**.** *Let $i\colon X \hookrightarrow Y$ be a closed embedding, with normal cone N and deformation homomorphism $\sigma\colon K_0(X) \to K_0(N)$. Let $f\colon Y \to Z$ be a flat morphism such that the composite $f \circ i$ is flat. Then we have*

$$\sigma \circ f^* = q^*,$$

where q is the flat morphism $N \to X \hookrightarrow Y \xrightarrow{f} Z$.

Proof. We have a commutative diagram, where each square is cartesian

$$\begin{array}{ccccc}
N & \longrightarrow & W & \longleftarrow & \mathbb{G}_m \times Y \\
{\scriptstyle q}\downarrow & & {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\times f} \\
Z & \longrightarrow & \mathbb{A}^1 \times Z & \longleftarrow & \mathbb{G}_m \times Z
\end{array}$$

Here W is the deformation scheme, and schemes on the right hand side are open complements of the schemes on the left hand side. The vertical arrows are flat morphisms by [EKM08, Lemma 51.8]. By Theorem A.4, the diagram

$$\begin{array}{ccccc}
K_0(X) & \xrightarrow{\{t\}\boxtimes -} & K_1(\mathbb{G}_m \times X) & \xrightarrow{\partial} & K_0(N) \\
{\scriptstyle f^*}\uparrow & & \uparrow{\scriptstyle (\mathrm{id}\times f)^*} & & \uparrow{\scriptstyle q^*} \\
K_0(Z) & \xrightarrow{\{t\}\boxtimes -} & K_1(\mathbb{G}_m \times Z) & \xrightarrow{\partial} & K_0(Z)
\end{array}$$

is commutative. The bottom composite is the identity by Lemma A.6, and the top composite is the deformation homomorphism σ. The formula follows. □

Proposition A.8 ([EKM08, Proposition 51.6]). *Let*

$$\begin{array}{ccc}
X' & \xrightarrow{i'} & Y' \\
\downarrow & & \downarrow{\scriptstyle j} \\
X & \xrightarrow[i]{} & Y
\end{array}$$

*be a cartesian square of closed embeddings, N (*resp. *N') be the normal cone of i (*resp. *i'), and $\sigma\colon K_0(Y) \to K_0(N)$ (*resp. *$\sigma'\colon K_0(Y') \to K_0(N')$) be the associated deformation homomorphism. Let $J\colon N' \hookrightarrow N$ be the induced closed embedding. Then*

$$\sigma \circ j_* = J_* \circ \sigma'.$$

Proof. Let W (*resp.* W') be the deformation scheme associated with i (*resp.* i'). Then we have a commutative diagram

$$\begin{array}{ccccc}
N' & \longrightarrow & W' & \longleftarrow & \mathbb{G}_m \times Y' \\
{\scriptstyle J}\downarrow & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}\times j} \\
N & \longrightarrow & W & \longleftarrow & \mathbb{G}_m \times Y
\end{array}$$

where each square is cartesian, horizontal arrows are closed embeddings, and schemes on the right hand side are open complements of the schemes on the left hand side. By Theorem A.4, we have a commutative diagram

$$\begin{array}{ccccc}
K_0(Y') & \xrightarrow{\{t\}\boxtimes -} & K_1(\mathbb{G}_m \times Y') & \xrightarrow{\partial} & K_0(N') \\
{\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle (\mathrm{id}\times j)_*} & & \downarrow{\scriptstyle J_*} \\
K_0(Y) & \xrightarrow{\{t\}\boxtimes -} & K_1(\mathbb{G}_m \times Y) & \xrightarrow{\partial} & K_0(N)
\end{array}$$

The composite on the top is σ', and the composite on the bottom is σ. This proves the formula. □

Corollary A.9 ([EKM08, Corollary 51.7]). *Let $i\colon X \hookrightarrow Y$ be a closed embedding, $s\colon X \hookrightarrow N$ the zero section of its normal cone. Then we have*

$$\sigma \circ i_* = s_*.$$

Proof. This follows from Proposition A.8 applied with $i = j$, hence $i' = \mathrm{id}_Y$, $J = s$ and $\sigma' = \mathrm{id}_{K_0(X)}$ by Lemma A.6. □

Lemma A.10 ([EKM08, Proposition 52.7]). *Let $i\colon X \hookrightarrow Y$ be a regular closed embedding with normal bundle N, and deformation homomorphism $\sigma\colon K_0(X) \to K_0(N)$. Let Z be a closed subvariety of Y, and form the fiber square*

$$\begin{array}{ccc} A & \longrightarrow & Z \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y. \end{array}$$

Let C be the normal cone of the closed embedding $A \hookrightarrow Z$, and $j\colon C \hookrightarrow N$ the natural closed embedding (see page 35). Then we have in $K_0(N)$

$$\sigma[\mathcal{O}_Z] = j_*[\mathcal{O}_C].$$

Proof. Let $\sigma_Z\colon K_0(Z) \to K_0(C)$ be the deformation homomorphism associated with the closed embedding $A \hookrightarrow Z$. Let $c\colon C \to \mathrm{Spec}(F)$ be the structure morphism. Applying Lemma A.7 to the closed embedding $A \hookrightarrow Z$ and the flat morphism $z\colon Z \to \mathrm{Spec}(F)$ we see that

$$\sigma_Z[\mathcal{O}_Z] = \sigma_Z \circ z^*(1) = c^*(1) = [\mathcal{O}_C].$$

By Proposition A.8 we have

$$j_*[\mathcal{O}_C] = j_* \circ \sigma_Z[\mathcal{O}_Z] = \sigma \circ i_*[\mathcal{O}_Z],$$

which proves the formula. □

We get the same formula as in Proposition I.2.29, which proves that the two constructions of the deformation homomorphism for K_0 coincide.

Appendix B

Virtual tangent bundles

In this section, we recall some of the properties of the virtual tangent bundle of a regular morphism. This leads to the notion of virtual tangent bundle of a regular variety, an element of the Grothendieck group, which will be for us a convenient generalization of the notion of the class of the tangent bundle of a smooth variety to the case of regular varieties.

Definition B.1. A *regular morphism* $f\colon X \to Y$ is morphism which can be factored as $i \circ p$, where i is a regular closed embedding and p a smooth morphism.

Proposition B.2. *Let $f\colon X \to Y$ be a regular morphism and $i \circ p, i' \circ p'$ two factorizations of f, with i, i' regular closed embeddings with normal bundle N_i and $N_{i'}$, and p, p' smooth morphism with tangent bundles $T_p, T_{p'}$. Then we have in $K^0(X)$ the equality*

$$i^*[T_p] - [N_i] = i'^*[T_{p'}] - [N_{i'}].$$

Proof. Let $X \hookrightarrow M \to Y$ and $X \hookrightarrow M' \to Y$ be the two factorizations of f. Then f also factors as $X \hookrightarrow M \times_Y M' \to Y$ where the first map is the diagonal embedding (i, i') over Y. This embedding factors as

$$X \xrightarrow{j} M \times_Y X \xrightarrow{k} M \times_Y M'.$$

We have a diagram with cartesian squares

$$\begin{array}{ccccc} M \times_Y X & \xrightarrow{k} & M \times_Y M' & \longrightarrow & M \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[i']{} & M' & \xrightarrow[p']{} & Y. \end{array}$$

The map j is a regular closed embedding by [Gro67, Corollaire 17.12.3] since it has a smooth section g. By [Gro67, Proposition 17.2.5], its normal bundle N_j is isomorphic to i^*T_p.

On the other hand, since h is flat, the embedding k is also regular, with normal bundle $N_k = g^*N_{i'}$.

By [Gro67, Proposition 19.1.5] the diagonal (i, i') is a regular closed embedding, and we have an exact sequence

$$0 \to j^*N_k \to N \to N_j \to 0,$$

that is

$$0 \to N_{i'} \to N \to i^*T_p \to 0.$$

where N is the normal bundle of (i, i').

A symmetric reasoning yields an exact sequence

$$0 \to N_i \to N \to i'^*T_{p'} \to 0,$$

which proves the statement. □

Definition B.3. Given a regular morphism $f: X \to Y$, factored as $i \circ p$, with i a regular closed embedding and p a smooth morphism, we define its *virtual tangent bundle* as

$$\mathcal{T}_f = i^*[T_p] - [N_i] \in K^0(X).$$

It follows from Proposition B.2 that this element does not depend on the choice of the factorization.

Sometimes we will rather consider the *virtual normal bundle*

$$\mathcal{N}_f = -\mathcal{T}_f = [N_i] - i^*[T_p] \in K^0(X).$$

Let X be a regular variety. Since X is quasi-projective, there is a smooth variety M and a closed embedding $i: X \hookrightarrow M$. As any closed embedding of regular schemes, i is a regular closed embedding ([Bou07, Proposition 2, §5, N°3, p.65]). It follows that the structure map $x: X \to \mathrm{Spec}(F)$ is a regular morphism. We define the *virtual tangent bundle* of X as

$$\mathcal{T}_X = \mathcal{T}_x \in K(X).$$

In particular, when X is a smooth variety, we can choose $M = X$ and the identity as closed embedding. It follows that in $K(X)$ we have the equality

$$\mathcal{T}_X = [T_X]$$

when X is smooth.

Lemma B.4. *Let X and Y be regular varieties over a common field such that $X \times Y$ is a regular variety. Then we have in $K(X \times Y)$*

$$\mathcal{T}_{X \times Y} = \mathcal{T}_X \times \mathcal{T}_Y.$$

Proof. Let $i_X\colon X \hookrightarrow M_X$ and $i_Y\colon Y \hookrightarrow M_Y$ be (regular) closed embeddings into smooth varieties. Then we have a closed embedding $i_X \times i_Y\colon X \times Y \hookrightarrow M_X \times M_Y$, which is regular since $X \times Y$ is assumed to be regular. By [EKM08, Proposition 104.7] we have

$$N_{i_X \times i_Y} \simeq N_{i_X} \times N_{i_Y},$$

and by [EKM08, Corollary 104.8] we have

$$T_{M_X \times M_Y} \simeq T_{M_X} \times T_{M_Y}.$$

The lemma now follows from the linearity of the map $K(X) \times K(Y) \to K(X \times Y)$ sending (x, y) to $x \times y$. □

Lemma B.5. *Let $f\colon X \to Y$ be a F-morphism of regular varieties over a field F. Then f is regular and*

$$\mathcal{T}_f = \mathcal{T}_X - f^*\mathcal{T}_Y.$$

Proof. Let $i\colon X \hookrightarrow \Gamma$ be a closed embedding into a smooth variety. Then f factors as

$$X \xrightarrow{(i,f)} \Gamma \times_F Y \to Y.$$

Since Γ is smooth over F and Y is regular, the product $\Gamma \times_F Y$ is regular, by [Bou07, Proposition 7, §6, N°4, p.76]. Hence the closed embedding (i, f) is regular ([Bou07, Proposition 2, §5, N°3, p.65]). The second projection $p\colon \Gamma \times_F Y \to Y$ is smooth, being a base change of the structure map of Γ. The first statement is proven.

Choose a regular closed embedding $j\colon Y \hookrightarrow \Omega$ with normal bundle N_j, and with Ω smooth.

Let $N_{(i,j\circ f)}$ be the normal bundle of the regular closed embedding $(i, j \circ f)\colon X \hookrightarrow \Gamma \times_F \Omega$. Then

$$\mathcal{T}_X = (i, j \circ f)^*[T_\Gamma \times T_\Omega] - [N_{(i,j\circ f)}] = i^*[T_\Gamma] + f^* \circ j^*[T_\Omega] - [N_{(i,j\circ f)}].$$

Let N be the normal bundle of the regular closed embedding $(i, f)\colon X \hookrightarrow \Gamma \times_F Y$. Then the closed embedding $\Gamma \times_F Y \hookrightarrow \Gamma \times_F \Omega$ has normal bundle bundle $\Gamma \times_F N_j$ and we have an exact sequence, by [Gro67, Proposition 19.1.5]

$$0 \to N \to N_{(i,j\circ f)} \to (i, f)^*(\Gamma \times_F N_j) = f^*N_j \to 0.$$

Hence

$$\mathcal{T}_X = i^*[T_\Gamma] + f^* \circ j^*[T_\Omega] - [N] - f^*[N_j] = i^*[T_\Gamma] - [N] + f^*\mathcal{T}_Y.$$

The tangent bundle T_p of $p\colon \Gamma \times_F Y \to Y$ is isomorphic to $T_\Gamma \times_F Y$, hence $(i, f)^*T_p \simeq i^*T_\Gamma$. We get

$$\mathcal{T}_f = (i, f)^*[T_{\Gamma\times Y}] - [N] = i^*[T_\Gamma] - [N] = \mathcal{T}_X - f^*\mathcal{T}_Y,$$

as required. □

Proposition B.6. *Let $f\colon X \to Y$ be a regular morphism of integral varieties. Then we have*

$$\operatorname{rank} \mathcal{T}_f = \dim X - \dim Y.$$

Proof. Choose a factorization $p \circ i$ of the morphism f , with $i\colon X \hookrightarrow M$ a regular closed embedding with normal bundle N, and $p\colon M \to Y$ a smooth morphism with tangent bundle T_p. We can assume that M is connected. Then N is a vector bundle of rank $\dim M - \dim X$ over X, and T_p a vector bundle of rank $\dim M - \dim Y$, and we obtain the formula. □

Corollary B.7. *Let X be a connected regular variety. We have*

$$\operatorname{rank} \mathcal{T}_X = \dim X.$$

Bibliography

[Ada61] John F. Adams. On Chern characters and the structure of the unitary group. *Proc. Cambridge Philos. Soc.*, 57:189–199, 1961.

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572 (electronic), 2002.

[Ati66] Michael F. Atiyah. Power operations in K-theory. *Quart. J. Math. Oxford Ser. (2)*, 17:165–193, 1966.

[Bas68] Hyman Bass. *Algebraic K-theory.* W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[BGS88] Jean-Michel Bismut, Henri Gillet, and Christophe Soulé. Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion. *Comm. Math. Phys.*, 115(1):49–78, 1988.

[Bou06a] Nicolas Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitres 5 à 7.* Springer-Verlag, Berlin, 2006. Reprint of the 1985 original.

[Bou06b] Nicolas Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitres 8 et 9.* Springer-Verlag, Berlin, 2006. Reprint of the 1983 original.

[Bou07] Nicolas Bourbaki. *Éléments de mathématique. Algèbre commutative. Chapitre 10.* Springer-Verlag, Berlin, 2007. Reprint of the 1998 original.

[Bro03] Patrick Brosnan. Steenrod operations in Chow theory. *Trans. Amer. Math. Soc.*, 355(5):1869–1903 (electronic), 2003.

[CP08] Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic. I. Reduction to local uniformization on Artin-Schreier and purely inseparable coverings. *J. Algebra*, 320(3):1051–1082, 2008.

[CP09] Vincent Cossart and Olivier Piltant. Resolution of singularities of threefolds in positive characteristic II. *J. Algebra*, 321(7):1836 – 1976, 2009.

[Dem74] Michel Demazure. Désingularisation des variétés de Schubert généralisées. *Ann. Sci. École Norm. Sup. (4)*, 7:53–88, 1974. Collection of articles dedicated to Henri Cartan on the occasion of his 70th birthday, I.

[EKM08] Richard Elman, Nikita Karpenko, and Alexander Merkurjev. *The algebraic and geometric theory of quadratic forms*, volume 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.

[FL85] William Fulton and Serge Lang. *Riemann-Roch algebra*, volume 277 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1985.

[Ful98] William Fulton. *Intersection theory*, volume 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 1998.

[Gab] Ofer Gabber. Finiteness theorems in étale cohomology. *Abstract of a talk given at the IHES on January 12, 2009.* `http://www.ihes.fr/document?id=1713&id_attribute=48`.

[Gil81] Henri Gillet. Riemann-Roch theorems for higher algebraic K-theory. *Adv. in Math.*, 40(3):203–289, 1981.

[Gil05] Henri Gillet. K-theory and intersection theory. In *Handbook of K-theory. Vol. 1, 2*, pages 235–293. Springer, Berlin, 2005.

[Gro58] Alexander Grothendieck. La théorie des classes de Chern. *Bull. Soc. Math. France*, 86:137–154, 1958.

[Gro61] Alexander Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[Gro65] Alexander Grothendieck. Éléments de géométrie algébrique. IV: Étude locale des schémas et des morphismes de schémas. (Seconde partie.). *Inst. Hautes Études Sci. Publ. Math.*, (24):231, 1965.

[Gro67] Alexander Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[GS87] Henri Gillet and Christophe Soulé. Intersection theory using Adams operations. *Invent. Math.*, 90(2):243–277, 1987.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[Hir66] Friedrich Hirzebruch. *Topological methods in algebraic geometry.* Third enlarged edition. New appendix and translation from the second German edition by R. L. E. Schwarzenberger, with an additional section by A. Borel. Die Grundlehren der Mathematischen Wissenschaften, Band 131. Springer-Verlag New York, Inc., New York, 1966.

[Kah09] Bruno Kahn. *Formes quadratiques sur un corps.* 2009. http://www.math.jussieu.fr/~kahn/preprints/exp.html.

[Kar90] Nikita A. Karpenko. Algebro-geometric invariants of quadratic forms. *Algebra i Analiz*, 2(1):141–162, 1990.

[Kar98] Nikita A. Karpenko. Codimension 2 cycles on Severi-Brauer varieties. *K-Theory*, 13(4):305–330, 1998.

[Kar03] Nikita A. Karpenko. On the first Witt index of quadratic forms. *Invent. Math.*, 153(2):455–462, 2003.

[Köc91] Bernhard Köck. Chow motif and higher Chow theory of G/P. *Manuscripta Math.*, 70(4):363–372, 1991.

[Köc00] Bernhard Köck. Riemann-Roch for tensor powers. *Math. Z.*, 233(4):755–801, 2000.

[Kun69] Ernst Kunz. Characterizations of regular local rings of characteristic p. *American Journal of Mathematics*, 91(3):772–784, 1969.

[Lev07] Marc Levine. Steenrod operations, degree formulas and algebraic cobordism. *Pure Appl. Math. Q.*, 3(1, part 3):283–306, 2007.

[LM07] Marc Levine and Fabien Morel. *Algebraic cobordism.* Springer Monographs in Mathematics. Springer, Berlin, 2007.

[LP09] Marc Levine and Rahul Pandharipande. Algebraic cobordism revisited. *Invent. Math.*, 176(1):63–130, 2009.

[Mer03] Alexander Merkurjev. Steenrod operations and degree formulas. *J. Reine Angew. Math.*, 565:13–26, 2003.

[Mer09] Alexander Merkurjev. Adams operations and the Brown-Gersten-Quillen spectral sequence. *preprint*, 2009. http://www.math.ucla.edu/~merkurev/publicat.htm.

[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbf{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Pan04] Ivan Panin. Riemann-Roch theorems for oriented cohomology. In *Axiomatic, enriched and motivic homotopy theory*, volume 131 of *NATO Sci. Ser. II Math. Phys. Chem.*, pages 261–333. Kluwer Acad. Publ., Dordrecht, 2004.

[PSZ08] Viktor Petrov, Nikita Semenov, and Kirill Zainoulline. J-invariant of linear algebraic groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 41(6):1023–1053, 2008.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[Ros96] Markus Rost. Chow groups with coefficients. *Doc. Math.*, 1:No. 16, 319–393 (electronic), 1996.

[Ros98] Markus Rost. The motive of a Pfister form. *preprint*, 1998. `http://www.math.uni-bielefeld.de/~rost/motive.html`.

[Ros06] Markus Rost. On the basic correspondence of a splitting variety. *preprint*, 2006. `http://www.math.uni-bielefeld.de/~rost/basic-corr.html`.

[Ros08] Markus Rost. On Frobenius, K-theory, and characteristic numbers. *preprint*, 2008. `http://www.math.uni-bielefeld.de/~rost/frobenius.html`.

[Ser78] Jean-Pierre Serre. *Représentations linéaires des groupes finis*. Hermann, Paris, revised edition, 1978.

[sga71] *Théorie des intersections et théorème de Riemann-Roch*. Lecture Notes in Mathematics, Vol. 225. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois-Marie 1966–1967 (SGA 6), Dirigé par P. Berthelot, A. Grothendieck et L. Illusie. Avec la collaboration de D. Ferrand, J. P. Jouanolou, O. Jussila, S. Kleiman, M. Raynaud et J. P. Serre.

[Sou85] Christophe Soulé. Opérations en K-théorie algébrique. *Canad. J. Math.*, 37(3):488–550, 1985.

[Sou92] Christophe Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.

[Sri96] Vaseduvan Srinivas. *Algebraic K-theory*, volume 90 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, second edition, 1996.

[Tho93] Robert W. Thomason. Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire. *Invent. Math.*, 112(1):195–215, 1993.

[Vis04] Alexander Vishik. Motives of quadrics with applications to the theory of quadratic forms. In *Geometric methods in the algebraic theory of quadratic forms*, volume 1835 of *Lecture Notes in Math.*, pages 25–101. Springer, Berlin, 2004.

[Voe03] Vladimir Voevodsky. Reduced power operations in motivic cohomology. *Publ. Math. Inst. Hautes Études Sci.*, (98):1–57, 2003.

[VY07] Alexander Vishik and Nobuaki Yagita. Algebraic cobordisms of a Pfister quadric. *J. Lond. Math. Soc. (2)*, 76(3):586–604, 2007.

[Wei09] Charles Weibel. *The K-book: An introduction to algebraic K-theory*. A graduate textbook in progress, 2009. `http://www.math.rutgers.edu/~weibel/Kbook.html`.

Notations

$\mathcal{N}_f$	virtual normal bundle of the morphism f	148
Reg/F	category of regular varieties and flat morphisms	52
Res_H^G	restriction functor	86
Sch/F	category of varieties over the field F	52
Sm/F	category of smooth varieties over the field F	52
$\mathfrak{S}_k$	k-th symmetric group	86
$\mathcal{T}_X$	virtual tangent bundle of the regular variety X	148
$\mathcal{T}_f$	virtual tangent bundle of the morphism f	148
θ^n	Bott's cannibalistic class	74
$\mathsf{VB}(X)$	category of locally free sheaves on X	20
$\mathsf{VB}(X,G)$	category of G-vector bundles on a variety X	86
$\overline{\alpha}$	the restriction of α defined over a field F to $\overline{F}$	111
$\overline{\mathsf{G}}$	the image of $\mathsf{G}(-) \to \mathsf{G}(- \times_F \overline{F})$	111
$\cap$	cap product	79
ch	Chern character	63
ch_n	n-th component of the Chern character	63
$\cup$	cup product	78
δ	Poincare homomorphism	24
δ_G	graded Poincare homomorphism	50
$\mathrm{disc}\, X$	discriminant algebra of the quadric X	134
γ^i	i-th gamma operation	54
$\mathrm{gr}\, K$	graded group associated with the topological filtration for regular varieties	31
λ^i	i-th lambda operation	54
$\mathsf{R}^i f_*$	i-th higher direct image functor of the morphism f	21
ψ^n	n-th Adams operation	61
ψ_n	n-th homological Adams operation	103
rank	rank homomorphism	25
σ	deformation homomorphism	42
σ	deformation homomorphism (alternative definition)	142
τ_t	t-th Todd number	132
$w^{\mathrm{CH},n}$	a Todd homomorphism with value in CH	73
$w^{\gamma,n}$	a Todd homomorphism with value in $\mathrm{gr}_\gamma K^0$	73
$w^{K,n}$	a Todd homomorphism with value in $\mathrm{gr}\, K$	73
$e(L)$	Euler class of the line bundle L	41
$f^!$	refined Gysin map	38
h	hyperplane class in the Chow group of a quadric	134
l_i	class of a totally isotropic subspace in the Chow group of a quadric	134
$x \boxtimes y$	external product	23
$x \times y$	external product	23

Index

Olivier Haution
olivier.haution@gmail.com