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POLYHEDRAL APPROACH TO TOTAL MATCHING AND TOTAL COLORING PROBLEMS

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Notation

$G = (V, E)$ a generic graph

$\alpha(G)$ cardinality of a maximum stable set

$\nu(G)$ size of a maximum matching

$\alpha_T(G)$ size of a maximum total matching

$\chi[A]$ characteristic vector of a subset $A \subseteq G$

$P_{STAB}(G)$ Stable Set Polytope

$P_M(G)$ Matching Polytope

$P_T(G)$ Total Matching Polytope

$\omega(G)$ the size of a maximum clique

$\overline{\chi}(G)$ clique number of G

$\chi(G)$ chromatic number of G

$\chi'(G)$ chromatic index of G

$\chi_T(G)$ total chromatic number of G

$\mathbf{1}_n$ the vector of n ones

$\mathbb{R}_+$ space of real, non-negative n-dimensional vectors

$\mathcal{S}$ the set of all stable sets of a graph

$\mathcal{M}$ the set of all matchings of a graph

$\mathcal{T}$ the set of all total matchings of a graph

$\mathcal{K}$ the set of all cliques of a graph

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Abstract

A total matching of a graph $G = (V, E)$ is a subset of G such that its elements, i.e. vertices and edges, are pairwise not adjacent. In this context, the Total Matching Problem calls for a total matching of maximum size. This problem has been mainly studied in the literature from a graph theoretical point of view. However, due to the strong connection with well-known combinatorial problems of the Stable Set Problem and the Matching Problem, we focus on an integer programming point of view. In this Thesis, we present a polyhedral approach to the Total Matching Problem, and hence, we introduce the corresponding polytope, namely the Total Matching Polytope. To the best of our knowledge, we are the first to tackle the problem from a polyhedral perspective. We introduce several families of valid inequalities: vertex-clique inequalities based on standard clique inequalities of the Stable Set Polytope, congruent-$2k3$ cycle inequalities based on the parity of the vertex set induced by the cycle, even-clique inequalities induced by complete subgraphs of even order, and, balanced biclique and non-balanced lifted biclique inequalities based on complete bipartite graphs, where the balanced family has the partitions of the vertex set of equal size, whereas the second class each vertex partition has different size. We prove that congruent-$2k3$ cycle inequalities are facet-defining when $k = 4$, and for cubic graphs under certain conditions. The non-balanced lifted biclique inequalities are obtained by a lifting procedure and are facet-defining for bipartite graphs. While the vertex-clique, even-cliques, and balanced bicliques inequalities are always facet-defining. In addition, we provide a first linear complete description for trees and complete bipartite graphs. For the latter family, the complete characterization is obtained by projecting a higher-dimension polytope onto the original space. This leads to also give an extended formulation of small size for the Total Matching Polytope of complete bipartite graphs.

Another problem related to the Total Matching Problem is the Total Coloring Problem. Any partition of the elements into total matchings induces a coloring of G, that is, each total matching is assigned to a color class. Hence, a total coloring is an assignment of colors to vertices and edges such that neither two adjacent vertices nor two incident edges get the same color, and, for each edge, the endpoints and the edge itself receive different colors. In this Thesis, we propose Integer Linear Programming models for Total Coloring problems, and we study the strength of the corresponding Linear Programming relaxations. The total coloring is formulated as the problem of finding the minimum number of total matchings that cover all the graph elements. This covering formulation can be solved by a Column Generation algorithm, where the

pricing subproblem corresponds to the Weighted Total Matching Problem. Finally, we present computational results of a Column Generation algorithm for the Total Coloring Problem and a Cutting Plane algorithm for the Total Matching Problem.

Research Activities

Published Articles

- *Normal 5-edge-colorings of a family of Loupekhine snarks.*
  Ferrarini L., Mazzuoccolo G., Mkrtchyan V.V.,
  AKCE International Journal of Graphs and Combinatorics 17(3),(2020), 720–724.

- *Total Coloring and Total Matching: Polyhedra and Facets.*
  Ferrarini L., Gualandi S.,
  European Journal of Operational Research (2022), https://doi.org/10.1016/j.ejor.2022.02.025.

- *An Integer Linear Programming Model for Tilings.*
  Auricchio G., Ferrarini L., Lanzarotto G.,
  Journal of Mathematics and Music (2023), https://doi.org/10.1080/17459737.2023.2180812

In Conferences

- *A SAT Encoding to Compute Aperiodic Tiling Rhythmic Canons.*
  Auricchio G., Ferrarini L., Gualandi S., Lanzarotto G., Pernazza L.,
  Integration of Constraint Programming, Artificial Intelligence, and Operations Research. CPAIOR 2022. Lecture Notes in Computer Science, vol 13292. Springer, Cham. https://doi.org/10.1007/978-3-031-08011-1_2

Under revision articles

- *The Total Matching Polytope of Complete Bipartite Graphs*
  Faenza Y., Ferrarini L.,
  Preprint, https://arxiv.org/abs/2303.00328.

Talks

- EURO 2022, Espoo (Finland), 3rd July - 6th July, Talk Presentation "A Polyhedral Approach to the Total Matching Problem".

- CPAIOR 2022, Los Angeles, 20th June - 23th June, Paper Presentation "A SAT Encoding to Compute Aperiodic Tiling Rhtymic Canons".

- MIP 2022, 23th May -26th June, 2022 DIMACS, Rutgers University, Poster Presentation "The Total Matching Problem: A Polyhedral Perspective".

- OR 63, September 15th, Poster Session “An integer Linear model for Mathemusical Problem”

- 31st European Conference on Operational Research, EuroAthens 2021, 11 July - 14 July, speaker in the session of MILP for graph coloring. Title of the talk “A Polyhedral Approach to the Total Coloring Problem”.

- Mathematical and Computational Models in Music, held in Pavia and hybrid form, June 28 - July 2, title of the talk “An ILP Model for Tiling Problems”.

- AIRO PhD School 2021 and 5th AIRO Workshop “Optimization and Data Science: Trends and Applications”, 8-12 February 2021, Naples. Title of the talk: “A Polyhedral Approach to the Total Coloring Problem”.

- 4th AIROYoung Workshop: “New Advances in Optimization, Machine Learning and Data Science”, 5-7 February 2020, Bolzano. Speaker of the session: “ Challenges in Public Transportation and Graph Coloring”.

Introduction

Consider a simple and undirected graph $G = (V, E)$ and let $D = V \cup E$ be the set of its elements. We say that a pair of elements $a, b \in D$ are *adjacent* if a and b are adjacent vertices, or if they are incident edges, or if a is an edge incident to a vertex b. If two elements $a, b \in D$ are not adjacent, they are *independent*. The Total Matching Problem (TMP) asks for a subset of the elements of G which yields an independent set of maximum size. The TMP generalizes both the Matching Problem, where we look for an independent set of edges [22], and the Stable Set Problem, where instead we look for an independent set of vertices [69, 68]. The first work on the TMP appeared in [1], where the authors derive lower and upper bounds on the size of a maximum total matching. In [57], Manlove provides a survey of the algorithmic complexities of the decision problems related to graph parameters. The author reports that $\alpha_T(G)$ is NP-complete for bipartite, planar, and arbitrary graphs. Despite the strong connection with the Matching Problem, the TMP is less studied in the operations research literature. In particular, significant results are obtained only for structured graphs, such as cycles, paths, full binary trees, hypercubes, and complete graphs, see [45]. This thesis aims to present the first polyhedral study of the TMP deriving several facet-defining inequalities for its polytope.

A problem strictly related to the TMP is the Total Coloring Problem (TCP). Given a set of colors $K = \{1, \dots, k\}$, a k–total coloring of G is an assignment $\phi : D \to K$ such that $\phi(a) \neq \phi(b)$ for every pair of adjacent elements $a, b \in D$. Each subset of elements assigned to the same color by ϕ defines a *total matching*, that is, a subset $T \subseteq D$ where the elements are pairwise independent. Hence, a k-total coloring induces a partition of the elements in D into k disjoint total matchings. The minimum value of k such that G admits a k-total coloring is called the total chromatic number, and it is denoted by $\chi_T(G)$.

The TCP consists of finding $\chi_T(G)$. It is an NP-hard problem [71], which is studied mainly in graph theory [78] for the conjecture attributed independently to Vizing [76] and Behzad [5] that relates $\chi_T(G)$ to the maximum degree $\Delta(G)$ of the nodes in G. The conjecture states that $\chi_T(G) \leq \Delta(G) + 2$. Observe that $\Delta(G) + 1$ is a lower bound, we need $\Delta(G)$ colors for the edges incident the vertex of maximum degree and one more color for the vertex itself. While the conjecture holds for specific classes of graphs (e.g., see [75]), the conjecture is still open for general graphs. In particular, Vizing's conjecture was proved for cubic graphs, and hence, the total chromatic number of a cubic graph is either 4 or 5. In [7], the authors pose the question of whether a cubic graph of Type 2 (that is, with $\chi_T(G) = \Delta(G) + 2$) and with girth greater than 4 exists, where the girth is defined as the length of the smallest cycle in the graph. So far, the question is still open. The TCP generalizes both the Vertex Coloring Problem, where we have to color only the vertices of G, and the Edge Coloring Problem, where instead we have to color only the edges. The Vertex Coloring Problem was tackled in the literature by many exact polyhedral approaches (for a recent survey, see [56]). The most effective exact approaches to the Vertex Coloring Problem are based on set covering formulations [59, 34, 55, 36], where each set of the covering represents a subset of vertices taking the same color, corresponding hence to a (maximal) stable set of G. Similarly, the best polyhedral approach to the Edge Coloring Problem is based on a set covering formulation, where the edges are covered by (maximal) matchings of G [62, 43]. An innovative alternative formulation for the Edge Coloring Problem is based on a different ILP model based on a binary encoding of the problem variables [44]. While in the literature there are other interesting approaches to graph coloring problems (e.g., branch-and-cut [61], semidefinite programming [42], decision diagrams [37], constraint satisfiability [35], memetic algorithms [51]), and other interesting types of coloring problems (e.g., equitable coloring [47, 17], graph multicoloring [32], sum coloring [18], selective graph coloring [19]), in this thesis, we focus on a polyhedral approach to the TCP.

The TCP has several practical applications, for instance, in Match Scheduling [41], Network Task Efficiency, and Math Art [45]. As an example of match scheduling, consider the martial art tournament problem, which can be formulated using a *tournament graph* $G = (V, E)$ and a set of colors K defined as follows. We introduce a vertex i to V for each player, and an edge $\{i, j\}$ to E for each match. Then, we associate a color in K to each time period of the tournament. The assignment of a color $k \in K$ to an edge $\{i, j\}$ represents the scheduled time period of the match between players i and j. The assignment of a color $k \in K$ to a vertex i represents a rest time for player i during the time period associated with color k. Given this graph formulation, no pair of incident edges get the same color because no player can be in two matches at once; no vertex can be incident to an edge with the same color as the vertex because no player should have a match during his rest time; no pair of adjacent vertices should get the same color because no two matched players can leave the stage simultaneously. Hence, a proper total coloring of the tournament graph represents a feasible scheduling of the tournament, and the total chromatic number represents the minimum number of time periods to schedule the tournament.

Our contributions The main results of this thesis are summarized as follows:

1. The definition of families of valid inequalities that we call the *vertex-clique* inequalities based on classic clique inequalities of the Stable Set Polytope, the *congruent-$2k3$ cycle inequalities*, which are based on the parity of the cycle, and the *even-clique inequalities*, which are based on complete graphs of even cardinality, the *balanced biclique inequalities* and *non-balanced lifted biclique inequalities*. We prove that the vertex-clique and even-cliques are always facet-defining, while the congruent-$2k3$ cycle inequalities are facet-defining when $k = 4$. In particular, we show that the separation problem of a congruent-$2k3$ cycle can be solved with a network flow model or equivalently, using a sequence of shortest path problems in an auxiliary directed graph. For the latter family of biclique inequalities, we distinguish between balanced biclique inequalities, which are always facet-defining, and the non-balanced lifted biclique inequalities, obtained by a lifting procedure and facet-defining for bipartite graphs.

2. Complete description of the Total Matching Polytope for trees and complete bipartite graphs. For the latter class, we introduce an extended formulation of polynomial size. By projecting such formulation onto the original space, we derive the original description of the Total Matching Polytope for complete bipartite graphs. In particular, an irredundant description is provided by characterizing the extreme rays of the associated projection cone.

3. A set covering formulation of the Total Coloring Problem based on maximal total matchings, which can be solved by column generation, and which yields a lower bound at least as strong as the lower bound obtained with standard ILP formulation.

4. Computational results on the strength of the set covering relaxation with respect to the assignment relaxation, and on the computational strength of the valid inequalities introduced for the Total Matching Polytope.

1. Mathematical Background

1.1 Mathematical Preliminaries

In this Chapter, we report classical results on polyhedral theory as a self-contained survey. The results present in this chapter can be found in [13]. In particular, we recall polyhedral tools which reveal to be useful throughout the Thesis. First, we deal with polyhedra.

Definition 1. *A polyhedron is a set of the form*

$$P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^n$.

We refer to a *polytope* if P is bounded. Polyhedra are *convex set*.

Definition 2. *A set* $S \subseteq \mathbb{R}^n$ *is convex if for any* $x, y \in S$ *and any* $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in S$

Given a set $S \subseteq \mathbb{R}^n$ the convex hull of S, denoted with $\text{conv}(S)$ is the smallest convex set containing S. Given a polyhedron P, we recall the notion of valid inequalities.

Definition 3. *An inequality* $c^T x \leq \delta$ *is valid for* P *if it is satisfied by every point in* P.

Definition 4. *A face* F *of a polyhedron* P *is of the form*

$$F := P \cap \{x \mid c^T x = \delta\}$$

for a valid inequality $c^T x \leq \delta$ *of* P.

Given a polyhedron P, we say that a face F is a facet of P if it is inclusion-wise maximal, that is, F is not contained in any other faces.

Proposition 1. *Let* P *be a polyhedron, the following statements are equivalent:*

- F *is a facet of* P.
- $dim(F) = dim(P) - 1$.

In order to describe the inequalities representing a polyhedron P, we are mainly interested in finding an irredundant description of it. It results that such inequalities are associated with maximal faces of P.

Theorem 1. *Let $P \subseteq \mathbb{R}^n$ be a nonempty polyhedron.*

- *For each facet F of P, at least one of the inequalities defining F is necessary in any description $Ax \leq b$ of P.*
- *Inequalities defining faces of dimension less than $dim(P) - 1$ are not needed in the description of P and can be removed.*

In order to derive max-min relations we make use of the following two fundamental theorems in folklore's linear programming.

Theorem 2 (Weak Duality Theorem)**.** *Given a matrix $A \in \mathbb{R}^{n \times m}$ and vectors $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ and $D := \{u \in \mathbb{R}^m \mid u^T A \geq 0\} \neq \emptyset$. Consider the problems $\max\{c^T x \mid x \in P\}$ and $\min\{b^T y \mid y \in D\}$. Then,*

$$c^T x \leq b^T y$$

Theorem 3 (Strong Duality Theorem)**.** *Given a matrix $A \in \mathbb{R}^{n \times m}$ and vectors $c \in \mathbb{R}^n, b \in \mathbb{R}^m$ let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\} \neq \emptyset$ and $D := \{u \in \mathbb{R}^m \mid u^T A \geq 0\} \neq \emptyset$. Then*

$$\max\{c^T x \mid Ax \leq 0, x \geq 0\} = \min\{b^T y \mid u^T A = c, u \geq 0\}$$

Proposition 2 (Farkas' Lemma)**.** *A system of linear inequalities $Ax \leq b$ is infeasible if and only if the system $u^T A = 0, u \geq 0$ is feasible.*

A vector $x \in \mathbb{R}^n$ is a conic combination of vectors $u_1, u_2, \ldots, u_r$ if there exist scalar $\alpha_1, \alpha_2, \ldots, \alpha_r \geq 0$ such that

$$x = \sum_{i=1}^{k} \alpha_i u_i,$$

Definition 5. *A set $C \subseteq \mathbb{R}^n$ is a cone if $0 \in C$ and for every $x \in C$ and $c \geq 0$, cx also belongs to C. In other terms, C is a cone if and only if it contains the origin and, for every $x \in C \setminus \{0\}$, C contains the half line starting from the origin and passing through x.*

In particular, a cone C is *convex* if every conic combination of vectors in C lies on C. Given a nonempty set $S \subseteq \mathbb{R}^n$, we denote as cone(S) the smallest convex cone containing S. From now on, we refer to a cone C as a convex cone.

Definition 6. *A polyhedral cone is a set of the form $C := \{x \in \mathbb{R}^n \mid Ax \leq 0\}$, that is, it is the intersection of finitely many halfspaces containing the origin on their boundary.*

Theorem 4 (Minkowski-Weyl Theorem)**.** *For a set $C \subseteq \mathbb{R}^n$ the following two conditions are equivalent:*

- *There exists a matrix A such that $C = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$.*
- *There exists a matrix R such that $C = \{x \in \mathbb{R}^n \mid x = Ru \text{ for some } u \geq 0\}$.*

In other words, the Minkowski-Weyl Theorem states that a convex cone is *finitely generated*, that is, it is the conic combination of a finite number of vectors, if and only if it is a polyhedral cone. We write as $C = \text{cone}(r_1, r_2, \ldots, r_k)$ the cone generated by the vectors $u_1, u_2, \ldots, u_r$, namely *extreme rays* of C. Thus, we are interested in identifying the generators of a cone. The next Theorem characterizes the extreme rays of a cone. A *pointed cone* is a cone without lines.

Theorem 5. *Let $C := \{x \in \mathbb{R}^n : Ax \leq 0\}$ be a pointed cone, and let $\bar{u}$ be a ray of C. The following statements are equivalent.*

- *$\bar{u}$ is an extreme ray of C,*
- *$\bar{u}$ satisfies at equality $n-1$ linearly independent inequalities of $Ax \leq 0$,*
- *$\bar{u}$ is not a proper conic combination of two distinct rays in C.*

We have two equivalent ways to represent a polyhedron. We refer to the V-description of a polyhedron if it can be expressed as a convex combination of its vertices, whereas H-description is the intersection of finitely many halfspaces. Given a two subsets $V, Q \subseteq \mathbb{R}^n$ the *Minkowski sum* is the set $P := V + U = \{x \in \mathbb{R}^n \mid x = v + u, v \in V, u \in U\}$

Theorem 6 (Minkowski-Weyl Theorem Polyhedra)**.** *For a subset $P \subseteq \mathbb{R}^n$ the two following conditions are equivalent:*

- *P is a polyhedron, that is, there exists a matrix A such that P can be expressed as $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$,*
- *There exists $v_1, v_2, \ldots, v_p \in \mathbb{R}^n$, $r_1, r_2, \ldots, r_q \in \mathbb{R}^n$ such that P can be written as the sum $P = conv(\{v_1, v_2, \ldots, v_p\}) + cone(\{r_1, r_2, \ldots, r_q\})$.*

Now, we turn to the question of when a polyhedron has only integral vertices. This ensures that the corresponding LP problem over the polyhedron has an optimal integral solution when it is finite. We introduce the following definition.

Definition 7. *A polyhedron P is integral if every nonempty face contains an integral point.*

As a consequence, if we optimize over an integral polyhedron P we obtain integrality for free. In fact, let $z^* = \max\{c^T x \mid x \in P\}$ which is associated to the face $F := \{x \in P \mid c^T x = z^*\}$. This holds for every face, in particular, there exists an optimal solution that is integral

$$\max\{c^T x \mid x \in P\} = \max\{c^T x \mid x \in P \cap \mathbb{Z}^n\}$$

We resort to the following Theorem.

Theorem 7. *Let $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ be a rational pointed polyhedron. Then, the following statements are equivalent.*

- *P is an integral polyhedron.*
- *The LP $\max\{c^T x : x \in P\}$ has an integral optimal solution for every $c \in \mathbb{R}^n$ where the value is finite.*
- *The LP $\max\{c^T x : x \in P\}$ has an integral optimal solution for every $c \in \mathbb{Z}^n$ where the value is finite.*
- *$P = conv(P \cap \mathbb{Z}^n)$.*

It is useful to focus on the structure of the constraint matrix defining a polytope. When the matrix has a specific form, the vertices of the polytope are integral. This fact occurs when we deal with totally unimodular matrices.

Definition 8. *A matrix A is said to be totally unimodular if every square submatrix has the determinant $\in \{0, -1, +1\}$.*

We recall the most important related to the concept of totally unimodular matrix

Theorem 8. *[38]Let A be a $m \times n$ totally unimodular matrix and $b \in \mathbb{Z}^m$. Then,*

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

is an integral polytope.

Ghouila-Houri in [30] proposes a characterization of totally unimodular matrices in terms of partition of rows and columns of the corresponding matrix

Theorem 9. *The following statements are equivalent*

- *A is totally unimodular*

- *For every $J \subseteq N = \{1, 2, \ldots, n\}$ there exists a partition of J_1, J_2 of J such that*

$$\left| \sum_{j \in J_2} a_{ij} - \sum_{j \in J_1} a_{ij} \right| \leq 1$$

Corollary 1. *The following statements are equivalent*

- *A is totally unimodular.*
- *A^T is totally unimodular.*
- *$[A|I]$ is totally unimodular.*

Consider the problem $\{c^T x \mid Ax \leq b\}$, where A and b are rational and $c \in \mathbb{Z}^n$. A linear system $Ax \leq b, x \geq 0$ is called *totally dual integral*, abbreviated as TDI, if for every integral vector $c \in \mathbb{Z}^n$, the LP problem $\max\{c^T x \mid x \in Ax \leq b\}$ has a dual problem that admits an integer optimal solution. Thanks to Edmonds and Giles we have the following characterization.

Theorem 10. *[23] The linear system $Ax \leq b, x \geq 0$ is TDI and $b \in \mathbb{Z}^m$, then the polyhedron $P := \{x \in \mathbb{R}^n_+ \mid Ax \leq b\}$ is integral.*

Valid inequalities can be made stronger by a lifting procedure. Specifically, the extension of a valid inequality for a polytope P to a valid inequality for a higher dimensional polytope Q is called *lifting*. Consider a mixed integer set $S := \{x \in \mathbb{Z}^n_+ \times \mathbb{R}^q_+ : Ax \leq b\}$. Given a subset T of $N := \{1, ..., n+q\}$, and a valid inequality $\sum_{j \in C} \alpha_j x_j \leq \beta$ for $\text{conv}(S) \cap \{x \in \mathbb{R}^{n+q} : x_j = 0, j \in N \setminus T\}$, an inequality $\sum_{j=1}^{n+q} \alpha_j x_j \leq \beta$ is a lifting of $\sum_{j \in T} \alpha_j x_j \leq \beta$ if it is valid for $\text{conv}(S)$.

Proposition 3. *Let $S \subseteq \{0,1\}^n$ such that $S \cap \{x : x_n = 1\} = \emptyset$, and let $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ be a valid inequality for $S \cap \{x : x_n = 0\}$. Then*

$$\alpha_n := \beta - \max\left\{\sum_{i=1}^{n-1} \alpha_i x_i : x \in S, x_n = 1\right\}$$

is the largest coefficient such that $\sum_{i=1}^{n-1} \alpha_i x_i + \alpha_n x_n \leq \beta$ is valid for S. Furthermore, if $\sum_{i=1}^{n-1} \alpha_i x_i \leq \beta$ defines a d-dimensional face of $conv(S) \cap \{x_n = 0\}$, then $\sum_{i=1}^{n} \alpha_i x_i \leq \beta$ defines a face of $conv(S)$ of dimension at least $d+1$.

1.2 Graph Theory

In this part, we provide graph-theoretical notions and notations used throughout the thesis. A graph $G = (V(G), E(G))$ is a pair $G = (V(G), E(G))$, where $V(G)$ is a set and $E(G) \subseteq \binom{V(G)}{2}$. The elements of $V(G)$ are called *vertices* (or equivalently *nodes*), and the elements of $E(G)$ are called *edges*, for simplicity we will write a graph as $G = (V, E)$ and $D = V \cup E$ as the whole set. Given a graph $G = (V, E)$, throughout the thesis, define $n = |V|$ and $m = |E|$. We will denote an edge $e = \{v, w\} \in E$, where v and w are the *end-points* of e. Two vertices v and w are adjacent if $e = \{v, w\} \in E$ and two edges e_1 and e_2 are *incident* if they share a vertex. For a vertex $v \in V$, we denote by $\delta(v)$ the set of edges *incident* to v and by $N_G(v)$ the set of vertices *adjacent* to v. The degree of a vertex is $|\delta(v)|$, in particular, we denote by $\Delta(G) := \max\{|\delta(v)| \mid v \in V\}$.

If two elements $a, b \in D$ are not adjacent, they are *independent*. An element $a \in D$ is said to be covered by a set $C \subseteq D$ if it is adjacent to at least one element of C. A *stable set* is an independent set of vertices, instead, a *matching* is an independent set of edges. A *total matching* is a subset $T \subseteq D$ where the elements are pairwise independent. A subset $C \subseteq V \cup E$ is a *total cover* of G if it covers all the elements of G. In particular, a matching of G is *perfect*, if it contains $|V(G)|/2$ edges.

It is possible also to have more than one edge between two vertices, or, to have an edge starting and ending at the same vertex, let e_1 and e_2 be two edges. e_1 and e_2 are *parallel* if $e_1 = \{u, v\} \in E$ and $e_2 = \{u, v\} \in E$ for two vertices u and v, and, an edge forms a loop if it starts and ends at the same vertex. A graph G is called simple if it does not contain loops.

Graphs considered in this thesis are simple. A graph G is k-regular if $|\delta(v)| = k$ for all $v \in V$; if $k = 3$, G is called *cubic*. U is a subgraph of G if $V(U) \subseteq V(G)$ and $E(U) \subseteq E(G)$. A subgraph U is said to be *induced* if $E(U)$ consists of all edges of G having both end-points in U; in particular, U is called a *spanning subgraph* if $V(U) = V(G)$. If $U \subseteq V$, we denote by $G[U]$ the induced subgraph having U as set of vertices and we use the notation $E[U]$ to indicate the edges of $G[U]$. We define $\delta(U) := \{e \in E \mid e = \{u, v\}, u \in U, v \in V \setminus U\}$ a *cut* of G.

A path of length $n \in \mathbb{N}$ is a graph $P = (V, E)$ such that $V = \{v_0, \ldots, v_n\}$, $E = \{v_0v_1, \ldots, v_kv_{k+1}, \ldots, v_{n-1}v_n\}$, where $v_i \neq v_j$, for $i \neq j$. A cycle C is a connected graph in which every vertex has degree 2. A cycle of length $n \in \mathbb{N}$ is a graph C_n that can be written as $P \cup \{v_n, v_0\}$ where $P = v_0v_1 \ldots v_n$ is a path.

Given a subset $W \subseteq V$, the graph $G \setminus W$ denotes the graph obtained by deleting all vertices in W. We write $G \setminus \{v\}$ as the graph obtained with the deletion of vertex v. A graph G is *connected* if, for every pair of vertices, there is a path connecting them. A *cutset* of G is a set $X \subseteq V(G)$ of the form $G \setminus X$ is not connected. If X is a cutset, then $G \setminus X$ can be partitioned into *connected components*. We refer to an *odd connected component* if the cardinality is odd, and even, otherwise. A k-regular spanning subgraph of G is called a k-factor, thus G has a 1-factor if G has a perfect matching.

A clique $K \subseteq V(G)$ is a graph in which every vertex in K is pairwise adjacent. We denote as $\omega(G)$ the size of a maximum clique in G. The complement of a graph G is the graph $\overline{G} = (\overline{V}, \overline{E})$ having the same vertex set of $V(G)$ and there is an edge $e = \{v, w\} \in \overline{E}$ if and only if v and w are not adjacent in G. The line graph $L(G)$ of a graph G has the vertex set the edges of G, and two vertices are adjacent in $L(G)$ if and only the corresponding edges in the original graph are incident to the same vertex. A graph is *chordal* if every cycle of length greater or equal to four has a *chord*, that is, there is an edge connecting two non-consecutive vertices of the cycle. An acyclic graph is called a *forest*. If the forest is connected then the graph is a *tree*. We recall a structural Theorem to characterize the class of Trees.

Theorem 11. *[21] A connected graph with n nodes is a tree if and only if it has $n-1$ edges.*

A graph $G = (V, E)$ is *bipartite* if the V can be partitioned into two sets A and B such that A and B form stable sets, that is, no edge occurs in the same set of vertices. The following theorem provides a complete characterization of bipartite graphs.

Theorem 12. *[21] A graph is bipartite if and only if G contains no odd cycles.*

We conclude the Chapter by introducing the following parameters which occur frequently throughout the thesis. We define $\alpha_T(G) := \max\{|T| : T \text{ is a total matching}\}$, $\nu(G) := \max\{|M| : M \text{ is a matching}\}$ and $\alpha(G) := \max\{|S| : S \text{ is a stable set}\}$.

2. Review of known results

This Chapter aims to give an overview of known results for the Stable Set and Matching problems, since the Total Matching Problem is strictly related to these ones. We recall relevant polyhedral properties of the associated polytopes, namely the Stable Set Polytope and the Matching Polytope.

2.1 Stable Set Polytope and Matching Polytope

Specifically, we denote by $P_M(G)$ the Matching Polytope, that is, the convex hull of all incidence vectors of matchings of a graph G. Thanks to Edmonds [22], we have a first complete linear description of the Matching Polytope, which can be described by the following inequalities

$$\Bigg\{ y \in \mathbb{R}_+^{|E|} : \sum_{e \in \delta(v)} y_e \le 1 \qquad\qquad \forall v \in V, \tag{2.1}$$

$$\sum_{e \in E[U]} y_e \le \frac{|U|-1}{2} \qquad\qquad \forall U \subseteq V,\ |U|\ is \text{ odd} \Bigg\}. \tag{2.2}$$

The inequalities (2.2) are called *blossom inequalities* and are induced by subgraphs of odd cardinality. Although the number of blossom inequalities is exponential in the size of the graph, for any point not lying on $P_M(G)$, the Padberg-Rao separation algorithm [72] gives a polynomial time algorithm for finding a violated blossom inequality. If the constraint (2.1) is modified with the equality we have the Perfect Matching Polytope denoted by $P_{PM}(G)$.

The Stable Set Polytope is defined as the convex hull of all incidence vectors of stable sets. Thus, given a stable set S, the incidence vector of S is the following vector in the $|V|$-dimensional space.

$$\chi[S] = \begin{cases} x_v = 1 & \text{if } v \in S \subseteq V, \\ x_v = 0 & \text{otherwise.} \end{cases}$$

Then we define the Stable Set Polytope as

$$P_{STAB}(G) = \text{conv}\{\chi[S] \mid S \subseteq V,\ S \text{ is a stable set}\}$$

The Stable Set Problem is an NP-hard problem in general, thus it is most unlikely to derive a compact description of it. On the other hand, for certain classes of graphs, many valid and facet-defining inequalities have been investigated for the Stable Set Polytope, see [46, 66, 11, 65, 25, 69, 29, 68, 26, 27]. In particular, complete linear descriptions have been obtained for classes of graphs as line graphs and quasi-line graphs, see [25, 22]. We will give an overview of the most well-known class of inequalities that characterize the Stable Set Polytope. The basic formulation for the Stable Set Problem is based on the edge inequalities

$$\begin{aligned} \alpha(G) = \max \sum_{v \in V} x_v & \\ x_i + x_j \leq 1 & \qquad \forall e = \{i, j\} \in E, \\ x_v \in \{0, 1\} & \qquad \forall v \in V. \end{aligned}$$

It is natural to consider its continuous relaxation

$$P_{FSTAB}(G) := \left\{ x \in [0,1]^{|V|} \mid x_i + x_j \leq 1, \forall e = \{i, j\} \in E(G) \right\}$$

this polytope is known as the Fractional Stable Set Polytope. While it is easy to optimize over $P_{FSTAB}(G)$, this provides very weak upper bounds on $\alpha(G)$. In fact, the vector $x^* = \mathbf{1}^T \frac{1}{2}$ yields an optimal solution to the problem. On other hand, A relevant fact about the structure of $P_{FSTAB}(G)$ is that the vertices of this polytope are *half-integral.*

Theorem 13. *[63] The vertices of $P_{FSTAB}(G)$ lie on $\{0, \frac{1}{2}, 1\}$.*

Proof. Let x be a vertex of $P_{FSTAB}(G)$ and define the two following subsets:

$$\begin{aligned} U &:= \left\{ v_i \mid 0 < x_i < \frac{1}{2}, 1 \leq i \leq n \right\} \\ W &:= \left\{ w_j \mid \frac{1}{2} < x_j < 1, 1 \leq j \leq n \right\} \end{aligned}$$

We aim to prove that the sets are empty. Define the two vectors

$$z_i := \begin{cases} x_i - \delta & \text{if } i \in U, \\ x_i + \delta & \text{if } i \in W, \\ x_i & \text{otherwise.} \end{cases}$$

$$t_i := \begin{cases} x_i + \delta & \text{if } i \in U, \\ x_i - \delta & \text{if } i \in W, \\ x_i & \text{otherwise.} \end{cases}$$

where $\delta > 0$ can be chosen arbitrarily small enough in such a way that $z, t \in P_{FSTAB}$. Assume now that U and W are not both empty. Observe that, by construction, the vertex x can be written as $x = \frac{z+t}{2}$. We show now that both the vectors z and t satisfy the inequalities characterizing P_{FSTAB}. First, by definition each component of z and t is non-negative. Because $z_i + z_j \leq 1$ for every edge $e = \{i, j\}$, the unique possible cases are listed as follows

- $z_i + z_j - 2\delta$, if $(i, j) \in U \times U$
- $z_i + z_j$, if $(i, j) \in U \times W$ or $(i, j) \in W \times U$, or $(i, j) \notin U \times W$
- $z_i + z_j - \delta$, if $i \notin (U \cup W), j \in V$, or if $j \notin (U \cup W), i \in V$
- $z_i + \delta$, if $i \in U, j \notin U \cup W$, or $j \in U, i \notin U \cup W$

The same procedure holds for the vector t, thus the vectors satisfy the edge inequalities. Since x is an extreme point and it is different from z and t, this implies that $U = W = \emptyset$. □

Nemhauser and Trotter provided a graphical characterization of the vertices for the Fractional Stable Set Polytope induced by subgraphs of G. The next Proposition establishes the structure of the vertices for $P_{FSTAB}(G)$ in correspondence to specific subgraphs of G.

Proposition 4. *Let $U \subseteq V$ and suppose that $x \in P_{FSTAB}(G)$ is a vertex defined by $x_u = \frac{1}{2}, \forall u \in U$. Then, x is a vertex if and only if $G[U]$ contains an odd cycle.*

2.1.1 Matching Polytopes

Let $P_{FPM}(G)$ be the Fractional Matching Polytope defined by the following constraints:

$$P_{FPM}(G) := \Big\{ y \in \mathbb{R}^{|E|} : \sum_{e \in \delta(v)} y_e = 1 \qquad \forall v \in V,$$
$$y_e \geq 0 \qquad \forall e \in E \Big\}.$$

Analogously to the Fractional Stable Set polytope, the vertices of this polytope reflect the same behavior.

Theorem 14. *$y \in P_{FPM}(G)$ is a vertex if and only if $y_e \in \{0, \frac{1}{2}, 1\}$. Moreover, $y_e = \frac{1}{2}$ form node disjoint odd-cycles.*

Proof. " $\Longleftarrow$ " First, a vertex can be seen as the unique intersection between a supporting hyperplane H and the polytope itself, i.e. $\{y'\} = P_{FPM}(G) \cap H$. Suppose we are given a point y' which is half-integral. Define the weights of a supporting hyperplane H as $\alpha_e = -1$, if $y'_e = 0$, $\alpha_e = 0$, if $y'_e > 0$ and define the following set $R := P_{FPM}(G) \cap \{\alpha^T y = 0\}$. We want to prove that R contains a unique point corresponding to a vertex of $P_{FPM}(G)$. On the contrary, suppose we have another point $\tilde{y} \neq y'$ such that $\tilde{y} \in R$. Define the set of edges $E_0 := \{e \in E \mid y_e = 0\}$ and $E_1 := \{e \in E \mid y_e > 0\}$. Since $\tilde{y}_e \in R$, this implies that $\tilde{y}_e = 0, \forall e \in E_0$ and $y'_e = \tilde{y}_e = \frac{1}{2}$ for every edge that lie on a cycle. Observe that any other edge that does not belong to a cycle must be set to 1. This proves that $y'_e = \tilde{y}_e$.

" $\Longrightarrow$ " Suppose that we have a vertex $y \in P_{FPM}(G)$. We apply a transformation of G into a new graph G' in such a way that if $e = \{u, w\} \in E$, then the new nodes u', u'', w', w'' are the end-points of $e' = \{u', w''\} \in E(G')$ and $e'' = \{u'', w'\} \in E(G')$. Hence, an edge e in G corresponds to edges e' and e'' in G'. Note that $y_e = \frac{1}{2}(y_{e'} + y_{e''})$. Observe that the graph G' is bipartite by construction. Thus, this implies that incidence vectors of matchings are the vertices of the matching polytope, i.e. it has only $0, 1$ vertices. If it corresponds to an odd cycle, exactly one of $y_{e'}$ and $y_{e''}$ will be set to one. Because $y_e = \frac{1}{2}(y_{e'} + y_{e''})$, it is necessarily that $y_e = \frac{1}{2}$. Now, it is easy to prove that if $y_e = \frac{1}{2}$, it must be part of an odd cycle. This concludes the proof. $\square$

Now we start the treatment of the complete linear descriptions for the Matching Polytope, and, for the Perfect Matching Polytope.

Theorem 15. *For any graph G the $P_{PM}(G)$ coincides with:*

$$Q := \Bigg\{ y \in \mathbb{R}_+^{|E|} : \sum_{e \in \delta(v)} y_e = 1, \qquad \forall v \in V, \tag{2.3}$$

$$\sum_{e \in \delta(U)} y_e \geq 1, \qquad \forall U \subseteq V, |U| \text{ is odd} \Bigg\}. \tag{2.4}$$

Proof. It is easy to verify that $P_{PM}(G) \subseteq Q$, since the characteristic vector of a perfect matching satisfies the constraints in Q. Now, we prove that $Q \subseteq P_{PM}(G)$. To this end, assume that G is a minimum counterexample to the statement, where G is chosen to be one that minimizes $|V| + |E|$. Hence, there exists a point $y \in Q$, but $y \notin P_{PM}(G)$. By minimality, G must be a connected graph and $0 < y_e < 1, \forall e \in E$. In fact, suppose that G is not connected, then there are at least two connected components with fewer edges than G and thus one of its components represents a counterexample, a contradiction. If $y_e = 0$, then we can delete e, yielding a smaller counterexample $G \setminus e$. In the other case, if $y_e = 1$ for an edge $e = \{u, v\}$, the edges variables incident to end-points u and v must set to zero, thus the edge e must be disconnected from the graph. But also in this case $G \setminus \{u, v\}$ creates a counterexample. We prove then that $|E(G)| \geq |V(G)| + 1$. If G contains a vertex v of degree 1, then the unique edge e incident to v must satisfy $y_e = 1$, a contradiction. Then suppose that every vertex of G has degree 2, in this case, G must be a disjoint union of cycles. We can assume that the cardinality of each cycle is even, for otherwise $Q = \emptyset$ and consequently $P_{PM}(G) = \emptyset$. So, let C an even cycle, then there are two perfect matchings M_1 and M_2 in C, and there exists $\alpha \in (0, 1)$ such that $y = \alpha\chi[M_1] + (1 - \alpha)\chi[M_2]$, a contradiction to the fact that y is an extreme point. This implies that there exists a vertex of degree strictly greater than 2 and therefore condition $|E| > |V|$ holds. Since y is a vertex of Q it satisfies $|E|$ linearly independent constraints at equality. Notice that this system of equalities must be selected from the constraints in Q. Since $|E| > |V|$ and by what we argued before the non-negative constraint must not be satisfied at equality, this imposes that there must exist at least an odd component $U \subseteq V$ such that $\sum_{e \in \delta(U)} y_e = 1$. Now, look at the partition resulting from the cut $(U, \overline{U})$. Denote by H' and H'' the graphs obtained by contracting respectively $G \setminus U$ to a single vertex w, and $G \setminus \overline{U}$ to a single vertex w'. Define the corresponding new edges variables $y'_e \in \mathbb{R}^{E(H')}$ and $y''_e \in \mathbb{R}^{E(H'')}$ the restriction of y with respect to the edges that are still present in the contracted graphs. Notice that by construction $|\delta(U)| = |\delta(w)| = |\delta(w')|$, hence $\sum_{e \in \delta(w)} y'_e = \sum_{e \in \delta(U)} y_e = 1$, this certifies that all the vertex degree and blossom constraints are satisfied showing that $y'_e \in P_{PM}(H')$ and $y''_e \in P_{PM}(H'')$, by inductive hypothesis. Hence, y' and can be written as a convex combination of a list of perfect matchings in H', and y'' as a convex combination of other perfect matchings in H''. Note that y is rational since it is a point of the polytope determined by the inequalities $(2.3) - (2.4)$, so y' and y'' are also rational points. Thus, there exists an integer r such that

$$y' = \frac{1}{r} \sum_{i=1}^{r} \chi[M'_i]$$

$$y'' = \frac{1}{r} \sum_{i=1}^{r} \chi[M''_i]$$

where $M'_1, M'_2, \ldots, M'_r$ are perfect matchings in H' and $M''_1, M''_2, \ldots, M''_r$ perfect matchings in H''. Now, since w belongs to every perfect matchings and using $\sum_{e \in \delta(w)} y_e = 1$, we have that $\forall e = \{w, v\}, v \in U$, e appears in exactly $ry'_e = ry''_e = ry_e$ perfect matchings among $M'_1, M'_2, \dots, M'_r$, and similarly in r perfect matchings of $M''_1, M''_2, \dots, M''_r$. Hence, for every edge $e \in \delta(U)$ there exist perfect matchings M'_i and M''_i sharing exactly the edge e. Therefore $M'_i \cup M''_i$ forms a perfect matching in G. Assume w.l.o.g. that $M_i := M'_i \cup M''_i, i = 1, 2, \dots, r$ are perfect matchings in G. This implies that:

$$y = \frac{1}{r}\sum_{i=1}^{r}\chi[M_i]$$

This concludes the proof, since we have a contradiction to our assumption. □

We use the Theorem 15 to show a complete polyhedral characterization of the Matching Polytope.

Theorem 16. *For any graph G, $P_M(G)$ is completely determined by:*

$$P_M(G) := \Bigg\{ y \in \mathbb{R}_+^{|E|} : \sum_{e\in\delta(v)} y_e \leq 1 \qquad \forall v \in V, \tag{2.5}$$

$$\sum_{e\in E[U]} y_e \leq \frac{|U|-1}{2} \qquad \forall U \subseteq V,\ |U| \ is\ odd \Bigg\}. \tag{2.6}$$

Proof. Construct a new graph $H' = (V', E')$ as follows. Take a copy $G' = (V', E')$ of G and add edges $e = \{v, v'\}$ such that $v \in V$ and v' is the copy of v in G'. For every point $x \in P_M(G)$ we construct a vector $y \in P_{PM}(H)$ such that y has weight x_e for every edge $e \in E$, and the corresponding copy e' of the edge e, for each edge $e = \{v, v'\}$ is defined as $y'_e := 1 - \sum_{e\in\delta(v)} x_e$. It is easy to see that y belongs to $P_{PM}(H)$. In fact, by construction of y we have $\sum_{e\in\delta(v)} y_e = \sum_{e\in\delta(v')} y_e = \sum_{e\in\delta(v)\setminus\{v,v'\}} y_e + y_{vv'} = 1$. Then, we have to check that y satisfies also the third constraint. Consider a subset $U \subseteq V(H)$ of odd cardinality, and let $W = U \cap V$ and $X' = U \cap V'$, where X' is the copy of X in V'. It is easy to see that $\sum_{e\in\delta(U)} y_e \geq \sum_{e\in\delta(W\setminus X)} y_e + \sum_{e\in\delta(X'\setminus W')} y_e$. Hence, we may assume that $|W|$

is odd and thus $X' = \emptyset$. Then, we obtain the following relations.

$$
\begin{aligned}
\sum_{\delta(U)} y_e = & \sum_{e\in\delta(W)} y_e \\
& = \sum_{e\in\delta(v), v\in W} y_e - 2 \sum_{e\in E(G[W])} y_e \\
& = |W| - \sum_{e\in E(G[W])} y_e \geq |W| - 2\frac{|W|-1}{2} \\
& \geq 1
\end{aligned}
$$

We have verified that $y \in P_{PM}(H)$, and therefore x lies to $P_M(G)$. This concludes the proof. $\square$

It is worthy of mention that, by using a LP duality technique, we can derive a min-max relation in the following formula. Let $\mathcal{O}(V(G))$ be the set of all subgraphs of G of odd cardinality.

Corollary 2. *Let $G = (V, E)$ be a graph and $w \in \mathbb{R}^{|E|}$ be a weighted function on the edges of G. Then, the maximum weight of a matching is equal to*

$$
\begin{aligned}
\min \quad & \sum_{v\in V} y_v + \sum_{u\in\mathcal{O}(V)} z_U \frac{|U|-1}{2} \\
s.\ t. \quad & y_v + y_w + \sum_{U\in\mathcal{O}(V): e\in U} z_U \geq w_e, \qquad \forall e = \{v, w\} \in E,
\end{aligned}
$$

where $y \in \mathbb{R}_+^{|V|}$ and $z \in \mathbb{R}_+^{|\mathcal{O}(V)|}$

Cunningham and Marsh,in [72], show that the defining inequalities of the matching polytope are totally dual integral. Thus, if we restrict the value of w to be integral, this allows defining the corresponding dual values y and z integers accordingly. Later on, Edmonds and Pulleyblank found a minimal description of the Matching Polytope by characterizing all the facets. They used the following concept: A graph H is called *hypomatchable* if, for all nodes v of H, the subgraph $H \setminus \{v\}$ admits a perfect matching.

Theorem 17. *[24] If $H = (V, E)$ is a 2-connected hypomatchable graph, then*

$$\sum_{e\in E(H)} y_e \leq \frac{|V|-1}{2}$$

is a facet of $P_M(H)$.

Furthermore, Edmonds and Pulleyblank in [24] characterized which hypomatchable subgraphs of G produce facets of $P_M(G)$. Thus we can restate the Theorem in terms of all facet-defining inequalities characterizing $P_M(G)$. We show in the next Proposition when the degree constraints are facet-defining for $P_M(G)$.

Proposition 5. *An inequality* $\sum_{e\in\delta(v)} y_e \leq 1$ *is a facet of* $P_M(G)$ *if and only if* $|\delta(v)| \geq 3$*, or* v *does not belong to a triangle.*

Proof. " $\Longleftarrow$ " Suppose $N_G(v) = \{u, w\}$ and that $\{u, w\} \in E$. Then every characteristic vector that satisfies $\sum_{e\in\delta(v)} y_e = 1$, also must satisfy at equality the blossom inequality associated with $S = \{u, v, w\}$. But the reverse is not true. Thus the face induced by the edge inequality is not maximal.

" $\Longrightarrow$ " For each edge $e \in \delta(v)$, let $M_e = \{e\}$, and for each edge $e \in E \setminus \delta(v)$, let $M_e = \{e, f\}$, where f is an edge in $\delta(v)$ not incident to e. M_e is a matching in G for all $e \in E$. The set of incidence vectors of M_e, $e \in E$, are affinely-independent, so by definition $\sum_{e\in\delta(v)} y_e \leq 1$ induces a facet of $P_M(G)$. □

The following Lemma turns out to be helpful in proving the next Proposition.

Lemma 1. *Let* $G = (V, E)$ *be a 2-vertex connected graph and hypomatchable graph, and let* W *be a proper subset of* V *with* $|W| \geq 3$ *odd. Then* G *has a matching of size* $\left\lfloor \frac{1}{2}|V| \right\rfloor$ *containiting less edges that* $\left\lfloor \frac{1}{2}|W| \right\rfloor$*.*

Proposition 6. *If* $H \subseteq G$ *is a 2-connected hypomatchable subgraph of* G*, then*

$$\sum_{e\in E(H)} y_e \leq \frac{|V(H)| - 1}{2}, \tag{2.7}$$

is a facet of $P_M(G)$*.*

Proof. Let F be the face associated with an inequality of type (2.7) and F' be a facet. By hypothesis F' must be one of the faces induced by the inequalities describing $P_M(G)$, that is, F is one of the inequalities of type (2.5). Suppose that $F \subseteq F'$. First, assume that F' is determined by $y_e = 0$ for some $e \in E$. If e is not covered by any vertex in H, there is a $v \in H$ such that e is not incident to $H \setminus \{v\}$. Let M be a perfect matching of $G[H] \setminus \{v\}$. Then $\chi[M \cup \{e\}] \in F \setminus F'$, a contradiction. If e is covered by H, choose v incident to the edge e and, let M be a perfect matching of $G[H] \setminus \{v\}$. Let $f \in M$ such that e and f share a vertex and define $M' := (M \setminus \{f\}) \cup \{e\}$. Then $\chi[M'] \in F \setminus F'$, a contradiction. Next assume that F' is determined by $\sum_{e\in\delta(v)} y_e = 1$ for some v such that $|\delta(v)| \geq 3$. Then, as $G[H]$ is hypomatchable, there exists a matching M with $|M \cap E[H]| = \frac{|H|-1}{2}$ and $|M \cap \delta(v)| = 0$. So $\chi[M] \in F \setminus F'$, a contradiction. If $H' \nsubseteq H$, there is a matching M with $|M \cap E[H]| = \lfloor \frac{1}{2}|H| \rfloor$ missing at least two vertices in H' and hence $|M \cap E[H']| < \lfloor \frac{1}{2}|H'| \rfloor$. Then $\chi[M] \in F \setminus F'$, a contradiction. So $H' \subset H$. By Lemma 1, $G[H]$ has a matching M of size $\lfloor \frac{1}{2}|H| \rfloor$ such that less than $\lfloor \frac{1}{2}|H'| \rfloor$ edges in M are spanned by H'. Then $\chi[M] \in F \setminus F'$, a contradiction. □

2.1.2 Class of valid inequalities for the Stable Set Polytope

In the following, we provide an overview of some of the most relevant valid inequalities studied for $P_{STAB}(G)$. For a clique K of a graph G the following inequality reads as follows

$$\sum_{v \in K} x_v \leq 1, \tag{2.8}$$

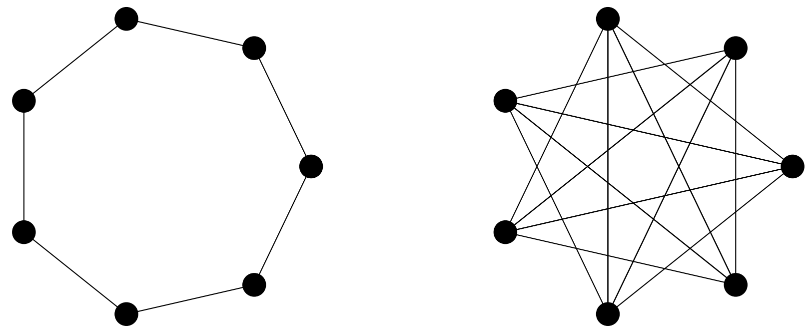
it is clearly valid for $P_{STAB}(G)$ since for any stable set S of a graph G we have $|S \cap K| \leq 1$. This family of inequalities known as *clique inequalities* was first introduced by Padberg, see [66]. The clique inequalities generalize the edge inequalities, and moreover, they are facet-defining if and only if the corresponding clique K is maximal. In the same paper, Padberg introduced the so-called *odd-hole inequalities* which are based on induced subgraphs that correspond to odd-holes, and the *odd-antihole* inequalities, which refer to induced subgraphs that are the complement of an odd-hole. Since it can be easily observed that $\alpha(G[H]) = \frac{|H|-1}{2}$, and similarly $\alpha(G[\overline{H}]) = 2$ for an induced odd-hole $H \subseteq G$, the corresponding odd-hole inequality reads as

$$\sum_{v \in H} x_v \leq \frac{|H|-1}{2},$$

and the odd-antihole is written as follows

$$\sum_{v \in \overline{H}} x_v \leq 2.$$

It is proven to be useful the process of lifting a valid inequality in the context of the Stable Set Polytope. The *sequential lifting* procedure is used to derive the coefficients of facet-defining inequalities for $P_{STAB}(G)$ of a graph G, starting with an inequality that it is facet-defining for the Stable Set Polytope of an induced subgraph of G. Let $\sum_{v \in H} a_v x_v \leq b$ be a facet-defining inequality of $P_{STAB}(H)$ where $H \subseteq V$, and consider a



(a) (b)

Figure 2.1: (a) An odd-hole C_7 and (b) the complement of C_7.

vertex $u \in V \setminus H$. The coefficient for the vertex u is chosen by solving the following optimization problem:

$$\begin{aligned} a_u = \quad & b - \max \sum_{v \in V \setminus H} a_v x_v \\ & x \in P_{STAB}(G[H \cup \{u\}]), \\ & x_u = 1. \end{aligned}$$

Then, it can be shown that a_u is the largest coefficient chosen such that $\sum_{v \in H} a_v x_v + a_u x_u \leq b$ is facet-defining for $P_{STAB}(G[H \cup \{u\}])$. By iteratively applying a sequential lifting procedure $|V \setminus H|$ times, we end up with an inequality $\sum_{v \in V} a_v x_v \leq \beta$ which is facet-defining for $P_{STAB}(G)$. In the same work [66], Padberg exploited the lifting argument approach to come up with an interesting family of inequalities based on an *odd-wheel*. Let $H \subseteq V$ be an odd-hole, and, let c be a vertex $c \in V$ adjacent to all the nodes of H. The subgraph $H \cup \{c\}$ is called an *odd-wheel*, see Figure 2.2. Since the odd-hole inequality is facet-defining for $P_{STAB}(H)$, the lifting coefficient for c can be achieved by applying a lifting procedure to the central vertex of the wheel to obtain the corresponding inequality:

$$\sum_{v \in H} x_v + \frac{|H| - 1}{2} x_c \leq \frac{|H| - 1}{2}.$$

We call these inequalities the *odd-wheel* inequalities.

Let p and q be two integers with $p \geq 2q + 1$. Trotter in [73] introduced the web inequalities $\sum_{i \in W} x_i \leq q$, where $W \subseteq V$ induces a web $W(p, q)$ of G, that is, a subgraph with p vertices $\{1, \ldots, p\}$ with, adopting modulo p arithmetic, an edge between every two vertices i and $j \in \{i + q, \ldots, i - q\}$. In the same paper [73], Trotter introduced the

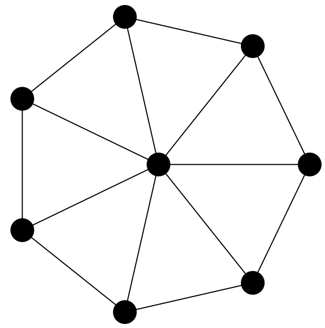


Figure 2.2: Odd wheel W_7.

antiweb inequalities $\sum_{i \in \overline{W}} x_i \leq \left\lfloor \frac{p}{q} \right\rfloor$, where $\overline{W}(p,q)$ is the complement of $W(p,q)$, that is, a web $W(p,q)$ of $\overline{G}$.

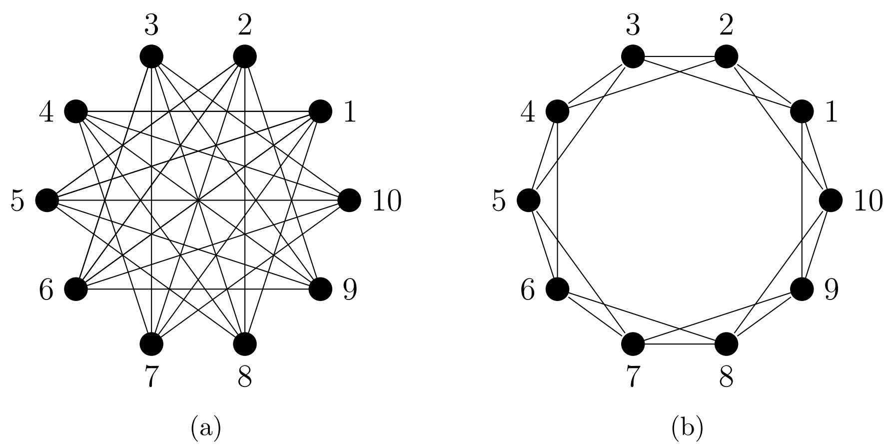


Figure 2.3: (a) Web $W(10,3)$, and (b) an Antiweb $\overline{W}(10,3)$.

All the previous classes of inequalities fall into the family of the so-called *rank-inequalities*, as shown in [15]. For any induced subgraph $U \subseteq G$, we can pick at most the cardinality of a maximum stable set on U, thus this allows to define the rank inequalities as

$$\sum_{v \in U} x_v \leq \alpha(G[U]). \tag{2.9}$$

By construction, all the coefficients are 0,1 and the right-hand-side corresponds to the maximum value of a stable set on U. Now, we summarize the results obtained from the classes of graphs investigated in the literature such that a complete linear description has been derived. We underline the most well-known classes of graphs based on the inequalities presented in this Section.

Bipartite Graphs. We start with the class of bipartite graphs. It turns out that edge inequalities and non-negative inequalities are sufficient to describe completely $P_{STAB}(G)$ if and only if G is bipartite, we report the statement in the following Theorem.

Theorem 18. *[72] G is a bipartite graph if and only if $P_{FSTAB}(G)$ coincides with $P_{STAB}(G)$.*

Proof. " $\Longrightarrow$ " Suppose first that G is not bipartite. Then, G contains at least an odd cycle C. Consider the vector x with values $x_v = \frac{1}{2}, \forall v \in C$ and $x_v = 0, \forall v \in V \setminus C$. It results that x satisfies the edge inequalities and non-negativity constraints, but x does not lie in $P_{STAB}(G)$. In fact, suppose by contradiction that x can be written as a convex combination of incidence vectors of stable sets of G. Let $\mathcal{S}$ be the usual set of all stable sets, and denote with $\mathcal{S}(C)$ the set of stable sets containing only vertices of C. Then, we have the following situation

$$x = \sum_{S \in \mathcal{S}} \lambda_S \chi[S] = \sum_{S \in \mathcal{S}(C)} \lambda_S \chi[S] + \sum_{S \notin \mathcal{S}(C)} \lambda_S \chi[S] = \sum_{S \in \mathcal{S}(C)} \lambda_S \chi[S],$$

Where $\sum_{S \in \mathcal{S}} \lambda_S = 1, \lambda_S \geq 0$. Notice that $\lambda_S = 0, \forall S \notin \mathcal{S}(C)$, since $\sum_{S \notin \mathcal{S}(C)} \lambda_S = 0$. Then,

$$x^T \mathbf{1}_V = \sum_{S \in \mathcal{S}(C)} \lambda_S (\chi[S]^T \mathbf{1}_V) = \sum_{S \in \mathcal{S}(C)} \lambda_S |S| \leq \sum_{S \in \mathcal{S}(C)} \lambda_S \frac{|C|-1}{2} = \frac{|C|-1}{2}.$$

The inequality says that any stable set intersects C in at most $(|C|-1)/2$ vertices, and, the last equality is obtained by using the convexity constraint $\sum_{S \in \mathcal{S}} \lambda_S = 1$. On the other hand,

$$x^T \mathbf{1}_V = \sum_{v \in V(C)} x_v = \frac{|C|}{2}.$$

We get a contradiction. Thus, we infer that x cannot be written as a convex combination of stable sets of G.

" $\Longleftarrow$ " Since $P_{FSTAB}(G) = P_{STAB}(G)$ every vertex in the Fractional Stable Set Polytope has $0, 1$ coordinates and corresponds to a characteristic vector of a stable set. By using Proposition 4, there does not exist an odd-cycle. This concludes the proof. $\square$

Some considerations are made in order. Observe that the maximum stable set problem can be solved efficiently via a standard LP formulation using the simplex method. The previous result can also be proved directly by exploiting the structure of the incidence matrix A of a bipartite graph G, which results in being totally unimodular. Then, it follows immediately that each vertex is integer and corresponds to a characteristic vector of a stable set. Since the incidence-node matrix of a graph is totally unimodular, then A^T is totally unimodular. This fact has important polyhedral consequences. Consider the computation of the maximum matching on a bipartite graph.

$$
\begin{aligned}
\max \sum_{e \in E} & y_e \\
\sum_{e \in \delta(v)} & y_e \leq 1 && \forall v \in V, \\
& y_e \geq 0 && \forall e \in E.
\end{aligned}
$$

By LP-duality relation the dual problem consists of finding the minimum value of

$$
\begin{aligned}
\min \sum_{v \in V} & y_v \\
& y_v + y_w \geq 1 && \forall e = \{v, w\} \in E, \\
& y_v \geq 0 && \forall v \in V.
\end{aligned}
$$

Denote with $P_{VC}(G)$ the convex hull of all vertices associated with the feasible region attached to the latter problem, it turns out that $P_{VC}(G)$ is the *vertex cover polytope* of G, that is, the convex hull of all incidence vectors of vertex covers of G. We have the following important consequence due to the properties of the total unimodularity of the matrix.

Theorem 19. *For any bipartite graph G, we have:*

$$P_{VC} := \left\{ y \in \mathbb{R}_+^{|V|} : \; y_v + y_w \geq 1, \forall e = \{v, w\} \in E \right\}.$$

Proof. The constraint matrix is totally unimodular since it is the transpose of the incidence-node matrix of a bipartite graph. □

With this useful observation, one may derive easily the following well-known Theorem which relates the minimum size of a vertex cover to the cardinality of a maximum matching in a bipartite graph by using LP-duality.

Theorem 20 (König's theorem)**.** *For any bipartite G*

$$\tau(G) = \nu(G).$$

that is the size of a minimum vertex cover equals to the size of a maximum matching.

Proof. By LP-duality we have $\max\left\{\sum_{e\in E} x_e \;\middle|\; \sum_{e\in E} x_e \leq 1, x \geq 0\right\} = \min\left\{\sum_{v\in V} y_v \;\middle|\; y_i + y_j \geq 1, \forall e = \{i,j\} \in E, y \geq 0\right\}$. Both the problems have integer optimal solutions y^* and x^*, by the total unimodularity of the defining constraint matrix. Clearly, y^* corresponds to an incidence vector of a matching of maximum size, whereas x^* to the characteristic vector of a minimum vertex cover. This concludes the proof. □

Perfect Graphs. The graphs for which the non-negativity constraints and clique inequalities are sufficient to describe completely the Stable Set Polytope are called *perfect*. Berge in 1960 introduced this class of graphs and proposed two conjectures related to them. The first is known as *the weak perfect graph theorem*, which asserts that a graph is perfect if and only if its complement is. This conjecture was later proven to be true by Lovàsz in [48]. The second conjecture referred to as the *strong perfect graph conjecture*, characterizes the class of perfect graphs in terms of forbidden induced subgraphs. Berge noticed that odd holes and their complements constitute the unique forbidden graphs, in fact, this conjecture was proven to be true in the seminal paper by Chudnovsky, Robertson, Seymour, and Thomas, in [9]. The strong interest in studying perfect graphs has brought important combinatorial consequences. We report the well-known Theorem which characterizes the perfect graphs and we will make use of it in the course of the thesis for the main results obtained.

Theorem 21 (Strong Perfect Graph Theorem, [9])**.** *A graph is perfect if and only if it does not contain an odd cycle of length at least five, or its complement, as an induced subgraph.*

For a graph G, we recall the following lower bound

$$\chi(G) \geq \omega(G)$$

For perfect graphs, it holds that $\chi(G') = \omega(G')$ for every induced subgraph $G' \subseteq G$. To this end, we introduce parameters that are strictly related to the structure of perfect graphs. A *clique cover* is a partition of the vertex set of a graph into cliques. The minimum number of cliques to cover the vertex set is denoted with $\overline{\chi}(G)$. Moreover, since a stable set in a graph is converted into a clique in the complement of G, we infer immediately that $\alpha(G) = \omega(\overline{G})$ and $\chi(\overline{G}) = \overline{\chi}(G)$. In particular, Lovàsz noticed the following.

Lemma 2 (Replication Lemma)**.** *[48] Let G be a perfect graph and $v \in V(G)$. Add a new vertex v' and connect it to v and to all the vertices in $N_G(v)$. Then, the new graph G' obtained from this operation is still perfect.*

Proof. Let $v \in V$ and its corresponding replicated vertex $v' \in V'$. Let H' be an induced subgraph of G'. We may assume that H' contains both the vertices v and v', for otherwise, H is an induced subgraph of a perfect graph, and thus is perfect. Consider the graph $H := H' \backslash \{v\}$ and Let K be a maximum clique of H, and, consider any vertex coloring of H with $\omega(H)$ colors. Suppose first that v belongs to a maximum clique K of H. By construction, the neighbors of v' are the vertices in K, hence $K' := K \cup \{v'\}$ forms a clique such that $|K'| = \omega(H') = \omega(H) + 1$. Use a new color for the vertex v', and clearly $\chi(H') = \omega(H')$ follows. Now suppose v does not belong to any maximum clique of H. Let S be the color class containing v. Then, $\omega(H \setminus (S \setminus \{v\})) = \omega(H) - 1$, since each maximum clique of H must receive $\omega(H)$ colors, and thus each maximum clique meets $S \setminus \{v\}$. By the perfection of H, the graph $H \backslash (S \backslash \{v\})$ can be colored with $\omega(H) - 1$ colors. Since v' is not adjacent to any other nodes in a maximum clique, we can use an additional color for the nodes $(S \backslash \{v\}) \cup \{v'\}$. Hence, we obtain a coloring of H' with $\omega(H')$ colors, and this concludes the proof. □

Theorem 22. *Let G be a perfect graph. Then, the linear system*

$$\sum_{v \in S} x_v \leq 1 \quad \forall S \in \mathcal{S},$$

is totally dual integral.

Proof. Let $w \in \mathbb{Z}^{|V|}$ and consider the primal-dual pair problems

$$\begin{array}{llll}
\max \sum_{v \in V} w_v x_v & & \min \sum_{S \in \mathcal{S}} y_S & \\
\sum_{v \in S} x_v \leq 1 & \forall S \in \mathcal{S}, & \sum_{S \in \mathcal{S}: v \in S} y_S \geq w_v & \forall v \in V, \\
x_v \geq 0 & \forall v \in V, & y_S \geq 0 & \forall S \in \mathcal{S}.
\end{array}$$

If we restrict the x variables to be integral, the problem calls for a maximum weighted clique. Thus, let $\omega(G, w)$ be a maximum weighted clique. First, given feasible solutions x and y respectively for the primal and for the dual, by weak duality we have $\sum_{S \in \mathcal{S}} y_s \geq \sum_{v \in V} w_v x_v$. Observe that $\omega(G, w)$ is an integer feasible solution to the primal, hence, we derive that $\omega(G, w) \leq \sum_{v \in V} w_v x_v^*$ for an optimal solution x^*. Then, by iteratively applying the replication Lemma 2 w_v times to each vertex v, we construct a new graph G'. Since each new vertex is adjacent to all the neighbors in a maximum clique we have $\omega(G, w) = \omega(G')$. The new graph is perfect in view of Lemma 2, thus $\chi(G') = \omega(G')$ and there exist $\chi(G')$ stable sets $S_1, S_2, \ldots, S_{\omega(G')}$ such that the union of these sets covers $V(G')$. Let y_S be the number of copies of the sable set S when is counted back in G, i.e., the restriction of a stable set S' in G' to S with respect to the times that replication procedure is applied to generate S. Hence, $\sum_{S \in \mathcal{S}} y_S = \chi(G') = \omega(G') = \omega(G, w) \leq \sum_{v \in V} w_v x_v^*$. This concludes the proof. $\square$

Corollary 3. *Let G be a perfect graph. Then,*

$$\sum_{v \in K} x_v \leq 1, \quad \forall K \in \mathcal{K} \subseteq G$$

is totally dual integral.

Proof. Observe that the constraint matrix has the characteristic vectors of cliques of G as rows. By taking the complement of G the constraint matrix coincides with the stable set matrix of $\overline{G}$, defined as the matrix having the incidence vectors of stable sets in each row. Hence, by applying the Theorem 22 we get the desired result. $\square$

In light of the observations made so far, it is useful to derive combinatorial consequences and how they are linked to each other. Consider the following polytope

$$P_{QSTAB}(G) := \left\{ x \in \mathbb{R}_+^{|V|} \middle| \sum_{v \in K} x_v \leq 1, \forall\ K \in \mathcal{K} \subseteq V(G), \forall v \in V \right\}$$

In general, $P_{STAB}(G) \subseteq P_{QSTAB}(G)$ by what we argued before, but for the family of *perfect graphs*, the equality holds. This allows giving a polyhedral characterization for this type of graphs. In particular, the graph G is perfect if and only if $P_{QSTAB}(G) = P_{STAB}(G)$.

Theorem 23. *[31, 10] For any graph G the following are equivalent:*

(i) G is perfect, that is, $\omega(G') = \chi(G')$ for any induced subgraph of $G' \subseteq G$.

(ii) $\omega(G', c) = \chi(G', c)$ for any weighted function $c : V \longrightarrow \mathbb{Z}_+^{|V|}$.

(iii) $P_{STAB}(G) = P_{QSTAB}(G)$.

(iv) The complement $\overline{G}$ is perfect.

Proof. Clearly, $(iv) \implies (i)$ since the complement of $\overline{G}$ coincides with the original graph G.

$(i) \implies (ii)$. By Lemma 2 replicating a node in a perfect graph yields a still perfect graph. Starting from a weighted graph (G, c), where c is the weighting function associated with G, for each node $i \in V(G, c)$ by replicating c_i times we obtain unweighted graph G'. This implies that $\omega(G') = \omega(G, c)$ and $\chi(G') = \chi(G, c)$

$(ii) \implies (iii)$. Let $y \in \mathbb{Q}^{|V|}$ be a vector that lies in $P_{QSTAB}(G)$. In order to show that $y \in P_{STAB}(G)$, we write y as a convex combination of incidence vectors of stable sets of G. Let $q \in \mathbb{Z}$ be the least common denominator of the entries of y. Then $qy \in \mathbb{Z}_+^{|V|}$, and due to $y \in P_{QSTAB}(G)$ the following holds for a clique K of maximum cardinality $\omega(G)$:

$$\omega(G, qy) = q \sum_{v \in K} y_v \leq q$$

By (ii), we have $\chi(G, qy) \leq q$. Hence, there exists a family q stable sets $S_1, S_2, \ldots, S_q$ such that each vertex is contained in exactly qy_i each of them.

$$qy = \sum_{i=1}^{q} \chi[S_i]$$

this implies that $y = \frac{1}{q} \sum_{i=1}^{q} \chi[S_i]$, that is, y is written as convex combination of vertices of $P_{STAB}(G)$. This concludes the proof since we have shown that $P_{QSTAB}(G) \subseteq P_{STAB}(G)$.

$(iii) \implies (iv)$ If $P_{STAB}(G)$ is completely determined by clique constraints and non-negative constraints, this holds also for every induced subgraph G' of G. It is enough therefore to show that G can be partitioned into $\alpha(G)$ cliques, since $\omega(\overline{G}) = \alpha(G)$. We use induction on $|V|$. Let $F := \{x \in P_{STAB}(G) \mid \sum_{v \in V(G)} x_v = \alpha(G)\}$, that is, the face induced by the convex combination of all the characteristic vectors of stable sets of size $\alpha(G)$. Since the clique inequalities are facets for $P_{STAB}(G)$, there does exist a clique K such that the face induced by the clique K contains F. But, since K intersects all the stable sets of G, this implies that, for each maximum stable set S of G, $|S \cap K| = \chi[S]^T \mathbf{1}_K = 1$ and thus, $\alpha(G \setminus K) = \alpha(G) - 1$, which is equivalent to say that $\omega(\overline{G} \setminus K) = \omega(\overline{G}) - 1$. By the inductive hypothesis, $G \setminus K$ can be partitioned into $\alpha(G \setminus K)$ cliques. Adding K to this family of cliques, we obtain the clique cover of G using $\alpha(G)$ cliques, or equivalently, an $\omega(\overline{G})$-coloring of $\overline{G}$.

□

Since we have a complete polyhedral characterization of the Stable Set Polytope for perfect graphs, it is natural to consider the following chain of inequalities, which provide typical tools to relate integer and linear programming problems.

$$\begin{aligned}
\alpha(G,w) &= \max\left\{ w^T x \middle| x \in P_{STAB}(G) \right\} \\
&= \max\left\{ w^T x \middle| x \in P_{STAB}(G), \sum_{v\in K} x_v \leq 1, \forall K \in \mathcal{K} \subseteq V, x \in \{0,1\}^{|V|} \right\} \\
&\leq \max\left\{ w^T x \middle| \sum_{v\in K} x_v \leq 1, \forall\ K \in \mathcal{K} \subseteq V, x_v \geq 0, \forall v \in V \right\} \\
&= \min\left\{ \sum_{K\in\mathcal{K}} y_K \middle| \sum_{K\in\mathcal{K}:v\in K} y_K \geq c_v, \forall v \in V, y_K \geq 0, \forall K \in \mathcal{K} \subseteq V \right\} \\
&\leq \min\left\{ \sum_{K\in\mathcal{K}} y_K \middle| \sum_{K\in\mathcal{K}:v\in K} y_K \geq c_v, \forall v \in V, y_K \in \mathbb{Z}_+, \forall K \in \mathcal{K} \subseteq V \right\} \\
&= \overline{\chi}(G,w)
\end{aligned}$$

Observe that the dual problem attached to the LP relaxation of the maximum stable set problem reads as the fractional clique covering problem, thus its integer programming version asks for a clique cover of minimum weight. It follows directly from Theorem [31] that the equality holds throughout the previous chain if and only if the graph G is perfect since the duality gap is zero. As mentioned, perfect graphs brought important combinatorial properties in terms of matrices and structural properties. Consider the following polytope defined by a $0,1$ matrix A.

$$P := \{x \mid Ax \leq \mathbf{1}, x \geq 0\}.$$

A $0,1$ matrix A is *perfect* if the polytope $P := \{x \mid Ax \leq \mathbf{1}, x \geq 0\}$ is integral. In particular, Chvàtal in [10] proved that $\{x \geq 0, Ax \leq \mathbf{1}\}$ is an integral polytope if and only if A is the *clique-node incidence matrix* of a perfect graph, where the clique-node incidence matrix of a graph G is the $0,1$ matrix whose columns are indexed by the nodes of G and whose rows are the characteristic vectors of maximal cliques of G. Thus, the study of perfect matrices boils down to the study of perfect graphs and the latter serves to recognize properties of the associated matrices

Theorem 24. *[10] Let A be a $0,1$ matrix with at least a 1 in each column. Then $P = \{x \geq 0, Ax \leq \mathbf{1}\}$ is integral if and only if A is the clique-node incidence matrix of a perfect graph.*

Proof. " $\Longleftarrow$ " This is a direct consequence of the previous Theorem 23.

" $\Longrightarrow$ " We have to show that if P is an integral polytope, then A is the clique-node incidence matrix of a graph. Define $G = (V, E)$ the graph with $V = \{1, 2, \ldots, n\}$ such that $\{i,j\} \in E$ if and only if $a_{ki} = a_{kj} = 1$, for some row $k \in \{1,2,\dots,m\}$ of A. Suppose by contradiction that A is not a clique-node incidence matrix of a graph. Then, there exists a clique S , denote with $j_1, j_2, \dots, j_{|S|}$ the columns to the vertices of S, such that $a_{kl} = 0$ for every row k of A, for some $l \in j_1, j_2, \dots, j_{|S|}$. Let c be the incidence vector of S, and consider the linear programming problems $z_{LP} := \max\{c^T x \mid Ax \leq \mathbf{1}, x \geq 0\}$ and $z_{IP} := \max\{c^T x \mid Ax \leq \mathbf{1}, x \in \{0,1\}^n\}$. Define the vector $\overline{x}$

$$\overline{x} = \begin{cases} \overline{x}_v = \frac{1}{|S|-1} & \text{if } v \in S, \\ \overline{x}_v = 0 & \text{otherwise.} \end{cases}$$

It is easy to see that by construction $\overline{x} \in P$. In particular, we have that $z_{LP} = c^T\overline{x} = \frac{|S|}{|S|-1} > 1$, whereas $z_{IP} \leq 1$, since S is a clique. But this is a contradiction since we suppose that $z_{LP} = z_{IP}$. $\square$

Theorem 25. *[49] For a 0,1 matrix A, the following statements are equivalent:*

- *The linear system $\{x \geq 0, Ax \leq \mathbf{1}\}$ is TDI.*
- *The matrix A is perfect.*

Proof. " $\implies$ " For every integral vector c let $z_{LP} := \{c^T x \mid Ax \leq \mathbf{1}, x \geq 0\}$. Let x^* be an optimal solution to the problem, then by Strong Duality $c^T x^* = z_{LP} = \mathbf{1}^T y$ for some integral solution y to the dual. This implies that the primal optimal solution x^* is integer, and hence, the result follows.

" $\impliedby$ " The matrix A is perfect if and only if it coincides with the incidence clique-node matrix of a perfect graph by Theorem 24. This is equivalent to say that the corresponding constraint matrix consists of the following constraints

$$\sum_{v \in K} x_v \leq 1, \ \forall K \in \mathcal{K} \subseteq V(G).$$

By the previous Corollary 3, the linear system is totally dual integral. This concludes the proof. $\square$

t-Graphs This class of graphs is based on edge inequalities, non-negativity constraints, and odd-hole constraints. We have $P_{STAB}(G) = P_{CSTAB}(G)$, where

$$P_{CSTAB}(G) := \Bigg\{ x \in \mathbb{R}_+^{|V|} : \quad \begin{aligned} &\sum_{v \in C} x_v \leq \frac{|C|-1}{2} && \forall C \subseteq V, \ |C| \text{ odd cycle} \\ &x_v + x_w \leq 1 && \forall e = \{v,w\} \in E \end{aligned} \Bigg\}.$$

We summarize the most well-known classes of graphs that fall into this family.

- The series-parallel graphs fall into this category. A graph is *series-parallel* if the graph is obtained from disjoint cycle-free subgraphs by repeated application of the following two operations: adding a new edge parallel to an existing edge and subdividing edges, i.e., replacing edges by a path. It is easy to check whether a graph is series-parallel. The author in [53] proved that if G is a series-parallel graph, then $P_{STAB}(G)$ coincides with the Stable Set Polytope of G.

- *Almost-bipartite graphs* have a node v such that the deletion of v makes the graph bipartite.

- *Strongly t-perfect graphs*, that are graphs having no subgraph obtained from subdividing edges of a K_4 such that all four cycles corresponding to the triangles of the K_4 are odd.

The separation problem of an odd-hole constraint calls for a vector $y \in \mathbb{Q}^{|V|}$ which certificates that $y \in P_{CSTAB}(G)$ or outputs an odd-hole inequality violated by y. Such a problem is polynomial solvable as we show in the next Proposition.

Proposition 7. *[31] The separation problem for an odd-hole constraint is polynomial-time solvable.*

Proof. Define for each edge $e = \{v, w\} \in E$ the weight $y_e := 1 - x_v - x_w$. Then, the corresponding vector is non-nonnegative and by summing up all the values along an odd cycle C we obtain:

$$\sum_{e \in E(C)} y_e = |C| - 2 \sum_{v \in C} x_v$$

note that the odd-hole inequality induced by C can be stated in an equivalent way:

$$\sum_{e \in C} y_e \geq 1$$

Hence, the separation problem asks for the minimum length of an odd hole, where the weights on each edge $e \in E$ are defined by y_e. This can be done in polynomial time. Construct an auxiliary weighted directed bipartite graph $H = (V_A \cup V_B, E_H, c)$ in the following way. Split each node $v \in V$ into two nodes $v_a \in V_A$ and $v_b \in V_B$, and for each edge $e = \{u, v\}$, make new arcs $e_1 = \{v_a, u_b\} \in E_H$ and $e_2 = \{u_a, v_b\} \in E_H$ such that $c(e_1) = c(e_2) := y_e$. By construction H is a bipartite graph, since there is no edge between any pair of vertices in V_A and V_B. Then, for all nodes $u \in V$, compute a minimum weighted path from u_a to u_b in the graph H. Observe that the corresponding path has odd cardinality, since u_a and u_b do not lie in the same set of bipartition of vertices. This concludes the proof, since finding a minimum weighted path is polynomial time solvable, and such procedure is repeated $|V|$ times. □

Theorem 26. *[31] The maximum weighted stable set problem in a t-perfect graph can be found in polynomial time.*

Line Graphs. Line graphs fall into the category of rank-perfect graphs due to the findings of Edmonds and Pulleyblank in [24]. They characterize completely the matching polytope by finding a minimal linear description of the Matching Polytope in terms of all facet-defining inequalities. By construction, given a graph G and its line graph $L(G)$, a stable set in $L(G)$ corresponds to a matching in the original graph G. This implies that we have also a complete characterization of the Stable Set Polytope of $L(G)$. Observe that, for each vertex $v \in G$, $\delta(v)$ induces a clique of $L(G)$ which is maximal if and only if v is a vertex of degree greater than 3, or it does not belong to a triangle. Let $Q \subseteq V(G)$ be an induced 2-connected hypomatchable subgraph of G. Then, the subgraph $H \subseteq L(G)$ induced by the set of vertices $E[Q]$ yields a facet-inducing inequality and $\alpha(H) = \nu(Q) = \frac{|Q|-1}{2}$. We can therefore list the facets of the Stable Set Polytope of a connected line graph L with at least two vertices. Let $\mathcal{H}$ be the set of all line graphs of hypomatchable graphs that are 2-connected.

Corollary 4. *Let G be a graph. Then, the following inequalities characterize completely $P_{STAB}(L(G))$*

$$P_{STAB}(L(G)) = \Big\{ x \in \mathbb{R}_+^{|V|} : \quad \sum_{v \in K} x_v \leq 1 \qquad \forall K \in \mathcal{K},\ K \text{ maximal}$$
$$\sum_{v \in H} x_v \leq \alpha(H) \qquad \forall H \subseteq L(G),\ H \in \mathcal{H} \Big\}.$$

2.1.3 Composition of stable set polytopes

We recall some polyhedral aspects associated with the decompositions of graphs, in particular, we examine the decomposition procedure in relation to its polyhedral counterpart. Chvàtal in his seminal paper [10] proved that if a graph has a clique separator, that is, if we delete the clique from the graph we obtain a disconnected graph, then a complete linear description of the Stable Set Polytope is the union of the stable sets of its components. In the proof that follows, we will make use of the following principle stated by Edmonds in [22].

Proposition 8. *[22] Let S be a finite set of solutions of*

$$S \subseteq P := \left\{ x \in \mathbb{R}_+^{|I|} \middle| \sum_{i \in I} a_{ji} x_i \le b_j, \forall j \in J \right\}$$

Then, $conv(S) = P$ if and only if for every vector $c \in \mathbb{Z}^{|I|}$ we have

$$\max\{c^T x : x \in S\} = \min \left\{ \sum_{j \in J} u_j b_j \middle| \sum_{j \in J} u_j a_{ji} \ge c_i, \forall i \in I, u \in \mathbb{R}_+^{|J|} \right\}$$

Theorem 27. *[11] Let $G = (V, E)$ be a connected graph. Let $K \subseteq V$ be a clique separator of G and $V_1, V_2 \subseteq V \setminus K$ be a partition of $V \setminus K$. Then,*

$$P_{STAB}(G) := \left\{ x \in \mathbb{R}^{|V|} : x_{V_1 \cup K} \in P_{STAB}(G[V_1 \cup K]), x_{V_2 \cup K} \in P_{STAB}(G[V_2 \cup K]) \right\}$$

Proof. Let $c \in \mathbb{Z}^{|V|}$ and denote by $U_1 := V_1 \cup K$ and $U_2 := V_2 \cup K$, and $\alpha := \max\{c^T x \mid x \in P_{STAB}(G)\}$. Then, for every vertex $v \in K$ and $i = 1, 2$, define the following values:

$$\begin{aligned} \beta_v^i = \max \sum_{v \in U_i} & c_v x_v \\ & x_v \in P_{STAB}(G[U_i]) && \forall v \in U_i, \\ & x_v = 1. \end{aligned}$$

This problem corresponds to the maximum weighted stable set containing a node from K and the following

$$\begin{aligned} \beta_0^i = \max \sum_{v \in U_i} & c_v x_v \\ & x_v \in P_{STAB}(G[U_i]) && \forall v \in U_i, \\ & x_v = 0 && \forall v \in K. \end{aligned}$$

corresponds to the maximum weighted stable set problem induced on $U_i, i = 1, 2$. Now, consider the following maximization problem

$$\begin{aligned} \alpha_1 = \max \sum_{v \in U_i} c_v x_v + \sum_{v \in K} \beta_0^i x_v - \sum_{v \in K} \beta_v^i x_v & \\ x_v \in P_{STAB}(G[U_i]) & \qquad \forall v \in U_i, i \in \{1,2\}. \end{aligned}$$

We distinguish two cases. Let S^* be a maximum stable set that yields an optimal solution. If $v \notin S^*, \forall v \in K$ then, by definition we have $\alpha_1 = \beta_0^i$. Now, suppose that there is a vertex in K which belongs to S^*. Since any stable set intersects a clique in at most one node, let $\overline{v} \in K$ be such a vertex, we derive the following

$$\sum_{v \in V_i} c_v x_v^* + \sum_{v \in K} c_v x_v^* + \sum_{v \in K} \beta_0^i x_v^* - \sum_{v \in K} \beta_v^i x_v^* = \sum_{v \in S^* \setminus \{\overline{v}\}} c_v + c_{\overline{v}} + \beta_0^i - \beta_{\overline{v}}^i = \beta_0^i$$

Where the last equality is implied by the definition of $\beta_{\overline{v}}^i$, that equals the weight of S^*. As a consequence, the maximum is attained at $\alpha_1 = \beta_0^i$. Hence, given a complete description of $P_{STAB}(G[U_i]), i = 1, 2$, where $\mathcal{J}_i$ represents the indices of the rows of the constraint matrices for the two descriptions, and, a^i the row vector of the constraint matrix

$$\begin{aligned} \sum_{v \in U_i} a_v^i x_v \leq b_i & \qquad \forall i \in \mathcal{J}_i, i \in \{1,2\}, \\ -x_v \leq 0 & \qquad \forall v \in U_i, i \in \{1,2\}. \end{aligned}$$

Then, by applying Proposition 8, there are non-negative reals $\gamma^i, i \in \mathcal{J}_i, i = 1, 2$, such that $\sum_{i \in \mathcal{J}_i} \gamma^i b_i = \beta_0^i$, and the following holds

$$\begin{aligned} \sum_{i \in \mathcal{J}_i} \gamma^i a_v^i \geq c_v & \qquad \forall v \in V_i, i \in \{1,2\}, \\ \sum_{i \in \mathcal{J}_i} \gamma^i a_v^i \geq c_v + \beta_0^i - \beta_v^i & \qquad \forall v \in K, i \in \{1,2\}. \end{aligned}$$

If we set $\beta_0 = \beta_0^1 + \beta_0^2$ and $\forall v \in K$, $\beta_v = \beta_v^1 + \beta_v^2 - c_v$ this implies that $\alpha = \max\{\beta_0, \beta_{v \in K}\}$.

In this case the maximum of

$$\alpha_2 = \max \sum_{v \in K} (\alpha - \beta_0) x_v$$
$$x_v \in P_{STAB}(K) \qquad\qquad \forall v \in K,$$

is equal to $\alpha - \beta_0$. Then, for every vertex $v \in V_1$ the following holds:

$$\sum_{i \in \mathcal{J}_1} \tilde{\gamma}^i a_v^i \geq 0, \qquad\qquad \forall v \in V_1$$
$$\sum_{i \in \mathcal{J}_1} \tilde{\gamma}^i a_v^i \geq \alpha - \beta_0, \qquad\qquad \forall v \in K$$

Define now

$$\overline{\gamma} = \begin{cases} \tilde{\gamma}^i + \gamma^i, & \text{for } i \in \mathcal{J}_1 \\ \gamma^i, & \text{for } i \in \mathcal{J}_2 \end{cases}$$

We shall prove that $\sum\limits_{i \in \mathcal{J}_1 \cup \mathcal{J}_2} \overline{\gamma}^i b_i = \alpha$, and for every $v \in V$, we must have $\sum\limits_{i \in \in \mathcal{J}_1 \cup \mathcal{J}_2} \gamma^i a_v^i \geq c_v$. Then, combining together the previous cases into a more general form we have the following three cases based on the partition of vertices.

- $\forall v \in V_1$, $\sum\limits_{k \in \mathcal{J}_1 \cup \mathcal{J}_2} \overline{\gamma}^k a_v^k = \sum\limits_{i \in \mathcal{J}_1} \tilde{\gamma}^i a_v^i + \sum\limits_{i \in \mathcal{J}_1} \gamma^i a_v^i \geq c_v$,
- $\forall v \in K$, $\sum\limits_{k \in \mathcal{J}_1 \cup \mathcal{J}_2} \overline{\gamma}^k a_v^k = \sum\limits_{i \in \mathcal{J}_1} \tilde{\gamma}^i a_v^i + \sum\limits_{i \in \mathcal{J}_1} \gamma^i a_v^i + \sum\limits_{i \in \mathcal{J}_2} \gamma^i a_v^i \geq (\alpha - \beta_0) + (c_v + \beta_0^1 - \beta_v^1) + (c_v + \beta_0^2 - \beta_v^2) = \alpha - \beta_v + c_v \geq c_v$,
- $\forall v \in V_2$, $\sum\limits_{k \in \mathcal{J}_1 \cup \mathcal{J}_2} \overline{\gamma}^k a_v^k = \sum\limits_{i \in \mathcal{J}_2} \gamma^i a_v^i \geq c_v$.

Finally,

$$\sum_{i \in \mathcal{J}_1 \cup \mathcal{J}_2} \overline{\gamma}^i b_i = \sum_{i \in \mathcal{J}_1} \tilde{\gamma}^i b_i + \sum_{i \in \mathcal{J}_1} \gamma^i b_i + \sum_{i \in \mathcal{J}_2} \gamma^i b_i = \beta_0^1 + (\alpha - \beta_0) + \beta_0^2 = \alpha$$

and this concludes the proof. □

In this Section, we have discussed the Stable Set Polytope of well-known classes of graphs. In the next Section, we will provide a survey on the formulation for the Graph Coloring Problems. Specifically, we report the standard formulation used in the literature and we compare it to other models.

2.2 Graph Coloring Problems

The Vertex Coloring Problem is to assign colors to the elements of a graph in such a way that no two adjacent elements have the same color while minimizing the number of colors used. In the following, we briefly summarize the integer programming formulations proposed in the literature to tackle the problem. We distinguish between the coloring restricted to the vertices as the Vertex Coloring Problem (VCP) and to the edges as the Edge Coloring Problem (ECP).

Vertex Coloring The basic formulation of the VCP is known in the literature as the *Assignment Model*. In this model, given a set K of colors where $|K| > \Delta(G)$, we introduce binary variables x_{vk} for every vertex $v \in V$ and every color $k \in K$, where $x_{vk} = 1$ if the color k is assigned to vertex v and 0 otherwise, and correspondingly the variable z_k indicates if the color k is used or not. The *Assignment Model* reads as:

$$\chi(G) = \min \sum_{k \in K} z_k \tag{2.10}$$

$$\sum_{k \in K} x_{vk} = 1 \qquad \forall v \in V, \tag{2.11}$$

$$x_{vk} + x_{wk} \leq z_k \qquad \forall \{v, w\} \in E, \forall k \in K, \tag{2.12}$$

$$x_{vk} \in \{0, 1\} \tag{2.13}$$

$$z_k \in \{0, 1\}. \tag{2.14}$$

Constraints (2.11) state that every vertex is assigned to exactly one color. Constraints (2.12) impose both that two adjacent vertices v and w cannot receive the same color k. Then, the objective function aims to minimize the number of colors used. Gualandi and Malucelli in [34], raise two shortcomings related to this simple model. The first one is the inherent symmetry present in the ILP Assignment model, as there are $\binom{n}{k}$ ways to select a subset of $\chi(G)$ out of K colors, leading to an exponential number of equivalent solutions. Additionally, the continuous LP-relaxation is quite weak, as it has always a feasible solution of value 2 regardless of the specific graph under consideration. This can be achieved by setting $\forall v \in V$, $x_{v1} = x_{v2} = \frac{1}{2}$ and $x_{vj} = 0$ for $j = 3, \ldots, |K|$, and $z_1 = z_2 = 1$ and all other $z_k = 0$. To address the issue imposed by the symmetry of the problem, Mendez-Diaz and Zabala [61] propose adding the following additional set of constraints:

$$z_k \leq \sum_{v \in V} x_{vk} \qquad \forall k = 1, \ldots, |K|, \tag{2.15}$$

$$z_k \leq z_{k-1} \qquad \forall k = 2, \ldots, |K|. \tag{2.16}$$

The new constraints ensure that the color k is only assigned to some vertex, if color $k-1$ is already assigned to another one. Moreover, they present several sets of constraints that arose from their studies of the polytope. In order to solve the new strengthened ILP model, they developed a branch-and-cut algorithm.

Since any k-coloring in G defines a partition of the nodes V into k stable sets, Mehrotra and Trick [59] proposed the so-called set covering formulation. Let $\mathcal{S}$ be the collection of all stable sets in G. Moreover, let x_S be a binary variable for each stable set $S \in \mathcal{S}$ with value 1 if and only if the stable set S belongs to the partition, and 0 otherwise, then an alternative formulation of VCP is given by

$$\chi(G) = \min \sum_{S \in \mathcal{S}} x_S \tag{2.17}$$

$$\sum_{S \in \mathcal{S} : v \in S} x_S \geq 1 \qquad \forall v \in V, \tag{2.18}$$

$$x_S \in \{0, 1\}, \qquad \forall S \in \mathcal{S}. \tag{2.19}$$

The objective function asks for the minimum number of stable sets to cover all the vertices in V, and, the constraints impose that each vertex belongs to at least to a stable set. One issue linked to this formulation resides in the number of variables, which is exponential in the number of nodes in G. In order to face this issue Mehrotra and Trick proposed a branch-and-price algorithm, starting with a suitable subset of variables and then adding new ones using a column generation scheme. Along this approach, Gualandi and Malucelli [34] proposed a method employing branch-and-price enhanced by constraint programming to compute exact solutions for the graph coloring problem.

Since each vertex in a clique must receive a different color, a natural lower bound for the chromatic number is the following

Proposition 9. *Let G be a graph. Then, $\chi(G) \geq \omega(G)$.*

By using LP-duality we derive the following relation. Let $\chi_f(G)$ be the optimal value of the continuous relaxation of the model (2.17)–(2.19). Clearly, $\chi_f(G)$, known as the *fractional chromatic number* is a lower bound of the $\chi(G)$. In particular, $y = \frac{1}{\alpha(G)} \mathbf{1}_V$ the point belongs to the feasible region associated to the dual, by using weak duality we derive $\chi_f(G) \geq \mathbf{1}^T \frac{1}{\alpha(G)}$, and in particular

$$\chi_f(G) \geq \max_{H \subseteq G} \frac{|V(H)|}{\alpha(H)}$$

Finally, we have the following relation

$$\chi(G) \geq \chi_f(G) = \omega_f(G) \geq \omega(G).$$

Edge Coloring Problem. We refer to an edge coloring of a graph when we color the edges of a graph. The parameter $\chi'(G)$ is known as the *chromatic index* of the graph G and indicates the minimum number of colors used to cover the edges of the graph. Holyer in [39] proved that it is NP-complete to determine the chromatic index of a graph. The following Theorem, known as Vizing's Theorem, shows a lower and upper bound in order to establish the chromatic index.

Theorem 28 (Vizing's Theorem,[76]). *Let G be a graph. Then,* $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$

Hence, to solve the problem we have to distinguish between two integer values. On the other hand, we can compute exactly the value $\chi'(G)$ when G is a bipartite graph.

Theorem 29 (König's Theorem,[21]). *Let G be a bipartite graph. Then,* $\chi'(G) = \Delta(G)$.

The most effective approach to tackle the problem is based on covering the edges of the graph into as few as possible subsets of edges that receive the same color. As for the stable set, the edges in the same color class form a matching. Let $\mathcal{M}$ be the set of all maximal matchings in $G = (V, E)$ and let x_M be the binary variable such that $x_M = 1$ if the matching M is selected and 0 otherwise. We have the following ILP:

$$\chi'(G) = \min \sum_{M \in \mathcal{M}} x_M \tag{2.20}$$

$$\sum_{M \in \mathcal{M} : e \in M} x_M \geq 1 \qquad \forall e \in E, \tag{2.21}$$

$$x_M \in \{0, 1\}, \qquad \forall M \in \mathcal{M}. \tag{2.22}$$

Denote with (MP) the continuous relaxation of the model, and let $\chi'_f(G)$ be the optimal value, known as the fractional chromatic index. By the weak duality Theorem notice that $\chi'_f(G)$ is a lower bound for $\chi'(G)$. Consider now the dual of the Problem MP:

$$\max \sum_{e \in E} y_e \tag{2.23}$$

$$\sum_{e \in M} y_e \leq 1 \qquad \forall M \in \mathcal{M}, \tag{2.24}$$

$$y_e \geq 0 \qquad \forall e \in E. \tag{2.25}$$

The constraints of the dual impose to find non-negative edge weights such that the sum over each matching is not more than one. It is important to notice that a feasible solution to the dual problem yields a lower bound for the optimal solution of the MP. To this end, we have:

Proposition 10. *$\chi'_f(G) \geq \Delta(G)$.*

Proof. Consider a vertex $v \in V(G)$ of degree $\Delta(G)$. In the dual problem, $\Delta(G)$ yields a feasible value for which the associated solution is obtained by setting $y_e = 1$ for each edge $e \in \delta(v)$ over the matching containing the edge e, and 0 for the edges of the same matching. By the weak duality Theorem, we conclude that the $\Delta(G)$ is a lower bound for $\chi'_f(G)$. □

Since we only have to distinguish between the values $\Delta(G)$ and $\Delta(G) + 1$ and since the LP-relaxation gives a lower bound to the value $\chi(G)$, we do not care about the exact optimum, if we know that it exceeds $\Delta(G)$. If $\chi'_f(G) > \Delta(G)$, we have $\chi'(G) = \Delta(G) + 1$, or if we have found an optimal solution to MP that is integral, then we also have found $\chi'(G)$. We summarize this idea in the following Proposition.

Proposition 11. *If $\chi'_f(G) > \Delta(G)$, then $\chi'(G) = \Delta(G) + 1$ and if $\chi'_f(G) = \Delta(G)$ and there is an integral optimal solution to MP, then $\chi'(G) = \Delta(G)$.*

We derive a non-trivial lower bound for the fractional chromatic index. Suppose that G has a k-edge-coloring with the color classes $E_1, E_2, E_3, \ldots, E_k$ where $k = \chi'(G)$. Since each color class corresponds to a matching, we have that $|E_i| \leq \left\lfloor \frac{|V(G)|}{2} \right\rfloor, \forall i \in \{1, 2, \ldots, k\}$. Thus, $|E(G)| \leq \chi'(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor$ and rearranging the terms it says that $\chi'(G) \geq \frac{|E(G)|}{\left\lfloor \frac{|V(G)|}{2} \right\rfloor}$. In particular, for every induced subgraph $H \subseteq G$ this implies that

$$\chi'(G) \geq \chi'(H) \geq \frac{|E(H)|}{\left\lfloor \frac{|V(H)|}{2} \right\rfloor}.$$

We take the maximum overall induced subgraphs $H \subseteq G$. Observe that this value is attained at an odd-induced subgraph. Finally, we get the lower bound

$$\chi'(G) \geq \max_{H \subseteq G, |V(H)| \geq 3, odd} \left\lceil \frac{2|E(H)|}{|V(H)| - 1} \right\rceil$$

The following is a direct consequence of the characterization of the Matching Polytope, thus $\chi'_f(G)$ can be computed in polynomial time.

Proposition 12. *[8] For any graph G, $\chi'_f(G) \geq \max \left\{ \Delta(G), \max_{H \subseteq G, |V(H)| \geq 3, H odd} \left\lceil \frac{2|E(H)|}{|V(H)|-1} \right\rceil \right\}$*

Proof. Consider the linear program defined in (2.20)–(2.21) where the inequality is substituted with equality, and let β be the optimal solution. By dividing the objective function by β we obtain

$$\frac{1}{\beta} \sum_{M \in \mathcal{M}} y_M \chi[M] = \mathbf{1}_E$$

this implies that $(\frac{1}{\beta}, \frac{1}{\beta}, \ldots, \frac{1}{\beta}) \in \mathbb{R}^{|E(G)|}$ is a convex combination of incidence vectors of matchings, thus $\frac{1}{\beta}\mathbf{1}_E \in P_M(G)$. This imposes $|\delta(v)| \leq \beta$ and $(1/\beta) \sum_{e \in E[S]} y_e^* \leq \frac{|S|-1}{2}$, $\forall S \subseteq V$ $|S|$ *odd*. The latter condition amounts to requiring that

$$\beta \geq \max_{S} \frac{|E[S]|}{|S| - 1} \tag{2.26}$$

where the maximum is taken over all odd subsets $S \subseteq V$. Hence, β must be the largest value between $\Delta(G)$ and (2.26). □

Total Coloring. Given a set of k colors and a graph G, a k–total coloring of G is a mapping $\phi : D \to K$ such that $\phi(a) \neq \phi(b)$ for every pair of incident elements $a, b \in D$. The minimum number of colors needed in any total coloring of G is called the total chromatic number, and it is denoted by the parameter $\chi_T(G)$. Hence, the Total Coloring Problem (TCP) asks for finding $\chi_T(G)$. In what follows, we show the total coloring for certain known classes of graphs. We focus more on detail from an integer programming perspective in Chapter 5.

First, we report the following important observation. For any graph G, $\Delta(G) + 1$ is a trivial lower bound on the $\chi_T(G)$. We can pick a vertex v of degree $\Delta(G)$, we need $\Delta(G)$ colors for the edges and one more additional color for the vertex v. Hence, we have the following important consequence

Remark 1. $\chi_T(G) \geq \Delta(G) + 1.$

We recall the Total Coloring Conjecture stated by Behzad and Vizing which estimates an upper bound for the total chromatic number $\chi_T(G)$.

Conjecture 1. *[5, 76]* $\Delta(G) + 1 \leq \chi_T(G) \leq \Delta(G) + 2$.

A graph G is of type 1 if $\chi_T(G) = \Delta(G) + 1$ and type 2 if $\chi_T(G) = \Delta(G) + 2$. In the next Theorem, we show that the *upper bound* is tight for K_n when n is even.

Theorem 30. *Let K_n be the complete graph on n vertices. Then,*

$$\chi_T(K_n) = \begin{cases} n & \text{if } n \text{ odd,} \\ n+1 & \text{otherwise.} \end{cases}$$

Proof. **Case 1** (n odd): We want to obtain a partition of K_n with the minimum number of total matchings, i.e., $V(K_n) \cup E(K_n) = T_1 \cup T_2 \cdots \cup T_k$ where k is the minimum number. Label the vertices $v_0, v_1, \ldots, v_{n-1}$. Define the total matchings $T_i = \{v_i, e_i \mid i = 0, \ldots, \lfloor \frac{n-1}{2} \rfloor \mod n\}$, for $i = 0, \ldots, n-1$ where $e_i = \{v_{i+1}, v_{i-1}\}$. Note that the total matchings T_i are maximal and in particular $\bigcup_{i=1}^{n} T_i = V \cup E$. Since $|V \cup E| = \frac{n(n+1)}{2}$ and $|T| = \frac{n+1}{2}$, we deduce that $\chi_T(K_n) = n$.
Case 2 (n even): Let K_{n+1} be the complete graph obtained by adding a new vertex v to K_n and the edges $e_j = \{v, v_j\}$ for $j = 0, \ldots n-1$. Notice that K_{n+1} is a complete graph of odd order and thus $\chi_T(K_{n+1}) = n+1$. Since K_n is an induced subgraph of K_{n+1}, we have that $\chi_T(K_n) \leq \chi_T(K_{n+1})$. Now, since the partition into total matchings must cover $\frac{n(n+1)}{2}$ elements, every total matching cannot contain more than $\frac{n}{2}$, so $\chi_T(K_n) \geq n+1$. This completes the proof. $\square$

Theorem 31. *Let $K_{r,s}$ be the complete bipartite graph. Then,*

$$\chi_T(K_{r,s}) = \begin{cases} \Delta(K_{r,s}) + 2 & \text{if } r = s \\ \Delta(K_{r,s}) + 1 & \text{if } r \neq s. \end{cases}$$

Proof. First, suppose that $r = s$. Observe that no total matching can contain more than r elements, and since $|V(K_{r,s}) \cup E(K_{r,s})| = r^2 + 2r$, this implies that $\chi_T(K_{r,s}) \geq r + 2$. Now, since $K_{r,r}$ is a bipartite graph we know that $\chi(K_{r,r}) = 2$, that is, we can color one side of the vertex bipartition with one color and a different color for the other partition. Now, by applying König's Theorem 29 we obtain $\chi'(K_{r,r}) = \Delta(K_{r,r}) = r$. Notice that, since $K_{r,r}$ is r-regular, each vertex must be incident to exactly r colored edges. Hence, it is possible to choose r colors for the edges and two more distinct colors for the vertices. Combining the two colorings we obtain that $\chi_T(K_{r,r}) = \chi'(K_{r,r}) + \chi(K_{r,r}) = \Delta(K_{r,r}) + 2$, see Figure 2.4. Now suppose that $r < s$, and let $V(K_{r,s}) = R \cup S$. Also in this case

König's Theorem guarantees a valid s-edge-coloring, then use one more distinct color for the vertices in R. In this case from the partial coloring defined, there are $s-r$ colors missing to each $v \in S$ from the s colors selected for the edges. Let C_v be the set of colors not incident $v \in S$. Now, for each $v \in S$ assign a color $k \in C_v$. Since S is a stable set, this provides a proper total coloring with $\Delta(K_{r,s})+1$ and this concludes the proof in view of Remark 1. $\square$

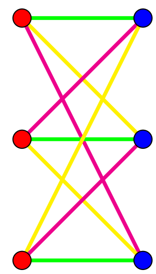


Figure 2.4: Total Coloring of $K_{3,3}$

2.3 Extended formulations and Projections

Most of the integral polytopes associated with combinatorial problems have a huge number of irredundant inequalities. Since it is difficult to handle such representations, it is convenient to introduce a polynomial number of additional variables and a polynomial number of constraints such that the corresponding polytope can be obtained as a projection of a higher dimensional polytope with a reduced number of defining inequalities. For instance, see the Figure 2.5, the Polytope Q has fewer facets than its projection P, thus it is more tractable to optimize a linear problem over Q rather than P. This permits us to solve the problem efficiently in the extended space. We recall the definition of an extended formulation for a polyhedron P.

Definition 9. *Given a polyhedron $P := \{x \in \mathbb{R}^n \mid Ax \leq b\}$ an extended formulation for P is a system $Bx + Cz \leq d$ such that $P := \{x \in \mathbb{R}^n : \exists z \in \mathbb{R}^p \mid Bx + Cz \leq d\}$.*

Given a polyhedron $Q := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^p \mid Bx + Cz \leq d\}$ the *projection of Q* onto the subspace of x is defined as follows:

$$\text{Proj}_x(Q) := \{x \in \mathbb{R}^n \mid \exists u \in \mathbb{R}^p \text{ such that } (x, z) \in Q\}$$

With abuse of notation, we refer to Q as an extended formulation of P. The *size* of the polyhedron Q is the number of inequalities describing Q. Thus, we are mostly interested in extended formulations of polynomial size, which we call *compact* formulations. In

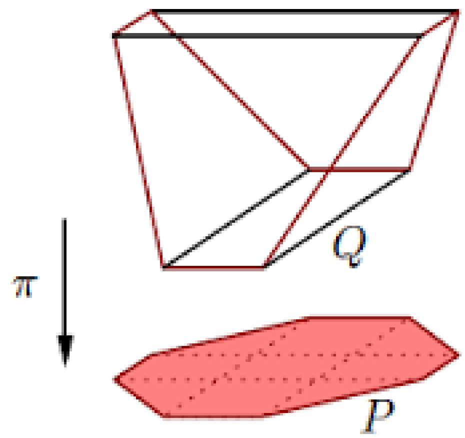


Figure 2.5: A polytope $Q \subseteq \mathbb{R}^3$ with 6 facets and its projection P with 8 facets.

order to project out the additional variables from the extended formulation we make use of the *projection cone*:

$$C_P := \{u \in \mathbb{R}^m \mid u^T C = 0, u \geq 0\}$$

The next Theorem underlines the role of the Projection cone.

Theorem 32 (Projection cone,[3])**.**

$$Proj_x(Q) := \{x \in P \mid (u^T B)x \leq u^T d, \forall u \in C_P\}$$

Proof. " $\Longleftarrow$ " Consider a valid inequality $\alpha^T x \leq \beta$ of Q. By definition of projection, we deduce that $\alpha^T x \leq \beta$ is valid also for $\mathrm{Proj}_x(Q)$. Since we consider vectors $u \geq 0$ such that $u^T C = 0$, it follows that $u^T Bx \leq u^T d$ yields a valid inequality of Q, and hence for $\mathrm{Proj}_x(Q)$.

" $\Longrightarrow$ " Consider a point $\overline{x} \notin \mathrm{Proj}_x(Q)$. Observe that, $\overline{x}$ does not belong to $\mathrm{Proj}_x(Q)$ if and only if the corresponding system $Cz \leq b - B\overline{x}$ is infeasible. Then, by Farkas' Lemma, there exists $u \geq 0, u^T C = 0$ and $u^T Bx > u^T d$. Thus $u^T Bx \leq u^T d$ is a valid inequality for $\mathrm{Proj}_x(Q)$, but it is violated by $\overline{x}$. $\square$

Observe that since $C_P \subseteq \mathbb{R}^m_+$, the projection cone is a pointed polyhedral cone. This implies that C_P is described by a conic combination of finite number rays

$$C_P := \left\{ x \in \mathbb{R}^m \middle| x = \sum_{i=1}^{r} \alpha_i u_i, \alpha_i \geq 0, i = 1, 2, \ldots, r \right\}$$

where $u_1, u_2, \ldots, u_r$ are the extreme rays. In light of this observation, we state the following

Proposition 13. *Let* $Q := \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^p \mid Bx + Cz \leq d\}$*, and, let* $u_1, u_2, \ldots, u_r$ *be the extreme rays of* C_P*. Then* $Proj_x(Q) = \{x \in P \mid (u_i^T B)x \leq u_i^T d, i = 1, 2, \ldots, r\}$*. Hence,* $Proj_x(Q)$ *is a polyhedron.*

We stress the fact the $\text{Proj}_x(Q)$ as given in Theorem 32 may have several redundant inequalities, even if they come from the extreme rays of projection cone C_P. This implies that in general there is not a one-to-one mapping between facet-defining inequalities of the projection and the extreme rays of C_P. Another relevant fact is that if $x \in \mathbb{R}^n$ is a vertex of $\text{Proj}_x(Q)$, then there exists $u \in \mathbb{R}^p$ such that $(x, z) \in Q$ is a vertex. We have the following consequence

Proposition 14. *[3] If* Q *is an integral polyhedron, then* $Proj_x(Q)$ *is an integral polyhedron.*

Now we recall important dimensional aspects of projections. We have the following Proposition.

Proposition 15. *Let* P *be a polyhedron and* F *be a face of* P*. If* Q *is an extended formulation for* P*, then there exists a face* F' *of* Q *such that* $F = Proj_x(F')$*.*

Proof. We denote as $F := \{x \in P \mid \alpha^T x = \beta\}$ the face associated with the inequality $\alpha^T x \leq \beta$ of P. Clearly, $\alpha^T x \leq \beta$ is also a valid inequality for Q, thus $\alpha^T x = \beta$ is associated with a face F' of Q. By definition of extended formulation we obtain the following:

$$F' := \{(x, y) \in Q \mid \alpha^T x = \beta\}$$

Then,

$$\begin{aligned} Proj_x(F') &= \{x \in P \mid \exists y \text{ s.t. } (x, y) \in F'\} \\ &= \{x \in P \mid \exists y \text{ s.t. } (x, y) \in Q, \alpha^T x = \beta\} \\ &= F \end{aligned}$$

□

It is important to point out that, the projection of a face does not correspond to a face of the projection, that is, given a face F of Q, $\text{Proj}_x(F)$ is not always a face of $\text{Proj}_x(Q)$. For instance, as shown in Figure 2.5, the projection of the base of the polytope is no longer a face of the corresponding projection. But, if Q has a face F_β defined by $\beta^T x \leq \beta_0$ then, $\text{Proj}_x(F)$ is face of $\text{Proj}_x(Q)$. Thus, we can state the following:

Proposition 16. *If F is a face of Q defined by an inequality of type $\beta^T x \leq \beta_0$, then $Proj_x(F)$ is face of $Proj_x(Q)$*

Proof. Suppose that $F := \{(x,z) \in Q \mid \beta^T x \leq \beta_0\}$. Since $\beta^T x \leq \beta_0$ is valid for $\mathrm{Proj}_x(Q)$. This implies that $\mathrm{Proj}_x(F) = \mathrm{Proj}_x(Q) \cap \{x \in P \mid \beta^T x \leq \beta_0\}$, and this is by definition a face of $\mathrm{Proj}_x(Q)$. □

This implies that, in general, there is a bijection between faces of Q and faces of $\mathrm{Proj}_x(Q)$ if and only if every face of Q can be defined by a valid inequality of the type $\beta^T x \leq \beta_0$.

Suppose now we deal with more than one polytope $P_1, P_2, \ldots P_r$, it may be useful sometimes to derive an extended formulation from the convex hull of the union of these polytopes. The following example will clarify the motivation behind the use of this technique. Let $N := \{1, 2, \ldots, n\}$ and consider the set $S_{even} := \{x \in \{0,1\}^n \mid \sum_{i \in S} x_i \equiv 0 \mod 2, \forall S \subseteq N\}$. A linear complete description for the convex hull of the set S_{even} is provided, the interested reader can find more details in [14]

$$P_{even} := \left\{ x \in \mathbb{R}^n_+ \middle| \sum_{i \in S} x_i - \sum_{i \in N \setminus S} x_i \leq |S| - 1, \quad \forall S \subseteq N \right\}.$$

Such a formulation is not compact. Now, we introduce the set $S_k = \left\{ x \in \{0,1\}^n \middle| \sum_{i \in S} x_i = k, \quad \forall S \subseteq N \right\}$. We show that a complete linear formulation associated with S_k has a linear number of inequalities.

Proposition 17. *Let $P_k = conv(S_k)$. Then:*

$$P_k = \left\{ x \in \mathbb{R}^n : \sum_{i \in N} x_i = k \right.$$
$$\left. 0 \leq x_i \leq 1, \forall i \in N \right\}.$$

Proof. This is a direct consequence of the total unimodularity of the constraint matrix that has two ones per column. This concludes the proof since we have shown that the polytope is integral. □

Denote with $N_{even} := \{0 \leq k \leq n, k \equiv 0 \mod 2\}$, then we write $S_{even} := \bigcup_{k \in N_{even}} S_k$. Our aim is to obtain a compact extended formulation in terms of the union of the polytopes $\mathrm{conv}(\bigcup_{k \in N_{even}} P_k) = \mathrm{conv}(S_{even}) = P_{even}$. To this end, we make use of the next important Theorem due to Balas.

Theorem 33 (Union of Polytopes, [2]). *Given $P_1, P_2, \dots, P_k \subseteq \mathbb{R}^n$ polytopes and $P = conv(\bigcup_{i=1}^k P_i)$. The following is an extended formulation of P:*

$$
\begin{aligned}
Q = \Big\{ (\lambda, x, \{z_i\}_{1 \le i \le k}) \in \mathbb{R}^{k+n+nk} : \; & x = \sum_{i=1}^{k} z_i \\
& A_i z_i \le \lambda_i b_i && \forall i \in \{1, \dots, k\}, \\
& \sum_{i=1}^{k} \lambda_i = 1 \\
& 0 \le z_i \le \lambda_i u_i && \forall i \in \{1, \dots, k\}, \\
& \lambda \ge 0 && \Big\}.
\end{aligned}
$$

Proof. First, we prove that $P \subseteq Proj_x(Q)$. Fix a point $x \in P$, we have to show that there exist suitable vectors $\lambda, \{z_i\}_{i=1,\dots,k}$ such that $(\lambda, x, \{z_i\}_{i=1,\dots,k}) \in Q$. Since $x \in P$, we can write x as convex combination of points in P, thus $x = \sum\limits_i \lambda_i z_i \in P$, and $\sum\limits_i \lambda_i = 1, \lambda \ge 0$. It is straightforward to see that the point satisfies $A_i \lambda_i z_i \le \lambda_i b_i$ and $0 \le \lambda_i z_i \le u_i$, therefore $(\lambda, x, \{\lambda_i z_i\}_i) \in Q$. For the other direction, let $u := (\lambda, x, \{z_i\}_{1 \le i \le k}) \in Q$, we have to show that $x \in P$. Suppose that $\lambda \in \{0,1\}^k$. Then, since $\sum\limits_i \lambda_i = 1$ there is exactly one index j such that $\lambda_j = 1$ and the others are set to zero. This implies that $x = z_j \in P_j$. Now assume that $\lambda \notin \{0,1\}^k$. We want to write $u = (\lambda, x, \{z_i\}_{1 \le i \le k}) \in Q$ as convex combination of points $\{\tilde{u}^j\}_{1 \le j \le k} \in Q$ as follows. The vectors $\tilde{u}^j := (\tilde{\lambda}^j, \tilde{x}^j, \{\tilde{z}_i^j\}_{1 \le i \le k})$ are defined as $\tilde{\lambda}^j \in \{0,1\}^k$ where $\tilde{\lambda}_i^j = 1$ if $i = j$ and 0 otherwise, $\tilde{z}_i^j = \frac{z_j}{\lambda_j}$, for $i = j$ and 0 otherwise, then we have $\tilde{x}^j = \frac{z_j}{\lambda_j}$,if $i = j$. By plugging these values into the system we deduce that $\tilde{u}^j \in Q, \forall j = 1, \dots, k$. Then, by simple calculations we conclude that $u = \sum\limits_j \lambda_j u^j$. □

Hence, we are able to derive an extended formulation of P_{even}. Let $|N_{even}| = r$ and define

$$
\begin{aligned}
Q := \Big\{ (\lambda, x, \{z^k\}_{k \in N_{even}}) \in \mathbb{R}_+^{r+n+nr} : \; & x = \sum_{k \in N_{even}} z^k \\
& \sum_{i \in N} z_i^k = k \lambda_k && i \in N, k \in N_{even} \\
& \sum_{k \in N_{even}} \lambda_k = 1 \\
& z_i^k \le \lambda_k && i \in N, k \in N_{even} \Big\}.
\end{aligned}
$$

Then, $\mathrm{Proj}_x(Q) = P_{even}$. The Balas' Union Theorem turns out to be one of the most useful tools when we deal with compact formulations in order to derive extended formulations. We shall make use of this Theorem to derive one of the main results exposed in Chapter 4.

3. Total Matching Polytope

The Chapter presents a polyhedral study of the Total Matching Problem (TMP). Part of the new results obtained are shown in [28], and, we extend further with new contributions. Given a graph G, we recall that a total matching is a subset $T \subseteq D$ where the elements are pairwise independent. Hence, stable sets and matchings are total matchings, but the converse may not be true. The Total Matching Problem asks for a total matching of maximum size. Thus, we introduce the Total Matching Polytope, defined as the convex hull of all incidence vectors of all total matchings. In the specific, we present families of valid inequalities for the corresponding polytope based on induced subgraphs. We show that certain classes of the new inequalities are facet-defining for the Total Matching Polytope, and, we provide the corresponding complexity of the separation problems. We point out that to the best of our knowledge, we are the first to propose a polyhedral approach to the problem.

Before presenting the main contributions of this Chapter, we briefly summarize some properties related to this problem and we introduce the relationships between graph parameters linked to the TMP. We recall the following parameters, $\alpha_T(G) := \max\{|T| : T \text{ is a total matching}\}$, $\nu(G) := \max\{|M| : M \text{ is a matching}\}$, $\rho(G) := \min\{|F| : F \text{ is an edge cover}\}$ and $\gamma'(G) := \min\{|F| : F \text{ is an edge dominating set}\}$, where an edge dominating set is a subset $F \subseteq E$ such that each edge not in F is covered by at least one of the elements of F. In [1, 64], the authors show that:

$$\alpha_T(G) \geq \max\{\alpha(G), \nu(G)\}$$

In [1], the authors find a relation between $\alpha_T(G)$ and $\tau_C(G) := \min\{|C| : C \text{ is a total cover}\}$, indeed they show that:

$$\tau_C(G) \leq \alpha_T(G)$$

In [57], Manlove gives an overview of the algorithmic complexities of the decision problems related to the previous parameters. In particular, the author states that $\alpha_T(G)$ can be computed in polynomial time for Trees and it is NP-complete already for bipartite and planar graphs. The Total Matching Problem is less studied in the operations research literature, even though some significant results are obtained for specific classes of graphs, such as cycles, paths, full binary trees, hypercubes, and complete graphs, [45]. Given a graph G, as for the Stable Set Problem and the Matching Problem we know that there exists a relationship between $\alpha(G)$ and $\tau(G)$, and similarly, $\nu(G)$ is linked to $\rho(G)$, it is natural to ask whether there may exist a similar relation between the total matching number and other graph parameters. We report the following Theorem.

Theorem 34 (Gallai-Theorem,[72]). *For any graph G we have*

$$\nu(G) + \rho(G) = n = \alpha(G) + \tau(G)$$

We start by noticing the following observation

Proposition 18. *Let $G = (V, E)$ be a graph and let $T \subseteq V \cup E$ be a total matching. Then T is a maximal total matching if and only if T is a total cover.*

Proof. The proof is by contradiction. Let us suppose that T is maximal total matching and there is an element a not covered by T. Then, $T' = T \cup \{a\}$ is a total matching strictly greater than T. For the other direction, a similar argument applies. This concludes the proof. $\square$

As we show in the next Theorem, we have a relationship between the minimum edge-dominating set of a graph and the total matching number. Before proving the Theorem, we need the following Lemma.

Lemma 3. *A set M is a maximal matching if and only if M is an independent edge dominating set for G.*

Proof. " $\implies$ " The proof is by contradiction. Let us suppose that M is a maximal matching and M is not an edge-dominating set. Then, there exists an edge $e \in E$ not adjacent to any edges of M. Thus, $M' = M \cup \{e\}$ is matching, but we get a contradiction by definition of M.

" $\impliedby$ " Suppose that M is an independent edge dominating set. If M is not maximal then we have a missed edge. We get again a contradiction by definition. $\square$

Theorem 35. *[80] Let G be a graph of order n*

$$\gamma'(G) + \alpha_T(G) = n$$

that is, the sum of the size of the minimum edge dominating and the size of the maximum total matching equals to n.

Proof. For simplicity, given a subset $F \subseteq E(G)$ denote with $V(F)$ the set of the endpoints of F. Let D be a minimum edge dominating set. Observe that $V \setminus V(D)$ forms a stable set of G, and thus $D \cup (V \setminus V(D))$ is a total matching. Therefore, $|D| + |D \cup (V \setminus V(D))| \leq \gamma'(G) + \alpha_T(G)$. Then, let $T = V_T \cup E_T$ be a total matching of maximum size with $|E_T|$ as large as possible. We claim that $V_T \cup V(E_T) = V$. On the converse, there exists a vertex $v \in V \setminus (V_T \cup V(E_T))$, and $w \in V_T$ such that $e := \{v, w\} \in E$. Notice that e must not be incident to a vertex $v \in V(E_T)$, otherwise, let $f := \{u,z\} \in E_T$ and $e = \{u,v\}$, define $E'_T := (E_T \setminus \{f\}) \cup \{u,v\}$. Hence, $\overline{T} := E'_T \cup V_T$ is a maximum total matching containing all the nodes of G. Now, define $T' := (T \setminus \{w\}) \cup \{e\}$. T' is a maximum total matching since $|T'| = |T|$, but $|T \cap E_T| > |E_T|$. This contradicts the choice of E_T. Observe that E_T is an independent edge-dominating set in view of Lemma 3. We conclude since $\gamma'(G) + \alpha_T(G) \leq |E_T| + |T| = n$ and the result follows. $\square$

This Theorem links the complexity of computing the parameter $\gamma'(G)$ to $\alpha_T(G)$. In [57], The author reports that the minimum edge domination parameter γ' remains NP-complete for planar or bipartite graphs of maximum degree 3, planar bipartite graphs, their subdivision, line, and total graphs, perfect claw-free graphs, planar cubic graphs. Whereas the problem of computing $\gamma'(G)$ is polynomial-time solvable for bipartite permutation graphs and cotriangulated graphs, trees, k-outerplanar graphs, and a number of other classes of graphs including claw-free chordal graphs.

From now on, we show the main results of this Chapter.

3.1 Facet inequalities

In this Section, we study the feasible region of the Total Matching Polytope, and we provide nontrivial valid inequalities induced by particular subgraphs and facet-defining inequalities for the corresponding polytope. Given a total matching T, the corresponding characteristic vector is defined as follows.

$$\chi[T] := \begin{cases} z_a = 1 & \text{if } a \in T \subseteq D = V \cup E, \\ z_a = 0 & \text{otherwise.} \end{cases}$$

where $z = (x, y) \in \{0,1\}^{n+m}$, x corresponds to the vertex variables and y to the edges variables. The **Total Matching Polytope** of a graph $G = (V, E)$ is defined as:

$$P_T(G) := \text{conv}\{\chi[T] \subseteq \mathbb{R}^{n+m} \mid T \subseteq D = V \cup E \text{ is a total matching}\}.$$

The following proposition implies that the valid inequalities that are facet-defining are nonredundant, and, hence, they represent a minimal system defining $P_T(G)$.

Proposition 19. *$P_T(G)$ is full-dimensional, that is,* $\dim(P_T(G)) = n + m$.

Proof. We have that the origin, the unit vectors $\chi[\{v\}]$ for every $v \in V$ and $\chi[\{e\}]$ for every $e \in E$ belong to $P_T(G)$, and clearly they are linearly independent. Thus, we have $n + m + 1$ affinely independent points. $\square$

We establish now an important connection between total matchings of a graph and the stable sets of the *total graph*, defined as in [6]. Consider a graph G and its line graph $L(G)$. Starting from $L(G)$, we construct a new graph $H = (V \cup V(L(G)), E(L(G)) \cup E')$, namely, *line-full* graph, where E' is the set of edges connecting the vertices of $L(G)$ to vertices of G, if and only if $v \in V(L(G))$ is an edge of G. Given an edge $e = \{v, w\} \in E$, we call a *doubling* of e the operation that adds a new edge between the end-points v and w.

Definition 10. *Let G be a graph and H its corresponding line-full graph. The graph W obtained from H applying a doubling of an edge for every pair of vertices $\{v, w\} \in V(H) \setminus V(L(G))$ such that $e = \{v, w\} \in E(G)$, is called the* **total graph** *of G.*

Observe that in the line graph, if $|\delta(v)| = r \in \mathbb{N}$, then we have a corresponding clique K_r. In addition, by doubling the edges, we can create triangles in the total graph. Hence, as shown in Figure 3.1, the total graph can be described as the union of cliques K_3 and general cliques. We can prove that total matchings of G correspond to stable sets of its total graph W.

Proposition 20. *Let G be a graph and W its total graph. Then, $P_T(G) = P_{STAB}(W)$.*

Proof. The characteristic vectors of the stable sets of W correspond to the characteristic vectors of total matchings of G, and, hence, the vertices of $P_{STAB}(W)$ are the vertices of $P_T(G)$. $\square$

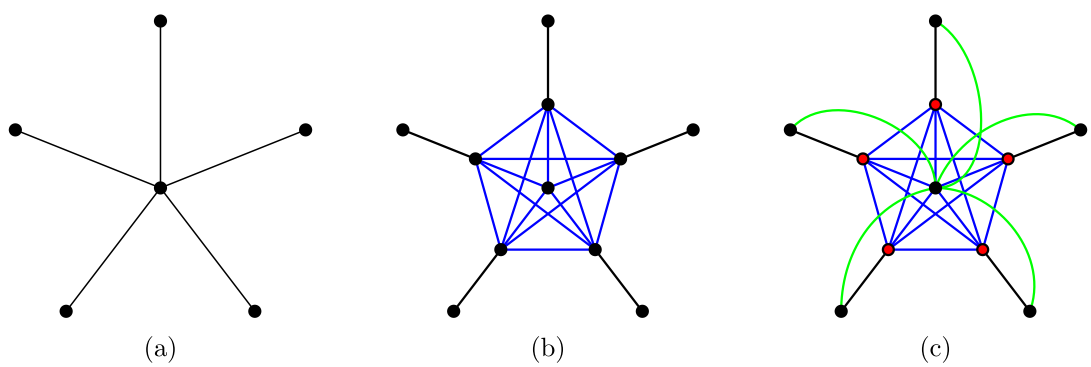


Figure 3.1: (a) The star graph S, (b) the line-full graph of S, (c) the total graph of S.

We start the treatment of the inequalities describing the feasible region of the Total Matching Polytope. We prove that the following inequalities are facet-defining.

Proposition 21. *Let G be a graph. Then, the inequalities*

$$x_v + \sum_{e \in \delta(v)} y_e \leq 1 \qquad \forall v \in V \tag{3.1}$$

$$x_v + x_w + y_e \leq 1 \qquad \forall e = \{v, w\} \in E \tag{3.2}$$

$$x_v, y_e \geq 0 \qquad \forall v \in V, \forall e \in E. \tag{3.3}$$

are facet-defining for $P_T(G)$.

Proof. Let $W = (V', E')$ be the total graph associated to G. Now, consider a vertex $v \in V$ and an edge $e := \{u, t\} \in E$. By construction of W, the subgraphs $W[\delta(v) \cup \{v\}]$ and $W[e \cup \{u, t\}]$ in W correspond to cliques $K_{|\delta(v)|+1}$ and K_3 respectively. Moreover, they are maximal cliques. Then, since $P_{STAB}(W) = P_T(G)$ by Proposition 20 and using the fact that maximal cliques and nonnegativity constraints are facet-defining inequalities for $P_{STAB}(W)$ (e.g., see [66]), we get the result. $\square$

Throughout the thesis, we call inequalities of type (3.1) the *total vertex inequalities* and (3.2) the *total edge inequalities*.

3.1.1 Perfect Total Matchings

A total matching is *perfect* if every vertex of the graph is covered by a total matching, that is, every vertex is either in the total matching or one of its incident edges belongs to the total matching. We prove next, that for any graph G, we can always find a perfect total matching.

Proposition 22. *Every graph G has a perfect total matching.*

Proof. If G has a perfect matching, it is trivial. Otherwise, let us suppose that G has no perfect matching. Given a subset of vertices $S \subseteq V$, let k be the number of odd components of G that is, the number of maximal connected components of odd order. We denote the odd components as $O_1, O_2, \ldots, O_k$. By applying the Tutte's theorem [74, 50], we have $k > |S|$. Notice that, since the maximum size of a matching in an odd component is $\frac{|V(O_i)|-1}{2}$ for $i = 1, 2, \ldots k$, there is a vertex that is not covered by a matching, we call it a left-out vertex. Instead, we have a perfect matching N that covers all the vertices in the even components.

Now, let T be a total matching of G. We show how to construct T so that all the vertices of G are covered by T. First, for each odd component we can construct a maximum matching M_i of size $\frac{|V(O_i)|-1}{2}$ for every $i = 1, 2, \ldots k$, in which we choose as a left-out vertex one of the vertices connecting an odd component to S. Let v_i

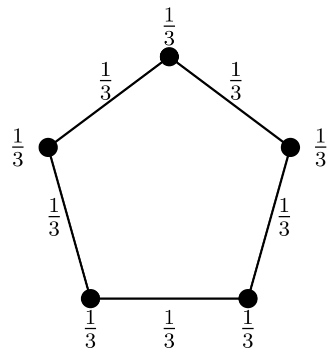


Figure 3.2: A vertex $\boldsymbol{z} = \frac{1}{3}\mathbf{1}$ of the cycle C_5.

be the left-out vertex by M_i of the component O_i for $i = 1, 2, \ldots, k$. Now, take one edge of $|S|$ odd components connecting v_i to the set S and consider the matching $S_O := \{e = \{v_i, s_i\} \mid s_i \in V(S) \text{ for } i = 1, 2, \ldots, |S|\}$. Since $k > |S|$, for each of the remaining components, we have a left-out vertex that cannot be covered by a matching M_i and in particular, in order to form an independent set of elements, we cannot choose an edge connecting S to the odd component. Consider the set L of these vertices and define $T := M_1 \cup M_2 \cdots \cup M_k \cup S_O \cup L \cup N$. Since every vertex is covered by T by construction, the assertion follows. □

The previous proposition allows us to define the Perfect Total Matching Polytope. Let $P_{PT}(G)$ be the convex hull of all perfect total matchings of G.

Proposition 23. *Let G be a graph. The following inequalities are valid for $P_{PT}(G)$.*

$$x_v + \sum_{e \in \delta(v)} y_e = 1 \qquad \forall v \in V \tag{3.4}$$

$$x_v + x_w + y_e = 1 \qquad \forall e = \{v, w\} \in E \tag{3.5}$$

$$x_v, y_e \geq 0 \qquad \forall v \in V, \forall e \in E. \tag{3.6}$$

In practice, for any perfect total matching, the inequalities describing the feasible region of TMP are all tight. In the following section, we introduce nontrivial facet-defining inequalities for the Total Matching Polytope.

3.2 Clique inequalities

In the previous Section, we have proved that all the basic inequalities defining the feasible region of the TMP are facet-defining. In this Section, we introduce some families of nontrivial valid inequalities. First, we will show that the result obtained by Padberg in [66] for a maximal clique on the intersection graph of a set packing problem can be extended to the Total Matching Polytope. This is because the total graph can be seen equivalently as the intersection graph having as ground set all the elements of G. In particular, Padberg shows that any maximal clique on the intersection graph induces a facet-defining inequality for the corresponding set packing polytope. From the above observation, any maximal clique on the total graph induces a facet-defining inequality for $P_T(G)$. However, in the following, we propose direct novel proofs using only the original graph G to show that certain families of valid inequalities that we propose are facet-defining.

Theorem 36. *Let G be a graph and let K_h be a maximal clique of G where $h \geq 3$. Then, the vertex-clique inequality*

$$\sum_{v \in V(K_h)} x_v \leq 1 \tag{3.7}$$

is facet-defining for $P_T(G)$.

Proof. Let G be a graph and let $K_h \subseteq G$ be a maximal clique. We have to exhibit $n+m$ affinely independent points which belong to the face F induced by the inequality (3.7).We know that, since K_h is maximal, by Theorem 2.4 in [66] we can easily construct n of such points belonging to F. Now, fix three distinct vertices u, v, w in $V(K_h)$ and define the total matchings $T_e^v := \{v, e\}$, $\forall e \notin \delta(v)$, the total matchings $T_e^w := \{w, e\}, \forall e \in \delta(v) \setminus \{\{v, w\}\}$, and $T_{v,w}^u := \{u, \{v, w\}\}$. It is easy to see that all the characteristic vectors of the total matchings constructed belong to F. In particular, we have found in total $n+m$ affinely independent points, since the matrix having the columns the characteristic vectors found assumes the following form:

$$\left[\begin{array}{c|c} A_v & B \\ \hline \mathbf{0} & I_e \end{array}\right],$$

where A_v represents the vertex components of the n points and the columns of the matrix B correspond to the restriction of the incidence vectors of the total matchings constructed to the vertex entries indexed by the clique K_h. This completes the proof. $\square$

Observe that we cannot drop the cardinality constraint in Theorem 36. In fact, suppose that there exists a maximal clique K_2 induced by an edge $e := \{v, w\}$. Then, the corresponding inequality reads as $x_v + x_w \leq 1$, which is always dominated by the facet-defining inequality $x_v + x_w + y_e \leq 1$.

3.3 Congruent-$2k3$ cycle inequalities

We proved that the inequalities (3.1)–(3.3) are facet-defining for the Total Matching Polytope, but as we expected due to the complexity of the associated problem, they do not describe the complete convex hull. For instance, Figure 3.2 shows that using only those inequalities, we have that for a cycle C of length 5, the point $z_a = \frac{1}{3}$ for all $a \in V(C) \cup E(C)$ belongs to $P_T(C)$ and it is a vertex. In [45], the author shows that the size of a maximum total matching in a cycle of cardinality $k \in \mathbb{N}$ is equal to $\lfloor \frac{2k}{3} \rfloor$. Hence, we introduce an inequality that cuts off these nonintegral solutions for cycles, which we call the **congruent-$2k3$ cycle inequality**.

Proposition 24. *Let C_k be an induced cycle. Then, if $k \equiv 1 \mod 3$ or $k \equiv 2 \mod 3$, the* **congruent-$2k3$ cycle inequality** *defined as*

$$\sum_{v \in V(C_k)} x_v + \sum_{e \in E(C_k)} y_e \leq \left\lfloor \frac{2k}{3} \right\rfloor \tag{3.8}$$

is facet-defining for $P_T(C_k)$.

Proof. Let $F := \{z \in P_T(C_k) \mid \lambda^T z = \lambda_0\}$ be a facet of $P_T(C_k)$ such that $\tilde{F} := \{z \in P_T(C_k) \mid \tilde{\lambda}^T z = \tilde{\lambda_0}\} \subseteq F$ where the inequality $\tilde{\lambda}^T z \leq \tilde{\lambda_0}$ corresponds to the inequality (3.8). We want to prove that there exists $a \in \mathbb{R}$ such that $\lambda = a\tilde{\lambda}$ and $\lambda_0 = a\tilde{\lambda_0}$. We distinguish two cases based on the parity of the cycle. We label the vertices $V(C_k) := \{v_0, \dots, v_{k-1}\}$, so that v_i is adjacent to v_{i-1} for $i = 0, 1, \dots, k-1 \mod k$, and the edges $E(C_k) := \{e_0, \dots, e_{k-1}\}$, so that $e_i = \{v_i, v_{i+1}\}$ for $i = 0, 1, \dots, k-1 \mod k$.

Case 1: ($k \equiv 1 \mod 3$). Consider the total matching $T_0 := \{v_i, e_{i+1} \mid 0 \leq i \leq k-4, \text{ for } i \equiv 0 \mod 3\}$. This is a maximal total matching, since every element in T_0 is mutually nonadjacent. The number of elements of T_0 is twice the numbers of integers i satisfying the condition, that is, $|T_0| = \frac{2(k-1)}{3}$, and, hence, $\chi[T_0] \in \tilde{F}$ and, in particular, $\chi[T_0] \in F$. Note that the set $\{v_{k-2}, e_{k-2}\}$ is not contained in T_0, because of our description of T_0. Now, consider the total matchings $T_0^- := (T_0 \setminus \{e_{k-3}\}) \cup \{v_{k-2}\}$ and $T_0^+ := (T_0 \setminus \{e_{k-3}\}) \cup \{e_{k-2}\}$. In this way, we obtain two distinct total matchings with the same cardinality, whose characteristic vectors belong to $\tilde{F}$. Since $\chi[T_0^+] \in F$ and $\chi[T_0^-] \in F$, then $\lambda^T\chi[T_0] = \lambda^T\chi[T_0^+]$ and $\lambda^T\chi[T_0] = \lambda^T\chi[T_0^-]$, thus $\lambda_{e_{k-3}} = \lambda_{v_{k-2}} = \lambda_{e_{k-2}}$, where λ_{v_i} is the cost coefficient corresponding to the vertex v_i and λ_{e_i} is the coefficient relative to the edge $e_i = \{v_i, v_{i+1}\}$. Now, consider the function $\sigma : C \longrightarrow C$ such that $\sigma(v_i) = v_{i+1}$ and $\sigma(e_i) = e_{i+1}$. Indeed, σ shifts every element to the next position with respect to the ordering of the vertices and the edges. Composing $k-1$ times the shifting function on T_0, we obtain the following total

matchings $\sigma(T_0), \sigma^2(T_0), \ldots, \sigma^{k-1}(T_0)$. For a fixed i, denote $\sigma^i(T_0) := T_i$. These are still total matchings and each characteristic vector $\chi[T_i] \in \tilde{F}$. Notice also that T_i does not contain $\{v_{i-2}, e_{i-2}\}$, for $i = 1$ the corresponding set is $\{v_{k-1}, e_{k-1}\}$. So, following the same previous procedure, we deduce that $\lambda_{e_{i-3}} = \lambda_{v_{i-2}} = \lambda_{e_{i-2}}$for all $i = 1, \ldots, k-1 \mod k$. This implies that there exists $a \in \mathbb{R}$ such that $\lambda = a\mathbf{1}$. Then, since $\chi[T_i] \in \tilde{F}$, we have that $\lambda^T\chi[T_i] = a(\mathbf{1}^T\chi[T_i]) = a\tilde{\lambda_0}$. We conclude that, since $(\lambda, \lambda_0) = a(\mathbf{1}, \tilde{\lambda_0})$, $\lambda^T z \leq \lambda_0$ is a scalar multiple of the cycle inequality.
Case 2: $(k \equiv 2 \mod 3)$. Consider the total matching $T_0 := \{v_i, e_{i+1} \mid 0 \leq i \leq k-5,$ for $i \equiv 0 \mod 3\} \cup \{v_{k-2}\}$. Notice that now $e_{k-2} \notin T_0$. Also in this case $\chi[T_0] \in \tilde{F}$, since $|T_0| = \frac{2(k-2)}{3} + 1 = \lfloor \frac{2k}{3} \rfloor$. We can construct other two total matchings with the same cardinality $\hat{T_0} := (T_0 \setminus \{v_{k-2}\}) \cup \{e_{k-2}\}$ and $\tilde{T_0} := (\hat{T_0} \setminus \{e_{k-4}\}) \cup \{v_{k-3}\}$. Note that $\chi[\hat{T_0}], \chi[\tilde{T_0}] \in \tilde{F}$, and so they also lie in F. Thus, $\lambda^T\chi[\hat{T_0}] = \lambda^T\chi[T_0]$ and $\lambda^T\chi[\hat{T_0}] = \lambda^T\chi[\tilde{T_0}]$. From the first equality, we deduce that $\lambda_{v_{k-2}} = \lambda_{e_{k-2}}$ and for the second one, $\lambda_{v_{k-3}} = \lambda_{e_{k-4}}$. We conclude as in the Case 1 by applying the shifting function σ, so we have a scalar multiple of the cycle inequality. □

A different proof of this result was first given by Trotter in [73]. By construction, it is possible to notice that the total graph $T(C_k)$ of a congruent-$2k3$ cycle is an antiweb $\overline{W}(p, 3)$, where $p = 2k$ with $k \in \mathbb{N}$, and $q = 3$. If G is itself an antiweb, and if p and q are relatively prime, then the antiweb inequalities are facet-defining. In our case, $p = 2k$ and $q = 3$, thus $gcd(p, q) = 1$ if and only if $k \equiv 1, 2 \mod 3$. We give a direct proof of Proposition (24), without using the total graph, since in the total graph we have a loss in structure, in the sense that we cannot any longer distinguish among vertices and edges of the original graph G. Using another direct proof, we proceed in proving the following important observation.

Proposition 25. *Let G be a graph and let C_4 be the induced cycle of four vertices. Then, the inequality:*

$$\sum_{v \in V(C_4)} x_v + \sum_{e \in E(C_4)} y_e \leq 2 \tag{3.9}$$

is facet-defining for $P_T(G)$.

Proof. Let F be the face induced by the inequality (3.9). Denote as $V(C_4) := \{v_0, v_1, v_2, v_3\}$ and given a vertex $v \in V(C_4)$, let $\overline{\delta(v)} := \delta(V(C_4)) \cap \delta(v)$. By Proposition (24), we can find a set $\mathcal{C}$ of $|V(C_4)| + |E(C_4)|$ affinely independent points belonging to the face F. Now, consider a perfect matching M on C_4, and define the total matchings $T_a := M \cup \{a\}, \forall a \notin \delta(V(C_4)) \cup C_4$, $T_{0,2} := \{v_0, v_2\} \cup \{e\}, \forall e \in \overline{\delta(v_1)} \cup \overline{\delta(v_3)}$, and

$T_{1,3} := \{v_1, v_3\} \cup \{f\}, \forall f \in \overline{\delta(v_0)} \cup \overline{\delta(v_2)}$, and, let $\mathcal{S}$ be the set of their incidence vectors. By construction, each of the total matchings constructed contains exactly two elements, this implies that the corresponding characteristic vectors belong to F. Then, $\mathcal{M} := \mathcal{C} \cup \mathcal{S}$ forms a set of $n+m$ affinely independent points, since the matrix having as columns vectors from $\mathcal{S}$ corresponds to the identity matrix whose support is not contained in the elements of C_4. This completes the proof. □

Dahl in [16] provides an irredundant complete linear description of the Stable Set Polytope for the class of antiwebs $\overline{W}(n,3)$. He shows that besides the clique inequalities and antiweb inequalities, other classes of facets can be obtained from the partition of the vertices into *intervals*, which are defined as subsets of consecutive vertices. To this end, we recall some definitions from his paper to derive facet-defining inequalities in our setting. A 1-interval set is a subset $U \subseteq V(\overline{W}(n,3))$ being the union of intervals $I_1, \dots, I_t$ separated by just one node; for instance, if $n=5$ the set $I = \{\{1,2\},\{4\}\}$ is a 1-interval set. Dahl uses the notation $U = I_1 + \cdots + I_t$ to indicate such a union of intervals.

Theorem 37. *[16] Let $U = I_1 + I_2 + \cdots + I_t \subseteq V(\overline{W}(n,3))$ be a strict subset of V (where I_s are disjoint intervals). Then, the inequality $\sum_{v \in U} x_v \leq \alpha(G[U])$ defines a rank facet of $P_{STAB}(\overline{W}(n,3))$ if and only if*

- *U is a 1-interval set*
- $|I_s| \equiv 1 \mod 3$, *$\forall s = 1,2,\dots,t$*
- *t is odd and $t \geq 3$.*

As mentioned earlier, when we deal with total matchings of a graph, we can strengthen the result by showing that the class of facet-defining inducing inequalities for the antiweb are facet-defining for the entire graph due to the structure of the initial graph with respect to its corresponding total graph.

Proposition 26. *Let G be any graph and let $C_k \subseteq G$ be an induced cycle where k is odd and k is not a multiple of 3. Then, the inequality*

$$\sum_{e \in E(C_k)} y_e \leq \frac{k-1}{2} \tag{3.10}$$

is facet-defining for $P_T(G)$.

Proof. Let F be the face induced by (3.10). Consider the total graph $T(C_k)$, which coincides as observed to $\overline{W}(2k,3)$. Now, define the set U of vertices the edges $e_1, e_2, e_3, \dots, e_k$

corresponding to the nodes of $T(C_k)$, that is, $I_s = \{e_s\}, \forall s = 1, 2, \ldots, k$. Clearly, U forms a 1-interval set satisfying the hypothesis of Theorem 37, thus the inequality (3.10) is facet-defining for $P_T(C_k)$. This guarantees that there are $|V(C_k)| + |E(C_k)|$ affinely independent points belonging to F. Let M be any matching such that $\chi[M] \in F$. Consider an element $a \in D \setminus \delta(V(C_k))$, then $T := M \cup \{a\}$ is a total matching such that $\chi[T] \in F$. Now, consider an element $e \in \delta(V(C_k))$. Since we can pick at most $\frac{k-1}{2}$ edges from C_k, there is a vertex v not covered by M, let $e \in \delta(v) \cap \delta(V(C_k))$. Then, define $T := M \cup \{e\}$. This procedure can be extended to any vertex of the cycle C_k, and therefore this holds for every edge $e \in \delta(V(C_k))$. It is straightforward to see that the matrix having as columns the incidence vectors of total matchings defined has maximum rank. This completes the proof. □

The result can be further generalized to any subset $U \subseteq C_k$ which satisfies the properties of Theorem 37. By taking any subset of edges, (that must be consecutive in light of the Previous Theorem), the same property holds. The proof runs along the same argument.

Proposition 27. *Let U be a 1-interval set satisfying the hypotheses of theorem 37 and corresponding to a subset of edges of C_k. Then,*

$$\sum_{e \in U} y_e \leq \frac{|U| - 1}{2}$$

is facet-defining for $P_T(G)$.

In particular, we succeed in proving that if the graph is cubic, then the congruent-$2k3$ cycle is facet-defining for the Total Matching Polytope of the entire graph. We are mainly interested in the study of these graphs due to the conjectures attributed to them. In what follows, the cubic graphs of our interest do not have trivial vertices, that is, we may assume that every vertex out of an induced hole C has at most two neighbors in C. This assumption allows stating the following Theorem.

Theorem 38. *Let G be a cubic graph and consider an induced cycle $C_k, k \equiv 1, 2 \mod 3$. Then,*

$$\sum_{v \in V(C_k)} x_v + \sum_{e \in E(C_k)} y_e \leq \left\lfloor \frac{2k}{3} \right\rfloor$$

is facet-defining for $P_T(G)$.

Proof. We label the vertices and the edges as in the proof 24. Let $\lambda^T z \leq \lambda_0$ be a congruent-$2k3$ cycle inequality and we denote as F the corresponding face associated with it. Notice that, since $\lambda^T z \leq \lambda_0$ is facet-defining for $P_T(C_k)$ by Propositon 24, this implies that we have $|V(C_k)| + |E(C_k)|$ affinely independent points belonging to F. We recall that $N_G(V(C_k))$ is the set of neighbors of vertex set of C_k and, $\delta(V(C_k))$ is the set of edges with an end-point in $V(C_k)$ and the other in $V \setminus V(C_k)$. First, let T be a maximum total matching of C_k and fix an element $a \notin C_k \cup \delta(V(C_k)) \cup N_G(V(C_k))$. Then, $T_a := T \cup \{a\}$ is a total matching such that $\chi[T_a] \in F$. We distinguish the cases based on the parity of the cycle.

Case 1: ($k \equiv 1 \mod 3$). As shown in the proof of Proposition 24, for every vertex $v \in V(C_k)$ there exists a total matching excluding v. Denote by T_{v_i} the total matching which does not include the vertex v_i. Then, we can construct the total matchings $T_{v_i}^f := T_{v_i} \cup \{f\}, f \in \delta(v_i) \cap \delta(V(C_k)), \forall i \in \{0, 1, \ldots, k-1\}$, such that the corresponding characteristic vectors belong to F. Notice that G is cubic and by assumption $|N_G(v) \cap V(C_k)| \leq 2, \forall v \in V \setminus V(C_k)$. If a vertex $u \in N_G(V(C_k))$ has exactly one neighbor $v_s \in V(C_k)$, then consider the total matching T_{v_s} excluding the vertex v_s and construct $T := T_{v_s} \cup \{u\}$. It is easy to see that $\chi[T_u] \in F$, since u is arbitrary, this holds for all the vertices in $N_G(V(C_k))$ with exactly one neighbor in C_k. Now, consider the other case. Let $\{v_i, v_j\}$ be the set of neighbors of a vertex $z \in N_G(V(C_k))$. By the previous observation, there exists a maximum total matching excluding the vertex v_i, for an arbitrary index i. Denote with $d(v, w)$ the distance between two nodes of C_k with respect to the ascending ordering of the vertices in C_k, that is, the length of the path between v and w. If $d(v_i, v_j) \equiv 0, 2 \mod 3$, construct a maximum total matching $T_{v_i} := \{v_{i+(3j+1)}, e_{i+(3j+2)} \mid 0 \leq j \leq \frac{k-4}{3} \mod k\}$, otherwise if $d(v_i, v_j) \equiv 1 \mod 3$ define $T_{v_i} := \{e_{i+(3j+1)}, v_{i+(3j+3)} \mid 0 \leq j \leq \frac{k-4}{3} \mod k\}$. By our description of T_{v_i}, $v_j \notin T_{v_i}$, hence $T_z := T_{v_i} \cup \{z\}$ is a total matching of G. Since T_{v_i} is a maximum total matching of C_k, we have that $\chi[T_z] \in F$, and this clearly holds $\forall z \in N_G(V(C_k))$ by applying the same argument. Now, observe that the matrix composed of the incidence vectors of total matchings introduced previously and the affinely independent points relative to C_k has maximum rank.

Case 2: ($k \equiv 2 \mod 3$). The proof runs along a similar argument to the previous case. Following the proof in 24 there exists a maximum total matching of C_k excluding an edge or a vertex. Let T_{v_i} be a total matching missing the vertex v_i, and, T_{e_i} be a total matching missing an edge e_i. For all the edges $e \in \delta(V(C_k))$, by construction we can apply the same argument of the previous case, we can define a total matching $T \cup \{e\}$, where T is a maximum total matching of C_k. Now, let $z \in N_G(V(C_k))$ and $\{v_i, v_j\} \in V(C_k)$ be the set of neighbors of u. If $d(v_i, v_j) \equiv 0, 2 \mod 3$ consider the total matching $T_i := \{e_{i+(3j+2)}, v_{i+(3j+4)} \mid 0 \leq j \leq \frac{k-5}{3} \mod k\} \cup \{e_i\}$. If $d(v_i, v_j) \equiv 1 \mod 3$ construct the total matching $T_i := \{v_{i+(3j+2)}, e_{i+(3j+3)} \mid 0 \leq j \leq \frac{k-5}{3} \mod k\} \cup \{e_i\}$.

Both are maximum total matchings of C_k, then $T_z := T_i \cup \{z\}$. This implies that $\chi[T_z] \in F$, since z is arbitrary this holds for all the vertices. We conclude in the same way as the previous case. This concludes the proof. $\square$

In the following, we derive other valid inequalities induced by standard well-known subgraphs for the stable set polytope. We start first to derive the maximum size of a total matching of the antiholes. In the following Proposition we only consider non-degenerate antiholes, that is, no complement of C_3 and C_4 is allowed.

Proposition 28. *Let H be an antihole of k vertices where $k \geq 5$. Then:*

$$\alpha_T(H) = \left\lfloor \frac{k}{2} \right\rfloor + 1$$

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_k\}$ be the vertices of H. Let T be a total matching of maximum size. Because every node is incident to each other vertex except for the previous and the next node with respect to the sequence of vertices, T can contain at most two nodes. W.l.o.g. suppose that $v_1, v_2 \in T$. Now, fixing two nodes in T, observe that a matching can contain at most $\frac{k-2}{2}$ edges if k is even and $\frac{k-3}{2}$ edges if k is odd. Assume that k is even, then there exists a matching $M := \{\{v_{k-i}, v_{3+i}\} \mid i \geq 0, i = 0, 1, \ldots, \frac{k}{2} - 4\} \cup \{\{v_{\frac{k}{2}}, v_{\frac{k+4}{2}}\}, \{v_{\frac{k+2}{2}}, v_{\frac{k+6}{2}}\}\}$, otherwise define $M' := \{\{v_{k-i}, v_{3+i}\} \mid i = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor - 2\}$. Thus, if k is even $|T| = \frac{k-2}{2} + 2 = \frac{k+2}{2}$, instead if k is odd $|T| = \frac{k-3}{2} + 2 = \frac{k+1}{2}$. This completes the proof since the bound obtained is the best possible by construction. $\square$

Proposition 29. *Let H be an antihole of a graph G of k vertices. Then,*

$$\sum_{v \in V(H)} x_v + \sum_{e \in E(H)} y_e \leq \left\lfloor \frac{k}{2} \right\rfloor + 1$$

is a valid inequality for $P_T(G)$.

Proposition 30. *Let W_k be a wheel with $k \equiv 1, 2 \mod 3$. Then:*

$$\alpha_T(W_k) = \left\lfloor \frac{2k}{3} \right\rfloor + 1$$

Proof. Let $V(W_k) = \{v_1, v_2, \ldots, v_k\} \cup \{v_c\}$ the nodes of W_k, where v_c represents the central vertex of W. Since C_k is an induced subgraph of W_k, then a maximum total matching T contains at most $\frac{2k}{3}$ elements from C_k, hence $|T| \geq \lfloor \frac{2k}{3} \rfloor$. Construct a total matching as in the proof 24, then it results that there exists a vertex $v_i \in C_k$ which is not part of T. Since the central vertex is incident to all the vertices of the hole, $T' := T \cup \{v_c, v_i\}$ forms a total matching of size $\lfloor \frac{2k}{3} \rfloor + 1$. This concludes the proof. $\square$

Proposition 31. *Let W_k be a wheel with $k \equiv 1, 2 \mod 3$. Then the odd total wheel inequality:*

$$\sum_{v\in V(W_k)} x_v + \sum_{e\in E(W_k)} y_e \leq \left\lfloor \frac{2k}{3} \right\rfloor + 1$$

is a valid inequality for $P_T(G)$.

Notice that, the odd total wheel inequality is valid for the Total Matching Polytope but it is not facet in any case, since it can be written as the sum of the facet-defining inequality

$$x_c + \sum_{e\in\delta(c)} y_e \leq 1,$$

and

$$\sum_{v\in V(C_k)} x_v + \sum_{e\in E(C_k)} y_e \leq \left\lfloor \frac{2k}{3} \right\rfloor .$$

3.3.1 Separation of congruent-$2k3$ cycle inequalities

In this part, we deal with the problem of separating the inequalities given by the class of the congruent-$2k3$ cycle inequalities. Given a fractional optimal solution of the LP relaxation of the Total Matching Problem, the separation for the congruent-$2k3$ cycle inequalities consists of either finding an inequality in this class that is violated by a cycle inequality or proving that all inequalities are satisfied. To this end, we propose an Integer Linear Programming formulation for solving this separation problem.

Let (c_v, w_e) be the fractional optimal value to the current LP problem, and let x_v and y_e denote the decision variables of the problem of finding a congruent-$2k3$ cycle in a graph G. The separation problem involves maximizing the following value

$$\alpha := \sum_{v\in V} c_v x_v + \sum_{e\in E} w_e y_e - \left\lfloor \frac{2k}{3} \right\rfloor , \tag{3.11}$$

where k is the cardinality of the cycle induced by the variables x_v and y_e. Hence, we want to detect a maximum weighted cycle, where node and edge weights are (c_v, w_e), and the cycle contains a number of nodes that is not a multiple of three. Whenever $\alpha > 0$, we have found a violated cycle. Otherwise, all the congruent-$2k3$ cycle inequalities are satisfied. Since $k \equiv 1, 2 \mod 3$, we can express $k = 3z + t$ where $z \in \mathbb{Z}$ and $t \in \{1, 2\}$, and we can rewrite the floor expression in (3.11) as follows

$$\left\lfloor \frac{2k}{3} \right\rfloor = \left\lfloor \frac{2(3z+t)}{3} \right\rfloor = \begin{cases} 2z & \text{if } t = 1 \\ 2z+1 & \text{if } t = 2, \end{cases}$$

and, hence, we get

$$\left\lfloor \frac{2k}{3} \right\rfloor = 2z + t - 1. \tag{3.12}$$

Another important element of our ILP model for the separation of congruent-$2k3$ cycle inequalities resides in the connectivity constraints, which we formulate exploiting the ideas presented in [54], by setting a network flow model. Given the original graph $G = (V, E)$ the flow networks is defined as $H = (V, A)$, where $A := \bigcup_{\{i,j\} \in E} \{(i, j), (j, i)\}$. The network H has a single source node that introduces all the flow, while every node that belongs to the cycle is the sink of a single unit of flow. However, we do not fix in advance the source node, and we let variables $s_i \in \{0, 1\}$ for $i = 1, \ldots, n$ to indicate which node of H is the source. Then, we introduce the variables $u_i \in \mathbb{Z}_+$ for every vertex $v_i \in V$ to indicate the overall amount of flow originated at the only source node i having $s_i = 1$. Indeed, we have that $u_i > 0$ only for the sink node. The complete ILP model for the separation of congruent-$2k3$ cycle inequalities is the following:

$$\begin{aligned}
\max \quad & \sum_{v \in V} c_v x_v + \sum_{e \in E} w_e y_e - (2z + t - 1) && && (3.13)\\
\text{s.t.} \quad & \sum_{e \in \delta(v)} y_e = 2x_v && \forall v \in V, && (3.14)\\
& \sum_{v \in V} x_v = 3z + t && && (3.15)\\
& x_i + \sum_{(i,j) \in A} f_{ij} = u_i + \sum_{(j,i) \in A} f_{ji} && \forall i \in V, && (3.16)\\
& \sum_{i=1}^{n} s_i = 1 && \forall i \in V, && (3.17)\\
& u_i \leq n \cdot s_i && \forall i \in V, && (3.18)\\
& f_{ij} \leq n \cdot y_e && \forall (i, j) \in A, && (3.19)\\
& y_e, x_v \in \{0, 1\} && \forall v \in V, \forall e \in E, && (3.20)\\
& u_i \in \mathbb{Z}_+ && \forall i \in V, && (3.21)\\
& z \in \mathbb{Z}_+, t \in \{1, 2\}. && && (3.22)
\end{aligned}$$

The objective function (3.13) includes the relation specified in (3.12). Constraints (3.14) ensure that the subgraph induced by the variables x_v and y_e is a union of disjoint cycles, since every node has either degree zero or two. Constraints (3.15) impose the congruence on the length of the cycle, which cannot be a multiple of three. Constraints (3.16) impose the flow conservation at every node, and constraints (3.17) impose that a single vertex is the origin of the flow. Constraints (3.18) impose that all the vertices but the source have $u_i = 0$, that is, they do not originate any unit of flow. For every flow variable f_{ij}, constraints (3.19) set the capacity of the flow variables to zero whenever $y_e = 0$, that is, whenever arc e is not included in the cycle.

The ILP model (3.13)–(3.22) permits us to look for the most violated congruent-$2k3$ cycle inequality by solving a single problem. Alternatively, we could solve a simplified version of the separation problem by fixing in advance both the source node s_i and the value of variable t. In this way, to find the most violated inequality, we have to solve two (easier) subproblems for every node, for a total of $6n$ subproblems. However, each subproblem reduces to a Shortest Path Problem defined on an auxiliary directed graph having nonnegative weights, as shown in the proof of the following proposition.

Proposition 32. *The separation problem of the congruent-$2k3$ cycle inequality is in polynomial time solvable.*

Proof. The separation problem consists of a sequence of $2n$ Minimum Weighted s,t-Path problems from a source node s to the target node t of an auxiliary graph. Let $G = (V, E)$ be a weighted graph where (c_v, w_e) are the optimal values of the current LP relaxation. Starting from $G = (V, E)$, we construct a weighted directed graph $H = (N, A)$ in the following way. For every vertex $v \in V$, we introduce three nodes labelled as v_0, v_1, v_2 in N. Now, for each edge $e = \{v, w\} \in E$, we introduce three arcs $a_i \in A$ with respect to the permutation $\sigma = (012)$, that is, $a_i = (v_i, w_{\sigma(i)})$, with $i = 0, 1, 2$. Observe that a path from v_0 to v_1 gives a path P_k of size $k \equiv 1 \mod 3$, and a path from v_0 to v_2 gives a path P_k of size $k \equiv 2 \mod 3$. Next, we distinguish the two cases.

Case 1: ($k \equiv 1 \mod 3$) In this case, we have $\lfloor \frac{2k}{3} \rfloor = \frac{2(k-1)}{3}$, and the separation problem reads as follows:

$$\exists C_k \ : \ \frac{2}{3}|C_k| - \sum_{v \in V(C_k)} c_v x_v - \sum_{e \in E(C_k)} w_e x_e < \frac{2}{3}.$$

Since we look for the most violated inequality, for each node $v \in V$, the separation problem is equivalent to a Minimum Weighted s,t-Path Problem where the source is v_0 and the target is v_1. Now, we define the costs on the arcs as $l_{a=(i,j)} := \frac{2}{3} - c_i - w_{e=\{i,j\}} + 1$, for every $a = (i, j) \in A$. We know that $c_i + c_j + w_{e=\{i,j\}} \leq 1$ due to feasibility of constraints (3.2) and, hence, the costs are positive. Let P_1 be a minimum weighted path in H from v_0 to v_1. By construction, the path P_1 in H corresponds to a cycle C_k in G of length $k \equiv 1 \mod 3$, where for each node $v_i \in N$ we consider the corresponding node $v \in V$. If we sum up all the costs on the path P_1, we obtain:

$$l(P_1) := \sum_{(i,j)\in A(P_1)} l_{(i,j)} = \frac{2}{3}|P_1| - \sum_{i\in V(C_k)} c_i - \sum_{e\in E(C_k)} w_e + |P_1|. \tag{3.23}$$

Hence, the path P_1 yields a violated congruent-$2k3$ cycle C_k in G if and only if $l(P_1) - |P_1| < \frac{2}{3}$.

Case 2: $(k \equiv 2 \mod 3)$ In this case, we have $\left\lfloor \frac{2k}{3} \right\rfloor = \frac{2(k-2)}{3}$, and the separation problem reads as follows:

$$\exists C_k \ : \ \frac{2}{3}|C_k| - \sum_{v\in V(C_k)} c_v x_v - \sum_{e\in E(C_k)} w_e x_e < \frac{4}{3}.$$

Hence, we have to find a minimum weighted path P_2 from v_0 to v_2 for each node v in V. We define the arc costs l_a as before, and we get a maximum violated cycle if and only if $l(P_2) - |P_2| < \frac{4}{3}$.

In conclusion, by solving $2n$ shortest path problems on a directed graph with positive weights, we get the most violated congruent-$2k3$ cycle inequalities in polynomial time.

□

In our case we have a generalization of the odd cycle inequalities by paths in tripartite graphs, instead of bipartite graphs, see [62].

3.4 Even and odd clique inequalities

At this point, we focus on valid inequalities that can be derived by complete subgraphs K_h of G, with $h \leq n$. This leads to consider the following valid inequality.

Proposition 33. *Let G be a graph, and let K_h a clique of order $h \leq n$ of G. Then,*

$$\sum_{v\in V(K_h)} x_v + \sum_{e\in E(K_h)} y_e \leq \left\lceil \frac{h}{2} \right\rceil \tag{3.24}$$

is a valid inequality for $P_T(G)$.

In particular, when the subgraph K_h has even cardinality, we get the following result.

Proposition 34. *Let K_h be a complete graph, where $h \in \mathbb{N}$ is an even number. Then, the* **even-clique inequality** *defined as*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h}{2} \tag{3.25}$$

is facet-defining for the Total Matching Polytope $P_T(K_h)$.

Proof. Let $G = K_h$ be a complete graph, where $h = 2l$ for $l \in \mathbb{N}$ and let $V(K_h) := \{v_1, v_2, \ldots, v_{2l}\}$ and $E(K_h) := \{e_{i,j} = \{v_i, v_j\} \mid \forall i, j \in \{1, 2, \ldots, 2l\}, i \neq j\}$. First, we show that the even-clique inequalities are valid for $P_T(G)$. Since K_h is a complete graph of even order, it admits a perfect matching M. Notice that any stable set S intersects K_h in at most one vertex, thus a maximum total matching T can be obtained by a perfect matching, or by deleting from a perfect matching an edge $e = \{i, j\}$ and adding one of its endpoints. This implies that $|T| \leq l$. Next, we prove that the face induced by an even-clique inequality is facet-defining. To this end, consider a face $F := \{z \in P_T(G) \mid \lambda^T z = \lambda_0\}$ and let $F' := \{z \in P_T(G) \mid \tilde{\lambda}^T z = \tilde{\lambda}_0\}$, where $\tilde{\lambda}^T z \leq \tilde{\lambda}_0$ corresponds to the even-clique inequality. Suppose that $F' \subseteq F$, we want to show that every inequality of F is a scalar multiple of the even-clique inequality. Place the vertices $v_1, v_2, \ldots, v_{2l-1}$ at equal distances on a circle and place v_{2l} in the center. Starting from this configuration, we show a decomposition of K_h into disjoint union of perfect matchings, such that $E(K_h) = M_1 \cup M_2 \cup \cdots \cup M_{h-1}$. Notice also that a perfect matching M can be naturally identified as a total matching. Now, fix an index i and consider the edge that connects a vertex v_i to the center v_{2l} of the circle, we call $c_i = \{v_i, v_{2l}\}$ the *central edge*, and consider the set of edges $E_i := \{e_{i+j,i-j} = \{v_{i+j}, v_{i-j}\} \mid \forall j \in \{1, \ldots, \frac{h}{2} - 1\}\}$, where the indexes run modulo $h - 1$. It turns out that $M_i := E_i \cup \{c_i\}$ is a perfect matching. In this way, repeating the same construction we can form $h - 1$ distinct perfect matchings M_i, with $\chi[M_i] \in F'$, for all $i \in \{1, 2, \ldots, 2l - 1\}$. Now, we can construct a total matching with the same cardinality of the perfect matchings just constructed. Consider an edge $e = \{v_j, v_k\} \in M_i$ of a fixed perfect matching M_i. Then, $T_k := (M_i \setminus \{e_{j,k}\}) \cup \{v_k\}$ and $T_j := (M_i \setminus \{e_{j,k}\}) \cup \{v_j\}$ are total matchings. Observe that $\chi[T_j] \in F'$ and $\chi[T_k] \in F'$, in particular these characteristic vectors lie on F. This implies that $\lambda_{v_j} = \lambda_{v_k} = \lambda_{e_{j,k}}$, since $\lambda^T \chi[T_j] = \lambda^T \chi[T_k] = \lambda^T \chi[M_i]$, where we denote as λ_a the cost coefficient for the element $a \in D = V \cup E$. In particular, we apply this construction for all the edges of the same perfect matching M_i. Repeating the same argument for all the perfect matchings in the decomposition, we obtain that $\lambda_v = \lambda_e$ for $e \in \delta(v)$, $\forall v \in V$, and since the cost coefficients for the endpoints of each edge are the same by construction, and we consider only perfect matchings (we can touch each vertex), we deduce that there exists $a \in \mathbb{R}$ such that $\lambda = a\mathbf{1}$. Thus, this implies that

$\lambda_0 = a\frac{h}{2}$. We conclude that $\lambda^T z \leq \lambda_0$ is a scalar multiple of the even-clique inequality since $(\lambda, \lambda_0) = a(\mathbf{1}, \frac{h}{2})$. This completes the proof. $\square$

Now, we are ready to prove the main theorem of this section.

Theorem 39. *Let G be a graph, and let K_h be a complete subgraph of G, where h is even. Then, the* **even-clique inequality** *defined as*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h}{2}$$

is facet-defining for the Total Matching Polytope $P_T(G)$.

Proof. Let K_h be a complete subgraph of even order of G. We denote as F the face induced by the even-clique inequality. By Proposition 34, we can find $|V(K_h)|+|E(K_h)|$ affinely independent points satisfying at equality the even-clique inequality. Now, fix a perfect matching M of $G[V(K_h)]$. Since $M \cap \{u\} = \emptyset$ for every $u \notin V(K_h)$, $T_u := M \cup \{u\}$ is a total matching. Observe that, $\chi[T_u] \in F$. Thus, the set of characteristic vectors $\{\chi[T_u] \mid \forall u \notin V(K_h)\}$ is contained in F and the corresponding $|V \setminus V(K_h)|$ points are affinely independent. Clearly, it is easy to see that they are still affinely independent with respect to the previous points, so we have n points up to now. Similarly, $T_e := M \cup \{e\}$ for every $e \notin \delta(V(K_h)) \cup E(K_h)$ is a total matching, since $M \cap \{e\} = \emptyset$. Consequently, also the set of vectors $\{\chi[T_e] \mid \forall e \notin \delta(V(K_h) \cup E(K_h)\}$ is contained in F, and the corresponding points are affinely independent. Now, let $S := \{v \in V(K_h) \mid \delta(V(K_h)) \neq \emptyset\}$. We can construct a total matching $T_{\overline{s}} := (M_s \setminus \{e\}) \cup \{\overline{s}\}$, where $e = \{s, \overline{s}\} \in E(K_h)$, $s \in S$ and M_s is a perfect matching of $G[V(K_h)]$ with one end-point in s. Then, $T_{\overline{e}_s} := T_{\overline{s}} \cup \{\overline{e}_s\}$ for every $\overline{e}_s \in \delta(V(K_h)) \cap \delta(s)$, is a total matching whose characteristic vector lies on F. Repeating the same construction for all the edges $e_s \in \delta(V(K_h))$, we can obtain distinct total matchings for every $s \in S$ whose characteristic vectors belong to F, where the corresponding points are affinely independent. In this way, we have found $n + m$ affinely independent points belonging to F, since we can rearrange the rows of the matrix having as columns these points in such a way that we get the following form:

$$\left[\begin{array}{c|c|c} A_{K_h} & B_{K_h} & C_{K_h} \\ \hline \mathbf{0} & \widetilde{I}_v & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \widetilde{I}_e \end{array}\right],$$

where the matrices $A_{K_h}, B_{K_h}, C_{K_h}$ have dimension $|V(K_h)| \times |E(K_h)|$ and correspond to the vertex and edge components of K_h. The rest of the blocks are the zero and identity matrices of the remaining vertex and edge components. This completes the proof. $\square$

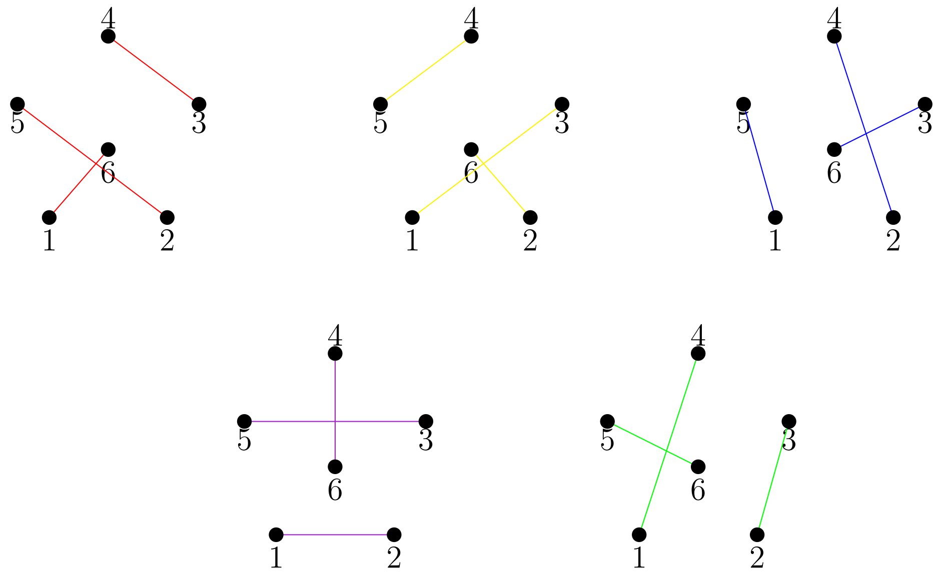


Figure 3.3: Five perfect matchings of K_6

Proposition 35. *Let G be a graph, and let K_h be a complete subgraph of G, where h is odd. Then, the* **odd clique inequality** *defined as*

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h+1}{2} \tag{3.26}$$

is valid for the Total Matching Polytope $P_T(G)$, but it is not facet-defining.

Proof. Let K_h be a clique of odd order. Since in a total matching of K_h we can pick at most one vertex, and the size of the largest matching is $\frac{h-1}{2}$, we can take at most $\frac{h-1}{2} + 1 = \frac{h+1}{2}$ elements of a total matching, as shown in Figure 3.4. Therefore, this implies that the odd clique inequality is valid for $P_T(G)$. Now, we prove that it is not facet-defining. Adding a vertex u to the clique K_h, we can form a clique of even order $K_{h+1} := (V(K_{h+1}), E(K_{h+1}))$, where $V(K_{h+1}) := V(K_h) \cup \{u\}$ and $E(K_{h+1}) := E(K_h) \cup \{e = \{u, v\} \mid v \in V(K_h)\}$. Then, the inequality

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h+1}{2}$$

is dominated by

$$\sum_{v \in V(K_{h+1})} x_v + \sum_{e \in E(K_{h+1})} y_e \leq \frac{h+1}{2}$$

This completes the proof. $\square$

We stress that, even if the odd clique inequality is maximal it remains not facet-

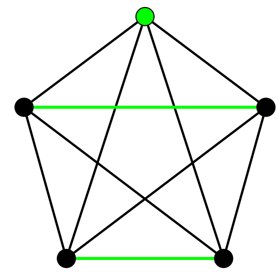


Figure 3.4: A complete K_5 graph. In green, a possible maximal total matching.

defining. Indeed, suppose that K_h is a maximal clique of odd order. We know that

$$\sum_{v \in V(K_h)} x_v \leq 1$$

is facet-defining, and it is easy to notice that the following is a valid inequality for the Total Matching Polytope

$$\sum_{e \in E(K_h)} y_e \leq \frac{h-1}{2}.$$

Thus, the sum of the two inequalities gives the odd-clique inequality.

3.4.1 Separation for the even-clique inequalities

We propose the following ILP model to detect a maximum violated even-clique, which is based on the maximum edge weighted clique model discussed in [70]:

$$\max \quad \sum_{v \in V} c_v x_v + \sum_{e \in E} w_e y_e - z \tag{3.27}$$

$$\text{s.t.} \quad x_v + x_w \leq 1 \qquad \forall \{v, w\} \in \overline{E}, \tag{3.28}$$

$$\sum_{v \in V} x_v = 2z, \tag{3.29}$$

$$y_e \leq x_v \qquad \forall e = \{u, v\} \in E, \tag{3.30}$$

$$y_e \leq x_u \qquad \forall e = \{u, v\} \in E, \tag{3.31}$$

$$x_v + x_u \leq y_e + 1 \qquad \forall e = \{u, v\} \in E, \tag{3.32}$$

$$x_v, y_e \in \{0, 1\} \qquad \forall v \in V, \forall e \in E, \tag{3.33}$$

$$z \in \mathbb{Z}, \tag{3.34}$$

where we recall that $\overline{E}$ represents the complement of $E(G)$. Since we want to detect a clique of even order, we introduce the integer variable $z \in \mathbb{Z}$. If the optimal solution is greater than zero, we get a maximally violated even-clique inequality. The constraints (3.28) are equivalent to imposing the condition that we can select at most one vertex from a maximal stable set in the clique found. Whereas, the constraints (3.30)–(3.32) ensure that if both the end-points of an edge are selected in the solution, the corresponding edge must be included in the clique. In [70], it is proven that finding a maximum weighted edge clique is NP-hard. Consequently, the problem (3.27)–(3.33) is NP-hard in general, and it contains the maximum edge weighted clique as a special case.

In the following Chapter, we focus more in detail on the facial structure of the Total Matching Polytope for specific classes of graphs for which the corresponding linear description has been derived. We succeeded in proving the results by providing new families of facet-defining inequalities for the graph considered.

4. Total Matching Polytope for bipartite graphs

Motivated by the study of the Stable Set Polytope and the Matching Polytope for bipartite graphs, we want to study the Total Matching Polytope for these classes of graphs [1]. For this reason, we mainly focus on the facial structure of $P_T(G)$ for bipartite graphs. In this Chapter, we introduce two new families of facet-defining inequalities for the Total Matching Polytope, and we derive complete characterizations for trees and complete bipartite graphs.

4.1 Total Matching Polytope for trees

Since computing $\alpha_T(G)$ is polynomial for trees, we study the linear description of such graphs. Together with a result from [77], Proposition 20 already allows us to characterize $P_T(G)$ when G is a tree.

Theorem 40. *Let G be a tree. Then a complete and irredundant description of $P_T(G)$ is given by*

$$P_T(G) := \left\{ (x,y) \in \mathbb{R}_+^{|V|+|E|} : \begin{array}{ll} x_v + \sum_{e \in \delta(v)} y_e \leq 1 & \forall v \in V, \\ x_v + x_w + y_e \leq 1 & \forall e = \{v,w\} \in E \end{array} \right\}.$$

Hence, the optimization problem on $P_T(G)$ for a tree graph can be solved in polynomial time.

Proof. Consider the total graph $T(G)$ of G. In [77, Theorem 5], it is proved that a graph is a tree if and only if its total graph is chordal. Hence, we have

$$P_T(G) = P_{STAB}(T(G)) = \left\{ x \in \mathbb{R}^{|V(T(G))|} : \begin{array}{ll} \sum_{v \in K} x_v \leq 1 & \forall K \in \mathcal{K} \subseteq V(T(G)), \\ x_v \geq 0 & \forall v \in V(T(G)) \end{array} \right\},$$

where the first equality follows by Proposition 20 and the second since chordal graphs are perfect graphs (see, e.g., [72]). Let K be a maximal clique in $T(G)$. Since G has no cycles, the preimage of K corresponds in G to either a node and its neighborhood,

[1] The results of this Chapter are obtained in collaboration with Professor Yuri Faenza during my visiting research period at Columbia University.

or to an edge and its endpoints. Hence, a maximal clique inequality corresponds to a total vertex inequality (3.1) or to a total edge inequality (3.2), completing the proof of the first statement. For the second, observe that the total vertex inequalities and total edge inequalities are $O(|V| + |E|)$. To observe that the description is non-redundant, we recall that we have shown in Proposition 21 that the total vertex inequalities and the total edge inequalities define facets. □

Theorem 40 gives an alternative polyhedral proof of the fact that a maximum total matching in a tree can be found in polynomial time, and in fact shows that even the weighted version of the problem (with weights defined over the elements of the tree) can be solved in polynomial time.

4.2 Balanced biclique inequalities

In this Section, we introduce a new class of facets based on complete bipartite graphs, where each vertex bipartition has the same size. Our treatment starts with the observation that the total edge facet-defining inequality (3.2) can be seen as induced by a balanced biclique $K_{1,1}$. We next derive a generalization of these inequalities, and we show that they are facet-defining for any graph.

Theorem 41. *Let G be a graph and $K_{r,r}$ be an induced balanced biclique of G. Then, the balanced biclique inequality:*

$$\sum_{v \in V(K_{r,r})} x_v + \sum_{e \in E(K_{r,r})} y_e \leq r \tag{4.1}$$

is facet-defining for $P_T(G)$.

Proof. Let $V(K_{r,r})$ be the union of the disjoint sets R, S, where $R := \{v_1, \dots, v_r\}$ and $S := \{w_1, \dots, w_r\}$. The validity of the inequality follows from $\nu_T(K_{r,r}) = r$ [45]. Let $\tilde{F} = \{z \in P_T(G) \mid \pi^T z = \pi_0\}$ be the face of $P_T(G)$ defined by (4.1), and let $F = \{z \in P_T(G) \mid \lambda^T z = \lambda_0\}$ be a facet of $P_T(G)$ such that $\tilde{F} \subseteq F$. We want to prove that there exists $a \in \mathbb{R}$ such that $(\lambda, \lambda_0) = a(\pi, \pi_0)$.

Let $e = \{u, v\} \in E[K_{r,r}]$ and define the total matchings $T_v := (S \setminus \{v\}) \cup \{e\}$ and $T_u := (R \setminus \{u\}) \cup \{e\}$. Note that, since $|T_u| = |T_v| = |S|$ and $\chi[S] \in \tilde{F} \subseteq F$, we have $\chi[T_u], \chi[T_v] \in \tilde{F} \subseteq F$. Hence, $\lambda^T \chi[S] = \lambda^T \chi[T_u] = \lambda^T \chi[T_v] = \lambda_0$. We deduce therefore that $\lambda_u = \lambda_v = \lambda_e$. Since $e \in E[K_{r,r}]$ arbitrarily, we deduce $\lambda_u = \lambda_v = \lambda_e$ $\forall u \in R, v \in S, e \in E[K_{r,r}]$

Now, consider $w \notin (R \cup S \cup E[K_{r,r}])$ to be an element of G. Let M be a perfect matching of $K_{r,r}$. Note that at least one of $T_1 := R \cup \{w\}$ and $T_2 := S \cup \{w\}$, and

$T_3 := M \cup \{w\}$ is a total matching. Without loss of generality, assume that T_1 is a total matching. $\chi[T_1]$ is a point of $P_T(G)$ that dominates $\chi[R]$ componentwise. Observe that $P_T(G)$ is a full-dimensional down-monotone polytope and F is not induced by a nonnegative inequality since λ has at least two non-zero coefficients. Since $\chi[R] \in F$, we have $\chi[T_1] \in F$, [72]. This implies that $\lambda_w = 0$. This completes the proof since we have proved that $(\lambda, \lambda_0) = a(\pi, \pi_0)$. □

Separation of balanced biclique inequalities

We next address the problem of separating balanced biclique inequalities of fixed cardinality. The following problem is NP-Complete [67].

Name: Weighted Edge Biclique Decision Problem (`WEBDP`).
Input: A complete bipartite graph G with edge weights $u \in \mathbb{Z}^{|E|}$, a number $k \in \mathbb{N}$.
Decide: If there exists a subgraph of G that is a biclique with vertex partition (R, S) such that $\sum_{e \in R \times S} u(e) \geq k$.

The previous result implies that the following problem is NP-Complete.

Name: Weighted Edge Biclique Decision Problem of fixed Cardinality (`WEBDPC`).
Input: A complete bipartite graph $G(V, E)$ with edge weights $u \in \mathbb{N}^{|E|}$, a number $k \in \mathbb{N}$ and $q \leq |V|$.
Decide: If there exists a subgraph of G that is a biclique with vertex partition (R, S), $|R| = |S| = q$, such that $\sum_{e \in R \times S} u(e) \geq k$.

Indeed, `WEBDPC` is clearly in NP. Suppose we want to solve `WEBDP` on input G, u, k. Let $u' \in \{\mathbb{N} \cup \{0\}\}^{|E|}$ defined as $u'_e = u_e + \|u\|_\infty$ for $e \in E(G)$. For $q = 1, \ldots, |V(G)|$ solve an instance of `WEBDPC` with input $G, u', k + q^2\|u\|_\infty, q$. Suppose any of those is a yes-instance. Then G has a biclique with vertex partition (R, S), $|R| = |S| = q$, that satisfies

$$\sum_{e \in R \times S} u'(e) \geq k + q^2\|u\|_\infty \;\Leftrightarrow\; \cancel{q^2\|u\|_\infty} + \sum_{e \in R \times S} u(e) \geq k + \cancel{q^2\|u\|_\infty}.$$

Hence, G, u, k is a yes-instance for `WEBDP` if and only if one of the $|V(G)|$ instances of `WEBDPC` defined above is a yes-instance, concluding the proof.

We next show that NP-completeness of `WEDPC` implies NP-completeness of the following.

Name: Separation Problem for Balanced Biclique Inequalities of a Given Size in complete bipartite graphs (`SPBBIGS`)
Input: A complete bipartite graph $G(R' \cup S', E)$, a point $(x^*, y^*) \in \mathbb{Q}_+^{|V|+|E|}$, $r \in \mathbb{N}$, $r \leq |R'|$.

Decide: If there exists a violated biclique inequality of $P_T(G)$ of G with r nodes on each side of the partition, that is, there exist $R \subseteq R', S \subseteq S', |R| = |S| = r$ such that:

$$\sum_{v \in R \cup S} x_v^* + \sum_{e \in R \times S} y_e^* > r. \tag{4.2}$$

Clearly `SPBBIGS` belongs to NP. Suppose we have an algorithm for `SPBBIGS`, we show an algorithm for `WEBDPC`. Let G, u, k, q be an input to `WEBDPC`, where G has vertex bipartition (R', S'). Define $y_e^* = u_e \frac{q}{k-1}$ for each edge e of G, $x^* = 0$, and run the algorithm for `SPBBIGS` on G, x^*, y^*, $r = q$. Suppose there exists an inequality of the form (4.2) separating (x^*, y^*). Then for some $R \subseteq R'$, $S \subseteq S'$, we have

$$\frac{q}{k-1} \sum_{e \in R \times S} u_e = \sum_{e \in R \times S} y_e^* = \sum_{v \in R \cup S} x_v^* + \sum_{e \in R \times S} y_e^* > r = q \Leftrightarrow \sum_{e \in R \times S} u_e > k-1 \Leftrightarrow \sum_{e \in R \times S} u_e \geq k,$$

thus giving a positive answer to `WEBDPC`. Similarly, if no inequality of the form (4.2) is violated, we have $\sum_{e \in R \times S} u_e < k$, for all $R \subseteq R', S \subseteq S'$, $|R| = |S| = r = q$.

4.3 Non-balanced biclique inequalities

Now, consider a general non-balanced biclique $K_{r,s}$, with $s > r$, of a graph G. By mimicking (4.1), it is natural to guess that

$$\sum_{v \in V(K_{r,s})} x_v + \sum_{e \in E(K_{r,s})} y_e \leq s. \tag{4.3}$$

defines a facet. This inequality is indeed valid but not facet-defining. In fact, since $K_{r,r}$ is an induced subgraph of $K_{r,s}$, the inequality can be written as the sum of

$$\sum_{v \in V(K_{r,r})} x_v + \sum_{e \in E(K_{r,r})} y_e \leq r.$$

and of the following inequalities

$$\sum_{e \in \delta(w)} y_e + x_w \leq 1, \qquad \forall w \in V(K_{r,s}) \setminus V(K_{r,r}).$$

We next give a strengthening of (4.3) and show that it defines a facet when G is a bipartite graph. From now on, let $K_{r,s}$ be a non-balanced biclique such that $V(K_{r,s}) = R \cup S$ where $R := \{v_1, v_2, \ldots, v_r\}$ and $S := \{w_1, w_2, \ldots, w_s\}$ be the partition of the vertices, with $s > r$.

Proposition 36. *Let $K_{r,s}$ with $s > r > 1$ be a non-balanced biclique. Then, for all $t \in R$, the non-balanced lifted biclique inequality:*

$$\sum_{i \in R} \alpha_i x_i + \sum_{j \in S} x_j + \sum_{e \in E(K_{r,s})} y_e \leq s \tag{4.4}$$

where:

$$\alpha_i = \begin{cases} s - (r-1) & \text{if } t = v_i, \\ 1 & \text{otherwise.} \end{cases}$$

is facet-defining for $P_T(K_{r,s})$.

Proof. W.l.o.g., assume $t = v_1$. Let N be the vertices of $P_T(K_{r,s})$. We show that lifting the (clearly) valid inequality $\sum_{v \in S} x_v + \sum_{e \in E(K_{r,s})} y_e \leq s$ of $P' := P_T(K_{r,s}) \cap \{x_v = 0, v \in R\}$ we obtain (4.4), thus showing that (4.4) is valid. We perform a sequential lifting [62, 79] of the coefficients of $\{x_v\}_{v \in R}$ following the ordering $1, 2, \ldots, r$ of subscripts of nodes in R. Now, consider the largest coefficient relative to x_{v_1}:

$$\begin{aligned} \alpha_{v_1} := s - \max & \Bigg\{ \sum_{i \in S} x_i + \sum_{e \in E(K_{r,s})} y_e \Bigg\} \\ & \text{s.t.} \quad x_{v_1} = 1, z \in N, \\ & x_j = 0, j \in R \setminus \{v_1\}. \end{aligned}$$

$\alpha_{v_1} = s - (r-1)$, since we can take a matching of size $r - 1$, while any total matching T with $v_1 \in T$ does not contain any vertex from S or edge incident to v_1. Now, we claim that $\alpha_i = 1$ for $i \in \{v_2, v_3, \ldots, v_r\}$.

$$\begin{aligned} \alpha_{v_2} := s - \max & \Bigg\{ \sum_{j \in S} x_j + \sum_{e \in E(K_{r,s})} y_e + (s - r + 1) x_{v_1} \Bigg\} \\ & \text{s.t.} \quad x_{v_2} = 1, z \in N, \\ & x_j = 0, j \in R \setminus \{v_1, v_2\}. \end{aligned}$$

Now, fixing the vertex v_2, it is easy to see that α_{v_2} is achieved by setting $(x, y) = \chi[M_2 \cup \{v_1\}]$, where M_2 is a matching of size $r-2$ in $G[R \cup S \setminus \{v_1, v_2\}]$. Thus $\alpha_{v_2} = 1$. Iteratively, at the step i of the sequence we have:

$$\alpha_{v_i} := s - \max\Big\{\sum_{j\in S} x_{w_j} + \sum_{e\in E(K_{r,s})} y_e + (s-r+1)x_{v_1} + \sum_{\ell=2}^{i-1} x_{v_\ell}\Big\}$$
$$\text{s.t.} \quad x_{v_i} = 1, z \in N,$$
$$x_j = 0, j \in R \setminus \{v_1, \ldots, v_{i-1}\}.$$

Repeating the same reasoning we obtain that $\alpha_{v_i} = 1$, and conclude that (4.4) is valid for $P_T(G)$.

Now let $F = \{z \in P_T(G) \mid \lambda^T z = \lambda_0\}$ be a facet of $P_T(G)$ such that $\tilde{F} \subseteq F$. By repeating the same argument as in the proof of Theorem 41, we deduce that $\lambda_v = \lambda_w = \lambda_e$ for all $v \in R \setminus \{v_1\}, w \in S, e \in E[K_{r,r}]$.

For the coefficient λ_{v_1}, $\chi[R] = \chi[S]$ implies that $\lambda_{v_1} + \sum_{i=2}^{r} \lambda_{v_i} = \sum_{j=1}^{s} \lambda_{w_j}$, thus $\lambda_{v_1} = (s-r+1)\lambda_{v_2}$. This completes the proof. □

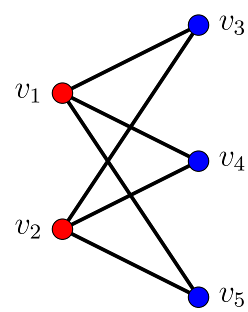


Figure 4.1: Biclique $K_{2,3}$

Now, we can see an easy direct application for the biclique $K_{2,3}$, see Figure 4.1. The corresponding inequalities read as follows:

$$2x_{v_1} + x_{v_2} + x_{v_3} + x_{v_4} + x_{v_5} + \sum_{e\in E(K_{2,3})} y_e \leq 3,$$

$$x_{v_1} + 2x_{v_2} + x_{v_3} + x_{v_4} + x_{v_5} + \sum_{e\in E(K_{2,3})} y_e \leq 3.$$

The following Proposition shows that, in a bipartite graph G, the lifting procedure exposed in Proposition 36 is exhaustive and maximal, that is, it generates all the possible facet-defining induced biclique inequalities.

Theorem 42. *Let G be a bipartite graph. Then, the non-balanced lifted biclique inequalities* (4.4) *are facet-defining for $P_T(G)$.*

Proof. Let $V(G) := A_1 \cup A_2$ be the partition of the vertices of G and $V(K_{r,s}) := R \cup S$ a non-balanced biclique of G. We denote by F the face induced by a non-balanced lifted biclique inequality. By Proposition 36, we have a set $\mathcal{S}$ of $|V(K_{r,s})| + |E(K_{r,s})|$ affinely independent points that lie in F whose support is contained in the elements of $K_{r,s}$. We show that, for each element d of G with $d \notin Q := R \cup S \cup E[K_{r,r}]$, there is a total matching M_d such that $\chi[M_d] \in F$ and $\mathcal{S} \cup \{M_d\}_{d \in (V \cup E) \setminus Q}$ is linearly independent.

Let d be an element of G with $d \notin Q$. If d is not adjacent to S, let $M_d = S \cup \{d\}$. Else, d is not adjacent to R since G is bipartite, and we let $M_d = R \cup \{d\}$. The matrix having as columns vectors from $\mathcal{S} \cup_{d \in (V \cup E) \setminus Q} M_d$ has the following form:

$$\left[\begin{array}{c|c|c} A_{K_{r,s}} & B_{K_{r,s}} & C_{K_{r,s}} \\ \hline \mathbf{0} & \widetilde{I}_v & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \widetilde{I}_e \end{array}\right],$$

where $A_{K_{r,s}}, B_{K_{r,s}}, C_{K_{r,s}}$ represent the components of elements of $K_{r,s}$, and $\widetilde{I}_v, \widetilde{I}_e$ are identity matrices of appropriate size. Since the matrix has maximum rank, the thesis follows. □

4.4 Complete Description for the Total Matching Polytope of complete bipartite graphs

In this Section, we show that the inequalities from Theorem 41 and Theorem 42, together with nonnegative inequalities and total vertex and total edge inequalities give a complete and non-redundant description of $P_T(G)$ when G is a complete bipartite graph. Our argument is as follows. In Section 4.4.1 we give a simple algorithm for solving the maximum weighted total matching problem on a complete bipartite graph G. In Section 4.4.2, we use this algorithm and Balas' classical theorem on the convex hull of the union of polytopes to give an extended formulation $Q_T(G)$ for $P_T(G)$. Then, in Section 4.4.3, we study the projection cone associated to $Q_T(G)$ to deduce the main result.

Theorem 43. *A complete and non-redundant description of $P_T(K_{r,s})$ is defined by:*

- *Total vertex and total edge inequalities* (3.1) – (3.3),
- *Balanced biclique inequalities* (4.1),
- *Non-balanced lifted biclique inequalities* (4.4).

4.4.1 Algorithm

A total matching containing a vertex from R (resp. S) cannot contain any vertex from S (resp. R). Thus, a total matching T of $K_{r,s}$ satisfies at least one of $T \cap R = \emptyset$ and $T \cap S = \emptyset$. For $U \in \{R, S\}$, let $T(K_{r,s}) \setminus U$ be the subgraph of $T(K_{r,s})$ obtained by removing edges corresponding to U.

We can solve the maximum weighted total matching problem by solving the maximum weighted stable set on $T(K_{r,s}) \setminus U$ for $U \in \{R, S\}$, and selecting the solution of maximum weight. The next lemma shows that such graphs have a special structure.

Lemma 4. *Let $U \in \{R, S\}$. The graph $T(K_{r,s}) \setminus U$ is perfect.*

Proof. Suppose w.l.o.g. that $U = S$. We denote by (r_i, s_j) the edge-vertex associated to the edge $e = \{v_i, w_j\}$ of the original graph $K_{r,s}$ and with r_i the node-vertex corresponding to the vertex v_{r_i}. We prove that there neither $T(K_{r,s}) \setminus S$ nor $\overline{T(K_{r,s}) \setminus S}$ contain an odd cycle with 5 or more nodes. The statement then follows from the well-known characterization of perfect graphs [9].

We start with $T(K_{r,s}) \setminus S$. By construction, every vertex r_i $\forall i = 1, \dots r$ lies in exactly one clique and it is not adjacent to any other node. Thus, no odd cycle C with at least 5 nodes contains a vertex r_i. Hence, C contains only vertices of the kind (r_i, s_j). We call r_i (resp. s_j) the *first* (resp. *second*) component of the vertex. Note that no three consecutive vertices of C can share the same first or second component; on the other hand, two consecutive vertices of C must share the first or the second component. Hence, if we let $C = \{v_0, \dots, v_{k-1}\}$, we can assume that, for i odd, v_i shares the first component with v_{i+1} and the second component with v_{i-1} (indices are taken modulo k). However, this contradicts k being odd.

We now focus on $\overline{T(K_{r,s}) \setminus S}$. Let $C = \{v_0, \dots, v_{k-1}\}$, $k \geq 5$ be an odd cycle in $\overline{T(K_{r,s}) \setminus S}$. First, observe that $V(C) \cap R = \emptyset$. Indeed, suppose by contradiction that $r_i \in V(C) \cap R$, and let wlog $r_i = v_0$. Then $v_{\lceil \frac{k}{2} \rceil} = (r_i, s_j)$ and $v_{\lfloor \frac{k}{2} \rfloor} = (r_i, s_\ell)$ for some indices j, ℓ. Then $v_{\lceil \frac{k}{2} \rceil}$ and $v_{\lfloor \frac{k}{2} \rfloor}$ are not adjacent in $\overline{T(K_{r,s}) \setminus S}$, a contradiction.

Hence, $V(C) \cap R = \emptyset$, and let wlog $v_0 = (r_1, s_1)$. We distinguish two cases.

First, assume that $k = 5$. Since v_2, v_3 are not adjacent to v_0 but they are adjacent to each other, we can assume wlog that $v_2 = (r_1, s_2)$, $v_3 = (r_2, s_1)$. Since v_1 is adjacent to v_0 but not to v_3, we must have $v_1 = (r_2, s_t)$, with $t \neq 1$. Since v_1 is adjacent to v_2, $t \neq 2$. Symmetrically, $v_4 = (r_p, s_2)$ with $p \neq 1, 2$. On the other hand, v_1 and v_4 are not adjacent, hence they must share one of their two components. Hence either $t = 2$ or $p = 2$, a contradiction.

Now assume $k \geq 7$. Similarly to the above, $v_{\lfloor \frac{k}{2} \rfloor} = (r_1, s_2)$, $v_{\lceil \frac{k}{2} \rceil} = (r_2, s_1)$. Since $v_{\lfloor \frac{k}{2} \rfloor - 1}$ is not adjacent to v_0 or $v_{\lceil \frac{k}{2} \rceil}$, we must have $v_{\lfloor \frac{k}{2} \rfloor - 1} = (r_t, s_1)$ for $t \neq 1, 2$. Symmetrically, $v_{\lceil \frac{k}{2} \rceil+1} = (r_1, s_p)$ for $p \neq 1, 2$. Again, using the fact that $v_{\lfloor \frac{k}{2} \rfloor-1}$ and $v_{\lceil \frac{k}{2} \rceil+1}$ are not adjacent, we deduce $t = 1$ or $p = 1$, a contradiction. $\square$

Lemma 4 allows us to use classical semidefinite techniques [31] to solve the maximum weighted stable set problem on $T(K_{r,s}) \setminus U$ for $U \in \{R, S\}$. However, in our case we do not need to employ semidefinite programming, because of the following.

Observation 1. *Let $U \in \{R, S\}$. The cliques of $T(K_{r,s}) \setminus U$ correspond in G either to a node in $\{R, S\} \setminus \{U\}$ and the edges incident to it, or to edges incident to a node in U. In particular, $T(K_{r,s}) \setminus U$ has $O(|R| + |S|)$ maximal cliques.*

As a consequence, we are able to solve the problem by means of linear programming techniques.

We stress the fact that if we consider $T(K_{r,s})$ instead of $T(K_{r,s}) \setminus U$ for $U \in \{R, S\}$, the graph is no longer perfect. For instance, the total graph $T(K_{2,2})$ contains an odd-hole. As shown in Figure 4.2 the cycle induced by the red vertices corresponds to an induced odd-hole C_5.

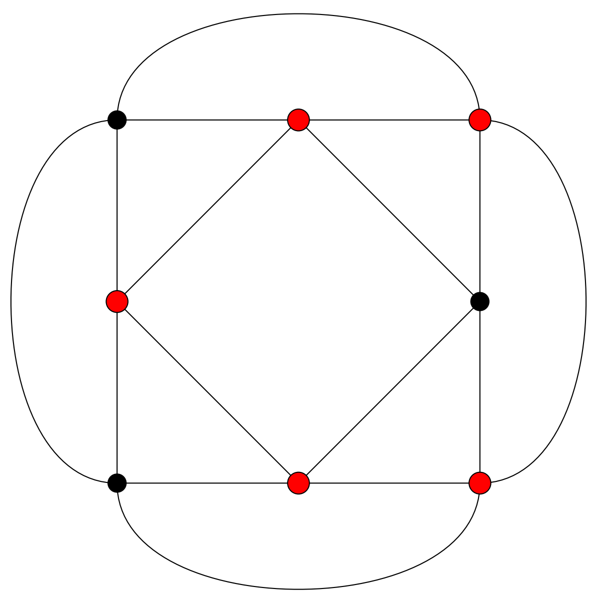


Figure 4.2: Total graph of $K_{2,2}$

4.4.2 Extended formulation

Define the two polytopes

$$P_R := \{z \in P_T(K_{r,s}) : x_v = 0 \text{ for } v \in S\},$$
$$P_S := \{z \in P_T(K_{r,s}) : x_v = 0 \text{ for } v \in R\}.$$

Following the discussion from Section 4.4.1, we can write $P = \mathrm{conv}(P_R \cup P_S)$. Using Lemma 4, we can describe P_R completely using cliques inequalities:

$$P_R = \left\{x \geq 0 : \sum_{v \in K} x_v \leq 1, \quad \forall K \in \mathcal{K} \subseteq V(T(K_{r,s}) \setminus S)\right\}.$$

In turns, Observation 1 allows for a simple description of P_R (and symmetrically, of P_S). We deduce the following.

Corollary 5.

$$P_R := \left\{(x, y) \in \mathbb{R}_+^{|R|+|E|} : \qquad x_v + \sum_{e \in \delta(v)} y_e \leq 1, \quad \forall v \in R; \qquad \sum_{e \in \delta(w)} y_e \leq 1, \quad \forall w \in S\right\},$$

$$P_S := \left\{(x, y) \in \mathbb{R}_+^{|S|+|E|} : \qquad x_w + \sum_{e \in \delta(w)} y_e \leq 1, \quad \forall w \in S; \qquad \sum_{e \in \delta(v)} y_e \leq 1, \quad \forall v \in R\right\}.$$

Balas showed that the convex hull of the union of two polytopes has an extended formulation that can be described in terms of the original formulations of the polytopes.

Theorem 44. *[2] Let $P_1 = \{x \in \mathbb{R}^n : A^1 x \leq b^1\}$, $P_2 = \{x \in \mathbb{R}^n : A^2 x \leq b^2\}$ be polytopes. Then $P := conv(P_1 \cup P_2)$ satisfies $P = Proj_x(Q)$, where*

$$Q := \left\{(x, x^1, x^2, \lambda^1, \lambda^2) \in \mathbb{R}^{3n+2} : \begin{aligned} & A^1 x^1 \leq \lambda^1 b^1, \\ & A^2 x^2 \leq \lambda^2 b^2, \\ & x = x^1 + x^2, \\ & \lambda^1 + \lambda^2 = 1, \\ & \lambda^1, \lambda^2 \geq 0. \end{aligned}\right\}$$

When applied to P_R, P_S defined as above, Theorem 44 gives the following extended formulation for $P_T(K_{r,s})$, where, for later usage, we also report the dual multipliers. Note that below we used the fact that node variables x_v for $v \in R$ (resp. x_w for $w \in S$)

do not appear in P_S (resp. P_R), allowing us to reduce the number of variables. Edge variables y_e for $e \in E$ appear in both P_R and P_S, but we use the equality $y_e = y_e^1 + y_e^2$ to project out y_e^2 for $e \in E$. Similarly, we use $\lambda_1 + \lambda_2 = 1$ to project out λ_2.

Corollary 6.

$$
\begin{aligned}
Q := \Big\{ (x, y, \lambda_1, \lambda_2, y_e^1) \in \mathbb{R}_+^{|V|+|E|+1+1+|E|} : x_v + & \sum_{e \in \delta(v)} y_e^1 - \lambda_1 \le 0 && \forall v \in R, \quad [u_v^1] \\
& \sum_{e \in \delta(w)} y_e^1 - \lambda_1 \le 0 && \forall w \in S, \quad [u_w^1] \\
x_w + & \sum_{e \in \delta(w)} (y_e - y_e^1) + \lambda_1 \le 1 && \forall w \in S, \quad [u_w^2] \\
& \sum_{e \in \delta(v)} (y_e - y_e^1) + \lambda_1 \le 1 && \forall v \in R, \quad [u_v^2] \\
& - y_e^1 \le 0 && \forall e \in E, \quad [u_e^{1,\ge}] \\
& - y_e + y_e^1 \le 0 && \forall e \in E, \quad [u_e^{2,\ge}] \\
& - \lambda_1 \le 0, && \quad [u^{\lambda_1}] \\
& \lambda_1 \le 1 && \quad [u^{\lambda_2}] \Big\}.
\end{aligned}
$$

4.4.3 Projection

In order to project the extended formulation defined in the previous Section, we study the associated projection cone. Let us start by recalling the connection between the projection cone and the description in the original space.

Theorem 45. *[14, Theorem 2.1] Let $Q := \{(x, z) \in \mathbb{R}_+^n \times \mathbb{R}^p \mid Ax + Bz \le d\}$ where A, B have m rows, and define its projection cone $C_P := \{u \in \mathbb{R}^m \mid u^T B = 0, u \ge 0\}$. The projection of Q onto the x-space is*

$$Proj_x(Q) = \{x \in \mathbb{R}_+^n \mid u^T Ax \le u^T b, \forall u \in C_P\}.$$

By applying Theorem 45 to Corollary 6, we can obtain a description of P in the original space.

Lemma 5. *Let $P := P_T(K_{r,s})$. Then*

$$P = \{(x, y) \in \mathbb{R}_+^{n+m} : \sum_{v \in R} u_v^1 x_v + \sum_{w \in S} u_w^2 x_w + \sum_{e=\{v,w\} \in E} \min_{j=1,2} (u_v^j + u_w^j) y_e \le \max_{j=1,2} \sum_{w \in V} u_w^j, \forall u \in Y\}$$

where Y is the set of vectors $u \in \mathbb{R}_+^{2(|R|+|S|)}$ that satisfy $2(|R|+|S|)-1$ linearly independent constraints from the set

$$\begin{aligned} u &= 0 \\ u_v^1 + u_w^1 &= u_v^2 + u_w^2 \quad \text{for } v \in R, w \in S. \\ \sum_{v \in V} u_v^1 &= \sum_{v \in V} u_v^2. \end{aligned} \tag{4.5}$$

Proof. We first claim that the projection cone associated to the extended formulation for P as in Corollary 6 is given by

$$C_P := \Big\{ u : u_v^1 + u_w^1 - u_e^{1,\geq} = u_v^2 + u_w^2 - u_e^{2,\geq}, \forall e = \{v,w\} \in E \tag{4.6}$$

$$\sum_{v \in V} u_v^1 + u^{\lambda_1} = \sum_{v \in V} u_v^2 + u^{\lambda_2} \tag{4.7}$$

$$u \geq 0 \Big\}. \tag{4.8}$$

Consider the constraint matrix B of the description associated with the additional variables. Since we want to project out the additional variables, we multiply the dual multipliers (which correspond to the variables of the projection cone) with each row of B. By summing up along the same column corresponding to a fixed edge variable y_e^1, we obtain (4.6). Similarly, the same procedure holds for projecting the variable λ_1 out along the corresponding column, as to obtain (4.7). The claim then follows by rearranging.

Using Theorem 45, all valid inequalities in the description of P have the following form:

$$\sum_{v \in R} u_v^1 x_v + \sum_{w \in S} u_w^2 x_w + \sum_{e=\{v,w\} \in E} (u_v^2 + u_w^2 - u_e^{2,\geq}) y_e \leq \sum_{w \in V} u_w^2 + u^{\lambda_2} \tag{4.9}$$

where u is an extreme ray of C_P. Note that by (4.6), we have $u_v^2 + u_w^2 - u_e^{2,\geq} = u_v^1 + u_w^1 - u_e^{1,\geq}$ for $e \in E$ and by (4.7), we have $\sum_{w \in V} u_w^2 + u^{\lambda_2} = \sum_{w \in V} u_w^1 + u^{\lambda_1}$.

We first claim that, we can assume that, for $e \in E$, at least one of $u_e^{1,\geq}, u_e^{2,\geq}$ is equal to 0. Indeed, since they are nonnegative, if they are both strictly positive we can decrease both by $\min\{u_e^{1,\geq}, u_e^{2,\geq}\} > 0$ and obtain a stronger inequality (4.9). Similarly, at most one of $u^{\lambda_1}, u^{\lambda_2} = 0$. Hence, we can rewrite (4.9) as

$$\sum_{v \in R} u_v^1 x_v + \sum_{w \in S} u_w^2 x_w + \sum_{e=\{v,w\} \in E} \min_{j=1,2} (u_v^j + u_w^j) y_e \leq \max_{j=1,2} \sum_{w \in V} u_w^j. \tag{4.10}$$

Let $\overline{u}$ be an extreme ray of C_P. We first claim that the vector obtained from $\overline{u}$ by projecting out $\{u_e^{1,\geq}, u_e^{2,\geq}\}_{e\in E}, u^{\lambda_1}, u^{\lambda_2}$ is a nonnegative vector that satisfies $2(|R|+|S|)-1$ linearly independent constraints from (4.5). By construction, $\overline{u}$ satisfies at equality a set $\mathcal{S}$ of $2(|R|+|S|)+2|E|+1$ linearly independent constraints from (4.6)–(4.8). By basic linear algebra, any set of linearly independent constraints from (4.6)–(4.8) tight at $\overline{u}$ can be enlarged to a linearly independent set of inequalities tight at $\overline{u}$ of maximum cardinality. Hence we can assume w.l.o.g. that $\mathcal{S}$ contains the following set $\mathcal{S}'$ of linearly independent constraints. For $e \in E$, if $\overline{u}_e^{2,\geq} = \overline{u}_e^{1,\geq} = 0$, then constraints $u_e^{2,\geq} = 0$, $u_e^{1,\geq} = 0$ belong to $\mathcal{S}'$. Else, from what argued above, we have $\overline{u}_e^{j,\geq} > 0$ and $\overline{u}_e^{3-j,\geq} = 0$ for some $j \in \{1,2\}$, and $\mathcal{S}'$ contains $u_e^{j,\geq} = 0$ and $u_v^1 + u_w^1 - u_e^{1,\geq} = u_v^2 + u_w^2 - u_e^{2,\geq}$. Similarly, either $u^{\lambda_1} = 0$ and $u^{\lambda_2} = 0$ belong to $\mathcal{S}'$, or one of them and $\sum_{v\in V} u_v^1 + u^{\lambda_1} = \sum_{v\in V} u_v^2 + u^{\lambda_2}$ belong to $\mathcal{S}'$. It is easy to see that constraints in $\mathcal{S}'$ are linearly independent, hence $\mathcal{S} \setminus \mathcal{S}'$ is a set of $2(|R|+|S|)-1$ linearly independent constraints.

Note that an inequality (4.6) belongs to $\mathcal{S} \setminus \mathcal{S}'$ only if both the variables $u_e^{1,\geq}$ and $u_e^{2,\geq}$ appearing in its support are set to 0. In particular $\overline{u}$ satisfies $\overline{u}_v^1 + \overline{u}_w^1 = \overline{u}_v^2 + \overline{u}_w^2$. Similarly, constraint (4.7) belongs to $\mathcal{S} \setminus \mathcal{S}'$ only if $\sum_{v\in V} \overline{u}_v^1 = \sum_{v\in V} \overline{u}_v^2$. Hence, $\overline{u}$ satisfies at equality $2(|R|+|S|)-1$ linearly independent constraints from (4.5), and the claim follows.

Conversely, any nonnegative vector in the components $\{u_v^1, u_v^2\}_{v\in V}$ that satisfies any set of constraints from (4.5) can be extended to a vector of C_P by appropriately adding components $\overline{u}_e^{1,\geq}, \overline{u}_e^{2,\geq}, \overline{u}^{\lambda_1}, \overline{u}^{\lambda_2}$, concluding the proof. □

We call a vector u that belongs to the set Y defined in Lemma 5 *valid*, and an inequality that is obtained in the description of P given in Lemma 5 from a valid u *legal*.

We moreover call a valid u *minimal* if the following properties hold:

- All its components are integer numbers with gcd 1.
- The inequality associated to u (as in Lemma 5) defines a facet of P different from (3.1)–(3.3), and
- There is no valid u' whose associated inequality is equivalent up to nonnegative scaling to the one associated to u, but the support of u' is strictly contained in the support of u.

We say that a set of $2(|R|+|S|)-1$ linearly independent constraints from (4.5) *support* a valid u if they are tight at u.

For a set $\mathcal{S}$ supporting a valid vector u, we let $G(\mathcal{S})$ be the graph that contains all vertices of $K_{r,s}$, colors a vertex v blue (resp. red) if $u_v^1 = 0$ (resp., $u_v^2 = 0$) belongs to $\mathcal{S}$, and contains edge vw if $u_v^1 + u_w^1 = u_v^2 + u_w^2$ belongs to $\mathcal{S}$. Note that a node can be colored both blue and red in $G(\mathcal{S})$ - we call such nodes *bicolored*. A node that is colored with exactly one of red and blue is *monochromatic*. A connected component of $G(\mathcal{S})$ is non-trivial if it contains at least two nodes.

Let $\mathcal{S}$ be the set of constraints supporting a valid, minimal vector Y, and let $a^T x \leq b$ be the inequality associated to u. $\mathcal{S}$ is called *canonical* if:

- It maximizes the number of colors used for nodes,
- subject to the previous condition, it maximizes the number of edges of $G(\mathcal{S})$.

Notice that, to obtain a valid description of P, it is enough to describe canonical sets of inequalities associated to minimal, valid vectors, and add to those inequalities (3.3)–(3.2). For simplicity of notation, in the treatment of the following Lemma we denote equivalently an edge $e = \{v, w\}$ or vw.

Lemma 6. *Let $u \in Y$ be minimal, let $\mathcal{S}$ be a canonical set supporting it, and $\mathcal{I}$ be the set of isolated nodes of $G(\mathcal{S})$. Then:*

1. *$G(\mathcal{S})$ does not have cycles.*
2. *$G(\mathcal{S})$ contains at least one edge.*
3. *Let $u_v^1 = 0$ (resp. $u_v^2 = 0$) for some $v \in V$. Then v is colored blue (resp. red).*
4. *For each edge $e = \{v, w\}$ of $G(\mathcal{S})$, v and w are monochromatic and colored with opposite colors.*
5. *If $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ does not belong to $\mathcal{S}$, there is exactly one connected component in $G(\mathcal{S})$, all its nodes are monochromatic, and all nodes from $\mathcal{I}$ are bicolored.*
6. *If $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ belongs to $\mathcal{S}$, there is exactly one connected component in $G(\mathcal{S})$, all its nodes are monochromatic, and all nodes from $\mathcal{I}$ are bicolored, except one that is monochromatic.*

Proof. 1. Since $K_{r,s}$ is bipartite, any cycle C in $G(\mathcal{S})$ must be even. Alternatively summing and subtracting the equalities corresponding to edges of C, we obtain $0 = 0$, contradicting linear independence.

2. Suppose the thesis does not hold. Then the only constraints in $\mathcal{S}$ are nonnegativity constraints and, possibly, $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$. Suppose first that $\sum_{v\in V} u_v^1 =$

$\sum_{v\in V} u_v^2$ is not contained in $\mathcal{S}$. Then all nodes of $G(\mathcal{S})$ except one are bicolored. Let $v \in V$ be the only node that is not bicolored, and assume w.l.o.g. that v is colored red. If $u_v^1 = 0$, then u is the zero vector, which is clearly not minimal. Hence, assume $u_v^1 > 0$. Then the inequality corresponding to u is dominated by the total edge inequality corresponding to vw for some w in the neighborhood of w, hence showing that u is not minimal.

Hence, assume that $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ belongs to $\mathcal{S}$. Then all nodes are bicolored except 2. Since u is not the zero vector and $\mathcal{S}$ is canonical, there must be nodes v, v' (possibly $v = v'$) with $u_v^1 = u_{v'}^2 > 0$. If v are on the opposite side of the bipartition, then we can replace $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ with the constraint corresponding to edge vv', showing that $\mathcal{S}$ is not canonical. Hence, assume that v, v' are on the same side of the bipartition. Then the inequality corresponding to u is again dominated by the edge inequality corresponding to vw for some w in the neighborhood of w, hence showing that u is not minimal.

3. Suppose w.l.o.g. that $u_v^1 = 0$ but v is not colored blue. By part 2, $G(\mathcal{S})$ has at least one edge. If $u_v^1 = 0$ is linearly independent from constraints in $\mathcal{S}$, then we can replace some edge constraint in $\mathcal{S}$ with $u_v^1 = 0$, contradicting the canonicity of $\mathcal{S}$. Hence $u_v^1 = 0$ can be generated by constraints in $\mathcal{S}$. Note that a minimal set of equations generating $u_v^1 = 0$ must contain either an edge constraint incident to v, or $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$, since those are the only other constraints whose support contains u_v^1. Hence, we can replace the one among those that appears in $\mathcal{S}$ with $u_v^1 = 0$, contradicting the canonicity of $\mathcal{S}$.

4. We first show that v, w, cannot be colored with the same color. Suppose w.l.o.g. both v, w are colored blue. In particular, $u_v^1 = u_w^1 = 0$. Since vw is an edge of $G(\mathcal{S})$, we have $u_v^2 + u_w^2 = 0$, which by nonnegativity of u implies $u_v^2 = u_w^2 = 0$. By part 3, $u_v^1 = 0$, $u_v^2 = 0$, $u_w^1 = 0$, $u_w^2 = 0$ belong to $\mathcal{S}$. By hypothesis, $u_w^1 + u_w^1 = u_w^2 + u_w^2$ also belongs to $\mathcal{S}$, contradicting the fact that $\mathcal{S}$ is linearly independent.

We conclude the proof by showing that both v, w are colored. Suppose by contradiction that this is not the case. Let $\mathcal{C}$ be the connected component of v, w in $G(\mathcal{S})$. By part 1, $\mathcal{C}$ contains $k \in \mathbb{N}$ nodes and $k-1$ edges. $\mathcal{S}$ contains at least $2k-1$ constraints whose support intersects the variables associated to nodes of $\mathcal{C}$. Suppose first $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ does not belong to $\mathcal{S}$. Then the only such constraints are the $k-1$ edge constraints, and the nonnegativity constraint associated to nodes of $\mathcal{C}$. Since a node of $\mathcal{C}$ is not colored, there must be some node of $\mathcal{C}$ that is bicolored, call it v'. Then $u_{v'}^1 = u_{v'}^2 = 0$. Let w' be a node adjacent to v' in $\mathcal{C}$. Since we showed above that two adjacent nodes cannot be colored with the same color, w' is not colored. Using the constraints on edges of $\mathcal{C}$, we obtain $u_{w'}^1 = u_{w'}^2$. If $u_{w'}^1 = u_{w'}^2 = 0$, using part 2 we deduce that w' is colored, a contradiction. Hence, $u^1_{w'} = u^2_{w'} > 0$. Let u' be obtained from u by setting $(u')^1_{w'} = 0$. Then the inequality associated to u is dominated by a conic combination of the inequality associated to u' and inequality (3.1) associated to w', a contradiction.

Now suppose $\sum_{v\in V} u^1_v = \sum_{v\in V} u^2_v$ belongs to $\mathcal{S}$, and let v be uncolored. If $u^1_v = 0$ or $u^2_v = 0$, the thesis follows by part 3 above. Hence, $u^1_v, u^2_v > 0$. Let $\alpha = \min\{u^1_v, u^2_v\}$, and let u' be the vector obtained from u by decreasing u^1_v, u^2_v by α. Then the inequality associated to u can be obtained as a conic combination of the total vertex inequality associated to v and the inequality associated to u'.

5. We know by part 4, that G has a non-trivial connected component $\mathcal{C}$, and that all its nodes are monochromatic. Let k be the number of nodes of $\mathcal{C}$. By part 1, $\mathcal{C}$ has $k-1$ edges, and by part 2, the total number of colors used in nodes from $\mathcal{C}$ is exactly k. Since $\sum_{v\in V} u^1_v = \sum_{v\in V} u^2_v$ does not belong to $\mathcal{S}$, there are at most $2k-1$ constraints from $\mathcal{S}$ supported over some of the $2k$ variables indexed over nodes of $\mathcal{C}$. Hence, no other non-trivial connected component of $G(\mathcal{S})$ exists. Since $\mathcal{S}$ contains $2(|V|) - 1 - (2k-1) = 2(|V| - k)$ more constraints, every node not in $\mathcal{C}$ must be isolated and bicolored.

6. By applying an argument similar to part 5 above, we deduce that, for each connected component $\mathcal{C}$ of $G(\mathcal{S})$ with k nodes, there are at most $2k-1$ constraints from $\mathcal{S}$ whose support is contained on the set of variables corresponding to nodes from $\mathcal{C}$. There is one more constraint in $\mathcal{S}$ that involves variables associated to nodes in $\mathcal{C}$, and this is $\sum_{v\in V} u^1_v = \sum_{v\in V} u^2_v$. Hence, we distinguish the following cases:

a) There are exactly two connected components in $G(\mathcal{S})$, all its nodes are monochromatic, and all nodes from $\mathcal{I}$ are bicolored.

b) There is exactly one connected component in $G(\mathcal{S})$, all its nodes are monochromatic, and all nodes from $\mathcal{I}$ are bicolored, except one that is monochromatic.

We conclude the proof by showing that a) cannot happen. Indeed, let C_α and C_β be the two connected components. Since all nodes of each connected component are monochromatic, all non-zero variables associated to vertices of C_α (resp. C_β) have the same value α (resp. β). Let R_α, S_α (resp. R_β, S_β) be the two sides of the bipartition of component C_α (resp. C_β). We assume without loss of generality that $\alpha = 1 \leq \beta$ and $|R_\beta| \leq |S_\beta|$.

Recall that nodes from R_α, S_α (resp. R_β, S_β) have distinct colors. Using

$$\sum_{v\in C_\alpha\cup C_\beta} u^1_v = \sum_{v\in V} u^1_v = \sum_{v\in V} u^2_v = \sum_{v\in C_\alpha\cup C_\beta} u^2_v,$$

the inequalities associated to the different possible colorings are as follows. If both nodes from R_α, R_β are red, then the inequality corresponding to u is

$$\sum_{v\in R_\alpha\cup S_\alpha} x_v + \sum_{v\in R_\beta\cup S_\beta} \beta x_v + \sum_{vw\in((S_\alpha\cup S_\beta)\times(R_\alpha\cup R_\beta))\setminus(S_\beta\times R_\beta)} y_{vw} + \sum_{vw\in S_\beta\times R_\beta} \beta y_{vw} \le |R_\alpha| + \beta|R_\beta|. \tag{4.11}$$

If both nodes from R_α, R_β are blue, then the inequality corresponding to u is

$$\sum_{vw\in((S_\alpha\cup S_\beta)\times(R_\alpha\cup R_\beta))\setminus(S_\beta\times R_\beta)} y_{vw} + \sum_{vw\in S_\beta\times R_\beta} \beta y_{vw} \le |R_\alpha| + \beta|R_\beta|, \tag{4.12}$$

which is clearly dominated by (4.11). If nodes from R_α are red and nodes from R_β are blue, then the inequality corresponding to u is

$$\sum_{v\in R_\alpha\cup S_\alpha} x_v + \sum_{vw\in S_\alpha\times R_\alpha} y_{vw} + \sum_{vw\in S_\beta\times R_\beta} \beta y_{vw} \le |R_\alpha| + \beta|S_\beta|, \tag{4.13}$$

which is also dominated by (4.11) since $|R_\beta| \le |S_\beta|$. The last case is when nodes from R_α are blue and nodes from R_β are red, and the inequality corresponding to u is

$$\sum_{v\in R_\beta\cup S_\beta} \beta x_v + \sum_{vw\in S_\alpha\times R_\alpha} y_{vw} + \sum_{vw\in S_\beta\times R_\beta} \beta y_{vw} \le |R_\alpha| + \beta|S_\beta| \tag{4.14}$$

which is again dominated by (4.11).

Hence, we can assume that nodes from $R_\alpha \cup R_\beta$ are colored red. Moreover, $|R_\beta| \le |S_\beta|$ with $\beta \ge 1$ and $|R_\alpha| + \beta|R_\beta| = |S_\alpha| + \beta|S_\beta|$ implies $|R_\alpha| + |R_\beta| > |S_\alpha| + |S_\beta|$. Clearly, $|R_\alpha| > |S_\alpha|$ (and $|R_\beta| < |S_\beta|$), that is, each connected component has not the same size with the respect to the same partition of vertices. Suppose $\beta > 1$, then the inequality associated to u is as follows

$$\sum_{v\in R_\alpha\cup S_\alpha} x_v + \sum_{v\in R_\beta\cup S_\beta} \beta x_v + \sum_{vw\in(R_\alpha\cup R_\beta)\times(S_\alpha\cup S_\beta)} y_{vw} \le |R_\alpha| + \beta|R_\beta| \tag{4.15}$$

Let F_β be the face induced by the inequality (4.15), and consider a non-balanced lifted biclique inequality defined as follows. Consider a vertex $\overline{w} \in S_\beta$ such that its coefficient is $\theta := |R_\alpha| + |R_\beta| - (|S_\alpha| + |S_\beta|) + 1$.

$$\sum_{v\in R_\alpha\cup S_\alpha} x_v + \sum_{v\in R_\beta\cup(S_\beta\setminus\{\overline{w}\})} x_v + \theta x_{\overline{w}} + \sum_{vw\in(R_\alpha\cup R_\beta)\times(S_\alpha\cup S_\beta)} y_{vw} \le |R_\alpha| + |R_\beta| \tag{4.16}$$

Let F_θ be the face induced by (4.16).

We claim that $F_\beta \subseteq F_\theta$ but the equality does not hold, that is, F_β is strictly contained in F_θ, and this proves that the inequality is not facet-defining. Let $F_\beta := \{z \in P_T(G) : \lambda^T z = \lambda_0\}$ and $F_\theta := \{z \in P_T(G) : \tilde{\lambda}^T z = \tilde{\lambda}_0\}$. Let z be an integer vector in F_β.

Then, z must be the incidence vector of a maximal total matching of the form $T_R := R_\beta \cup T_\alpha$, $T_S := S_\beta \cup T_\beta$, where T_α and T_β are total matchings induced respectively by $G[R_\alpha \cup E[S_\alpha \cup S_\beta]]$ and $G[E[R_\alpha \cup R_\beta] \cup S_\alpha]$. Since each of the incidence vectors defined includes either the vertex $\overline{w}$ and it is maximal, or corresponds to a maximal total matching in a non-balanced biclique, we have that $z \in F_\theta$.

Now, consider the incidence vector of the maximal total matching $T := \overline{S}_\beta \cup M_\beta$, with $\overline{w} \in \overline{S}_\beta$, where M_β is a perfect matching induced on $G[R_\beta \cup S_\beta]$ (this exists since $|S_\beta| > |R_\beta|$), and $\overline{S}_\beta$ is the set of vertices not covered in S_β by M_β. Then, $\chi[T] \in F_\theta$, but $\chi[T] \notin F_\beta$. This concludes the proof, since we have shown that case (a) does not hold.

□

We can now complete the proof of Theorem 43.

Let $\mathcal{S}$ be a canonical set. Assume first $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ does not belong to $\mathcal{S}$. Then by Lemma 6, part 5, $G(\mathcal{S})$ has exactly one non-trivial connected component (whose nodes are all monochromatic) and all other nodes are bicolored. By Lemma 6, part 4, nodes from $\mathcal{C}$ from the opposite side of the bipartition are colored with different colors. Hence, for some $j \in \{1,2\}$, we have $u_v^j = u_w^{3-j} = 1$ and $u_v^{3-j} = u_w^j = 0$ for all $v \in R' := R \cap V(\mathcal{C}), w \in S' := S \cap V(\mathcal{C})$. If $j = 2$, the inequality we obtain is

$$\sum_{e=\{v,w\}, v\in R', w\in S'} y_e \leq \max\{|R'|, |S'|\},$$

while if $j = 1$ the inequality we obtain is

$$\sum_{v\in R'} x_v + \sum_{w\in S'} x_w + \sum_{e=\{v,w\}, v\in R', w\in S'} y_e \leq \max\{|R'|, |S'|\}.$$

All such inequalities coincide or are dominated by the balanced and non-balanced biclique inequalities associated to the pair R', S'. Now, assume that $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ belongs to $\mathcal{S}$. By Lemma 6, part 6, $G(\mathcal{S})$ has exactly one nontrivial component $\mathcal{C}$ induced by all monochromatic nodes, and let $\overline{v}$ be the monochromatic node in the trivial component. Let $R' := R \cap V(\mathcal{C})$ and $S' := S \cap V(\mathcal{C})$. W.l.o.g. assume that $|R'| < |S'|$ and all nodes in R' are colored red (consequently, the other vertex bipartition S' receives color blue). Similarly as the previous case, for some $j \in \{1,2\}$, we have $u_v^j = u_w^{3-j} = 1$ and $u_v^{3-j} = u_w^j = 0$ for all $v \in R', w \in S'$. We may assume that $\overline{v}$ is colored red, for otherwise, by using $\sum_{v\in V} u_v^1 = \sum_{v\in V} u_v^2$ we obtain $u = 0$, a contradiction. Then, by a simple computation we derive that $u_{\overline{v}}^1 = (|S'| - |R'|)$. If $j = 2$ we conclude as the previous case. If $j = 1$, the vector associated to u is

$$\sum_{v\in R'} x_v + (|S'| - |R'|)x_{\overline{v}} + \sum_{w\in S'} x_w + \sum_{e=\{v,w\},v\in R',w\in S'} y_e \leq \max\{|R'|, |S'|\} = |S'|,$$

which is exactly the non-balanced lifted biclique inequality induced on $G[(R'\cup\{\overline{v}\})\cup S']$ such that the coefficient of $\overline{v}$ equal to $|S'| - |R'|$.

We have shown therefore that all the inequalities induced by the vectors u associated with a canonical set $\mathcal{S}$ are dominated, or, correspond to the list of balanced and non-balanced lifted biclique inequalities of Theorem 43.

4.4.4 An ILP model for the Separation of balanced biclique inequalities

We conclude the Chapter by proposing an Integer Linear Programming model for the separation of a balanced biclique inequality. Since we mostly refer to bipartite graphs in this Chapter, we want to specialize the formulation for bipartite graphs.

Let (c_v, w_e) be the fractional optimal value to the current LP problem, and let x_v and y_e denote the decision variables of the problem of finding a balanced biclique inequality in a graph G. The separation problem asks for maximizing the following quantity

$$\beta := \sum_{v\in V} c_v x_v + \sum_{e\in E} w_e y_e - k, \tag{4.17}$$

where k is the cardinality of the balanced biclique induced by the variables x_v and y_e. Thus, we want to detect a maximum weighted balanced biclique, where node and edge weights are (c_v, w_e). Whenever $\beta > 0$, we have found a violated balanced biclique. Otherwise, all the balanced biclique inequalities are satisfied. Given a bipartite graph $G = (A \cup B, E)$ we introduce binary variables $x_v, \forall v \in V$ and $y_e, \forall e \in E$ to denote whether they are part of the selected biclique. The complete model reads as follows.

$$
\begin{aligned}
\max \quad & \sum_{v \in V} c_v x_v + \sum_{e \in E} w_e y_e - k && && (4.18)\\
& \sum_{v \in A} x_v = \sum_{w \in B} x_w, && && (4.19)\\
& \sum_{v \in A} x_v = k, && && (4.20)\\
& x_v + x_w \leq 1 && \forall v \in A, \forall w \notin A \cup N_G(v), && (4.21)\\
& y_e \leq x_v && \forall e = \{v, w\} \in E, && (4.22)\\
& y_e \leq x_w && \forall e = \{v, w\} \in E, && (4.23)\\
& x_v + x_w \leq y_e + 1 && \forall e = \{v, w\} \in E, && (4.24)\\
& k \in \mathbb{Z}^+. && && (4.25)
\end{aligned}
$$

Constraint (4.19) ensures that the cardinality of the selected biclique is balanced, that is, the size of each vertex bipartition is the same. The constraints of type (4.21) impose that the graph induces a biclique, the constraints (4.22)–(4.24) assure that if the two end-points are selected then the edge connecting them must be selected. Let β be the optimal solution to this model. Finally, if $\beta > 0$ we have found a violated biclique inequality.

5. Total Coloring and Computational Results

In this Chapter, we present the Total Coloring Problem from a polyhedral point of view. We introduce two Integer Linear Programming models for the Total Coloring Problem, the first based on an assignment model, and, the latter is a set covering formulation based on the idea of covering the elements of the graph by the minimum number of maximal total matchings.

5.1 Total Coloring: Assignment model

Let $G = (V, E)$ be a graph and let K be the set of available colors, with $|K| \geq \Delta(G)+1$. We introduce binary variables $x_{vk} \in \{0, 1\}$ for every vertex v and binary variables $y_{ek} \in \{0, 1\}$ for every edge e to denote whether they get assigned color k. Besides, we introduce the binary variables z_k to indicate whether any element uses color k. Using these variables, our assignment ILP model for the TCP is as follows.

$$\chi_T(G) := z_{IP}^{(A)} = \min \quad \sum_{k \in K} z_k \tag{5.1}$$

$$\text{s.t.} \quad \sum_{k \in K} x_{vk} = 1 \qquad \forall v \in V, \tag{5.2}$$

$$\sum_{k \in K} y_{ek} = 1 \qquad \forall e \in E, \tag{5.3}$$

$$x_{vk} + \sum_{e \in \delta(v)} y_{ek} \leq z_k \qquad \forall v \in V, \forall k \in K, \tag{5.4}$$

$$x_{vk} + x_{wk} + y_{ek} \leq z_k \qquad \forall e = \{v, w\} \in E, \forall k \in K, \tag{5.5}$$

$$x_{vk} \in \{0, 1\} \qquad \forall v \in V, \forall k \in K, \tag{5.6}$$

$$y_{ek} \in \{0, 1\} \qquad \forall e \in E, \forall k \in K. \tag{5.7}$$

The objective function (5.1) minimizes the number of used colors. Constraints (5.2)–(5.3) ensure that every vertex and every edge get assigned a color. Constraint (5.4) enforces that all edges e incident to a vertex v, and the vertex v itself, take a different color; at the same time, the constraints guarantee that the corresponding variable z_k is set to 1 whenever color k is used by at least an element of G. Constraint (5.5) imposes that for each edge $e = \{i, j\}$ at most one element among $\{e, i, j\}$ can take color k, and it sets the corresponding variable z_k accordingly. If we relax the integrality constraints (5.6) and (5.7), we get a Linear Programming relaxation. We denote the optimal value of the LP relaxation by $z_{LP}^{(A)}$.

The LP relaxation of model (5.1)–(5.7) yields the following lower bound.

Proposition 37. *Let $G = (V, E)$ be a graph. Then, we have $\chi_T(G) \geq z_{LP}^{(A)} \geq \Delta + 1$.*

Proof. Let $x_{vk} = y_{ek} = \frac{1}{\Delta+1}$ for $k = 1, \ldots, \Delta + 1$, $z_k = 1$ for $k = 1, 2, \ldots, \Delta + 1$ and $x_{vk} = 0, y_{ek} = 0$ for $k > \Delta + 1, \forall v \in V, \forall e \in E$. Notice that this assignment gives a feasible solution for the LP relaxation of (5.1)–(5.7). Since $|K| \geq \Delta + 1$, the assertion follows immediately.

$\square$

This Section aims to analyze the polyhedral properties of the relaxation of the assignment formulation just introduced. Let $P_{TC}(G)$ be the *total coloring polytope* defined as the convex hull of all feasible solutions associated with the assignment model. To understand the properties of the $P_{TC}(G)$, we start by finding a minimal equation system for $P_{TC}(G)$ and determining its dimension.

Let also $P_{VC}(G)$ be the *vertex coloring polytope*, defined as the convex hull of all characteristic vectors of proper vertex colorings induced by the standard assignment model on the vertices known in the literature, see the model (2.10) in Chapter 2. Since there is a one-to-one mapping between total matchings of G and stable sets of the total graph $T(G)$, a proper vertex coloring corresponds to a total coloring of the initial graph G. Hence, we derive the following observation.

Proposition 38. *Let G be a graph. Then, $P_{TC}(G) = P_{VC}(T(G))$.*

The authors in [12] provide a minimal defining system for $P_{VC}(G)$, which is given by the equality constraints of type (5.2). It follows that $dim(P_{VC}(G)) = n^2$. Hence, we derive the following Corollary.

Corollary 7. *The dimension of $P_{TC}(G)$ is $(n + m)^2$.*

In the same paper, the authors show the following

Proposition 39. *[60] Let K be a maximal clique of G. Then, for a fixed color k_0*

$$\sum_{v \in V} x_{vk_0} \leq z_{k_0}$$

is facet-defining for $P_{VC}(G)$.

Using this Proposition we prove the following

Proposition 40. *The inequalities:*

$$x_{vk} + \sum_{e \in \delta(v)} y_{ek} \leq z_k \qquad \forall v \in V, \forall k \in K,$$

$$x_{vk} + x_{wk} + y_{ek} \leq z_k \qquad \forall e \in E, \forall k \in K.$$

are facet-defining for $P_{TC}(G)$.

Proof. Consider the total graph $T(G)$ of the original graph G. Now, consider a vertex $v \in V$ and an edge $e := \{u, w\} \in E$. By construction of T, the subgraphs $T[\delta(v) \cup \{v\}]$ and $T[e \cup \{u, w\}]$ in T correspond to cliques $K_{|\delta(v)|+1}$ and K_3 respectively. It is easy to see that they correspond to maximal cliques of $T(G)$. Now, by Proposition 39, we deduce that they are facet-defining for $P_{TC}(G)$. □

5.2 Total Coloring: Set Covering model

The assignment model (5.1)–(5.7) is easy to write, but it suffers from symmetry issues: any permutation of the color classes indexed by k generates the same optimal solution [58, 40]. To overcome this issue and to get a stronger LP lower bound, we introduce a set covering formulation based on maximal total matchings. A total matching is (inclusion-wise) maximal if it is not a subset of any other total matching. Note that the number of maximal total matchings in a graph is strictly less than the number of total matchings.

Let $\mathcal{T}$ be the set of all maximal total matchings of G. Let λ_t be a binary decision variable indicating if the matching $t \subset \mathcal{T}$ is selected (or not) for representing a color class. The 0–1 parameter a_{vt} indicates if vertex v is contained in the total matching t. Similarly, the 0–1 parameter $b_{et} = 1$ indicates if edge e is contained in the total matching t. The following set covering model is a valid formulation for the TCP.

$$\chi_T(G) = z_{IP}^{(C)} := \min \quad \sum_{t \in \mathcal{T}} \lambda_t \tag{5.8}$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{T}} a_{vt} \lambda_t \geq 1 \qquad \forall v \in V \tag{5.9}$$

$$\sum_{t \in \mathcal{T}} b_{et} \lambda_t \geq 1 \qquad \forall e \in E \tag{5.10}$$

$$\lambda_t \in \{0, 1\} \qquad \forall t \in \mathcal{T}. \tag{5.11}$$

Given an optimal solution of the previous problem, whenever an element of G appears

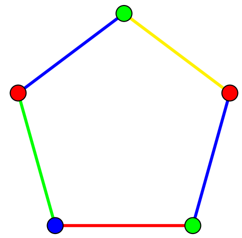


Figure 5.1: A total coloring of a cycle of length 5 with $k = 4 = \Delta(G) + 2$ colors. The optimal value of the LP relaxation of the assignment model (5.1)–(5.7) is equal to $z_{LP}^{(A)} = 3$, while the optimal value of the LP relaxation of the set covering model (5.8)–(5.11) is equal to $z_{LP}^{(C)} = \frac{10}{3}$.

in $t > 1$ maximal total matchings, it is always possible to recover a proper total coloring by removing the element from $t-1$ of those total matchings. Note that the covering model has an exponential number of variables, one for each maximal total matching in G. We denote by $z_{LP}^{(C)}$ the optimum value of the LP relaxation of problem (5.8)–(5.11).

If we introduce the dual variables α_v for constraints (5.9) and the variables β_e for constraints (5.10), we can write the dual of the set covering LP relaxation as follows.

$$
\begin{aligned}
z_{LP}^{(C)} := \min \quad & \sum_{t \in \mathcal{T}} \lambda_t \quad \text{s.t. } (5.9)\text{–}(5.10), \lambda_t \geq 0, \forall t \in \mathcal{T} && (Primal) \quad (5.12)\\
= \max \quad & \sum_{v \in V} \alpha_v + \sum_{e \in E} \beta_e && (Dual) \quad (5.13)\\
\text{s.t.} \quad & \sum_{v \in V} a_{vt}\alpha_v + \sum_{e \in E} b_{et}\beta_e \leq 1 && \forall t \in \mathcal{T} \quad (5.14)\\
& \alpha_v, \beta_e \geq 0 && \forall v \in V, \forall e \in E. \quad (5.15)
\end{aligned}
$$

For this LP covering relaxation, the following proposition holds.

Proposition 41. *Let $G = (V, E)$ be a graph. Then, we have $\chi_T(G) \geq z_{LP}^{(C)} \geq \Delta(G)+1$.*

Proof. Consider a vertex v of maximum degree, and let $\Delta(G) = k$, where $N_G(v) := \{v_1, ..., v_k\}$ and $\delta(v) := \{e_1, ..., e_k\}$. Consider the total matching $T_0 := \{v\}$, and the additional k distinct total matchings $T_i := \{v_i, e_{i+1}\}$ for all $i = 1, \ldots, k-1$ and $T_k := \{v_k, e_1\}$. Hence, we have $k+1$ total matchings, which can be used to define a feasible dual solution: we set $\alpha_v = 1$, $\alpha_{v_i} = \beta_{e_{i+1}} = \frac{1}{2}$ for all $i = 1, \ldots, k-1$ and $\alpha_{v_k} = \beta_{e_1} = \frac{1}{2}$. Thus, summing up all these dual values in the dual objective function, we get the valid lower bound result $z_{LP}^{(C)} \geq \Delta(G) + 1$. □

The example in Figure 5.1 shows that the optimal value of the LP relaxation of the set covering model can be tighter than the value of the LP assignment relaxation. Next, we prove that the LP covering relaxation always provides a lower bound at least as strong as that of the LP assignment relaxation. Our proof uses the equivalence of the set covering relaxation $z_{LP}^{(C)}$ with a set partitioning relaxation, where the inequality constraints (5.9)–(5.10) are replaced with equality constraints. Herein, we denote by $z_{LP}^{(P)}$ the optimal value of the LP partitioning relaxation. The proof of the following result is straightforward.

Lemma 7. $z_{LP}^{(C)} = z_{LP}^{(P)}$.

We are now ready to prove the following proposition.

Proposition 42. *Let $G = (V, E)$ be a graph. Then, we have $\chi_T(G) \geq z_{LP}^{(C)} \geq z_{LP}^{(A)} \geq \Delta + 1$.*

Proof. It suffices to prove that the set covering model (5.8)–(5.11) can be obtained by applying the Dantzig-Wolfe reformulation of the assignment model (5.1)–(5.7). We exploit the block structure of the constraint matrix. First, we group the variables by a fixed color $k \in K$. Let $\mathbf{u_k} := (x_{v_1k}, x_{v_2k}, \ldots, x_{v_nk}, y_{e_1k}, y_{e_2k}, \ldots, y_{e_mk}, z_k)$ be the vector associated to the decision variable of the assignment model and let $\mathbf{z} := (z_1, \ldots, z_k)$. We notice that the constraint matrix has the following block structure:

$$
\begin{array}{rllllll}
\min & \mathbf{1}^T\mathbf{z} & & & & & \\
 & A_1\mathbf{u}_1 + & A_2\mathbf{u}_2 + & \ldots & +A_{|K|}\mathbf{u}_{|K|} & = \mathbf{1} \\
 & B_1\mathbf{u}_1 & & & & \leq \mathbf{0} \\
\text{s.t.} & & B_2\mathbf{u}_2 & & & \leq \mathbf{0} \\
 & & & \ddots & & \\
 & & & & B_{|K|}\mathbf{u}_{|K|} & \leq \mathbf{0} \\
 & \multicolumn{5}{l}{\mathbf{u}_k \in \{0,1\}^{n+m+1}, \forall k = 1, 2, \ldots, |K|}
\end{array}
$$

The corresponding sub-block matrices can be written as:

$$
A_j = \left[\begin{array}{c|c} I_{v,j} & \mathbf{0}_{n\times(m+1)} \\ \hline \mathbf{0}_{m\times n} & I_{e,j} \end{array}\right], \qquad B_j = \left[\begin{array}{c|c} I_v & B_{v,j} \\ \hline B_{e,j} & I_e \end{array}\right].
$$

The blocks A_j for $j = 1, \ldots, |K|$ correspond to the constraint matrices of (5.2)–(5.3), where $I_{v,j}$ is the identity matrix relative to the vertex components, and $I_{e,j}$ is the identity matrix relative to the edge components with one more column with all zeros. The blocks B_j for $j = 1, \ldots, |K|$ correspond to constraints (5.4)–(5.5), where $B_{v,j}$ and $B_{e,j}$ are the edge-vertex incidence matrix and the vertex-edge incidence matrix respectively, both with one more column of all minus ones indicating the color j. Notice that the blocks $A_1 = A_2 = \cdots = A_{|K|}$ and $B_1 = B_2 = \cdots = B_{|K|}$ are identical, since they are incidence matrices of the same graph. Now, define $P_t := \{\mathbf{w}_t \in \{0,1\}^{n+m+1} \mid$

$B_t\mathbf{w}_t \leq \mathbf{0}\}$ for $t = 1, \ldots, |K|$. Thus, for a fixed $k \in K$, we can express $\mathbf{u}_t = \sum_{j \in P_k} \lambda_j^t \mathbf{w}_j$ such that $\sum_{j \in P_t} \lambda_j^t = 1$, where the variables λ_j^t for $j = 1, \ldots, |P_t|$ correspond to the convexity coefficients with respect to the points of $\operatorname{conv}(P_t)$. In order to guarantee the integrality of the solution and to select exactly one of the feasible solution, we impose that $\lambda_j^t \in \{0, 1\}$ for $j = 1, \ldots, |K|$. Since we cannot distinguish between colors and the blocks B_j are the same, we have the same feasible regions, and thus, we can define $P := P_1 = P_2 = \cdots = P_{|K|}$. Let $A_{v,j}$ be the upper block matrix corresponding to the vertex components of A_j and $A_{e,j}$ be the below block matrix corresponding to the edge components, then we can rewrite the model as:

$$\begin{aligned}
\min \quad & \sum_{j=1}^{|K|} \sum_{t \in P} \lambda_t^j \mathbf{w}_t && && (5.16)\\
\text{s.t.} \quad & \sum_{j=1}^{|K|} \sum_{t \in P} (A_{v,j}\mathbf{w}_t)\lambda_t^j = \mathbf{1} && && (5.17)\\
& \sum_{j=1}^{|K|} \sum_{t \in P} (A_{e,j}\mathbf{w}_t)\lambda_t^j = \mathbf{1} && && (5.18)\\
& \sum_{t \in P} \lambda_t^j = 1 && \forall j \in K && (5.19)\\
& \lambda_t^j \in \{0, 1\} && \forall t \in P, \forall j \in K. && (5.20)
\end{aligned}$$

Notice that $A_{v,j}\mathbf{w}_t = A_{v,t}$ and $A_{e,j}\mathbf{w}_t = A_{e,t}$, where $A_{v,t} = (a_{vt}, \mathbf{0}_m)_{v \in V}$ and $A_{e,t} = (\mathbf{0}_n, a_{et})_{e \in E}$ are characteristic vectors of a total matching t, since each element belonging to the same color class corresponds to a total matching. From the previous observation, we can replace for every $t \in P$, $\lambda_t := \sum_{j=1}^{|K|} \lambda_t^j$. In addition, since we can select at most one total matching $t \in P$ for every color class $j \in K$, we replace constraint (5.19) with $\lambda_t \in \{0, 1\}$. The final integer program with the Dantzig-Wolfe reformulation becomes:

$$\begin{aligned}
\min \quad & \sum_{t \in P} \lambda_t && && (5.21)\\
\text{s.t.} \quad & \sum_{t \in P} a_{vt}\lambda_t = 1 && \forall v \in V, && (5.22)\\
& \sum_{t \in P} a_{et}\lambda_t = 1 && \forall e \in E, && (5.23)\\
& \lambda_t \in \{0, 1\} && \forall t \in P. && (5.24)
\end{aligned}$$

where P represents the set of all possible total matchings. □

We remark that the set partitioning model has been obtained by applying Dantzig-Wolfe decomposition to the assignment formulation, keeping constraints (2) and (3) in the master and moving constraints (4) and (5) to the subproblem. For a detailed survey of this technique, we refer the reader to [4].

We can solve problem (5.12), or equivalently its dual (5.13)–(5.15), by considering a subset $\bar{\mathcal{T}} \subset \mathcal{T}$, and by applying a Column Generation algorithm, where looking for a primal negative reduced cost variables corresponds to look for a violated dual constraint [52, 4, 20, 33]. Given a dual feasible solution $\bar{\alpha}$ and $\bar{\beta}$, the separation problem of the dual constraints (5.14) reduces to the following Maximum Weighted Total Matching.

$$\begin{aligned}
\alpha_T(G, \bar{\alpha}, \bar{\beta}) := \max \quad & \sum_{v \in V} \bar{\alpha}_v x_v + \sum_{e \in E} \bar{\beta}_e y_e && && (5.25)\\
\text{s.t.} \quad & x_v + \sum_{e \in \delta(v)} y_e \leq 1 && \forall v \in V, && (5.26)\\
& x_v + x_w + y_e \leq 1 && \forall e = \{v, w\} \in E, && (5.27)\\
& x_v, y_e \in \{0, 1\} && \forall v \in V, \forall e \in E. && (5.28)
\end{aligned}$$

Note that constraints (5.26) and (5.27) together define the valid constraints for total matchings of G. In addition, whenever the optimal value $\alpha_T(G, \bar{\alpha}, \bar{\beta}) > 1$, the corresponding total matching gives a violated constraint (5.14). That is, problem (5.25)–(5.28) is the pricing subproblem for solving our set covering model by Column Generation.

In the next section, we provide computational results based on the Integer Linear Programming model introduced and we discuss new line of future research.

5.3 Computational results

This Section presents computational results for the two relaxations of TCP presented in Section 5.1 and for the relaxations of TMP based on different valid (facet) inequalities discussed in Sections 3.1 and 3.2. The results for the TCP are obtained with a Column Generation algorithm based on model (5.13)–(5.15), which are compared with the results provided by the LP assignment model (5.1)–(5.7). The results for the TMP aim to compare the strength of the families of valid (facet) inequalities discussed in this paper. For both problems, the goal of the computational tests is to compare the bound strengths, that are, lower bounds for TCP and upper bounds for TMP.

Datasets First, we tested our algorithms on a few named graphs: the cycle C_5, the complete graph K_{12}, and the classical Petersen, Chvatal, Tutte, and Watkins graphs.

Then, after running preliminary tests on a large set of graphs from the literature, we decided to focus on a small set of graphs to evaluate the relaxations of TCP. In practice, we observed that most of the graphs from the literature are of Type-1, that is, they have $\chi_T(G) = \Delta(G)+1$ (see, [78]). For this class of graphs, the naive lower bound equal to $\Delta(G) + 1$ is tight, and the contribution of any LP relaxation is minimal. Hence, we have selected 11 cubic graphs of Type-1 and 11 Snark graphs of Type-2, those having $\chi_T(G) > \Delta(G) + 1$, downloaded from the House of Graph library[1], first introduced in [7], and named `graph_N`. The Snarks graphs are cyclically 4-edge-connected graphs with $\chi'(G) = 4$. For Type-2 graphs, we show that the set covering model (5.13)–(5.15) provides better lower bounds than the assignment model (5.1)–(5.7). Finally, we consider random cubic graphs of different sizes and random sparse graphs with 80 vertices but different edge densities for evaluating the total matching relaxations.

Implementation Details We have implemented in Python a Column Generation algorithm for the TCP and a Cutting Plane algorithm for the TMP. We use Gurobi 9.1.1 for solving both the master, the pricing, and the different cut-separation subproblems. The experiments are run on a Dell Workstation with a Intel Xeon W-2155 CPU with 10 physical cores at 3.3GHz and 32 GB of RAM. The source code and the dataset is freely available on GitHub at `https://github.com/stegua/total-matching`.

5.3.1 Total Coloring Lower Bounds

Table 5.1 reports our results for comparing the bound strength achieved by the LP relaxation of model (5.8)–(5.11) (LP, 6-*th* column) and the LP relaxation of the set covering model (SC-LP, 7-*th* columnn). The first columns give for each graph the number of vertices n and of edges m, the maximum degree $\Delta(G)$, and the total chromatic number $\chi_T(G)$. The last two columns report the number of column generation iterations (CG iter) and the total runtime in seconds. For all Type-1 graphs, the trivial lower bound $\Delta(G) + 1$ is already equal to $\chi_T(G)$, and the contribution of the LP relaxation in terms of bounds is null. On the contrary, for all Type-2 graphs, the lower bounds provided by the column generation algorithm are higher than those obtained with the assignment model: this provides computational evidence that the inequality $z_{LP}^{(C)} \geq z_{LP}^{(A)}$ in Proposition 3 is not always tight.

[1] See `https://hog.grinvin.org/`

Graph Name	n	m	Type	$\Delta(G)$	$\chi_T(G)$	LP	SC-LP	CG iter	Runtime
Cycle C_5	5	5	Type-2	2	4	3.00	**3.33**	22	0.00
Complete K_{12}	12	66	Type-2	11	13	12.00	**13.00**	156	0.38
Petersen	10	15	Type-1	3	4	4.00	4.00	61	0.00
Chvatal	12	24	Type-1	4	5	5.00	5.00	95	0.01
Tutte	46	69	Type-1	3	4	4.00	4.00	777	11.35
Watkins	50	75	Type-1	3	4	4.00	4.00	686	9.10
graph_6921	20	30	Type-1	3	4	4.00	4.00	164	0.08
graph_1008	22	33	Type-1	3	4	4.00	4.00	219	0.17
graph_1012	22	33	Type-1	3	4	4.00	4.00	207	0.16
graph_3334	26	39	Type-1	3	4	4.00	4.00	279	0.37
graph_20015	30	45	Type-1	3	4	4.00	4.00	314	0.56
graph_3383	36	54	Type-1	3	4	4.00	4.00	438	1.50
graph_22470	38	57	Type-1	3	4	4.00	4.00	459	1.83
graph_25159	44	66	Type-1	3	4	4.00	4.00	583	6.44
graph_1338	50	75	Type-1	3	4	4.00	4.00	691	7.01
graph_1427	50	75	Type-1	3	4	4.00	4.00	704	7.30
graph_1389	60	90	Type-1	3	4	4.00	4.00	1010	21.15
graph_6630	22	31	Type-2	3	5	4.00	**4.08**	185	0.14
graph_6710	40	60	Type-2	3	5	4.00	**4.08**	459	2.70
graph_6714	40	60	Type-2	3	5	4.00	**4.08**	429	2.19
graph_6720	40	60	Type-2	3	5	4.00	**4.08**	444	2.89
graph_6724	40	60	Type-2	3	5	4.00	**4.08**	441	2.60
graph_6728	40	60	Type-2	3	5	4.00	**4.08**	458	2.85
graph_6708	40	60	Type-2	3	5	4.00	**4.08**	456	3.03
graph_6712	40	60	Type-2	3	5	4.00	**4.08**	439	2.67
graph_6718	40	60	Type-2	3	5	4.00	**4.08**	445	2.66
graph_6722	40	60	Type-2	3	5	4.00	**4.08**	435	2.51
graph_6726	40	60	Type-2	3	5	4.00	**4.08**	456	2.80

Table 5.1: Comparing lower bounds of TCP obtained with two different relaxations: LP refers to the relaxation of (5.8)–(5.11), while SC-LP refers to (5.13)–(5.15).

5.3.2 Total Matchings Lower Bounds on Snark Graphs

Table 5.2 presents the computational results on the total matching problem for 11 snark (cubic) graphs of Type-2. For each graph, the table reports the number of vertices n and of edges m, the matching number $\nu(G)$, the stable set number $\alpha(G)$, and the total matching number $\alpha_T(G)$. Then, we report the upper bounds obtained with the *basic inequalities* (3.1)–(3.3) (column *Basic*), the upper bounds obtained separating only vertex-clique inequalities (3.7) (column *Clique*), and separating only congruent-$2k3$ cycle inequality (3.8) (column *Cycle*-$2k3$). For the latter inequality, we also report the

	Snark Graphs					Upper bounds				Number of cuts	
Name	n	m	$\nu(G)$	$\alpha(G)$	$\alpha_T(G)$	Basic	Clique	Cycle-$2k3$	Percentage	Clique	Cycle-$2k3$
graph_6630	22	31	11	9	13	15.23	15.23	14.24	9.5%	0	19
graph_6710	40	60	20	17	26	28.00	28.00	27.24	4.8%	0	24
graph_6714	40	60	20	16	26	28.00	28.00	27.13	4.3%	0	26
graph_6720	40	60	20	16	26	28.00	28.00	27.23	4.7%	0	25
graph_6724	40	60	20	16	26	28.00	28.00	27.21	4.6%	0	27
graph_6728	40	60	20	16	26	28.00	28.00	27.20	4.6%	0	22
graph_6708	40	60	20	17	26	28.00	28.00	27.20	4.6%	0	29
graph_6712	40	60	20	16	26	28.00	28.00	27.20	4.6%	0	29
graph_6718	40	60	20	16	26	28.00	28.00	27.19	4.6%	0	24
graph_6722	40	60	20	16	26	28.00	28.00	27.14	4.4%	0	27
graph_6726	40	60	20	16	26	28.00	28.00	27.21	4.7%	0	32

Table 5.2: Total Matching results for 11 Snark (cubic) graphs: Upper bounds and number of generated cuts.

percentage of the optimality gap, computed as $\frac{UB-\alpha_T(G)}{\alpha_T(G)} \times 100$. Since Snark graphs have no cliques violated, the only inequalities that improve the bounds are the congruent-$2k3$ cycle inequalities. Notice that we did not even try to separate even-clique inequalities for this family of graphs since Snark graphs are a special case of cubic graphs, and they cannot have cliques with cardinality larger than three. The results presented in the next paragraphs will show for which type of graphs the vertex-clique and the even-clique inequalities begin to play a role.

5.3.3 Total Matchings Lower Bounds on Cubic Graphs

Table 5.3 presents the computational results on 6 random cubic graphs. For each graph, the table gives the number of nodes n and of edges m, the matching number $\nu(G)$, the stable set number $\alpha(G)$, and the total matching number $\alpha_T(G)$. From the 6-th to the 9-th columns we report the upper bound obtained with the LP relaxation (3.1)–(3.3) (column *Basic*), starting with (3.1)–(3.3) and separating only constraints (3.7) (column *Clique*), starting with (3.1)–(3.3) and separating only constraints (3.8) (column *Cycle-$2k3$*), and starting with (3.1)–(3.3) and separating both (3.7) and (3.8) (column *All*). For the same combinations of inequalities, the last four columns report the percentage optimality gap computed with respect to $\alpha_T(G)$.

We remark that on random cubic graphs, which are very sparse graphs, the congruent-$2k3$ cycle inequalities close a larger fraction of the optimality gap than the vertex-clique inequalities.

Table 5.4 reports the average results for our algorithm based on vertex-clique con-

Cubic Graphs					Upper bounds				Percentage optimality gap			
n	m	$\nu(G)$	$\alpha(G)$	$\alpha_T(G)$	Basic	Clique	Cycle-$2k3$	All	Basic	Clique	Cycle-$2k3$	All
50	75	25	22	34	35.00	34.92	34.41	34.33	2.94%	2.70%	1.20%	0.98%
60	90	30	26	40	42.00	42.00	41.27	41.27	5.00%	5.00%	3.18%	3.18%
70	105	35	30	47	49.00	48.95	48.28	48.25	4.26%	4.15%	2.71%	2.66%
80	120	40	35	54	56.00	55.95	55.51	55.46	3.70%	3.61%	2.79%	2.70%
90	135	45	39	61	63.00	62.78	62.29	62.05	3.28%	2.91%	2.12%	1.73%
100	150	50	44	68	70.00	69.85	69.08	69.03	2.94%	2.72%	1.58%	1.52%

Table 5.3: Total Matching results for six cubic graphs: Comparison of upper bounds and of the percentage optimality gaps.

Cubic Graphs			Mean of violated cuts			Mean of percentage gap			
n	m	density	Clique	Cycle-$2k3$	All	Basic	Clique	Cycle-$2k3$	All
50	75	6.1%	0.9	24.9	25.1	3.88%	3.75%	2.14%	2.06%
60	90	5.1%	1.3	25.3	27.4	3.72%	3.54%	2.20%	2.10%
70	105	4.3%	2.1	27.3	27.6	4.04%	3.81%	2.79%	2.65%
80	120	3.8%	1.8	28.1	29.9	3.70%	3.52%	2.55%	2.44%
90	135	3.4%	1.5	30.1	32.3	3.45%	3.32%	2.21%	2.11%
100	150	3.0%	1.8	30.3	33.0	3.09%	2.95%	2.05%	1.96%

Table 5.4: Comparison of maximal clique inequality and congruent-$2k3$ cycle inequalities: Average results of the number of violated cuts and percentage optimality gap. Each row reports the average over 10 random instances of the same size.

straints and congruent-$2k3$ cycle inequalities. The purpose of the table is to show the average strength in terms of bound strength for both families of inequalities. For each row of the table, we report the average over 10 random instances of the number of violated cuts identified by our algorithm and of the percentage gap with respect to the optimal solution. We remark that the number of vertex-cliques is limited, with an average of violated cuts ranging 0.9 to to 2.1. The number of violated congruent-$2k3$ cycle inequalities is in average much larger, but the percentage gap of the upper bound is only slightly better. Again, combining both vertex-cliques and congruent-$2k3$ cycle inequalities, the average percentage gap of the upper bound is always stronger than for the two families taken separately.

5.3.4 Total Matchings Lower Bounds on Random Graphs

We finally present the results of TMP comparing the LP relaxations based on different valid inequalities. We use random graphs with 80 vertices and an edge density ranging from 5% up to 25%. We do not report results for larger edge density because the cut strength has no more impact. Table 5.5 reports in the first column the percentage graph density. In the remaining columns, the table gives the numbers for $\nu(G)$, $\alpha(G)$, and

Random Graphs				Upper bounds					Percentage optimality gap				
den	$\nu(G)$	$\alpha(G)$	$\alpha_T(G)$	Basic	Clique	Cycle-2k3	Even-C	All	Basic	Clique	Cycle-2k3	Even-C	All
5%	39	40	58	58.90	58.89	58.79	58.90	58.78	1.54%	1.54%	1.37%	1.54%	1.34%
10%	40	28	54	58.12	55.56	57.14	58.12	55.52	7.63%	2.89%	5.82%	7.63%	2.81%
15%	40	21	50	58.95	53.23	57.76	58.90	53.23	17.90%	6.47%	15.51%	17.80%	6.47%
20%	40	18	49	59.28	51.16	58.49	59.04	51.16	20.98%	4.42%	19.36%	20.49%	4.42%
25%	40	16	48	59.41	49.97	58.78	59.02	49.97	23.78%	4.11%	22.46%	22.96%	4.11%

Table 5.5: Total Matching results for five random graphs with 80 vertices and different edge density (column *den*): Comparison of upper bounds and of percentage optimality gaps.

$\alpha_T(G)$. Regarding the upper bounds and the percentage optimality gap, in addition to the vertex-clique and the congruent-$2k3$ inequalities, we also consider the even-clique inequalities (39) (column *Even-C*). We notice that on very sparse graphs, the congruent-$2k3$ cycle inequalities close a large fraction of the optimality gap. However, as soon as the graphs become denser, the vertex-clique inequalities play a crucial role in reducing the upper bounds (and hence reducing the percentage optimality gap). On dense graphs, the even-clique inequalities reduce the upper bounds, but only marginally, and they are not as effective as the vertex-clique inequalities, despite being facet-defining.

Table 5.6 reports more extensive results for random graphs: in each row, the table gives the average over 10 random graphs with the same density the number of violated cuts and the percentage optimality gap. For very sparse graphs, combining several families of inequalities pays off in terms of upper bound. However, on dense random graphs, it appears that only the vertex-clique inequalities are very effective in reducing the optimality gap. For instance, on graphs with an edge density of 25% (last row of Table 5.6), by separating a mean of 131.1 cuts, we get a mean optimality gap of 6.3%. At the same time, we have an average of 831.7 violated congruent-$2k3$ cycle inequalities achieving an optimality gap of 25.22%, and an average of 232.5 even-clique violated inequalities for a gap of 25.64%. Indeed, vertex-clique inequalities are the most effective in reducing the upper bounds.

5.4 Hunting of cubic graphs of Type 2

We dedicate this Section to introducing a new method based on an Integer Linear Programming model to generate cubic graphs which are of interest for the conjectures in the total coloring's folklore. In [7], the authors raise the question of whether a cubic graph of Type 2, that is, with $\chi_T(G) = \Delta(G) + 2$, and with girth greater than 4 exists, where we recall that the girth is defined as the length of the smallest cycle in the graph.

	Mean of violated cuts				Mean of percentage gap				
den	Clique	Cycle-2k3	Even-C	All	Basic	Clique	Cycle-2k3	Even-C	All
5%	2.3	14.8	0.0	12.1	1.08%	0.86%	0.86%	1.08%	0.86%
10%	44.1	166.6	0.8	112.5	9.52%	4.61%	8.06%	9.52%	4.55%
15%	95.5	590.8	15.9	137.9	17.93%	5.74%	15.56%	17.68%	5.74%
20%	117.0	669.1	75.3	142.6	20.96%	4.61%	19.31%	20.48%	4.61%
25%	131.1	831.7	232.5	153.0	26.48%	6.30%	25.22%	25.64%	6.30%

Table 5.6: Comparison of the strength of valid inequalities for total matching: Average results of the number of violated cuts and percentage optimality gaps for random graphs with 80 vertices and different edge density. Each row reports the average over 10 random instances of the same size.

In the same paper, they show by means of computational results that every every known Type 2 cubic graph contains a square or a triangle, and a candidate Type 2 cubic graph with a girth greater than 4 must have at least 34 vertices. For this reason, we want to investigate if there exists a possible cubic graph with these properties. To try to answer the question, we propose a MILP model to construct such cubic graphs. Given a set V of vertices, with $|V| = n$, we introduce the binary variables $y_e, \forall e \in E$ to denote whether the edge e is selected or not in the candidate cubic graph. The feasibility model is the following:

$$\sum_{e \in \delta(v)} y_e = 3 \qquad \forall v \in V, \tag{5.29}$$

$$\sum_{(v_0,j) \in \delta(v_0)} f_{v_0 j} = n - 1, \tag{5.30}$$

$$f_{ij} \leq (n-1) y_e \qquad \forall e = \{i, j\} \in E, \tag{5.31}$$

$$\sum_{(i,j) \in \delta(v)} f_{ij} = \sum_{(j,i) \in \delta(v)} f_{ji} - 1 \qquad \forall v \in V \setminus \{v_0\}, \tag{5.32}$$

$$y_{ij} + y_{jk} + y_{ki} \leq 2 \qquad \forall i, j, k \in V, \tag{5.33}$$

$$y_{ij} + y_{jk} + y_{kl} + y_{li} \leq 3 \qquad \forall i, j, k, l \in V, \tag{5.34}$$

$$y_e \in \{0, 1\} \qquad \forall e \in E, \tag{5.35}$$

$$f_{ij} \geq 0 \qquad \forall e \in E, \forall k \in K. \tag{5.36}$$

The constraints (5.29) assure that every node has degree 3. Then, we impose the connectivity constraints as a network flow model $D = (N, A)$. We add the variables flow f_{ij} representing the flow of whatever it is from i to j across the edge between them. We choose an arbitrary vertex v_0 as a source node and every other vertex is a sink with demand equals 1. Hence, constraint (5.30) imposes that the overall flow is distributed on each node of the graph and the flow on each edge cannot exceed the capacity equal to the $n-1$, see (5.31). We then impose the flow conservation constraint on each node, by taking into account that we spend a unit of flow by passing through each vertex. The constraints (5.33)–(5.34) impose that the girth must be at least 5. In fact, to eliminate cycles of length 3, for every tuple of 3 vertices we can pick at most two edges, for cycles of length 4 whereas we pick at most 3 edges among the four candidates. Hence, a feasible solution to this model represents a cubic graph with girth at least 5. We stress the fact that the model proposed is innovative, in the sense that, we are able to generate (new) cubic graphs with a fixed cardinality of vertices.

The tests are conducted as follows. First, the instances are provided through the model, and then, we test if there exists a cubic graph of type 2 from the list generated. Since we deal with graphs of small dimensions, it is not time-consuming and for this reason, we decide to test each graph with the Assignment Model proposed in the previous Section. We have implemented the model in Python using the Gurobi Solver 9.1.1. The enumeration carried out by Gurobi for the ILP model has been customized setting specific parameters of the Gurobi Optimizer. We rely on the `PoolSolutions` parameter to limit the size of solutions to collect and retain during the process. This allows us to have more control over the quality of searching. In our case, we want to generate 200 instances. We then impose the search approach in retrieving the solutions with the parameter `PoolSearchmode`. We have chosen to do a systematic search for the number of desired best solutions setting the parameter equals to 2. Notice that when the parameter is set to 2 the MIP solver succeeded in finding the desired number of best solutions, or it proved that the model does not have many distinct feasible solutions. Moreover, we restrict the search space by setting a gap for the worst possible solution found along the process. We generate 200 random instances of cubic graphs with girth 5 with 38, 40 and 42 vertices. All these graphs are of type 1, that is, $\chi_T(G) = 4$.

Conclusions

In this thesis, we have proposed polyhedral approaches to Total Matching and Total Coloring Problems. For the Total Matching Problem, our contribution includes the characterization of several families of valid inequalities for the Total Matching Polytope. We have shown in Chapter 3 that certain classes of inequalities introduced are facet-defining for the Total Matching Polytope. In particular, in Chapter 4 we have found a complete linear description for trees and complete bipartite graphs, and, we have introduced families of facet-defining inequalities which characterize completely the Total Matching Polytope for the latter class of graphs. Such a result is obtained by using extended formulation techniques and the projection of the corresponding higher dimension polytope onto the original space. Furthermore, the new extended formulation introduced has a polynomial number of constraints. We have introduced two ILP models for the Total Coloring Problem, the assignment model and the set covering model. The latter is based on maximal total matchings, and we have shown how to get the second model by applying a Dantzig-Wolfe reformulation to the first.

As future work, we plan to give a complete linear description of the Total Matching Polytope for further classes of graphs for which the TMP can be computed in polynomial time, such as the class of bipartite permutation graphs. Roughly speaking, the family of bipartite permutation graphs can be described as the union of complete bipartite graphs. Such a decomposition would suggest a nice complete characterization of the $P_T(G)$ when G is a bipartite permutation graph. Furthermore, since we have a complete description of the Stable Set Polytope for chordal graphs, our research direction that we intend to pursue is to study new facet-defining inequalities that will completely describe the Total Matching Polytope for chordal, and likely, for other known classes as quasi-line and claw-free graphs. Computationally, as future development, it will be of interest to implement a complete branch-and-price algorithm for the Total Coloring Problem and a complete branch-and-cut algorithm for the Total Matching Problem.

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