Stability of equilibrium points of projected dynamical systems*

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Abstract. We present a survey of the main results about asymptotic stability, exponential stability and monotone attractors of locally and globally projected dynamical systems, whose stationary points coincide with the solutions of a corresponding variational inequality. In particular, we show that the global monotone attractors of locally projected dynamical systems are characterized by the solutions of a corresponding Minty variational inequality. Finally, we discuss two special cases: when the domain is a polyhedron, the stability analysis for a locally projected dynamical system, at regular solutions to the associated variational inequality, is reduced to one of a standard dynamical system of lower dimension; when the vector field is linear, some global stability results, for locally and globally projected dynamical systems, are proved if the matrix is positive definite (or strictly copositive when the domain is a convex cone).

Keywords. Variational inequality, projected dynamical system, equilibrium solution, stability analysis.

1 Introduction

Equilibrium is a central concept in the study of complex and competitive systems. Examples of well-known equilibrium problems include oligopolistic market equilibrium and traffic network equilibrium problems. For these problems many variational formulations have been introduced in the last years, nevertheless their analysis is focused on the static study of the equilibrium, while it is also of interest to analyse the time evolution of adjustment processes for these equilibrium problems. Recently, two models for studying dynamic behaviour of such systems have been proposed: they are based on constrained dynamical systems involving projection operators. The first one is the so-called locally projected dynamical system (first proposed in [5]), the other is known as globally projected dynamical system (introduced in [6]). The most important connection between these dynamical models and variational models lies is the possibility of characterizing stationary points for dynamical models by solutions of a particular variational inequality. The locally projected dynamical systems have had recently important applications in economics (see [3], [13] and [14]) and in the traffic networks equilibrium problems (see [16], [17] and [18]); the globally projected dynamical systems have been introduced as a model for describing network disequilibria (see [6]) and they have been applied to neural networks for solving a class of optimization problems (see [20]).

The main purpose of this paper is to give, to the best of my knowledge, a survey of the main results about the stability of these two types of projected dynamical systems (see [12] and [15], [19] and [20]). The paper is organized as follows. In Section 2, we recall the definitions of locally and globally projected dynamical systems and we show the equivalence between their equilibrium points and the solutions of a suitable associated variational inequality. In Section 3, we recall some stability definitions needed throughout the paper (monotone attractor, asymptotic stability, exponential stability); we show some stability results

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for a locally projected dynamical system under monotonicity assumptions on the vector field; we state the correspondence between the global monotone attractors of a locally projected dynamical system and the solutions to a related Minty variational inequality; we give asymptotic and exponential stability results for a globally projected dynamical system when the jacobian matrix of the vector field is symmetric; finally we provide a stability result, for both the projected dynamical systems, similar to the nonlinear sink theorem for standard dynamical systems. Section 4 is dedicated to the stability analysis in two special cases. When the domain is a convex polyhedron, the stability of a locally projected dynamical system, at regular solutions to the associated variational inequality, is essentially the same as that of a standard dynamical system of lower dimension, the so-called minimal face flow. When the vector field is linear and the matrix is positive definite (or strictly copositive if the domain is a convex cone), the global exponential stability for locally and globally projected dynamical systems is proved. Finally, some suggestions for future research in the linear case are described.

2 Variational and Dynamical Models

Throughout this paper K denotes a closed convex subset of R^n and $F : R^n \to R^n$ a vector field. The following monotonicity definitions will be needed for our later discussions.

Definition 2.1. *F is said to be locally pseudomonotone at x^* if there is a neighborhood $N(x^*)$ of x^* such that*

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \Longrightarrow \langle F(x), x - x^* \rangle \geq 0, \quad \forall\, x \in N(x^*),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in R^n;

F is said to be pseudomonotone on K if

$$\langle F(y), x - y \rangle \geq 0, \quad \Longrightarrow \langle F(x), x - y \rangle \geq 0, \quad \forall\, x, y \in K;$$

F is said to be locally strictly pseudomonotone at x^* if there is a neighborhood $N(x^*)$ of x^* such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \Longrightarrow \langle F(x), x - x^* \rangle > 0, \quad \forall\, x \in N(x^*);$$

F is said to be strictly pseudomonotone on K if

$$\langle F(y), x - y \rangle \geq 0, \quad \Longrightarrow \langle F(x), x - y \rangle > 0, \quad \forall\, x, y \in K;$$

F is said to be monotone on K if

$$\langle F(x) - F(y), x - y \rangle \geq 0, \quad \forall\, x, y \in K;$$

F is said to be locally strongly monotone at x^* if there is a neighborhood $N(x^*)$ of x^* and $\eta > 0$ such that

$$\langle F(x) - F(x^*), x - x^* \rangle \geq \eta \|x - x^*\|^2, \quad \forall\, x \in N(x^*);$$

F is said to be strongly monotone on K if there is $\eta > 0$ such that

$$\langle F(x) - F(y), x - y \rangle \geq \eta \|x - y\|^2, \quad \forall\, x, y \in K.$$

We recall that a variational inequality of Stampacchia-type SVI(F,K) consists in determining a vector $x^* \in K$, such that

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall\, x \in K;$$

the associated Minty variational inequality MVI(F,K) consists in finding a vector $x^* \in K$, such that

$$\langle F(x), x^* - x \rangle \leq 0, \quad \forall\, x \in K.$$

It is well-known that if F is continuous on K, then each solution to MVI(F,K) is a solution to SVI(F,K); whereas if F is pseudomonotone on K, then each solution to SVI(F,K) is also a solution to MVI(F,K).

The first dynamical model we consider in this context is the so-called locally projected dynamical system (first introduced in [5]), denoted by LPDS(F,K), which is defined by the following ordinary differential equation

$$\dot{x} = P_{T_K(x)}(-F(x)),$$

where $T_K(x)$ denotes the tangent cone to K at x, P_S denotes the usual projection on a closed convex subset S:

$$P_S(x) = \arg\min_{z \in S} \|x - z\|,$$

and $\|\cdot\|$ is the euclidean norm on R^n.

We remark that if the vector field F is continuous on K, then the right-hand side of LPDS(F,K) is also continuous on the relative interior of K, and it can be discontinuous elsewhere. By a solution to the LPDS(F,K) we mean an absolutely continuous function $x : [0, +\infty) \to K$ such that

$$\dot{x}(t) = P_{T_K(x(t))}(-F(x(t))),$$

for all $t \geq 0$ save on a set of Lebesgue measure zero.

We are interested in the equilibrium (or stationary) points of LPDS(F,K), i.e. the vectors $x^* \in K$ such that

$$P_{T_K(x^*)}(-F(x^*)) = 0;$$

that is once a solution of the LPDS(F,K) is at x^*, it will remain at x^* for all future times. The first connection between the locally projected dynamical systems and the variational inequalities is that the stationary points of the LPDS(F,K) coincide with the solutions of SVI(F,K). This equivalence allows us to carry out a stability analysis of a solution to SVI(F,K) respect to the dynamical system LPDS(F,K), as we will see in Section 3.

Since the main purpose of this paper is to analyse the stability of the equilibrium points, we confine ourself to cite only the following result about the existence, uniqueness and continuous dependence on the initial value of solutions to LPDS(F,K), in the special case where K is a polyhedron (see [15]).

Theorem 2.1. *Let K be a polyhedron. If there exists a constant $M > 0$ such that*

$$\|F(x)\| \leq M\,(1 + \|x\|), \quad \forall\, x \in K,$$

and also

$$\langle F(y) - F(x), x - y \rangle \leq M\, \|x - y\|^2, \quad \forall\, x, y \in K,$$

then

- *for any $x_0 \in K$, there exists a unique solution $x_0(t)$ to LPDS(F,K), such that $x_0(0) = x_0$;*
- *if $x_n \to x_0$ as $n \to +\infty$, then $x_n(t)$ converges to $x_0(t)$ uniformly on every compact set of $[0, +\infty)$.*

We note that Lipschitz continuity of F on K is a sufficient condition for the properties stated in Theorem 2.1.

The second dynamical model we consider is the so-called globally projected dynamical system, denoted by GPDS(F,K,α), which is defined as the following ordinary differential equation

$$\dot{x} = P_K(x - \alpha\, F(x)) - x,$$

where α is a positive constant. If the vector field F is continuous on K, then the right-hand side of GPDS(F,K,α) is also continuous on K, but it can be different from $-\alpha\, F(x)$ even if x is an interior point to K. Hence the solutions of GPDS(F,K,α) and LPDS(F,K) are different in general.

The following result about global existence and uniqueness of solutions to GPDS(F,K,α) follows on the ordinary differential equations theory (see [20]).

Theorem 2.2. *If F is locally Lipschitz continuous and there exists a constant $M > 0$ such that*

$$\|F(x)\| \leq M\,(1 + \|x\|), \qquad \forall\, x \in K,$$

then for any $x_0 \in K$ there exists a unique solution $x_0(t)$ for GPDS(F,K,α), such that $x_0(0) = x_0$, that is defined for all $t \in R$.

It can be proved that, as in the case of LPDS(F,K), a solution to GPDS(F,K,α) starting from a point in K has to remain in K (see [20] and [19]).

The equilibrium (or stationary) points of GPDS(F,K,α) are naturally defined as the vectors $x^* \in K$ such that

$$P_K(x^* - \alpha\, F(x^*)) = x^*.$$

It is easy to check that they also coincide with the solutions to SVI(F,K). Hence, we can analyze the stability of the solutions to SVI(F,K) also respect to the dynamical system GPDS(F,K,α).

3 Stability Analysis

We have seen that LPDS(F,K) and GPDS(F,K,α) have the same stationary points, but, in general, their solutions are different. In this section we analyze the stability of these equilibrium points, namely we wish to know the behavior of the solutions, of LPDS(F,K) and GPDS(F,K,α) respectively, which start near an equilibrium point. Since we are mainly focused on the stability issue, we can assume the property of existence and uniqueness of solutions to the Cauchy problems corresponding to locally and globally projected dynamical systems. In the following, $B(x^*, r)$ denotes the open ball with center x^* and radius r.

Now we recall some definitions on stability.

Definition 3.1. *Let x^* be a stationary point of LPDS(F,K) and GPDS(F,K,α).*

x^* is called stable *if for any $\epsilon > 0$, there exists $\delta > 0$ such that for every solution $x(t)$, with $x(0) \in B(x^*, \delta) \cap K$, one has $x(t) \in B(x^*, \epsilon)$ for all $t \geq 0$;*

x^* is said asymptotically stable *if x^* is stable and $\lim_{t \to +\infty} x(t) = x^*$ for every solution $x(t)$, with $x(0) \in B(x^*, \delta) \cap K$; x^* is said* globally asymptotically stable *if it is stable and $\lim_{t \to +\infty} x(t) = x^*$ for every solution $x(t)$ with $x(0) \in K$.*

We recall also that x^* is called monotone attractor *if there exists $\delta > 0$ such that, for every solution $x(t)$ with $x(0) \in B(x^*, \delta) \cap K$, the euclidean distance between $x(t)$ and x^*, that is $\|x(t) - x^*\|$, is a nonincreasing function of t; whereas x^* is said* strictly monotone attractor *if $\|x(t) - x^*\|$ is decreasing to zero in t. Moreover x^* is a* (strictly) global monotone attractor *if the same properties hold for any solution $x(t)$ such that $x(0) \in K$.*

Finally, x^* is a finite-time attractor *if there is $\delta > 0$ such that, for every solution $x(t)$, with $x(0) \in B(x^*, \delta) \cap K$, there exists some $T < +\infty$ such that $x(t) = x^*$ for all $t \geq T$.*

It is trivial to remark that the monotone attractors are stable equilibrium points, whereas the strictly monotone attractors and the finite-time attractors are asymptotically stable ones.

It is easy to check that the stability of a locally (or globally) projected dynamical system can differ from the stability of a standard dynamical system in the same vector field (see examples in [12]).

The pseudo-monotonicity property of F is directly related to the monotone attractors of LPDS(F,K), as shown in the following theorem which is a direct generalization of a result proved in [12].

Theorem 3.1. *Let x^* be a stationary point of LPDS(F,K). If F is locally (strictly) pseudomonotone at x^*, then x^* is a (strictly) monotone attractor for LPDS(F,K); if F is (strictly) pseudomonotone on K, then x^* is a (strictly) global monotone attractor for LPDS(F,K).*

However, the monotonicity of F is not sufficient to prove the stability for an equilibrium point of a globally projected dynamical system, as the following example shows (for the details see [20]).

Example 3.1. We consider $K = \{x \in R^3 : \ -10 \leq x_i \leq 10\}$, $\alpha = 1$ and $F(x) = Ax + b$, where

$$A = \begin{pmatrix} 0.1 & 0.1 & -0.5 \\ 0.1 & 0.1 & 0.5 \\ 0.5 & -0.5 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} -1 \\ 1 \\ -0.5 \end{pmatrix}.$$

The affine vector field F is monotone, because A is positive semidefinite, but the unique equilibrium point of GPDS(F,K,α), i.e. $x^* = (0.5, -0.5, -2)^T$, is not stable.

In addition to the monotonicity of F, the symmetry of the jacobian matrix of F, denoted by JF, is necessary in order to achieve the asymptotic stability of a stationary point of a globally projected dynamical system (see [20]).

Theorem 3.2. *Let x^* be the unique stationary point of GPDS(F,K,α). If F is monotone on K and the jacobian matrix JF is symmetric on an open convex set including K, then x^* is globally asymptotically stable for GPDS(F,K,α) for any $\alpha > 0$.*

We remark that under the assumptions of Theorem 3.2, the vector field F is the gradient map of a real convex function on K.

Now we go back to the monotone attractors. When the vector filed F is continuous, there is a further connection between locally projected dynamical systems and variational inequalities: the global monotone attractors of LPDS(F,K) are equivalent to the solutions of the Minty variational inequality MVI(F,K) (see [19]).

Theorem 3.3. *Let F be continuous on K. Then $x^* \in K$ is a global monotone attractor for LPDS(F,K) if and only if it is solution to MVI(F,K).*

We remark that Theorem 3.3 does not hold for globally projected dynamical systems: the following example (see [19]) shows that the solutions to MVI(F,K) are not necessarily monotone attractors of GPDS(F,K,α), even if the vector field F is continuous on K.

Example 3.2. We consider $K = R^2_+$ and the vector field

$$F(x) = \begin{pmatrix} x_2 + ((x_1 - 1)^2 + x_2^2 - 1)^2 \\ -x_1 + ((x_1 - 1)^2 + x_2^2 - 1)^2 \end{pmatrix}.$$

The point $x^* = (0, 0)^T$ is solution to MVI(F,K), but it is not a monotone attractor for GPDS(F,K,α) for any fixed $\alpha > 0$.

Another stability type we consider is the so-called exponential stability.

Definition 3.2. *Let x^* be a stationary point of LPDS(F,K) and GPDS(F,K,α). It is said* exponentially stable, *if the solutions starting from points close to x^* are convergent to x^* with exponential rate, that is if there is $\delta > 0$ and two constants $a > 0$ and $C > 0$ such that for every solution $x(t)$, with $x(0) \in B(x^*, \delta) \cap K$, one has*

$$\|x(t) - x^*\| \leq C \, \|x(0) - x^*\| \, e^{-at} \quad \forall \ t \geq 0; \tag{1}$$

x^* is globally exponentially stable *if (1) holds for all solutions $x(t)$ such that $x(0) \in K$.*

We remark that a strictly monotone attractor is not necessarily exponentially stable and vice versa (see examples in [19]).

The exponential stability of a stationary point of LPDS(F,K) is proved under the strong monotonicity assumption of the vector field F (see [12] and [15]).

Theorem 3.4. *Let x^* be a stationary point of LPDS(F,K). If F is locally strongly monotone at x^*, then x^* is a strictly monotone attractor and exponentially stable for LPDS(F,K); if F is strongly monotone on K, then x^* is a strictly global monotone attractor and globally exponentially stable for LPDS(F,K).*

The strong monotonicity and the Lipschitz continuity of F give the exponential stability of a stationary point of GPDS(F,K,α), provided that α is small enough (see [19]).

Theorem 3.5. *Let x^* be a stationary point of GPDS(F,K,α). If F is locally strongly monotone at x^* with constant η, and locally Lipschitz continuous at x^* with constant L, then x^* is a strictly monotone attractor and exponentially stable for GPDS(F,K,α), provided that $\alpha < 2\eta/L^2$; if F is strongly monotone on K with constant η and locally Lipschitz continuous on K with constant L, then x^* is a strictly global monotone attractor and globally exponentially stable for GPDS(F,K,α), provided that $\alpha < 2\eta/L^2$.*

The global exponential stability for GPDS(F,K,α), where α is small enough, has been proved even when the jacobian matrix JF is symmetric and positive definite (see [20]).

Theorem 3.6. *Let x^* be a stationary point of GPDS(F,K,α). If JF is symmetric and uniformly positive definite in R^n, $\|JF\|$ has an upper bound, then x^* is globally exponentially stable for GPDS(F,K,α), provide that $\alpha < 2/\max_{x \in R^n} \|JF(x)\|$.*

A further result on the exponential stability for globally projected dynamical systems can be proved when the jacobian matrix JF is not symmetric, but the domain K is bounded (see [20]).

Theorem 3.7. *Let x^* be a stationary point of GPDS(F,K,α). If K is bounded, F is continuously differentiable on K and JF is positive definite on K, then there exists $\alpha_0 > 0$ such that x^* is globally exponentially stable for GPDS(F,K,α) for any $\alpha < \alpha_0$.*

Now we present a stability result for LPDS(F,K) analogous to the nonlinear sink theorem for classical (i.e. not projected) dynamical systems (see [7]); we will assume a stronger condition on F, that is the jacobian matrix of F in x^* is positive definite, instead of having its eigenvalues positive real parts, but we also obtain a stronger result, i.e. x^* is a strictly monotone attractor and exponentially stable (see [19]), instead of only exponentially stable.

Theorem 3.8. *Let x^* be a stationary point of LPDS(F,K). If F is continuously differentiable on a neighborhood of x^* and the jacobian matrix $JF(x^*)$ is positive definite, then x^* is a strictly monotone attractor and exponentially stable for LPDS(F,K).*

A result similar to Theorem 3.8 can be proved also for GPDS(F,K,α), provided that α is small enough (see [19]).

Theorem 3.9. *Let x^* be a stationary point of GPDS(F,K,α). If F is continuously differentiable on a neighborhood of x^* and the jacobian matrix $JF(x^*)$ is positive definite, then there exists $\alpha_0 > 0$ such that x^* is a strictly monotone attractor and exponentially stable for GPDS(F,K,α) for any $\alpha < \alpha_0$.*

4 Special Cases

This section is devoted to the stability analysis in two special cases: when the domain K is a convex polyhedron and when the vector field F is linear.

We remarked that the stability for a locally projected dynamical system is generally different from that of a standard dynamical system; however, when K is a convex polyhedron, many local stability properties for LPDS(F,K) follow on that of a classical dynamical system in lower dimension, under suitable assumption on the regularity of the stationary points of LPDS(F,K) (see [12]).

We assume that K is specified by

$$K = \{x \in R^n : \ Bx \leq b\},$$

where B is an $m \times n$ matrix, with rows B_i, and $b \in R^n$. We recall that a face of K is the intersection of K and a number of hyperplanes that support K, and the minimal face of K containing a point x, denoted by $E(x)$, is the intersection of all the faces of K containing x. If we denote

$$I(x) = \{i : \ B_i x = b_i\} \quad \text{and} \quad S(x) = \{x \in R^n : \ B_i x = 0, \ \forall i \in I(x)\},$$

then $E(x) = (S(x) + x) \cap K$. We assume that $S(x) = R^n$, when $I(x) = \emptyset$.

Let x^* be a stationary point of LPDS(F,K), with $\dim S(x^*) \geq 1$, then there is a $\delta > 0$ such that

$$z + x^* \in E(x^*), \quad \forall \ z \in S(x^*) \cap B(0, \delta).$$

The following ordinary differential equation defined on the subspace $S(x^*)$:

$$\dot{z} = P_{S(x^*)}(-F(z + x^*)), \qquad z \in S(x^*);$$

is called the minimal face flow and it is denoted by MFF(F,K,x^*). Note that if F is locally Lipschitz continuous, then so is the right hand side of MFF(F,K,x^*), hence, for any $z_0 \in S(x^*)$, there is a unique solution $z_0(t)$ to MFF(F,K,x^*), defined in a neighborhood of 0, such that $z_0(0) = z_0$. Moreover, it is clear that $0 \in S(x^*)$ is a stationary point of MFF(F,K,x^*). The stability of $0 \in S(x^*)$ for MFF(F,K,x^*) assures the stability of x^* for LPDS(F,K), under some regularity condition on x^*, which we now introduce. Since x^* solves the variational inequality SVI(F,K), we have

$$-F(x^*) \in N_K(x^*),$$

where $N_K(x^*) = \{y \in R^n : \ \langle y, x - x^* \rangle \leq 0, \ \ \forall \ x \in K\}$ is the normal cone of K at x^*. We say that x^* is a *regular solution* of SVI(F,K) if

$$-F(x^*) \in \mathrm{ri} N_K(x^*),$$

where $\mathrm{ri} N_K(x^*)$ denotes the relative interior of $N_K(x^*)$. Note that any interior solution of SVI(F,K) is regular if we assume $\mathrm{ri}\{0\} = \{0\}$; moreover, when x^* is a solution of SVI(F,K) that lies on an (n-1)-dimensional face of K, it is regular if and only if $F(x^*) \neq 0$.

Now we show two stability results proved in [12]. First, a regular solution to SVI(F,K) has the strongest stability when it is an extreme point of K.

Theorem 4.1. *If x^* is a regular solution to SVI(F,K) and $S(x^*) = \{0\}$, then it is a finite-time attractor for LPDS(F,K) and there are $\gamma > 0$ and $\delta > 0$ such that, for any solution $x(t)$, with $x(0) \in B(x^*, \delta) \cap K$*

$$\frac{d}{dt}\|x(t) - x^*\| \leq -\gamma.$$

The stability results in the general case are summarized in the following theorem.

Theorem 4.2. *If x^* is a regular solution to SVI(F,K) and* $\dim S(x^*) \geq 1$*, then*

- *if 0 is stable for MFF(F,K,x^*), then x^* is stable for LPDS(F,K);*
- *if 0 is asymptotically stable for MFF(F,K,x^*), then x^* is asymptotically stable for LPDS(F,K);*
- *if 0 is a finite-time attractor for MFF(F,K,x^*), then x^* is a finite-time attractor for LPDS(F,K).*

We remark that the local stability of a stationary point x^* of LPDS(F,K) depends on the combination of the regularity of x^* and the local stability of MFF(F,K,x^*). In the extreme case $S(x^*) = \{0\}$ the stability for LPDS(F,K) is implied only by the regularity condition, in the other extreme case, $S(x^*) = R^n$, MFF(F,K,x^*) is just a translation of LPDS(F,K) from x^* to the origin, hence they enjoy the same stability.

We now proceed to consider the special case of F being a linear vector field, that is $F(x) = Ax$, where A is a real n-dimensional matrix. Under this assumption, the existence and uniqueness property for the solutions to the Cauchy problems associated to LPDS(F,K) and GPDS(F,K,α) holds for any closed convex domain K (see [5] and [20]).

We first remark that when the matrix A is positive definite, the local stability properties obtained for LPDS(F,K), by Theorem 3.8, and for GPDS(F,K,α), by Theorem 3.9, become global properties, as shown by the following result.

Proposition 4.1. *Assume that $F(x) = Ax$ and x^* is a stationary point of LPDS(F,K) and GPDS(F,K,α). If A is a positive semidefinite matrix, then x^* is a global monotone attractor for LPDS(F,K). If A is positive definite, then x^* is the unique stationary point for LPDS(F,K) and GPDS(F,K,α); moreover there is $\alpha_0 > 0$ such that x^* is a strictly global monotone attractor and globally exponentially stable for LPDS(F,K) and for GPDS(F,K,α) for any $\alpha < \alpha_0$.*

Proof: It easy to prove that F is strongly monotone on K with constant

$$\eta = \min_{\|x\|=1} \langle Ax, x\rangle > 0;$$

therefore the variational inequality SVI(F,K) has a unique solution x^*, which is also a stationary point for LPDS(F,K) and GPDS(F,K,α). By Theorem 3.4, x^* is a strictly global monotone attractor and globally exponentially stable for LPDS(F,K). Moreover F is Lipschitz continuous on K with constant $\|A\|$. By Theorem 3.5, x^* is a strictly global monotone attractor and globally exponentially stable for GPDS(F,K,α), provided that $\alpha < 2\eta/\|A\|^2$.

In addition, if we assume that the domain K is a closed convex cone, then it is well-known that SVI(F,K) is equivalent to a generalized complementarity problem. In this case the origin is a trivial stationary point for LPDS(F,K) and GPDS(F,K,α), and we can easily prove some stability properties for it, under the weaker assumption that the matrix A is (strictly) copositive with respect to the cone K.

Proposition 4.2. *Assume that K be a closed convex cone and $F(x) = Ax$. If A is a copositive matrix with respect to K, that is $\langle x, Ax\rangle \geq 0$ for all $x \in K$, then $x^* = 0$ is a global monotone attractor for LPDS(F,K). If A is strictly copositive with respect to K, that is $\langle x, Ax\rangle > 0$ for all $x \in K$, then $x^* = 0$ is the unique* *stationary point for LPDS(F,K) and GPDS(F,K,α), and there exists $\alpha_0 > 0$ such that $x^* = 0$ is a strictly global monotone attractor and globally exponentially stable for LPDS(F,K) and for GPDS(F,K,α), for any $\alpha < \alpha_0$.*

The stability analysis in the linear case is still open; future research might be carried on to study suitable conditions on the matrix A providing stability for any closed and convex domain. Also, it might be of interest to check if, when F is an affine vector field and $K = R^n_+$, the classes of matrices needed for the study of existence and uniqueness of the solutions to the linear complementarity problem are sufficient to guarantee some stability properties.

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