# Brezis pseudomonotone bifunctions and quasi equilibrium problems via penalization

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**Abstract**

In this paper we investigate quasi equilibrium problems in a real Banach space under the assumption of Brezis pseudomonotonicity of the function involved. To establish existence results under weak coercivity conditions we replace the quasi equilibrium problem with a sequence of penalized usual equilibrium problems. To deal with the non compact framework, we apply a regularized version of the penalty method. The particular case of set-valued quasi variational inequalities is also considered.

**Key words**: quasi equilibrium problem; set-valued quasi variational inequality; Brezis pseudomonotonicity; regularized penalty method; coercivity conditions.

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## 1 Introduction

Given a nonempty set $C$ and a bifunction $f : C \times C \to \mathbb{R}$, the equilibrium problem (in the sequel EP, for short) is defined as follows: find $\overline{x} \in C$ such that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in C.$$

It seems that the earliest mathematical formulation of the problem above belongs to Nikaido and Isoda [21]: they used it as an auxiliary problem to study the existence of solutions of Nash equilibrium problem. The first existence results on (EP) date back to the seventies and are attributed to Fan [14] and Brezis, Nirenberg and Stampacchia [6]. The term *equilibrium problem* was coined by Muu and Oettli in 1992 ([20]), while in 1994 Blum and Oettli [5] adopted this terminology perhaps because it is equivalent to find the equilibrium point of several problems, namely, minimization problems, saddle point problems, Nash equilibrium problems, variational inequality problems, fixed point problems, and so forth.

Related to (EP) in literature has been considered the so-called *quasi equilibrium problem* (QEP, for short), that is an equilibrium problem with a constraint set depending on the

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current point. More precisely, given a set-valued map $\Phi : C \rightrightarrows C$, (QEP) requires to find a point $\overline{x} \in \Phi(\overline{x})$ such that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in \Phi(\overline{x}).$$

While there is an extensive literature on existence results, stability of the solutions, and solution methods concerning equilibrium problems (for a recent survey see [4]), the investigation of quasi equilibrium problems, initially introduced by Mosco in 1976 (see [19]), received some attention only quite recently. These problems arise from quasi variational inequalities, which are well-known tools to model equilibria in several frameworks (see, for instance, [16]). Quasi equilibrium problems showed to be of interest in various fields of application, like, in particular, the generalized Nash equilibrium problems (see e.g. [1], [8] and references therein).

Most of the results concerning (QEP) are stated in a finite-dimensional framework, on a compact set $C$, and they involve generalized monotonicity assumptions on $f$ together with upper semicontinuity of the set-valued map $\Phi$ which describes the constraint, as well as lower semicontinuity of this map with the additional assumption of the closedness of the set of fixed points (see, for instance, [9]). In case of an unbounded set $C$, an additional coercivity condition is required (for a recent result, see [12]).

In the literature, to prove existence results for (QEP) the key step goes through standard fixed-point techniques. On the other hand, Konnov ([18]) proposed to apply in a finite-dimensional setting a regularized version of the penalty method to establish existence results for the general quasi equilibrium problem under weak coercivity conditions by replacing the quasi equilibrium problem with a sequence of usual equilibrium problems.

We apply these techniques to (QEP) in real Banach spaces, under the assumption of *topological*, or *Brezis pseudomonotonicity* of $f$ and weak lower semicontinuity of the constraint map. Our approach does not require *a priori* properties of the fixed points, even if, at least in the compact case, their existence is entailed by the assumptions of $\Phi$ (in case of a separable Banach space, see Corollary 3.1 in [8]).

The paper is organized as follows. First, in Section 2 we describe the B-pseudomonotone bifunctions and recall some well-known existence results concerning equilibrium problems. Section 3 is devoted to the study of (EP) for the sum of bifunctions. In Section 4 we prove existence results under B-pseudomonotonicity of the bifunction involved for a quasi equilibrium problem by replacing it with a sequence of penalized and regularized (EP). In particular, in Theorem 4 we prove an extension of Theorem 4.1 in [18] to infinite dimensional spaces under a boundedness assumption on the equilibrium bifunction. Finally, in Section 5 we deal with set-valued quasi variational inequalities, providing a set-valued version of Theorem 4.4 in [16].

## 2 Preliminaries on EP and B-pseudomonotone bifunctions

Let us recall a well-known existence result for (EP) that holds in the general setting of Hausdorff topological vector spaces:

**Theorem 1.** (see [6], Theorem 1) Let $C$ be a nonempty, closed and convex subset of a Hausdorff topological vector space $E$, and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

i. $f(x, x) \geq 0$ for all $x \in C$;

ii. for every $x \in C$, the set $\{y \in C : f(x, y) < 0\}$ is convex;

iii. for every $y \in C$, the function $f(\cdot, y)$ is upper semicontinuous on the intersection of $C$ with any finite dimensional subspace $Z$ of $E$;

iv. whenever $x, y \in C$, $x_n \in C$, $x_n \to x$ and $f(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0, 1]$ and for all $n$, then $f(x, y) \geq 0$;

v. there exists a compact subset $K$ of $E$, and $y_0 \in K \cap C$ such that $f(x, y_0) < 0$ for every $x \in C \setminus K$.

Then, there exists $\overline{x} \in C \cap K$ such that

$$f(\overline{x}, y) \geq 0 \quad \text{for all } y \in C.$$

From now on we will assume $E = X$, with $X$ real Banach space. In this framework, we will introduce a coercivity condition weaker than condition v. in Theorem 1.

Let $\mu : X \to \mathbb{R}$ and $B_\mu(r) = \{x \in X : \mu(x) \leq r\}$. The function $\mu$ is said to be *coercive* with respect to $C \subseteq X$ if

$$\lim_{\|x\| \to +\infty,\, x \in C} \mu(x) = +\infty.$$

This is trivially equivalent to say that $B_\mu(r) \cap C$ is bounded for every $r > \inf_C \mu$.

Given a bifunction $f : C \times C \to \mathbb{R}$, we will denote by **(C)** the following coercivity condition:

**(C)** There exist a convex and lower semicontinuous function $\mu : X \to \mathbb{R}$, which is coercive with respect to the set $C$, and a number $r \in \mathbb{R}$ such that, for any point $x \in C \setminus B_\mu(r)$ there is a point $z \in C$ with

$$\min\{f(x, z), \mu(z) - \mu(x)\} < 0 \quad \text{and} \quad \max\{f(x, z), \mu(z) - \mu(x)\} \leq 0.$$

Note that, if **(C)** is fulfilled for some $r \in \mathbb{R}$, then it holds true for any $\rho \geq r$.

**Remark 1.** i. It is easy to see that **(C)** weakens condition v. in Theorem 1. Indeed, set $\mu(x) = \|x\|$ and $r$ such that $K \subseteq B_\mu(r)$, $z = y_0 \in C \cap K$. Then, for every $x \in C \setminus B_\mu(r)$ we have $f(x, y_0) < 0$ and $\|z\| = \|y_0\| \leq r < \|x\|$, and therefore **(C)** is fulfilled.

ii. If the bifunction $f$ satisfies condition **(C)** over $C$ for suitable $\mu$ and $r$, and for every $x \in C \setminus B_\mu(r)$ there exists $z \in C$ such that $f(x, z) < 0$, then the solution set of (EP) is contained in $B_\mu(r)$.

Taking into account the previous discussion, we provide an existence result on unbounded sets. Despite the line of proof is often employed in literature, we insert it for the reader's convenience.

**Theorem 2.** Let $C$ be a nonempty, closed and convex subset of a Banach space $X$ and $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

i. $f(x, x) = 0$ for all $x \in C$;

ii. $f(x, \cdot)$ is convex for every $x \in C$;

iii. for every $y \in C$, the function $f(\cdot, y)$ is upper semicontinuous on the intersection of $C$ with any finite dimensional subspace $Z$ of $E$;

iv. whenever $x, y \in C$, $x_n \in C$, $x_n \to x$ and $f(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0, 1]$ and for all $n$, then $f(x, y) \geq 0$;

v. if $C$ is unbounded, the coercivity condition **(C)** holds.

Then, (EP) is solvable, i.e. there exists $\overline{x} \in C$ such that

$$f(\overline{x}, y) \geq 0 \quad \text{for all } y \in C.$$

*Proof.* If $C$ is bounded, the assertion follows from Theorem 1. Let us suppose that $C$ is unbounded. We will show that there exists $\rho \geq r$ such that any solution $x_\rho$ of (EP) on the bounded set $B_\mu(\rho) \cap C$ is indeed a solution on the whole set $C$.

Let us first note that $r \geq r(m)$, where $r(m) = \min_C \mu$. Indeed, assume by contradiction that $r < r(m)$, i.e. $C \setminus B_\mu(r) = C$, and denote by $\overline{x}$ a solution of the (EP) on the nonempty and bounded set $B_\mu(r(m)) \cap C$. By applying the coercivity condition **(C)**, with $x = \overline{x}$, there exists $z \in C$ such that

$$\min\{f(\overline{x}, z), \mu(z) - \mu(\overline{x})\} < 0 \quad \text{and} \quad \max\{f(\overline{x}, z), \mu(z) - \mu(\overline{x})\} \leq 0.$$

Since $\mu(z) \geq \mu(\overline{x})$, for every $z \in C$, we have that

$$f(\overline{x}, z) < 0, \qquad \mu(z) = \mu(\overline{x}),$$

thereby contradicting that $\overline{x}$ is a solution of (EP) on $B_\mu(r(m)) \cap C$.

In addition, let us show that given any $\rho \geq r$, if there exists $w \in B_\mu(\rho) \cap C$ such that $\mu(w) < \rho$ and $f(x_\rho, w) = 0$, then $x_\rho$ is a solution on $C$. By contradiction, suppose that $f(x_\rho, y) < 0$ for some $y \in C \setminus B_\mu(\rho)$, and take $x_t = (1-t)w + ty \in C$. Then, by the assumptions on $f$, we have

$$f(x_\rho, x_t) \leq (1-t)f(x_\rho, w) + tf(x_\rho, y) = tf(x_\rho, y) < 0, \quad \forall t \in (0, 1). \tag{1}$$

Since, for $t$ positive and small enough,

$$\mu(x_t) \leq (1-t)\mu(w) + t\mu(y) < \rho,$$

we have that $x_t \in B_\mu(\rho)$ for $t$ small, and thus (1) is a contradiction.

Let us now consider the following two cases:

• $\mu(x_{\overline{\rho}}) < \overline{\rho}$, for some $\overline{\rho} \geq r$. In this case, by taking $w = x_{\overline{\rho}}$, since $f(x_{\overline{\rho}}, x_{\overline{\rho}}) = 0$, from the argument above $x_{\overline{\rho}}$ is a solution on $C$.

• $\mu(x_\rho) = \rho$, for every $\rho \geq r$. Take any $\rho > r$. Then, $x_\rho \notin B_\mu(r)$. By condition **(C)**, and taking $x = x_\rho$, we can easily see that the inequalities

$$f(x_\rho, z) < 0, \quad \text{and} \quad \mu(z) - \mu(x_\rho) = \mu(z) - \rho \leq 0$$

contradict that $x_\rho$ is a solution on $B_\mu(\rho) \cap C$. Therefore,

$$f(x_\rho, z) = 0, \quad \text{and} \quad \mu(z) - \rho < 0$$

hold true. Thus, taking $w = z$, again the argument above shows that $x_\rho$ is a solution on $C$. □

In order to fruitfully apply Theorem 2, we look for sufficient conditions for a bifunction $f$ to satisfy condition iv.

**Proposition 1.** Let $C \subseteq X$ be nonempty, closed and convex, and let $f : C \times C \to \mathbb{R}$ satisfy the assumptions:

i. $f(x, x) = 0$ for every $x \in C$, and $f$ is monotone (i.e., $f(x, y) + f(y, x) \leq 0$ for every $x, y \in C$);

ii. $f(x, \cdot)$ is weakly sequentially lower semicontinuous and convex, for every $x \in C$;

iii. $f$ is upper sign continuous, i.e., for every $x, y \in C : f((1-t)x + ty, y) \geq 0 \ \forall t \in (0, 1)$ implies $f(x, y) \geq 0$ (see [2]).

Then $f$ satisfies assumption iv. in Theorem 2.

*Proof.* Take any $x, y \in C$, $\{x_n\} \subset C$ with $x_n \rightharpoonup x$, such that $f(x_n, (1-t)x + ty) \geq 0$ for every $n$ and for every $t \in [0, 1]$. We will show that $f(x, y) \geq 0$. Set $z_t = (1-t)x + ty$. From the monotonicity, we have that

$$f(z_t, x_n) \leq -f(x_n, z_t),$$

and, by the weak sequential lower semicontinuity in the second variable,

$$f(z_t, x) \leq \liminf_{n\to+\infty} f(z_t, x_n) \leq \liminf_{n\to+\infty} (-f(x_n, z_t)) = -\limsup_{n\to+\infty} f(x_n, z_t) \leq 0 \qquad \forall t \in [0, 1].$$

Then, by the convexity in the second variable,

$$f(z_t, x) \leq 0 = f(z_t, z_t) \leq (1-t)f(z_t, x) + tf(z_t, y) \leq tf(z_t, y), \qquad \forall t \in (0, 1], \qquad (2)$$

and therefore

$$f(z_t, y) \geq 0, \qquad \forall t \in (0, 1].$$

By the upper sign continuity, this implies that $f(x, y) \geq 0$. □

**Remark 2.** A simple adjustment of the previous proof shows that the result holds also in case the assumption of convexity in ii. is replaced by the weaker assumption of explicit quasiconvexity, i.e. $f(x, \cdot)$ is quasiconvex and semistrictly quasiconvex. In this case, starting from (2), and by the quasiconvexity, it follows that

$$0 = f(z_t, z_t) \leq \max\{f(z_t, x), f(z_t, y)\}.$$

If $f(z_{t'}, x) > f(z_{t'}, y)$ for some $t' \in (0, 1]$, then $f(z_{t'}, x) = 0$, $f(z_{t'}, y) < 0$, and, by the semistrict quasiconvexity, $f(z_{t'}, z_{t'}) < 0$, a contradiction. Thus, $\max\{f(z_t, x), f(z_t, y)\} = f(z_t, y) \geq 0$, for every $t \in (0, 1]$, and the assertion follows again from the upper sign continuity.

Another sufficient condition for iv., that will be intensively exploited in the sequel, relies upon the following property of a bifunction $f$ (see [15]):

**Definition 1.** Let $C$ be a nonempty, closed and convex subset of $X$. A bifunction $f : C \times C \to \mathbb{R}$ is said to be *topologically*, or *Brezis pseudomotone* (B-pseudomonotone, for short) if for every $x_n \rightharpoonup x$ in $C$ such that $\liminf_{n\to\infty} f(x_n, x) \geq 0$ it follows that

$$f(x, y) \geq \limsup_{n\to\infty} f(x_n, y) \quad \forall y \in C.$$

**Remark 3.** i. Note that if $f(\cdot, y)$ is either weakly sequentially upper semicontinuous, or sequentially upper semicontinuous and of type $S_+$, i.e.

$$x_n \rightharpoonup x \text{ and } \liminf_{n\to\infty} f(x_n, x) \geq 0 \Rightarrow x_n \to x,$$

(see [11]), then $f$ is B-pseudomonotone.

ii. It is easy to prove that the sum of two B-pseudomonotone bifunctions, that are non-positive on the diagonal, is B-pseudomonotone too (see Proposition 2.1 in [10]).

**Proposition 2.** Every B-pseudomonotone bifunction satisfies assumption iv. in Theorem 1 (with $E = X$), or 2.

*Proof.* Let $x, y \in C$, $x_n \in C$, $x_n \to x$ and $f(x_n, (1-t)x + ty) \geq 0$ for all $t \in [0, 1]$; in particular, $f(x_n, x) \geq 0$ and $f(x_n, y) \geq 0$. Then, $\liminf_{n\to\infty} f(x_n, x) \geq 0$ and, since $x_n \to x$ implies that $x_n \rightharpoonup x$, by B-pseudomonotonicity $f(x, y) \geq \limsup_{n\to\infty} f(x_n, y) \geq 0$, for all $y \in C$, that is $f(x, y) \geq 0$. □

Let us provide an interesting example showing that the assumptions in Proposition 1 do not imply B-pseudomonotonicity.

**Example 1.** Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be defined as follows:

$$f(x, y) = \begin{cases} x - y, & x \neq 0 \\ -2y, & x = 0 \end{cases}$$

Easy computations show that the assumptions i.-iii. hold true. Take now the sequence $x_n = \frac{1}{n} \to 0$ ($x = 0$). From $f(x_n, x) = f(x_n, 0) = \frac{1}{n}$, we have

$$\liminf_{n\to+\infty} f(x_n, 0) = \lim_{n\to+\infty} \frac{1}{n} = 0.$$

Take now $y = 1$ : we have

$$f(0,1) = -2 < -1 = \lim_{n\to+\infty} f\left(\frac{1}{n}, 1\right) = \frac{1}{n} - 1 = \limsup_{n\to+\infty} f\left(\frac{1}{n}, 1\right),$$

and therefore $f$ is not B-pseudomonotone.

# 3 Equilibrium problems for the sum of bifunctions

In this short section existence results for equilibrium problems associated to a sum of bifunctions are proved. While in case of monotone and generalized monotone bifunctions some results in this framework can be found in [5], we will focus instead on the B-pseudomonotonicity property of the bifunctions. We will show first an existence result in case of a weakly compact set $C$, and then in case of $C$ unbounded.

**Proposition 3.** Let $C$ be a nonempty, convex and weakly compact subset of a Banach space $X$, $f, g : C \times C \to \mathbb{R}$ bifunctions such that:

i. $f(x,x) = 0$, $g(x,x) = 0$ for all $x \in C$;

ii. $f(x,\cdot)$, $g(x,\cdot)$ are convex for every $x \in C$;

iii. the function $f(\cdot, y)$ is upper semicontinuous on the intersection of $C$ with any finite dimensional subspace $Z$ of $X$, and $g(\cdot, y)$ is weakly sequentially upper semicontinuous, for every $y \in C$;

iv. $f$ is B-pseudomonotone.

Then, there exists $\overline{x} \in C$ such that

$$f(\overline{x}, y) + g(\overline{x}, y) \geq 0 \quad \text{for all } y \in C.$$

*Proof.* Let $\varphi(x,y) = f(x,y) + g(x,y)$. Then, conditions i.-iii. of Theorem 1 are trivially satisfied. Moreover, according to Remark 3, the B-pseudomonotonicity of $\varphi$ follows from the B-pseudomonotonicity of $f$, and therefore, via Proposition 2, condition iv. of Theorem 1 is fulfilled. □

In case the set $C$ is unbounded, in order to prove an existence result, we will assume the following coercivity condition:

**(C′)** there exists a convex and lower semicontinuous function $\mu : X \to \mathbb{R}$ which is coercive with respect to the set $C$, and a positive number $r$ such that, for any point $x \in C \setminus B_\mu(r)$ there is a point $z \in C$ such that

$$g(x, z) \leq 0, \text{ and}$$

$$\min\{f(x, z), \mu(z) - \mu(x)\} < 0 \quad \text{and} \quad \max\{f(x, z), \mu(z) - \mu(x)\} \leq 0.$$

**Proposition 4.** Let $C$ be a nonempty, convex and closed subset of a Banach space $X$, $f, g : C \times C \to \mathbb{R}$ bifunctions such that $f(x, x) = g(x, x) = 0$. Suppose that conditions ii.-iv. of Proposition 3 are satisfied and that the coercivity condition **(C′)** holds. Then, there exists $\overline{x} \in C$ such that

$$f(\overline{x}, y) + g(\overline{x}, y) \geq 0 \quad \text{for all } y \in C.$$

*Proof.* Note that the bifunction $\varphi(x, y) = f(x, y) + g(x, y)$ does satisfy **(C)**. Indeed, if $x \in C \setminus B_\mu(r)$, there exists $z \in C$ such that

$$f(x, z) < 0 \text{ and } \mu(z) \leq \mu(x), \quad \text{or} \quad f(x, z) \leq 0 \text{ and } \mu(z) < \mu(x).$$

In both cases, since $g(x, z) \leq 0$, we have that

$$\min\{f(x, z) + g(x, z), \mu(z) - \mu(x)\} < 0 \quad \text{and} \quad \max\{f(x, z) + g(x, z), \mu(z) - \mu(x)\} \leq 0.$$

Thus, the assertion follows by Theorem 2 applied to the bifunction $\varphi$. □

## 4 Existence results for quasi equilibrium problems

In this section we will deal with a quasi equilibrium problem in the framework of a real Banach space $X$. More precisely, given a nonempty, closed and convex subset $C$ of $X$, a set-valued map $\Phi : C \rightrightarrows C$ with nonempty, closed and convex values, and a bifunction $f : C \times C \to \mathbb{R}$ such that $f(x, x) = 0$ for every $x \in C$, the quasi equilibrium problem requires to find a point $\overline{x} \in \Phi(\overline{x})$ such that

$$f(\overline{x}, y) \geq 0, \quad \forall y \in \Phi(\overline{x}).$$

Unlike the standard approach, usually based on properties of the fixed points of the map $\Phi$, in the following we will provide an existence result for (QEP) via *penalization*.

From now on let $P : C \times C \to \mathbb{R}$ be a nonnegative bifunction such that $P(x, y) = 0$ if and only if $y \in \Phi(x)$, and denote by ($\mathrm{EP}_k$), with $k \in \mathbb{N}$, the following equilibrium problem: find $x \in C$ such that

$$f(x, y) + k(P(x, y) - P(x, x)) \geq 0, \qquad \forall y \in C \tag{$\mathrm{EP}_k$}$$

(see, for instance, [18]).

Let us recall the following notion for the continuity of a map:

**Definition 2.** A set-valued map $\Phi : C \rightrightarrows C$ is said to be weakly lower semicontinuous at $x \in C$ if for every $x_n \rightharpoonup x$, $\{x_n\} \subset C$, and for every $y \in \Phi(x)$, there exists a subsequence $\{x_{n_k}\}$ and $y_k \in \Phi(x_{n_k})$ such that $y_k \to y$.

First of all, we state the following

**Proposition 5.** Suppose that (EP$_k$) has a solution $\overline{x}_k$, and that $\overline{x}$ is a weak limit point of $\{\overline{x}_k\}$. Under the following assumptions:

i. $f$ is B-pseudomonotone;

ii. $|f(z, y) - f(z, x)| \leq h(z)||y - x||$, for every $x, y, z \in C$, where $h : C \to \mathbb{R}$ is positive and bounded on bounded sets;

iii. $t \to P(t, t)$ is weakly sequentially lower semicontinuous, for every $t \in C$;

iv. $\Phi$ is weakly lower semicontinuous at every $x \in C$,

$\overline{x}$ is a solution of (QEP).

*Proof.* Assume, without loss of generality, that $\overline{x}_k \rightharpoonup \overline{x}$. Let $\overline{y}$ be a point in $\Phi(\overline{x})$. Thus, by Definition 2, again without loss of generality there exists $\{\tilde{x}_k\}$ with $\tilde{x}_k \in \Phi(\overline{x}_k)$, and $\tilde{x}_k \to \overline{y}$. Then,

$$f(\overline{x}_k, \tilde{x}_k) - kP(\overline{x}_k, \overline{x}_k) \geq 0.$$

This implies that

$$0 \leq P(\overline{x}_k, \overline{x}_k) \leq \frac{1}{k} f(\overline{x}_k, \tilde{x}_k) \leq \frac{1}{k} h(\overline{x}_k)||\overline{x}_k - \tilde{x}_k||.$$

If $k \to +\infty$, from the boundedness of $h$ and iii., we get $P(\overline{x}, \overline{x}) = 0$, i.e. $\overline{x} \in \Phi(\overline{x})$.

Let us now take a sequence $\{\tilde{x}_k\}$, with $\tilde{x}_k \in \Phi(\overline{x}_k)$, and $\tilde{x}_k \to \overline{x}$. Since $f(\overline{x}_k, \tilde{x}_k) \geq 0$, we get

$$0 \leq f(\overline{x}_k, \tilde{x}_k) = f(\overline{x}_k, \tilde{x}_k) + f(\overline{x}_k, \overline{x}) - f(\overline{x}_k, \overline{x}) \leq f(\overline{x}_k, \overline{x}) + h(\overline{x}_k)||\tilde{x}_k - \overline{x}||$$

and therefore $\liminf_{k \to +\infty} f(\overline{x}_k, \overline{x}) \geq 0$. From the B-pseudomonotonicity of $f$,

$$f(\overline{x}, y) \geq \limsup_{k \to +\infty} f(\overline{x}_k, y), \qquad \forall y \in \Phi(\overline{x}). \tag{3}$$

For any $\overline{y} \in \Phi(\overline{x})$, take $y_k \in \Phi(\overline{x}_k)$, $y_k \to \overline{y}$. Then,

$$0 \leq f(\overline{x}_k, y_k) = f(\overline{x}_k, y_k) + f(\overline{x}_k, \overline{y}) - f(\overline{x}_k, \overline{y}) \leq f(\overline{x}_k, \overline{y}) + h(\overline{x}_k)||y_k - \overline{y}||.$$

This implies that $\limsup_{k \to +\infty} f(\overline{x}_k, y) \geq 0$, for every $y \in \Phi(\overline{x})$. The assertion follows by (3). □

In case $C$ is weakly compact, the following existence result for a (QEP) problem easily follows:

**Theorem 3.** Let $C$ be a nonempty, weakly compact and convex subset of a real Banach space $X$, $f, P : C \times C \to \mathbb{R}$ bifunctions, $\Phi : C \rightrightarrows C$ a set-valued map with nonempty closed and convex values. Suppose that

i. $f$ is B-pseudomonotone, and $f(x, x) = 0$ for all $x \in C$;

ii. $|f(z, y) - f(z, x)| \leq h(z)||y - x||$, for every $x, y, z \in C$, where $h : C \to \mathbb{R}$ is positive and bounded on bounded sets;

iii. $f(x, \cdot)$ and $P(x, \cdot)$ are convex, for every $x \in C$;

iv. $t \to P(t, t)$ is weakly sequentially lower semicontinuous, for every $t \in C$;

v. the function $f(\cdot, y)$ is upper semicontinuous on the intersection of $C$ with any finite dimensional subspace $Z$ of $X$, and $P(\cdot, y)$ is weakly sequentially upper semicontinuous, for every $y \in C$;

vi. $\Phi$ is weakly lower semicontinuous at every $x \in C$.

Then, problem ($\mathrm{EP}_k$) is solvable, and any weak limit point $\overline{x}$ of a sequence of solutions $x_k$ of problem ($\mathrm{EP}_k$) is a solution of (QEP).

*Proof.* The assumptions give the solvability of Problem ($\mathrm{EP}_k$) via Proposition 3 setting $g(x, y) = P(x, y) - P(x, x)$. Moreover, the weak compactness of the set $C$ entails the existence of limit points for every sequence in $C$. Therefore the sequence of solutions $x_k$ of problems ($\mathrm{EP}_k$) admits weak limit points, and the assertion follows by Proposition 5. □

In case $C$ is not weakly compact, we introduce the following *regularized* problem ($\mathrm{EP}'_k$): find $x \in C$ such that

$$f(x, y) + k(P(x, y) - P(x, x)) + \frac{1}{k}(\mu(y) - \mu(x)) \geq 0, \qquad \forall y \in C, \qquad (\mathrm{EP}'_k)$$

where $\mu : X \to \mathbb{R}$ is convex, coercive with respect to $C$, and continuous with respect to the strong topology. Note that, in particular, $\mu$ is weakly lower semicontinuous.

We can now state the following result, under the stronger assumption that $X$ is a reflexive Banach space:

**Theorem 4.** Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $X$, $f, P : C \times C \to \mathbb{R}$ bifunctions, $\Phi : C \rightrightarrows C$ a set-valued map with nonempty closed and convex values. Suppose that the assumptions i.-vi. of Theorem 3 hold, together with the coercivity condition $(\mathbf{C}')$, where $g(x, y) = P(x, y) - P(x, x)$.

Then, problem ($\mathrm{EP}'_k$) is solvable and any sequence of solutions $x_k$ of problem ($\mathrm{EP}'_k$) has weak limit points that are, in particular, solutions of (QEP).

*Proof.* Condition **(C′)** with $g(x, y) = P(x, y) - P(x, x)$, implies that for every $x \in C \setminus B_\mu(r)$ there exists $z \in C$ such that

$$k(P(x, z) - P(x, x)) + \frac{1}{k}(\mu(z) - \mu(x)) \leq 0.$$

Therefore the assumptions in Proposition 4 are satisfied setting, in this case,

$$g(x, y) = k(P(x, y) - P(x, x)) + \frac{1}{k}(\mu(y) - \mu(x)),$$

and ($\mathrm{EP}'_k$) is solvable. Moreover, it is easy to show that for every $x \in C \setminus B_\mu(r)$, there exists $z \in C$ such that

$$f(x, z) + k(P(x, z) - P(x, x)) + \frac{1}{k}(\mu(z) - \mu(x)) < 0,$$

and therefore all the solutions of the regularized problem necessarily belong to the bounded set $B_\mu(r)$ (see Remark 1 ii). Thus, by the reflexivity of $X$, for any sequence of solutions $\{x_k\}$ of the problems ($\mathrm{EP}'_k$) there exists a weak limit point $\overline{x} \in C$. Without loss of generality, we will assume that $x_k \rightharpoonup \overline{x}$.

From the lower semicontinuity of $\mu$ we have that $\mu(\overline{x}) \leq \liminf_{k\to+\infty} \mu(x_k)$, which implies that, for every $\epsilon > 0$, $\mu(x_k) > \mu(\overline{x}) - \epsilon$ for $k$ big enough, i.e., $\frac{\mu(x_k)}{k} > \frac{\mu(\overline{x})-\epsilon}{k}$. Therefore,

$$\liminf_{k\to+\infty} \frac{\mu(x_k)}{k} \geq 0. \tag{4}$$

Obviously, taking $k^2$ instead of $k$ within the argument above, we obtain from (4) $\liminf_{k\to+\infty} \frac{\mu(x_k)}{k^2} \geq 0$, which implies in particular that

$$\liminf_{k\to+\infty} \left(-\frac{\mu(x_k)}{k^2}\right) = -\limsup_{k\to+\infty} \frac{\mu(x_k)}{k^2} \leq 0 \tag{5}$$

From the inequality

$$f(x_k, y) + k(P(x_k, y) - P(x_k, x_k)) + \frac{1}{k}(\mu(y) - \mu(x_k)) \geq 0, \qquad \forall y \in C, \tag{6}$$

we get that

$$0 \leq P(x_k, x_k) \leq \frac{1}{k} f(x_k, y) + P(x_k, y) + \frac{1}{k^2}(\mu(y) - \mu(x_k)), \qquad \forall y \in C. \tag{7}$$

Let $y' \in \Phi(\overline{x})$. From vi. of Theorem 3, there exists $y_k \in \Phi(x_k)$ such that $y_k \to y'$. Take $y = y_k$ in (7). Then, from the assumptions, we have

$$\begin{aligned}
0 \leq P(\overline{x}, \overline{x}) &\leq \liminf_{k\to+\infty} P(x_k, x_k) \\
&\leq \liminf_{k\to+\infty} \left(\frac{1}{k} f(x_k, y_k) + \frac{1}{k^2}(\mu(y_k) - \mu(x_k))\right) \\
&\leq \liminf_{k\to+\infty} \left(\frac{1}{k} h(x_k)\|y_k - x_k\| + \frac{1}{k^2}(\mu(y_k) - \mu(x_k))\right) \\
&= 0.
\end{aligned}$$

This implies that $P(\overline{x}, \overline{x}) = 0$, i.e., $\overline{x} \in \Phi(\overline{x})$. Again from vi. of Theorem 3, there exists $\{z_k\}$ such that $z_k \in \Phi(x_k)$ and $z_k \to \overline{x}$. From (6) with $y = z_k$, we get

$$f(x_k, z_k) + \frac{1}{k}(\mu(z_k) - \mu(x_k)) \geq 0.$$

This implies that

$$0 \leq f(x_k, z_k) + \frac{1}{k}(\mu(z_k) - \mu(x_k)) = f(x_k, \overline{x}) + (f(x_k, z_k) - f(x_k, \overline{x})) + \frac{1}{k}(\mu(z_k) - \mu(x_k)). \quad (8)$$

Then,

$$\begin{aligned}
0 &\leq \liminf_{k\to+\infty}(f(x_k, \overline{x}) + (f(x_k, z_k) - f(x_k, \overline{x})) + \frac{1}{k}(\mu(z_k) - \mu(x_k))) \\
&\leq \liminf_{k\to+\infty}(f(x_k, \overline{x}) + h(x_k)\|z_k - \overline{x}\| + \frac{1}{k}(\mu(z_k) - \mu(x_k))) \\
&= \liminf_{k\to+\infty}(f(x_k, \overline{x}) - \frac{1}{k}\mu(x_k)) \\
&\leq \liminf_{k\to+\infty} f(x_k, \overline{x}) - \liminf_{k\to+\infty} \frac{1}{k}\mu(x_k) \\
&\leq \liminf_{k\to+\infty} f(x_k, \overline{x})
\end{aligned}$$

(the last inequality follows by (4)). From the assumption of B-pseudomonotonicity, we get $f(\overline{x}, y) \geq \limsup_{k\to+\infty} f(x_k, y)$, for every $y \in C$.

Let now $z' \in \Phi(\overline{x})$. Then there exists $\{z_k\}$ such that $z_k \in \Phi(x_k)$, and $z_k \to z'$. As before we have

$$\begin{aligned}
0 &\leq \liminf_{k\to+\infty}((f(x_k, z_k) - f(x_k, z')) + f(x_k, z') + \frac{1}{k}(\mu(z_k) - \mu(x_k))) \\
&\leq \liminf_{k\to+\infty}(h(x_k)\|z_k - z'\| + f(x_k, z') + \frac{1}{k}(\mu(z_k) - \mu(x_k))) \\
&\leq \liminf_{k\to+\infty} f(x_k, z').
\end{aligned}$$

This implies that $\limsup_{k\to+\infty} f(x_k, z') \geq 0$. Again from the B-pseudomonotonicity,

$$f(\overline{x}, z') \geq \limsup_{k\to+\infty} f(x_k, z') \geq 0.$$

Since $z'$ is arbitrarily chosen in $\Phi(\overline{x})$, we can conclude that $\overline{x}$ is a solution of (QEP). □

**Remark 4.** If $\Phi : C \rightrightarrows C$ denotes the set-valued map of the constraint sets of the (QEP), a possible choice for the bifunction $P$ could be $P(x, y) = d_{\Phi(x)}y$, where $d_A y$ is the usual distance function of the point $y$ from the set $A$ (see [16]). In order to satisfy the assumptions of Theorem 3 and Theorem 4, taking into account Lemma 4.3 in [16], the set-valued map $\Phi$ should satisfy some additional requirements. In case $C$ is weakly compact, it is enough to assume the following: for every $x_n \rightharpoonup x$, if the sequence $\{y_n\}$, with $y_n \in \Phi(x_n)$, has a weak limit point $y$, then $y \in \Phi(x)$. Note that this condition, together with the lower semicontinuity in Definition 2, gives the well-known weak Mosco continuity of $\Phi$. In case $C$ is not weakly compact, one must ask for the supplementary condition that $\Phi(C)$ is relatively weakly compact. Finally, let us point out that the coercivity condition **(C′)** requires, in fact, that $\Phi(x)$ is not a singleton if $x$ is a fixed point in $C \setminus B_\mu(r)$. This is not an effective restriction, since these possible fixed points would be trivial solutions of (QEP) since the equilibrium bifunction is null on the diagonal.

## 5 Set-valued quasi variational inequalities

The purpose of this section is to apply the previous results to the particular case of quasi variational inequalities. Let $T : X \rightrightarrows X^*$ be an operator. One can naturally associate to $T$ the representative bifunction $G_T : X \times X \to \mathbb{R} \cup \{+\infty\}$, given by

$$G_T(x, y) = \sup_{x^* \in T(x)} \langle x^*, y - x\rangle.$$

The quasi equilibrium problem associated to the bifunction $G_T$ gives rise to the *set-valued quasi variational inequality* $QVI(T, \Phi)$: find $\overline{x} \in \Phi(\overline{x})$ such that

$$\sup_{x^* \in T(\overline{x})} \langle x^*, y - \overline{x}\rangle \geq 0 \quad \text{for all } y \in \Phi(\overline{x}). \tag{9}$$

It is well known that $G_T$ inherits the monotonicity property from the operator $T$ and viceversa. We are now interested in studying the relationship in case of B-pseudomonotonicity.

Let us first recall that $T$ is *B-pseudomonotone* on a nonempty subset $C$ of $\text{dom}(T)$ if, for every $\{x_n\}$ in $C$ such that $x_n \rightharpoonup x \in C$, and for every $x_n^* \in T(x_n)$ with

$$\liminf_{n\to+\infty} \langle x_n^*, x - x_n\rangle \geq 0,$$

one has that, for every $y \in C$, there exists $x^*(y) \in T(x)$ such that

$$\langle x^*(y), y - x\rangle \geq \limsup_{n\to+\infty} \langle x_n^*, y - x_n\rangle.$$

It is easy to prove the following

**Proposition 6.** Let $T : X \rightrightarrows X^*$ be a weakly compact-valued operator, $C \subset \text{dom}(T)$, and consider the bifunction $G_T : C \times C \to \mathbb{R}$. If $G_T$ is B-pseudomonotone on $C \times C$, then $T$ is B-pseudomonotone on $C$.

*Proof.* Let us assume that $G_T$ is B-pseudomonotone on $C\times C$. Take any sequence $\{x_n\} \subset C$, such that $x_n \rightharpoonup x \in C$, and $\{x_n^*\}$, with $x_n^* \in T(x_n)$ and $\liminf_{n\to+\infty} \langle x_n^*, x - x_n\rangle \geq 0$. This implies that $\liminf_{n\to+\infty} G_T(x_n, x) \geq 0$. By the B-pseudomonotonicity of $G_T$, we have that

$$G_T(x, y) \geq \limsup_{n\to+\infty} G_T(x_n, y) \geq \limsup_{n\to+\infty} \langle x_n^*, y - x_n\rangle, \qquad \forall y \in C.$$

By the weak compactness of $T(x)$, for every $y \in C$ there exists $x^*(y) \in T(x)$ such that $G_T(x, y) = \langle x^*(y), y - x\rangle$, and thus

$$\langle x^*(y), y - x\rangle \geq \limsup_{n\to+\infty} \langle x_n^*, y - x_n\rangle, \qquad \forall y \in C,$$

proving the B-pseudomonotonicity of $T$. □

On the contrary, the question whether B-pseudomonotonicity of $T$ implies B-pseudomonotonicity of $G_T$ is still open. In [3] the authors proved that if $T$ is B-pseudomonotone, then $G_T$ satisfies condition iv. in Theorem 1 which, according to Proposition 2 and Example 1, is weaker than B-pseudomonotonicity of $G_T$. However, assuming that $X$ is a reflexive Banach space, if we strengthen the assumptions on $T$, it is possible to achieve $B$-pseudomonotonicity of $G_T$ which can be useful in dealing with set-valued quasi variational inequalities.

First, let us introduce the concept of $S_+$ set-valued operators as a natural extension of the scalar notion introduced by Browder in [7] for single-valued operators and which plays an important role in the theory of Galerkin approximations.

**Definition 3.** The operator $T : X \rightrightarrows X^*$ is said to be of type $S_+$ on a nonempty subset $C$ of $\mathrm{dom}(T)$ if, for every $\{x_n\}$ in $C$ such that $x_n \rightharpoonup x \in C$, and for every $x_n^* \in T(x_n)$, with

$$\limsup_{n\to+\infty} \langle x_n^*, x_n - x\rangle \leq 0,$$

it follows that $x_n \to x$ in $C$.

In what follows we shall denote by $\mathrm{gph}(T)$ the graph of the operator $T$. Recall that $T$ is said to be *s-w-closed* on $C$ if, for any $(x_n, x_n^*) \in \mathrm{gph}(T|_C)$ such that $x_n \to x$ and $x_n^* \rightharpoonup x^*$, one has that $(x, x^*) \in \mathrm{gph}(T|_C)$. Furthermore, $T$ is said to be bounded if it maps bounded subsets of its domain into bounded sets.

We are now in the position to prove the following

**Proposition 7.** Let $X$ be a reflexive Banach space, $T : X \rightrightarrows X^*$, and $C \subset \mathrm{dom}(T)$ be a nonempty, closed and convex set. Suppose that $T$ is convex-valued, s-w-closed, bounded and of type $S_+$ on $C$. Then $G_T$ is of type $S_+$, and $G_T(\cdot, y)$ is sequentially upper semicontinuous for every $y \in C$. In particular, $G_T$ is B-pseudomonotone on $C \times C$.

*Proof.* Under the assumptions, the set $T(x)$ is convex and bounded, for every $x \in X$. In addition, $T(x)$ is closed. Indeed, for any $x_n^* \to x^*$, with $x_n^* \in T(x)$ we also have $x_n^* \rightharpoonup x^*$; thus, since $(x, x_n^*) \in \mathrm{gph}(T)$ and $T$ is s-w-closed, it follows that $x^* \in T(x)$. Then, by reflexivity, $T(x)$ is weakly compact.

To prove that $G_T$ is of type $S_+$, take any sequence $\{x_n\} \subset C$, such that $x_n \rightharpoonup x \in C$, and $\liminf_{n\to\infty} G_T(x_n, x) \geq 0$. This means, by the definition of $G_T$, that

$$\liminf_{n\to\infty} \sup_{x_n^* \in T(x_n)} \langle x_n^*, x - x_n\rangle \geq 0.$$

By the weak compactness of $T(x_n)$, there exists $z_n^* \in T(x_n)$ such that

$$\sup_{x_n^* \in T(x_n)} \langle x_n^*, x - x_n \rangle = \langle z_n^*, x - x_n \rangle$$

and, therefore, $\limsup_{n\to\infty} \langle z_n^*, x_n - x \rangle \le 0$. Since $T$ is of type $S_+$, we have that $x_n \to x$, and hence $G_T$ is of type $S_+$.

To prove the sequential upper semicontinuity of $G_T(\cdot, y)$ for every $y \in C$, take any sequence $\{x_n\} \subset C$, such that $x_n \to x \in C$. By contradiction, there exists $\overline{y} \in C$ such that

$$G_T(x, \overline{y}) < \limsup_{n\to+\infty} G_T(x_n, \overline{y}) = \lim_{k\to+\infty} G_T(x_{n_k}, \overline{y}), \tag{10}$$

where $\{x_{n_k}\}$ is a suitable subsequence of $\{x_n\}$. By the weak compactness of $T(x_{n_k})$, there exists $x_{n_k}^* \in T(x_{n_k})$ such that

$$G_T(x_{n_k}, \overline{y}) = \langle x_{n_k}^*, \overline{y} - x_{n_k} \rangle.$$

Since $T$ is bounded, let us assume, without loss of generality, that $x_{n_k}^* \rightharpoonup x^*$. Furthermore, by the s-w closedness of $T$ on $C$, $x^* \in T(x)$. Therefore,

$$\lim_{k\to+\infty} G_T(x_{n_k}, \overline{y}) = \lim_{k\to+\infty} \langle x_{n_k}^*, \overline{y} - x_{n_k} \rangle = \langle x^*, \overline{y} - x \rangle \le G_T(x, \overline{y}),$$

thereby contradicting (10). Finally, according to Remark 3 i., $G_T$ is B-pseudomonotone on $C \times C$. □

**Remark 5.** Note that, from Proposition 6, an operator satisfying the assumptions in Proposition 7 is $B$-pseudomonotone.

By taking advantage of the previous discussion that links properties of $T$ and $G_T$, we are able to prove existence results for $QVI(T, \Phi)$.

In case $C$ is weakly compact, we underline that an existence result can be provided solely via properties enjoyed by the operator $T$, meaning that $G_T$ and/or $\mu$ are not involved in the assumptions.

The following result holds:

**Theorem 5.** Let us assume that $P$ and $\Phi$ satisfy the assumptions of Theorem 3; moreover, suppose that $T$ is convex-valued, bounded, of type $S_+$ and s-w–closed on $C$. Then the problem $QVI(T, \Phi)$ is solvable.

*Proof.* In order to apply Theorem 3 with $f(x, y) = G_T(x, y)$, taking into account Proposition 7, we need only to show that ii. is satisfied, i.e.,

$$|G_T(z, y) - G_T(z, x)| \le h(z)||y - x||,$$

where $h : C \to \mathbb{R}$ is positive and bounded on bounded sets. By the weak compactness of $T(z)$, there exists $u^* \in T(z)$ such that $G_T(z, y) = \langle u^*, y - z \rangle$. Then

$$G_T(z, y) - G_T(z, x) \le \langle u^*, y - z \rangle - \langle u^*, x - z \rangle = \langle u^*, y - x \rangle \le \sup_{v^* \in T(z)} ||v^*|| \cdot ||x - y||$$

and, analogously, $G_T(z,x) - G_T(z,y) \leq \sup_{v^* \in T(z)} ||v^*|| \cdot ||x-y||$. Since $T$ is bounded, the function $h(z) = \sup_{v^* \in T(z)} ||v^*||$ is bounded on bounded sets and the assertion follows. $\square$

In case $C$ is unbounded, let us consider the following coercivity condition for a bifunction $f : C \times C \to \mathbb{R}$ (see [3] for a similar condition): there exists $y_0 \in C$ such that, for some $s > 0$,

$$\limsup_{\|x\| \to +\infty,\, x \in C} \frac{f(x, y_0)}{\|x - y_0\|^s} < +\infty. \tag{11}$$

The following result holds:

**Theorem 6.** Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $X$, and $T : X \rightrightarrows X^*$ be convex-valued, bounded, of type $S_+$ and s-w–closed on $C$. Let us assume that $P$ and $\Phi$ satisfy the assumptions of Theorem 3; moreover, $P$ and $G_T$ satisfy, for the same $y_0 \in C$ and $s > 0$, the condition of the bifunction $f$ in (11).

Then, for every positive $s' > s$, $s' \geq 1$, the regularized problem

$$G_{T,k}(x, y) = G_T(x, y) + k(P(x, y) - P(x, x)) + \frac{1}{k}(\|y\|^{s'} - \|x\|^{s'}) \geq 0, \qquad \forall y \in C, \tag{12}$$

is solvable, and any sequence of solutions $x_k$ of the problem above has weak limit points that are, in particular, solutions of (QVI).

*Proof.* Let us show that $G_{T,k}$ satisfies the coercivity condition v. of Theorem 1, with $E = X$ equipped with the weak topology. Indeed, taking into account (11) and the sign of $P$, we get

$$\begin{aligned}
&\limsup_{\|x\| \to +\infty,\, x \in C} \frac{G_{T,k}(x, y_0)}{\|x - y_0\|^s} \\
&\quad = \limsup_{\|x\| \to +\infty,\, x \in C} \left( \frac{G_T(x, y_0)}{\|x - y_0\|^s} + k\frac{P(x, y_0) - P(x, x)}{\|x - y_0\|^s} + \frac{1}{k}\frac{\|y_0\|^{s'} - \|x\|^{s'}}{\|x - y_0\|^s} \right) \\
&\quad \leq \limsup_{\|x\| \to +\infty,\, x \in C} \frac{G_T(x, y_0)}{\|x - y_0\|^s} + k \limsup_{\|x\| \to +\infty,\, x \in C} \frac{P(x, y_0) - P(x, x)}{\|x - y_0\|^s} + \\
&\qquad\qquad + \frac{1}{k} \limsup_{\|x\| \to +\infty,\, x \in C} \frac{\|y_0\|^{s'} - \|x\|^{s'}}{\|x - y_0\|^s} \\
&\quad = -\infty.
\end{aligned}$$

This implies that there exists $M > \|y_0\|$, such that

$$G_{T,k}(x, y_0) < 0, \quad \forall x \in C,\ \|x\| > M,$$

i.e., $G_{T,k}(x, y_0) < 0$ for every $x \in C \setminus \overline{B}_X(0, M)$, where $\overline{B}_X(0, M)$ denotes the closed ball with centre 0 and radius $M$. Therefore v. holds choosing $K = \overline{B}_X(0, M)$. Furthermore, $G_{T,k}$ trivially satisfies assumptions i. and ii. of Theorem 1. Assumptions iii. and iv. follow from Proposition 7 and Proposition 2. Therefore, problem (12) is solvable and the solutions belong to $\overline{B}_X(0, M)$. To conclude the proof, we can now follow the line of Theorem 4, with $\mu(x) = \|x\|^{s'}$. □

**Remark 6.** Note that if we choose $P(x, y) = d_{\Phi(x)}y$, under the assumptions of Remark 4, the coercivity condition (11) holds. Indeed, by the relative weak compactness of $\Phi(C)$, $\Phi(x)$ is bounded and $P(x, y_0) = d_{\Phi(x)}y_0$ is bounded, too. With this choice of $P$, Theorem 6 can be considered an extension of Theorem 4.4. in [16] to the set-valued and non compact case.

Another possible approach to the unbounded case should be to introduce a coercivity condition in terms of the function $\mu$.

**Proposition 8.** Suppose that there exists a convex and lower semicontinuous function $\mu : X \to \mathbb{R}$, which is coercive with respect to the set $C$, and a point $y_0 \in C$ such that

$$\lim_{\mu(x)\to\infty,\, x\in C} \frac{\inf_{x^*\in T(x)}\langle x^*, x - y_0\rangle}{\mu(x)} = +\infty. \tag{13}$$

Then condition **(C)** is satisfied by $f(x, y) = G_T(x, y)$.

*Proof.* Let $R := \max\{0, \mu(y_0)\}$. By (13) there exists $r > R$ such that

$$\frac{\inf_{x^*\in T(x)}\langle x^*, x - y_0\rangle}{\mu(x)} \geq 1, \quad \forall x \in C,\, \mu(x) > r, \tag{14}$$

i.e.,

$$G_T(x, y_0) = \sup_{x^*\in T(x)} \langle x^*, y_0 - x\rangle \leq -\mu(x), \quad \forall x \in C,\, \mu(x) > r. \tag{15}$$

Hence for every $x \in C \setminus B_\mu(r)$ we have $\mu(x) > r > R \geq \mu(y_0)$ which, together with (15), shows that condition **(C)** is satisfied. □

We are now in the position to apply Theorem 2 to the regularized problem (12) and, with similar steps as Theorem 6, prove the following:

**Theorem 7.** Let $C$ be a nonempty, closed and convex subset of a reflexive Banach space $X$, and $T : X \rightrightarrows X^*$ be convex-valued, bounded, of type $S_+$ and s-w–closed on $C$. Let us assume that $P$ and $\Phi$ satisfy the assumptions of Theorem 3; moreover, there exists $y_0 \in C$ such that $T$ satisfy (13) for a suitable function $\mu$, and $P(x, y_0) \leq P(x, x)$ for all $x \in C \setminus B_\mu(r)$ where $r$ is fixed in the proof of Proposition 8.

Then, the regularized problem

$$G_{T,k}(x, y) = G_T(x, y) + k(P(x, y) - P(x, x)) + \frac{1}{k}(\mu(y) - \mu(x)) \geq 0, \qquad \forall y \in C, \tag{16}$$

is solvable, and any sequence of solutions $x_k$ of the problem above has weak limit points that are, in particular, solutions of (QVI).

**Remark 7.** Assumptions of type (13) are often considered in the theory of differential equations when dealing with coercivity conditions. Suppose that $0 \in C$. A typical particular instance is the case of elliptic operators, i.e., when $\mu(x) = \|x\|$ and there exists $\alpha > 0$ such that $\inf_{x^* \in T(x)} \langle x^*, x \rangle \geq \alpha \|x\|^2$ for all $x \in C$ (here $y_0 = 0$). For single-valued operators the relation above reduces to $\langle T(x), x \rangle \geq \alpha \|x\|^2$.

### Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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