

# CONSTRUCTING INFINITELY MANY HALF-ARC-TRANSITIVE COVERS OF TETRAVALENT GRAPHS

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ABSTRACT. We prove that, given a finite graph  $\Sigma$  satisfying some mild conditions, there exist infinitely many tetravalent half-arc-transitive normal covers of  $\Sigma$ . Applying this result, we establish the existence of infinite families of finite tetravalent half-arc-transitive graphs with certain vertex stabilizers, and classify the vertex stabilizers up to order  $2^8$  of finite connected tetravalent half-arc-transitive graphs. This sheds some new light on the longstanding problem of classifying the vertex stabilizers of finite tetravalent half-arc-transitive graphs.

*Key words:* half-arc-transitive; vertex stabilizer; normal quotient; normal cover; concentric group

*MSC2010:* 20B25, 05C20, 05C25

## 1. INTRODUCTION

Let  $\Gamma$  be a graph and let  $G$  be a subgroup of the automorphism group  $\text{Aut}(\Gamma)$  of  $\Gamma$ . We say that  $G$  is *vertex-transitive*, *edge-transitive* or *arc-transitive* if  $G$  acts transitively on the vertex set, edge set or the set of ordered pairs of adjacent vertices, respectively, of  $\Gamma$ . If  $G$  is vertex-transitive and edge-transitive but not arc-transitive, then we say that  $G$  is *half-arc-transitive*. The graph  $\Gamma$  is said to be half-arc-transitive if  $\text{Aut}(\Gamma)$  is half-arc-transitive.

Numerous papers have been published on half-arc-transitive graphs over the last half a century (see the survey papers [5, 11]), most of which are on those of valency 4, the smallest valency of half-arc-transitive graphs. However, somewhat surprisingly, not so many examples of tetravalent half-arc-transitive graphs are known in the literature (see [16]), compared with the considerable attention they have received.

For a graph  $\Gamma$  and a group  $N$  such that  $N$  is normal in  $G$  for some vertex-transitive subgroup  $G$  of  $\text{Aut}(\Gamma)$ , the *normal quotient*  $\Gamma/N$  is the graph whose vertex set  $V(\Gamma/N)$  is the set of  $N$ -orbits on the vertex set  $V(\Gamma)$  of  $\Gamma$ , with an edge of  $\Gamma/N$  between vertices  $\Delta$  and  $\Omega$  if and only if there is an edge of  $\Gamma$  between  $\alpha$  and  $\beta$  for some  $\alpha \in \Delta$  and  $\beta \in \Omega$ . Such a graph  $\Gamma$  is called a *normal cover* of the graph  $\Gamma/N$ . Broadly speaking, in this paper, given a graph  $\Sigma$  satisfying some mild conditions, we establish the existence of infinitely many tetravalent half-arc-transitive graphs that are normal covers of  $\Sigma$ .

Let  $p$  be a prime number. For a positive integer  $m$ , denote the largest power of  $p$  dividing  $m$  by  $m_p$ . Moreover, given a finite group  $X$ , let  $\mathbf{O}_p(X)$  denote the largest normal  $p$ -subgroup of  $X$ . Our main result is as follows.

**Theorem 1.1.** *Let  $\Sigma$  be a finite connected tetravalent graph and let  $T$  be a nonabelian simple half-arc-transitive subgroup of  $\text{Aut}(\Sigma)$ . Then, for each prime number  $p$ , such that  $p > |T|_2$  and  $p$  is coprime to  $|T|$ , there exists a finite connected tetravalent graph  $\Gamma$  satisfying the following:*

- (a)  $\Gamma$  is half-arc-transitive;

- (b)  $\text{Aut}(\Gamma)$  has vertex stabilizer isomorphic to that of  $T$ ;  
(c)  $\mathbf{O}_p(\text{Aut}(\Gamma)) \neq 1$ ,  $\text{Aut}(\Gamma)/\mathbf{O}_p(\text{Aut}(\Gamma)) \cong T$  and  $\Gamma/\mathbf{O}_p(\text{Aut}(\Gamma)) \cong \Sigma$ .

Although it is not hard to construct a graph  $\Gamma$  with a half-arc-transitive group  $G$  of automorphisms, it is in general not known whether  $\text{Aut}(\Gamma)$  is larger than  $G$  to possibly make  $\text{Aut}(\Gamma)$  arc-transitive on  $\Gamma$ . In this sense, the significance of Theorem 1.1 is asserting the existence (under some mild conditions) of infinitely many half-arc-transitive graphs which are normal covers of a given connected tetravalent graph, even if the given graph is not itself half-arc-transitive. Thus, with the help of Theorem 1.1, one can construct infinitely many connected tetravalent half-arc-transitive graphs with some exotic vertex stabilizers, and we will present some examples in this paper.

For a half-arc-transitive graph  $\Gamma$ , the vertex stabilizer in  $\text{Aut}(\Gamma)$  will be called the *vertex stabilizer* of  $\Gamma$ . It is not hard to construct half-arc-transitive graphs with abelian vertex stabilizers (see for instance [12]). However, half-arc-transitive graphs with nonabelian vertex stabilizers are much more elusive and the problem of constructing half-arc-transitive graphs with nonabelian vertex stabilizers has received extensive attention and considerable effort (see for instance [4, 5, 17, 18]). The first infinite family of half-arc-transitive graphs with nonabelian vertex stabilizers was only constructed very recently in [18]. The vertex stabilizers in [18] are isomorphic to  $D_8 \times D_8 \times C_2^{m-6}$  for integers  $m$  with  $m \geq 7$ .

In Example 3.3 we construct a finite connected tetravalent graph  $\Sigma_m$  for every integer  $m \geq 4$  such that  $\Sigma_m$  admits a half-arc-transitive action of the alternating group  $A_{2m}$  with vertex stabilizer  $D_8 \times C_2^{m-3}$ . Then, by applying Theorem 1.1 to the graphs in Example 3.3 and to the graphs in [18], we obtain the following result:

**Theorem 1.2.** *For every integer  $m \geq 4$ , there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer  $D_8 \times C_2^{m-3}$  and, for every integer  $m \geq 7$ , there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer  $D_8 \times D_8 \times C_2^{m-6}$ .*

A group  $H = \langle a_1, \dots, a_m \rangle$  is said to be *concentric* if  $|\langle a_i, \dots, a_j \rangle| = 2^{j-i+1}$  for all  $1 \leq i < j \leq m$  and there exists a group isomorphism

$$\varphi : \langle a_1, \dots, a_{m-1} \rangle \rightarrow \langle a_2, \dots, a_m \rangle$$

such that  $a_i^\varphi = a_{i+1}$  for  $i = 1, \dots, m-1$ . (Note in the definition that each  $a_i$  is necessarily an involution if  $m \geq 3$ .) The study of concentric groups dates back to Glauberman [8, 9] about 50 years ago and was made systematic by Marušič and Nedela [13] in 2001. It was proved in [13] that a group  $H$  is concentric if and only if there exist a connected tetravalent graph  $\Gamma$  and a subgroup  $G$  of  $\text{Aut}(\Gamma)$  such that  $G$  is half-arc-transitive with vertex stabilizer  $H$ . Moreover, Marušič and Nedela gave a characterization of concentric groups in terms of their defining relations [13, Theorem 5.5] and determined the concentric groups of order up to  $2^8$  [13, Theorem 6.3]. Let

$$\begin{aligned} \mathcal{H}_7 = \langle a_1, \dots, a_7 \mid a_i^2 = 1 \text{ for } i \leq 7, (a_i a_j)^2 = 1 \text{ for } |i - j| \leq 4, \\ (a_1 a_6)^2 = a_3, (a_2 a_7)^2 = a_4, (a_1 a_7)^2 = a_5 \rangle. \end{aligned}$$

**Theorem 1.3** (Glauberman-Marušič-Nedela). *The following are precisely the concentric groups of order at most  $2^8$ :*

$$\begin{aligned} C_2^m \text{ for } 1 \leq m \leq 8, \quad D_8 \times C_2^{m-3} \text{ for } 3 \leq m \leq 8, \\ D_8 \times D_8 \times C_2^{m-6} \text{ for } 6 \leq m \leq 8, \quad \mathcal{H}_7 \times C_2^{m-7} \text{ for } 7 \leq m \leq 8. \end{aligned}$$

Marušič [12] has shown that every nontrivial elementary abelian 2-group is the vertex stabilizer of a connected tetravalent half-arc-transitive graph. Similar results have been proved for  $D_8$  by Conder and Marušič [4] and for  $D_8 \times C_2$  by Conder, Potočnik and Šparl [5]. Moreover, the first author showed in [17] that  $D_8 \times D_8$  and  $\mathcal{H}_7$  are both vertex stabilizers of connected tetravalent half-arc-transitive graphs in a response to a problem posed in [13], and the second author recently proved in [18] that  $D_8 \times D_8 \times C_2^{m-6}$  is the vertex stabilizer of a connected tetravalent half-arc-transitive graph for every integer  $m \geq 7$ . In light of these results and Theorem 1.2, we see that the only concentric group of order at most  $2^8$  that is not known to be the vertex stabilizer of a connected tetravalent half-arc-transitive graph is  $\mathcal{H}_7 \times C_2$ . In Example 3.2, we apply Theorem 1.1 to construct connected tetravalent half-arc-transitive graphs with vertex stabilizer  $\mathcal{H}_7 \times C_2$ . This leads to the next theorem.

**Theorem 1.4.** *Every concentric group of order at most  $2^8$  is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs.*

We prove Theorem 1.1 in Section 2. Then in Section 3 we construct some connected tetravalent graphs admitting a half-arc-transitive nonabelian simple group action with vertex stabilizer  $\mathcal{H}_7 \times C_2$  and  $D_8 \times C_2^{m-3}$  for  $m \geq 3$ , respectively, which will be used in Section 4 to prove Theorems 1.2 and 1.4. In Section 5 we briefly discuss the relevance of our work and a conjecture of Džambić-Jones and Conder concerning faithful amalgams. We also include a natural open problem at the end of Section 5.

## 2. PROOF OF THEOREM 1.1

For a group  $X$ , let  $\text{Soc}(X)$  denote the socle of  $X$  and let  $\text{Rad}(X)$  denote the maximal normal solvable subgroup of  $X$ . Let  $\Gamma$  be a graph, let  $G$  be a vertex-transitive subgroup of  $\text{Aut}(\Gamma)$  and let  $N$  be a normal subgroup of  $G$ . Then the group  $G$  induces a vertex-transitive subgroup of  $\text{Aut}(\Gamma/N)$ . Denote by  $\alpha^N$  and  $\beta^N$  the  $N$ -orbits containing the vertices  $\alpha$  and  $\beta$ , respectively, of  $\Gamma$ . If  $\alpha^N$  and  $\beta^N$  are adjacent in  $\Gamma/N$ , then each vertex in  $\alpha^N$  is adjacent to the same number of vertices in  $\beta^N$  (because  $N$  is transitive on both sets). Moreover, the stabilizer in  $G$  of the vertex  $\alpha^N$  in  $\Gamma/N$  is  $G_\alpha N$ .

See [15, Subsection 2.2] for the definition of *regular covering projection*, *lift* and *group of covering transformations*.

*Proof of Theorem 1.1.* Let  $\Sigma$  and  $T$  be as in Theorem 1.1 and let  $p$  be a prime number such that  $p > |T|_2$  and  $p$  is coprime to  $|T|$ . Viewing [15, Corollary 8] and applying [15, Theorem 6] with the prime  $p$ , the graph  $\Sigma$  and the group of automorphisms  $T$ , we obtain a regular covering projection  $\varphi : \Gamma \rightarrow \Sigma$  such that the following hold:

- (i)  $\Gamma$  is finite;
- (ii) the maximal group that lifts along  $\varphi$  is  $T$ ;
- (iii) the group of covering transformations of  $\varphi$  is a  $p$ -group.

Let  $A = \text{Aut}(\Gamma)$ , let  $G$  be the subgroup of  $A$  that  $T$  lifts to along  $\varphi$ , and let  $P$  be the group of covering transformations of  $\varphi$ . Then conclusion (iii) shows that  $P$  is a  $p$ -group, and  $G/P \cong T$  is nonabelian simple. Since  $P$  is a normal solvable subgroup of  $G$ , it follows that  $P = \text{Rad}(G)$ . Moreover, we deduce from conclusion (ii) and [15, Lemma 1] that

$$\mathbf{N}_A(P) = G. \tag{1}$$

Since  $P = \text{Rad}(G)$  is characteristic in  $G$ , we derive that  $P$  is normal in  $\mathbf{N}_A(G)$ , that is,  $\mathbf{N}_A(G) \leq \mathbf{N}_A(P)$ . Thus it follows from (1) that

$$\mathbf{N}_A(G) = G. \quad (2)$$

We aim to prove that  $A = G$ , from which the proof of Theorem 1.1 immediately follows. Assume for a contradiction that  $A > G$ . Then  $G < B$  for some subgroup  $B$  of  $A$  such that  $G$  is maximal in  $B$ .

Let  $\alpha$  be a vertex of  $\Gamma$ . Since  $T$  is half-arc-transitive on  $\Sigma$ , the group  $G$  is half-arc-transitive on  $\Gamma$ . This implies that  $G_\alpha$  is a 2-group and  $B = GB_\alpha$  is edge-transitive and vertex-transitive on  $\Gamma$ . It follows that  $|B : G| = |GB_\alpha : G| = |B_\alpha : G_\alpha|$  divides  $|B_\alpha|$ . As  $B_\alpha$  is a  $\{2, 3\}$ -group and  $p > |T|_2 \geq 5$ , we infer that  $p$  is coprime to  $|B : G|$ . Since  $p$  is coprime to  $|T| = |G/P|$ , we see that  $P$  is a Sylow  $p$ -subgroup of  $B$ . According to Sylow's theorem, the number of Sylow  $p$ -subgroups of  $B$  is  $|B : \mathbf{N}_B(P)| \equiv 1 \pmod{p}$  and so  $p$  divides  $|B : \mathbf{N}_B(P)| - 1$ . By (1) we have  $\mathbf{N}_B(P) = G$ . Hence

$$p \mid (|B : G| - 1). \quad (3)$$

Let  $K$  be the core of  $G$  in  $B$ . Then  $K \trianglelefteq B$ ,  $K \leq G$ , and the action of  $B/K$  on the set  $\Omega$  of right cosets of  $G/K$  in  $B/K$  is faithful and primitive of degree  $|B : G|$ . Since both  $K$  and  $P$  are normal in  $G$ , the group  $KP$  is normal in  $G$ , which implies that  $KP/P$  is normal in  $G/P$ . As  $G/P \cong T$  is a simple group, we deduce that either  $G = KP$  or  $K \leq P$ .

**Case 1.**  $G = KP$ .

In this case,  $P \cap K$  is a normal subgroup of  $K$  with

$$K/(P \cap K) \cong KP/P = G/P \cong T$$

nonabelian simple. Since  $P \cap K$  is solvable, we conclude that

$$P \cap K = \text{Rad}(K)$$

is characteristic in  $K$ . As  $K$  is normal in  $B$ , it follows that

$$P \cap K \trianglelefteq B.$$

Note that  $|G/K| = |KP/K| = |P/(P \cap K)|$  is a power of  $p$  and  $G \neq K$  by (2). We have

$$|G/K| = p^n$$

for some positive integer  $n$ .

Suppose that  $|B_\alpha|$  is divisible by 3. Then  $B_\alpha$  is 2-transitive on the neighborhood of  $\alpha$  in  $\Gamma$ , and so it follows from a result of Gardiner (see for instance [7, Lemma 2.3]) that  $|B_\alpha|$  divides  $2^4 3^6$ . Now  $B/K$  is a primitive group of degree  $|B : G| = |B_\alpha : G_\alpha|$  dividing  $2^3 3^6$  such that the point stabilizer  $G/K$  is a  $p$ -group. We deduce from [10] that  $B/K$  is an affine group of degree  $3^k$  with  $3 \leq k \leq 6$ , and  $\text{Soc}(B/K)$  is the unique Sylow 3-subgroup of  $B/K$ . Since  $B/K = (G/K)(B_\alpha K/K)$  and  $|G/K|$  is coprime to 3, it follows that  $\text{Soc}(B/K) \trianglelefteq B_\alpha K/K \cong B_\alpha/K_\alpha$ . Note that

$$|\text{Soc}(B/K)| = 3^k = |B : G| = |B_\alpha : G_\alpha| = |B_\alpha|_3$$

as  $G_\alpha$  is a 2-group. We conclude that the Sylow 3-subgroup of  $B_\alpha$  is elementary abelian of order  $3^k \geq 3^3$ . The structure of the vertex stabilizer  $B_\alpha$  is described in [14, Table 1], which shows that  $B_\alpha$  cannot have an elementary abelian Sylow 3-subgroup of order at

least  $3^3$ , a contradiction. Thus  $B_\alpha$  is a 2-group, and so  $|B : G| = |B_\alpha : G_\alpha|$  is a power of 2, say,

$$|B : G| = 2^\ell.$$

Note that  $\ell > 1$  by (2).

Since  $|B : G| = 2^\ell$  and  $|G : K| = p^n$ , we see that  $B/K$  has order  $2^\ell p^n$  and thus is solvable. Moreover, as  $B/K$  is a primitive group of degree  $2^\ell$ , it follows that  $\text{Soc}(B/K)$  is an elementary abelian group of order  $2^\ell$ . Let  $H$  be the subgroup of  $B$  such that  $H/K = \text{Soc}(B/K)$ . The reader may find Figure 1 useful at this point.

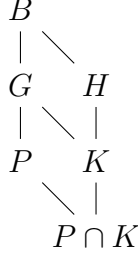


FIGURE 1. The structure of  $B$

Let  $\overline{B} = B/(P \cap K)$ ,  $\overline{H} = H/(P \cap K)$ ,  $\overline{K} = K/(P \cap K)$  and  $\overline{C} = \mathbf{C}_{\overline{H}}(\overline{K})$ . Then  $\overline{K} \cong T$ , and both  $\overline{H}$  and  $\overline{K}$  are normal in  $\overline{B}$ . It follows that  $\overline{C} = \overline{H} \cap \mathbf{C}_{\overline{B}}(\overline{K}) \trianglelefteq \overline{B}$ , and

$$\overline{H}/\overline{C} \lesssim \text{Aut}(\overline{K}) \cong \text{Aut}(T).$$

Moreover,

$$\overline{C}\overline{K}/\overline{C} \cong \overline{K}/(\overline{K} \cap \overline{C}) \cong \text{Inn}(\overline{K}) \cong \text{Inn}(T). \quad (4)$$

Thus  $\overline{H}/(\overline{C}\overline{K}) \lesssim \text{Out}(T)$ . Let  $C$  be the subgroup of  $H$  containing  $P \cap K$  such that  $C/(P \cap K) = \overline{C}$ . Then

$$C \trianglelefteq B$$

and  $H/(CK) \lesssim \text{Out}(T)$ . Now  $CK \trianglelefteq B$  and so  $CK/K \trianglelefteq B/K$ . As  $CK/K \leq H/K$  and  $H/K = \text{Soc}(B/K)$  is a minimal normal subgroup of the affine primitive group  $B/K$ , it follows that either  $CK/K = 1$  or  $CK/K = H/K$ . If  $CK/K = 1$ , then the elementary abelian 2-group  $H/K = H/(CK)$  is isomorphic to a subgroup of  $\text{Out}(T)$ , which implies that

$$|B : G| = 2^\ell = |H/K| \leq |\text{Out}(T)|_2 \leq |T|_2 < p,$$

contradicting (3). (Observe that the inequality  $|\text{Out}(T)|_2 \leq |T|_2$  follows by inspecting the list of finite simple groups.) Therefore,  $CK/K = H/K$  and hence  $H = CK$ . This in turn with (4) implies that

$$H/C = CK/C \cong \overline{C}\overline{K}/\overline{C} \cong T.$$

Note that  $T$  is the unique nonsolvable composition factor of  $H$  as  $H/K$  is solvable and  $K$  is a  $p$ -group extended by  $T$ . We then conclude that

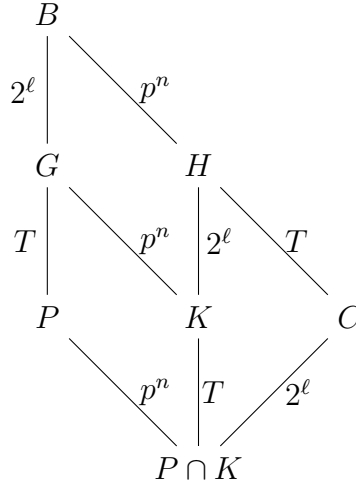
$$C = \text{Rad}(H).$$

Consequently,

$$C \cap K = \text{Rad}(H) \cap K = \text{Rad}(K) = P \cap K$$

and so

$$|C/(P \cap K)| = |C/(C \cap K)| = |CK/K| = |H/K| = 2^\ell.$$

FIGURE 2. More detailed structure of  $B$ 

The reader may find Figure 2 useful at this point.

Consider the quotient graph  $\Gamma/C$ . Let  $N$  be the kernel of  $B$  acting on  $V(\Gamma/C)$ . Since  $H$  is a normal subgroup of  $B$  with index  $p^n$  odd and  $B_\alpha$  is a 2-group, we have  $B_\alpha \leq H$ . Consequently,  $N = CN_\alpha \leq CB_\alpha \leq H$ . Moreover,  $N = CN_\alpha$  is a  $\{2, p\}$ -group and thus is solvable. Hence  $N \leq \text{Rad}(H) = C$ . This shows that the action of  $B/C$  on  $V(\Gamma/C)$  is faithful. Suppose that  $C_\alpha \neq 1$ . Then the number of orbits of  $C_\alpha$  on the neighborhood of  $\alpha$  in  $\Gamma$  is less than 4. It follows that the valency of  $\Gamma/C$  is less than 4 and so must be 1 or 2, being a divisor of 4. Thereby we conclude that  $B/C \leq \text{Aut}(\Gamma/C)$  is solvable, a contradiction. Thus  $C_\alpha = 1$ .

As  $C_\alpha = 1$ , the orbits of  $C$  on  $V(\Gamma)$  have size  $|C|$ . Since  $C$  is normal in  $B$  and  $B$  is transitive on  $V(\Gamma)$ , it follows that  $|C|$  divides  $|V(\Gamma)|$ . Hence  $|C|$  divides  $|G|$  as  $G$  is transitive on  $V(\Gamma)$ . In particular,  $|C|_2 \leq |G|_2$ . As  $|C|_2 = |C/(P \cap K)|_2 = 2^\ell$  and  $|G|_2 = |G/P|_2 = |T|_2$ , we then obtain  $2^\ell \leq |T|_2$ . This together with (3) implies that  $p < |B : G| = 2^\ell \leq |T|_2$ , contradicting our choice of  $p$ .

### Case 2. $K \leq P$ .

Let  $\overline{B} = B/K$ ,  $\overline{G} = G/K$ ,  $\overline{P} = P/K$  and  $\overline{H} = H/K = \text{Soc}(\overline{B})$ . Recall that  $\overline{B}$  acts primitively and faithfully on the set of right cosets of  $\overline{G}$  in  $\overline{B}$ , and

$$|\overline{B} : \overline{G}| = |B : G| = |GB_\alpha : G| = |B_\alpha : G_\alpha|.$$

As  $B_\alpha$  is a  $\{2, 3\}$ -group, we obtain  $|\overline{B} : \overline{G}| = 2^\ell 3^k$  for some nonnegative integers  $\ell$  and  $k$ . If  $|B_\alpha|$  is divisible by 3, then  $B_\alpha$  is 2-transitive on the neighborhood of  $\alpha$  in  $\Gamma$  and so [7, Lemma 2.3] shows that  $|B_\alpha|$  divides  $2^4 3^6$ . Consequently, either  $\ell \leq 3$  and  $1 \leq k \leq 6$ , or  $k = 0$ .

Since  $K$  is normal in  $B$ , we deduce from (1) that  $K \neq P$ . Hence  $K < P$  and so  $\overline{P}$  is a nontrivial  $p$ -group. This shows that  $\overline{G}$  is a nontrivial  $p$ -group extended by the nonabelian simple group  $G/P \cong T$ . Then as  $\overline{G}$  is a point stabilizer of the primitive group  $\overline{B}$  of degree  $|\overline{B} : \overline{G}| = 2^\ell 3^k$ , it follows from [10] that  $k = 0$  and  $\overline{B}$  is an affine primitive group of degree  $2^\ell$ . Hence  $|\overline{H}| = 2^\ell$ , and so  $H$  is a  $\{2, p\}$ -group.

Let  $R = PH$ . Then  $R$  is a  $\{2, p\}$ -group and thus is solvable. Moreover,  $R \trianglelefteq B$ , and as  $P \leq G$ , we have  $B = HG = HPG = RG$ . Hence

$$B/R = RG/R \cong G/(G \cap R) \cong (G/P)/((G \cap R)/P).$$

Since  $G/P \cong T$  is simple, it follows that either  $B/R = 1$  or  $B/R \cong T$ . Clearly,  $B \neq R$  as  $R$  is solvable and  $B$  is nonsolvable. Thus  $B/R \cong T$  is nonabelian simple, which implies

$$R = \text{Rad}(B).$$

Consider the quotient graph  $\Gamma/R$ . Let  $M$  be the kernel of  $B$  acting on  $V(\Gamma/R)$ . Then  $M = RM_\alpha$ . Since  $M_\alpha \leq B_\alpha$  is a 2-group, we see that  $M$  is a  $\{2, p\}$ -group as  $R$  is a  $\{2, p\}$ -group. Accordingly,  $M$  is solvable, and so  $M \leq \text{Rad}(B) = R$ . This shows that the action of  $B/R$  on  $V(\Gamma/R)$  is faithful. Suppose that  $R_\alpha \neq 1$ . Then the number of orbits of  $R_\alpha$  on the neighborhood of  $\alpha$  in  $\Gamma$  is less than 4. It follows that the valency of  $\Gamma/R$  is less than 4 and so must be 1 or 2 as it divides 4. Thereby we conclude that  $B/R \leq \text{Aut}(\Gamma/R)$  is solvable, a contradiction. Thus  $R_\alpha = 1$ .

As  $R_\alpha = 1$ , the orbits of  $R$  on  $V(\Gamma)$  have size  $|R|$ . Since  $R$  is normal in  $B$  and  $B$  is transitive on  $V(\Gamma)$ , it follows that  $|R|$  divides  $|V(\Gamma)|$ . Hence  $|R|$  divides  $|G|$  as  $G$  is transitive on  $V(\Gamma)$ . In particular,  $|R|_2 \leq |G|_2$ . As  $|R|_2 = |PH|_2 = |H|_2 = |H/K|_2 = 2^\ell$  and  $|G|_2 = |G/P|_2 = |T|_2$ , we then obtain  $2^\ell \leq |T|_2$ . This in conjunction with (3) implies that  $p < |G : B| = |\overline{B} : \overline{G}| = 2^\ell \leq |T|_2$ , contradicting our choice of  $p$ .  $\square$

### 3. EXAMPLES

Recall the standard construction of the *coset graph*  $\text{Cos}(X, Y, S)$  for a group  $X$  with a subgroup  $Y$  and an inverse-closed subset  $S$  of  $X \setminus Y$  such that  $S$  is finite union of double cosets of  $Y$  in  $X$ . Such a graph has vertex set  $[X : Y]$ , the set of right cosets of  $Y$  in  $X$ , and edge set  $\{\{Yt, Yst\} \mid t \in X, s \in S\}$ . It is easy to see that  $\text{Cos}(X, Y, S)$  has valency  $|S|/|Y|$ , and  $X$  acts by right multiplication on  $[X : Y]$  as a group of automorphisms of  $\text{Cos}(X, Y, S)$ . Moreover,  $\text{Cos}(X, Y, S)$  is connected if and only if  $X = \langle Y, S \rangle$ .

3.1. **Example  $D_8$ .** Let  $G = A_{10}$  and

$$H = \langle (1, 2, 3, 4)(5, 6, 7, 8), (1, 4)(2, 3)(5, 7)(9, 10) \rangle < G.$$

Clearly,  $H \cong D_8$ . Let

$$s = (1, 8, 10)(2, 7, 4, 6, 9, 3, 5) \in G.$$

It can be checked immediately by the computational algebra system MAGMA [1] that

$$\langle H, s \rangle = G, \quad |H : s^{-1}Hs| = 2 \quad \text{and} \quad s^{-1} \notin HsH.$$

Then letting

$$\Sigma = \text{Cos}(G, H, H\{s, s^{-1}\}H), \tag{5}$$

we see that

- $\Sigma$  is a connected tetravalent graph;
- $G$  acts faithfully and half-arc-transitively on  $\Sigma$ ;
- the vertex stabilizer in  $G$  is  $H \cong D_8$ .

3.2. **Example**  $\mathcal{H}_7 \times C_2$ . Let

$$H = \langle a_1, \dots, a_8 \mid a_i^2 = 1 \text{ for } i \leq 8, (a_i a_j)^2 = 1 \text{ for } |i - j| \leq 5, \\ (a_1 a_7)^2 = a_3, (a_2 a_8)^2 = a_4, (a_1 a_8)^2 = a_6 \rangle.$$

Then  $H = \langle a_1, a_2, a_3, a_4, a_6, a_7, a_8 \rangle \times \langle a_5 \rangle \cong \mathcal{H}_7 \times C_2$ . Let

$$B = \langle a_1, \dots, a_7 \rangle, \quad C = \langle a_2, \dots, a_8 \rangle$$

and let  $\varphi : B \rightarrow C$  be the group isomorphism defined by

$$a_i^\varphi = a_{i+1} \quad \text{for } i = 1, \dots, 7.$$

Then  $H = B \cup a_8 B = C \cup a_1 a_2 C$ . Let  $x$  be the permutation on  $H$  defined by

$$b^x = b^\varphi \quad \text{and} \quad (a_8 b)^x = a_1 a_2 b^\varphi \quad \text{for } b \in B.$$

Denote the right regular representation of  $H$  by  $R : H \rightarrow \text{Sym}(H)$ . It can be checked easily by the computational algebra system MAGMA [1] that

$$\langle R(H), x \rangle = \text{Alt}(H), \quad x^{-1} R(H) x = R(C) \quad \text{and} \quad x^{-1} \notin R(H) x R(H).$$

Then letting

$$\Pi = \text{Cos}(\text{Alt}(H), R(H), R(H)\{x, x^{-1}\}R(H)), \quad (6)$$

we see that

- $\Pi$  is a connected tetravalent graph;
- $\text{Alt}(H)$  acts faithfully and half-arc-transitively on  $\Pi$ ;
- the vertex stabilizer in  $\text{Alt}(H)$  is  $R(H) \cong H \cong \mathcal{H}_7 \times C_2$ .

3.3. **Example**  $D_8 \times C_2^{m-3}$ . Let  $m \geq 4$  be an integer,

$$H = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1 \rangle \times \langle c_1 \rangle \times \cdots \times \langle c_{m-3} \rangle,$$

where  $c_1, \dots, c_{m-3}$  are involutions. Clearly,  $H \cong D_8 \times C_2^{m-3}$ . Let  $h = a \prod_{i=0}^{\lfloor (m-5)/2 \rfloor} c_{2i+1}$  and

$$K = \langle a^2, b, c_1, \dots, c_{m-3} \rangle = \langle a^2 \rangle \times \langle b \rangle \times \langle c_1 \rangle \times \cdots \times \langle c_{m-3} \rangle.$$

Then  $K \cong C_2^{m-1}$  and  $H = K \cup aK = K \cup hK$ . For convenience, put  $c_i = 1$  for  $i \leq 0$ . Define  $x \in \text{Aut}(H)$  by letting

$$a^x = a^{-1}, \quad b^x = ab, \quad c_{2i+1}^x = c_{2i+1} \quad \text{and} \quad c_{2i+2}^x = a^2 c_{2i+1} c_{2i+2}$$

for  $0 \leq i \leq \lfloor (m-5)/2 \rfloor$  and letting  $c_{m-3}^x = a^2 c_{m-3}$  in addition if  $m$  is even. Define  $\tau \in \text{Aut}(K)$  by letting

$$(a^2)^\tau = b, \quad b^\tau = a^2, \quad c_{2i+1}^\tau = c_{2i-1} c_{2i} c_{2i+2} \quad \text{and} \quad c_{2i+2}^\tau = c_{2i-1} c_{2i} c_{2i+1}$$

for  $0 \leq i \leq \lfloor (m-5)/2 \rfloor$  and letting  $c_{m-3}^\tau = c_{m-3}$  in addition if  $m$  is even.

Note that  $x$  and  $\tau$  are automorphisms of  $H$  and  $K$  respectively as the images of generators under  $x$  and  $\tau$  are generators of  $H$  and  $K$  satisfying the defining relations. Let  $y$  be the permutation of  $H$  such that  $g^y = g^\tau$  and

$$(hg)^y = \begin{cases} hg^\tau & \text{if } m \text{ is odd,} \\ hg^\tau c_{m-3} & \text{if } m \text{ is even,} \end{cases}$$



for  $g \in K$ . Denote the right regular representation of  $H$  by  $R : H \rightarrow \text{Sym}(H)$ . It follows from [2, Lemmas 2.1 and 2.3] that  $x$  and  $y$  are both involutions and  $\langle x, y, R(H) \rangle \leq \text{Alt}(H)$ . Let

$$\Sigma_m = \text{Cos}(\text{Alt}(H), R(H), R(H)\{xy, yx\}R(H)). \quad (7)$$

Fix the notation of  $H$ ,  $R$ ,  $x$  and  $y$  in this subsection, and let

$$z = \begin{cases} R(h)yR(h^{-1}) & \text{if } m \text{ is odd,} \\ R(h)yR(h^{-1}c_{m-3}) & \text{if } m \text{ is even.} \end{cases}$$

According to [2, Lemma 2.1], the permutation  $z$  is an involution. Since  $z \in \langle x, y, R(H) \rangle$ , it follows from  $\langle x, y, R(H) \rangle \leq \text{Alt}(H)$  that  $z \in \text{Alt}(H)$ . As  $x$ ,  $y$  and  $z$  all fix  $1 \in H$ , we may also view them as elements of  $\text{Alt}(H \setminus \{1\})$  when they cause no confusion. Use  $\sqcup$  to denote a disjoint union of sets.

**Lemma 3.1.** *The following hold:*

- (a)  $R(H)xyR(H) = R(H)xy \sqcup R(H)xz$ ;
- (b)  $R(H)yxR(H) = R(H)yx \sqcup R(H)zx$ ;
- (c)  $R(H)\{xy, yx\}R(H) = R(H)xy \sqcup R(H)yx \sqcup R(H)xz \sqcup R(H)zx$ .

*Proof.* Note that  $x$ ,  $y$  and  $z$  are all involutions. It is straightforward to verify that  $x$  and  $y$  normalize  $R(H)$  and  $R(K)$ , respectively. If  $yR(H)x \cap R(H) = R(H)$ , then

$$\langle x, y, R(H) \rangle \leq \mathbf{N}_{\text{Alt}(H)}(R(H)) < \text{Alt}(H),$$

contrary to [2, Lemmas 3.6 and 3.11]. Thus

$$yxR(H)xy \cap R(H) = yR(H)y \cap R(H) \neq R(H).$$

Since

$$yxR(H)xy \cap R(H) = yR(H)y \cap R(H) \geq yR(K)y \cap R(K) = R(K)$$

and  $R(K)$  has index 2 in  $R(H)$ , we then deduce that  $yxR(H)xy \cap R(H) = R(K)$ . In particular,  $yxR(H)xy$  has index 2 in  $R(H)$ , whence

$$\frac{|R(H)xyR(H)|}{|R(H)|} = \frac{|R(H)|}{|yxR(H)xy \cap R(H)|} = 2.$$

Consequently,

$$|R(H)yxR(H)| = |(R(H)yxR(H))^{-1}| = |R(H)xyR(H)| = 2|R(H)| \quad (8)$$

and thus

$$|R(H)\{xy, yx\}R(H)| \leq |R(H)xyR(H)| + |R(H)yxR(H)| = 4|R(H)|. \quad (9)$$

Note from the definition of  $z$  that

$$xz \in xR(H)yR(H) = R(H)xyR(H).$$

Hence  $R(H)xz \subseteq R(H)xyR(H)$  and  $R(H)zx \subseteq R(H)yxR(H)$ . It is direct to verify that

$$(a^2)^{xy} = b, \quad (a^2)^{yx} = ab, \quad (a^2)^{xz} = a^2b, \quad (a^2)^{zx} = a^3b,$$

which shows that  $xy$ ,  $yx$ ,  $xz$  and  $zx$  are pairwise distinct. Then as  $xy, yx, xz, zx \in \text{Alt}(H)_1$  and  $\text{Alt}(H)_1$  forms a right transversal of  $R(H)$  in  $\text{Alt}(H)$ , it follows that  $R(H)xy$ ,  $R(H)yx$ ,  $R(H)xz$  and  $R(H)zx$  are pairwise disjoint. Therefore,

$$R(H)xyR(H) \supseteq R(H)xy \sqcup R(H)xz,$$

$$R(H)yxR(H) \supseteq R(H)yx \sqcup R(H)zx$$

and

$$R(H)\{xy, yx\}R(H) \supseteq R(H)xy \sqcup R(H)yx \sqcup R(H)xz \sqcup R(H)zx.$$

This combined with (8) and (9) yields the lemma.  $\square$

**Proposition 3.2.** *Let  $m \geq 4$  be an integer and let  $\Sigma_m$  be the graph defined in (7). Then  $\Sigma_m$  is a connected tetravalent graph admitting a half-arc-transitive action of  $A_{2^m}$  with vertex stabilizer  $D_8 \times C_2^{m-3}$ .*

*Proof.* Let  $S = \{xy, yx, xz, zx\} \subset \text{Alt}(H \setminus \{1\})$ . According to [2, Lemmas 4.1 and 4.2],  $\text{Cay}(\text{Alt}(H \setminus \{1\}), \{x, y, z\})$  is connected, which means that  $\text{Alt}(H \setminus \{1\}) = \langle x, y, z \rangle$ . Consider the subgroup  $W$  of even words of the generators  $x$ ,  $y$  and  $z$  in  $\text{Alt}(H \setminus \{1\})$ . Then  $W$  has index 1 or 2 in  $\text{Alt}(H \setminus \{1\})$ . Since  $\text{Alt}(H \setminus \{1\})$  is simple, it follows that  $W = \text{Alt}(H \setminus \{1\})$ . Moreover, as  $x$ ,  $y$  and  $z$  are involutions, we have

$$W = \langle xy, xz, yz \rangle = \langle xy, xz, (xy)^{-1}(xz) \rangle = \langle xy, xz \rangle.$$

Thus  $\text{Alt}(H \setminus \{1\}) = \langle xy, xz \rangle$ , and so  $\text{Cay}(\text{Alt}(H \setminus \{1\}), S)$  is connected.

Let  $\varphi: g \mapsto R(H)g$  be the mapping from  $\text{Alt}(H \setminus \{1\})$  to the vertex set of  $\Sigma_m$ . Since  $\text{Alt}(H \setminus \{1\})$  forms a right transversal of  $R(H)$  in  $\text{Alt}(H)$ ,  $\varphi$  is bijective. Moreover, for any  $u$  and  $v$  in  $\text{Alt}(H \setminus \{1\})$ ,  $u$  is adjacent to  $v$  in  $\text{Cay}(\text{Alt}(H \setminus \{1\}), S)$  if and only if

$$vu^{-1} \in S = \{xy, yx, xz, zx\},$$

which is equivalent to

$$R(H)vu^{-1} \in \{R(H)xy, R(H)yx, R(H)xz, R(H)zx\}.$$

By Lemma 3.1, this means that  $u$  and  $v$  are adjacent in  $\text{Cay}(\text{Alt}(H \setminus \{1\}), S)$  if and only if

$$R(H)vu^{-1} \subseteq R(H)S = R(H)\{xy, yx\}R(H),$$

or equivalently,  $R(H)u$  is adjacent to  $R(H)v$  in  $\Sigma_m$ . Therefore,  $\varphi$  is a graph isomorphism from  $\text{Cay}(\text{Alt}(H \setminus \{1\}), S)$  to  $\Sigma_m$ . As a consequence,  $\Sigma_m$  is a connected tetravalent graph.

Finally, Lemma 3.1 implies that  $\{R(H)xy, R(H)xz\}$  and  $\{R(H)yx, R(H)zx\}$  are the two orbits of the group  $R(H)$  acting on the neighborhood of the vertex  $R(H)$  in  $\Sigma_m$ . Thereby we conclude that the right multiplication action of  $\text{Alt}(H)$  on  $\Sigma_m$  is half-arc-transitive. Since  $R(H) \cong H \cong D_8 \times C_2^{m-3}$  and  $\text{Alt}(H) \cong A_{|H|} = A_{2^m}$ , this completes the proof.  $\square$

#### 4. PROOFS OF THEOREM 1.2 AND THEOREM 1.4

*Proof of Theorem 1.2.* Let  $m \geq 4$  be an integer and let  $\Sigma_m$  be the graph defined by (7). Then as Proposition 3.2 asserts,  $\Sigma_m$  is a connected tetravalent graph admitting a half-arc-transitive action of  $A_{2^m}$  with vertex stabilizer  $D_8 \times C_2^{m-3}$ . Thus, by Theorem 1.1, there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer  $D_8 \times C_2^{m-3}$ .

Let  $m \geq 7$  be an integer and let  $\Gamma_m$  be the graph defined in [18, Section 3]. Then [18, Theorem 1.2] asserts that  $\Gamma_m$  is a connected tetravalent half-arc-transitive graph whose automorphism group is isomorphic to  $A_{2^m}$  with vertex stabilizer  $D_8 \times D_8 \times C_2^{m-6}$ . Thus, by Theorem 1.1, there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer  $D_8 \times D_8 \times C_2^{m-6}$ .  $\square$

*Proof of Theorem 1.4.* According to [12, Theorem 1.1] and Theorem 1.2, each of the groups

$$C_2^m \text{ with } 1 \leq m \leq 8, \quad D_8 \times C_2^{m-3} \text{ with } 4 \leq m \leq 8, \quad D_8^2 \times C_2^{m-6} \text{ with } 7 \leq m \leq 8$$

is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs. By Example 3.1, [4], [17] and Example 3.2, each of the groups

$$D_8, \quad D_8 \times D_8, \quad \mathcal{H}_7, \quad \mathcal{H}_7 \times C_2$$

is the vertex stabilizer of a half-arc-transitive nonabelian simple group acting on a finite connected tetravalent graph. This implies that each of these groups is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs by Theorem 1.1. Hence every group in the list of Theorem 1.3 is the vertex stabilizer of infinitely many finite connected tetravalent half-arc-transitive graphs, and so Theorem 1.4 is true.  $\square$

## 5. CONCLUDING REMARKS

The work in this paper was inspired by some new ideas developed in [18]. We observe that there are many papers that recently keep the interest on half-arc-transitive graphs very high. For instance, in [20], the author made significant progress in the study of tetravalent non-normal half-arc-transitive Cayley graphs of prime power order, and answered two very important problems related to this topic. Similarly, in [19], the authors answered a long-standing problem regarding the existence of half-arc-transitive graphs of order twice a prime square.

The reader may have noticed that our key ingredient in this paper is [15]. The main results of [15] are rather general and apply to most actions of groups on graphs. Our application of [15] in our work is rather successful, in our opinion, as we consider normal covers of graphs

( $\dagger$ ) admitting a *nonabelian simple* group of automorphisms.

Under this extra hypothesis, the results in [15] can be combined with rather strong group-theoretic results based on CFSG and, as a consequence, we are able to obtain infinite families of graphs having exotic vertex stabilizers. As far as we are aware, this type of constructions is a novelty.

In light of the following conjecture originating from Džambić and Jones [6] and supported by Conder (see [3, Section 2]), the hypothesis ( $\dagger$ ) does not seem strong. This suggests that Theorem 1.1 could be applied to show the existence of tetravalent half-arc-transitive graphs with other vertex stabilizers as well, and thus sheds light on classifying the vertex stabilizers of finite tetravalent half-arc-transitive graphs.

**Conjecture 5.1** (Conder-Džambić-Jones). *If  $A$  and  $B$  are finite groups and  $C$  is a subgroup of  $A \cap B$  of index at least 2 in  $A$  and at least 3 in  $B$ , then all but finitely many alternating groups are homomorphic images of the amalgamated free product  $A *_C B$ .*

**Remark.** In fact, Marston Conder has a stronger conjecture:

**Conjecture 5.2** ([3]). *Let  $A$  and  $B$  be finite groups, let  $C$  be a subgroup of  $A \cap B$  of index at least 2 in  $A$  and at least 3 in  $B$ , and let  $K$  be the core of  $C$  in the amalgamated free product  $A *_C B$ . Then all but finitely many alternating groups occur as the image of  $A *_C B$  under some homomorphism that takes  $A$  and  $B$  to subgroups (of the alternating group) isomorphic to  $A/K$  and  $B/K$  respectively.*

We conclude this section by giving a natural generalization of the example in Subsection 3.2. Let  $m \geq 7$  be an integer and let

$$H = \langle a_1, \dots, a_m \mid a_i^2 = 1 \text{ for } i \leq m, (a_i a_j)^2 = 1 \text{ for } |i - j| \leq m - 3, \\ (a_1 a_{m-1})^2 = a_3, (a_2 a_m)^2 = a_4, (a_1 a_m)^2 = a_{m-2} \rangle.$$

Then  $H = \langle a_1, a_2, a_3, a_4, a_{m-2}, a_{m-1}, a_m \rangle \times \langle a_5, \dots, a_{m-3} \rangle \cong \mathcal{H}_7 \times C_2^{m-7}$ . Let

$$B = \langle a_1, \dots, a_{m-1} \rangle, \quad C = \langle a_2, \dots, a_m \rangle$$

and let  $\varphi : B \rightarrow C$  be the group isomorphism defined by

$$a_i^\varphi = a_{i+1} \quad \text{for } i = 1, \dots, m - 1.$$

Then  $H = B \cup a_m B = C \cup a_1 a_2 C$ . Let  $x$  be the permutation on  $H$  defined by

$$b^x = b^\varphi \quad \text{and} \quad (a_m b)^x = a_1 a_2 b^\varphi \quad \text{for } b \in B.$$

Denote the right regular representation of  $H$  by  $R$ . Inspired by [17] and the results in Subsection 3.2, we make the following conjecture.

**Conjecture 5.3.** *Let  $H$ ,  $R$  and  $x$  be as above. Then*

$$\text{Cos}(\text{Alt}(H), R(H), R(H)\{x, x^{-1}\}R(H))$$

*is a connected tetravalent graph on which the right multiplication action of  $\text{Alt}(H)$  is half-arc-transitive.*

If this conjecture is true then Theorem 1.1 will imply that for every integer  $m \geq 7$  there exist infinitely many finite connected tetravalent half-arc-transitive graphs with vertex stabilizer  $\mathcal{H}_7 \times C_2^{m-7}$ .

## REFERENCES

- [1] W. Bosma, J. Cannon, C. Playoust, The magma algebra system I: The user language, *J. Symbolic Comput.*, **24** (1997), no. 3–4, 235–265.
- [2] J. Chen, B. Xia, J.-X. Zhou, An infinite family of cubic nonnormal Cayley graphs on nonabelian simple groups, *Discrete Math.*, **341** (2018), no. 5, 1282–1293.
- [3] M. D. E. Conder, Simple group actions on arc-transitive graphs with prescribed transitive local action, *2017 MATRIX annals*, 327–335, MATRIX Book Ser., 2, Springer, Cham, 2019.
- [4] M. D. E. Conder, D. Marušič, A tetravalent half-arc-transitive graph with non-abelian vertex stabilizer, *J. Combin. Theory Ser. B*, **88** (2003), no. 1, 67–76.
- [5] M. D. E. Conder, P. Potočník, P. Šparl, Some recent discoveries about half-arc-transitive graphs, *Ars Math. Contemp.*, **8** (2015), no. 1, 149–162.
- [6] A. Džambić, G. A. Jones,  $p$ -adic Hurwitz groups, *J. Algebra*, **379** (2013), 179–207.
- [7] X. G. Fang, C. H. Li, M. Y. Xu, On edge-transitive Cayley graphs of valency four, *European J. Combin.*, **25** (2004), no. 7, 1107–1116.
- [8] G. Glauberman, Normalizers of  $p$ -subgroups in finite groups, *Pacific J. Math.*, **29** (1969), 137–144.
- [9] G. Glauberman, Isomorphic subgroups of finite  $p$ -groups. I, *Canad. J. Math.*, **23** (1971), 983–1022.
- [10] C. H. Li, X. Li, On permutation groups of degree a product of two prime-powers, *Comm. Algebra*, **42** (2014), no. 11, 4722–4743.
- [11] D. Marušič, Recent developments in half-transitive graphs, *Discrete Math.*, **182** (1998), no. 1–3, 219–231.
- [12] D. Marušič, Quartic half-arc-transitive graphs with large vertex stabilizers, *Discrete Math.*, **229** (2005), no. 1–3, 180–193.
- [13] D. Marušič, R. Nedela, On the point stabilizers of transitive groups with non-self-paired suborbits of length 2, *J. Group Theory*, **4** (2001), no. 1, 19–43.

- [14] P. Potočnik, A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index  $(4, 2)$ , *European J. Combin.*, **30** (2009), no. 5, 1323–1336.
- [15] P. Potočnik, P. Spiga, Lifting a prescribed group of automorphisms of graphs, *Proc. Amer. Math. Soc.*, **147** (2019), no. 9, 3787–3796.
- [16] P. Potočnik, P. Spiga, G. Verret, A census of 4-valent half-arc-transitive graphs and arc-transitive digraphs of valence two, *Ars Math. Contemp.*, **8** (2015), no. 1, 133–148.
- [17] P. Spiga, Constructing half-arc-transitive graphs of valency four with prescribed vertex stabilizers, *Graphs Combin.*, **32** (2016), no. 5, 2135–2144.
- [18] B. Xia, Tetravalent half-arc-transitive graphs with unbounded nonabelian vertex stabilizers, to appear in *J. Combin. Theory Ser. B*, <https://arxiv.org/abs/1908.09361>.
- [19] M.-M. Zhang, J.-X. Zhou, The classification of half-arc-regular bi-circulants of valency 6, *European J. Combin.*, **64** (2017) 45–56.
- [20] J.-X. Zhou, Tetravalent half-arc-transitive  $p$ -graphs, *J. Algebraic Combin.*, **44** (2016), no. 4, 947–971.

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