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β-Ensembles and higher genera Catalan numbers

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Abstract

We propose formulas for the large N expansion of the generating function of connected correlators of the β-deformed Gaussian and Wishart–Laguerre matrix models. We show that our proposal satisfies the known transformation properties under the exchange of β with $1/\beta$ and, using Virasoro constraints, we derive a recursion formula for the coefficients of the expansion. In the undeformed limit $\beta = 1$, these coefficients are integers and they have the combinatorial interpretation of generalized Catalan numbers. For generic β, we define the higher genus Catalan polynomials $C_{g,\nu}(\beta)$ whose coefficients are integer numbers.

Contents

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1 Introduction

It is by now a well-known fact that the generating functions of connected correlators of matrix models often admit a topological expansion [1] which contains information about some enumerative geometric problems, such as map enumeration, Hurwitz theory, intersection theory on moduli spaces and Gromov–Witten theory [2–5].

On the other hand, matrix models satisfy Virasoro (or W-algebra) constraints associated to reparametrization invariance of the integrals under certain infinitesimal deformations [5]. Such constraints arise as Ward identities for the correlations functions and can be recast in the form of linear differential equations for the formal generating function of all correlation functions. The generating function of connected correlators of the matrix model, being the logarithm of the generating function of all correlators, satisfies a related set of non-linear equations which follow directly from Virasoro constraints.

In this article, we will be studying the interplay between these two aspects of the theory of random matrix models; namely, we will be interested in exploiting Virasoro constraints to derive the topological expansion of the generating function of connected correlators. By a slight abuse of notation, we will refer to this generating function as the *time-dependent free energy* which is a formal power series in higher times.

More specifically, we consider β-deformed ensembles of random Hermitian matrices with polynomial potentials,

$$Z(N, \beta, \lambda, \mathbf{u}) = \frac{1}{N!} \int \prod_{i=1}^{N} \mathrm{d}x_i \prod_{i<j} |x_i - x_j|^{2\beta} \mathrm{e}^{-\sum_{i=1}^{N} V(x_i) + \frac{1}{N} \sum_{k=1}^{\infty} u_k \sum_{i=1}^{N} x_i^k} \tag{1.1}$$

where $V(x) = \frac{N}{m\lambda} x^m$ and β is an arbitrary complex deformation parameter which allows to interpolate between the various types of standard ensembles such as orthogonal ($\beta = 1/2$), unitary ($\beta = 1$) and symplectic ($\beta = 2$) [7–9]. This generating function provides an highest weight representation of the Virasoro algebra whose generators are represented by differential operators in the *higher times* variables $\mathbf{u} = \{u_1, u_2, \ldots\}$, such that (1.1) is annihilated by a parabolic subalgebra,

$$\left(\frac{N^2}{\lambda}\frac{\partial}{\partial u_{n+m}} - L_n\right) Z(N,\beta,\lambda,\mathbf{u}) = 0\,, \qquad n \geq 1-m \tag{1.2}$$

for certain operators L_n defined as in (3.2). In the case of $m = 1, 2$, the constraints admit a unique solution which can be recast either in the form of exponentials of W-operators [10, 11] or in the form of *superintegrability* formulas for Jack polynomials [11–14].

We restrict ourselves to considering these two cases, which correspond to the Gaussian β-ensemble (GβE) for $m = 2$ and the Wishart–Laguerre β-ensemble (WLβE) for $m = 1$. In order to derive the topological expansion of the time-dependent free energy, we make use of the following key ingredients:

- nonlinear Virasoro constraints for the generating function

$$F(N,\beta,\lambda,\mathbf{u}) := \log\frac{Z(N,\beta,\lambda,\mathbf{u})}{Z(N,\beta,\lambda,0)}\,;$$

- superintegrability formulas for the average of characters [12], i.e., the property of some matrix models that

$$\langle \mathrm{Jack}_\lambda \rangle \sim \mathrm{Jack}_\lambda\,;$$

- symmetries of the constraint equations under the involution that sends β to $1/\beta$.

Combining these properties of the matrix model, we are led to the following *ansatz* for the time-dependent free energy,

$$\begin{aligned} F(N,\beta,\lambda,\mathbf{u}) = \sum_{\ell=1}^{\infty}\sum_{g\in\frac{1}{2}\mathbb{N}} \frac{N^{2-2g-2\ell}\beta^{1-\ell-2g}}{\ell!} \sum_{k_1,\dots,k_\ell=1}^{\infty} (\beta\lambda)^{\sum_{j=1}^{\ell}\frac{k_j}{m}} \\ \times \sum_{i_1+i_2=2g} (-\beta)^{i_1} C^{(i_1,i_2)}_{g,[k_1,\dots,k_\ell]} \prod_{j=1}^{\ell} u_{k_j} \end{aligned} \tag{1.3}$$

where the sum over half-integers g is interpreted as a *genus expansion*. Correspondingly, the coefficients $C^{(i_1,i_2)}_{g,[k_1,\dots,k_\ell]}$ take the role of enumerative invariants associated to the β-deformed model. By analogy with the undeformed case, we define the sum

$$C_{g,[k_1,\dots,k_\ell]}(\beta) := \sum_{i_1+i_2=2g} (-\beta)^{i_1} C^{(i_1,i_2)}_{g,[k_1,\dots,k_\ell]} \tag{1.4}$$

to be the *Catalan polynomial* of genus g associated to the tuple $[k_1,\dots,k_\ell]$. Plugging the ansatz into the Virasoro constraint equations gives a set of (*cut-and-join*) recursion relations for the polynomials $C_{g,[k_1,\dots,k_\ell]}(\beta)$ which can be solved uniquely. Remarkably, the coefficients $C^{(i_1,i_2)}_{g,[k_1,\dots,k_\ell]}$ are all *integer* numbers and they provide a refinement of the ordinary higher genus Catalan numbers described in [15, 16].

The organization of the article is as follows.

- In Sects. 2 to 7 we give the definition of the generating function of (connected) correlators of the GβE as a function of higher times, the rank N, the coupling λ and the deformation parameter β. We derive the Virasoro constraints satisfied by the time-dependent free energy, and we observe the symmetry of these objects under the involution that exchanges β and $1/\beta$. Making use of these properties, we then derive an ansatz for the $1/N$ expansion of the time-dependent free energy, and we identify the coefficients $C_{g,\nu}(\beta)$ as β-deformations of the higher genus Catalan numbers that appear in the topological expansion of the ordinary Gaussian matrix model. We show that the β-dependence of the Catalan polynomials $C_{g,\nu}(\beta)$ can be reabsorbed into Schur polynomials of two variables $s_\lambda(1, -\beta)$ and this leads to the definition of a secondary set of integer invariants, the $n_{\nu,\lambda}$. A cut-and-join recursion formula for the Catalan polynomials is then obtained from Virasoro constraints and the undeformed limit $\beta \to 1$ is discussed.
- In Sect. 8 we repeat the analysis for the case of linear potential, i.e., the WLβE matrix model, and we obtain similar formulas for the genus expansion and the recursion relations.
- In Sect. 9 we comment on our results and identify some of the open questions that deserve further investigation.

Finally, in Appendix A we collect some useful facts about Schur polynomials in two variables, in Appendix B we provide a formula to relate the Catalan polynomials of the GβE to the marginal b-polynomials that appear in the b-conjectures of Goulden and Jackson, and in Appendix C we tabulate some of the polynomials $C_{g,\nu}(\beta)$ and integer invariants $n_{\nu,\lambda}$ obtained by solving the recursion up to finite order in the genus and in higher times.

2 The β-deformed Gaussian ensemble

Recall that the classical Gaussian unitary ensemble (GUE) is defined by the matrix integral

$$\int_{N\times N} \mathrm{d}M \; \mathrm{e}^{-\frac{N}{2\lambda}\mathrm{Tr} M^2} \tag{2.1}$$

where the integration is over the space of Hermitian $N \times N$ matrices and the parameter N is the rank. Here we consider a quadratic potential with coupling constant λ such that $\Re(\lambda) > 0$.

The measure and the potential are invariant under the adjoint action of the group $U(N)$; therefore, it is possible to rewrite this matrix integral as an integral over eigenvalues x_i as

$$\frac{1}{N!}\int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i \prod_{i<j} |x_i - x_j|^2 \, \mathrm{e}^{-\frac{N}{2\lambda}\sum_i x_i^2} \tag{2.2}$$

where $\prod_{i<j}(x_i - x_j)$ is the Vandermonde determinant. The β-deformation of the GUE is defined as a 1-parameter deformation of the eigenvalue integral by substituting the Vandermonde determinant with its β power. This is a very natural and well-studied deformation which is known to have a matrix integral representation via tridiagonal

matrices as shown in [17]. We refer to this matrix model as the Gaussian β-deformed ensemble.

The generating function of all polynomial expectation values is defined by the formal power series in higher times $\mathbf{u} = \{u_1, u_2, \dots\}$ as

$$Z(N, \beta, \lambda, \mathbf{u}) := \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}x_i \prod_{i<j} |x_i - x_j|^{2\beta}\, \mathrm{e}^{-\frac{N}{2\lambda}\sum_i x_i^2 + \frac{1}{N}\sum_{k=1}^{\infty} u_k \sum_i x_i^k} \tag{2.3}$$

This is the main object that we will study in the following. Its logarithm is the generating function of all connected correlation functions and is known as the time-dependent free energy of the matrix model. We denote this function as

$$F(N, \beta, \lambda, \mathbf{u}) := \log \frac{Z(N, \beta, \lambda, \mathbf{u})}{Z(N, \beta, \lambda, 0)} \tag{2.4}$$

where we normalized the time-dependent free energy so that $F(N, \beta, \lambda, 0) = 0$.

It is a well-known fact that the time-dependent free energy of a matrix model admits a simultaneous expansion in the parameter N and higher times u_k which can be interpreted as a genus expansion, i.e., a sum over contributions coming from surfaces of arbitrary genus g and punctures. In Sect. 5 we show that a similar expansion also exists after the β-deformation.

We conclude this section by observing that the dependence on the coupling parameter λ is quite simple, and it can be easily reabsorbed via a rescaling of times, as we show in the following lemma.

Lemma 2.1 *The function $F(N, \beta, \lambda, \mathbf{u})$ satisfies the following homogeneity equation*

$$F(N, \beta, \lambda, \mathbf{u}) = \lambda^{\frac{D_{\mathbf{u}}}{2}} F(N, \beta, 1, \mathbf{u}) \tag{2.5}$$

with the dilation operator defined as

$$D_{\mathbf{u}} = \sum_{k \geq 1} k u_k \frac{\partial}{\partial u_k}\,. \tag{2.6}$$

Proof This follows from the change of variables $x_i = \lambda^{\frac{1}{2}} y_i$ in the integral (2.2). □

3 Virasoro constraints

The generating function of a matrix model satisfies an infinite set of differential equations known as Virasoro constraints [6] which encode all the linear relations among the correlation functions. These constraints are a consequence of invariance of the integral under infinitesimal reparametrizations generated by the vector fields $\sum_i x_i^{n+1} \frac{\partial}{\partial x_i}$ for $n \in \mathbb{Z}$ which provide a representation of the Virasoro algebra.

In the case of the β-ensemble in (2.3), the constraints can be written explicitly as

$$\left(\frac{N^2}{\lambda}\frac{\partial}{\partial u_{n+2}} - L_n\right) Z(N,\beta,\lambda,\mathbf{u}) = 0\,, \qquad n \geq -1 \tag{3.1}$$

with the Virasoro generators L_n defined as

$$\begin{aligned} L_{n>0} &= 2\beta N^2 \frac{\partial}{\partial u_n} + \beta N^2 \sum_{i+j=n} \frac{\partial^2}{\partial u_i \partial u_j} + (1-\beta)N(n+1)\frac{\partial}{\partial u_n} + \sum_{k>0} k u_k \frac{\partial}{\partial u_{k+n}} \\ L_0 &= \beta N^2 + (1-\beta)N + \sum_{k>0} k u_k \frac{\partial}{\partial u_k} \\ L_{-1} &= u_1 + \sum_{k>0} k u_k \frac{\partial}{\partial u_{k-1}}\,, \end{aligned} \tag{3.2}$$

While these constraints are linear for the function $Z(N,\beta,\lambda,\mathbf{u})$, they are nonlinear and non-homogeneous for $F(N,\beta,\lambda,\mathbf{u})$; in fact, we have

$$\begin{aligned} &\Big(\frac{N^2}{\lambda}\frac{\partial}{\partial u_{n+2}} - \Big(2\beta N^2 + (1-\beta)N(n+1)\Big)\frac{\partial}{\partial u_n} - \beta N^2 \sum_{i+j=n}\frac{\partial^2}{\partial u_i \partial u_j} \\ &\quad - \sum_{k>0} k u_k \frac{\partial}{\partial u_{k+n}}\Big) F(N,\beta,\lambda,\mathbf{u}) \\ &\quad - \beta N^2 \sum_{i+j=n} \frac{\partial F(N,\beta,\lambda,\mathbf{u})}{\partial u_i}\frac{\partial F(N,\beta,\lambda,\mathbf{u})}{\partial u_j} \\ &= \Big(\beta N^2 + (1-\beta)N\Big)\delta_{n,0} + u_1 \delta_{n,-1} \end{aligned} \tag{3.3}$$

Later, we will use these equations to derive a recursion relation for the coefficients of the series expansion of $F(N,\beta,\lambda,\mathbf{u})$.

Remark 3.1 The rank N of a matrix model, even β-deformed, is by definition the number of eigenvalues (integration variables) and therefore it is a positive integer number. We remark, however, that the Virasoro constraint Eq. (3.3) make sense not just for positive integer rank but can be analytically continued to arbitrary complex values. The corresponding solutions will then interpolate between actual matrix models and more general functions that can be interpreted as analytic continuations away from integer rank N. From now on, we will therefore assume that N is just another complex variable on which the coefficients of the generating function $F(N,\beta,\lambda,\mathbf{u})$ depend analytically.

4 Symmetries of the GβE

Before discussing the dependence of $F(N, \beta, \lambda, \mathbf{u})$ on the parameter N as a power series, we pause to observe that the β-ensemble has a non-trivial symmetry under the exchange of β and β^{-1}, [17]. This is not a symmetry which is manifest at the level of the integral representation of the model, and in fact, it is better understood as a non-perturbative duality known as Langlands duality [17]. Namely, one notices that the Virasoro constraints (3.1) are invariant under the duality transformation

$$\beta \mapsto 1/\beta, \tag{4.1}$$

together with the rescalings

$$N \mapsto -\beta N, \qquad u_k \mapsto u_k, \qquad \lambda \mapsto \beta^2\lambda \tag{4.2}$$

Since the solution of the constraints is unique (up to normalization), it must follow that the (normalized) generating function is invariant under this symmetry. This produces the identity

$$F(-\beta N, \beta^{-1}, \beta^2\lambda, \mathbf{u}) = F(N, \beta, \lambda, \mathbf{u}) \tag{4.3}$$

for the time-dependent free energy. We will use this identity to constrain the form of $F(N, \beta, \lambda, \mathbf{u})$ as a power series in N.

In addition to the inversion symmetry of β, the GβE just like the GUE is symmetric under the transformation $x_i \mapsto -x_i$. Then it is well-known that there is a corresponding Ward identity that sets all odd correlation functions to zero, namely

$$\frac{\partial}{\partial u_{k_1}} \cdots \frac{\partial}{\partial u_{k_n}} Z(N, \beta, \lambda, \mathbf{u}) = 0 \tag{4.4}$$

when $k_1 + \cdots + k_n$ is odd. Correspondingly, $F(N, \beta, \lambda, \mathbf{u})$ must be a formal power series in times such that each monomial is even w.r.t. the degree induced by the dilation operator $D_{\mathbf{u}}$ in (2.6).

5 Large N behavior and genus expansion

The Virasoro constraints of the GβE induce a recursion for the correlation functions, and this recursion is known to have a unique solution [18] up to normalization. Using the superintegrability formula for averages of Jack polynomials [10, 12, 13, 17],

$$\frac{\langle \mathrm{JackP}_\mu(x_1, \ldots, x_N)\rangle_{\mathrm{G}\beta\mathrm{E}}}{\langle 1\rangle_{\mathrm{G}\beta\mathrm{E}}} = \frac{\mathrm{JackP}_\mu(p_k = N)\mathrm{JackP}_\mu\big(p_k = (\frac{\beta\lambda}{N})^{\frac{k}{2}}\delta_{k,2}\big)}{\mathrm{JackP}_\mu\left(p_k = \delta_{k,1}\right)} \tag{5.1}$$

together with Cauchy's identity for Jack polynomials

$$\exp\Big(\beta\sum_{k\geq 1}\frac{p_k t_k}{k}\Big)=\sum_{\mu}\mathrm{JackP}_{\mu}(p_k)\mathrm{JackQ}_{\mu}(t_k) \tag{5.2}$$

we can rewrite the generating function as follows

$$\frac{Z(N,\beta,\lambda,\mathbf{u})}{Z(N,\beta,\lambda,0)}=\sum_{\mu}\frac{\mathrm{JackP}_{\mu}(p_k=N)\mathrm{JackP}_{\mu}(p_k=\delta_{k,2})}{\mathrm{JackP}_{\mu}(p_k=\delta_{k,1})}\mathrm{JackQ}_{\mu}\Big(p_k=(\tfrac{\beta\lambda}{N})^{\frac{k}{2}}\frac{ku_k}{\beta N}\Big) \tag{5.3}$$

This implies that $Z(N,\beta,\lambda,\mathbf{u})$ admits a Taylor series expansion in N around ∞ and, after taking the logarithm, the same is also true of $F(N,\beta,\lambda,\mathbf{u})$, so that we can write the series expansion

$$F(N,\beta,\lambda,\mathbf{u})=\sum_{s=0}^{\infty}N^{-s}F_s(\beta,\lambda,\mathbf{u})\ . \tag{5.4}$$

The Virasoro constraints for the coefficients $F_s(\beta,\lambda,\mathbf{u})$ read

$$\begin{aligned}&\Big(\lambda^{-1}\frac{\partial}{\partial u_{n+2}}-2\beta\frac{\partial}{\partial u_n}-\beta\sum_{n_1+n_2=n}\frac{\partial^2}{\partial u_{n_1}\partial u_{n_2}}\Big)F_s(\beta,\lambda,\mathbf{u})\\&=(1-\beta)(n+1)\frac{\partial}{\partial u_n}F_{s-1}(\beta,\lambda,\mathbf{u})+\sum_{k>0}ku_k\frac{\partial}{\partial u_{k+n}}F_{s-2}(\beta,\lambda,\mathbf{u})\\&\quad+\beta\sum_{\substack{n_1+n_2=n\\ s_1+s_2=s}}\frac{\partial F_{s_1}(\beta,\lambda,\mathbf{u})}{\partial u_{n_1}}\frac{\partial F_{s_2}(\beta,\lambda,\mathbf{u})}{\partial u_{n_2}}\\&\quad+\beta\delta_{n,0}\delta_{s,0}+(1-\beta)\delta_{n,0}\delta_{s,1}+u_1\delta_{n,-1}\delta_{s,2}\end{aligned} \tag{5.5}$$

Let us assume we know the functions $F_0(\beta,\lambda,\mathbf{u}),\ldots,F_{s-1}(\beta,\lambda,\mathbf{u})$ and we want to solve the constraints with respect to $F_s(\beta,\lambda,\mathbf{u})$. Then (5.5) gives the following

$$\begin{aligned}&\Big(\lambda^{-1}\frac{\partial}{\partial u_{n+2}}-2\beta\frac{\partial}{\partial u_n}-\beta\sum_{n_1+n_2=n}\frac{\partial^2}{\partial u_{n_1}\partial u_{n_2}}-2\beta\sum_{n_1+n_2=n}\frac{\partial F_0(\beta,\lambda,\mathbf{u})}{\partial u_{n_1}}\frac{\partial}{\partial u_{n_2}}\Big)\\&\quad F_s(\beta,\lambda,\mathbf{u})\\&=B_{s,n}(\beta,\lambda,\mathbf{u})\end{aligned} \tag{5.6}$$

where $B_{s,n}(\beta, \lambda, \mathbf{u})$ is given by

$$B_{s,n}(\beta, \lambda, \mathbf{u}) := (1-\beta)(n+1)\frac{\partial F_{s-1}(\beta, \lambda, \mathbf{u})}{\partial u_n} + \sum_{k>0} k u_k \frac{\partial F_{s-2}(\beta, \lambda, \mathbf{u})}{\partial u_{k+n}} + \beta \sum_{a=1}^{n-1}\sum_{j=1}^{s-1} \frac{\partial F_j(\beta, \lambda, \mathbf{u})}{\partial u_a}\frac{\partial F_{s-j}(\beta, \lambda, \mathbf{u})}{\partial u_{n-a}} + \beta\delta_{n,0}\delta_{s,0} + (1-\beta)\delta_{n,0}\delta_{s,1} + u_1\delta_{n,-1}\delta_{s,2} \tag{5.7}$$

Observe that (5.6) as an equation for $F_s(\beta, \lambda, \mathbf{u})$ is now non-homogeneous but linear (for $s > 0$). This fact now allows us to solve the Virasoro constraints in a fashion similar to that of [20] in the case of matrix models with boundaries. A formal solution can be derived as follows. We multiply (5.6) by $(n+2)u_{n+2}$ and sum over $n = -1, 0, \ldots,$ to get

$$\Big(D_{\mathbf{u}} - (\beta\lambda)W\Big) F_s(\beta, \lambda, \mathbf{u}) = \lambda \sum_{n=1}^{\infty} n u_n B_{s,n-2}(\beta, \lambda, \mathbf{u}) \tag{5.8}$$

with

$$W := 2\sum_{n=1}^{\infty}(n+2)u_{n+2}\frac{\partial}{\partial u_n} + \sum_{n_1,n_2=1}^{\infty}(n_1+n_2+2)u_{n_1+n_2+2} \times \left(\frac{\partial^2}{\partial u_{n_1}\partial u_{n_2}} + 2\frac{\partial F_0(\beta, \lambda, \mathbf{u})}{\partial u_{n_1}}\frac{\partial}{\partial u_{n_2}}\right) \tag{5.9}$$

and $D_{\mathbf{u}}$ is the dilation operator in (2.6). Since both $F_s(\beta, \lambda, \mathbf{u})$ and the function in the r.h.s. have no constant terms in $\mathbf{u}$, we can invert the operator $D_{\mathbf{u}}$ to get

$$F_s(\beta, \lambda, \mathbf{u}) = \sum_{k=0}^{\infty}(\beta\lambda)^k (D_{\mathbf{u}}^{-1}W)^k \lambda D_{\mathbf{u}}^{-1}\sum_{n=1}^{\infty} n u_n B_{s,n-2}(\beta, \lambda, \mathbf{u}) \tag{5.10}$$

With enough computational power, this formula allows to derive recursively all the functions $F_s(\beta, \lambda, \mathbf{u})$ by repeatedly applying the operators W and $D_{\mathbf{u}}^{-1}$. Unfortunately, we do not know how to use (5.10) to give a closed formula for the solution of the constraints. In the next sections, however, we will make use of this formal expression to argue some properties about the polynomial dependence on the times $\mathbf{u}$. In fact, equation (5.10) is instrumental in the proof of proposition 5.1.

5.1 Order zero

The order-zero term $F_0(\beta, \lambda, \mathbf{u})$ can be easily computed by solving Virasoro constraints in that limit. The constraints (5.5) for $s = 0$ yield the relations

$$\Big((\beta\lambda)^{-1}\frac{\partial}{\partial u_{n+2}} - 2\frac{\partial}{\partial u_n} - \sum_{n_1+n_2=n}\frac{\partial^2}{\partial u_{n_1}\partial u_{n_2}}\Big)F_0(\beta, \lambda, \mathbf{u}) \\ = \sum_{n_1+n_2=n}\frac{\partial F_0(\beta, \lambda, \mathbf{u})}{\partial u_{n_1}}\frac{\partial F_0(\beta, \lambda, \mathbf{u})}{\partial u_{n_2}} + \delta_{n,0} \tag{5.11}$$

In order to solve this equation, we first come up with an ansatz for the function $F_0(\beta, \lambda, \mathbf{u})$ and we plug it in the constraint. If we can fix the parameters of the ansatz so that the constraints are satisfied, then it must follow that the ansatz is correct, since we know that the solution is unique. Let us consider the ansatz

$$F_0(\beta, \lambda, \mathbf{u}) = \sum_{k=1}^{\infty}(\beta\lambda)^{\frac{k}{2}}F_{0,[k]}(\beta)\,u_k \tag{5.12}$$

with $F_{0,[k]}(\beta)$ some coefficients to be determined. From the Virasoro constraints, we get the recursion relation

$$F_{0,[n+2]}(\beta) = 2F_{0,[n]}(\beta) + \sum_{n_1+n_2=n}F_{0,[n_1]}(\beta)F_{0,[n_2]}(\beta) + \delta_{n,0} \tag{5.13}$$

which admits the unique solution

$$F_{0,[k]}(\beta) = \frac{(1+(-1)^k)}{2}\frac{1}{k/2+1}\binom{k}{k/2} \tag{5.14}$$

for $k \geq 1$. We then have that, at order zero, the coefficients $F_{0,[k]}(\beta)$ are constant that do not depend on the deformation parameter β.

Observe that naively it would appear that $F_0(\beta, \lambda, \mathbf{u})$ is not polynomial in β because of the fractional power in the term $(\beta\lambda)^{\frac{k}{2}}$. However, by explicitly solving the recursion (5.13), we obtain that $F_{0,[k]}(\beta) = 0$ if k is odd, so that $F_0(\mathbf{u})$ is indeed polynomial in β (at each order in the expansion in times $\mathbf{u}$). It then follows that the numbers $F_{0,[2k]}(\beta)$ coincide with the ordinary Catalan numbers.

5.2 Order one

The order-one term $F_1(\beta, \lambda, \mathbf{u})$ satisfies the constraints

$$\Big(\lambda^{-1}\frac{\partial}{\partial u_{n+2}} - 2\beta\frac{\partial}{\partial u_n} - \beta\sum_{n_1+n_2=n}\frac{\partial^2}{\partial u_{n_1}\partial u_{n_2}} - 2\beta\sum_{n_1+n_2=n}\frac{\partial F_0(\beta,\lambda,\mathbf{u})}{\partial u_{n_1}}\frac{\partial}{\partial u_{n_2}}\Big) F_1(\beta,\lambda,\mathbf{u}) = (1-\beta)(n+1)\frac{\partial}{\partial u_n}F_0(\beta,\lambda,\mathbf{u}) + (1-\beta)\delta_{n,0} \tag{5.15}$$

which also involve the order-zero function $F_0(\beta, \lambda, \mathbf{u})$. In this case we consider the following ansatz

$$F_1(\beta,\lambda,\mathbf{u}) = \beta^{-1}\sum_{k=1}^{\infty}(\beta\lambda)^{\frac{k}{2}}F_{1,[k]}(\beta)\,u_k \tag{5.16}$$

with $F_{1,[k]}(\beta)$ some polynomial function of β. Then from the Virasoro constraints, we get the recursion relation

$$F_{1,[n+2]}(\beta) = 2F_{1,[n]}(\beta) + (1-\beta)(n+1)F_{0,[n]}(\beta) + 2\sum_{n_1+n_2=n}F_{0,[n_1]}(\beta)F_{1,[n_2]}(\beta) + (1-\beta)\delta_{n,0} \tag{5.17}$$

which admits the unique solution

$$F_{1,[k]}(\beta) = (1-\beta)\frac{(1+(-1)^k)}{2}\left(2^{k-1} - \binom{k-1}{k/2}\right) \tag{5.18}$$

for $k \geq 1$. As in the previous case, the ansatz allows to solve the recursion uniquely, and therefore, it is the correct formula for the function $F_1(\beta, \lambda, \mathbf{u})$.

It would appear at this point that $F_s(\beta, \lambda, \mathbf{u})$ is always a polynomial of degree 1 in the times (w.r.t. the grading operator $\sum_{k\geq 1} u_k \frac{\partial}{\partial u_k}$); however, this is not true for $s \geq 2$, as we will show in the next section.

5.3 Higher orders

Next, we want to compute all higher-order terms in the $1/N$ expansion. While the constraint equations for the leading order are self-contained and can be solved independently of other orders, the constraints for higher-order functions $F_s(\beta, \lambda, \mathbf{u})$ do depend explicitly on lower orders as well. Nevertheless, we would like to use the same strategy to solve the constraints at all orders. Namely, we come up with an ansatz for the full time-dependent free energy, and we use the constraints to fix the coefficients. If a solution exists, then it must follow that the ansatz gives the correct answer for $F(N, \beta, \lambda, \mathbf{u})$. Moreover, the Virasoro constraints will give a recursive definition of the coefficients. By analogy with the order-zero case, we name these coefficients β-*deformed generalized Catalan numbers*. This is in fact compatible with the definition of (undeformed) generalized Catalan numbers of [15, 16] when $\beta = 1$.

We first want to fix the polynomial dependence on times of the functions $F_s(\beta, \lambda, \mathbf{u})$. Making use of the formal solution (5.10), we obtain the following.

Proposition 5.1 *The function $F_s(\beta, \lambda, \mathbf{u})$ is polynomial in the time variables $\mathbf{u}$ of degree at most $\lfloor s/2 \rfloor + 1$ w.r.t. the grading operator $\sum_{k>0} u_k \frac{\partial}{\partial u_k}$.*

Proof The proposition can be proven by induction on s. The functions $F_0(\beta, \lambda, \mathbf{u})$ and $F_1(\beta, \lambda, \mathbf{u})$ are both of degree 1 in times as shown in the previous sections; hence, they satisfy the statement of the proposition. For $s \geq 2$, we assume that $F_r(\beta, \lambda, \mathbf{u})$ is of degree at most $\lfloor r/2 \rfloor + 1$ for $0 \leq r < s$; then, from (5.7) one observes that the function

$$\sum_{n=1}^{\infty} n u_n B_{s,n-2}(\beta, \lambda, \mathbf{u}) \tag{5.19}$$

is of degree at most $\lfloor s/2 \rfloor + 1$. [1] Since the operator $D_{\mathbf{u}}$ is degree 0 and W is the sum of a degree 0 and a degree -1 operator, it follows from (5.10) that $F_s(\beta, \lambda, \mathbf{u})$ is a polynomial of degree at most $\lfloor s/2 \rfloor + 1$. □

Next, we want to fix the λ and β dependence. The former is dictated by the homogeneity property in lemma 2.1, while the latter follows from the fact that $F_s(\beta, \lambda, \mathbf{u})$ is polynomial in β at every order in times. We can then write an explicit formula for $F_s(\beta, \lambda, \mathbf{u})$,

$$F_s(\beta, \lambda, \mathbf{u}) = \sum_{\ell=1}^{\lfloor s/2 \rfloor + 1} \frac{\beta^{m(s,\ell)}}{\ell!} \sum_{k_1,\dots,k_\ell=1}^{\infty} (\beta\lambda)^{\frac{k_1+\dots+k_\ell}{2}} F_{s,[k_1,\dots,k_\ell]}(\beta) \prod_{j=1}^{\ell} u_{k_j} \tag{5.20}$$

where the coefficients $F_{s,[k_1,\dots,k_\ell]}(\beta)$ and the exponent $m(s, \ell)$ are yet to be determined (via the constraints). Our analysis now indicates that the coefficients $F_{s,[k_1,\dots,k_\ell]}(\beta)$ are polynomial functions of β. Moreover, the exponent $m(s, \ell)$ is an integer-valued function of s, ℓ such that $m(s, \ell) \geq 0$ if $F_{s,[k_1,\dots,k_\ell]}(\beta) \neq 0$[2].

At this point of our derivation, we would like to find a concrete formula for $m(s, \ell)$. As this is not fully specified by just polynomiality or symmetry arguments, we propose the following ansatz.

Ansatz 5.1 *The integer function $m(s, \ell)$ is given by*

$$m(s, \ell) = \ell - s - 1 \tag{5.21}$$

A proof that this ansatz is indeed the correct choice for the function $m(s, \ell)$ will follow from the recursion relations that the Virasoro constraint impose on the functions $F_{s,[k_1,\dots,k_\ell]}(\beta)$.

[1] To see this, we need to use the inequality $\lfloor i/2 \rfloor + \lfloor j/2 \rfloor \leq \lfloor s/2 \rfloor$ for all $0 < i, j < s$ such that $i + j = s$.

[2] This positivity constraint on $m(s, \ell)$ follows from the requirement that $F_s(\beta, \lambda, \mathbf{u})$ must be polynomial in β.

Let us make sure that our formulas satisfy the symmetry discussed in Sect. 4. In the new variables, the symmetry acts trivially on the times $\mathbf{u}$, while it transforms the coupling as

$$\lambda \mapsto \beta^2 \lambda, \tag{5.22}$$

so that from (4.3) it must follow that

$$F_s(\beta^{-1}, \beta^2\lambda, \mathbf{u}) = (-\beta)^s F_s(\beta, \lambda, \mathbf{u}) \tag{5.23}$$

This can be checked explicitly for $s = 0, 1$ from (5.12) and (5.16). More generally, we get the non-trivial relation

$$F_{s,[k_1,\ldots,k_\ell]}(\beta^{-1}) = (-\beta)^{2\ell-s-2} F_{s,[k_1,\ldots,k_\ell]}(\beta) \tag{5.24}$$

Let $\deg F_{s,[k_1,\ldots,k_\ell]}(\beta)$ denote the polynomial degree in β, then we can write

$$F_{s,[k_1,\ldots,k_\ell]}(\beta) = \sum_{i=0}^{\deg F_{s,[k_1,\ldots,k_\ell]}(\beta)} (-\beta)^i F_{s,[k_1,\ldots,k_\ell],i} \tag{5.25}$$

for some constant coefficients $F_{s,[k_1,\ldots,k_\ell],i}$. Then the symmetry constraint (5.24) implies the following:

$$\deg F_{s,[k_1,\ldots,k_\ell]}(\beta) = 2 - 2\ell + s \tag{5.26}$$

$$F_{s,[k_1,\ldots,k_\ell],2-2\ell+s-i} = F_{s,[k_1,\ldots,k_\ell],i} \tag{5.27}$$

where the second equation can be equally stated as saying that $F_{s,[k_1,\ldots,k_\ell]}(\beta)$ is a palindromic polynomial in $-\beta$.

Remark 5.1 Observe that any polynomial $P(x, y) = \sum_{i=0}^{r} P_i x^i y^{r-i}$ such that the coefficients are palindromic (i.e., $P_{r-i} = P_i$), is symmetric in the exchange of x and y, and therefore can be written as a linear combination of homogeneous degree-r symmetric polynomials in two variables. The space of such symmetric functions has a special linear basis consisting of Schur polynomials $s_\lambda(x, y)$ associated to partitions λ with r boxes (see appendix A). If the coefficients P_i are integers, then the coefficients in the expansion of $P(x, y)$ over the Schur basis are also integer numbers. The polynomial functions $F_{s,[k_1,\ldots,k_\ell]}(\beta)$ can be regarded as special cases of polynomials $P(x, y)$ where $x = -\beta$, $y = 1$ and $r = 2 - 2\ell + s$.

Lemma 5.1 *If s is odd, the polynomial $F_{s,[k_1,\ldots,k_\ell]}(\beta)$ has a zero at $\beta = 1$.*

Proof Let us use the symbol ν to indicate the tuple $[k_1, \ldots, k_\ell]$. If s is odd, we can assume that there is an integer h such that $2 - 2\ell + s = 2h + 1$. Then we can use the identity (5.27) to write

$$\begin{aligned} F_{s,\nu}(\beta) &= \sum_{i=0}^{2h+1} (-\beta)^i F_{s,\nu,i} \\ &= \sum_{i=0}^{h} (-\beta)^i F_{s,\nu,i} + \sum_{i=h+1}^{2h+1} (-\beta)^i F_{s,\nu,i} \\ &= \sum_{i=0}^{h} (-\beta)^i F_{s,\nu,i} + \sum_{i=0}^{h} (-\beta)^{2h+1-i} F_{s,\nu,2h+1-i} \\ &= \sum_{i=0}^{h} (-\beta)^i F_{s,\nu,i} \left(1 - \beta^{1+2(h-i)}\right) \\ &= (1-\beta) \sum_{i=0}^{h} (-\beta)^i F_{s,\nu,i} \frac{1-\beta^{1+2(h-i)}}{1-\beta} \end{aligned} \tag{5.28}$$

where $\frac{1-\beta^{1+2(h-i)}}{1-\beta} = 1 + \beta + \cdots + \beta^{2(h-i)}$ is a polynomial for all i s.t. $0 \leq i \leq h$. □

Putting everything together, we get the expansion

$$\begin{aligned} F(N, \beta, \lambda, \mathbf{u}) = \sum_{s=0}^{\infty} N^{-s} \sum_{\ell=1}^{\lfloor s/2 \rfloor + 1} \frac{\beta^{\ell-s-1}}{\ell!} \\ \sum_{k_1,\ldots,k_\ell=1}^{\infty} (\beta\lambda)^{\sum_{j=1}^{\ell} \frac{k_j}{2}} \sum_{i=0}^{2-2\ell+s} (-\beta)^i F_{s,[k_1,\ldots,k_\ell],i} \prod_{j=1}^{\ell} u_{k_j} \end{aligned} \tag{5.29}$$

5.4 Genus expansion

Recall that for any function $f(s, \ell)$ we have the identity

$$\sum_{s=0}^{\infty} N^{-s} \sum_{\ell=1}^{\lfloor s/2 \rfloor + 1} f(s, \ell) = \sum_{\ell=1}^{\infty} \sum_{g \in \frac{1}{2}\mathbb{N}} N^{2-2g-2\ell} f(2g - 2 + 2\ell, \ell) \tag{5.30}$$

which is just a reorganization of the sum on the left. Hence, we can rewrite the time-dependent free energy as

$$\begin{aligned} F(N, \beta, \lambda, \mathbf{u}) = \sum_{\ell=1}^{\infty} \sum_{g \in \frac{1}{2}\mathbb{N}} \frac{N^{2-2g-2\ell} \beta^{1-\ell-2g}}{\ell!} \sum_{k_1,\ldots,k_\ell=1}^{\infty} (\beta\lambda)^{\sum_{j=1}^{\ell} \frac{k_j}{2}} \\ \times \sum_{i=0}^{2g} (-\beta)^i F_{2g-2+2\ell,[k_1,\ldots,k_\ell],i} \prod_{j=1}^{\ell} u_{k_j} \end{aligned} \tag{5.31}$$

where, by analogy with the undeformed case, we regard the sum over the half-integer g as the genus expansion of the time-dependent free energy.

Definition 5.1 Let ν be an integer partition. We define the genus-g Catalan polynomial associated to ν as

$$C_{g,\nu}(\beta) := F_{2g-2+2\ell(\nu),\nu}(\beta) \tag{5.32}$$

where $\ell(\nu)$ is the length of the partition and the polynomial in the r.h.s. is defined as in eq. (5.20). Moreover, from (5.24) it follows that

$$C_{g,\nu}(\beta^{-1}) = (-\beta)^{-2g} C_{g,\nu}(\beta) \tag{5.33}$$

and we can use this symmetry to write

$$C_{g,\nu}(\beta) = \sum_{i_1+i_2=2g} C_{g,\nu}^{(i_1,i_2)}(1)^{i_1}(-\beta)^{i_2} \tag{5.34}$$

with coefficients $C_{g,\nu}^{(i_1,i_2)}$ symmetric in i_1, i_2.

In the next section, we will argue that Catalan polynomials satisfy the two crucial properties:

- the coefficients of $C_{g,\nu}(\beta)$ are integers;
- the polynomials $C_{g,\nu}(\beta)$ evaluate to the generalized Catalan numbers of [15] at $\beta = 1$.

Observe that the summation label g in (5.31) can now be interpreted as the genus of some surface. Remarkably, the sum over g ranges over both integer and half-integer (positive) numbers, which suggests that the combinatorial interpretation of the time-dependent free energy is not as simple as in the undeformed case. Namely, the fact that the genus is allowed to take half-integer values is an indication that what the GβE is counting are not just orientable maps but more generally all locally orientable[3] maps as already recognized in works of Goulden and Jackson (see [23, 24]). We provide a more in-depth discussion of the combinatorial interpretation of our results in Appendix B.

In order to solve for the coefficients $C_{g,\nu}(\beta)$, we will now simplify the genus expansion of the time-dependent free energy as much as possible. To this end, we consider the change of time variables[4]

$$u_k = \beta N^2 (\beta\lambda)^{-\frac{k}{2}} v_k \tag{5.35}$$

[3] Observe that the genus of an orientable surface with Euler characteristic χ is defined as $g = \frac{1}{2}(2-\chi)$ where χ is always an even integer. For a non-orientable surface, however, the Euler number χ can be odd, therefore $\frac{1}{2}(2-\chi)$ can be half-integer.

[4] Under the inversion symmetry $\beta \mapsto 1/\beta$, the new times $\mathbf{v}$ transform trivially, as the combinations of parameters βN^2 and $\beta\lambda$ are both invariant.

which leads to the following expression

$$\begin{aligned}
&F(N,\beta,\lambda,\{u_k=\beta N^2(\beta\lambda)^{-\frac{k}{2}}v_k\})\\
&=\sum_{\ell=1}^{\infty}\sum_{g\in\frac{1}{2}\mathbb{N}}\frac{(\beta N^2)^{1-g}\beta^{-g}}{\ell!}\sum_{k_1,\dots,k_\ell=1}^{\infty}C_{g,[k_1,\dots,k_\ell]}(\beta)\prod_{j=1}^{\ell}v_{k_j}\\
&=\beta N^2\sum_{g\in\frac{1}{2}\mathbb{N}}(\beta N)^{-2g}\sum_{\nu\neq\emptyset}\frac{1}{|\mathrm{Aut}(\nu)|}C_{g,\nu}(\beta)\,p_\nu(\mathbf{v}),
\end{aligned}\tag{5.36}$$

where $p_\nu(\mathbf{v})=\prod_{k\in\nu}v_k$ are power-sum polynomials in times $\mathbf{v}$, and we used the combinatorial identity

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell!}\sum_{k_1,\dots,k_\ell=1}^{\infty}f_{[k_1,\dots,k_\ell]}=\sum_{\nu\neq\emptyset}\frac{1}{|\mathrm{Aut}(\nu)|}f_\nu\,,\tag{5.37}$$

with $|\mathrm{Aut}(\nu)|=p_\nu(\{\frac{\partial}{\partial v_k}\})p_\nu(\mathbf{v})$ being the order of the automorphism group of the partition ν.

Let us define the function

$$G(z,\beta,\mathbf{v}):=\beta z^2F((\beta z)^{-1},\beta,\lambda,\{u_k=(\beta z^2)^{-1}(\beta\lambda)^{-\frac{k}{2}}v_k\})\tag{5.38}$$

then we have that

$$G(z,\beta,\mathbf{v})=\sum_{r=0}^{\infty}z^rG_r(\beta,\mathbf{v})=\sum_{r=0}^{\infty}z^r\sum_{\nu\neq\emptyset}\frac{1}{|\mathrm{Aut}(\nu)|}C_{r/2,\nu}(\beta)\,p_\nu(\mathbf{v})\tag{5.39}$$

is the generating function of all Catalan polynomials, with z being the genus counting variable. The Virasoro constraints for the function $G(z,\beta,\mathbf{v})$ become

$$\begin{aligned}
&\Bigg(\frac{\partial}{\partial v_{n+2}}-(2+z(1-\beta)(n+1))\,\frac{\partial}{\partial v_n}\\
&-z^2\beta\sum_{n_1+n_2=n}\frac{\partial^2}{\partial v_{n_1}\partial v_{n_2}}-\sum_{k>0}kv_k\frac{\partial}{\partial v_{k+n}}\Bigg)G(z,\beta,\mathbf{v})\\
&-\sum_{n_1+n_2=n}\frac{\partial G(z,\beta,\mathbf{v})}{\partial v_{n_1}}\frac{\partial G(z,\beta,\mathbf{v})}{\partial v_{n_2}}=(1+z(1-\beta))\,\delta_{n,0}+v_1\delta_{n,-1},
\end{aligned}\tag{5.40}$$

and expanding in powers of z, we get

$$\Big(\frac{\partial}{\partial v_{n+2}} - 2\frac{\partial}{\partial v_n} - \sum_{k>0} k v_k \frac{\partial}{\partial v_{k+n}}\Big) G_r(\beta, \mathbf{v}) = (1-\beta)(n+1)\frac{\partial G_{r-1}(\beta, \mathbf{v})}{\partial v_n}$$
$$+\beta \sum_{n_1+n_2=n} \frac{\partial^2 G_{r-2}(\beta, \mathbf{v})}{\partial v_{n_1} \partial v_{n_2}} + \sum_{n_1+n_2=n} \sum_{r_1+r_2=r} \frac{\partial G_{r_1}(\beta, \mathbf{v})}{\partial v_{n_1}} \frac{\partial G_{r_2}(\beta, \mathbf{v})}{\partial v_{n_2}}$$
$$+(\delta_{n,0} + v_1 \delta_{n,-1})\delta_{r,0} + (1-\beta)\delta_{n,0}\delta_{r,1} \,. \tag{5.41}$$

Remark 5.2 Let us consider the case when $\beta = -\frac{\epsilon_2}{\epsilon_1}$. Then, we can write

$$\epsilon_1^{2g} C_{g,\nu}(-\epsilon_2/\epsilon_1) = \sum_{i_1+i_2=2g} \epsilon_1^{i_1} \epsilon_2^{i_2} C_{g,\nu}^{(i_1,i_2)} \tag{5.42}$$

which is a homogeneous degree-$2g$ symmetric polynomial in ϵ_1, ϵ_2, due to the property (5.33). Since any such symmetric polynomial can be expressed as an integer linear combination of Schur polynomials $s_\lambda(\epsilon_1, \epsilon_2)$ for $|\lambda| = 2g$ (see appendix A), we can rewrite (5.42) as

$$\epsilon_1^{2g} C_{g,\nu}(-\epsilon_2/\epsilon_1) = \sum_{\lambda \vdash 2g} n_{\nu,\lambda} s_\lambda(\epsilon_1, \epsilon_2) \,, \tag{5.43}$$

where $n_{\nu,\lambda} \in \mathbb{Z}$. Then we can define the generating function of all $n_{\nu,\lambda}$ as

$$n(\epsilon_1, \epsilon_2, \mathbf{v}) := G(\epsilon_1, -\epsilon_2/\epsilon_1, \mathbf{v}) \tag{5.44}$$

or, equivalently, we can write $G(z, \beta, \mathbf{v}) = n(z, -\beta z, \mathbf{v})$. The constraints for the function $n(\epsilon_1, \epsilon_2, \mathbf{v})$ are

$$\Big(\frac{\partial}{\partial v_{n+2}} - (2 + (\epsilon_1+\epsilon_2)(n+1))\frac{\partial}{\partial v_n} + (\epsilon_1\epsilon_2) \sum_{n_1+n_2=n} \frac{\partial^2}{\partial v_{n_1} \partial v_{n_2}}$$
$$- \sum_{k>0} k v_k \frac{\partial}{\partial v_{k+n}}\Big) n(\epsilon_1, \epsilon_2, \mathbf{v})$$
$$- \sum_{n_1+n_2=n} \frac{\partial n(\epsilon_1, \epsilon_2, \mathbf{v})}{\partial v_{n_1}} \frac{\partial n(\epsilon_1, \epsilon_2, \mathbf{v})}{\partial v_{n_2}} = (1 + (\epsilon_1+\epsilon_2))\,\delta_{n,0} + v_1 \delta_{n,-1} \tag{5.45}$$

Observe that Schur functions of two variables vanish identically for $\ell(\lambda) > 2$; therefore, the sum over λ collapses to a sum over partitions of the form $\lambda = [h+d, d]$

for $h, d \geq 0$. We then have

$$\begin{aligned} n(\epsilon_1, \epsilon_2, \mathbf{v}) &= \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} \sum_{r=0}^{\infty} \sum_{d=0}^{\lfloor r/2 \rfloor} n_{\nu,[r-d,d]} s_{[r-d,d]}(\epsilon_1, \epsilon_2) \\ &= \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} \sum_{h=0}^{\infty} \sum_{d=0}^{\infty} n_{\nu,[h+d,d]} s_{[h+d,d]}(\epsilon_1, \epsilon_2) \\ &= \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} \sum_{i,j,d \geq 0} n_{\nu,[i+j+d,d]} \epsilon_1^{d+i} \epsilon_2^{d+j} \\ &= \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} \sum_{a,b \geq 0} C^{(a,b)}_{\frac{a+b}{2},\nu} \epsilon_1^a \epsilon_2^b \end{aligned} \tag{5.46}$$

where we used the combinatorial identity

$$\sum_{r=0}^{\infty} \sum_{d=0}^{\lfloor r/2 \rfloor} f(r, d) = \sum_{h=0}^{\infty} \sum_{d=0}^{\infty} f(h + 2d, d) \tag{5.47}$$

together with (A.4). The genus in the formula for $n(\epsilon_1, \epsilon_2, \mathbf{v})$ is given by half of the monomial degree in the ϵ_i's, i.e., the eigenvalue of the operator $\frac{1}{2}(\epsilon_1 \frac{\partial}{\partial \epsilon_1} + \epsilon_2 \frac{\partial}{\partial \epsilon_2})$. We have the relation of coefficients

$$C^{(i,2g-i)}_{g,\nu} = \sum_{d=0}^{\min(i,2g-i)} n_{\nu,[2g-d,d]} \tag{5.48}$$

which suggests that the integers $n_{\nu,\lambda}$ might have a more fundamental role than the coefficients of the Catalan polynomials $C_{g,\nu}(\beta)$. Moreover, explicit computations of some of the $n_{\nu,\lambda}$ indicate that these coefficients are always positive integers, see Table 2. While this is strong evidence that these numbers are the solution to some counting problem, we are not aware of possible combinatorial or enumerative geometric interpretations of the integers $n_{\nu,\lambda}$. It would be interesting to investigate this further.

6 Recursion formula

Having fixed the general form of the series expansion of the time-dependent free energy, and the closely related function $G(z, \beta, \mathbf{v})$, we now need to fix the coefficients of $C_{g,[k_1,\ldots,k_\ell]}(\beta)$. To this end, we make use again of the Virasoro constraints as written in (5.41). By plugging the formula for the generating function $G(z, \beta, \mathbf{v})$ into the constraint equations, we get non-linear relations between the coefficients. Provided we fix some appropriate initial conditions, these relations determine a recursion on the Catalan polynomials, which can be solved uniquely. This recursion formula is completely equivalent to the Virasoro constraints themselves, and it is in fact also interpreted as a topological recursion formula for the genus expansion of the time-dependent free energy. Similar topological recursion formulas for the GβE were already considered in [21].

6.1 Initial conditions

We consider first the constraint equations following from the Virasoro constraints for $n = -1$ and $n = 0$ as they allow to define the initial conditions for the recursion.

For $n = -1$ we have the differential equation

$$\boxed{n=-1} \qquad \Big(\frac{\partial}{\partial v_1} - \sum_{k>0} k v_k \frac{\partial}{\partial v_{k-1}}\Big) G_r(\beta, \mathbf{v}) = v_1 \delta_{r,0} \tag{6.1}$$

which translates to the following recursion relation for the Catalan polynomials

$$C_{g,[k_1,\ldots,k_\ell,1]}(\beta) = \sum_{j=1}^{\ell} k_j C_{g,[k_1,\ldots,k_{j-1},k_j-1,k_{j+1},\ldots,k_\ell]}(\beta) + \delta_{[k_1,\ldots,k_\ell],[1]}\delta_{g,0} \tag{6.2}$$

Similarly, for $n = 0$ we have the differential equation

$$\boxed{n=0} \qquad \Big(\frac{\partial}{\partial v_2} - \sum_{k>0} k v_k \frac{\partial}{\partial v_k}\Big) G_r(\beta, \mathbf{v}) = \delta_{r,0} + (1-\beta)\delta_{r,1} \tag{6.3}$$

which translates to

$$C_{g,[k_1,\ldots,k_\ell,2]}(\beta) = \sum_{j=1}^{\ell} k_j C_{g,[k_1,\ldots,k_\ell]}(\beta) + \left(\delta_{g,0} + (1-\beta)\delta_{g,\frac{1}{2}}\right)\delta_{[k_1,\ldots,k_\ell],[]}\,. \tag{6.4}$$

Observe that both equations only contain contributions $G_r(\beta, \mathbf{v})$ for a fixed r, which means that these relations are independent of all other genera (differently from what happens for $n \geq 1$).

6.2 Recursion for $n \geq 1$

In general, we can write the recursion relation for an arbitrary polynomial $C_{g,[k_1,\ldots,k_\ell]}(\beta)$ by expanding (5.41) in all possible monomials in times $\mathbf{v}$. For $n \geq 1$, the equation corresponding to the monomial $p_{[k_1,\ldots,k_\ell]}(\mathbf{v}) = v_{k_1}\cdots v_{k_\ell}$ gives the recursion

$$C_{g,[k_1,\ldots,k_\ell,n+2]}(\beta) = 2C_{g,[k_1,\ldots,k_\ell,n]}(\beta) + \sum_{j=1}^{\ell} k_j C_{g,[k_1,\ldots,k_{j-1},k_j+n,k_{j+1},\ldots,k_\ell]}(\beta)$$
$$+(1-\beta)(n+1)C_{g-\frac{1}{2},[k_1,\ldots,k_\ell,n]}(\beta) + \beta \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} C_{g-1,[k_1,\ldots,k_\ell,n_1,n_2]}(\beta)$$
$$+ \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} \sum_{\substack{I_1\cup I_2=[k_1,\ldots,k_\ell] \\ I_1\cap I_2=\emptyset}} C_{g_1,I_1\cup[n_1]}(\beta)\, C_{g_2,I_2\cup[n_2]}(\beta) \tag{6.5}$$

where it is understood that the polynomials $C_{g,[k_1,\ldots,k_\ell]}(\beta)$ are identically zero whenever one or more of the labels k_j are ≤ 0. This recursion relation is sometimes also known as *cut-and-join recursion*, and it corresponds to the β-deformed version of the recursion in [15, (6)]. Observe that symmetry of the equation (6.5) under $\beta \mapsto 1/\beta$ is a straightforward consequence of (5.33).

Expanding further in powers of β, we get

$$C^{(i,2g-i)}_{g,[k_1,\ldots,k_\ell,n+2]} = 2C^{(i,2g-i)}_{g,[k_1,\ldots,k_\ell,n]} + \sum_{j=1}^{\ell} k_j C^{(i,2g-i)}_{g,[k_1,\ldots,k_{j-1},k_j+n,k_{j+1},\ldots,k_\ell]}$$
$$+(n+1)\left(C^{(i-1,2g-i)}_{g-\frac{1}{2},[k_1,\ldots,k_\ell,n]} + C^{(i,2g-i-1)}_{g-\frac{1}{2},[k_1,\ldots,k_\ell,n]}\right) - \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} C^{(i-1,2g-i-1)}_{g-1,[k_1,\ldots,k_\ell,n_1,n_2]}$$
$$+ \sum_{\substack{i_1+i_2=i \\ i_1,i_2\geq 0}} \sum_{\substack{n_1+n_2=n \\ n_1,n_2\geq 1}} \sum_{\substack{g_1+g_2=g \\ g_1,g_2\geq 0}} \sum_{\substack{I_1\cup I_2=[k_1,\ldots,k_\ell] \\ I_1\cap I_2=\emptyset}} C^{(i_1,2g_1-i_1)}_{g_1,I_1\cup[n_1]}\, C^{(i_2,2g_2-i_2)}_{g_2,I_2\cup[n_2]}$$
$$+\left(\delta_{g,0} + (\delta_{i,1} + \delta_{2g-i,1})\delta_{g,\frac{1}{2}}\right)\delta_{[k_1,\ldots,k_\ell],[]}\delta_{n,0} + \delta_{[k_1,\ldots,k_\ell],[1]}\delta_{g,0}\delta_{n,-1}$$

One can check explicitly that this recursion admits a unique solution and, moreover, that the coefficients $C^{(i,2g-i)}_{g,\nu}$ that solve the recursion are integer numbers. Solving the equations for the first few Catalan polynomials gives Table 1.

7 The undeformed limit

In the limit $\beta \to 1$, the polynomials $C_{g,[k_1,\ldots,k_\ell]}(\beta)$ reduce to alternating sums of coefficients $C^{(a,b)}_{g,[k_1,\ldots,k_\ell]}$ as follows

$$C_{g,[k_1,\ldots,k_\ell]}(1) = \sum_{i=0}^{2g} (-1)^i C^{(i,2g-i)}_{g,[k_1,\ldots,k_\ell]} =: C^{\mathrm{CLPS}}_{g,[k_1,\ldots,k_\ell]} \tag{7.1}$$

where $C^{\mathrm{CLPS}}_{g,[k_1,\ldots,k_\ell]}$ are the coefficients of [16].

Observe also that, in the limit $\beta \to 1$ (i.e., $\epsilon_1 + \epsilon_2 = 0$), the Schur polynomials $s_\lambda(\epsilon_1, \epsilon_2)$ evaluate to either 0 or ± 1 (see A.4):

$$\lim_{\beta \to 1} s_{[h+d,d]}(z, -\beta z) = z^{h+2d}(-1)^d \frac{1+(-1)^h}{2} \tag{7.2}$$

This implies that

$$\lim_{\beta\to 1} G(z,\beta,\mathbf{v}) = \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} \sum_{h=0}^{\infty}\sum_{d=0}^{\infty} z^{2(h+d)}(-1)^d n_{\nu,[2h+d,d]} \tag{7.3}$$

which means that the genus expansion of $G(z, 1, \mathbf{v})$ only contains integer genera contributions.

Similarly, the time-dependent free energy simplifies to

$$\lim_{\beta\to 1} F(N,\beta,\lambda,\mathbf{u}) = \sum_{\ell=1}^{\infty}\sum_{g\in\mathbb{N}} \frac{1}{\ell!} N^{2-2g-2\ell} \sum_{k_1,\ldots,k_\ell=1}^{\infty} \lambda^{\sum_{j=1}^{\ell} k_j/2} C^{\mathrm{CLPS}}_{g,[k_1,\ldots,k_\ell]} \prod_{j=1}^{\ell} u_{k_j} \tag{7.4}$$

so that

$$C^{\mathrm{CLPS}}_{g,\nu} = \sum_{d=0}^{g} (-1)^d n_{\nu,[2g-d,d]}\,. \tag{7.5}$$

Observe that from Lemma 5.1 we have that the sum in (7.1) vanishes identically when $2g$ is odd. The sum over the genus g then collapses to a sum over only positive integers, and we recover the known genus expansion of the time-dependent free energy of the GUE as described in [15].

8 Wishart–Laguerre β-ensemble

There is another example of β-deformed ensemble that has the property that Virasoro constraints admit a unique solution. This is the Wishart–Laguerre β-ensemble which corresponds to the deformation of the matrix model with linear potential $V(x) = \frac{N}{\lambda}x$. Differently from the case of the Gaussian matrix model, one can additionally introduce a logarithmic term in the potential without spoiling the solvability of Virasoro constraints [11]. We will denote as α the coupling corresponding to such logarithmic interaction. After exponentiation of the potential, this gives rise to a determinant insertion of the form $\prod_{i=1}^{N} x_i^\alpha$. The generating function of all polynomial expectation values can then be defined as

$$\frac{1}{N!}\int_{\mathbb{R}_+^N} \prod_{i=1}^{N} \mathrm{d}x_i \prod_{i<j} |x_i - x_j|^{2\beta} \prod_{i=1}^{N} x_i^\alpha \, \mathrm{e}^{-\frac{N}{\lambda}\sum_i x_i + \frac{1}{N}\sum_{k=1}^{\infty} u_k \sum_i x_i^k} \tag{8.1}$$

Convergence of the integral imposes that the contour be restricted to the positive orthant in real N-dimensional space, together with the conditions $\Re(\alpha) > -1$ and $\Re(\lambda^{-1}) > 0$.

8.1 Symmetries

The time-dependent free energy of the WLβE satisfies the following symmetry properties. First, both the normalized generating function and its logarithm are invariant under the involution

$$\beta \mapsto \tfrac{1}{\beta}\ ,\ N \mapsto -\beta N\ ,\ \lambda \mapsto \beta^2\lambda\ ,\ \alpha \mapsto -\tfrac{1}{\beta}\alpha \tag{8.2}$$

For later convenience, we introduce a new parameter ϕ defined by the equation

$$\alpha = (1-\beta)(\phi - 1) \tag{8.3}$$

which implies $\phi = 1 + \frac{\alpha}{1-\beta}$. Observe that under the symmetry (8.2), the new parameter ϕ is trivially mapped to itself. We then have

$$F(-\beta N, \beta^{-1}, \beta^2\lambda, \phi, \mathbf{u}) = F(N, \beta, \lambda, \phi, \mathbf{u}) \tag{8.4}$$

Second, we observe the homogeneity equation

$$F(N, \beta, \lambda, \phi, \mathbf{u}) = \lambda^{D_{\mathbf{u}}} F(N, \beta, 1, \phi, \mathbf{u}) \tag{8.5}$$

which follows from the change of variables $x_i = \lambda y_i$ in the integral (8.1). This is analogous to the equation (2.1) in the Gaussian case, with the caveat that the homogeneity degree of the time-dependent free energy is different in the two cases.

8.2 Superintegrability

Similarly to the Gaussian case, the WLβE is a matrix model that exhibits superintegrability for the averages of characters; namely, it can be shown [11, 14, 28] that the ensemble average of Jack polynomials satisfies

$$\begin{aligned} &\frac{\langle \mathrm{JackP}_\mu(x_1, \ldots, x_N)\rangle_{\mathrm{WL}\beta\mathrm{E}}}{\langle 1\rangle_{\mathrm{WL}\beta\mathrm{E}}} \\ &\quad = \frac{\mathrm{JackP}_\mu(p_k = N)\mathrm{JackP}_\mu(p_k = (\frac{\beta\lambda}{N})^k(N + \beta^{-1}\phi(1-\beta)))}{\mathrm{JackP}_\mu(p_k = \delta_{k,1})} \end{aligned} \tag{8.6}$$

This formula then can be used to define the generating function of the matrix model

$$\frac{Z(N,\beta,\lambda,\phi,\mathbf{u})}{Z(N,\beta,\lambda,\phi,0)} = \sum_{\mu} \frac{\mathrm{JackP}_{\mu}(p_k = N)\mathrm{JackP}_{\mu}(p_k = N + \beta^{-1}\phi(1-\beta))}{\mathrm{JackP}_{\mu}(p_k = \delta_{k,1})}$$
$$\mathrm{JackQ}_{\mu}\Big(p_k = (\tfrac{\beta\lambda}{N})^k \frac{k u_k}{\beta N}\Big) \tag{8.7}$$

and can be used to argue for the polynomiality of the time-dependent free energy in the parameters β and ϕ.

8.3 Virasoro constraints

By analogy with the GβE, we are led to consider the change of time variables

$$u_k = \beta N^2 (\beta\lambda)^{-k} v_k \tag{8.8}$$

where the power of $(\beta\lambda)$ has no factors of $\frac{1}{2}$ due to the different dependence on the coupling λ as observed in (8.5).

A similar analysis to that of Sect. 5 then suggests that one should consider the function

$$G(z,\beta,\phi,\mathbf{v}) := \beta z^2 F\Big((\beta z)^{-1}, \beta, \lambda, \phi, \{u_k = (\beta z^2)^{-1}(\beta\lambda)^{-k} v_k\}\Big) \tag{8.9}$$

for which the Virasoro constraints [5] become

$$\Big(\frac{\partial}{\partial v_{n+1}} - (2 + z(1-\beta)(n+\phi))\frac{\partial}{\partial v_n} - z^2\beta \sum_{n_1+n_2=n} \frac{\partial^2}{\partial v_{n_1}\partial v_{n_2}}$$
$$- \sum_{k=1}^{\infty} k v_k \frac{\partial}{\partial v_{k+n}}\Big) G(z,\beta,\phi,\mathbf{v})$$
$$- \sum_{n_1+n_2=n} \frac{\partial G(z,\beta,\phi,\mathbf{v})}{\partial v_{n_1}} \frac{\partial G(z,\beta,\phi,\mathbf{v})}{\partial v_{n_2}} = (1 + z(1-\beta)\phi)\delta_{n,0} \tag{8.10}$$

with $n \geq 0$.

[5] See [11] for a derivation of the constraints for the generating function $Z(N,\beta,\lambda,\phi,\mathbf{u})$ of the WLβE.

The function $G(z, \beta, \phi, \mathbf{v})$ then admits the following genus expansion in z,

$$\begin{aligned} G(z, \beta, \phi, \mathbf{v}) &= \sum_{r=0}^{\infty} z^r G_r(\beta, \phi, \mathbf{v}) \\ &= \sum_{r=0}^{\infty} z^r \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} C_{r/2,\nu}(\beta, \phi) \\ &= \sum_{\nu \neq \emptyset} \frac{p_\nu(\mathbf{v})}{|\mathrm{Aut}(\nu)|} \sum_{\lambda} z^{|\lambda|} n_{\nu,\lambda}(\phi) s_\lambda(1, -\beta) \end{aligned} \tag{8.11}$$

where $C_{g,\nu}(\beta, \phi)$ are polynomial functions in β and ϕ,

$$C_{g,\nu}(\beta, \phi) \equiv \sum_{i,j=0}^{2g} (-\beta)^i \phi^j C_{g,\nu,j}^{(i,2g-i)} \tag{8.12}$$

with coefficients $C_{g,\nu,j}^{(i,2g-i)}$ to be fixed by the constraints. We shall call $C_{g,\nu}(\beta, \phi)$ the *Catalan polynomials* in genus g. The symmetry in (8.2) imposes that $C_{g,\nu}(\beta, \phi)$ be palindromic polynomials in $(-\beta)$; however, there does not seem to be such a symmetry w.r.t. the variable ϕ.

The equations in (8.10) then can be expanded in powers of z and monomials in times $\mathbf{v}$, to obtain a recursion on the Catalan polynomials $C_{g,\nu}(\beta, \phi)$. Namely, we have

$$\begin{aligned} C_{g,[k_1,\ldots,k_\ell,n+1]}(\beta, \phi) &= 2C_{g,[k_1,\ldots,k_\ell,n]}(\beta, \phi) + \sum_{j=1}^{\ell} k_j C_{g,[k_1,\ldots,k_{j-1},k_j+n,k_{j+1},\ldots,k_\ell]}(\beta, \phi) \\ &+ (1-\beta)(n+\phi) C_{g-\frac{1}{2},[k_1,\ldots,k_\ell,n]}(\beta, \phi) + \beta \sum_{\substack{n_1+n_2=n \\ n_1,n_2 \geq 1}} C_{g-1,[k_1,\ldots,k_\ell,n_1,n_2]}(\beta, \phi) \\ &+ \sum_{\substack{n_1+n_2=n \\ n_1,n_2 \geq 1}} \sum_{\substack{g_1+g_2=g \\ g_1,g_2 \geq 0}} \sum_{\substack{I_1 \cup I_2 = [k_1,\ldots,k_\ell] \\ I_1 \cap I_2 = \emptyset}} C_{g_1, I_1 \cup [n_1]}(\beta, \phi)\, C_{g_2, I_2 \cup [n_2]}(\beta, \phi) \\ &+ \left(\delta_{g,0} + (1-\beta)\phi\, \delta_{g,\frac{1}{2}} \right) \delta_{[k_1,\ldots,k_\ell],[]} \delta_{n,0} \end{aligned} \tag{8.13}$$

for $n \geq 0$. We recall that the $n = -1$ constraint is not well-defined in the WLβE case; however, the recursion for non-negative n can still be solved uniquely.

These recursion relations imply that the coefficients of $C_{g,[k_1,\ldots,k_\ell]}(\beta, \phi)$ are integer numbers. Equivalently, we have that $n_{\nu,\lambda}(\phi)$ are polynomials in ϕ of degree $2g = |\lambda|$ and coefficients in $\mathbb{Z}$. Explicit computations of these integer numbers by means of the recursion suggest that they are always positive; however, we do not have a proof of this claim. See Tables 4 and 5 for lists of some of these coefficients for arbitrary ϕ and for $\phi = 1$, respectively.

9 Outlook

In this article we have analyzed the dependence of the free energy (logarithm of the generating function of correlators) of the GβE and WLβE on the parameters of the matrix model, most importantly the rank N and the deformation parameter β.

We exploited Virasoro constraints for the (time-dependent) free energy to derive an ansatz for the genus expansion (5.31) with coefficients defined as β-deformations of generalized higher genus Catalan numbers. The ansatz is motivated both by symmetry arguments and by showing that it leads to a (topological) recursion relation on the coefficients, whose solution is unique.

Upon suitable change of time variables, we have been able to also define the function $G(z, \beta, \mathbf{v})$ which naturally admits a genus expansion in the variable z, with coefficients given by the higher genus Catalan polynomials $C_{g,\nu}(\beta)$. This led us to reformulate the genus expansion in terms of more fundamental integer invariants $n_{\nu,\lambda}$ and their generating function $n(\epsilon_1, \epsilon_2, \mathbf{v})$, which is manifestly symmetric in the exchange of ϵ_1 and ϵ_2, equivalent to the involution sending β to $1/\beta$. Quite remarkably, all the invariants $n_{\nu,\lambda}$ that we have been able to compute turn out to be positive integers, strongly suggesting that they should have a specific combinatorial interpretation.

As a last comment, we observe that our results hold for arbitrary values of the deformation parameter β, and that, by specializing to the values $\beta = \frac{1}{2}$, 1 and 2 we immediately obtain the case of the orthogonal, unitary and symplectic ensembles, respectively. This implies that the $C_{g,\nu}(\beta)$ provide an analytic continuation of higher genus Catalan numbers for all those classical matrix ensembles.

A number of question arise naturally from our discussion.

- While integrality of the $n_{\nu,\lambda}$ follows from the cut-and-join recursion relations, their positivity is not immediately obvious; however, Tables 2 and 5 give strong evidence for this claim. A possible explanation of this fact could follow from some combinatorial interpretation of the positive integers $n_{\nu,\lambda}$ as dimensions of certain vector spaces. Comparison with the results of [23, 24, 29] would suggest that they should be related to counts of locally orientable maps as discussed in appendix B; however, the precise details of the identification remain unclear. Another possible connection is with the theory of monotone Hurwitz numbers, as in recent works of [30, 31].
- Can one use similar techniques to derive a genus expansion for the time-dependent free energy of q, t-deformed matrix models as those considered in [11, 32, 33]? What kind of topological recursion relation can one obtain from q-Virasoro constraints? What is the combinatorial meaning of the coefficients of the corresponding genus expansion?
- It would appear from our analysis that one could come up with an ansatz for the time-dependent free energy of any β-deformed matrix model with polynomial potential $V(x) = \frac{N}{m\lambda}x^m$. The natural generalization of the ansatz for the genus expansion would be

$$F(N,\beta,\lambda,\mathbf{u}) = \sum_{\ell=1}^{\infty} \sum_{g\in\frac{1}{2}\mathbb{N}} \frac{N^{2-2g-2\ell}\beta^{1-\ell-2g}}{\ell!} \sum_{k_1,\dots,k_\ell=1}^{\infty} (\beta\lambda)^{\sum_{j=1}^{\ell}\frac{k_j}{m}} \\ \times \sum_{i_1+i_2=2g} (-\beta)^{i_1} C^{(i_1,i_2)}_{g,[k_1,\dots,k_\ell]} \prod_{j=1}^{\ell} u_{k_j} \tag{9.1}$$

for some appropriate coefficients $C^{(i_1,i_2)}_{g,[k_1,\dots,k_\ell]}$. However, it is known from [10, 11, 20] for example, that Virasoro constraints for $m \geq 3$ do not admit a unique solution. For this reason, the polynomials $C_{g,[k_1,\dots,k_\ell]}(\beta)$ are not well-defined in this case, and we cannot guarantee that the ansatz (9.1) gives the correct genus expansion of the time-dependent free energy. It would be interesting to study this case further, as it seems to require different techniques from those used in this article.

- In the case of the GβE, we are able to give an explicit identification between Catalan polynomials and marginal b-polynomials as discussed in appendix B. It is not known to us whether marginal b-polynomials can be defined also in the case of the WLβE or not. Nevertheless, an analogue of the rooted map series (B.5) can be defined as

$$M(N,b,\phi,\mathbf{y}) := H(\{x_k = N\},\mathbf{y},\{z_k = N+b\phi\};b) \\ = \sum_{\mu,\nu,\lambda} c_{\mu,\nu,\lambda}(b) N^{\ell(\mu)} (N+b\phi)^{\ell(\lambda)} p_\nu(\mathbf{y}) \tag{9.2}$$

by using superintegrability (8.7). The obvious question now is: what is the function $M(N,b,\phi,\mathbf{y})$ counting? Moreover, what is the combinatorial meaning of the parameter ϕ? The answer might be related to a β-deformed version of the analysis in [34, 35] which relates moments of the WLβE at $\beta = 1$ and double monotone Hurwitz numbers.
- Superintegrability of averages of Jack polynomials gives a closed formula for the character expansion of the generating function $Z(N,\beta,\lambda,\mathbf{u})$. Is there an equivalent reformulation of superintegrability for the function $F(N,\beta,\lambda,\mathbf{u})$? The existence of such a formula would give a closed form expression for all Catalan polynomials $C_{g,\nu}(\beta)$ and integer invariants $n_{\nu,\lambda}$. A possible way to relate superintegrability to the topological expansion could be via the W-representation of the matrix model as discussed in [36].

We leave the investigation of these and other questions to future research.

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Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

A Schur polynomials in two variables

We recall some properties of Schur symmetric polynomials in two variables. See [37] for a thorough discussion on symmetric functions.

Schur polynomials constitute a linear basis of the space of symmetric functions in n variables. They are labeled by integer partitions λ, which correspond to irreducible representations of gl_n. Whenever the length of the partition λ is greater than the number of variables n, the corresponding Schur polynomial s_λ vanishes identically. In the case $n = 2$, the only nonzero Schur polynomials are those associated to partitions of length at most two. Any such partition can be represented by an ordered set of two nonnegative integers, $[\lambda_1, \lambda_2]$ with $\lambda_1 \geq \lambda_2 \geq 0$. We parametrize such partitions as $\lambda = [h + d, d]$ for $h, d \in \mathbb{N}$.

Let ϵ_1, ϵ_2 be the two arguments of the Schur polynomial. Then one of the Pieri rule can be stated as

$$s_{[h+d,d]}(\epsilon_1, \epsilon_2) = s_{[1,1]}(\epsilon_1, \epsilon_2)^d s_{[h]}(\epsilon_1, \epsilon_2) \tag{A.1}$$

where

$$s_{[1,1]}(\epsilon_1, \epsilon_2) = \epsilon_1\epsilon_2 \tag{A.2}$$

This implies that knowing $s_{[h]}(\epsilon_1, \epsilon_2)$ for any h is enough to determine uniquely all other Schur polynomials. Using the coproduct rule

$$s_\lambda(\epsilon_1, \epsilon_2) = \sum_{\nu\subset\lambda} s_{\lambda/\nu}(\epsilon_1)s_\nu(\epsilon_2), \tag{A.3}$$

we obtain the closed formula

$$s_{[h+d,d]}(\epsilon_1, \epsilon_2) = \sum_{i+j=h} \epsilon_1^{d+i}\epsilon_2^{d+j} = (\epsilon_1\epsilon_2)^d \frac{\epsilon_1^{h+1} - \epsilon_2^{h+1}}{\epsilon_1 - \epsilon_2} \tag{A.4}$$

As a corollary, we observe that

$$\frac{s_{[2h+1+d,d]}(\epsilon_1, \epsilon_2)}{s_{[1]}(\epsilon_1, \epsilon_2)} = (\epsilon_1\epsilon_2)^d \frac{\epsilon_1^{2(h+1)} - \epsilon_2^{2(h+1)}}{\epsilon_1^2 - \epsilon_2^2} \tag{A.5}$$

where the r.h.s. is a symmetric polynomial in ϵ_1, ϵ_2 for all values of $d, h \in \mathbb{N}$. In other words, $s_{[1]}(\epsilon_1, \epsilon_2)$ always divides $s_{[2h+1+d,d]}(\epsilon_1, \epsilon_2)$. This implies that if $|\lambda|$ is odd and $\epsilon_1 + \epsilon_2 = 0$, then $s_\lambda(\epsilon_1, \epsilon_2) = 0$.

B Hypermap counts and the *b*-conjecture

In this section, we review some known results regarding counting hypermaps and the so-called b-conjecture [23, 29]. Our reference on the subject will be [24]. After introducing the rooted hypermap series H, rooted map series M and the marginal b-polynomials $d_{\ell,\nu}(b)$, we will give a formula to relate the polynomials $d_{\ell,\nu}(b)$ to the Catalan polynomials $C_{g,\nu}(\beta)$ that we have defined in Sect. 5.

Throughout this section, we adopt the notations of [24], with the understanding that the parameters b and β are related by the equation

$$1 + b = \beta^{-1} \tag{B.1}$$

Let us introduce the series

$$\Phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; \beta) := \sum_\lambda \frac{\mathrm{JackP}_\lambda(\mathbf{x})\mathrm{JackQ}_\lambda(\mathbf{y})\mathrm{JackP}_\lambda(\mathbf{z})}{\mathrm{JackP}_\lambda(p_k = \delta_{k,1})} \tag{B.2}$$

for three independent sets of times $\mathbf{x} = \{x_1, x_2, \dots\}$, $\mathbf{y} = \{y_1, y_2, \dots\}$ and $\mathbf{z} = \{z_1, z_2, \dots\}$.

Definition B.1 The rooted Hypermap series is defined as

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}; b) := (1 + b) D_{\mathbf{y}} \log \Phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; \tfrac{1}{1+b}) \tag{B.3}$$

with $D_{\mathbf{y}} = \sum_{k>0} k y_k \frac{\partial}{\partial y_k}$ being the degree operator in the $\mathbf{y}$ times.

Then we can express $H(\mathbf{x}, \mathbf{y}, \mathbf{z}; b)$ in the power-sum basis

$$H(\mathbf{x}, \mathbf{y}, \mathbf{z}; b) = \sum_{\substack{\mu,\nu,\lambda \\ |\mu|=|\nu|=|\lambda|}} c_{\mu,\nu,\lambda}(b)\, p_\mu(\mathbf{x})\, p_\nu(\mathbf{y})\, p_\lambda(\mathbf{z}) \tag{B.4}$$

with coefficients $c_{\mu,\nu,\lambda}(b)$ being the b-polynomials implicitly defined by this expansion. Here, $p_\mu(\mathbf{x}) = \prod_{k\in\mu} x_k$ and similarly for $p_\nu(\mathbf{y})$, $p_\lambda(\mathbf{z})$.

Definition B.2 The rooted map series $M(N, b, \mathbf{y})$ is defined as a specialization of $H(\mathbf{x}, \mathbf{y}, \mathbf{z}; b)$,

$$\begin{aligned} M(N, b, \mathbf{y}) :=& H(\{x_k = N\}, \mathbf{y}, \{z_k = \delta_{k,2}\}; b) \\ =& \sum_{n=0}^{\infty} \sum_{\mu,\nu \vdash 2n} c_{\mu,\nu,[2^n]}(b) N^{\ell(\mu)} p_\nu(\mathbf{y}) \\ =& \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} N^\ell \sum_{\nu\vdash 2n} d_{\ell,\nu}(b) p_\nu(\mathbf{y}) \end{aligned} \tag{B.5}$$

where the coefficients

$$d_{\ell,\nu}(b) = \sum_{\substack{\mu \vdash 2n \\ \ell(\mu)=\ell}} c_{\mu,\nu,[2^n]}(b) \tag{B.6}$$

are the *marginal b-polynomials*.

The marginal b-conjecture (proven in [24, Corollary 4.17]) states the following.

Theorem B.1 (La Croix) *The $d_{\ell,\nu}(b)$ are polynomials with non-negative integer coefficients. Moreover, the coefficient of b^k in $d_{\ell,\nu}(b)$ is the number of rooted maps with ℓ vertices, face-degree partition ν, and a total of k twisted edges.*

We now claim that marginal polynomials $d_{\ell,\nu}(b)$ and Catalan polynomials $C_{g,\nu}(\beta)$ are related (up to overall combinatorial factors) by an appropriate identification of labels and variables as in (B.1). In order to show this, we first identify the corresponding generating functions. From (5.3), we can identify the time-dependent free energy of the GβE with a specialization of the logarithm of $\Phi(\mathbf{x}, \mathbf{y}, \mathbf{z}; \beta)$,

$$F(N, \beta, \lambda, \mathbf{u}) = \log \Phi\Big(\{x_k = N\}, \Big\{y_k = (\tfrac{\beta\lambda}{N})^{\frac{k}{2}} \frac{k u_k}{\beta N}\Big\}, \{z_k = \delta_{k,2}\}; \beta\Big) \tag{B.7}$$

which, together with (5.38), implies

$$D_{\mathbf{v}} G(z, \beta, \mathbf{v}) = (\beta z)^2 M\Big((\beta z)^{-1}, \beta^{-1} - 1, \{y_k = (\beta z)^{\frac{k}{2}-1} k v_k\}\Big) \tag{B.8}$$

Expanding both sides as power series as in (5.39) and (B.5), we can identify the coefficients to finally obtain the relation

$$C_{g,\nu}(\beta) = |\mathrm{Aut}(\nu)| \, \frac{\prod_{k\in\nu} k}{|\nu|} \beta^{2g} \, d_{2-2g-\ell(\nu)+\frac{|\nu|}{2},\nu}(\beta^{-1} - 1)\,. \tag{B.9}$$

This identification of polynomials is evidence of an underlying relation between the invariants $n_{\nu,\lambda}$ with the counting of locally orientable maps; however, the precise nature of this relation is unclear to us.

C Tables of Catalan polynomials

C.1 The GβE case

See Tables 1 and 2

Table 1 Polynomials $C_{g,\nu}(\beta)$ for the GβE

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[1, 1]	1	0	0	0	0
[2]	1	$1-\beta$	0	0	0
[1, 1, 1, 1]	0	0	0	0	0
[2, 1, 1]	2	0	0	0	0
[2, 2]	2	$2-2\beta$	0	0	0
[3, 1]	3	$3-3\beta$	0	0	0
[4]	2	$5-5\beta$	$3\beta^2-5\beta+3$	0	0
[1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 1, 1]	8	0	0	0	0
[2, 2, 2]	8	$8-8\beta$	0	0	0
[3, 1, 1, 1]	6	0	0	0	0
[3, 2, 1]	12	$12-12\beta$	0	0	0
[3, 3]	12	$27-27\beta$	$15\beta^2-27\beta+15$	0	0
[4, 1, 1]	12	$12-12\beta$	0	0	0
[4, 2]	8	$20-20\beta$	$12\beta^2-20\beta+12$	0	0
[5, 1]	10	$25-25\beta$	$15\beta^2-25\beta+15$	0	0
[6]	5	$22-22\beta$	$32\beta^2-54\beta+32$	$-15\beta^3+32\beta^2-32\beta+15$	0
[1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 2, 1, 1]	48	0	0	0	0
[2, 2, 2, 2]	48	$48-48\beta$	0	0	0

Table 1 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[3, 1, 1, 1, 1, 1]	0	0	0	0	0
[3, 2, 1, 1, 1]	36	0	0	0	0
[3, 2, 2, 1]	72	$72-72\beta$	0	0	0
[3, 3, 1, 1]	72	$72-72\beta$	0	0	0
[3, 3, 2]	72	$162-162\beta$	$90\beta^2-162\beta+90$	0	0
[4, 1, 1, 1, 1]	24	0	0	0	0
[4, 2, 1, 1]	72	$72-72\beta$	0	0	0
[4, 2, 2]	48	$120-120\beta$	$72\beta^2-120\beta+72$	0	0
[4, 3, 1]	72	$168-168\beta$	$96\beta^2-168\beta+96$	0	0
[4, 4]	36	$152-152\beta$	$212\beta^2-364\beta+212$	$-96\beta^3+212\beta^2-212\beta+96$	0
[5, 1, 1, 1]	60	$60-60\beta$	0	0	0
[5, 2, 1]	60	$150-150\beta$	$90\beta^2-150\beta+90$	0	0
[5, 3]	45	$180-180\beta$	$240\beta^2-420\beta+240$	$-105\beta^3+240\beta^2-240\beta+105$	0
[6, 1, 1]	60	$150-150\beta$	$90\beta^2-150\beta+90$	0	0
[6, 2]	30	$132-132\beta$	$192\beta^2-324\beta+192$	$-90\beta^3+192\beta^2-192\beta+90$	0
[7, 1]	35	$154-154\beta$	$224\beta^2-378\beta+224$	$-105\beta^3+224\beta^2-224\beta+105$	0
[8]	14	$93-93\beta$	$234\beta^2-398\beta+234$	$-260\beta^3+565\beta^2-565\beta+260$	$105\beta^4-260\beta^3+331\beta^2-260\beta+105$
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 2, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 2, 2, 1, 1]	384	0	0	0	0
[2, 2, 2, 2, 2]	384	$384-384\beta$	0	0	0

Table 1 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[3, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[3, 2, 1, 1, 1, 1, 1]	0	0	0	0	0
[3, 2, 2, 1, 1, 1]	288	0	0	0	0
[3, 2, 2, 2, 1]	576	$576-576\beta$	0	0	0
[3, 3, 1, 1, 1, 1]	216	0	0	0	0
[3, 3, 2, 1, 1]	576	$576-576\beta$	0	0	0
[3, 3, 2, 2]	576	$1296-1296\beta$	$720\beta^2-1296\beta+720$	0	0
[3, 3, 3, 1]	648	$1458-1458\beta$	$810\beta^2-1458\beta+810$	0	0
[4, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[4, 2, 1, 1, 1, 1]	192	0	0	0	0
[4, 2, 2, 1, 1]	576	$576-576\beta$	0	0	0
[4, 2, 2, 2]	384	$960-960\beta$	$576\beta^2-960\beta+576$	0	0
[4, 3, 1, 1, 1]	504	$504-504\beta$	0	0	0
[4, 3, 2, 1]	576	$1344-1344\beta$	$768\beta^2-1344\beta+768$	0	0
[4, 3, 3]	432	$1656-1656\beta$	$2124\beta^2-3780\beta+2124$	$-900\beta^3+2124\beta^2-2124\beta+900$	0
[4, 4, 1, 1]	576	$1344-1344\beta$	$768\beta^2-1344\beta+768$	0	0
[4, 4, 2]	288	$1216-1216\beta$	$1696\beta^2-2912\beta+1696$	$-768\beta^3+1696\beta^2-1696\beta+768$	0
[5, 1, 1, 1, 1, 1]	120	0	0	0	0
[5, 2, 1, 1, 1]	480	$480-480\beta$	0	0	0
[5, 2, 2, 1]	480	$1200-1200\beta$	$720\beta^2-1200\beta+720$	0	0
[5, 3, 1, 1]	540	$1290-1290\beta$	$750\beta^2-1290\beta+750$	0	0
[5, 3, 2]	360	$1440-1440\beta$	$1920\beta^2-3360\beta+1920$	$-840\beta^3+1920\beta^2-1920\beta+840$	0
[5, 4, 1]	360	$1480-1480\beta$	$2020\beta^2-3500\beta+2020$	$-900\beta^3+2020\beta^2-2020\beta+900$	0

Table 1 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[5, 5]	180	$1075-1075\beta$	$2450\beta^2-4300\beta+2450$	$-2500\beta^3+5725\beta^2-5725\beta+2500$	$945\beta^4-2500\beta^3+3275\beta^2-2500\beta+945$
[6, 1, 1, 1, 1]	360	$360-360\beta$	0	0	0
[6, 2, 1, 1]	480	$1200-1200\beta$	$720\beta^2-1200\beta+720$	0	0
[6, 2, 2]	240	$1056-1056\beta$	$1536\beta^2-2592\beta+1536$	$-720\beta^3+1536\beta^2-1536\beta+720$	0
[6, 3, 1]	360	$1476-1476\beta$	$2016\beta^2-3492\beta+2016$	$-900\beta^3+2016\beta^2-2016\beta+900$	0
[6, 4]	144	$912-912\beta$	$2184\beta^2-3768\beta+2184$	$-2316\beta^3+5172\beta^2-5172\beta+2316$	$900\beta^4-2316\beta^3+2988\beta^2-2316\beta+900$
[7, 1, 1, 1]	420	$1050-1050\beta$	$630\beta^2-1050\beta+630$	0	0
[7, 2, 1]	280	$1232-1232\beta$	$1792\beta^2-3024\beta+1792$	$-840\beta^3+1792\beta^2-1792\beta+840$	0
[7, 3]	168	$1029-1029\beta$	$2394\beta^2-4158\beta+2394$	$-2478\beta^3+5607\beta^2-5607\beta+2478$	$945\beta^4-2478\beta^3+3213\beta^2-2478\beta+945$
[8, 1, 1]	280	$1232-1232\beta$	$1792\beta^2-3024\beta+1792$	$-840\beta^3+1792\beta^2-1792\beta+840$	0
[8, 2]	112	$744-744\beta$	$1872\beta^2-3184\beta+1872$	$-2080\beta^3+4520\beta^2-4520\beta+2080$	$840\beta^4-2080\beta^3+2648\beta^2-2080\beta+840$
[9, 1]	126	$837-837\beta$	$2106\beta^2-3582\beta+2106$	$-2340\beta^3+5085\beta^2-5085\beta+2340$	$945\beta^4-2340\beta^3+2979\beta^2-2340\beta+945$
[10]	42	$386-386\beta$	$1450\beta^2-2480\beta+1450$	$-2750\beta^3+6050\beta^2-6050\beta+2750$	$2589\beta^4-6545\beta^3+8395\beta^2-6545\beta+2589$
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 2, 2, 1, 1, 1, 1]	0	0	0	0	0
[2, 2, 2, 2, 2, 1, 1]	3840	0	0	0	0
[2, 2, 2, 2, 2, 2]	3840	$3840-3840\beta$	0	0	0
[3, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[3, 2, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[3, 2, 2, 1, 1, 1, 1, 1]	0	0	0	0	0

Table 1 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[3, 2, 2, 2, 1, 1, 1]	2880	0	0	0	0
[3, 2, 2, 2, 2, 1]	5760	$5760-5760\beta$	0	0	0
[3, 3, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[3, 3, 2, 1, 1, 1, 1]	2160	0	0	0	0
[3, 3, 2, 2, 1, 1]	5760	$5760-5760\beta$	0	0	0
[3, 3, 2, 2, 2]	5760	$12960-12960\beta$	$7200\beta^2-12960\beta+7200$	0	0
[3, 3, 3, 1, 1, 1]	5184	$5184-5184\beta$	0	0	0
[3, 3, 3, 2, 1]	6480	$14580-14580\beta$	$8100\beta^2-14580\beta+8100$	0	0
[3, 3, 3, 3]	5184	$19116-19116\beta$	$23652\beta^2-42768\beta+23652$	$-9720\beta^3+23652\beta^2-23652\beta+9720$	0
[4, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[4, 2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[4, 2, 2, 1, 1, 1, 1]	1920	0	0	0	0
[4, 2, 2, 2, 1, 1]	5760	$5760-5760\beta$	0	0	0
[4, 2, 2, 2, 2]	3840	$9600-9600\beta$	$5760\beta^2-9600\beta+5760$	0	0
[4, 3, 1, 1, 1, 1, 1]	1440	0	0	0	0
[4, 3, 2, 1, 1, 1]	5040	$5040-5040\beta$	0	0	0
[4, 3, 2, 2, 1]	5760	$13440-13440\beta$	$7680\beta^2-13440\beta+7680$	0	0
[4, 3, 3, 1, 1]	6048	$13896-13896\beta$	$7848\beta^2-13896\beta+7848$	0	0
[4, 3, 3, 2]	4320	$16560-16560\beta$	$21240\beta^2-37800\beta+21240$	$-9000\beta^3+21240\beta^2-21240\beta+9000$	0
[4, 4, 1, 1, 1, 1]	4032	$4032-4032\beta$	0	0	0
[4, 4, 2, 1, 1]	5760	$13440-13440\beta$	$7680\beta^2-13440\beta+7680$	0	0
[4, 4, 2, 2]	2880	$12160-12160\beta$	$16960\beta^2-29120\beta+16960$	$-7680\beta^3+16960\beta^2-16960\beta+7680$	0
[4, 4, 3, 1]	4320	$16896-16896\beta$	$22080\beta^2-38976\beta+22080$	$-9504\beta^3+22080\beta^2-22080\beta+9504$	0

Table 1 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[4, 4, 4]	1728	$10592-10592\beta$	$24512\beta^2-42688\beta+24512$	$-25152\beta^3+57248\beta^2-57248\beta+25152$	$9504\beta^4-25152\beta^3+32736\beta^2-25152\beta+9504$
[5, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0
[5, 2, 1, 1, 1, 1, 1]	1200	0	0	0	0
[5, 2, 2, 1, 1, 1]	4800	$4800-4800\beta$	0	0	0
[5, 2, 2, 2, 1]	4800	$12000-12000\beta$	$7200\ \beta^2$-$12000\ \beta$ +7200	0	0
[5, 3, 1, 1, 1, 1]	3960	$3960-3960\beta$	0	0	0
[5, 3, 2, 1, 1]	5400	$12900-12900\beta$	$7500\ \beta^2$-$12900\ \beta$ +7500	0	0
[5, 3, 2, 2]	3600	$14400-14400\beta$	$19200\beta^2-33600\beta+19200$	$-8400\beta^3+19200\beta^2-19200\beta+8400$	0
[5, 3, 3, 1]	4320	16920-16920 β	22140 β^2-39060 β +22140	-9540 β^3+22140 β^2-22140 β +9540	0
[5, 4, 1, 1, 1]	5040	11880-11880 β	6840 β^2-11880 β +6840	0	0
[5, 4, 2, 1]	3600	14800-14800 β	20200 β^2-35000 β +20200	-9000 β^3+20200 β^2-20200 β +9000	0
[5, 4, 3]	2160	12540-12540 β	27720 β^2-48960 β +27720	-27420 β^3+63840 β^2-63840 β +27420	10080 β^4-27420 β^3+36120 β^2-27420 β +10080
[5, 5, 1, 1]	3600	14800-14800 β	20200 β^2-35000 β +20200	-9000 β^3+20200 β^2-20200 β +9000	0
[5, 5, 2]	1800	10750-10750 β	24500 β^2-43000 β +24500	-25000 β^3+57250 β^2-57250 β +25000	9450 β^4-25000 β^3+32750 β^2-25000 β +9450
[6, 1, 1, 1, 1, 1, 1]	720	0	0	0	0
[6, 2, 1, 1, 1, 1]	3600	3600-3600 β	0	0	0
[6, 2, 2, 1, 1]	4800	12000-12000 β	7200 β^2-12000 β +7200	0	0
[6, 2, 2, 2]	2400	10560-10560 β	15360 β^2-25920 β +15360	-7200 β^3+15360 β^2-15360 β +7200	0
[6, 3, 1, 1, 1]	4680	11340-11340 β	6660 β^2-11340 β +6660	0	0
[6, 3, 2, 1]	3600	14760-14760 β	20160 β^2-34920 β +20160	-9000 β^3+20160 β^2-20160 β +9000	0
[6, 3, 3]	2160	12582-12582 β	27900 β^2-49140 β +27900	-27648 β^3+64206 β^2-64206 β +27648	10170 β^4-27648 β^3+36306 β^2-27648 β +10170
[6, 4, 1, 1]	3600	14784-14784 β	20184 β^2-34968 β +20184	-9000 β^3+20184 β^2-20184 β +9000	0
[6, 4, 2]	1440	9120-9120 β	21840 β^2-37680 β +21840	-23160 β^3+51720 β^2-51720 β +23160	9000 β^4-23160 β^3+29880 β^2-23160 β +9000

Table 1 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[6, 5, 1]	1800	11010-11010 β	25620 β^2-44640 β +25620	-26580 β^3+60210 β^2-60210 β +26580	10170 β^4-26580 β^3+34590 β^2-26580 β +10170
[6, 6]	600	5184-5184 β	18276 β^2-31752 β +18276	-32616 β^3+73812 β^2-73812 β +32616	29094 β^4-76338 β^3+99258 β^2-76338 β +29094
[7, 1, 1, 1, 1, 1]	2520	2520-2520 β	0	0	0
[7, 2, 1, 1, 1]	4200	10500-10500 β	6300 β^2-10500 β +6300	0	0
[7, 2, 2, 1]	2800	12320-12320 β	17920 β^2-30240 β +17920	-8400 β^3+17920 β^2-17920 β +8400	0
[7, 3, 1, 1]	3360	14028-14028 β	19488 β^2-33516 β +19488	-8820 β^3+19488 β^2-19488 β +8820	0
[7, 3, 2]	1680	10290-10290 β	23940 β^2-41580 β +23940	-24780 β^3+56070 β^2-56070 β +24780	9450 β^4-24780 β^3+32130 β^2-24780 β +9450
[7, 4, 1]	1680	10500-10500 β	24864 β^2-43008 β +24864	-26124 β^3+58632 β^2-58632 β +26124	10080 β^4-26124 β^3+33768 β^2-26124 β +10080
[7, 5]	700	5810-5810 β	19810 β^2-34720 β +19810	-34440 β^3+78960 β^2-78960 β +34440	30135 β^4-80185 β^3+104895 β^2-80185 β +30135
[8, 1, 1, 1, 1]	3360	8400-8400 β	5040 β^2-8400 β +5040	0	0
[8, 2, 1, 1]	2800	12320-12320 β	17920 β^2-30240 β +17920	-8400 β^3+17920 β^2-17920 β +8400	0
[8, 2, 2]	1120	7440-7440 β	18720 β^2-31840 β +18720	-20800 β^3+45200 β^2-45200 β +20800	8400 β^4-20800 β^3+26480 β^2-20800 β +8400
[8, 3, 1]	1680	10464-10464 β	24768 β^2-42816 β +24768	-26064 β^3+58416 β^2-58416 β +26064	10080 β^4-26064 β^3+33648 β^2-26064 β +10080
[8, 4]	560	4912-4912 β	17576 β^2-30392 β +17576	-31776 β^3+71376 β^2-71376 β +31776	28632 β^4-74528 β^3+96552 β^2-74528 β +28632
[9, 1, 1, 1]	2520	11088-11088 β	16128 β^2-27216 β +16128	-7560 β^3+16128 β^2-16128 β +7560	0
[9, 2, 1]	1260	8370-8370 β	21060 β^2-35820 β +21060	-23400 β^3+50850 β^2-50850 β +23400	9450 β^4-23400 β^3+29790 β^2-23400 β +9450
[9, 3]	630	5400-5400 β	18936 β^2-32832 β +18936	-33642 β^3+75996 β^2-75996 β +33642	29871 β^4-78435 β^3+101853 β^2-78435 β +29871
[10, 1, 1]	1260	8370-8370 β	21060 β^2-35820 β +21060	-23400 β^3+50850 β^2-50850 β +23400	9450 β^4-23400 β^3+29790 β^2-23400 β +9450
[10, 2]	420	3860-3860 β	14500 β^2-24800 β +14500	-27500 β^3+60500 β^2-60500 β +27500	25890 β^4-65450 β^3+83950 β^2-65450 β +25890
[11, 1]	462	4246-4246 β	15950 β^2-27280 β +15950	-30250 β^3+66550 β^2-66550 β +30250	28479 β^4-71995 β^3+92345 β^2-71995 β +28479
[12]	132	1586-1586 β	8178 β^2-14046 β +8178	-22950 β^3+50945 β^2-50945 β +22950	36500 β^4-93612 β^3+120692 β^2-93612 β +36500

Table 2 Integers $n_{\nu,\lambda}$ for the GβE

ν	λ								
	[]	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[1, 1]	1	0	0	0	0	0	0	0	0
[2]	1	1	0	0	0	0	0	0	0
[1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 1, 1]	2	0	0	0	0	0	0	0	0
[2, 2]	2	2	0	0	0	0	0	0	0
[3, 1]	3	3	0	0	0	0	0	0	0
[4]	2	5	2	3	0	0	0	0	0
[1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 1, 1]	8	0	0	0	0	0	0	0	0
[2, 2, 2]	8	8	0	0	0	0	0	0	0
[3, 1, 1, 1]	6	0	0	0	0	0	0	0	0
[3, 2, 1]	12	12	0	0	0	0	0	0	0
[3, 3]	12	27	12	15	0	0	0	0	0
[4, 1, 1]	12	12	0	0	0	0	0	0	0
[4, 2]	8	20	8	12	0	0	0	0	0
[5, 1]	10	25	10	15	0	0	0	0	0
[6]	5	22	22	32	17	15	0	0	0
[1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 2, 1, 1]	48	0	0	0	0	0	0	0	0
[2, 2, 2, 2]	48	48	0	0	0	0	0	0	0
[3, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 2, 1, 1, 1]	36	0	0	0	0	0	0	0	0
[3, 2, 2, 1]	72	72	0	0	0	0	0	0	0
[3, 3, 1, 1]	72	72	0	0	0	0	0	0	0
[3, 3, 2]	72	162	72	90	0	0	0	0	0
[4, 1, 1, 1, 1]	24	0	0	0	0	0	0	0	0
[4, 2, 1, 1]	72	72	0	0	0	0	0	0	0
[4, 2, 2]	48	120	48	72	0	0	0	0	0
[4, 3, 1]	72	168	72	96	0	0	0	0	0
[4, 4]	36	152	152	212	116	96	0	0	0
[5, 1, 1, 1]	60	60	0	0	0	0	0	0	0
[5, 2, 1]	60	150	60	90	0	0	0	0	0
[5, 3]	45	180	180	240	135	105	0	0	0
[6, 1, 1]	60	150	60	90	0	0	0	0	0
[6, 2]	30	132	132	192	102	90	0	0	0

Table 2 continued

ν	λ								
	[]	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[7, 1]	35	154	154	224	119	105	0	0	0
[8]	14	93	164	234	305	260	71	155	105
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 2, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 2, 2, 1, 1]	384	0	0	0	0	0	0	0	0
[2, 2, 2, 2, 2]	384	384	0	0	0	0	0	0	0
[3, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 2, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 2, 2, 1, 1, 1]	288	0	0	0	0	0	0	0	0
[3, 2, 2, 2, 1]	576	576	0	0	0	0	0	0	0
[3, 3, 1, 1, 1, 1]	216	0	0	0	0	0	0	0	0
[3, 3, 2, 1, 1]	576	576	0	0	0	0	0	0	0
[3, 3, 2, 2]	576	1296	576	720	0	0	0	0	0
[3, 3, 3, 1]	648	1458	648	810	0	0	0	0	0
[4, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[4, 2, 1, 1, 1, 1]	192	0	0	0	0	0	0	0	0
[4, 2, 2, 1, 1]	576	576	0	0	0	0	0	0	0
[4, 2, 2, 2]	384	960	384	576	0	0	0	0	0
[4, 3, 1, 1, 1]	504	504	0	0	0	0	0	0	0
[4, 3, 2, 1]	576	1344	576	768	0	0	0	0	0
[4, 3, 3]	432	1656	1656	2124	1224	900	0	0	0
[4, 4, 1, 1]	576	1344	576	768	0	0	0	0	0
[4, 4, 2]	288	1216	1216	1696	928	768	0	0	0
[5, 1, 1, 1, 1, 1]	120	0	0	0	0	0	0	0	0
[5, 2, 1, 1, 1]	480	480	0	0	0	0	0	0	0
[5, 2, 2, 1]	480	1200	480	720	0	0	0	0	0
[5, 3, 1, 1]	540	1290	540	750	0	0	0	0	0
[5, 3, 2]	360	1440	1440	1920	1080	840	0	0	0
[5, 4, 1]	360	1480	1480	2020	1120	900	0	0	0
[5, 5]	180	1075	1850	2450	3225	2500	775	1555	945
[6, 1, 1, 1, 1]	360	360	0	0	0	0	0	0	0
[6, 2, 1, 1]	480	1200	480	720	0	0	0	0	0
[6, 2, 2]	240	1056	1056	1536	816	720	0	0	0
[6, 3, 1]	360	1476	1476	2016	1116	900	0	0	0
[6, 4]	144	912	1584	2184	2856	2316	672	1416	900
[7, 1, 1, 1]	420	1050	420	630	0	0	0	0	0
[7, 2, 1]	280	1232	1232	1792	952	840	0	0	0

Table 2 continued

ν	λ								
	[]	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[7, 3]	168	1029	1764	2394	3129	2478	735	1533	945
[8, 1, 1]	280	1232	1232	1792	952	840	0	0	0
[8, 2]	112	744	1312	1872	2440	2080	568	1240	840
[9, 1]	126	837	1476	2106	2745	2340	639	1395	945
[10]	42	386	1030	1450	3300	2750	1850	3956	2589
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 2, 2, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[2, 2, 2, 2, 2, 1, 1]	3840	0	0	0	0	0	0	0	0
[2, 2, 2, 2, 2, 2]	3840	3840	0	0	0	0	0	0	0
[3, 1, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 2, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 2, 2, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 2, 2, 2, 1, 1, 1]	2880	0	0	0	0	0	0	0	0
[3, 2, 2, 2, 2, 1]	5760	5760	0	0	0	0	0	0	0
[3, 3, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[3, 3, 2, 1, 1, 1, 1]	2160	0	0	0	0	0	0	0	0
[3, 3, 2, 2, 1, 1]	5760	5760	0	0	0	0	0	0	0
[3, 3, 2, 2, 2]	5760	12960	5760	7200	0	0	0	0	0
[3, 3, 3, 1, 1, 1]	5184	5184	0	0	0	0	0	0	0
[3, 3, 3, 2, 1]	6480	14580	6480	8100	0	0	0	0	0
[3, 3, 3, 3]	5184	19116	19116	23652	13932	9720	0	0	0
[4, 1, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[4, 2, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[4, 2, 2, 1, 1, 1, 1]	1920	0	0	0	0	0	0	0	0
[4, 2, 2, 2, 1, 1]	5760	5760	0	0	0	0	0	0	0
[4, 2, 2, 2, 2]	3840	9600	3840	5760	0	0	0	0	0
[4, 3, 1, 1, 1, 1, 1]	1440	0	0	0	0	0	0	0	0
[4, 3, 2, 1, 1, 1]	5040	5040	0	0	0	0	0	0	0
[4, 3, 2, 2, 1]	5760	13440	5760	7680	0	0	0	0	0
[4, 3, 3, 1, 1]	6048	13896	6048	7848	0	0	0	0	0
[4, 3, 3, 2]	4320	16560	16560	21240	12240	9000	0	0	0
[4, 4, 1, 1, 1, 1]	4032	4032	0	0	0	0	0	0	0
[4, 4, 2, 1, 1]	5760	13440	5760	7680	0	0	0	0	0
[4, 4, 2, 2]	2880	12160	12160	16960	9280	7680	0	0	0
[4, 4, 3, 1]	4320	16896	16896	22080	12576	9504	0	0	0

Table 2 continued

ν	λ								
	[]	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[4, 4, 4]	1728	10592	18176	24512	32096	25152	7584	15648	9504
[5, 1, 1, 1, 1, 1, 1, 1]	0	0	0	0	0	0	0	0	0
[5, 2, 1, 1, 1, 1, 1]	1200	0	0	0	0	0	0	0	0
[5, 2, 2, 1, 1, 1]	4800	4800	0	0	0	0	0	0	0
[5, 2, 2, 2, 1]	4800	12000	4800	7200	0	0	0	0	0
[5, 3, 1, 1, 1, 1]	3960	3960	0	0	0	0	0	0	0
[5, 3, 2, 1, 1]	5400	12900	5400	7500	0	0	0	0	0
[5, 3, 2, 2]	3600	14400	14400	19200	10800	8400	0	0	0
[5, 3, 3, 1]	4320	16920	16920	22140	12600	9540	0	0	0
[5, 4, 1, 1, 1]	5040	11880	5040	6840	0	0	0	0	0
[5, 4, 2, 1]	3600	14800	14800	20200	11200	9000	0	0	0
[5, 4, 3]	2160	12540	21240	27720	36420	27420	8700	17340	10080
[5, 5, 1, 1]	3600	14800	14800	20200	11200	9000	0	0	0
[5, 5, 2]	1800	10750	18500	24500	32250	25000	7750	15550	9450
[6, 1, 1, 1, 1, 1, 1]	720	0	0	0	0	0	0	0	0
[6, 2, 1, 1, 1, 1]	3600	3600	0	0	0	0	0	0	0
[6, 2, 2, 1, 1]	4800	12000	4800	7200	0	0	0	0	0
[6, 2, 2, 2]	2400	10560	10560	15360	8160	7200	0	0	0
[6, 3, 1, 1, 1]	4680	11340	4680	6660	0	0	0	0	0
[6, 3, 2, 1]	3600	14760	14760	20160	11160	9000	0	0	0
[6, 3, 3]	2160	12582	21240	27900	36558	27648	8658	17478	10170
[6, 4, 1, 1]	3600	14784	14784	20184	11184	9000	0	0	0
[6, 4, 2]	1440	9120	15840	21840	28560	23160	6720	14160	9000
[6, 5, 1]	1800	11010	19020	25620	33630	26580	8010	16410	10170
[6, 6]	600	5184	13476	18276	41196	32616	22920	47244	29094
[7, 1, 1, 1, 1, 1]	2520	2520	0	0	0	0	0	0	0
[7, 2, 1, 1, 1]	4200	10500	4200	6300	0	0	0	0	0
[7, 2, 2, 1]	2800	12320	12320	17920	9520	8400	0	0	0
[7, 3, 1, 1]	3360	14028	14028	19488	10668	8820	0	0	0
[7, 3, 2]	1680	10290	17640	23940	31290	24780	7350	15330	9450
[7, 4, 1]	1680	10500	18144	24864	32508	26124	7644	16044	10080
[7, 5]	700	5810	14910	19810	44520	34440	24710	50050	30135
[8, 1, 1, 1, 1]	3360	8400	3360	5040	0	0	0	0	0
[8, 2, 1, 1]	2800	12320	12320	17920	9520	8400	0	0	0

Table 2 continued

ν	λ								
	[]	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[8, 2, 2]	1120	7440	13120	18720	24400	20800	5680	12400	8400
[8, 3, 1]	1680	10464	18048	24768	32352	26064	7584	15984	10080
[8, 4]	560	4912	12816	17576	39600	31776	22024	45896	28632
[9, 1, 1, 1]	2520	11088	11088	16128	8568	7560	0	0	0
[9, 2, 1]	1260	8370	14760	21060	27450	23400	6390	13950	9450
[9, 3]	630	5400	13896	18936	42354	33642	23418	48564	29871
[10, 1, 1]	1260	8370	14760	21060	27450	23400	6390	13950	9450
[10, 2]	420	3860	10300	14500	33000	27500	18500	39560	25890
[11, 1]	462	4246	11330	15950	36300	30250	20350	43516	28479
[12]	132	1586	5868	8178	27995	22950	27080	57112	36500

C.2 The WLβE case

See Tables 3, 4, and 5

Table 3 Polynomials $C_{g,\nu}(\beta,\phi)$ for the WLβE at $\phi = 1$

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[1]	1	$1-\beta$	0	0	0
[1, 1]	1	$1-\beta$	0	0	0
[2]	2	$4-4\beta$	$2\beta^2-4\beta+2$	0	0
[1, 1, 1]	2	$2-2\beta$	0	0	0
[2, 1]	4	$8-8\beta$	$4\beta^2-8\beta+4$	0	0
[3]	5	$16-16\beta$	$17\beta^2-33\beta+17$	$-6\beta^3+17\beta^2-17\beta+6$	0
[1, 1, 1, 1]	6	$6-6\beta$	0	0	0
[2, 1, 1]	12	$24-24\beta$	$12\beta^2-24\beta+12$	0	0
[2, 2]	18	$56-56\beta$	$58\beta^2-114\beta+58$	$-20\beta^3+58\beta^2-58\beta+20$	0
[3, 1]	15	$48-48\beta$	$51\beta^2-99\beta+51$	$-18\beta^3+51\beta^2-51\beta+18$	0
[4]	14	$64-64\beta$	$110\beta^2-210\beta+110$	$-84\beta^3+230\beta^2-230\beta+84$	$24\beta^4-84\beta^3+120\beta^2-84\beta+24$
[1, 1, 1, 1, 1]	24	$24-24\beta$	0	0	0
[2, 1, 1, 1]	48	$96-96\beta$	$48\beta^2-96\beta+48$	0	0
[2, 2, 1]	72	$224-224\beta$	$232\beta^2-456\beta+232$	$-80\beta^3+232\beta^2-232\beta+80$	0
[3, 1, 1]	60	$192-192\beta$	$204\beta^2-396\beta+204$	$-72\beta^3+204\beta^2-204\beta+72$	0
[3, 2]	72	$318-318\beta$	$528\beta^2-1020\beta+528$	$-390\beta^3+1092\beta^2-1092\beta+390$	$108\beta^4-390\beta^3+564\beta^2-390\beta+108$
[4, 1]	56	$256-256\beta$	$440\beta^2-840\beta+440$	$-336\beta^3+920\beta^2-920\beta+336$	$96\beta^4-336\beta^3+480\beta^2-336\beta+96$
[5]	42	$256-256\beta$	$630\beta^2-1190\beta+630$	$-780\beta^3+2090\beta^2-2090\beta+780$	$484\beta^4-1640\beta^3+2320\beta^2-1640\beta+484$
[1, 1, 1, 1, 1, 1]	120	$120-120\beta$	0	0	0
[2, 1, 1, 1, 1]	240	$480-480\beta$	$240\beta^2-480\beta+240$	0	0

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[2, 2, 1, 1]	360	$1120-1120\beta$	$1160\,\beta^2-2280\beta+1160$	$-400\beta^3+1160\beta^2-1160\beta+400$	0
[2, 2, 2]	432	$1864-1864\beta$	$3024\,\beta^2-5888\beta+3024$	$-2184\beta^3+6208\beta^2-6208\beta+2184$	$592\,\beta^4-2184\beta^3+3184\beta^2-2184\beta+592$
[3, 1, 1, 1]	300	$960-960\beta$	$1020\,\beta^2-1980\beta+1020$	$-360\beta^3+1020\beta^2-1020\beta+360$	0
[3, 2, 1]	360	$1590-1590\beta$	$2640\,\beta^2-5100\beta+2640$	$-1950\beta^3+5460\beta^2-5460\beta+1950$	$540\,\beta^4-1950\beta^3+2820\beta^2-1950\beta+540$
[3, 3]	300	$1746-1746\beta$	$4098\,\beta^2-7848\beta+4098$	$-4842\beta^3+13308\beta^2-13308\beta+4842$	$2874\,\beta^4-10080\beta^3+14448\beta^2-10080\beta+2874$
[4, 1, 1]	280	$1280-1280\beta$	$2200\,\beta^2-4200\beta+2200$	$-1680\beta^3+4600\beta^2-4600\beta+1680$	$480\,\beta^4-1680\beta^3+2400\beta^2-1680\beta+480$
[4, 2]	280	$1648-1648\beta$	$3912\,\beta^2-7464\beta+3912$	$-4672\beta^3+12760\beta^2-12760\beta+4672$	$2800\,\beta^4-9744\beta^3+13920\beta^2-9744\beta+2800$
[5, 1]	210	$1280-1280\beta$	$3150\,\beta^2-5950\beta+3150$	$-3900\beta^3+10450\beta^2-10450\beta+3900$	$2420\,\beta^4-8200\beta^3+11600\beta^2-8200\beta+2420$
[6]	132	$1024-1024\beta$	$3360\,\beta^2-6300\beta+3360$	$-5952\beta^3+15712\beta^2-15712\beta+5952$	$5980\,\beta^4-19812\beta^3+27832\beta^2-19812\beta+5980$
[1, 1, 1, 1, 1, 1, 1]	720	$720-720\beta$	0	0	0
[2, 1, 1, 1, 1, 1]	1440	$2880-2880\beta$	$1440\,\beta^2-2880\beta+1440$	0	0
[2, 2, 1, 1, 1]	2160	$6720-6720\beta$	$6960\,\beta^2-13680\beta+6960$	$-2400\beta^3+6960\beta^2-6960\beta+2400$	0
[2, 2, 2, 1]	2592	$11184-11184\beta$	$18144\,\beta^2-35328\beta+18144$	$-13104\beta^3+37248\beta^2-37248\beta+13104$	$3552\,\beta^4-13104\beta^3+19104\beta^2-13104\beta+3552$
[3, 1, 1, 1, 1]	1800	$5760-5760\beta$	$6120\,\beta^2-11880\beta+6120$	$-2160\beta^3+6120\beta^2-6120\beta+2160$	0

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[3, 2, 1, 1]	2160	$9540-9540\beta$	$15840\,\beta^2-30600\beta+15840$	$-11700\beta^3+32760\beta^2-32760\beta+11700$	$3240\,\beta^4-11700\beta^3+16920\beta^2-11700\beta+3240$
[3, 2, 2]	2160	$12336-12336\beta$	$28392\,\beta^2-54648\beta+28392$	$-32880\beta^3+91296\beta^2-91296\beta+32880$	$19128\,\beta^4-68112\beta^3+98136\beta^2-68112\beta+19128$
[3, 3, 1]	1800	$10476-10476\beta$	$24588\,\beta^2$-$47088\,\beta+24588$	$-29052\beta^3+79848\,\beta^2-79848\beta+29052$	$17244\,\beta^4-60480\beta^3+86688\,\beta^2-60480\beta+17244$
[4, 1, 1, 1]	1680	$7680-7680\beta$	$13200\,\beta^2-25200\beta+13200$	$-10080\beta^3+27600\,\beta^2-27600\beta+10080$	$2880\,\beta^4-10080\beta^3+14400\,\beta^2-10080\beta+2880$
[4, 2, 1]	1680	$9888-9888\beta$	$23472\,\beta^2-44784\beta+23472$	$-28032\beta^3+76560\,\beta^2-76560\beta+28032$	$16800\,\beta^4-58464\beta^3+83520\,\beta^2-58464\beta+16800$
[4, 3]	1200	$8856-8856\beta$	$27600\,\beta^2-52440\beta+27600$	$-46440\beta^3+125616\,\beta^2-125616\beta+46440$	$44400\,\beta^4-152016\beta^3+216168\,\beta^2-152016\beta+44400$
[5, 1, 1]	1260	$7680-7680\beta$	$18900\,\beta^2-35700\beta+18900$	$-23400\beta^3+62700\,\beta^2-62700\beta+23400$	$14520\,\beta^4-49200\beta^3+69600\,\beta^2-49200\beta+14520$
[5, 2]	1080	$8100-8100\beta$	$25660\,\beta^2-48520\beta+25660$	$-43860\beta^3+117660\,\beta^2-117660\beta+43860$	$42540\,\beta^4-144160\beta^3+204160\,\beta^2-144160\beta+42540$
[6, 1]	792	$6144-6144\beta$	$20160\,\beta^2-37800\beta+20160$	$-35712\beta^3+94272\,\beta^2-94272\beta+35712$	$35880\,\beta^4-118872\beta^3+166992\,\beta^2-118872\beta+35880$
[7]	429	$4096-4096\beta$	$17094\,\beta^2-31878\beta+17094$	$-40320\beta^3+105280\,\beta^2-105280\beta+40320$	$57841\,\beta^4-188496\beta^3+263431\,\beta^2-188496\beta+57841$
[1, 1, 1, 1, 1, 1, 1, 1]	5040	$5040-5040\beta$	0	0	0
[2, 1, 1, 1, 1, 1, 1]	10080	$20160-20160\beta$	$10080\,\beta^2-20160\beta+10080$	0	0
[2, 2, 1, 1, 1, 1]	15120	$47040-47040\beta$	$48720\,\beta^2-95760\beta+48720$	$-16800\beta^3+48720\,\beta^2-48720\beta+16800$	0

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[2, 2, 2, 1, 1]	18144	$78288-78288\beta$	$127008\beta^2-247296\beta+127008$	$-91728\beta^3+260736\beta^2-260736\beta+91728$	$24864\beta^4-91728\beta^3+133728\beta^2-91728\beta+24864$
[2, 2, 2, 2]	18144	$101568-101568\beta$	$229008\beta^2-443280\beta+229008$	$-259776\beta^3+729216\beta^2-729216\beta+259776$	$148080\beta^4-535584\beta^3+776016\beta^2-535584\beta+148080$
[3, 1, 1, 1, 1, 1]	12600	$40320\text{-}40320\beta$	$42840\beta^2-83160\beta+42840$	$-15120\beta^3+42840\beta^2-42840\beta+15120$	0
[3, 2, 1, 1, 1]	15120	$66780-66780\beta$	$110880\beta^2-214200\beta+110880$	$-81900\beta^3+229320\beta^2-229320\beta+81900$	$22680\beta^4-81900\beta^3+118440\beta^2-81900\beta+22680$
[3, 2, 2, 1]	15120	$86352-86352\beta$	$198744\beta^2-382536\beta+198744$	$-230160\beta^3+639072\beta^2-639072\beta+230160$	$133896\beta^4-476784\beta^3+686952\beta^2-476784\beta+133896$
[3, 3, 1, 1]	12600	$73332-73332\beta$	$172116\beta^2-329616\beta+172116$	$-203364\beta^3+558936\beta^2-558936\beta+203364$	$120708\beta^4-423360\beta^3+606816\beta^2-423360\beta+120708$
[3, 3, 2]	10800	$77688-77688\beta$	$235728\beta^2-450720\beta+235728$	$-385920\beta^3+1056744\beta^2-1056744\beta+385920$	$358992\beta^4-1250856\beta^3+1789776\beta^2-1250856\beta+358992$
[4, 1, 1, 1, 1]	11760	$53760-53760\beta$	$92400\beta^2-176400\beta+92400$	$-70560\beta^3+193200\beta^2-193200\beta+70560$	$20160\beta^4-70560\beta^3+100800\beta^2-70560\beta+20160$
[4, 2, 1, 1]	11760	$69216-69216\beta$	$164304\beta^2-313488\beta+164304$	$-196224\beta^3+535920\beta^2-535920\beta+196224$	$117600\beta^4-409248\beta^3+584640\beta^2-409248\beta+117600$
[4, 2, 2]	10080	$73248-73248\beta$	$224560\beta^2-428080\beta+224560$	$-371328\beta^3+1011504\beta^2-1011504\beta+371328$	$348624\beta^4-1206784\beta^3+1722256\beta^2-1206784\beta+348624$
[4, 3, 1]	8400	$61992-61992\beta$	$193200\beta^2-367080\beta+193200$	$-325080\beta^3+879312\beta^2-879312\beta+325080$	$310800\beta^4-1064112\beta^3+1513176\beta^2-1064112\beta+310800$

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[4, 4]	4900	$44224-44224\beta$	$174152\beta^2-329304\beta+174152$	$-387616\beta^3+1038000\beta^2-1038000\beta+387616$	$525700\beta^4-1771920\beta^3+2507180\beta^2-1771920\beta+525700$
[5, 1, 1, 1]	8820	$53760-53760\beta$	$132300\beta^2-249900\beta+132300$	$-163800\beta^3+438900\beta^2-438900\beta+163800$	$101640\beta^4-344400\beta^3+487200\beta^2-344400\beta+101640$
[5, 2, 1]	7560	$56700-56700\beta$	$179620\beta^2-339640\beta+179620$	$-307020\beta^3+823620\beta^2-823620\beta+307020$	$297780\beta^4-1009120\beta^3+1429120\beta^2-1009120\beta+297780$
[5, 3]	4725	$42900-42900\beta$	$169950\beta^2-320790\beta+169950$	$-380400\beta^3+1015680\beta^2-1015680\beta+380400$	$518505\beta^4-1741560\beta^3+2460735\beta^2-1741560\beta+518505$
[6, 1, 1]	5544	$43008-43008\beta$	$141120\beta^2-264600\beta+141120$	$-249984\beta^3+659904\beta^2-659904\beta+249984$	$251160\beta^4-832104\beta^3+1168944\beta^2-832104\beta+251160$
[6, 2]	4158	$38448-38448\beta$	$155172\beta^2-291444\beta+155172$	$-353664\beta^3+936144\beta^2-936144\beta+353664$	$490230\beta^4-1628688\beta^3+2291658\beta^2-1628688\beta+490230$
[7, 1]	3003	$28672-28672\beta$	$119658\beta^2-223146\beta+119658$	$-282240\beta^3+736960\beta^2-736960\beta+282240$	$404887\beta^4-1319472\beta^3+1844017\beta^2-1319472\beta+404887$
[8]	1430	$16384-16384\beta$	$84084\beta^2-156156\beta+84084$	$-251904\beta^3+652288\beta^2-652288\beta+251904$	$480238\beta^4-1545456\beta^3+2151226\beta^2-1545456\beta+480238$
[1, 1, 1, 1, 1, 1, 1, 1, 1]	40320	$40320-40320\beta$	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1]	80640	$161280-161280\beta$	$80640\beta^2-161280\beta+80640$	0	0

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[2, 2, 1, 1, 1, 1, 1]	120960	$376320-376320\,\beta$	$389760\,\beta^2-766080\,\beta+389760$	$-134400\,\beta^3+389760\,\beta^2-389760\,\beta+134400$	0
[2, 2, 2, 1, 1, 1]	145152	$626304-626304\,\beta$	$1016064\,\beta^2-1978368\,\beta+1016064$	$-733824\,\beta^3+2085888\,\beta^2-2085888\,\beta+733824$	$198912\,\beta^4-733824\,\beta^3+1069824\,\beta^2-733824\,\beta+198912$
[2, 2, 2, 2, 1]	145152	$812544-812544\,\beta$	$1832064\,\beta^2-3546240\,\beta+1832064$	$-2078208\,\beta^3+5833728\,\beta^2-5833728\,\beta+2078208$	$1184640\,\beta^4-4284672\,\beta^3+6208128\,\beta^2-4284672\,\beta+1184640$
[3, 1, 1, 1, 1, 1, 1]	100800	$322560-322560\,\beta$	$342720\,\beta^2-665280\,\beta+342720$	$-120960\,\beta^3+342720\,\beta^2-342720\,\beta+120960$	0
[3, 2, 1, 1, 1, 1]	120960	$534240-534240\,\beta$	$887040\,\beta^2-1713600\,\beta+887040$	$-655200\,\beta^3+1834560\,\beta^2-1834560\,\beta+655200$	$181440\,\beta^4-655200\,\beta^3+947520\,\beta^2-655200\,\beta+181440$
[3, 2, 2, 1, 1]	120960	$690816-690816\,\beta$	$1589952\,\beta^2-3060288\,\beta+1589952$	$-1841280\,\beta^3+5112576\,\beta^2-5112576\,\beta+1841280$	$1071168\,\beta^4-3814272\,\beta^3+5495616\,\beta^2-3814272\,\beta+1071168$
[3, 2, 2, 2]	103680	$733440-733440\,\beta$	$2186784\,\beta^2-4197600\,\beta+2186784$	$-3515472\,\beta^3+9702192\,\beta^2-9702192\,\beta+3515472$	$3209952\,\beta^4-11315808\,\beta^3+16256064\,\beta^2-11315808\,\beta+3209952$
[3, 3, 1, 1, 1]	100800	$586656-586656\,\beta$	$1376928\,\beta^2-2636928\,\beta+1376928$	$-1626912\,\beta^3+4471488\,\beta^2-4471488\,\beta+1626912$	$965664\,\beta^4-3386880\,\beta^3+4854528\,\beta^2-3386880\,\beta+965664$
[3, 3, 2, 1]	86400	$621504-621504\,\beta$	$1885824\,\beta^2-3605760\,\beta+1885824$	$-3087360\,\beta^3+8453952\,\beta^2-8453952\,\beta+3087360$	$2871936\,\beta^4-10006848\,\beta^3+14318208\,\beta^2-10006848\,\beta+2871936$
[3, 3, 3]	54000	$472608-472608\,\beta$	$1802016\,\beta^2-3430944\,\beta+1802016$	$-3879576\,\beta^3+10528488\,\beta^2-10528488\,\beta+3879576$	$5088024\,\beta^4-17479152\,\beta^3+24897168\,\beta^2-17479152\,\beta+5088024$

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[4, 1, 1, 1, 1, 1]	94080	$430080-430080\,\beta$	$739200\,\beta^2-1411200\,\beta+739200$	$-564480\,\beta^3+1545600\,\beta^2-1545600\,\beta+564480$	$161280\,\beta^4-564480\,\beta^3+806400\,\beta^2-564480\,\beta+161280$
[4, 2, 1, 1, 1]	94080	$553728-553728\,\beta$	$1314432\,\beta^2-2507904\,\beta+1314432$	$-1569792\,\beta^3+4287360\,\beta^2-4287360\,\beta+1569792$	$940800\,\beta^4-3273984\,\beta^3+4677120\,\beta^2-3273984\,\beta+940800$
[4, 2, 2, 1]	80640	$585984-585984\,\beta$	$1796480\,\beta^2-3424640\,\beta+1796480$	$-2970624\,\beta^3+8092032\,\beta^2-8092032\,\beta+2970624$	$2788992\,\beta^4-9654272\,\beta^3+13778048\,\beta^2-9654272\,\beta+2788992$
[4, 3, 1, 1]	67200	$495936-495936\,\beta$	$1545600\,\beta^2-2936640\,\beta+1545600$	$-2600640\,\beta^3+7034496\,\beta^2-7034496\,\beta+2600640$	$2486400\,\beta^4-8512896\,\beta^3+12105408\,\beta^2-8512896\,\beta+2486400$
[4, 3, 2]	50400	$445056-445056\,\beta$	$1712640\,\beta^2-3253200\,\beta+1712640$	$-3721008\,\beta^3+10055904\,\beta^2-10055904\,\beta+3721008$	$4922880\,\beta^4-16819008\,\beta^3+23908080\,\beta^2-16819008\,\beta+4922880$
[4, 4, 1]	39200	$353792-353792\,\beta$	$1393216\,\beta^2-2634432\,\beta+1393216$	$-3100928\,\beta^3+8304000\,\beta^2-8304000\,\beta+3100928$	$4205600\,\beta^4-14175360\,\beta^3+20057440\,\beta^2-14175360\,\beta+4205600$
[5, 1, 1, 1, 1]	70560	$430080-430080\,\beta$	$1058400\,\beta^2-1999200\,\beta+1058400$	$-1310400\,\beta^3+3511200\,\beta^2-3511200\,\beta+1310400$	$813120\,\beta^4-2755200\,\beta^3+3897600\,\beta^2-2755200\,\beta+813120$
[5, 2, 1, 1]	60480	$453600-453600\,\beta$	$1436960\,\beta^2-2717120\,\beta+1436960$	$-2456160\,\beta^3+6588960\,\beta^2-6588960\,\beta+2456160$	$2382240\,\beta^4-8072960\,\beta^3+11432960\,\beta^2-8072960\,\beta+2382240$
[5, 2, 2]	45360	$406560-406560\,\beta$	$1588400\,\beta^2-3004880\,\beta+1588400$	$-3502400\,\beta^3+9397840\,\beta^2-9397840\,\beta+3502400$	$4697760\,\beta^4-15906080\,\beta^3+22532480\,\beta^2-15906080\,\beta+4697760$

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[5, 3, 1]	37800	$343200-343200\,\beta$	$1359600\,\beta^2-2566320\,\beta+1359600$	$-3043200\,\beta^3+8125440\,\beta^2-8125440\,\beta+3043200$	$4148040\,\beta^4-13932480\,\beta^3+19685880\,\beta^2-13932480\,\beta+4148040$
[5, 4]	19600	$211780-211780\,\beta$	$1023120\,\beta^2-1925840\,\beta+1023120$	$-2885080\,\beta^3+7656000\,\beta^2-7656000\,\beta+2885080$	$5186080\,\beta^4-17243440\,\beta^3+24289120\,\beta^2-17243440\,\beta+5186080$
[6, 1, 1, 1]	44352	$344064-344064\,\beta$	$1128960\,\beta^2-2116800\,\beta+1128960$	$-1999872\,\beta^3+5279232\,\beta^2-5279232\,\beta+1999872$	$2009280\,\beta^4\text{-}6656832\,\beta^3+9351552\,\beta^2-6656832\,\beta+2009280$
[6, 2, 1]	33264	$307584-307584\,\beta$	$1241376\,\beta^2-2331552\,\beta+1241376$	$-2829312\,\beta^3+7489152\,\beta^2-7489152\,\beta+2829312$	$3921840\,\beta^4-13029504\,\beta^3+18333264\,\beta^2-13029504\,\beta+3921840$
[6, 3]	18480	$201546-201546\,\beta$	$982800\,\beta^2-1845144\,\beta+982800$	$-2796012\,\beta^3+7387320\,\beta^2-7387320\,\beta+2796012$	$5066064\,\beta^4-16756128\,\beta^3+23554008\,\beta^2-16756128\,\beta+5066064$
[7, 1, 1]	24024	$229376-229376\,\beta$	$957264\,\beta^2-1785168\,\beta+957264$	$-2257920\,\beta^3+5895680\,\beta^2-5895680\,\beta+2257920$	$3239096\,\beta^4-10555776\,\beta^3+14752136\,\beta^2-10555776\,\beta+3239096$
[7, 2]	16016	$178038-178038\,\beta$	$885248\,\beta^2-1654072\,\beta+885248$	$-2567124\,\beta^3+6725544\,\beta^2-6725544\,\beta+2567124$	$4735920\,\beta^4-15495536\,\beta^3+21694456\,\beta^2-15495536\,\beta+4735920$
[8, 1]	11440	$131072-131072\,\beta$	$672672\,\beta^2-1249248\,\beta+672672$	$-2015232\,\beta^3+5218304\,\beta^2-5218304\,\beta+2015232$	$3841904\,\beta^4-12363648\,\beta^3+17209808\,\beta^2-12363648\,\beta+3841904$
[9]	4862	$65536-65536\,\beta$	$403260\,\beta^2-746460\,\beta+403260$	$-1483776\,\beta^3+3816960\,\beta^2-3816960\,\beta+1483776$	$3586726\,\beta^4-11428560\,\beta^3+15857842\,\beta^2-11428560\,\beta+3586726$
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	362880	$362880-362880\,\beta$	0	0	0

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[2, 1, 1, 1, 1, 1, 1, 1, 1]	725760	$1451520-1451520\,\beta$	$725760\,\beta^2-1451520\,\beta+725760$	0	0
[2, 2, 1, 1, 1, 1, 1, 1]	1088640	$3386880-3386880\,\beta$	$3507840\,\beta^2-6894720\,\beta+3507840$	$-1209600\,\beta^3+3507840\,\beta^2-3507840\,\beta+1209600$	0
[2, 2, 2, 1, 1, 1, 1]	1306368	$5636736-5636736\,\beta$	$9144576\,\beta^2-17805312\,\beta+9144576$	$-6604416\,\beta^3+18772992\,\beta^2-18772992\,\beta+6604416$	$1790208\,\beta^4-6604416\,\beta^3+9628416\,\beta^2-6604416\,\beta+1790208$
[2, 2, 2, 2, 1, 1]	1306368	$7312896-7312896\,\beta$	$16488576\,\beta^2-31916160\,\beta+16488576$	$-18703872\,\beta^3+52503552\,\beta^2-52503552\,\beta+18703872$	$10661760\,\beta^4-38562048\,\beta^3+55873152\,\beta^2-38562048\,\beta+10661760$
[2, 2, 2, 2, 2]	1119744	$7782912-7782912\,\beta$	$22783488\,\beta^2-43923456\,\beta+22783488$	$-35944320\,\beta^3+100041600\,\beta^2-100041600\,\beta+35944320$	$32205312\,\beta^4-114919680\,\beta^3+165799680\,\beta^2-114919680\,\beta+32205312$
[3, 1, 1, 1, 1, 1, 1, 1]	907200	$2903040-2903040\,\beta$	$3084480\,\beta^2-5987520\,\beta+3084480$	$-1088640\,\beta^3+3084480\,\beta^2-3084480\,\beta+1088640$	0
[3, 2, 1, 1, 1, 1, 1]	1088640	$4808160-4808160\,\beta$	$7983360\,\beta^2-15422400\,\beta+7983360$	$-5896800\,\beta^3+16511040\,\beta^2-16511040\,\beta+5896800$	$1632960\,\beta^4-5896800\,\beta^3+8527680\,\beta^2-5896800\,\beta+1632960$
[3, 2, 2, 1, 1, 1]	1088640	$6217344-6217344\,\beta$	$14309568\,\beta^2-27542592\,\beta+14309568$	$-16571520\,\beta^3+46013184\,\beta^2-46013184\,\beta+16571520$	$9640512\,\beta^4-34328448\,\beta^3+49460544\,\beta^2-34328448\,\beta+9640512$
[3, 2, 2, 2, 1]	933120	$6600960-6600960\,\beta$	$19681056\,\beta^2-37778400\,\beta+19681056$	$-31639248\,\beta^3+87319728\,\beta^2-87319728\,\beta+31639248$	$28889568\,\beta^4-101842272\,\beta^3+146304576\,\beta^2-101842272\,\beta+28889568$
[3, 3, 1, 1, 1, 1]	907200	$5279904-5279904\,\beta$	$12392352\,\beta^2-23732352\,\beta+12392352$	$-14642208\,\beta^3+40243392\,\beta^2-40243392\,\beta+14642208$	$8690976\,\beta^4-30481920\,\beta^3+43690752\,\beta^2-30481920\,\beta+8690976$

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[3, 3, 2, 1, 1]	777600	$5593536-5593536\,\beta$	$16972416\,\beta^2-32451840\,\beta+16972416$	$-27786240\,\beta^3+76085568\,\beta^2-76085568\,\beta+27786240$	$25847424\,\beta^4-90061632\,\beta^3+128863872\,\beta^2-90061632\,\beta+25847424$
[3, 3, 2, 2]	583200	$5031360-5031360\,\beta$	$18894528\,\beta^2-36078624\,\beta+18894528$	$-40031568\,\beta^3+109283472\,\beta^2-109283472\,\beta+40031568$	$51631920\,\beta^4-178968672\,\beta^3+255695040\,\beta^2-178968672\,\beta+51631920$
[3, 3, 3, 1]	486000	$4253472-4253472\,\beta$	$16218144\,\beta^2-30878496\,\beta+16218144$	$-34916184\,\beta^3+94756392\,\beta^2-94756392\,\beta+34916184$	$45792216\,\beta^4-157312368\,\beta^3+224074512\,\beta^2-157312368\,\beta+45792216$
[4, 1, 1, 1, 1, 1, 1]	846720	$3870720-3870720\,\beta$	$6652800\,\beta^2-12700800\,\beta+6652800$	$-5080320\,\beta^3+13910400\,\beta^2-13910400\,\beta+5080320$	$1451520\,\beta^4-5080320\,\beta^3+7257600\,\beta^2-5080320\,\beta+1451520$
[4, 2, 1, 1, 1, 1]	846720	$4983552-4983552\,\beta$	$11829888\,\beta^2-22571136\,\beta+11829888$	$-14128128\,\beta^3+38586240\,\beta^2-38586240\,\beta+14128128$	$8467200\,\beta^4-29465856\,\beta^3+42094080\,\beta^2-29465856\,\beta+8467200$
[4, 2, 2, 1, 1]	725760	$5273856-5273856\,\beta$	$16168320\,\beta^2-30821760\,\beta+16168320$	$-26735616\,\beta^3+72828288\,\beta^2-72828288\,\beta+26735616$	$25100928\,\beta^4-86888448\,\beta^3+124002432\,\beta^2-86888448\,\beta+25100928$
[4, 2, 2, 2]	544320	$4739712-4739712\,\beta$	$17969088\,\beta^2-34225536\,\beta+17969088$	$-38427840\,\beta^3+104441280\,\beta^2-104441280\,\beta+38427840$	$50001792\,\beta^4-172334208\,\beta^3+245686464\,\beta^2-172334208\,\beta+50001792$
[4, 3, 1, 1, 1]	604800	$4463424-4463424\,\beta$	$13910400\,\beta^2-26429760\,\beta+13910400$	$-23405760\,\beta^3+63310464\,\beta^2-63310464\,\beta+23405760$	$22377600\,\beta^4-76616064\,\beta^3+108948672\,\beta^2-76616064\,\beta+22377600$
[4, 3, 2, 1]	453600	$4005504-4005504\,\beta$	$15413760\,\beta^2-29278800\,\beta+15413760$	$-33489072\,\beta^3+90503136\,\beta^2-90503136\,\beta+33489072$	$44305920\,\beta^4-151371072\,\beta^3+215172720\,\beta^2-151371072\,\beta+44305920$

Table 3 continued

ν	$g = 0$	$g = 1/2$	$g = 1$	$g = 3/2$	$g = 2$
[4, 3, 3]	252000	$2636676-2636676\,\beta$	$12314880\,\beta^2-23335200\,\beta+12314880$	$-33536232\,\beta^3+90142848\,\beta^2-90142848\,\beta+33536232$	$58192272\,\beta^4-197053776\,\beta^3+279341280\,\beta^2-197053776\,\beta+58192272$
[4, 4, 1, 1]	352800	$3184128-3184128\,\beta$	$12538944\,\beta^2-23709888\,\beta+12538944$	$-27908352\,\beta^3+74736000\,\beta^2-74736000\,\beta+27908352$	$37850400\,\beta^4-127578240\,\beta^3+180516960\,\beta^2-127578240\,\beta+37850400$
[4, 4, 2]	235200	$2480224-2480224\,\beta$	$11678976\,\beta^2-22090752\,\beta+11678976$	$-32068928\,\beta^3+85911296\,\beta^2-85911296\,\beta+32068928$	$56101696\,\beta^4-189108096\,\beta^3+267643840\,\beta^2-189108096\,\beta+56101696$
[5, 1, 1, 1, 1, 1]	635040	$3870720-3870720\,\beta$	$9525600\,\beta^2-17992800\,\beta+9525600$	$-11793600\,\beta^3+31600800\,\beta^2-31600800\,\beta+11793600$	$7318080\,\beta^4-24796800\,\beta^3+35078400\,\beta^2-24796800\,\beta+7318080$
[5, 2, 1, 1, 1]	544320	$4082400-4082400\,\beta$	$12932640\,\beta^2-24454080\,\beta+12932640$	$-22105440\,\beta^3+59300640\,\beta^2-59300640\,\beta+22105440$	$21440160\,\beta^4-72656640\,\beta^3+102896640\,\beta^2-72656640\,\beta+21440160$
[5, 2, 2, 1]	408240	$3659040-3659040\,\beta$	$14295600\,\beta^2-27043920\,\beta+14295600$	$-31521600\,\beta^3+84580560\,\beta^2-84580560\,\beta+31521600$	$42279840\,\beta^4-143154720\,\beta^3+202792320\,\beta^2-143154720\,\beta+42279840$
[5, 3, 1, 1]	340200	$3088800-3088800\,\beta$	$12236400\,\beta^2-23096880\,\beta+12236400$	$-27388800\,\beta^3+73128960\,\beta^2-73128960\,\beta+27388800$	$37332360\,\beta^4-125392320\,\beta^3+177172920\,\beta^2-125392320\,\beta+37332360$
[5, 3, 2]	226800	$2405070-2405070\,\beta$	$11388960\,\beta^2-21508680\,\beta+11388960$	$-31440900\,\beta^3+84007320\,\beta^2-84007320\,\beta+31440900$	$55271040\,\beta^4-185713200\,\beta^3+262510920\,\beta^2-185713200\,\beta+55271040$
[5, 4, 1]	176400	$1906020-1906020\,\beta$	$9208080\,\beta^2-17332560\,\beta+9208080$	$-25965720\,\beta^3+68904000\,\beta^2-68904000\,\beta+25965720$	$46674720\,\beta^4-155190960\,\beta^3+218602080\,\beta^2-155190960\,\beta+46674720$

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[5, 5]	79380	$1005050-1005050\,\beta$	$5798500\,\beta^2-10881600\,\beta+5798500$	$-20004300\,\beta^3+52753600\,\beta^2-52753600\,\beta+20004300$	$45424140\,\beta^4-149581200\beta^3+210045500\,\beta^2-149581200\beta+45424140$
[6, 1, 1, 1, 1]	399168	$3096576-3096576\,\beta$	$10160640\,\beta^2-19051200\beta+10160640$	$-17998848\beta^3+47513088\,\beta^2-47513088\beta+17998848$	$18083520\,\beta^4-59911488\beta^3+84163968\,\beta^2-59911488\beta+18083520$
[6, 2, 1, 1]	299376	$2768256-2768256\,\beta$	$11172384\,\beta^2-20983968\,\beta+11172384$	$-25463808\beta^3+67402368\,\beta^2-67402368\beta+25463808$	$35296560\,\beta^4-117265536\beta^3+164999376\,\beta^2-117265536\beta+35296560$
[6, 2, 2]	199584	$2153016-2153016\,\beta$	$10375008\,\beta^2-19508160\beta+10375008$	$-29136144\beta^3+77252064\,\beta^2-77252064\beta+29136144$	$52049952\,\beta^4-173181408\beta^3+243897888\,\beta^2-173181408\beta+52049952$
[6, 3, 1]	166320	$1813914-1813914\,\beta$	$8845200\,\beta^2-16606296\beta+8845200$	$-25164108\beta^3+66485880\,\beta^2\text{-}66485880\,\beta+25164108$	$45594576\,\beta^4-150805152\beta^3+211986072\,\beta^2-150805152\beta+45594576$
[6, 4]	77616	$986352-986352\,\beta$	$5711544\,\beta^2-10707408\beta+5711544$	$-19772256\beta^3+52053024\,\beta^2-52053024\beta+19772256$	$45034176\,\beta^4-147999024\,\beta^3+207658416\,\beta^2-147999024\,\beta+45034176$
[7, 1, 1, 1]	216216	$2064384-2064384\,\beta$	$8615376\,\beta^2-16066512\,\beta+8615376$	$-20321280\,\beta^3+53061120\,\beta^2-53061120\beta+20321280$	$29151864\,\beta^4-95001984\beta^3+132769224\,\beta^2-95001984\beta+29151864$
[7, 2, 1]	144144	$1602342-1602342\,\beta$	$7967232\,\beta^2-14886648\,\beta+7967232$	$-23104116\beta^3+60529896\beta^2-60529896\beta+23104116$	$42623280\,\beta^4-139459824\beta^3+195250104\,\beta^2-139459824\beta+42623280$
[7, 3]	72072	$926226-926226\,\beta$	$5424090\,\beta^2-10138548\beta+5424090$	$-18980220\beta^3+49717920\,\beta^2-49717920\beta+18980220$	$43654548\,\beta^4-142589076\beta^3+199597776\,\beta^2-142589076\beta+43654548$

Table 3 continued

ν	$g=0$	$g=1/2$	$g=1$	$g=3/2$	$g=2$
[8, 1, 1]	102960	$1179648-1179648\ \beta$	$6054048\ \beta^2-11243232\beta+6054048$	$-18137088\beta^3+46964736\beta^2-46964736\beta+18137088$	$34577136\ \beta^4-111272832\beta^3+154888272\beta^2-111272832\beta+34577136$
[8, 2]	61776	$809312-809312\ \beta$	$4833568\ \beta^2\text{-}8994464\ \beta+4833568$	$-17246016\beta^3+44816128\beta^2-44816128\beta+17246016$	$40408080\ \beta^4-130622848\beta^3+182155568\ \beta^2-130622848\beta+40408080$
[9, 1]	43758	$589824-589824\ \beta$	$3629340\ \beta^2-6718140\beta+3629340$	$-13353984\beta^3+34352640\beta^2-34352640\beta+13353984$	$32280534\ \beta^4-102857040\beta^3+142720578\beta^2-102857040\beta+32280534$
[10]	16796	$262144-262144\ \beta$	$1896180\ \beta^2-3500640\ \beta+1896180$	$-8355840\beta^3+21381120\beta^2-21381120\beta+8355840$	$24771604\ \beta^4-78301080\beta^3+108368260\beta^2-78301080\beta+24771604$

Table 4 Polynomials $n_{\nu,\lambda}(\phi)$ for the WLβE

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[1]	1	ϕ	0	0	0	0	0	0	0
[1, 1]	1	ϕ	0	0	0	0	0	0	0
[2]	2	$3\phi+1$	$\phi^2+\phi$	$\phi^2+\phi$	0	0	0	0	0
[1, 1, 1]	2	2ϕ	0	0	0	0	0	0	0
[2, 1]	4	$6\phi+2$	$2\phi^2+2\phi$	$2\phi^2+2\phi$	0	0	0	0	0
[3]	5	$10\phi+6$	$6\phi^2+9\phi+1$	$6\phi^2+9\phi+2$	$2\phi^3+6\phi^2+3\phi$	$\phi^3+3\phi^2+2\phi$	0	0	0
[1, 1, 1, 1]	6	6ϕ	0	0	0	0	0	0	0
[2, 1, 1]	12	$18\phi+6$	$6\phi^2+6\phi$	$6\phi^2+6\phi$	0	0	0	0	0
[2, 2]	18	$36\phi+20$	$22\phi^2+30\phi+4$	$22\phi^2+30\phi+6$	$8\phi^3+20\phi^2+10\phi$	$4\phi^3+10\phi^2+6\phi$	0	0	0
[3, 1]	15	$30\phi+18$	$18\phi^2+27\phi+3$	$18\phi^2+27\phi+6$	$6\phi^3+18\phi^2+9\phi$	$3\phi^3+9\phi^2+6\phi$	0	0	0
[4]	14	$35\phi+29$	$30\phi^2+58\phi+12$	$30\phi^2+58\phi+22$	$20\phi^3+70\phi^2+51\phi+5$	$10\phi^3+35\phi^2+33\phi+6$	$2\phi^4+12\phi^3+17\phi^2+5\phi$	$3\phi^4+18\phi^3+28\phi^2+11\phi$	$\phi^4+6\phi^3+11\phi^2+6\phi$
[1, 1, 1, 1, 1]	24	24ϕ	0	0	0	0	0	0	0
[2, 1, 1, 1]	48	$72\phi+24$	$24\phi^2+24\phi$	$24\phi^2+24\phi$	0	0	0	0	0
[2, 2, 1]	72	$144\phi+80$	$88\phi^2+120\phi+16$	$88\phi^2+120\phi+24$	$32\phi^3+80\phi^2+40\phi$	$16\phi^3+40\phi^2+24\phi$	0	0	0
[3, 1, 1]	60	$120\phi+72$	$72\phi^2+108\phi+12$	$72\phi^2+108\phi+24$	$24\phi^3+72\phi^2+36\phi$	$12\phi^3+36\phi^2+24\phi$	0	0	0
[3, 2]	72	$180\phi+138$	$156\phi^2+276\phi+60$	$156\phi^2+276\phi+96$	$108\phi^3+336\phi^2+234\phi+24$	$54\phi^3+168\phi^2+144\phi+24$	$12\phi^4+60\phi^3+78\phi^2+24\phi$	$18\phi^4+90\phi^3+126\phi^2+48\phi$	$6\phi^4+30\phi^3+48\phi^2+24\phi$

Table 4 continued

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[4, 1]	56	$140\,\phi+116$	$120\,\phi^2+232\,\phi+48$	$120\,\phi^2+232\,\phi+88$	$80\,\phi^3+280\,\phi^2+204\,\phi+20$	$40\,\phi^3+140\,\phi^2+132\,\phi+24$	$8\,\phi^4+48\,\phi^3+68\,\phi^2+20\,\phi$	$12\,\phi^4+72\,\phi^3+112\,\phi^2+44\,\phi$	$4\,\phi^4+24\,\phi^3+44\,\phi^2+24\,\phi$
[5]	42	$126\,\phi+130$	$140\,\phi^2+325\,\phi+95$	$140\,\phi^2+325\,\phi+165$	$140\,\phi^3+560\,\phi^2+520\,\phi+90$	$70\,\phi^3+280\,\phi^2+330\,\phi+100$	$30\,\phi^4+190\,\phi^3+315\,\phi^2+135\,\phi+10$	$45\,\phi^4+285\,\phi^3+515\,\phi^2+285\,\phi+26$	$15\,\phi^4+95\,\phi^3+200\,\phi^2+150\,\phi+24$

Table 5 Polynomials $n_{\nu,\lambda}(\phi)$ for the WLβE evaluated at $\phi = 1$

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[1]	1	1	0	0	0	0	0	0	0
[1, 1]	1	1	0	0	0	0	0	0	0
[2]	2	4	2	2	0	0	0	0	0
[1, 1, 1]	2	2	0	0	0	0	0	0	0
[2, 1]	4	8	4	4	0	0	0	0	0
[3]	5	16	16	17	11	6	0	0	0
[1, 1, 1, 1]	6	6	0	0	0	0	0	0	0
[2, 1, 1]	12	24	12	12	0	0	0	0	0
[2, 2]	18	56	56	58	38	20	0	0	0
[3, 1]	15	48	48	51	33	18	0	0	0
[4]	14	64	100	110	146	84	36	60	24
[1, 1, 1, 1, 1]	24	24	0	0	0	0	0	0	0
[2, 1, 1, 1]	48	96	48	48	0	0	0	0	0
[2, 2, 1]	72	224	224	232	152	80	0	0	0
[3, 1, 1]	60	192	192	204	132	72	0	0	0
[3, 2]	72	318	492	528	702	390	174	282	108
[4, 1]	56	256	400	440	584	336	144	240	96
[5]	42	256	560	630	1310	780	680	1156	484
[1, 1, 1, 1, 1, 1]	120	120	0	0	0	0	0	0	0
[2, 1, 1, 1, 1]	240	480	240	240	0	0	0	0	0
[2, 2, 1, 1]	360	1120	1120	1160	760	400	0	0	0
[2, 2, 2]	432	1864	2864	3024	4024	2184	1000	1592	592
[3, 1, 1, 1]	300	960	960	1020	660	360	0	0	0

Table 5 continued

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[3, 2, 1]	360	1590	2460	2640	3510	1950	870	1410	540
[3, 3]	300	1746	3750	4098	8466	4842	4368	7206	2874
[4, 1, 1]	280	1280	2000	2200	2920	1680	720	1200	480
[4, 2]	280	1648	3552	3912	8088	4672	4176	6944	2800
[5, 1]	210	1280	2800	3150	6550	3900	3400	5780	2420
[6]	132	1024	2940	3360	9760	5952	8020	13832	5980
[1, 1, 1, 1, 1, 1, 1]	720	720	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1]	1440	2880	1440	1440	0	0	0	0	0
[2, 2, 1, 1, 1]	2160	6720	6720	6960	4560	2400	0	0	0
[2, 2, 2, 1]	2592	11184	17184	18144	24144	13104	6000	9552	3552
[3, 1, 1, 1, 1]	1800	5760	5760	6120	3960	2160	0	0	0
[3, 2, 1, 1]	2160	9540	14760	15840	21060	11700	5220	8460	3240
[3, 2, 2]	2160	12336	26256	28392	58416	32880	30024	48984	19128
[3, 3, 1]	1800	10476	22500	24588	50796	29052	26208	43236	17244
[4, 1, 1, 1]	1680	7680	12000	13200	17520	10080	4320	7200	2880
[4, 2, 1]	1680	9888	21312	23472	48528	28032	25056	41664	16800
[4, 3]	1200	8856	24840	27600	79176	46440	64152	107616	44400
[5, 1, 1]	1260	7680	16800	18900	39300	23400	20400	34680	14520
[5, 2]	1080	8100	22860	25660	73800	43860	60000	101620	42540
[6, 1]	792	6144	17640	20160	58560	35712	48120	82992	35880
[7]	429	4096	14784	17094	64960	40320	74935	130655	57841
[1, 1, 1, 1, 1, 1, 1, 1]	5040	5040	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1]	10080	20160	10080	10080	0	0	0	0	0

Table 5 continued

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[2, 2, 1, 1, 1, 1]	15120	47040	47040	48720	31920	16800	0	0	0
[2, 2, 2, 1, 1]	18144	78288	120288	127008	169008	91728	42000	66864	24864
[2, 2, 2, 2]	18144	101568	214272	229008	469440	259776	240432	387504	148080
[3, 1, 1, 1, 1, 1]	12600	40320	40320	42840	27720	15120	0	0	0
[3, 2, 1, 1, 1]	15120	66780	103320	110880	147420	81900	36540	59220	22680
[3, 2, 2, 1]	15120	86352	183792	198744	408912	230160	210168	342888	133896
[3, 3, 1, 1]	12600	73332	157500	172116	355572	203364	183456	302652	120708
[3, 3, 2]	10800	77688	214992	235728	670824	385920	538920	891864	358992
[4, 1, 1, 1, 1]	11760	53760	84000	92400	122640	70560	30240	50400	20160
[4, 2, 1, 1]	11760	69216	149184	164304	339696	196224	175392	291648	117600
[4, 2, 2]	10080	73248	203520	224560	640176	371328	515472	858160	348624
[4, 3, 1]	8400	61992	173880	193200	554232	325080	449064	753312	310800
[4, 4]	4900	44224	155152	174152	650384	387616	735260	1246220	525700
[5, 1, 1, 1]	8820	53760	117600	132300	275100	163800	142800	242760	101640
[5, 2, 1]	7560	56700	160020	179620	516600	307020	420000	711340	297780
[5, 3]	4725	42900	150840	169950	635280	380400	719175	1223055	518505
[6, 1, 1]	5544	43008	123480	141120	409920	249984	336840	580944	251160
[6, 2]	4158	38448	136272	155172	582480	353664	662970	1138458	490230
[7, 1]	3003	28672	103488	119658	454720	282240	524545	914585	404887
[8]	1430	16384	72072	84084	400384	251904	605770	1065218	480238
[1, 1, 1, 1, 1, 1, 1, 1, 1]	40320	40320	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1]	80640	161280	80640	80640	0	0	0	0	0
[2, 2, 1, 1, 1, 1, 1]	120960	376320	376320	389760	255360	134400	0	0	0

Table 5 continued

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[2, 2, 2, 1, 1, 1]	145152	626304	962304	1016064	1352064	733824	336000	534912	198912
[2, 2, 2, 2, 1]	145152	812544	1714176	1832064	3755520	2078208	1923456	3100032	1184640
[3, 1, 1, 1, 1, 1, 1]	100800	322560	322560	342720	221760	120960	0	0	0
[3, 2, 1, 1, 1, 1]	120960	534240	826560	887040	1179360	655200	292320	473760	181440
[3, 2, 2, 1, 1]	120960	690816	1470336	1589952	3271296	1841280	1681344	2743104	1071168
[3, 2, 2, 2]	103680	733440	2010816	2186784	6186720	3515472	4940256	8105856	3209952
[3, 3, 1, 1, 1]	100800	586656	1260000	1376928	2844576	1626912	1467648	2421216	965664
[3, 3, 2, 1]	86400	621504	1719936	1885824	5366592	3087360	4311360	7134912	2871936
[3, 3, 3]	54000	472608	1628928	1802016	6648912	3879576	7418016	12391128	5088024
[4, 1, 1, 1, 1, 1]	94080	430080	672000	739200	981120	564480	241920	403200	161280
[4, 2, 1, 1, 1]	94080	553728	1193472	1314432	2717568	1569792	1403136	2333184	940800
[4, 2, 2, 1]	80640	585984	1628160	1796480	5121408	2970624	4123776	6865280	2788992
[4, 3, 1, 1]	67200	495936	1391040	1545600	4433856	2600640	3592512	6026496	2486400
[4, 3, 2]	50400	445056	1540560	1712640	6334896	3721008	7089072	11896128	4922880
[4, 4, 1]	39200	353792	1241216	1393216	5203072	3100928	5882080	9969760	4205600
[5, 1, 1, 1, 1]	70560	430080	940800	1058400	2200800	1310400	1142400	1942080	813120
[5, 2, 1, 1]	60480	453600	1280160	1436960	4132800	2456160	3360000	5690720	2382240
[5, 2, 2]	45360	406560	1416480	1588400	5895440	3502400	6626400	11208320	4697760
[5, 3, 1]	37800	343200	1206720	1359600	5082240	3043200	5753400	9784440	4148040
[5, 4]	19600	211780	902720	1023120	4770920	2885080	7045680	12057360	5186080
[6, 1, 1, 1]	44352	344064	987840	1128960	3279360	1999872	2694720	4647552	2009280
[6, 2, 1]	33264	307584	1090176	1241376	4659840	2829312	5303760	9107664	3921840
[6, 3]	18480	201546	862344	982800	4591308	2796012	6797880	11690064	5066064

Table 5 continued

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[7, 1, 1]	24024	229376	827904	957264	3637760	2257920	4196360	7316680	3239096
[7, 2]	16016	178038	768824	885248	4158420	2567124	6198920	10759616	4735920
[8, 1]	11440	131072	576576	672672	3203072	2015232	4846160	8521744	3841904
[9]	4862	65536	343200	403260	2333184	1483776	4429282	7841834	3586726
[1, 1, 1, 1, 1, 1, 1, 1, 1, 1]	362880	362880	0	0	0	0	0	0	0
[2, 1, 1, 1, 1, 1, 1, 1, 1]	725760	1451520	725760	725760	0	0	0	0	0
[2, 2, 1, 1, 1, 1, 1, 1]	1088640	3386880	3386880	3507840	2298240	1209600	0	0	0
[2, 2, 2, 1, 1, 1, 1]	1306368	5636736	8660736	9144576	12168576	6604416	3024000	4814208	1790208
[2, 2, 2, 2, 1, 1]	1306368	7312896	15427584	16488576	33799680	18703872	17311104	27900288	10661760
[2, 2, 2, 2, 2]	1119744	7782912	21139968	22783488	64097280	35944320	50880000	82714368	32205312
[3, 1, 1, 1, 1, 1, 1, 1]	907200	2903040	2903040	3084480	1995840	1088640	0	0	0
[3, 2, 1, 1, 1, 1, 1]	1088640	4808160	7439040	7983360	10614240	5896800	2630880	4263840	1632960
[3, 2, 2, 1, 1, 1]	1088640	6217344	13233024	14309568	29441664	16571520	15132096	24687936	9640512
[3, 2, 2, 2, 1]	933120	6600960	18097344	19681056	55680480	31639248	44462304	72952704	28889568
[3, 3, 1, 1, 1, 1]	907200	5279904	11340000	12392352	25601184	14642208	13208832	21790944	8690976
[3, 3, 2, 1, 1]	777600	5593536	15479424	16972416	48299328	27786240	38802240	64214208	25847424
[3, 3, 2, 2]	583200	5031360	17184096	18894528	69251904	40031568	76726368	127336752	51631920
[3, 3, 3, 1]	486000	4253472	14660352	16218144	59840208	34916184	66762144	111520152	45792216
[4, 1, 1, 1, 1, 1, 1]	846720	3870720	6048000	6652800	8830080	5080320	2177280	3628800	1451520
[4, 2, 1, 1, 1, 1]	846720	4983552	10741248	11829888	24458112	14128128	12628224	20998656	8467200
[4, 2, 2, 1, 1]	725760	5273856	14653440	16168320	46092672	26735616	37113984	61787520	25100928
[4, 2, 2, 2]	544320	4739712	16256448	17969088	66013440	38427840	73352256	122332416	50001792
[4, 3, 1, 1, 1]	604800	4463424	12519360	13910400	39904704	23405760	32332608	54238464	22377600
[4, 3, 2, 1]	453600	4005504	13865040	15413760	57014064	33489072	63801648	107065152	44305920

Table 5 continued

ν	λ []	[1]	[1,1]	[2]	[2,1]	[3]	[2,2]	[3,1]	[4]
[4, 3, 3]	252000	2636676	11020320	12314880	56606616	33536232	82287504	138861504	58192272
[4, 4, 1, 1]	352800	3184128	11170944	12538944	46827648	27908352	52938720	89727840	37850400
[4, 4, 2]	235200	2480224	10411776	11678976	53842368	32068928	78535744	133006400	56101696
[5, 1, 1, 1, 1, 1]	635040	3870720	8467200	9525600	19807200	11793600	10281600	17478720	7318080
[5, 2, 1, 1, 1]	544320	4082400	11521440	12932640	37195200	22105440	30240000	51216480	21440160
[5, 2, 2, 1]	408240	3659040	12748320	14295600	53058960	31521600	59637600	100874880	42279840
[5, 3, 1, 1]	340200	3088800	10860480	12236400	45740160	27388800	51780600	88059960	37332360
[5, 3, 2]	226800	2405070	10119720	11388960	52566420	31440900	76797720	130442160	55271040
[5, 4, 1]	176400	1906020	8124480	9208080	42938280	25965720	63411120	108516240	46674720
[5, 5]	79380	1005050	5083100	5798500	32749300	20004300	60464300	104157060	45424140
[6, 1, 1, 1, 1]	399168	3096576	8890560	10160640	29514240	17998848	24252480	41827968	18083520
[6, 2, 1, 1]	299376	2768256	9811584	11172384	41938560	25463808	47733840	81968976	35296560
[6, 2, 2]	199584	2153016	9133152	10375008	48115920	29136144	70716480	121131456	52049952
[6, 3, 1]	166320	1813914	7761096	8845200	41321772	25164108	61180920	105210576	45594576
[6, 4]	77616	986352	4995864	5711544	32280768	19772256	59659392	102964848	45034176
[7, 1, 1, 1]	216216	2064384	7451136	8615376	32739840	20321280	37767240	65850120	29151864
[7, 2, 1]	144144	1602342	6919416	7967232	37425780	23104116	55790280	96836544	42623280
[7, 3]	72072	926226	4714458	5424090	30737700	18980220	57008700	98934528	43654548
[8, 1, 1]	102960	1179648	5189184	6054048	28827648	18137088	43615440	76695696	34577136
[8, 2]	61776	809312	4160896	4833568	27570112	17246016	51532720	90214768	40408080
[9, 1]	43758	589824	3088800	3629340	20998656	13353984	39863538	70576506	32280534
[10]	16796	262144	1604460	1896180	13025280	8355840	30067180	53529476	24771604

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