Inverse Spectral Theory and Kramers-Kronig Relations

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Abstract— Inverse problems aimed at locating cracks and voids inside a dielectric medium by means of electromagnetic waves involve knowledge of transmission eigenvalues. The subject has been actively investigated by many Authors for some years and results have been presented in articles and books. Unfortunately, the frequency dependence of dielectric permittivity and magnetic permeability of the material has not been taken into account. A theorem which provides existence and asymptotic properties of real transmission eigenvalues is shown to be physically inconsistent, because it ignores the Kramers-Kronig relations.

1. INTRODUCTION

Inverse problems, which consist of determining voids or cracks inside a dielectric from knowledge of one or more incident plane waves and the corresponding (complex valued) scattered waves in the far zone, involve the so-called transmission eigenvalues. In a scalar setting, which makes sense if the dielectric body has size, shape and physical properties which are translation-invariant, e.g., along the x_3 axis and either the $\vec{E}$ or the $\vec{H}$ fields of the incident waves are polarised accordingly, then one can state the following.

Pbm. (*interior, scalar transmission eigenvalue problem.*) Let the dielectric medium be enclosed inside a right cylinder of axis x_3 and cross section D with sufficiently smooth boundary, ∂D, and be surrounded by empty space. Find a non trivial pair, $\{w_1, w_2\}$, of transmission eigenfunctions and the corresponding free space, real wavenumber, k, complying with

$$w_2 - w_1 \in H_0^2(D) \tag{1}$$

and

$$\left(\Delta + \chi^2\right) w_2 = 0 \text{ in } D, \quad \left(\Delta + k^2\right) w_1 = 0 \text{ in } D, \quad w_2 = w_1 \text{ and } \partial_n w_2 = \partial_n w_1 \text{ on } \partial D. \tag{2}$$

Here the complex valued function $\chi[\cdot]$ describes the electric and magnetic properties of the medium, as well as their spatial dependence. Knowledge of transmission eigenvalues is relevant to two classes of inverse problems. If the so-called sampling methods are applied to reconstruct the support of the scatterer [1], incident waves shall have frequencies which do not correspond to transmission eigenvalues. In particular, the latter shall at most form a countable set. More recently, transmission eigenvalues have been said to play a role in the reconstruction of the scatterer from far zone measurements [2–4].

2. TYPICAL EXISTENCE RESULT FOR TRANSMISSION EIGENVALUES

A result believed to play a key role in the spectral theory of interior transmission problems involving inhomogeneous, isotropic dielectric media [5] has been stated for a function $\chi[\cdot]$ which factorises according to

$$\chi^2 := k^2 n^2[\vec{x}], \tag{3}$$

where the refractive index, $n[\cdot]$, is real valued and depends at most on $\vec{x} \in D$. In particular [5, Thm. 2.5], if either

$$\inf_D n^2[\vec{x}] > 1 + \alpha, \ \forall \vec{x} \in D, \tag{4}$$

with some $\alpha > 0$, fixed, or

$$\sup_D n^2[\vec{x}] < 1 - \beta, \ \forall \vec{x} \in D, \tag{5}$$

with some $\beta > 0$, fixed, then there exists an infinite, countable set of real transmission eigenvalues with $+\infty$ as the only accumulation point.

The need to impose Ineqs. (4) or (5) arises, because the pair of Helmholtz equations (with obvious notation) $\{\mathcal{K}_2 w_2 = 0; \mathcal{K}_1 w_1 = 0\}$, appearing in the transmission problem of Eq. (2) is transformed into

$$-\left(\mathcal{K}_2 \left(\mathcal{K}_2 - \mathcal{K}_1\right)^{-1} \mathcal{K}_1\right) [w_2 - w_1] = 0 \text{ in } D. \tag{6}$$

The operator $(\mathcal{K}_2 - \mathcal{K}_1)^{-1}$ is no longer differential. Instead, it acts as multiplication by $(n^2[\vec{x}] - 1)^{-1}$, which stays bounded, provided in Eqs. (4) or (5) hold. In the variational form of Eq. (6) the term $(n^2[\vec{x}] - 1)^{-1}$ appears in an integral over D.

The boundedness of $|n^2[\vec{x}] - 1|$ away from zero $\forall \vec{x} \in D$ is a recurrent requirement in other articles which address transmission eigenvalue problems [6].

3. ASYMPTOTIC INCONSISTENCY OF THE THEORY

Claim (*physical inconsistency of the hypothesis under which existence of transmission eigenvalues is derived.*) The above results, which pertain to existence and asymptotic properties of transmission eigenvalues, have been derived under hypotheses which do not take the physical properties of materials into account. Moreover, if ω stands for the angular frequency of the incident wave, the determination of transmission eigenvalues is asymptotically ill-posed for $\omega \to \infty$.

Proof. The control variable is angular frequency, ω. Transmission eigenvalues relate to wavenumbers. In order to attain $k \to \infty$, the dispersion relation of empty space, $\omega = ck$, implies $\omega \to \infty$. Even if, in many applications, magnetic permeability, μ, of the medium can be regarded as independent of frequency and equal to the permeability of vacuum, μ_0, the ω-dependence of relative dielectric permittivity, ϵ_r, cannot be ignored. In fact, it must be taken into account if asymptotic properties are to be investigated. Let a model of the Lorentz class be chosen for the dependence of ε_r on ω [7, p. 284 ff.], and let the material medium in D be uniformly distributed, resulting in N_{mol} molecules per unit volume, each of them with Z electrons. If there are Z_j electrons per molecule with binding frequency ω_j and damping constant γ_j, $1 \le j \le J$, then

$$\varepsilon_r[\omega] = 1 + 4\pi \frac{e^2}{m} N_{\mathrm{mol}} \sum_{j=i}^{J} \frac{Z_j}{\omega_j^2 - \omega^2 - \mathrm{i}\omega\gamma_j}, \tag{7}$$

subject to

$$\sum_{j=i}^{J} Z_j = Z. \tag{8}$$

In the limit, $\omega \to \infty$, neither in Eq. (4) nor in Eq. (5) are met by the $\varepsilon_r[\omega]$ of Eq. (7), hence $\lim_{\omega\to\infty}(\mathcal{K}_2 - \mathcal{K}_1) = 0$ and the existence proof of Sect. II collapses.

Rem. The same conclusion would be attained if the Lorentz model of Eq. (7) were replaced by any other **causal** relation between the displacement, $\vec{D}$, and the electric, $\vec{E}$, fields, as formalised by Kramers and Kronig [8, 9; 7, §7.10].

Rem. Absorbing media with frequency dependent dielectric permittivity have been recently addressed [10]. The type of wavenumber-dependence introduced into $\varepsilon_r[\cdot]$ is however the result of a low frequency approximation and, as such, still affected by asymptotic inconsistency.

ACKNOWLEDGMENT

The financial support by the following sources is gratefully acknowledged: *a*) Contract *W911NF 11-1-0277 R&D 1449-BC-01* granted to Università Milan-Bicocca by the *US Army RDECOM ACQ CTR*; *b*) the home University's *Fondo di Ateneo per la ricerca*, from 2013 onwards; *c*) the research agreement between the home University and *Digicom A.G.*, Gallarate, N. Italy.

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