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Unique solutions to hyperbolic conservation laws with a strictly convex entropy

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Abstract

Consider a strictly hyperbolic $n \times n$ system of conservation laws, where each characteristic field is either genuinely nonlinear or linearly degenerate. In this standard setting, it is well known that there exists a Lipschitz semigroup of weak solutions, defined on a domain of functions with small total variation. If the system admits a strictly convex entropy, we give a short proof that every entropy weak solution taking values within the domain of the semigroup coincides with a semigroup trajectory. The result shows that the assumptions of "Tame Variation" or "Tame Oscillation", previously used to achieve uniqueness, can be removed in the presence of a strictly convex entropy.

Keywords: Systems of conservation laws; Uniqueness of entropy solutions

1. Introduction

We consider the Cauchy problem for a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0, \tag{1.1}$$

$$u(0, x) = \bar{u}(x). \tag{1.2}$$

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As usual, $f : \Omega \to \mathbb{R}^n$ is the flux, defined on some open set $\Omega \subset \mathbb{R}^n$. We assume that each characteristic family is either genuinely nonlinear or linearly degenerate. In this setting, it is well known [8,10,11,16,20,23] that there exists a Lipschitz continuous semigroup $S : \mathcal{D} \times [0, +\infty[\mapsto \mathcal{D}$ of entropy weak solutions [8, Section 7.7], defined on a domain

$$\mathcal{D} = cl\,\Big\{u \in \mathbf{L}^1(\mathbb{R};\, \mathbb{R}^n);\, u \text{ is piecewise constant and } \mathbf{V}(u) + C_0\mathbf{Q}(u) < \delta_0\Big\} \tag{1.3}$$

containing all functions with sufficiently small total variation. Here $\mathbf{V}(u)$ and $\mathbf{Q}(u)$ are respectively the *total strength of waves* and the *interaction potential* of u defined in [8, (7.99)] and $C_0,\ \delta_0$ are two suitable positive constants. The trajectories of this semigroup are the unique limits of front tracking approximations, and also of Glimm approximations [7] and of vanishing viscosity approximations [6]. We recall that the semigroup is globally Lipschitz continuous w.r.t. the $\mathbf{L}^1$ distance. Namely, there exists a constant L such that

$$\big\|S_t\bar{u} - S_s\bar{v}\big\|_{\mathbf{L}^1} \;\leq\; L\Big(|t-s| + \|\bar{u} - \bar{v}\|_{\mathbf{L}^1}\Big) \qquad \text{for all } s, t \geq 0, \quad \bar{u}, \bar{v} \in \mathcal{D}. \tag{1.4}$$

Given any weak solution $u = u(t, x)$ of (1.1)-(1.2), various conditions have been derived in [12,14,15] which guarantee the identity

$$u(t) \;=\; S_t\bar{u} \qquad \text{for all } t \geq 0. \tag{1.5}$$

Since the semigroup S is unique, the identity (1.5) yields the uniqueness of solutions to the Cauchy problem (1.1)-(1.2). In addition to the standard assumptions, earlier results required some additional regularity conditions, such as “Tame Variation” or “Tame Oscillation”, controlling the behavior of the solution near a point where the variation is locally small.

Aim of the present note is to show that, if the $n \times n$ system (1.1) is endowed with a strictly convex entropy $\eta(\cdot)$, then every entropy-weak solution $t \mapsto u(t)$ taking values within the domain $\mathcal{D}$ of the semigroup satisfies (1.5). In other words, uniqueness is guaranteed without any further regularity assumption.

As in [12,14,15], the proof relies on the elementary error estimate

$$\big\|u(t) - S_t\bar{u}\big\|_{\mathbf{L}^1} \;\leq\; L \cdot \int_0^t \liminf_{h \to 0+} \frac{\big\|u(\tau + h) - S_h u(\tau)\big\|_{\mathbf{L}^1}}{h}\, d\tau\,. \tag{1.6}$$

Assuming that the system is endowed with a strictly convex entropy, we will prove that the integrand is zero for a.e. time $\tau \geq 0$. Following an argument introduced in [7], this is achieved by two estimates:

(i) In a neighborhood of a point (τ, y) where $u(\tau, \cdot)$ has a large jump, the weak solution u is compared with the solution to a Riemann problem.
(ii) In a region where the total variation is small, the weak solution u is compared with the solution to a linear system with constant coefficients.

The main difference is that here we estimate the lim-inf in (1.6) only at times τ which are Lebesgue points for a countable family of total variation functions $W^{\xi,\zeta}(\cdot)$, defined at (3.5).

To precisely state the result, we begin by collecting the main assumptions.

(A1) **(Conservation equations)** *The function $u = u(t, x)$ is a weak solution of the Cauchy problem (1.1)-(1.2) taking values within the domain of the semigroup.*
More precisely, $u : [0, T] \mapsto \mathcal{D}$ is continuous w.r.t. the $\mathbf{L}^1$ distance. The identity $u(0, \cdot) = \bar{u}$ holds in $\mathbf{L}^1$, and moreover

$$\iint \Big(u\varphi_t + f(u)\varphi_x \Big)\, dxdt = 0 \tag{1.7}$$

for every $\mathcal{C}^1$ function φ with compact support contained inside the open strip $]0, T[\, \times \mathbb{R}$.

Regarding the entropy conditions, we assume that the system (1.1) admits a $\mathcal{C}^2$ entropy function $\eta : \Omega \mapsto \mathbb{R}$ with entropy flux q, so that the equality $\nabla q(\omega) = \nabla\eta(\omega) Df(\omega)$ holds for all $\omega \in \Omega$. We also assume that the entropy η satisfies the strict convexity condition

$$\eta(\omega) \;\geq\; \eta(\overline{\omega}) + \nabla\eta(\overline{\omega}) \cdot (\omega - \overline{\omega}) + c_0|\omega - \overline{\omega}|^2, \tag{1.8}$$

for some $c_0 > 0$ and every couple of states $\omega, \ \overline{\omega} \in \Omega$. As usual, we say that a weak solution u is entropy-admissible if it satisfies:

(A2) **(Entropy admissibility condition)** *For every $\mathcal{C}^1$ function $\varphi \geq 0$ with compact support contained inside the open strip $]0, T[\, \times \mathbb{R}$, one has*

$$\iint \Big(\eta(u)\varphi_t + q(u)\varphi_x \Big)\, dxdt \;\geq\; 0. \tag{1.9}$$

Our result can be simply stated as:

Theorem 1.1. *Let (1.1) be a strictly hyperbolic $n \times n$ system, where each characteristic field is either genuinely nonlinear or linearly degenerate, and which admits a strictly convex entropy $\eta(\cdot)$ as in (1.8). Then every entropy-weak solution $u : [0, T] \mapsto \mathcal{D}$, taking values within the domain of the semigroup, coincides with a semigroup trajectory.*

The theorem will be proved in Section 3. We remark that, restricted to a class of 2×2 systems, a more elaborate proof of this result was recently given in [19].

In our view, the main interest in the above uniqueness theorem is that, combined with a compactness argument, it yields a uniform convergence rate for a very wide class of approximation algorithms. This will be better explained in the concluding remarks contained in Section 4.

2. Preliminary lemmas

Let M be an upper bound on the total variation of all functions in the domain $\mathcal{D}$ of the semigroup:

$$\text{Tot.Var.}\{u; \mathbb{R}\} \leq M, \qquad \text{for all } u \in \mathcal{D}. \tag{2.1}$$

Since by assumption our solution $u(t, \cdot) \in \mathcal{D}$, for sake of definiteness we shall assume that it is right continuous, namely $u(t, x) = \lim_{y \to x+} u(t, y)$. By [20, Theorem 4.3.1], we have the Lipschitz bound

$$\left\|u(t_2,\cdot)-u(t_1,\cdot)\right\|_{\mathbf{L}^1(\mathbb{R})} \leq C_M\,(t_2-t_1) \qquad \text{for all } 0\leq t_1\leq t_2\,, \tag{2.2}$$

for some constant $C_M > 0$ depending only on M and on the flux f.

We begin by reviewing the well known fact that the entropy has finite propagation speed. The proof relies on the notion of relative entropy, see [24] for an overview of the subject.

Lemma 2.1. *Let $u = u(t,x)$ be a function satisfying* **(A1)** *and* **(A2)**. *Then there exists two constants $\widehat{C}$, $\hat{\lambda} > 0$ such that the following holds. For any constant state $u^* \in \Omega$, any $a < b$, and any $0 \leq \tau < \tau'$ with $2\hat{\lambda}(\tau'-\tau) < b-a$, one has*

$$\int_{a+\hat{\lambda}(\tau'-\tau)}^{b-\hat{\lambda}(\tau'-\tau)} \left|u(\tau',x)-u^*\right|^2\,dx \;\leq\; \widehat{C}\int_a^b \left|u(\tau,x)-u^*\right|^2\,dx. \tag{2.3}$$

Proof. Given the constant state $u^* \in \Omega$, for all $\omega \in \Omega$ define the relative entropy $\eta\,(\omega\,|\,u^*)$ and the corresponding entropy flux $q\,(\omega\,|\,u^*)$ as

$$\begin{aligned} \eta\,(\omega\,|\,u^*) &= \eta\,(\omega)-\eta\,(u^*)-\nabla\eta\,(u^*)\,(\omega-u^*)\,,\\ q\,(\omega\,|\,u^*) &= q\,(\omega)-q\,(u^*)-\nabla\eta\,(u^*)\big(f\,(\omega)-f\,(u^*)\big). \end{aligned} \tag{2.4}$$

The equations (1.7) and (1.9) yield

$$\eta\left(u\,|\,u^*\right)_t + q\left(u\,|\,u^*\right)_x \;\leq\; 0, \tag{2.5}$$

while (1.8) implies

$$\eta\left(\omega\,|\,u^*\right) \;\geq\; c_0\left|\omega-u^*\right|^2, \qquad \text{for all } \omega, u^*\in\Omega. \tag{2.6}$$

By the $\mathcal{C}^2$ regularity of the functions η, q, there exists a constant C' such that

$$\eta\left(\omega\,|\,u^*\right) \;\leq\; C'\left|\omega-u^*\right|^2, \qquad \left|q\left(\omega\,|\,u^*\right)\right| \;\leq\; C'\left|\omega-u^*\right|^2, \qquad \text{for all } \omega, u^*\in\Omega. \tag{2.7}$$

In view of (2.6), there exists a sufficiently large constant $\hat{\lambda} > 0$ such that

$$-\hat{\lambda}\eta\left(\omega\,|\,u^*\right) \pm q\left(\omega\,|\,u^*\right) \;\leq\; 0, \qquad \text{for all } \omega, u^*\in\Omega. \tag{2.8}$$

Using (2.5), in connection with test functions that approximate the characteristic function of the trapezoid

$$\Gamma \;=\; \Big\{(t,x)\,;\;\tau<t<\tau',\;\; a+\hat{\lambda}(t-\tau)<x<b-\hat{\lambda}(t-\tau)\Big\},$$

and recalling (2.8), we obtain

$$\int_{a+\hat\lambda(\tau'-\tau)}^{b-\hat\lambda(\tau'-\tau)} \eta\left(u\,|\,u^*\right)\left(\tau',x\right)\,dx \;\le\; \int_a^b \eta\left(u\,|\,u^*\right)(\tau,x)\,dx$$
$$+\int_\tau^{\tau'}\Big(-\hat\lambda\eta\left(u\,|\,u^*\right)+q\left(u\,|\,u^*\right)\Big)\Big(t,a+\hat\lambda\,(t-\tau)\Big)\,dt$$
$$+\int_\tau^{\tau'}\Big(-\hat\lambda\eta\left(u\,|\,u^*\right)-q\left(u\,|\,u^*\right)\Big)\Big(t,b-\hat\lambda\,(t-\tau)\Big)\,dt$$
$$\le\;\int_a^b \eta\left(u\,|\,u^*\right)(\tau,x)\,dx.$$

Together with (2.6)-(2.7), this proves the lemma. □

Throughout the following, without loss of generality we shall always assume $\hat\lambda = 1$. We observe that this can always be achieved by a suitable rescaling of the time variable:

$$\tilde t \;=\; \kappa t.$$

Similarly, to simplify notation, we also assume that all wave speeds lie in the interval $[-1, 1]$.

Given any $\tau \ge 0$ and any bounded interval $]a,b[$ with $-\infty \le a < b \le +\infty$, we consider the open intervals

$$J(t)\;=\;\big]a+(t-\tau)\,,\;b-(t-\tau)\big[\,,\qquad\qquad \tau\le t<\tau+\frac{b-a}{2}\,. \tag{2.9}$$

Toward the proof of Theorem 1.1, in order to replace the "Tame Variation" condition, the main tool is provided by the following elementary lemma.

Lemma 2.2. *In the setting of Theorem 1.1, for some constant $C > 0$ the following holds. Let $u = u(t,x)$ be any entropy weak solution to (1.1). Then*

$$\int_{J(t)}\big|u(t,x)-u(\tau,x)\big|\,dx\;\le\;C(t-\tau)\cdot\text{Tot.Var.}\big\{u(\tau,\cdot)\,;\;]a,b[\big\}. \tag{2.10}$$

Proof. 1. We first consider the case where $-\infty < a < b < +\infty$. For notational simplicity, w.l.o.g. we assume that $\tau = 0$. Given a time $0 < t < \frac{b-a}{2}$, as shown in Fig. 1 we define the points x_k, the values u_k and the integer $N \ge 1$ such that

$$x_k\;=\;a+kt,\qquad\qquad u_k=u(0,x_k),\qquad\qquad x_N\le b<x_{N+1}\,. \tag{2.11}$$

For $k = 1, 2, \ldots, N-2$, we apply Lemma 2.1 with $u^* = u_k$ on the trapezoids

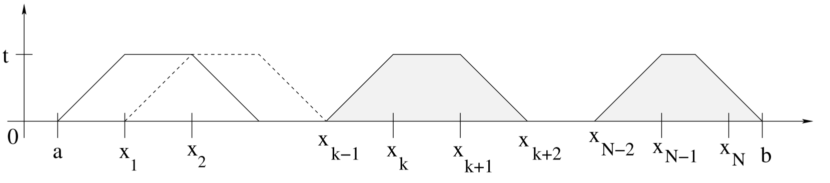

Fig. 1. The covering of the interval $[a, b]$ used in the proof of Lemma 2.2.

$$\Gamma_k \doteq \Big\{(s,x)\,;\;\; s\in[0,t],\;\; x_{k-1}+s<x<x_{k+2}-s\Big\}.$$

Then we apply the same lemma with $u^* = u_{N-1}$ on the domain

$$\widetilde{\Gamma}_N \doteq \Big\{(s,x)\,;\;\; s\in[0,t],\;\; x_{N-2}+t<x<b-t\Big\}.$$

This yields the estimates

$$\begin{aligned}\int_{x_k}^{x_{k+1}} \big|u(t,x)-u_k\big|^2\,dx \;&\le\; \widehat{C}\int_{x_{k-1}}^{x_{k+2}} \big|u(0,x)-u_k\big|^2\,dx\\ &\le\; \widehat{C}\cdot 3t\cdot\Big(\text{Tot.Var.}\{u(0,\cdot)\,;\;]x_{k-1},x_{k+2}[\}\Big)^2,\end{aligned} \tag{2.12}$$

$$\int_{x_{N-1}}^{b-t} \big|u(t,x)-u_{N-1}\big|^2\,dx \;\le\; \widehat{C}\cdot 3t\cdot\Big(\text{Tot.Var.}\{u(0,\cdot)\,;\;]x_{N-2},b[\}\Big)^2. \tag{2.13}$$

2. Define the piecewise constant approximation $\bar u : [a+t, b-t[\,\mapsto \Omega$ by setting

$$\bar u(x) = u_k \qquad \text{if } x\in\big[x_k, x_{k+1}\big[, \qquad\qquad k=1,\ldots,N-1. \tag{2.14}$$

Using Cauchy's inequality and the bounds (2.12)-(2.13), we obtain

$$\begin{aligned}\int_{a+t}^{b-t} \big|u(t,x)-\bar u(x)\big|\,dx \;&=\; \sum_{k=1}^{N-2}\int_{x_k}^{x_{k+1}} \big|u(t,x)-u_k\big|\,dx + \int_{x_{N-1}}^{b-t}\big|u(t,x)-u_{N-1}\big|\,dx\\ &\le\; \sum_{k=1}^{N-2}\sqrt{t}\left(\int_{x_k}^{x_{k+1}}\big|u(t,x)-u_k\big|^2dx\right)^{1/2} + \sqrt{t}\left(\int_{x_{N-1}}^{b-t}\big|u(t,x)-u_{N-1}\big|^2dx\right)^{1/2}\\ &\le\; \sqrt{t}\cdot\sqrt{\widehat{C}\cdot 3t}\cdot\left(\sum_{k=1}^{N-2}\text{Tot.Var.}\{u(0,\cdot)\,;\;]x_{k-1},x_{k+2}[\} + \text{Tot.Var.}\{u(0,\cdot)\,;\;]x_{N-2},b[\}\right).\end{aligned} \tag{2.15}$$

Observing that every point $x\in[a,b]$ is contained in at most three open intervals $]x_{k-1}, x_{k+2}[$, from (2.15) we conclude

$$\int_{a+t}^{b-t} \big|u(t,x)-\bar u(x)\big|\,dx \;\le\; \sqrt{3\widehat C}\cdot 3t\cdot \text{Tot.Var.}\big\{u(0,\cdot)\,;\;\,]a,b[\big\}. \tag{2.16}$$

3. Next, we compute

$$\begin{aligned}\int_{a+t}^{b-t} \big|u(0,x)-\bar u(x)\big|\,dx \;&=\; \sum_{k=1}^{N-2}\int_{x_k}^{x_{k+1}} \big|u(0,x)-u_k\big|\,dx + \int_{x_{N-1}}^{b-t}\big|u(0,x)-u_{N-1}\big|\,dx\\ &\le \sum_{k=1}^{N-2} t\cdot \text{Tot.Var.}\big\{u(0,\cdot)\,;\;]x_k,x_{k+1}[\,\big\} + t\cdot \text{Tot.Var.}\big\{u(0,\cdot)\,;\;]x_{N-1},b-t[\,\big\}\\ &\le\; t\cdot \text{Tot.Var.}\big\{u(0,\cdot)\,;\;\,]a,b[\big\}.\end{aligned} \tag{2.17}$$

Combining (2.16) with (2.17) we obtain a proof of the lemma for finite a and b.

Letting $a\to-\infty$ or $b\to+\infty$ we see that the same conclusion remains valid also for unbounded intervals, such as $]-\infty,b[$ or $]a,+\infty[$. □

3. Proof of the theorem

We are now ready to give a proof of Theorem 1.1, in several steps.

1. By the structure theorem for BV functions [1,22], [8, Theorem 2.6], there is a null set of times $\mathcal N\subset[0,T]$ such that the following holds.

Every point $(\tau,\xi)\in[0,T]\times\mathbb R$ with $\tau\notin\mathcal N$ has the following property: there exist states $u^-,u^+\in\mathbb R^n$ and a speed $\lambda\in\mathbb R$ such that, calling

$$U(t,x)\;\doteq\;\begin{cases} u^- & \text{if}\quad (x-\xi)<\lambda(t-\tau)\,,\\ u^+ & \text{if}\quad (x-\xi)>\lambda(t-\tau)\,,\end{cases} \tag{3.1}$$

there holds

$$\lim_{r\to0+}\frac{1}{r^2}\int_{-r}^{r}\int_{-r}^{r}\Big|u(\tau+t,\;\xi+x)-U(\tau+t,\xi+x)\Big|\,dxdt \;=\;0. \tag{3.2}$$

When (3.1), (3.2) hold with $u^-\neq u^+$ we say that (τ,ξ) is a point of approximate jump of the function u. If instead $u^-=u^+$ we say that u is approximately continuous at the point (τ,ξ) [8, Definition 2.1]. The conservation equations (1.7) imply that the piecewise constant function U must be a weak solution to the system of conservation laws (see [8, Theorem 4.1]), satisfying the Rankine-Hugoniot equations:

$$f(u^+)-f(u^-)\;=\;\lambda(u^+-u^-). \tag{3.3}$$

Moreover, the entropy condition (1.9) implies

$$q(u^+)-q(u^-)\;\le\;\lambda\big(\eta(u^+)-\eta(u^-)\big). \tag{3.4}$$

Next, we observe that, for every couple of rational points $\xi, \zeta \in \mathbb{Q}$, the scalar function

$$W^{\xi,\zeta}(t) \doteq \begin{cases} \text{Tot.Var.}\Big\{u(t)\,;\ ]\xi+t\,,\ \zeta-t[\Big\} & \text{if} \quad \xi+t<\zeta-t\,, \\ 0 & \text{otherwise,} \end{cases} \tag{3.5}$$

is bounded and measurable (indeed, it is lower semicontinuous). Therefore a.e. $t \in [0,T]$ is a Lebesgue point. We denote by $\mathcal{N}' \subset [0,T]$ the set of all times t which are NOT Lebesgue for at least one of the countably many functions $W^{\xi,\zeta}$. Of course, $\mathcal{N}'$ has zero Lebesgue measure.

In view of (1.6), we will prove the theorem by establishing the following claim.

(C) *For every $\tau \in [0,T] \setminus (\mathcal{N} \cup \mathcal{N}')$ and $\varepsilon > 0$, one has*

$$\limsup_{h\to 0+} \frac{1}{h} \Big\| u(\tau+h) - S_h u(\tau) \Big\|_{\mathbf{L}^1} \leq \varepsilon. \tag{3.6}$$

2. Assume $\tau \notin \mathcal{N} \cup \mathcal{N}'$. Since $u(\tau,\cdot)$ has bounded variation, we define points

$$-\infty = y_{-1} < y_0 < y_1 < \cdots < y_N < y_{N+1} = +\infty,$$

$$y_{k+1} = \sup\Big\{x > y_k\,;\quad \text{Tot.Var.}\Big\{u(\tau,\cdot)\,;\ \ ]y_k, x[\Big\} \leq \varepsilon\Big\}.$$

Since $u(\tau,\cdot)$ is right continuous, we have

$$\begin{cases} \text{Tot.Var.}\Big\{u(\tau,\cdot)\,;\ ]y_{k-1},y_k[\Big\} \leq \varepsilon, & \text{for } k=0,\ldots,N+1, \\ \text{Tot.Var.}\Big\{u(\tau,\cdot)\,;\ ]y_{k-1},y_k]\Big\} \geq \varepsilon, & \text{for } k=0,\ldots,N, \\ N \leq \frac{M}{\varepsilon}, \end{cases} \tag{3.7}$$

where M is an upper bound for the total variation of all functions $\bar{u} \in \mathcal{D}$, as in (2.1).

Then we choose points y'_k, y''_k such that

$$-\infty < y_0 < y''_0 \leq y'_1 < y_1 < y''_1 \leq y'_2 < y_2 < y''_2 < \cdots \leq y'_N < y_N < +\infty$$

$$\begin{cases} \text{Tot.Var.}\Big\{u(\tau,\cdot)\,;\ ]y'_k,y_k[\Big\} \leq \varepsilon^2, & k=1,\ldots N, \\ \text{Tot.Var.}\Big\{u(\tau,\cdot)\,;\ ]y_k,y''_k[\Big\} \leq \varepsilon^2, & k=0,\ldots N-1, \\ \text{all values } y'_k+\tau,\ y''_k-\tau \text{ are rational.} \end{cases} \tag{3.8}$$

3. For any given $y \in \mathbb{R}$, we denote by $U^\sharp = U^\sharp_{(u,\tau,y)}(t,x)$ the solution, for $t \geq \tau$, to the Riemann problem for (1.1) with initial data at $t=\tau$:

$$\bar{u}(x) = \begin{cases} u(\tau,y-) & \text{if } x<y, \\ u(\tau,y+) & \text{if } x>y. \end{cases} \tag{3.9}$$

Moreover, for every given $k=1,\ldots,N$ we denote by $U^\flat = U^\flat_{(u,\tau,k)}(t,x)$ the solution to the linear Cauchy problem with constant coefficients

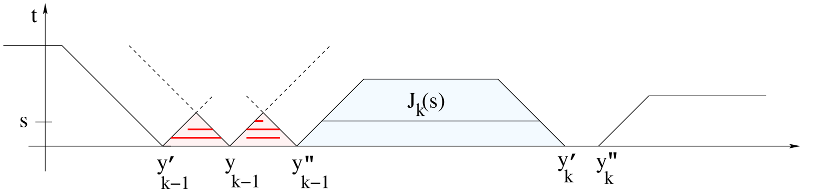

Fig. 2. The points $y'_k < y_k < y''_k$ constructed in the proof of the theorem. Typically, y_k is the location of a shock. Since $y_k \pm \tau$ need not be rational, the additional points y'_k, y''_k must be considered.

$$v_t + Av_x \;=\; 0, \qquad\qquad v(\tau, x) = u(\tau, x). \tag{3.10}$$

Here the $n \times n$ matrix A is the Jacobian matrix of f computed at the midpoint of the interval $[y''_{k-1}, y'_k]$. Namely,

$$A \;=\; Df\,(\widetilde{u}_k)\,, \qquad \widetilde{u}_k = u\left(\tau, \frac{y''_{k-1} + y'_k}{2}\right), \quad k = 1, \ldots, N.$$

With reference to Fig. 2, to estimate the lim-sup in (3.6), we need to estimate three types of integrals.

(I) The integral of $\big|u(t,x) - U^{\sharp}_{(u;\tau,y)}(t,x)\big|$ over the interval

$$\big[y - (t-\tau)\,,\; y + (t-\tau)\big],$$

for all points $y \in \big\{y_0, y''_0, y'_1, y_1, y''_1, \ldots, y'_N, y_N\big\}$.

(II) The integral of $\big|u(t,x) - U^{\flat}_{(u;\tau,k)}(t,x)\big|$ over the interval

$$J_k(t) \;=\; \big]y''_{k-1} + (t-\tau)\,,\; y'_k - (t-\tau)\big[, \quad k = 1, \ldots, N. \tag{3.11}$$

(III) The integral of $\big|u(t,x) - u(\tau,x)\big|$ over the intervals

$$\begin{cases} J'_k(t) \;=\; \big]y'_k + (t-\tau)\,,\; y_k - (t-\tau)\big[, \quad k = 1, \ldots, N, \\ J''_k(t) \;=\; \big]y_k + (t-\tau)\,,\; y''_k - (t-\tau)\big[, \quad k = 0, \ldots, N-1, \\ J_0(t) \;\doteq\;]-\infty, y_0 - (t-\tau)[\,, \\ J_{N+1} \;\doteq\;]y_N + (t-\tau)\,, +\infty[\,. \end{cases}$$

4. To estimate integrals of type (I), assuming that (τ, y) is either a point of approximate continuity or approximate jump of the function u, we obtain

$$\lim_{h\to 0+} \frac{1}{h} \int_{y-h}^{y+h} \Big|u(\tau+h,\; x) - U^{\sharp}_{(u;\tau,y)}(\tau+h, x)\Big|\, dx \;=\; 0. \tag{3.12}$$

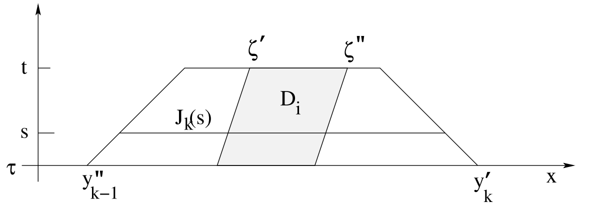

Fig. 3. The domain D_i considered at (3.15).

Indeed, by [8, Theorem 2.6], setting $u^\pm = u(\tau, y\pm)$, the function U defined in (3.1) satisfies (3.2) and consequently $u^\pm$ satisfy (3.3) and (3.4). It implies that $u(\tau, y\pm)$, when different, are connected by a single entropic shock whose speed is λ. Consequently $U^\sharp_{(u;\tau,y)}$ coincides with the piecewise constant function U defined in (3.1) so that (3.12) follows from [8, Theorem 2.6].

5. We now estimate the integrals of type (II). By construction, both values $y''_{k-1} - \tau$ and $y'_k + \tau$ are rational. Hence

$$y''_{k-1} = \xi + \tau, \qquad\qquad y'_k = \zeta - \tau,$$

for some $\xi, \zeta \in \mathbb{Q}$. This implies that τ is a Lebesgue point of the map

$$t \mapsto V(t) = W^{\xi,\zeta}(t). \tag{3.13}$$

Let $\tilde\lambda_i = \lambda_i(\tilde u_k)$, $\tilde l_i = l_i(\tilde u_k)$, $\tilde r_i = r_i(\tilde u_k)$, $i = 1, \dots, n$, be respectively the i-th eigenvalues and left and right eigenvectors of the matrix $A \doteq Df(\tilde u_k)$. We thus have

$$\tilde l_i \cdot U^\flat(t,x) = \tilde l_i \cdot U^\flat\big(\tau, x - (t-\tau)\tilde\lambda_i\big) = \tilde l_i \cdot u\big(\tau, x - (t-\tau)\tilde\lambda_i\big).$$

Following the proof of [8, Theorem 9.4], fix any two points $\zeta', \zeta'' \in J_k(t)$, $\zeta' < \zeta''$ and consider the quantity

$$\begin{aligned} E_i(\zeta', \zeta'') &\doteq \tilde l_i \cdot \int_{\zeta'}^{\zeta''} \big(u(t,x) - U^\flat(t,x)\big)\, dx \\ &= \tilde l_i \cdot \int_{\zeta'}^{\zeta''} \Big(u(t,x) - u\big(\tau,\ x - (t-\tau)\tilde\lambda_i\big)\Big)\, dx. \end{aligned} \tag{3.14}$$

We apply the divergence theorem to the vector $\big(u,\ f(u)\big)$ on the domain

$$D_i \doteq \Big\{(s,x);\ \ s \in [\tau,\ t],\ \ \zeta' - (t-s)\tilde\lambda_i \le x \le \zeta'' - (t-s)\tilde\lambda_i\Big\}, \tag{3.15}$$

shown in Fig. 3. Since u satisfies the conservation equation (1.1), the difference between the integral of u at the top and at the bottom of the domain D_i is thus measured by the inflow from the left side minus the outflow from the right side of D_i. From (3.14) it thus follows

$$
\begin{aligned}
E_i(\zeta',\zeta'') = & \int_\tau^t \tilde{l}_i \cdot \Big(\big(f(u)-\tilde\lambda_i u\big)(s,\ \zeta'-(t-s)\tilde\lambda_i)\Big)\,ds \\
& \qquad - \int_\tau^t \tilde{l}_i \cdot \Big(\big(f(u)-\tilde\lambda_i u\big)(s,\ \zeta''-(t-s)\tilde\lambda_i)\Big)\,ds \\
= & \int_\tau^t l_i(\tilde u_k)\cdot\Big(\big(f(u'(s))-\tilde\lambda_i\,(\tilde u_k)\,u'(s)\big)-\big(f(u''(s))-\lambda_i(\tilde u_k)u''(s)\big)\Big)\,ds \\
= & \int_\tau^t H\big(\widetilde{u}_k, u'(s), u''(s)\big)\,ds,
\end{aligned}
\tag{3.16}
$$

where we set

$$
u'(s) \doteq u\big(s,\ \zeta'-(t-s)\tilde\lambda_i\big), \qquad u''(s) \doteq u\big(s,\ \zeta''-(t-s)\tilde\lambda_i\big),
$$
$$
H(u,u_1,u_2) \doteq l_i(u)\cdot\Big(\big(f(u_1)-\lambda_i(u)u_1\big)-\big(f(u_2)-\lambda_i(u)u_2\big)\Big).
$$

Observing that

- $H(u,u_2,u_2)=0$,
- $D_{u_1}H(u,u_1,u_2)=l_i(u)\cdot(Df(u_1)-\lambda_i(u)I)$,
- $D_{u_1}H(u,u,u_2)=l_i(u)\cdot(Df(u)-\lambda_i(u)I)=0$,

we estimate

$$
\begin{aligned}
H(u,u_1,u_2) \ = & \ H(u,u_1,u_2)-H(u,u_2,u_2) \\
= & \int_0^1 D_{u_1}H(u,u_2+\sigma(u_1-u_2),u_2)\ d\sigma\cdot(u_1-u_2) \\
= & \int_0^1 \big[D_{u_1}H(u,u_2+\sigma(u_1-u_2),u_2)-D_{u_1}H(u,u,u_2)\big]\ d\sigma\cdot(u_1-u_2) \\
= & \ \mathcal{O}(1)\cdot\big(|u_1-u|+|u_2-u|\big)\cdot|u_1-u_2|\,.
\end{aligned}
$$

Here $\mathcal{O}(1)$ is any function bounded by a constant that depends only on the system, i.e. on the flux f. Therefore,

$$
E_i(\zeta',\zeta'') \ = \ \mathcal{O}(1)\cdot\int_\tau^t \big|u'(s)-u''(s)\big|\cdot\Big(\big|u'(s)-\widetilde{u}_k\big|+\big|u''(s)-\widetilde{u}_k\big|\Big)\,ds.
$$

Recalling (3.11) and (3.13), for any $x\in J_k(s)$ we now compute

$$
\big|u'(s)-\widetilde{u}_k\big| \le V(s)+\big|u\big(s,x\big)-u(\tau,x)\big|+|u(\tau,x)-\widetilde{u}_k| \le V(s)+\big|u\big(s,x\big)-u(\tau,x)\big|+V(\tau).
$$

Integrating w.r.t. x over the interval $J_k(s)$, dividing by its length and using (2.2) we obtain

$$\begin{aligned}\big|u'(s)-\widetilde u_k\big| \;&\leq\; V(s)+V(\tau)+\frac{1}{\mathrm{meas}\big(J_k(s)\big)}\int_{J_k(s)}\big|u(s,x)-u(\tau,x)\big|\,dx\\ &=\; V(s)+\varepsilon+\frac{C_M(s-\tau)}{\mathrm{meas}\big(J_k(s)\big)}\doteq g(s).\end{aligned}\tag{3.17}$$

An entirely similar estimate clearly holds for $\big|u''(s)-\widetilde u_k\big|$. Hence

$$\begin{aligned}E_i(\zeta',\zeta'') \;&=\; \mathcal{O}(1)\cdot\int_\tau^t\big|u'(s)-u''(s)\big|\cdot g(s)\,ds\\ &=\; \mathcal{O}(1)\cdot\int_\tau^t \mathrm{Tot.Var.}\Big\{u(s);\;]\zeta'-(t-s)\tilde\lambda_i,\;\zeta''-(t-s)\tilde\lambda_i]\Big\}\cdot g(s)\,ds\\ &=\; \mathcal{O}(1)\cdot\mu_i\big(]\zeta',\zeta'']\big).\end{aligned}$$

Here μ_i is the Borel measure defined by

$$\mu_i\big(]a,b[\big) \;=\; \int_\tau^t \mathrm{Tot.Var.}\Big\{u(s);\;]a-(t-s)\tilde\lambda_i,\; b-(t-s)\tilde\lambda_i\big[\Big\}\cdot g(s)\,ds,$$

for any open interval $]a,b[\,\subset J_k(t)$.

According to [8, Lemma 9.3], we now have

$$\begin{aligned}\int_{J_k(t)}\big|u(t,x)-U^\flat(t,x)\big|\;dx &= \mathcal{O}(1)\cdot\sum_{i=1}^n\int_{J_k(t)}\Big|\tilde l_i\cdot\big(u(t,x)-U^\flat(t,x)\big)\Big|\;dx\\ &= \mathcal{O}(1)\cdot\sum_{i=1}^n\mu_i\left(J_k(t)\right) \;=\; \mathcal{O}(1)\cdot\int_\tau^t V(s)\cdot g(s)\,ds.\end{aligned}$$

We now observe that, for all $\tau\leq s\leq t<\tau+\frac{1}{2}\left(y_k'-y_k''\right)$, the function g introduced at (3.17) satisfies

$$g(s)\;\leq\; V(s)+\varepsilon+\frac{C_M}{y_k'-y_{k-1}''-2\,(t-\tau)}\cdot(t-\tau)\,.\tag{3.18}$$

This implies

$$\frac{1}{t-\tau}\int_{J_k(t)}\big|u(t,x)-U^\flat(t,x)\big|\;dx \;=\; \frac{\mathcal{O}(1)}{t-\tau}\cdot\int_\tau^t V(s)\cdot\left(V(s)+\varepsilon\right)\,ds$$

$$+\frac{\mathcal{O}(1)\cdot C_M}{y'_k - y''_{k-1} - 2(t-\tau)}\cdot\int_\tau^t V(s)\,ds. \tag{3.19}$$

Since $t=\tau$ is a Lebesgue point for V, taking the limit of (3.19) as $t\to\tau+$ we thus obtain

$$\limsup_{t\to\tau+}\frac{1}{t-\tau}\int_{J_k(t)}\left|u(t,x)-U^\flat(t,x)\right|dx \;=\; \mathcal{O}(1)\cdot V(\tau)\left(V(\tau)+\varepsilon\right) \;=\; \mathcal{O}(1)\cdot\varepsilon^2. \tag{3.20}$$

6. Finally, regarding integrals of type (III), using Lemma 2.2 we obtain the bounds

$$\limsup_{h\to 0}\frac{1}{h}\int_{y'_k+h}^{y_k-h}\left|u(\tau+h,\ x)-u(\tau,x)\right|dx \;=\; \mathcal{O}(1)\cdot\varepsilon^2, \tag{3.21}$$

$$\limsup_{h\to 0}\frac{1}{h}\int_{y_k+h}^{y''_k-h}\left|u(\tau+h,\ x)-u(\tau,x)\right|dx \;=\; \mathcal{O}(1)\cdot\varepsilon^2, \tag{3.22}$$

$$\limsup_{h\to 0}\frac{1}{h}\int_{-\infty}^{y_0-h}\left|u(\tau+h,\ x)-u(\tau,x)\right|dx \;=\; \mathcal{O}(1)\cdot\varepsilon, \tag{3.23}$$

$$\limsup_{h\to 0}\frac{1}{h}\int_{y_N+h}^{+\infty}\left|u(\tau+h,\ x)-u(\tau,x)\right|dx \;=\; \mathcal{O}(1)\cdot\varepsilon. \tag{3.24}$$

7. On the other hand, it is well known [7,8] that semigroup trajectories satisfy entirely similar estimates. Indeed, at every point y the difference between the semigroup solution and the solution to a Riemann problem satisfies

$$\lim_{h\to 0+}\frac{1}{h}\int_{y-h}^{y+h}\left|\big(S_h u(\tau)\big)(x)-U^\sharp_{(u;\tau,y)}(\tau+h,x)\right|dx \;=\; 0. \tag{3.25}$$

Since the total variation of $u(\tau,\cdot)$ on the open interval $]y''_{k-1},\,y'_k[$ is $\le\varepsilon$, we have

$$\limsup_{h\to 0}\frac{1}{h}\int_{y''_{k-1}+h}^{y'_k-h}\left|\big(S_h u(\tau)\big)(x)-U^\flat_{(u,\tau,k)}(\tau+h,x)\right|dx \;=\; \mathcal{O}(1)\cdot\varepsilon^2. \tag{3.26}$$

Moreover, since the total variation of $u(\tau,\cdot)$ on the open intervals $]y'_k,\,y_k[$ and $]y_k,\,y''_k[$ is $\le\varepsilon^2$, we have

$$\limsup_{h\to 0} \frac{1}{h}\int_{y_k'+h}^{y_k-h} \Big|\big(S_h u(\tau)\big)(x) - u(\tau,x)\Big|\,dx \;=\; \mathcal{O}(1)\cdot\varepsilon^2, \tag{3.27}$$

$$\limsup_{h\to 0} \frac{1}{h}\int_{y_k+h}^{y_k''-h} \Big|\big(S_h u(\tau)\big)(x) - u(\tau,x)\Big|\,dx \;=\; \mathcal{O}(1)\cdot\varepsilon^2, \tag{3.28}$$

and similarly

$$\limsup_{h\to 0} \frac{1}{h}\int_{-\infty}^{y_0-h} \Big|\big(S_h u(\tau)\big)(x) - u(\tau,x)\Big|\,dx \;=\; \mathcal{O}(1)\cdot\varepsilon, \tag{3.29}$$

$$\limsup_{h\to 0} \frac{1}{h}\int_{y_N+h}^{\infty} \Big|\big(S_h u(\tau)\big)(x) - u(\tau,x)\Big|\,dx \;=\; \mathcal{O}(1)\cdot\varepsilon. \tag{3.30}$$

8. Combining all the previous estimates, and recalling that the total number of intervals is $N \le M\varepsilon^{-1}$, we establish the limit (3.6), proving the theorem. □

4. Concluding remarks

The present analysis opens the door to the study of convergence and a posteriori error estimates for a wide variety of approximation schemes.

Following [9], we say that $u = u(t,x)$ is an ε-approximate solution to (1.1) if, given the time step $\varepsilon = \Delta t$, the following holds.

(AL) Approximate Lipschitz continuity. For every $\tau, \tau' \ge 0$ one has

$$\|u(\tau,\cdot) - u(\tau',\cdot)\|_{\mathbf{L}^1} \;\le\; M\big(|\tau-\tau'|+\varepsilon\big)\cdot \sup_{t\in[\tau,\tau']} \text{Tot.Var.}\big\{u(t,\cdot)\big\}.$$

($\mathbf{P}_\varepsilon$) Approximate conservation law, and approximate entropy inequality.
For every strip $[\tau,\tau']\times\mathbb{R}$ with $\tau,\tau'\in\varepsilon\mathbb{N}$, and every test function $\varphi\in\mathcal{C}^1_c(\mathbb{R}^2)$, there holds

$$\begin{aligned}&\left|\int u(\tau,x)\varphi(\tau,x)\,dx - \int u(\tau',x)\varphi(\tau',x)\,dx + \int_\tau^{\tau'}\!\!\int \big\{u\varphi_t + f(u)\varphi_x\big\}\,dx\,dt\right| \\ &\qquad \le\; C\varepsilon\|\varphi\|_{W^{1,\infty}}\cdot(\tau'-\tau)\cdot \sup_{t\in[\tau,\tau']}\text{Tot.Var.}\big\{u(t,\cdot)\big\}.\end{aligned} \tag{4.1}$$

Moreover, given a uniformly convex entropy η with flux q, assuming $\varphi\ge 0$ one has the entropy inequality

$$\int \eta(u(\tau,x))\varphi(\tau,x)\,dx - \int \eta(u(\tau',x))\varphi(\tau',x)\,dx + \int_{\tau}^{\tau'}\!\!\int \big\{\eta(u)\varphi_t + q(u)\varphi_x\big\}\,dxdt$$
$$\geq\ -C\varepsilon\|\varphi\|_{W^{1,\infty}}\cdot(\tau'-\tau)\cdot \sup_{t\in[\tau,\tau']}\ \text{Tot.Var.}\big\{u(t,\cdot)\big\}. \tag{4.2}$$

In the above setting, the recent paper [9] has established **a posteriori** error estimates, assuming that the total variation of $u(t,\cdot)$ remains small, so that $u(t,\cdot)$ remains inside the domain of the semigroup. However, the estimates in [9] also required a "post processing algorithm", tracing the location of the large shocks in the approximate solution. We would like to achieve error estimates based solely on an a posteriori bound of the total variation. The possibility of such estimates is the content of the following corollary.

Corollary 4.1. *Let (1.1) be an $n\times n$ strictly hyperbolic system where each characteristic field is either genuinely nonlinear or linearly degenerate, and which admits a strictly convex entropy $\eta(\cdot)$ as in (1.8). Let S be the unique Lipschitz semigroup defined on a domain $\mathcal{D}$ of functions with small total variation. Then, given $T, R>0$, there exists a function $\varepsilon\mapsto\varrho(\varepsilon)$ with the following properties.*

(i) *ϱ is continuous, nondecreasing, with $\varrho(0)=0$.*
(ii) *Let $t\mapsto u(t)\in\mathcal{D}$ be an ε-approximate solution to (1.1), satisfying* **(AL)**-**(P$_\varepsilon$)** *and supported inside the interval $[-R,R]$. Then, calling $\bar u = u(0)$, one has*

$$\big\|u(t)-S_t\bar u\big\|_{\mathbf{L}^1}\ \leq\ \varrho(\varepsilon) \qquad \textit{for all } t\in[0,T]. \tag{4.3}$$

Proof. If the conclusion fails, there exists a sequence of ε_n-approximate solutions $(u_n)_{n\geq 1}$, all supported inside $[-R,R]$, with $\varepsilon_n\to 0$ but

$$\sup_{t\in[0,T]}\ \big\|u_n(t)-S_tu_n(0)\big\|_{\mathbf{L}^1}\ \geq\ \delta_0\ >\ 0 \qquad \text{for all } n\geq 1. \tag{4.4}$$

By compactness, taking a subsequence we achieve the $\mathbf{L}^1$-convergence $u_n(t)\to u(t)$, uniformly for $t\in[0,T]$. Setting $\bar u(x)\doteq u(0,x)$, the limit function u is thus an entropy weak solution of (1.1)-(1.2), distinct from the semigroup trajectory $S_t\bar u$. This contradicts the uniqueness stated in Theorem 1.1. □

We regard the function $\varrho(\cdot)$ as a **universal convergence rate** for approximate BV solutions to the hyperbolic system (1.1). Having proved the existence of such a function, the major open problem is now to provide an asymptotic estimate on $\varrho(\varepsilon)$, as $\varepsilon\to 0$. In some sense, starting from a uniqueness theorem and deriving a uniform convergence rate is a task analogous to the derivation of quantitative compactness estimates [2–4,21]. Based on the convergence estimates already available for the Glimm scheme [5,17] and for vanishing viscosity approximations [13,18], one might guess that $\varrho(\varepsilon)\approx\sqrt{\varepsilon}\,|\ln\varepsilon|$. We leave this as an open question for future investigation.

Data availability

No data was used for the research described in the article.

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