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# Three-way decisions with evaluative linguistic expressions

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ABSTRACT

The theory of three-way decisions (3WD) requires dividing a finite, non-empty universe into three disjoint sets called positive, negative, and boundary regions. Three types of decisions are then made on the objects in each region: acceptance, rejection, and abstention (or non-commitment), respectively. Until today, a large number of 3WD extensions and applications have been proposed; some of the most recent ones also include aspects of linguistics. In this article, we first propose an innovative linguistic interpretation of three-way decisions, where the positive, negative, and boundary regions are constructed by means of the so-called evaluative linguistic expressions. These are expressions of natural language, such as small, medium, very short, quite roughly strong, extremely good, etc., and they are described within a logical theory based on the formal system of higher-order fuzzy logic. Furthermore, in line with our linguistic 3WD approach, we introduce the novel notion of linguistic rough sets, thus contributing to the development of Rough Set Theory. Finally, we connect the theory of linguistic three-way decisions with the standard 3WD model based on probabilistic rough sets, establishing conditions under which the two approaches coincide. Our results highlight connections between two different research areas: three-way decisions and the theory of evaluative linguistic expressions.

## 1. Introduction

The theory of three-way decisions (3WD) divides a finite and non-empty universe into three disjoint sets, which are called positive, negative, and boundary regions. These regions respectively induce positive, negative, and boundary rules: a positive rule makes a decision of acceptance, a negative rule makes a decision of rejection, and a boundary rule makes an abstained or non-committed decision [1,2]. The concept of three-way decision was originally introduced in Rough Set Theory [1,3], and so far, it has been studied and employed to solve many decision-making problems (see [4–7] for some examples). Thus, different methods generating the three regions exist in literature [8]; the most important one is based on probabilistic rough sets (that are extensions of Pawlak rough sets [9]), where the three regions called probabilistic positive, negative, and boundary regions, are constructed using a pair of thresholds and the notion of conditional probability. The philosophy and the power of three-way decisions for building high-level conceptual models are discussed in [10].

Three-way decision theory with probabilistic rough sets has been extensively developed from a theoretical and applied perspective. To show the wide range of fields touched by these studies, let us mention [11], where a temporal-spatial three-way recommendation strategy (TS3WR) is proposed to realize a multi-step recommendation; in [12], an optimization-based three-way decision model is realized in the interval-valued intuitionistic fuzzy environment; in [13], the interplay of three-way decision theory

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and granular computing is explored in the context of cognitive science; in [14], three-way approximations of fuzzy sets are presented by using the concept of granular shadowed sets in the quotient space; in [15], a three-way multi-attribute decision-making model is constructed in incomplete fuzzy decision systems.

The contribution of this article is to provide a linguistic interpretation of the positive, negative, and boundary regions. Therefore, we propose a three-way decision method based on the concept of *evaluative linguistic expressions*, which are expressions of natural language such as *small, medium, very short, quite roughly strong, extremely good, etc*. In particular, we focus on the expressions involving the adjectives *small*, *medium*, and *big* that can be preceded by an adverb; examples are *very small*, *roughly medium*, *extremely big*, and so on.

In recent years, evaluative linguistic expressions have been studied in the fields of theoretical linguistics and mathematical logic [16–18], and have been considered in the majority of applications of fuzzy modelling. Particularly relevant for our study is the application of linguistic expressions in fuzzy decision-making. In this context, experts commonly give their evaluation using expressions of natural language, which are aggregated with selected operators and exploited following the classical decision scheme. In this case, the simplest evaluative linguistic expressions are modelled by fuzzy sets with trapezoidal/triangular-shaped membership functions. More complex evaluative linguistic expressions are modelled using several approaches such as the ones based on hesitant fuzzy linguistic term sets [19] or linguistic distributions [20].

Our model stands out from those usually considered in fuzzy decision-making for two aspects. Firstly, the mathematical instrument chosen to represent evaluative linguistic expressions. In fact, evaluative linguistic expressions involved in this work are described within a formal logical theory based on the formal system of higher-order fuzzy logic [21–24]. The second difference is the role of evaluative linguistic expressions in the decision process. Indeed, they are employed to evaluate the size of sets in order to obtain a tri-partition of an initial universe and they do not relate to expert opinions. As a consequence, the corresponding aggregation and exploitation steps need not be taken into account.

We notice that also various 3WD models include some aspects of linguistics: linguistic terms could appear for determining thresholds as in [25] or for defining a generalized formula of the conditional probability as in [26]. The substantial differences with our approach are two: in all these models, evaluative linguistic expressions are semantically described by means of mathematical tools different from our approach and the three regions are obtained using the concept of conditional probability. Therefore, evaluative linguistic expressions as introduced in [21] are applied to 3WD theory for the first time in this work. It is fundamental to underline that we adopt an abstract approach to 3WD because the thresholds $\alpha$ and $\beta$ are generic values of [0,1] satisfying the inequality $\beta < \alpha$ and they are not the solutions of specific equations as in [2].

Let us briefly describe how evaluative linguistic expressions act here. First of all, consider that evaluative linguistic expressions are characterized by the notions of intention, context, and extension. The meaning of an evaluative linguistic expression is modelled by its intention, which is a function assigning to each context another mapping called extension (see Subsection 2.2 for more details). In this article, since we usually deal with the standard context, the extension of an evaluative linguistic expression is a map $Ev : [0,1] \to [0,1]$. The positive, negative, and boundary regions of a non-empty and finite universe $U$ are defined starting from a subset $X$ of $U$, an equivalence relation $\mathcal{R}$ on $U$ (i.e. $\mathcal{R}$ is reflexive, symmetric and transitive), an evaluative linguistic expression represented by $Ev : [0,1] \to [0,1]$, and a pair of thresholds $(\alpha, \beta)$ with $0 \le \beta < \alpha \le 1$. Then, an object $x$ belongs to the positive region when *the size of* $[x]_{\mathcal{R}} \cap X$ *evaluated w.r.t.* $Ev$ *is at least* $\alpha$, where $[x]_{\mathcal{R}}$ is the equivalence class of $x$ w.r.t. $\mathcal{R}$. Analogously, $x$ belongs to the negative region when *the size of* $[x]_{\mathcal{R}} \cap X$ *evaluated w.r.t.* $Ev$ *is at most* $\beta$. Finally, the remaining elements form the boundary region. In order to obtain the three regions, the size of $X \cap [x]_{\mathcal{R}}$ is quantified using a fuzzy measure [27].

The role of evaluative linguistic expressions in the context of three-way decision can be better understood by the following example.

**Example 1.** Suppose that the number of buses between the University of Buenos Aires and the rest of the city has to be increased from 7 am to 8 am. Thus, we intend to understand which city areas need buses the most, as resources are limited. Let us denote the areas of the city with $A_1, \ldots, A_n$ and map each area $A_i$ with the set $S_{A_i}$ made of all students of the university who live in $A_i$. Thus, $S_{A_1}, \ldots, S_{A_n}$ can be seen as the equivalence classes w.r.t. the relation $\mathcal{R}$ on the set of all students of the University of Buenos Aires living in the city: $x\mathcal{R}y$ if and only if $x$ and $y$ live in the same area. Based on a survey, we also consider a set $X$ made of all students that usually take a bus to the university in the slot time [7 am, 8 am]. We also choose $(\alpha, \beta) = (0.6, 0.3)$ and $Ev = extremely\ big$. We construct three regions in the following way. The positive region is the union of $S_{A'_1}, \ldots, S_{A'_k}$ (with $\{A'_1, \ldots, A'_k\} \subseteq \{A_1, \ldots, A_n\}$) so that the number of students of $S_{A'_i}$ that take a bus from 7 am to 8 am is "extremely big" with a truth value of at least 0.6. Similarly, the negative region is the union of $S_{A^*_1}, \ldots, S_{A^*_h}$ (with $\{A^*_1, \ldots, A^*_h\} \subseteq \{A_1, \ldots, A_n\}$) so that the amount of students of $S_{A^*_i}$ that take a bus from 7 am to 8 am is extremely big with a truth value of at most 0.3. All other students form the boundary region. The final decision is immediate: the buses are certainly increased for the areas $A'_1, \ldots, A'_k$, but not for $A^*_1, \ldots, A^*_h$. Furthermore, the decision is postponed for the remaining areas (that is, for each $A_i \notin \{A'_1, \ldots, A'_k\} \cup \{A^*_1, \ldots, A^*_h\}$). In order to make a decision in those areas, for example, we could take into account the workers (besides the students) who need a bus in the slot of time [7 am, 8 am].

The choice of $Ev$ depends on the situation where the three regions are used. Indeed, in the previous example, we have chosen *extremely big* in order to select the areas where a large number of students catch the bus from 7 am to 8 am. However, if we focus on the inverse problem (namely we need to eliminate some existing bus rides), then we should identify the areas where there are fewer students taking the bus in the time slot [7 am, 8 am]. Therefore, in this case, the evaluative linguistic expression *extremely small* is more appropriate to construct the three regions.

The principal strengths of our linguistic 3WD model are:

- providing a linguistic and novel interpretation of the positive, negative, and boundary regions already determined with probabilistic rough sets. Consequently, the reasons for decisions of acceptance, rejection, and non-commitment can be explained in terms of expressions of natural language. Of course, the advantage is that non-technical users dealing with 3WD models can better understand the reliability of the procedures related to the final decisions. This is in line with the scope of *Explainable Artificial Intelligence (XAI)*, which is a new approach to AI emphasizing the ability of machines to give sound motivations about their decisions and behaviour [28];
- contributing to the development of Rough Set Theory. Indeed, the exploitation of the strict connection between 3WD and rough sets naturally leads to the notion of linguistic rough sets;
- allowing a probabilistic model (3WD model based on probabilistic rough sets) to be reinterpreted as a model involving fuzziness. In fact, in order to obtain the three regions, we compare the thresholds $\alpha$ and $\beta$ with truth degrees instead of probabilities of events;
- highlighting connections between two different research areas: three-way decisions and the theory of evaluative linguistic expressions.

The article is organized as follows. The next section reviews some basic notions regarding probabilistic three-way decisions and the concept of evaluative linguistic expressions. Also, the notion of fuzzy measure is recalled. Section 3 presents a new model of three-way decision based on the theory of evaluative linguistic expressions. As a consequence, a linguistic generalization of Pawlak rough sets is introduced. Finally, Section 4 connects the 3WD models based on evaluative linguistic expressions and probabilistic rough sets. In particular, confining to the evaluative linguistic expressions modelled by increasing functions, we find the class of thresholds so that the corresponding probabilistic positive, negative, and boundary regions are equal to those generated by a given evaluative linguistic expression.

## 2. Preliminaries

In the following, we consider a finite and non-empty universe $U$, a subset $X$ of $U$, and an equivalence relation $\mathcal{R}$ on $U$ (i.e. $\mathcal{R}$ is reflexive, symmetric, and transitive). Moreover, we indicate the equivalence class of $x \in U$ w.r.t. $\mathcal{R}$ with $[x]_{\mathcal{R}}$.

### 2.1. Three-way decision with probabilistic rough sets

This subsection recalls the fundamental notions of three-way decisions based on probabilistic rough sets.

Viewing $X$ and $[x]_{\mathcal{R}}$ as events of $U$, the symbol $Pr(X|[x]_{\mathcal{R}})$ denotes the *conditional probability* of $X$ given $[x]_{\mathcal{R}}$, i.e.

$$Pr(X|[x]_{\mathcal{R}}) = \frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|}. \tag{1}$$

Then, three special subsets of $U$ are determined by using (1) and a pair of thresholds as shown by the next definition.

**Definition 1.** Let $\alpha, \beta \in [0,1]$ such that $\beta < \alpha$, the $(\alpha,\beta)$-*probabilistic positive, negative* and *boundary regions* are respectively the following:

(i) $POS_{(\alpha,\beta)}(X) = \{x \in U \mid Pr(X|[x]_{\mathcal{R}}) \geq \alpha\}$,
(ii) $NEG_{(\alpha,\beta)}(X) = \{x \in U \mid Pr(X|[x]_{\mathcal{R}}) \leq \beta\}$,
(iii) $BND_{(\alpha,\beta)}(X) = \{x \in U \mid \beta < Pr(X|[x]_{\mathcal{R}}) < \alpha\}$.

We put

$$\mathcal{T}_{(\alpha,\beta)}(X) = \{POS_{(\alpha,\beta)}(X), NEG_{(\alpha,\beta)}(X), BND_{(\alpha,\beta)}(X)\}. \tag{2}$$

Trivially, $\mathcal{T}_{(\alpha,\beta)}(X)$ is a *tri-partition* of $U$.[1] Namely,

- the three regions of $\mathcal{T}_{(\alpha,\beta)}(X)$ are mutually disjoint: $A \cap B = \emptyset$ for each $A, B \in \mathcal{T}_{(\alpha,\beta)}(X)$ with $A \neq B$;
- they cover the universe $U$:

$$POS_{(\alpha,\beta)}(X) \cup NEG_{(\alpha,\beta)}(X) \cup BND_{(\alpha,\beta)}(X) = U. \tag{3}$$

In the context of three-way decision theory, the following rules are considered: let $x \in U$,

[1] By a tri-partition, we mean a partition of $U$ made of three equivalence classes. On the other hand, $\{POS_{(\alpha,\beta)}(X), NEG_{(\alpha,\beta)}(X), BND_{(\alpha,\beta)}(X)\}$ could collapse into a bi-partition or the whole universe when one or two of its sets are empty.

- if $x \in POS_{(\alpha,\beta)}(X)$, then $x$ is accepted;
- if $x \in NEG_{(\alpha,\beta)}(X)$, then $x$ is rejected;
- if $x \in BND_{(\alpha,\beta)}(X)$, then we abstain on $x$.

The values $Pr(X|[x]_{\mathcal{R}})$ represent the *accuracy* (or *confidence*) of the rules:

- the higher $Pr(X|[x]_{\mathcal{R}})$ is, the more confident we are that $x \in POS_{(\alpha,\beta)}(X)$ is correctly accepted,
- the lower $Pr(X|[x]_{\mathcal{R}})$ is, the more confident we are that $x \in NEG_{(\alpha,\beta)}(X)$ is correctly rejected.

Definition 1 is strictly related to the notion of probabilistic rough sets.

**Definition 2.** The $(\alpha,\beta)$*-probabilistic rough set* of $X$ is the pair

$$(\mathcal{L}_{(\alpha,\beta)}(X), \mathcal{U}_{(\alpha,\beta)}(X)),$$

where

$$\mathcal{L}_{(\alpha,\beta)}(X) = POS_{(\alpha,\beta)}(X) \text{ and } \mathcal{U}_{(\alpha,\beta)}(X) = POS_{(\alpha,\beta)}(X) \cup BND_{(\alpha,\beta)}(X),$$

which are respectively called $(\alpha,\beta)$– *lower and upper approximations* of $X$.

**Remark 1.** When $\alpha = 1$ and $\beta = 0$, $(\mathcal{L}_{(\alpha,\beta)}(X), \mathcal{U}_{(\alpha,\beta)}(X))$ is the rough set $(\mathcal{L}(X), \mathcal{U}(X))$ of $X$ defined by Pawlak in [3], namely

$$(\mathcal{L}(X), \mathcal{U}(X)) = (\{x \in U \mid [x]_R \subseteq X\}, \{x \in U \mid [x]_R \cap X \neq \emptyset\}). \tag{4}$$

The sets $\mathcal{L}(X)$ and $\mathcal{U}(X)$ are respectively called *lower* and *upper approximations* of $X$ w.r.t. $\mathcal{R}$.

### 2.2. Evaluative linguistic expressions

This subsection reviews concepts and results that are found in [21,24] and it recalls the notion of fuzzy measure.

*Evaluative linguistic expressions* are special expressions of natural language, which people commonly employ to evaluate, judge, estimate, and in many other situations. Examples of evaluative linguistic expressions are *small, medium, big, about twenty-five, roughly one hundred, very short, more or less deep, not very tall, roughly warm or medium-hot*, etc. For convenience, we will often omit the adjective “linguistic” and use only the term “evaluative expressions”. The simplest evaluative expressions are called *pure evaluative expressions* and have the following structure:

$\langle$*linguistic hedge*$\rangle\langle$*TE-adjective*$\rangle$,

where

- a linguistic hedge is an adverbial modification such as *very, roughly, approximately*, and *significantly*;
- a TE-adjective is an adjective such as *good, medium, big, short*, etc. TE stands for *trichotomous evaluative*, indeed TE-adjectives typically form pairs of antonyms like *small* and *big* completed by a middle member, which is *medium* in the case of *small* and *big*. Other examples are “*weak, medium-strong,* and *strong*” and “*soft, medium-hard,* and *hard*”.

The *empty linguistic hedge* is employed to deal with evaluative expressions made of only a TE-adjective; hence, *small*, *medium*, and *big* are considered evaluative expressions. Other pure evaluative expressions are the fuzzy numbers like *about twenty-five*. Two or more pure evaluative expressions can be connected to form *negative evaluative expressions* like *“NOT very small”* and *compound evaluative expressions* like *“very expensive AND extremely small”* and *“very expensive OR extremely small”*.

The semantics of evaluative expressions is based on the essential concepts of *context*, *intension*, and *extension*.

**Context.** The meaning of an evaluative expression in the natural language can change based on the context in which it appears. For instance, regarding the evaluative expression *big*, we spontaneously think of two different sizes if dealing with *big cats* or *big elephants*. Therefore, when handling an evaluative expression, it is always necessary to specify three values $s, m, b \in \mathbb{R}$ forming the interval $[s,m] \cup [m,b]$ and indicating what is meant by *small*, *medium*, and *big*. The triple $\omega = \langle s, m, b\rangle$ is called *context* of the considered evaluative expression. Let us show an example of two different contexts. Suppose that evaluative expressions are used to evaluate the size of apartments. If we are thinking of apartments for one person, then we could choose $\omega_1 = \langle 40, 70, 100\rangle$ as context, which means that flats measuring 40 $m^2$, 70 $m^2$, and 100 $m^2$ are typically considered small, medium and big, respectively. On the other hand, when thinking of apartments for a family of 5 people, the context $\omega_5 = \langle 70, 120, 160\rangle$ is intuitively more appropriate.

**Extension.** Evaluative expressions are employed in the natural language to evaluate something like the size of animals, apartments, and so on. Thus, an evaluative expression is associated with a universe of objects. In the example above, such a universe could be composed of some apartments $a_1, a_2, a_3$, and $a_4$ whose size you want to evaluate. Thus, the context is described by a mapping

$w : [0,1] \to U$ where let $u \in U$, $w^{-1}(a) = 0$, $w^{-1}(a) = 0.5$, and $w^{-1}(a) = 1$ if and only if $a$ is typically considered small, medium and big, respectively. Once the universe $U$ corresponding to an evaluative expression $E$ and the context are established, the *extension* of $E$ is a special fuzzy set of the set of the real numbers $\mathbb{R}$. Its role is to model $E$ in the fixed context. Taking up the previous example again, $w^{-1}(a_i)$ could be identified with the size $x_{a_i} \in \mathbb{R}$ of $a_i$. Then, once fixed the context $\omega_5$, the extension of the evaluative expression $small$ is a map $Sm_{\omega_5} : \mathbb{R} \to [0,1]$ so that $Sm_{\omega_5}(\omega_5^{-1}(a_i))$ is the truth degree to which "*$a_i$ is small in the context $\omega_5$*" (namely, "*$a_i$ is small for 5 people*"). For example, if the size of the apartment $a_1$ is 0.8, then the truth degree of the statement "*the size of the apartment $a_1$ is small in the context $w_5$*" (for 5 people) is 0.9, while if the size of $a_4$ is 140, then $a_4$ *is small in the same context $w_5$*" with the truth degree 0. Of course, the extension of an evaluative expression changes with the context. Indeed, it is easy to understand that if $Sm_{\omega_5}(\omega_5^{-1}(a_i)) = 0.8$ (the apartment $a_i$ is considered small for 5 people with the truth degree 0.8), then $Sm_{\omega_1}(\omega_1^{-1}(a_i)) < 0.8$ (the apartment $a_i$ cannot be considered equally small for only one person).

**Intension.** Since the extension of an evaluative expression $E$ depends on the context, a good mathematical description of $E$ includes a function assigning to each context the corresponding extension. Such mapping is called the *intension* of $E$ and it is denoted with $Int_E$. In the case of apartments, we get $Int_{small} : \omega_1 \mapsto Sm_{\omega_1}$ and $Int_{small} : \omega_5 \mapsto Sm_{\omega_5}$.

In this article, we confine to the TE adjectives *small*, *medium*, and *big* because we use evaluative expressions to evaluate the size of sets. Therefore, let $X$ be a subset of a universe $U$, we will say that the size of $X$ w.r.t. the size of $U$ is *very small*, *extremely big*, etc. Furthermore, we deal with the *standard context*, which is $\langle 0, 0.5, 1\rangle$. Finally, since sizes are expressed by means of a fuzzy measure (by Example 2, the measure of the size of a set $X$ is a value of $[0,1]$), the extensions of our evaluative expressions are functions from $[0,1]$ to $[0,1]$, which have a specific formula. The extension of an evaluative expression like $\langle linguist\ hedge\rangle\langle TE - adjective\rangle$ with $TE - adjective \in \{small, medium, big\}$ is obtained by composing two functions, one models the linguistic hedge and the other models the TE-adjective. The function describing a linguistic hedge depends on three parameters, which are experimentally estimated (see [24] for more details).

In what follows, we provide the formula of $\neg Sm : [0,1] \to [0,1]$, $BiVe : [0,1] \to [0,1]$, and $BiEx : [0,1] \to [0,1]$, which are the extensions of the evaluative expressions *not small*, *very big*, and *extremely big*, when the context $\langle 0, 0.5, 1\rangle$ is fixed.[2]

$$\neg Sm(x) = \begin{cases} 1 & \text{if } x \in [0.275, 1], \\ 1 - \dfrac{(0.275 - x)^2}{0.02305} & \text{if } x \in (0.16, 0.275), \\ \dfrac{(x - 0.0745)^2}{0.01714} & \text{if } x \in (0.0745, 0.16], \\ 0 & \text{if } x \in [0, 0.0745]. \end{cases} \tag{5}$$

$$BiVe(x) = \begin{cases} 1 & \text{if } x \in [0.9575, 1], \\ 1 - \dfrac{(0.9575 - x)^2}{0.00796} & \text{if } x \in [0.895, 0.9575), \\ \dfrac{(x - 0.83)^2}{0.00828} & \text{if } x \in (0.83, 0.895), \\ 0 & \text{if } x \in [0, 0.83]. \end{cases} \tag{6}$$

$$BiEx(x) = \begin{cases} 1 & \text{if } x \in [0.995, 1], \\ 1 - \dfrac{(0.995 - x)^2}{0.00495} & \text{if } x \in [0.95, 0.995), \\ \dfrac{(x - 0.885)^2}{0.00715} & \text{if } x \in (0.885, 0.95), \\ 0 & \text{if } x \in [0, 0.885]. \end{cases} \tag{7}$$

The graphics of the functions $\neg Sm$, $BiVe$, and $BiEx$ are depicted by Fig. 1.

**Remark 2.** The evaluative expressions $\neg Sm$, $BiVe$, and $BiEx$ have a special role: they are respectively used to construct the formula of fuzzy quantifiers *many*, *most*, and *almost all* [29].

We also consider the class of operators $\{\Delta_t : [0,1] \to \{0,1\} \mid t \in [0,1]\}$, where let $t \in [0,1]$,

$$\Delta_t(a) = \begin{cases} 1 & \text{if } a \geq t, \\ 0 & \text{otherwise}, \end{cases} \tag{8}$$

for each $a \in [0,1]$.

[2] We have got the formulas of $BiVe$ and $BiEx$ using the function $v_{a,b,c}(LH(\omega^{-1}))$ and Table 5.1 given in [24]. Concerning the formula of $\neg Sm$, we have considered that $\neg Sm(x) = 1 - Sm(x)$. After that, we have found the formula of $Sm$ using the function $v_{a,b,c}(RH(\omega^{-1}))$ and Table 5.1 given in [24].

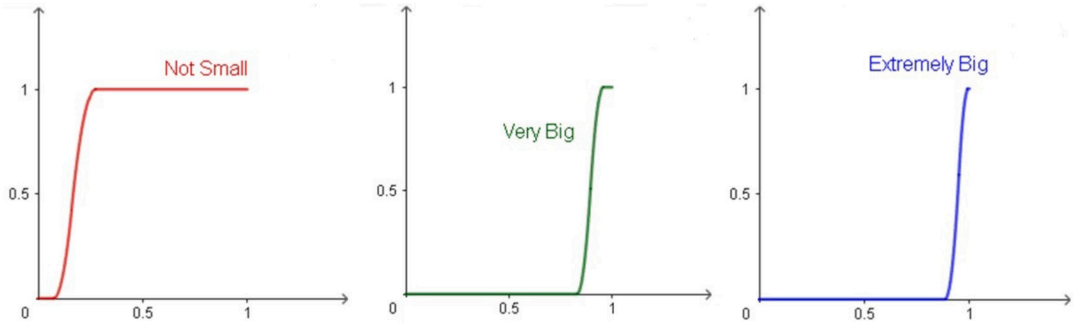


**Fig. 1.** The graphics of the functions $\neg Sm : [0,1] \to [0,1]$, $BiVe : [0,1] \to [0,1]$, and $BiEx : [0,1] \to [0,1]$, which are respectively the extensions of the evaluative expressions *not small*, *very big*, and *extremely big* in the context $\langle 0, 0.5, 1 \rangle$.

The operators defined by (8) are interpretations of logical formulas of the formal theory of evaluative linguistic expressions presented in [21].

**Remark 3.** $\Delta_1$ coincides with the evaluative expression *utmost* and it is related to the fuzzy quantifier *all* [29].

In the sequel, we need the notion of fuzzy measure [27].

**Definition 3.** Let $U$ be a finite universe, a mapping $\varphi : 2^U \to \mathbb{R}$ is called *fuzzy measure* if and only if

(a) $\varphi(\emptyset) = 0$;
(b) if $X \subseteq Y$ then $\varphi(X) \leq \varphi(Y)$, for each $X, Y \subseteq U$ (monotonicity).

We say that a fuzzy measure $\varphi$ is *normalized* or *regular* when $\varphi(U) = 1$.
In this paper, we focus on the normalized fuzzy measure given by the next example.

**Example 2.** Let $U$ be a finite universe, the function $f : 2^U \to \mathbb{R}$ that assigns $\dfrac{|Y|}{|U|}$ to each $Y \subseteq U$ is a fuzzy measure.
The value $\dfrac{|Y|}{|U|}$ belongs to [0,1] and measures *"how much $Y$ is large with respect to $U$ in the scale* $[0,1]$*"*.

Let us observe that in Probability theory $\dfrac{|Y|}{|U|}$ represents *"how likely the event $Y$ is to occur"*.

## 3. Three-way decisions with linguistic expressions

In this section, a novel 3WD model and the corresponding generalized rough sets are constructed using evaluative expressions.

### 3.1. Linguistic three-way decision model

This subsection proposes a novel 3WD model, which is based on the concept of evaluative linguistic expressions previously described.

In the sequel, we use the symbol $\mathcal{E}$ to denote the collection of the extensions of all evaluative expressions in the context $\langle 0, 0.5, 1 \rangle$. Notice that $\neg Sm$, $BiVe$, $BiEx$, and $\Delta_t$ belong to $\mathcal{E}$.

Therefore, let $Ev \in \mathcal{E}$, let $X \subseteq U$, and let $\alpha, \beta \in [0,1]$ with $\beta < \alpha$, three regions of $U$ are determined. In particular, the region of a given element $x \in U$ is found by taking into account the following steps:

1. computing $Ev\left(\dfrac{|X \cap [x]_\mathcal{R}|}{|[x]|_\mathcal{R}}\right)$, which is the evaluation of *the size of* $X \cap [x]_\mathcal{R}$ *w.r.t. the size of* $[x]_\mathcal{R}$ by using $Ev$;
2. comparing $Ev\left(\dfrac{|X \cap [x]_\mathcal{R}|}{|[x]|_\mathcal{R}}\right)$ with the thresholds $\alpha$ and $\beta$.

For example, regarding point 1, if *Ev* models the evaluative expression "*significantly big*", then $Ev\left(\dfrac{|X \cap [x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$ measures

*"how much the size of $X \cap [x]_\mathcal{R}$ is significantly big w.r.t. the size of $[x]_\mathcal{R}$".*

Equivalently, we are saying that

*"the size of the set of the elements of $[x]_\mathcal{R}$ that also belong to $X$ is significantly big (w.r.t. the size of $[x]_\mathcal{R}$) with the truth degree $Ev\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$".*

Observe that $\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}$ syntactically coincides with the conditional probability (see (1)), but here it has a different interpretation: it is the fuzzy measure specified by Example 2.

Formally, the three regions of $U$ determined by an evaluative expression are given by the following definition.

**Definition 4.** Let $Ev\in\mathcal{E}$, the $(\alpha,\beta)$-linguistic positive, negative, and boundary regions induced by $Ev$ are respectively the following:

(i) $POS^{Ev}_{(\alpha,\beta)}(X)=\left\{x\in U\mid Ev\left(\frac{|[x]_\mathcal{R}\cap X|}{|[x]_\mathcal{R}|}\right)\geq\alpha\right\}$;
(ii) $NEG^{Ev}_{(\alpha,\beta)}(X)=\left\{x\in U\mid Ev\left(\frac{|[x]_\mathcal{R}\cap X|}{|[x]_\mathcal{R}|}\right)\leq\beta\right\}$;
(iii) $BND^{Ev}_{(\alpha,\beta)}(X)=\left\{x\in U\mid \beta< Ev\left(\frac{|[x]_\mathcal{R}\cap X|}{|[x]_\mathcal{R}|}\right)<\alpha\right\}$.

We put

$$\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)=\{POS^{Ev}_{(\alpha,\beta)}(X),NEG^{Ev}_{(\alpha,\beta)}(X),BND^{Ev}_{(\alpha,\beta)}(X)\},\tag{9}$$

which is trivially a tri-partition of $U$. Thus,

- the three regions of $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)$ are mutually disjoint, i.e. $A\cap B=\emptyset$ for each $A,B\in\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)$ with $A\neq B$;
- they cover the universe $U$, i.e.

$$POS^{Ev}_{(\alpha,\beta)}(X)\cup NEG^{Ev}_{(\alpha,\beta)}(X)\cup BND^{Ev}_{(\alpha,\beta)}(X)=U.\tag{10}$$

**Remark 4.** Let us focus on the evaluative expressions *not small*, *very big*, *extremely big*, and *utmost*. The first three expressions are respectively modelled by (5), (6), and (7). According to Remark 2, $\neg Sm$, $BiVe$, and $BiEx$ appear in the formula of the fuzzy quantifiers *many*, *most*, and *almost all*, which are special functions $\mathcal{S}_{many},\mathcal{S}_{most}$, and $\mathcal{S}_{almost\ all}$ assigning a value of [0,1] to each pair of fuzzy sets of the initial universe. As explained in [30] (see Lemma 4.5), considering that $X$ and $[x]_\mathcal{R}$ are crisp set, $\neg Sm\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$, $BiVe\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$, and $BiEx\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$ exactly coincide with the formula of *many*, *most*, and *almost all*; in symbols, $\mathcal{S}_{many}([x]_\mathcal{R},X)=\neg Sm\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$, $\mathcal{S}_{most}([x]_\mathcal{R},X)=BiVe\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$, and $\mathcal{S}_{almost\ all}([x]_\mathcal{R},X)=BiEx\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$. Hence, they have the following meanings:

- $\neg Sm\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$ is the truth degree to which "*many* objects of $[x]_\mathcal{R}$ are in $X$",
- $BiVe\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$ is the truth degree to which "*most* objects of $[x]_\mathcal{R}$ are in $X$",
- $BiEx\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$ is the truth degree to which "*almost all* objects of $[x]_\mathcal{R}$ are in $X$".

Moreover, according to Remark 3, the function $\Delta_1$ (obtained by (8) with $t=1$), models the evaluative expression *utmost* and is used to construct the fuzzy quantifier *all*. Therefore, analogously to the previous evaluative expressions, since $[x]_\mathcal{R}$ and $X$ are crisp sets, $\Delta^1\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)$ is understood as the truth degree to which "*all* objects of $[x]_\mathcal{R}$ are in $X$".

Observe that here the universe of quantification coincides with $[x]_\mathcal{R}$, which is always non-empty. This is because $\{x\}\subseteq[x]_\mathcal{R}$, considering that $\mathcal{R}$ is reflexive. In mathematical logic, the assumption expressing that the universe of quantification must be non-empty is called *existential import* (or *presupposition*) [31].

**Remark 5.** Consider the operators represented by (8). Let $t\in[0,1]$, then $\Delta_t\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)\in\{0,1\}$. Furthermore,

$$\Delta_t\left(\frac{|X\cap[x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right)=\begin{cases}1 & \text{if "the size of the set of elements of } [x]_\mathcal{R} \\ & \text{that also belong to } X \text{ is at least as large as } t \\ & \text{(in the scale [0,1]) w.r.t. the size of } [x]_\mathcal{R}\text{";}\\ 0 & \text{otherwise.}\end{cases}$$

### 3.2. An illustrative example

In this subsection, we provide an example of how to use linguistic three-way decisions to provide recommendations based on users's profiles.

We consider a universe $U = \{u_1, \ldots, u_{32}\}$ made of users of online communities and the following equivalence relation $\mathcal{R}$ on $U$: let $x, y \in U$,

$$x\mathcal{R}y \text{ if and only if } x \text{ and } y \text{ belong to the same community.}$$

Therefore, $\mathcal{R}$ corresponds to the partition $\{C_1, \ldots, C_6\}$ of $U$, where $C_i$ is the set of users of $U$ belonging to the community $i$. In particular, we suppose that

$$C_1 = \{u_1, \ldots, u_5\},\ C_2 = \{u_6, \ldots, u_{10}\},\ C_3 = \{u_{11}, \ldots, u_{15}\},\ C_4 = \{u_{16}, \ldots, u_{20}\},\ C_5 = \{u_{21}, \ldots, u_{25}\}, \text{ and } C_6 = \{u_{26}, \ldots, u_{32}\}.$$

We use the symbol $X_T$ to denote the set of all users of $U$ interested in a specific topic $T$.

For example,

$$X_{Sport} = \{u_{10}, u_{11}, u_{12}, u_{18}, u_{19}, u_{20}, u_{21}, u_{22}, u_{23}, u_{24}, u_{26}\}$$

is the set of all users of $U$ interested in the topic *Sport*.

Employing our three-way decision model, we intend to select the most appropriate communities among $C_1, \ldots, C_6$ to which propose news related to the topic $T$.

If we choose $(\alpha, \beta) = (0.8, 0.2)$ and the evaluative expression $\neg Sm$ corresponding to the fuzzy quantifier *many*, then we decide to assign the news about the topic $T$ to the communities of $POS^{\neg Sm}_{(0.8,0.2)}(X_T)$. This is because $POS^{\neg Sm}_{(0.8,0.2)}(X_T)$ includes all the communities where the amount of users interested in $T$ is considered enough height for us. Indeed, $x \in POS^{\neg Sm}_{(0.8,0.2)}$ if and only if the truth degree to which

*"many users of the community of $x$ are interested in $T$"*

is greater than or equal to 0.8.

In the sequel, we determine the communities to which provide the news about $Sport$. Namely, we find $POS^{\neg Sm}_{(0.8,0.2)}(X_T)$, where $X_T = Sport$. To do this, we firstly compute the value $\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$ for each $x \in U$:

$$\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = \begin{cases} 0 & \text{if } x \in C_1, \\ 0.14 & \text{if } x \in C_6, \\ 0.2 & \text{if } x \in C_2, \\ 0.4 & \text{if } x \in C_3, \\ 0.6 & \text{if } x \in C_4, \\ 0.8 & \text{if } x \in C_5. \end{cases} \tag{11}$$

According to the definition of $\neg Sm$ that is given by (5), we get $\neg Sm(0) = 0$, $\neg Sm(0.14) = 0.25$, $\neg Sm(0.2) = 0.75$ and $\neg Sm(0.4) = \neg Sm(0.6) = \neg Sm(0.8) = 1$.

Consequently,

$$\neg Sm\left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) = \begin{cases} 0 & \text{if } x \in C_1, \\ 0.25 & \text{if } x \in C_6, \\ 0.75 & \text{if } x \in C_2 \\ 1 & \text{if } x \in C_3 \cup C_4 \cup C_5. \end{cases} \tag{12}$$

Then, the positive, negative and boundary regions induced by $(0.8, 0.2)$ and $\neg Sm$ are the following:

$$POS^{\neg Sm}_{(0.8,0.2)}(X_{Sport}) = \left\{ x \in U \mid \neg Sm\left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) \geq 0.8 \right\} = C_3 \cup C_4 \cup C_5,$$
$$NEG^{\neg Sm}_{(0.8,0.2)}(X_{Sport}) = \left\{ x \in U \mid \neg Sm\left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) \leq 0.2 \right\} = C_1,$$
$$BND^{\neg Sm}_{(0.8,0.2)}(X_{Sport}) = \left\{ x \in U \mid 0.2 < \neg Sm\left(\frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right) < 0.8 \right\} = C_2 \cup C_6.$$

The three regions lead the following decisions. Firstly, we choose to provide the news about the sport to the communities $C_3$, $C_4$, and $C_5$ that form the positive region, considering that *these contain many users interested in sport* with a truth degree that we think is enough high ($\geq 0.8$). Moreover, we require further analysis of the communities $C_2$ and $C_6$ forming the boundary region, before providing news about sports. For example, we could evaluate the interests of the users in $C_2 \cup C_6$ in the future or once new users join such communities. Finally, we surely do not provide sports news to $C_1$ because we think that not enough of its users are interested in sports topics, indeed we consider the truth degree to which *many users of $C_1$ are interested in sports* low ($\leq 0.2$).

### 3.3. Linguistic rough sets

Definition 4 also leads to a novel generalization of Pawlak rough sets.

**Definition 5.** Let $Ev \in \mathcal{E}$, the $(\alpha,\beta)$-linguistic rough set of $X$ determined by $\mathcal{R}$ and $Ev$ is the pair $(\mathcal{L}^{Ev}_{(\alpha,\beta)}(X), \mathcal{U}^{Ev}_{(\alpha,\beta)}(X))$, where

$$\mathcal{L}^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in X \mid Ev\left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|}\right) \geq \alpha \right\} \text{ and } \mathcal{U}^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in X \mid Ev\left(\frac{|[x]_{\mathcal{R}} \cap X|}{|[x]_{\mathcal{R}}|}\right) > \beta \right\}.$$

$\mathcal{L}^{Ev}_{(\alpha,\beta)}(X)$ and $\mathcal{U}^{Ev}_{(\alpha,\beta)}(X)$ are respectively called $(\alpha,\beta)$-*linguistic lower and upper approximations* of $X$ determined by $\mathcal{R}$ and $Ev$.

Equivalently, by Definition 4, we get

$$\mathcal{L}^{Ev}_{(\alpha,\beta)}(X) = POS^{Ev}_{(\alpha,\beta)}(X) \text{ and } \mathcal{U}^{Ev}_{(\alpha,\beta)}(X) = POS^{Ev}_{(\alpha,\beta)}(X) \cup BND^{Ev}_{(\alpha,\beta)}(X).$$

The following is an illustrative example.

**Example 3.** Consider the example presented by Subsection 3.2. Then, $X_{Sport}$ can be approximated by the $(0.8, 0.2)$-linguistic rough set[3]

$$(\mathcal{L}^{\neg Sm}_{(0.8,0.2)}(X_{Sport}), \mathcal{U}^{\neg Sm}_{(0.8,0.2)}(X_{Sport})) = (C_3 \cup C_4 \cup C_5, C_2 \cup C_3 \cup C_4 \cup C_5 \cup C_6) = (\{u_{11}, \dots, u_{25}\}, \{u_6, \dots, u_{32}\})\}.$$

Thus, in line with the meaning of rough sets, the set $X_{Sport}$ of all users interested in sport could be approximated by the pair $(\{u_1, \dots, u_{25}\}, \{u_6, \dots, u_{32}\})$, where

- $\{u_1, \dots, u_{25}\}$ is made of all users being in communities containing *many users interested in sport* with a truth degree at least or equal to 0.8;
- $\{u_6, \dots, u_{32}\}$ is made of all users being in communities containing *many users interested in sport* with a truth degree greater than 0.2.

## 4. Connection with thee-way decision methods

In this section, we find a link between the 3WD methods based on probabilistic rough sets and evaluative expressions. In particular, we fix a finite universe $U$, a subset $X$ of $U$, an equivalence relation $\mathcal{R}$ on $U$, and a pair of thresholds $(\alpha,\beta)$ such that $0 \leq \alpha < \beta \leq 1$, and we aim to determine for each evaluative expression $Ev$, the class of all pairs of thresholds like $(\alpha', \beta')$ so that $\mathcal{T}_{(\alpha',\beta')}(X)$ coincides with $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)$.

We confine to the class $\mathcal{E}^+ \subset \mathcal{E}$, which is made of all extensions that are increasing functions, i.e. let $Ev \in \mathcal{E}$, $Ev \in \mathcal{E}^+$ if and only if "$Ev(x) \leq Ev(y)$ for each $x, y \in [0,1]$ such that $x \leq y$". Examples of evaluative expressions so that their extension is an increasing function, are *not small*, *very big*, *extremely big*, and *utmost* (see (5), (6), (7), and (8)). However, there exist evaluative expressions like *small* that are represented by a decreasing function and others like *medium* that are represented by a non-monotonic function.

In order to obtain the results of this section, we need to define the values $\alpha_1^{Ev}$, $\alpha_2^{Ev}$, $\beta_1^{Ev}$, and $\beta_2^{Ev}$, which are associated with $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)$, where $Ev \in \mathcal{E}$.

**Definition 6.** Let $Ev \in \mathcal{E}$. If $POS^{Ev}_{(\alpha,\beta)}(X), NEG^{Ev}_{(\alpha,\beta)}(X), BND^{Ev}_{(\alpha,\beta)}(X) \neq \emptyset$, then we put

(i) $\alpha_1^{Ev} = \max\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND^{Ev}_{(\alpha,\beta)}(X) \right\}$,

(ii) $\alpha_2^{Ev} = \min\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in POS^{Ev}_{(\alpha,\beta)}(X) \right\}$,

(iii) $\beta_1^{Ev} = \max\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in NEG^{Ev}_{(\alpha,\beta)}(X) \right\}$,

(iv) $\beta_2^{Ev} = \min\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND^{Ev}_{(\alpha,\beta)}(X) \right\}$.

[3] Recall that in classical Rough Set Theory, the rough set $(\mathcal{L}(X), \mathcal{U}(X))$ represents an approximation of $X$.

**Example 4.** Consider the universe $U$, its subset $X_{Sport}$, and the pair of thresholds $(\alpha, \beta)$ that are defined by the example in Subsection 3.2. Then, the corresponding positive, negative, and boundary regions are the following:

$$POS^{\neg Sm}_{(0.8,0.2)}(X_{Sport}) = C_3 \cup C_4 \cup C_5,\ NEG^{\neg Sm}_{(0.8,0.2)}(X_{Sport}) = C_1,\ \text{ and }\ BND^{\neg Sm}_{(0.8,0.2)}(X_{Sport}) = C_2 \cup C_6.$$

Hence, by (11), we get[4]

$$\left\{ \frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in POS^{Ev}_{(\alpha,\beta)}(X_{Sport}) \right\} = \left\{ \frac{|X_{Sport} \cap C_3|}{|C_3|}, \frac{|X_{Sport} \cap C_4|}{|C_4|}, \frac{|X_{Sport} \cap C_5|}{|C_5|} \right\} = \{0.4, 0.6, 0.8\};$$

$$\left\{ \frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in NEG^{Ev}_{(\alpha,\beta)}(X_{Sport}) \right\} = \left\{ \frac{|X_{Sport} \cap C_1|}{|C_1|} \right\} = \{0\};$$

$$\left\{ \frac{|X_{Sport} \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND^{Ev}_{(\alpha,\beta)}(X_{Sport}) \right\} = \left\{ \frac{|X_{Sport} \cap C_2|}{|C_2|}, \frac{|X_{Sport} \cap C_6|}{|C_6|} \right\} = \{0.2, 0.14\}.$$

Finally, by Definition 6, $\alpha_1^{\neg Sm} = \max\{0.14, 0.2\} = 0.2$, $\alpha_2^{\neg Sm} = \min\{0.4, 0.6, 0.8\} = 0.4$, $\beta_1^{\neg Sm} = \max\{0\} = 0$, and $\beta_2^{\neg Sm} = \min\{0.14, 0.2\} = 0.14$.

If $Ev$ is an increasing function, namely $Ev \in \mathcal{E}^+$, then we can order $\beta_1^{Ev}$, $\beta_2^{Ev}$, $\alpha_1^{Ev}$, and $\alpha_2^{Ev}$ as shown in the following proposition.

**Proposition 1.** *Let $Ev \in \mathcal{E}^+$. If $POS^{Ev}_{(\alpha,\beta)}(X), NEG^{Ev}_{(\alpha,\beta)}(X), BND^{Ev}_{(\alpha,\beta)}(X) \neq \emptyset$, then $0 \le \beta_1^{Ev} < \beta_2^{Ev} \le \alpha_1^{Ev} < \alpha_2^{Ev} \le 1$.*

**Proof.** By Definition 6, it is trivial that $0 \le \beta_1^{Ev}, \beta_2^{Ev}, \alpha_1^{Ev}, \alpha_2^{Ev} \le 1$.

($\beta_1^{Ev} < \beta_2^{Ev}$). By Definition 6 ((iii) and (iv)), $\beta_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$ with $x_1 \in NEG^{Ev}_{(\alpha,\beta)}(X)$ and $\beta_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$ with $x_2 \in BND^{Ev}_{(\alpha,\beta)}(X)$. Then, by Definition 4 ((ii) and (iii)), $Ev(\beta_1^{Ev}) \le \beta$ and $\beta < Ev(\beta_2^{Ev}) < \alpha$. Hence, $Ev(\beta_1^{Ev}) < Ev(\beta_2^{Ev})$. Thus, considering that $Ev$ is an increasing function, we can conclude that $\beta_1^{Ev} < \beta_2^{Ev}$.

**($\alpha_1^{Ev} < \alpha_2^{Ev}$).** By Definition 6 ((i) and (ii)), $\alpha_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$ with $x_1 \in BND^{Ev}_{(\alpha,\beta)}(X)$ and $\alpha_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$ with $x_2 \in POS^{Ev}_{(\alpha,\beta)}(X)$. Thus, by Definition 4 ((iii) and (i)), $\beta < Ev(\alpha_1^{Ev}) < \alpha$ and $Ev(\alpha_2^{Ev}) \ge \alpha$. Thus, $Ev(\alpha_1^{Ev}) < Ev(\alpha_2^{Ev})$. Hence, considering that $Ev$ is an increasing function, $\alpha_1^{Ev} < \alpha_2^{Ev}$.

**($\beta_2^{Ev} \le \alpha_1^{Ev}$).** By Definition 6 ((i) and (iv)), $\alpha_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$ with $x_1 \in BND^{Ev}_{(\alpha,\beta)}(X)$ and $\beta_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$ with $x_2 \in BND^{Ev}_{(\alpha,\beta)}(X)$. Therefore, $\beta_2^{Ev}$ and $\alpha_1^{Ev}$ are respectively the minimum and the maximum of $\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND^{Ev}_{(\alpha,\beta)}(X) \right\}$, $\beta_2^{Ev} \le \alpha_1^{Ev}$ clearly holds. □

**Example 5.** In Example 4, we have shown that $\alpha_1^{\neg Sm} = 0.2$, $\alpha_2^{\neg Sm} = 0.4$, $\beta_1^{\neg Sm} = 0$, and $\beta_2^{\neg Sm} = 0.14$. Then, according to Proposition 1, $0 \le \beta_1^{Ev} < \beta_2^{Ev} \le \alpha_1^{Ev} < \alpha_2^{Ev} \le 1$.

The next theorems show that the three regions generated by $Ev \in \mathcal{E}^+$ can be also obtained by using the probabilistic approach and changing the initial thresholds. We separately analyze the following cases: all three regions are non-empty (Theorem 1) and only one of the three regions is empty (Theorems 2-4). The remaining case where only one region is non-empty (namely, one of the three regions coincides with the universe) is omitted because not significant.

**Theorem 1.** *Let $Ev \in \mathcal{E}^+$ such that $POS^{Ev}_{(\alpha,\beta)}(X), NEG^{Ev}_{(\alpha,\beta)}(X), BND^{Ev}_{(\alpha,\beta)}(X) \neq \emptyset$ and let $\alpha', \beta' \in [0,1]$ with $\beta' < \alpha'$. Then,*

$$\mathcal{T}^{Ev}_{(\alpha,\beta)}(X) = \mathcal{T}_{(\alpha',\beta')}(X) \text{ if and only if } \alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}] \text{ and } \beta' \in [\beta_1^{Ev}, \beta_2^{Ev}).$$

**Proof.** ($\Leftarrow$). Let $\alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}]$ and let $\beta' \in [\beta_1^{Ev}, \beta_2^{Ev})$, we need to prove that $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$, $NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$, and $BND^{Ev}_{(\alpha,\beta)}(X) = BND_{(\alpha',\beta')}(X)$.

**($POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$).** Let $\bar{x} \in POS^{Ev}_{(\alpha,\beta)}(X)$, then $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \ge \alpha_2^{Ev}$ from Definition 6 (ii). Moreover, $\alpha' \le \alpha_2^{Ev}$ because $\alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}]$. Consequently, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \ge \alpha'$. Finally, $\bar{x} \in POS_{(\alpha',\beta')}(X)$ from Definition 1 (i).

[4] Recall that the equivalence classes of $\{[x]_{\mathcal{R}} \mid x \in U\}$ are the sets $C_1, C_2, C_3, C_4, C_5$, and $C_6$.

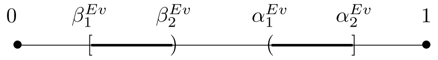


**Fig. 2.** Intervals $[\beta_1^{Ev}, \beta_2^{Ev})$ and $(\alpha_1^{Ev}, \alpha_2^{Ev}]$.

Let $\bar{x} \in POS_{(\alpha',\beta')}(X)$, then $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq \alpha'$ from Definition 1 (i). By the previous inequality and by $\alpha' > \alpha_1^{Ev}$, we get $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \alpha_1^{Ev}$. Hence, considering that $\alpha_1^{Ev}$ is the maximum of $\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in BND^{Ev}_{(\alpha,\beta)}(X) \right\}$ (see Definition 6(i)), we are sure that $\bar{x} \notin BND^{Ev}_{(\alpha,\beta)}(X)$. Furthermore, $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \alpha_1^{Ev}$ and $\beta_1^{Ev} < \alpha_1^{Ev}$ (see Proposition 1) imply that $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} > \beta_1^{Ev}$. Thus, considering that $\beta_1^{Ev}$ is the maximum of $\left\{ \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \mid x \in NEG^{Ev}_{(\alpha,\beta)}(X) \right\}$ (see Definition 6(iii)), we have $\bar{x} \notin NEG^{Ev}_{(\alpha,\beta)}(X)$. Ultimately, by (10), $\bar{x} \in POS^{Ev}_{(\alpha,\beta)}(X)$.

**($BND^{Ev}_{(\alpha,\beta)}(X) = BND_{(\alpha',\beta')}(X)$).** Let $\bar{x} \in BND^{Ev}_{(\alpha,\beta)}(X)$. By Definition 6 ((i) and (iv)), $\beta_2^{Ev} \leq \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \leq \alpha_1^{Ev}$. Moreover, by hypothesis, $\beta' < \beta_2^{Ev}$ and $\alpha' > \alpha_1^{Ev}$. Thus, we can conclude that $\beta' < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha'$, namely $\bar{x} \in BND_{(\alpha',\beta')}(X)$ from Definition 1 (iii).

Let $\bar{x} \in BND_{(\alpha',\beta')}(X)$, then $\beta' < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha'$ from Definition 1 (iii). By hypothesis, $\beta_1^{Ev} \leq \beta'$ and $\alpha' \leq \alpha_2^{Ev}$. Hence, we know that $\beta_1^{Ev} < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha_2^{Ev}$. By Definition 6 (iii), $\beta_1^{Ev} < \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|}$ implies that $\bar{x} \notin NEG^{Ev}_{(\alpha,\beta)}(X)$. Furthermore, by Definition 6 (ii), $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} < \alpha_2^{Ev}$ implies that $\bar{x} \notin POS^{Ev}_{(\alpha,\beta)}(X)$. Ultimately, by (10), $\bar{x} \in BND^{Ev}_{(\alpha,\beta)}(X)$.

**($NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$).** We have previously shown that $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$ and $BND^{Ev}_{(\alpha,\beta)}(X) = BND_{(\alpha',\beta')}(X)$. Thus, by (3) and (10), it is clear that $NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$.

($\Rightarrow$). Let $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X) = \mathcal{T}_{(\alpha',\beta')}(X)$, we intend to prove that $\beta_1^{Ev} \leq \beta' < \beta_2^{Ev}$ and $\alpha_1^{Ev} < \alpha' \leq \alpha_2^{Ev}$.

**($\alpha' \leq \alpha_2^{Ev}$).** Let $x_2 \in U$ such that $\alpha_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$. By Definition 6 (ii), $x_2 \in POS^{Ev}_{(\alpha,\beta)}(X)$. Hence, $\alpha' > \alpha_2^{Ev}$ means that $\frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|} < \alpha'$. Thus, $x_2 \notin POS_{(\alpha',\beta')}(X)$ from Definition 1 (i). This contradicts that $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$. Thus, it must be true that $\alpha' \leq \alpha_2^{Ev}$.

**($\alpha_1^{Ev} < \alpha'$).** Let $x_1 \in U$ such that $\alpha_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$. By Definition 6 (i), $x_1 \in BND^{Ev}_{(\alpha,\beta)}(X)$. If $\alpha_1^{Ev} \geq \alpha'$, then $\frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|} \geq \alpha'$. Therefore, $x_1 \in POS_{(\alpha',\beta')}(X)$ from Definition 1 (i). This contradicts that $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$. Thus, it must be true that $\alpha_1^{Ev} < \alpha'$.

**($\beta_1^{Ev} \leq \beta'$).** Let $x_1 \in U$ such that $\beta_1^{Ev} = \frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|}$. By Definition 6(iii), $x_1 \in NEG^{Ev}_{(\alpha,\beta)}(X)$. If $\beta_1^{Ev} > \beta'$, then $\frac{|X \cap [x_1]_{\mathcal{R}}|}{|[x_1]_{\mathcal{R}}|} > \beta'$, which implies that $x_1 \notin NEG_{(\alpha',\beta')}(X)$ from Definition 1(ii). This contradicts that $NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$. Thus, it must be true that $\beta_1^{Ev} \leq \beta'$.

**($\beta' < \beta_2^{Ev}$).** Let $x_2 \in U$ such that $\beta_2^{Ev} = \frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|}$. By Definition 6(iv), $x_2 \in BND^{Ev}_{(\alpha,\beta)}(X)$. Also, if $\beta' \geq \beta_2^{Ev}$, then $\frac{|X \cap [x_2]_{\mathcal{R}}|}{|[x_2]_{\mathcal{R}}|} \leq \beta'$, which implies that $x_2 \in NEG_{(\alpha',\beta')}(X)$ from Definition 1 (ii). This contradicts that $BND^{Ev}_{(\alpha,\beta)}(X) = BND_{(\alpha',\beta')}(X)$. Thus, it must be true that $\beta' < \beta_2^{Ev}$. □

**Remark 6.** Let us represent the intervals that contain $\alpha'$ and $\beta'$ (i.e. the values for generating $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)$ with probabilistic rough sets) by Fig. 2. As an immediate consequence of Definition 6, these intervals separate $POS^{Ev}_{(\alpha,\beta)}(X)$ from $BND^{Ev}_{(\alpha,\beta)}(X)$ and $NEG^{Ev}_{(\alpha,\beta)}(X)$ from $BND^{Ev}_{(\alpha,\beta)}(X)$. Indeed, the three regions can be rewritten as follows:

- $NEG^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \in [0, \beta_1^{Ev}) \right\}$ from Definition 6(iii);
- $BND^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \in [\beta_2^{Ev}, \alpha_1^{Ev}] \right\}$ from Definition 6 (i) and (iv);
- $POS^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \in (\alpha_2^{Ev}, 1] \right\}$ from Definition 6 (ii).

Consequently, $x \in U$ belongs to

- the $(\alpha, \beta)$-linguistic negative region when $\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$ falls at the left of $[\beta_1^{Ev}, \beta_2^{Ev})$;

- the $(\alpha,\beta)$-linguistic boundary region when $\dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$ falls at the right of $[\beta_1^{Ev},\beta_2^{Ev})$ and at the left of $(\alpha_1^{Ev},\alpha_2^{Ev}]$;
- the $(\alpha,\beta)$-linguistic positive region when $\dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}$ falls at the right of $(\alpha_1^{Ev},\alpha_2^{Ev}]$.

Therefore, connecting to the probabilistic 3WD model, we can view

- $POS^{Ev}_{(\alpha,\beta)}(X)$ as the collection of all elements so that the conditional probability of $X$ given $[x]_{\mathcal{R}}$ is less than $\beta_1^{Ev}$;
- $BND^{Ev}_{(\alpha,\beta)}(X)$ as the collection of all elements so that the conditional probability of $X$ given $[x]_{\mathcal{R}}$ is between or equal to $\beta_2^{Ev}$ and $\alpha_1^{Ev}$;
- $NEG^{Ev}_{(\alpha,\beta)}(X)$ as the collection of all elements so that the conditional probability of $X$ given $[x]_{\mathcal{R}}$ is greater than $\alpha_2^{Ev}$.

**Example 6.** Consider the example given by Subsection 3.2, $\neg Sm$ is an increasing function and all the three regions of $\mathcal{T}^{\neg Sm}_{(0.8,0.2)}(X_{Sport})$ are non-empty. In Example 4, we have found that $\alpha_1^{\neg Sm}=0.2$, $\alpha_2^{\neg Sm}=0.4$, $\beta_1^{\neg Sm}=0$, and $\beta_2^{\neg Sm}=0.14$. Therefore, according to Theorem 1, we get $\mathcal{T}^{\neg Sm}_{(0.8,0.2)}(X_{Sport})=\mathcal{T}_{(\alpha',\beta')}(X_{Sport})$ for each $(\alpha',\beta')$ such that $\alpha'\in(0.2,0.4]$ and $\beta'\in[0,0.14)$

For example, we can easily verify that $\mathcal{T}^{\neg Sm}_{(0.8,0.2)}(X_{Sport})=\mathcal{T}_{(0.3,0.1)}(X_{Sport})$. Indeed, by (11) and by Definition 1,

- $POS_{(0.3,0.1)}(X_{Sport})=\left\{x\in U \mid \dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\geq 0.3\right\}=C_3\cup C_4\cup C_5$,
- $NEG_{(0.3,0.1)}(X_{Sport})=\left\{x\in U \mid \dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\leq 0.1\right\}=C_1$, and
- $BND_{(0.3,0.1)}(X_{Sport})=\left\{x\in U \mid 0.1<\dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}<0.3\right\}=C_2\cup C_6$.

By Theorem 1, we can connect linguistic rough sets with classical rough sets. More precisely, the following corollary holds.

**Corollary 1.** *Let $Ev\in\mathcal{E}^+$ with $POS^{Ev}_{(\alpha,\beta)}(X), NEG^{Ev}_{(\alpha,\beta)}(X), BND^{Ev}_{(\alpha,\beta)}(X)\neq\emptyset$. Then,*

$$(\mathcal{L}^{Ev}_{(\alpha,\beta)}(X),\mathcal{U}^{Ev}_{(\alpha,\beta)}(X))=(\mathcal{L}(X),\mathcal{U}(X))^{5} \text{ if and only if } \beta_1^{Ev}=0 \text{ and } \alpha_2^{Ev}=1.$$

**Proof.** (⇒). Suppose that $(\mathcal{L}^{Ev}_{(\alpha,\beta)}(X),\mathcal{U}^{Ev}_{(\alpha,\beta)}(X))$ is the rough set of $X$ determined by $X$. Then, by (4), let $x\in U$, $x\in POS^{Ev}_{(\alpha,\beta)}(X)$ if and only if $[x]_{\mathcal{R}}\subseteq X$. The latter means that $\dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}=1$ for each $x\in POS^{Ev}_{(\alpha,\beta)}(X)$. Hence, by Definition 6 (ii), $\alpha_2^{Ev}=1$.

By (4), let $x\in U$, $x\in POS^{Ev}_{(\alpha,\beta)}(X)\cup BND^{Ev}_{(\alpha,\beta)}(X)$ if and only if $[x]_{\mathcal{R}}\cap X\neq\emptyset$. Since $POS^{Ev}_{(\alpha,\beta)}(X)\cup BND^{Ev}_{(\alpha,\beta)}(X)=U\setminus NEG^{Ev}_{(\alpha,\beta)}(X)$, we know that $\dfrac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}=0$ for each $x\in NEG^{Ev}_{(\alpha,\beta)}(X)$. Finally, by Definition 6 (iii), $\beta_1^{Ev}=0$.

(⇐). Suppose that $\alpha_2^{Ev}=1$ and $\beta_1^{Ev}=0$. Trivially, $\alpha_2^{Ev}\in(\alpha_1^{Ev},\alpha_2^{Ev}]$ and $\beta_1^{Ev}\in[\beta_1^{Ev},\beta_2^{Ev})$. Then, by Theorem 1, $(POS^{Ev}_{(\alpha,\beta)}(X), POS^{Ev}_{(\alpha,\beta)}(X)\cup BND^{Ev}_{(\alpha,\beta)}(X))=(POS_{(1,0)}(X),POS_{(1,0)}(X)\cup BND_{(1,0)}(X))$. Moreover, by Remark 1, $(\mathcal{L}_{(1,0)}(X),\mathcal{U}_{(1,0)}(X))=(POS_{(1,0)}(X), POS_{(1,0)}(X)\cup BND_{(1,0)}(X))$ coincides with the rough set $(\mathcal{L}(X),\mathcal{U}(X))$ given by (4). □

**Example 7.** Consider the universe $U=\{u_1,\dots,u_{20}\}$ and the evaluative expression *very big*, which is modelled by (6). We suppose that $U$ is partitionated into three equivalence classes: $C_1=\{u_1,\dots,u_5\}$, $C_2=\{u_6,\dots,u_{10}\}$, and $C_3=\{u_{11},\dots,u_{20}\}$. If $X=\{u_6,\dots,u_{19}\}$, we can simply prove that $(\mathcal{L}^{BiVe}_{(0.7,0.3)}(X),\mathcal{E}^{BiVe}_{(0.7,0.3)}(X))=(\mathcal{L}(X),\mathcal{U}(X))$. Indeed, we get

$$\frac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}=\begin{cases}0 & \text{if } x\in C_1,\\ 0.9 & \text{if } x\in C_3,\\ 1 & \text{if } x\in C_2.\end{cases} \text{ and } BiVe\left(\frac{|X\cap[x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|}\right)=\begin{cases}0 & \text{if } x\in C_1,\\ 0.59 & \text{if } x\in C_3,\\ 1 & \text{if } x\in C_2.\end{cases}$$

Thus,

$$POS^{BiVe}_{(0.7,0.3)}(X)=\bigcup\left\{C_i \mid i\in\{1,2,3\} \text{ and } BiVe\left(\frac{|X\cap C_i|}{|C_i|}\right)\geq 0.7\right\}=C_2;$$
$$NEG^{BiVe}_{(0.7,0.3)}(X)=\bigcup\left\{C_i \mid i\in\{1,2,3\} \text{ and } BiVe\left(\frac{|X\cap C_i|}{|C_i|}\right)\leq 0.3\right\}=C_1;$$
$$BND^{BiVe}_{(0.7,0.3)}(X)=\bigcup\left\{C_i \mid i\in\{1,2,3\} \text{ and } 0.3<BiVe\left(\frac{|X\cap C_i|}{|C_i|}\right)<0.7\right\}=C_3.$$

[5] Recall that $(\mathcal{L}^{Ev}_{(\alpha,\beta)}(X),\mathcal{U}^{Ev}_{(\alpha,\beta)}(X))$ is the linguistic rough set of $X$ given by Definition 5 and $(\mathcal{L}(X),\mathcal{U}(X))$ is the classical rough set of $X$ given by Eq. (4).

Also, by Definition 6, $\beta_1^{BiVe} = 0$, $\alpha_2^{BiVe} = 1$, $\alpha_1^{BiVe} = \beta_2^{BiVe} = 0.9$. Since the hypothesis of the previous corollary is satisfied, we expect that $(\mathcal{L}(X), \mathcal{U}(X)) = (\mathcal{L}^{BiVe}_{(0.7,0.3)}(X), \mathcal{U}^{BiVe}_{(0.7,0.3)}(X)) = (C_2, C_2 \cup C_3)$. We can immediately verify that this is true: $\mathcal{L}(X) = C_2$ because $C_2$ is the unique class among $C_1, C_2$, and $C_3$ that is included in $X$; moreover, $\mathcal{U}(X) = C_2 \cup C_3$ because $X \cap C_2 \neq \emptyset$ and $X \cap C_3 \neq \emptyset$, while $X \cap C_1 = \emptyset$.

We are now going to deal with the cases where one of $BND^{Ev}_{(\alpha,\beta)}$, $POS^{Ev}_{(\alpha,\beta)}$, $NEG^{Ev}_{(\alpha,\beta)}$ is empty.

**Theorem 2.** *Let $Ev \in \mathcal{E}^+$ such that $BND^{Ev}_{(\alpha,\beta)} = \emptyset$ and $POS^{Ev}_{(\alpha,\beta)}, NEG^{Ev}_{(\alpha,\beta)} \neq \emptyset$. Let $\alpha', \beta' \in [0,1]$ such that $\beta' < \alpha'$. Then,*

$$\mathcal{T}^{Ev}_{(\alpha,\beta)}(X) = \mathcal{T}_{(\alpha',\beta')}(X) \text{ if and only if } \alpha', \beta' \in [\beta_1^{Ev}, \alpha_2^{Ev}].$$

**Proof.** ($\Leftarrow$). Let $\alpha', \beta' \in [0,1]$ such that $\beta_1^{Ev} \le \beta' < \alpha' \le \alpha_2^{Ev}$, we need to prove that $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$, $NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$, and $BND^{Ev}_{(\alpha,\beta)}(X) = BND_{(\alpha',\beta')}(X)$.

($POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$). Let $\bar{x} \in POS^{Ev}_{(\alpha,\beta)}(X)$. Then, $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} \ge \alpha_2^{Ev}$ from Definition 6 (ii). By hypothesis, $\alpha' \le \alpha_2^{Ev}$. Finally, by the previous two inequalities, we obtain that $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} \ge \alpha'$. Namely, $\bar{x} \in POS_{(\alpha',\beta')}(X)$ from Definition 1 (i).

Let $\bar{x} \in POS_{(\alpha',\beta')}(X)$. Then, $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} \ge \alpha'$ by Definition 1 (i). Moreover, by hypothesis $\beta_1^{Ev} < \alpha'$ (notice that $\beta_1^{Ev} \le \beta' < \alpha'$). So, by the previous two inequalities, we get $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} > \beta_1^{Ev}$. Then, by Definition 6 (iii), $\bar{x} \notin NEG^{Ev}_{(\alpha,\beta)}(X)$. Lastly, by (10) and by the hypothesis $BND^{Ev}_{(\alpha,\beta)}(X) = \emptyset$, we can conclude that $\bar{x} \in POS^{Ev}_{(\alpha,\beta)}(X)$.

($NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$). Let $\bar{x} \in NEG^{Ev}_{(\alpha,\beta)}(X)$. Then, by Definition 6 (iii), $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} \le \beta_1^{Ev}$. Additionally, we know that $\beta_1^{Ev} \le \beta'$ from hypothesis. Then, $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} \le \beta'$, namely $\bar{x} \in NEG_{(\alpha',\beta')}(X)$ from Definition 1 (ii).

Let $\bar{x} \in NEG_{(\alpha',\beta')}(X)$, then $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} \le \beta'$ from Definition 1 (ii). By hypothesis, $\beta' < \alpha_2^{Ev}$ (notice that $\beta' < \alpha' \le \alpha_2^{Ev}$). By the last two inequalities, $\frac{|X \cap [\bar{x}]_\mathcal{R}|}{|[\bar{x}]_\mathcal{R}|} < \alpha_2^{Ev}$. Thus, by Definition 6 (ii), $\bar{x}$ cannot belong to $POS^{Ev}_{(\alpha,\beta)}(X)$. Therefore, by (10) and the hypothesis $BND^{Ev}_{(\alpha,\beta)}(X) = \emptyset$, we can deduce that $\bar{x} \in NEG^{Ev}_{(\alpha,\beta)}(X)$.

($BND^{Ev}_{(\alpha,\beta)}(X) = BND_{(\alpha',\beta')}(X)$). This equality follows from $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$ and $NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$, considering that the sets $POS^{Ev}_{(\alpha,\beta)}(X), NEG^{Ev}_{(\alpha,\beta)}(X)$, and $BND^{Ev}_{(\alpha,\beta)}(X)$ (as well as $POS_{(\alpha',\beta')}(X)$, $NEG_{(\alpha',\beta')}(X)$, and $BND_{(\alpha',\beta')}(X)$) cover the universe $U$ (see (3) and (10)).

($\Rightarrow$). Let $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X) = \mathcal{T}_{(\alpha',\beta')}(X)$, we intend to prove that $\beta_1^{Ev} \le \beta'$ and $\alpha' \le \alpha_2^{Ev}$.

($\beta_1^{Ev} \le \beta'$). Let $x_1 \in U$ such that $\beta_1^{Ev} = \frac{|X \cap [x_1]_\mathcal{R}|}{|[x_1]_\mathcal{R}|}$. Of course, $x_1 \in NEG^{Ev}_{(\alpha,\beta)}(X)$ from Definition 6 (iii).

It is clear that the inequality $\beta_1^{Ev} > \beta'$ leads to a contradiction:

- if $\beta_1^{Ev} > \beta'$, then $x_1 \notin NEG_{(\alpha',\beta')}(X)$ from Definition 1 (ii);
- but, this contradicts that $NEG^{Ev}_{(\alpha,\beta)}(X) = NEG_{(\alpha',\beta')}(X)$.

Thus, $\beta_1^{Ev} \le \beta'$ must hold.

($\alpha' \le \alpha_2^{Ev}$). Let $x_2 \in U$ such that $\alpha_2^{Ev} = \frac{|X \cap [x_2]_\mathcal{R}|}{|[x_2]_\mathcal{R}|}$. Then, $x_2 \in POS^{Ev}_{(\alpha,\beta)}(X)$ from Definition 6 (ii).

It is clear that the inequality $\alpha' > \alpha_2^{Ev}$ leads to a contradiction:

- if $\alpha' > \alpha_2^{Ev}$, then $x_2 \notin POS_{(\alpha',\beta')}(X)$ from Definition 1 (i);
- but, this contradicts that $POS^{Ev}_{(\alpha,\beta)}(X) = POS_{(\alpha',\beta')}(X)$.

Finally, $\alpha' \le \alpha_2^{Ev}$ must hold. □

Examples of evaluative expressions satisfying the hypothesis of Theorem 2 can be obtained from the class defined by (8). Indeed, let $t \in [0,1]$, $\Delta_t$ is trivially an increasing function (i.e. $\Delta_t \in \mathcal{E}^+$) and the boundary region determined by $\Delta_t$ is always empty as shown by the following proposition. In addition, in Proposition 2, the formula of the three regions that are related to $\Delta_t$ is rewritten so that the thresholds $\alpha$ and $\beta$ do not appear in it.

**Proposition 2.** *Let $t \in [0,1]$ and let $\alpha, \beta \in [0,1]$ such that $\beta < \alpha$, then*

*(a)* $POS^{\Delta_t}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq t \right\}$;

*(b)* $NEG^{\Delta_t}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t \right\}$;

*(c)* $BND^{\Delta_t}_{(\alpha,\beta)}(X) = \emptyset$.

**Proof.** (a). Let $\bar{x} \in U$. Thus, $\bar{x} \in POS^{\Delta_t}_{(\alpha,\beta)}(X)$ if and only if

$$\Delta_t \left( \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \right) \geq \alpha \tag{13}$$

from Definition 4 (i).

By (8), the inequality (13) is true if and only if $\frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \geq t$.

(b). Let $\bar{x} \in U$. Then, $\bar{x} \in NEG^{\Delta_t}_{(\alpha,\beta)}(X)$ if and only if

$$\Delta_t \left( \frac{|X \cap [\bar{x}]_{\mathcal{R}}|}{|[\bar{x}]_{\mathcal{R}}|} \right) \leq \beta \tag{14}$$

from Definition 4 (ii).

By (8), the inequality (14) is true if and only if $\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t$.

(c). Notice that $\left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq t \right\} \cup \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t \right\} = U$. Moreover, we have proved that $POS^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \geq t \right\}$ and $NEG^{Ev}_{(\alpha,\beta)}(X) = \left\{ x \in U \mid \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} < t \right\}$. Hence, since $\mathcal{T}^{Ev}_{(\alpha,\beta)}(X)$ is a tri-partition of $U$, $BND^{Ev}_{(\alpha,\beta)}(X)$ must be empty. □

**Example 8.** Let us focus on $\mathcal{T}^{\Delta_{0.5}}_{(\alpha,\beta)}(X)$, where $U$, $X$, and $\mathcal{R}$ are defined in Example 7. By Proposition 2, it is easy to verify that $POS^{\Delta_{0.5}}_{(\alpha,\beta)}(X) = C_2 \cup C_3$, $NEG^{\Delta_{0.5}}_{(\alpha,\beta)}(X) = C_1$, and $BND^{\Delta_{0.5}}_{(\alpha,\beta)}(X) = \emptyset$ for each $\alpha, \beta \in [0,1]$ with $\beta < \alpha$. Furthermore, according to Theorem 2, $POS_{(\alpha',\beta')}(X) = C_2 \cup C_3$, $NEG_{(\alpha',\beta')}(X) = C_1$, and $BND_{(\alpha',\beta')}(X) = \emptyset$, for each $\alpha', \beta' \in [0,1]$ such that $\beta_1^{\Delta_{0.5}} \leq \beta' < \alpha' \leq \alpha_2^{\Delta_{0.5}}$, where $\beta_1^{\Delta_{0.5}} = 0$ and $\alpha_2^{\Delta_{0.5}} = 0.9$. For example, if we choose $\alpha' = 0.2$ and $\beta' = 0.7$, we obtain $POS_{(0.7,0.2)}(X) = C_2 \cup C_3$, $NEG_{(0.7,0.2)}(X) = C_1$, and $BND_{(0.7,0.2)}(X) = \emptyset$.

Now, let us suppose that the negative region is empty.

**Theorem 3.** *Let $Ev \in \mathcal{E}^+$ such that $NEG^{Ev}_{(\alpha,\beta)} = \emptyset$ and $POS^{Ev}_{(\alpha,\beta)}, BND^{Ev}_{(\alpha,\beta)} \neq \emptyset$. Let $\alpha', \beta' \in [0,1]$ such that $\beta' < \alpha'$. Then,*

*$\mathcal{T}^{Ev}_{(\alpha,\beta)}(X) = \mathcal{T}_{(\alpha',\beta')}(X)$ if and only if $\beta' \in [0, \beta_2^{Ev})$ and $\alpha' \in (\alpha_1^{Ev}, \alpha_2^{Ev}]$.*

**Proof.** The proof is similar to that of Theorems 1 and 2. Therefore, it is omitted. □

**Example 9.** Let $U = \{u_1, \ldots, u_{30}\}$. We supposed that $U$ is divided into the following equivalence classes: $C_1 = \{u_1, \ldots, u_5\}$, $C_2 = \{u_6, \ldots, u_{10}\}$, and $C_3 = \{u_{11}, \ldots, u_{30}\}$.

Also, let $X = \{u_1, \ldots u_{28}\}$, we are interested in $\mathcal{T}^{BiVe}_{(0.8,0.4)}(X)$. Then,

$$\frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} = \begin{cases} 0.9 & \text{if } x \in C_3, \\ 1 & \text{if } x \in C_1 \cup C_2, \end{cases}$$

and

$$BiVe \left( \frac{|X \cap [x]_{\mathcal{R}}|}{|[x]_{\mathcal{R}}|} \right) = \begin{cases} 0.59 & \text{if } x \in C_3, \\ 1 & \text{if } x \in C_1 \cup C_2. \end{cases}$$

Thus, by Definition 4, $POS^{BiVe}_{(0.8,0.4)}(X) = C_1 \cup C_2$, $BND^{BiVe}_{(0.8,0.4)}(X) = C_3$, and $NEG^{BiVe}_{(0.8,0.4)}(X) = \emptyset$. Moreover, $\beta_2^{BiVe} = 0.9$, $\alpha_1^{BiVe} = 0.9$, and $\alpha_2^{BiVe} = 1$. Therefore, according to Theorem 3, $POS_{(\alpha',\beta')}(X) = C_1 \cup C_2$, $BND_{(\alpha',\beta')}(X) = C_3$, and $NEG_{(\alpha',\beta')}(X) = \emptyset$ for each $\beta' \in [0, 0.9)$ and $\alpha' \in (0.9, 1]$. For example, if $\alpha' = 0.95$ and $\beta' = 0.8$, then we can easily verify that $POS_{(0.95,0.8)}(X) = C_1 \cup C_2$, $BND_{(0.95,0.8)}(X) = C_3$, and $NEG_{(0.95,0.8)}(X) = \emptyset$.

Finally, the case of an empty positive region.

**Theorem 4.** *Let $Ev \in \mathcal{E}^+$ such that $POS^{Ev}_{(\alpha,\beta)} = \emptyset$ and $NEG^{Ev}_{(\alpha,\beta)}, BND^{Ev}_{(\alpha,\beta)} \neq \emptyset$. Let $\alpha', \beta' \in [0,1]$ such that $\beta' < \alpha'$. Then,*

$$\mathcal{T}^{Ev}_{(\alpha,\beta)}(X) = \mathcal{T}_{(\alpha',\beta')}(X) \text{ if and only if } \beta' \in [\beta_1^{Ev}, \beta_2^{Ev}) \text{ and } \alpha' \in (\alpha_1^{Ev}, 1].$$

**Proof.** The proof is similar to that of Theorems 1 and 2. For this reason, it is omitted. □

**Example 10.** Consider the universe $U$ and the equivalence classes $C_1, C_2$, and $C_3$, which are defined by Example 9. Let $X = \{u_1, u_6, u_{11}, \dots u_{28}\}$, we focus on $\mathcal{T}^{BiVe}_{(0.7,0.2)}(X)$. Then,

$$\frac{|X \cap [x]_\mathcal{R}|}{|[x]_\mathcal{R}|} = \begin{cases} 0.2 & \text{if } x \in C_1 \cup C_2, \\ 0.9 & \text{if } x \in C_3, \end{cases}$$

and

$$BiVe\left(\frac{|X \cap [x]_\mathcal{R}|}{|[x]_\mathcal{R}|}\right) = \begin{cases} 0.59 & \text{if } x \in C_3, \\ 0 & \text{if } x \in C_1 \cup C_2. \end{cases}$$

By Definition 4, $NEG^{BiVe}_{(0.7,0.2)}(X) = C_1 \cup C_2$, $BND^{BiVe}_{(0.7,0.2)}(X) = C_3$, and $POS^{BiVe}_{(0.7,0.2)}(X) = \emptyset$. Also, $\beta_1^{Ev} = 0.2$, $\beta_2^{Ev} = 0.9$, $\alpha_1^{Ev} = 0.9$. Thus, according to Theorem 4, $NEG_{(\alpha',\beta')}(X) = C_1 \cup C_2$, $BND_{(\alpha',\beta')}(X) = C_3$, and $POS_{(\alpha',\beta')}(X) = \emptyset$ for each $\beta' \in [0.2, 0.9)$ and $\alpha' \in (0.9, 1]$. For example, let $(\alpha', \beta') = (0.95, 0.6)$, we can easily verify that $NEG_{(0.95,0.6)}(X) = C_1 \cup C_2$, $BND_{(0.95,0.6)}(X) = C_3$, and $POS_{(0.95,0.6)}(X) = \emptyset$.

**Remark 7.** The inverse problem to that discussed in this section can be easily addressed. Then, given $\alpha, \beta \in [0,1]$ such that $\alpha < \beta$ and $Ev \in \mathcal{E}^+$, we can find a pair of intervals $(I_\alpha, I_\beta)$ so that $\mathcal{T}_{(\alpha,\beta)}(X) = \mathcal{T}^{Ev}_{(\alpha',\beta')}(X)$ for each $\alpha' \in I_\alpha$ and $\beta' \in I_\beta$.

**Remark 8.** As a possible application, we could use our model to compare the pairs of thresholds $(\alpha_{M_1}, \beta_{M_1})$ and $(\alpha_{M_2}, \beta_{M_2})$ determined by two methods $M_1$ and $M_2$, which generate different tri-partitions of the starting universe. In fact, we could find the evaluative expressions $Ev_1$ and $Ev_2$ together with a pair of thresholds $(\alpha, \beta)$ so that

$$\mathcal{T}_{(\alpha_{M_1},\beta_{M_1})}(X) = \mathcal{T}^{Ev_1}_{(\alpha,\beta)}(X) \text{ and } \mathcal{T}_{(\alpha_{M_2},\beta_{M_2})}(X) = \mathcal{T}^{Ev_2}_{(\alpha,\beta)}(X).$$

Therefore, we have the possibility to give a linguistic interpretation to the tri-partitions $\mathcal{T}_{(\alpha_{M_1},\beta_{M_1})}(X)$ and $\mathcal{T}_{(\alpha_{M_2},\beta_{M_2})}(X)$ using a unique pair of thresholds $(\alpha, \beta)$. In this way, it becomes more easy and interpretable to compare the two methods and consequently facilitate the decision to adopt $(\alpha_{M_1}, \beta_{M_1})$ or $(\alpha_{M_2}, \beta_{M_2})$.

In order to better explain this idea, let us come back to the context of Example of Subsection 3.2, where we deal with a set of communities $C_1, \dots, C_n$ of users and the set $X_T$ representing the users interested in sports topics. Recall that the aim is to choose the community to which to propose sports news.

Suppose that $(\alpha_{M_1}, \beta_{M_1}) = (0.7, 0.2)$ and $(\beta_{M_1}, \beta_{M_2}) = (0.9, 0.2)$ derive by two different methods and the corresponding probabilistic positive regions are respectively $C_i \cup C_j \cup C_k$ and $C_i \cup C_j$. Which of the two should we choose for our decision? Can we propose sports news to the community $C_k$? To answer these questions, assume that

$$POS_{(0.7,0.2)}(X_T) = POS^{Ev_1}_{(0.5,0.3)}(X_T) \text{ and } POS_{(0.9,0.2)}(X_T) = POS^{Ev_2}_{(0.5,0.3)}(X_T),$$

where $Ev_1 = \neg Sm$ and $Ev_2 = BiEx$ (i.e., $Ev_1$ and $Ev_2$ are respectively the extensions of the evaluative expressions *not small* and *extremely big*). Then, we take into account the tri-partition of $(0.7, 0.2)$ when we intend to propose sports news to the communities having many users interested in sports topics, whereas we take into account the tri-partition of $(0.9, 0.2)$ when we intend to propose sports news to the communities having almost all users interested in sports topics.

## 5. Conclusions and future directions

This work proposes a novel model for three-way decisions based on the concept of evaluative linguistic expressions. Thus, a new way is provided to divide the initial universe into three regions with the corresponding decision rules. Moreover, our results allow decision-makers to give a linguistic interpretation to the regions already obtained using the probabilistic approach.

As possible directions to continue this work, we firstly need to extend the results of Section 4 to the evaluative expressions that are not necessarily represented by increasing functions. Then, we want to deepen the study of linguistic regions by comparing our methods with those presented in [8]. In addition, we intend to understand how the decisions about the elements change using different evaluative expressions. Finally, we could analyze the logical relations between the linguistic regions determined by a given evaluative expression and investigate their consequences in terms of decisions by constructing a hexagon of opposition.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

No data was used for the research described in the article.

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