games

MDPI

Article

A Generalized Nash Equilibrium Approach to the Inverse Eigenvector Centrality Problem

Mauro Passacantando [1] and Fabio Raciti [2,*]

[1] Department of Business and Law, University of Milano-Bicocca, Via degli Arcimboldi 8, 20126 Milan, Italy; mauro.passacantando@unimib.it
[2] Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
* Correspondence: fabio.raciti@unict.it

Abstract

Eigenvector-based centrality captures recursive notions of importance in networks. While the direct problem computes centrality from given edge weights, the inverse eigenvector centrality problem seeks edge weights that reproduce a prescribed centrality profile; for directed multigraphs, this inverse task is typically non-unique and depends on the admissible arc structure. We study the direct and inverse problems on directed multigraphs and derive an explicit linear characterization of the set of admissible edge-weight vectors that are compatible with a given centrality target. On this feasible set, we formulate a generalized Nash equilibrium problem with shared centrality constraints, in which multiple agents select edge weights to maximize economically interpretable payoffs that incorporate arc-level competition effects. We provide conditions under which the induced game admits a concave potential function, yielding equilibrium existence and, under standard strict concavity assumptions, uniqueness. Finally, we illustrate the model on an airport network where nodes represent airports and parallel arcs represent airline-specific routes, showing that equilibrium selection produces a feasible and interpretable weight configuration that preserves the prescribed centrality.

Keywords: eigenvector centrality; multigraph; inverse problem; generalized Nash equilibrium; variational inequality; potential game

1. Introduction

Networks provide a natural and powerful framework for representing systems of interacting entities across a wide range of applications, including social, economic, technological, and engineered settings. In such systems, understanding the relative importance of agents is often crucial for analysis, prediction, and design. This has motivated the introduction of a wide variety of centrality measures that quantify the role of nodes within a network based on structural or dynamical considerations (Brin & Page, 1998; Freeman, 1979; Newman, 2010). Centrality notions are now routinely employed to identify influential agents, assess vulnerabilities, and support intervention or regulation strategies.

Among the many existing measures, eigenvector-based centralities are particularly appealing for their global nature and clear spectral interpretation. In the direct formulation, a weighted network induces a centrality profile through an eigenvalue problem associated with its adjacency structure (Bonacich, 1972, 1987). While this direct problem is well understood, an increasing number of applications call for an inverse perspective, in which

Check for updates

Academic Editor: Kjell Hausken

Received: 27 February 2026
Revised: 31 March 2026
Accepted: 2 April 2026
Published: 7 April 2026

a desired centrality profile is prescribed a priori and one seeks to determine edge weights compatible with it.

A fundamental feature of the inverse eigenvector centrality problem, for directed graphs, is its intrinsic non-uniqueness. For a fixed network topology and a given centrality vector, there typically exist infinitely many weighted networks that realize the same centrality profile. This lack of uniqueness naturally raises the issue of selection: how should one choose, among all admissible configurations, a meaningful, interpretable, or consistent weight assignment? Early investigations of this problem have shown that, for strongly connected directed networks, any prescribed centrality profile can be realized by assigning suitable edge weights (Nicosia et al., 2012). A natural way to cope with this non-uniqueness is to equip the admissible set of centrality-compatible weights with a selection rule, for instance by imposing additional structural requirements or by solving auxiliary selection problems. This viewpoint is pursued in (Passacantando & Raciti, 2026). Here, instead, we adopt a game-theoretic perspective: feasible weight configurations arise endogenously as equilibrium outcomes under shared centrality constraints. Specifically, we interpret the set of edge weights compatible with a prescribed centrality profile as a shared feasible set for a generalized Nash equilibrium problem. Within this framework, multiple agents interact over the same set of weights, each optimizing an individual utility function that reflects local incentives or competitive effects. The centrality constraint serves as a shared constraint that couples the players' strategies, naturally leading to a generalized Nash equilibrium formulation (Passacantando & Raciti, 2021). This perspective provides a structurally different mechanism for resolving the non-uniqueness of the inverse problem, grounded in strategic interaction rather than external optimality. To the best of our knowledge, this is the first paper that formulates and studies a game-theoretic model for the inverse eigenvector centrality problem, in which admissible edge-weight configurations arise endogenously as equilibrium outcomes under shared centrality constraints.

The structure of the paper is the following. First, we formalize the inverse eigenvector centrality problem on directed multigraphs and characterize the associated admissible set of weights. Second, we introduce a class of generalized Nash equilibrium problems defined on this set and show that, under suitable assumptions, the resulting game admits an exact potential, ensuring the existence and uniqueness of a variational equilibrium. Third, we illustrate the proposed framework through an application to an air transportation network, highlighting how distinct strategic interactions yield distinct weighted network structures while preserving the prescribed centrality profile.

2. The Direct and Inverse Eigenvector Centrality Problem

Let $G = (V, E)$ be a directed multigraph, where $V = \{1, \dots, n\}$ denotes the set of nodes and $E \subseteq V \times V \times \mathcal{K}$ is the set of directed edges. An element $(i, j, k) \in E$ represents the directed edge from node i to node j in the class k. For a given ordered pair (i, j), multiple parallel edges indexed by k may exist. Let $\mathcal{R} := \{(i, j) \in V \times V : \exists k \in \mathcal{K} \text{ such that } (i, j, k) \in E\}$ denote the set of ordered pairs associated with at least one edge of the directed multigraph. To each edge $(i, j, k) \in E$ we associate a non-negative weight $w_{ij}^k \geq 0$, which quantifies the intensity of the interaction carried by that edge. The weighted adjacency matrix of the multigraph is the matrix $A(w) \in \mathbb{R}^{n \times n}$ defined componentwise as

$$A_{ij}(w) = \sum_{\substack{k \in \mathcal{K}: \\ (i,j,k) \in E}} w_{ij}^k, \qquad i, j = 1, \dots, n. \tag{1}$$

If no directed edge from i to j exists, then $A_{ij}(w) = 0$. In the special case of a simple directed graph, the above sum reduces to a single term.

The eigenvector centrality associated with the weighted multigraph is defined as a positive vector $c \in \mathbb{R}^n_{++}$ satisfying the eigenvalue equation

$$A(w)^\top c = \rho\, c, \tag{2}$$

where $\rho > 0$ denotes the spectral radius of $A(w)$. Under standard assumptions of the non-negativity and irreducibility of $A(w)$, the Perron–Frobenius theorem (see, e.g., Horn & Johnson, 2013; Minc, 1988) guarantees that ρ is simple and that the corresponding eigenvector c is unique up to scaling. We recall that the irreducibility of $A(w)$ is equivalent to the strong connectedness of the multigraph. Writing the eigenvalue Equation (2) componentwise, for each node $j \in V$ we obtain

$$\sum_{i \in V} A_{ij}(w)\, c_i = \rho\, c_j, \qquad j \in V. \tag{3}$$

Substituting the definition of the weighted adjacency matrix yields the explicit form

$$\sum_{i \in V} \sum_{\substack{k \in \mathcal{K}: \\ (i,j,k) \in E}} w_{ij}^k\, c_i = \rho\, c_j, \qquad j \in V. \tag{4}$$

This formulation highlights how the centrality of each node depends on the aggregate contribution of all ingoing edges, weighted by the centrality of their origin nodes.

Let $m := |E|$ denote the total number of directed edges of the multigraph. We fix once and for all an ordering of the edges: $E = \{e_1, e_2, \dots, e_m\}$, where each edge e_ℓ corresponds to a triple $e_\ell = (i_\ell, j_\ell, k_\ell) \in V \times V \times \mathcal{K}$. Accordingly, we collect all edge weights into a single vector

$$w := \begin{pmatrix} w_{e_1} \\ \vdots \\ w_{e_m} \end{pmatrix} \in \mathbb{R}^m, \qquad w_{e_\ell} := w_{i_\ell j_\ell}^{k_\ell}. \tag{5}$$

Using this ordering, the eigenvector centrality equations can be rewritten in linear form with respect to the edge-weight vector w. Specifically, for a fixed centrality vector $c \in \mathbb{R}^n_{++}$, define the matrix $B(c) \in \mathbb{R}^{n \times m}$ by

$$B_{i\ell}(c) = \begin{cases} c_{j_\ell} & \text{if } i = i_\ell, \\ 0 & \text{otherwise}, \end{cases} \qquad i = 1, \dots, n, \quad \ell = 1, \dots, m. \tag{6}$$

With this definition, the componentwise eigenvector centrality Equation (4) can be compactly written as the linear system

$$B(c)\, w = \rho\, c. \tag{7}$$

This formulation highlights that, for a fixed centrality profile c and a scalar $\rho > 0$, the inverse eigenvector centrality problem consists in finding non-negative vectors $w \in \mathbb{R}^m$ such that the linear constraint $B(c)w = \rho c$ is satisfied. Once a scaling of c is chosen, the associated eigenvalue ρ is uniquely determined by the feasibility condition $B(c)w = \rho c$.

Fix a target centrality vector $c \in \mathbb{R}^n_{++}$ and a scalar $\rho > 0$. We consider the linear system $B(c)\, w = \rho\, c$. For a given lower bound $\varepsilon > 0$, we define the admissible set of edge-weight vectors as

$$\mathcal{W}_\varepsilon(c, \rho) := \{w \in \mathbb{R}^m :\ B(c)\, w = \rho\, c, \quad w \geq \varepsilon \mathbf{1}\}, \tag{8}$$

where **1** denotes the vector with all components equal to one and the use of positive ε prevents changes in the network topology.

Lemma 1. *For any target centrality vector $c \in \mathbb{R}^n_{++}$ and a scalar $\rho > 0$, the set $\mathcal{W}_\varepsilon(c, \rho) \neq \emptyset$ for all ε such that*

$$0 < \varepsilon < \min_{j \in V} \frac{\rho c_j}{\sum_{i \in V:\ (i,j) \in \mathcal{R}} c_i}. \tag{9}$$

Proof. We first discuss the solvability of the linear system $B(c)w = \rho c$ over $\mathbb{R}^m$, without imposing any sign constraints on w. For each node $j \in V$, the j-th row of $B(c)w = \rho c$ reads

$$\sum_{\ell:\ j_\ell = j} c_{i_\ell}\, w_{e_\ell} = \rho\, c_j. \tag{10}$$

Equivalently, in multi-index notation,

$$\sum_{i \in V} \sum_{\substack{k \in \mathcal{K}: \\ (i,j,k) \in E}} c_i\, w^k_{ij} = \rho\, c_j, \qquad j = 1, \ldots, n. \tag{11}$$

Hence, the linear system is solvable if and only if, for every node $j \in V$, there exists at least one edge pointing to j, i.e., $\forall j \in V, \exists (i,k)$ such that $(i,j,k) \in E$. Indeed, if node j has no ingoing edges, then the left-hand side of the j-th equation is identically zero for all w, whereas the right-hand side equals $\rho c_j > 0$, yielding an inconsistency. Conversely, if each row contains at least one variable, then each equation involves at least one unknown and thus admits infinitely many real solutions; collecting these solutions row by row yields a solution $w \in \mathbb{R}^m$.

We now address feasibility under positivity constraints. Assume that every node has at least one ingoing edge, as above. Fix an ordering of the ingoing edges of each node j and select one distinguished ingoing edge $e^*_j = (i^*_j, j, k^*_j) \in E$. Let $\varepsilon > 0$ and set

$$w^k_{ij} = \varepsilon \qquad \text{for all } (i,j,k) \in E \text{ with } (i,j,k) \neq (i^*_j, j, k^*_j). \tag{12}$$

Then the equality constraint for node j determines the remaining variable $w^{k^*_j}_{i^*_j j}$ as

$$w^{k^*_j}_{i^*_j j} = \frac{1}{c_{i^*_j}} \left(\rho c_j - \varepsilon \sum_{\substack{(i,j,k) \in E: \\ (i,j,k) \neq (i^*_j, j, k^*_j)}} c_i \right). \tag{13}$$

Since $c \in \mathbb{R}^n_{++}$, this quantity is strictly positive provided $\varepsilon > 0$ is sufficiently small. A sufficient condition is

$$0 < \varepsilon < \min_{j \in V} \frac{\rho c_j}{\sum_{i \in V:\ (i,j) \in \mathcal{R}} c_i},$$

which ensures that $\mathcal{W}_\varepsilon(c, \rho) \neq \emptyset$. $\square$

3. A Generalized Nash Equilibrium Problem over the Admissible Set of Weights

3.1. A Brief Recall on Generalized Nash Equilibrium Problems with Shared Constraints

Game-theoretic models on networks often involve decision makers whose feasible strategy sets and objective functions are coupled through shared resources, technological constraints, or global feasibility conditions. In such situations, the classical notion of Nash equilibrium is no longer adequate, and the appropriate modeling framework is provided by Generalized Nash Equilibrium Problems (GNEPs). In a GNEP, each player solves an individual optimization problem whose feasible set depends on the strategies chosen by the other players. The systematic study of concave games with coupled constraints can be traced back to the seminal work of Rosen (Rosen, 1965), where sufficient conditions for the existence and uniqueness of equilibria were established under suitable concavity and monotonicity assumptions. Since then, the theory has evolved substantially, particularly in connection with equilibrium problems arising in networks, economics, and engineering systems.

In this paper, we consider GNEPs with shared constraints, namely problems in which all players are subject to the same global feasibility set. This structure naturally arises in network settings where decisions are constrained by common balance equations, capacity limitations, or aggregate consistency requirements. Formally, consider a finite set of players indexed by $p = 1, \ldots, N$. Each player p controls a decision variable $x_p \in \mathbb{R}^{n_p}$ and seeks to solve

$$\max_{x_p} \; u_p(x_p, x_{-p}) \quad \text{subject to} \quad (x_p, x_{-p}) \in K, \tag{14}$$

where u_p denotes the utility function of player p, x_{-p} collects the strategies of all other players, and $K \subseteq \mathbb{R}^{n_1+\cdots+n_N}$ is a nonempty closed and convex set representing the shared constraints. A vector $x^* = (x_1^*, \ldots, x_N^*)$ is a generalized Nash equilibrium if no player can improve her payoff by a unilateral deviation while preserving feasibility.

An important equilibrium concept for GNEPs with shared constraints is the variational equilibrium. Roughly speaking, a variational equilibrium is a generalized Nash equilibrium that satisfies the Karush–Kuhn–Tucker conditions with a common multiplier associated with the shared constraints. This notion was introduced to single out equilibria with stronger stability and robustness properties and plays a central role in the modern analysis of GNEPs (Facchinei et al., 2007; Nabetani et al., 2011).

From a mathematical standpoint, variational equilibria admit a convenient reformulation in terms of variational inequalities. Defining the aggregated mapping (pseudogradient of the game)

$$F(x) := \big(-\nabla_{x_1} u_1(x), \ldots, -\nabla_{x_N} u_N(x) \big), \tag{15}$$

a variational equilibrium corresponds to a solution of the variational inequality problem $VI(F, K)$:

$$\text{find } x^* \in K \quad \text{such that} \quad \langle F(x^*), x - x^* \rangle \geq 0 \qquad \forall\, x \in K. \tag{16}$$

This formulation provides a powerful analytical and computational framework, allowing one to exploit the monotonicity properties of the operator F and convexity of the feasible set K. In particular, if F is strongly monotone on K, the existence and uniqueness of the variational equilibrium follow directly from standard results in variational inequality theory (Konnov, 2007; Nagurney, 1999). A more detailed overview of the relationship between the GNEPs with shared constraints and the variational inequality problem is provided in Appendix A.

The relevance of GNEPs with shared constraints in network applications has been highlighted in several recent contributions, where equilibrium problems are defined over network-induced feasible sets and analyzed through variational inequality techniques (see, e.g., Passacantando and Raciti (2021) for an application to Network Games). In the next section, we build upon this framework to formulate a GNEP on the set of edge weights compatible with a prescribed centrality profile.

3.2. The Model

We consider a GNEP defined over the admissible set (8). The constraint $B(c)w = \rho c$ is shared by all players and encodes compatibility with the prescribed eigenvector centrality profile. The set $\mathcal{K}$ of edge classes denotes the set of players. Each player $k \in \mathcal{K}$ controls the subset of variables

$$w^k := (w_{ij}^k)_{(i,j,k)\in E} \tag{17}$$

while the remaining components w^{-k} are controlled by the other players. For each ordered pair $(i,j) \in \mathcal{R}$, we introduce the aggregate quantity

$$y_{ij}(w) := \sum_{\substack{k\in\mathcal{K}:\\ (i,j,k)\in E}} w_{ij}^k, \tag{18}$$

which captures the total weight associated with that pair and models the interaction effects among players acting on the same ordered pair. The utility function of player k is defined as

$$u_k(w) := \sum_{\substack{(i,j)\in\mathcal{R}:\\ (i,j,k)\in E}} \left[r_{ij}^k w_{ij}^k - \frac{\mu_{ij}^k}{2} (w_{ij}^k)^2 \right] - \sum_{(i,j)\in\mathcal{R}} \frac{\alpha_{ij}}{2} \left[y_{ij}(w)\right]^2, \tag{19}$$

where $r_{ij}^k > 0$, $\mu_{ij}^k > 0$, and $\alpha_{ij} \geq 0$ are given parameters. The first term represents a linear benefit associated with the individual decision variables, penalized by strictly convex quadratic costs, while the second term introduces interaction effects through the aggregate variables y_{ij}. It is easy to note that u_k is a concave function of all variables w and strongly concave with respect to w^k. Each player k aims to solve the optimization problem

$$\max_{w^k} \; u_k(w^k, w^{-k}) \quad \text{subject to} \quad w \in \mathcal{W}_\varepsilon(c,\rho), \tag{20}$$

leading to a GNEP with shared constraints.

Example 1. *Consider a simple GNEP with two players on the multigraph shown in Figure 1. Player 1 controls the weights of red edges, while player 2 controls the weights of blue ones.*

Assuming that parameters $r_{ij}^k = \rho = 1$ and $\mu_{ij}^k = \alpha_{ij} = 2$, the utility functions of the two players are

$$\begin{aligned} u_1(w^1,w^2) &= w_{12}^1 - (w_{12}^1)^2 + w_{21}^1 - (w_{21}^1)^2 + w_{23}^1 - (w_{23}^1)^2 + w_{32}^1 - (w_{32}^1)^2 \\ &\quad - (w_{12}^1)^2 - (w_{13}^2)^2 - (w_{21}^1)^2 - (w_{23}^1)^2 - (w_{31}^2)^2 - (w_{32}^1 + w_{32}^2)^2, \end{aligned} \tag{21}$$

$$\begin{aligned} u_2(w^1,w^2) &= w_{13}^2 - (w_{13}^2)^2 + w_{31}^2 - (w_{31}^2)^2 + w_{32}^2 - (w_{32}^2)^2 \\ &\quad - (w_{12}^1)^2 - (w_{13}^2)^2 - (w_{21}^1)^2 - (w_{23}^1)^2 - (w_{31}^2)^2 - (w_{32}^1 + w_{32}^2)^2, \end{aligned} \tag{22}$$

and the admissible set $\mathcal{W}_\varepsilon(c,\rho)$ is defined as follows:

$$\begin{cases} w_{21}^1 c_2 + w_{31}^2 c_3 = c_1 \\ w_{12}^1 c_1 + w_{32}^1 c_3 + w_{32}^2 c_3 = c_2 \\ w_{13}^2 c_1 + w_{23}^1 c_2 = c_3 \\ w^1 \geq \varepsilon \mathbf{1} \\ w^2 \geq \varepsilon \mathbf{1} \end{cases} \tag{23}$$

We note that the variables of the two players are linked both in the utility functions (through the weights of edges from 3 to 2) and in the shared constraints, which ensure that c is an eigenvector centrality vector.

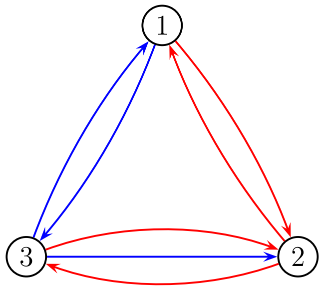

Figure 1. Multigraph of Example 1.

We now show that the game is an exact potential generalized Nash equilibrium problem according to Monderer and Shapley (1996).

Lemma 2. *Define the function* $\Phi : \mathbb{R}^m \to \mathbb{R}$ *as*

$$\Phi(w) := \sum_{(i,j,k)\in E} \left[r_{ij}^k \, w_{ij}^k - \frac{\mu_{ij}^k}{2} \, (w_{ij}^k)^2 \right] - \sum_{(i,j)\in\mathcal{R}} \frac{\alpha_{ij}}{2} \, \big[y_{ij}(w) \big]^2. \tag{24}$$

Then the pseudogradient F *of the game is actually the gradient of the potential function* Φ.

Proof. The following equality

$$\frac{\partial u_k}{\partial w_{ij}^k}(w) = r_{ij}^k - \mu_{ij}^k w_{ij}^k - \alpha_{ij} y_{ij}(w) = \frac{\partial \Phi}{\partial w_{ij}^k}(w), \tag{25}$$

holds for every player $k \in \mathcal{K}$ and every edge $(i,j,k) \in E$. Hence $F(w) = \nabla\Phi(w)$ for any $w \in \mathbb{R}^m$. $\square$

Theorem 1 (Variational equilibrium and potential maximization). *Let* $c \in \mathbb{R}^n_{++}$, $\rho > 0$, *and* $\varepsilon > 0$ *be fixed, and consider the admissible set (8). Assume that* $\mu_{ij}^k > 0$ *for all* $(i,j,k) \in E$ *and* $\alpha_{ij} \geq 0$ *for all* $(i,j) \in \mathcal{R}$. *Then the potential function* Φ *is strictly concave on* $\mathbb{R}^m$ *with a block-diagonal Hessian matrix given by*

$$\nabla^2\Phi(w) = -\,\mathrm{blockdiag}\left(D_{ij} + \alpha_{ij} \, \mathbf{1}_{m_{ij}} \mathbf{1}_{m_{ij}}^\top \right)_{(i,j)\in\mathcal{R}} \tag{26}$$

where $D_{ij} = \mathrm{diag}\left(\mu_{ij}^k\right)$. *Moreover, the optimization problem*

$$\max \; \Phi(w) \quad \textit{subject to} \quad w \in \mathcal{W}_\varepsilon(c,\rho) \tag{27}$$

admits a unique solution $w^\star$, *which coincides with the unique variational equilibrium of the generalized Nash equilibrium problem.*

Proof. The potential function Φ can be equivalently written as

$$\Phi(w) = \sum_{(i,j)\in\mathcal{R}} \left\{ \sum_{\substack{k\in\mathcal{K}: \\ (i,j,k)\in E}} \left[r_{ij}^k \, w_{ij}^k - \frac{\mu_{ij}^k}{2} (w_{ij}^k)^2 \right] - \frac{\alpha_{ij}}{2} \left(\sum_{\substack{k\in\mathcal{K}: \\ (i,j,k)\in E}} w_{ij}^k \right)^2 \right\}. \tag{28}$$

Fix an ordering of the variables w that groups together all components w_{ij}^k corresponding to the same ordered pair $(i,j) \in \mathcal{R}$. With this ordering, the Hessian matrix $\nabla^2\Phi \in \mathbb{R}^{m\times m}$ is block-diagonal with respect to the ordered pairs (i,j):

$$\nabla^2\Phi = \begin{bmatrix} \nabla^2\Phi_{i_1j_1} & 0 & \cdots & 0 \\ 0 & \nabla^2\Phi_{i_2j_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \nabla^2\Phi_{i_Rj_R} \end{bmatrix}. \tag{29}$$

For a fixed $(i,j) \in \mathcal{R}$, let $m_{ij} := |\{k \in \mathcal{K} : \ (i,j,k) \in E\}|$ denote the number of variables associated with that pair and let $\mathbf{1}_{m_{ij}} \in \mathbb{R}^{m_{ij}}$ denote the vector of all ones. Then, the corresponding block of the Hessian is given by

$$\nabla^2\Phi_{ij} = -D_{ij} - \alpha_{ij}\,\mathbf{1}_{m_{ij}}\mathbf{1}_{m_{ij}}^\top \in \mathbb{R}^{m_{ij}\times m_{ij}}, \tag{30}$$

where $D_{ij} = \operatorname{diag}(\mu_{ij}^k)$ is the diagonal matrix collecting the coefficients μ_{ij}^k associated with (i,j). For any vector $z_{ij} \in \mathbb{R}^{m_{ij}}$, it holds that

$$z_{ij}^\top \nabla^2\Phi_{ij} z_{ij} = -\sum_{\substack{k\in\mathcal{K}:\\ (i,j,k)\in E}} \mu_{ij}^k (z_{ij}^k)^2 - \alpha_{ij}\left(\sum_{\substack{k\in\mathcal{K}:\\ (i,j,k)\in E}} z_{ij}^k\right)^2. \tag{31}$$

Since $\mu_{ij}^k > 0$ and $\alpha_{ij} \geq 0$, the right-hand side is strictly negative for all $z_{ij} \neq 0$. Therefore, each block $\nabla^2\Phi_{ij}$ is negative definite and consequently $\nabla^2\Phi$ is negative definite on $\mathbb{R}^m$. This proves that Φ is strictly concave.

Since $\mathcal{W}_\varepsilon(c,\rho)$ is a nonempty convex set, strict concavity implies that the maximization problem admits a unique solution. Finally, by standard results on exact potential generalized Nash equilibrium problems with shared constraints, this unique maximizer coincides with the unique variational equilibrium of the game. □

The social welfare function is defined as the sum of the utility functions of players, i.e.,

$$S(w) := \sum_{k\in\mathcal{K}} u_k(w), \tag{32}$$

while the social optimum is the unique maximizer of S on the feasible set $\mathcal{W}_\varepsilon(c,\rho)$, i.e.,

$$w^{so} = \arg\max\{S(w) : \ w \in \mathcal{W}_\varepsilon(c,\rho)\}. \tag{33}$$

We now give an a priori estimate of the difference between the optimal value of the social welfare and the social welfare evaluated at the variational equilibrium.

Theorem 2. *Let $w^\star$ be the variational equilibrium of the game and w^{so} the social optimum. Then, the following inequalities hold:*

$$S(w^\star) \leq S(w^{so}) \leq S(w^\star) + (|\mathcal{K} - 1|) \sum_{(i,j)\in\mathcal{R}} \frac{\alpha_{ij}}{2}\left(\frac{\rho^2 c_j^2}{c_i^2} - |\mathcal{K}|^2\varepsilon^2\right). \tag{34}$$

Proof. The first inequality follows directly from the definition of w^{so}. We now prove the second inequality. If the congestion term in each utility function is denoted by

$$C(w) := \sum_{(i,j)\in\mathcal{R}} \frac{\alpha_{ij}}{2} \left[y_{ij}(w)\right]^2, \tag{35}$$

then the following relation between the social welfare and the potential holds:

$$S(w) = \Phi(w) - (|\mathcal{K}| - 1)C(w) \qquad \forall\, w \in \mathcal{W}_\varepsilon(c,\rho). \tag{36}$$

Moreover, for any $(i,j) \in \mathcal{R}$ and $w \in \mathcal{W}_\varepsilon(c,\rho)$ we have

$$y_{ij}(w) = \sum_{\substack{k\in\mathcal{K}:\\(i,j,k)\in E}} w_{ij}^k \geq |\mathcal{K}|\varepsilon, \tag{37}$$

and (4) implies that

$$c_i\, y_{ij}(w) = \sum_{\substack{k\in\mathcal{K}:\\(i,j,k)\in E}} c_i\, w_{ij}^k \leq \sum_{h\in V} \sum_{\substack{k\in\mathcal{K}:\\(h,j,k)\in E}} c_h\, w_{hj}^k = \rho\, c_j, \tag{38}$$

Hence, we have

$$|\mathcal{K}|^2\varepsilon^2 \leq [y_{ij}(w)]^2 \leq \frac{\rho^2 c_j^2}{c_i^2} \qquad \forall\, w \in \mathcal{W}_\varepsilon(c,\rho). \tag{39}$$

Therefore, the following chain of equalities and inequalities holds:

$$\begin{aligned} S(w^{so}) &= \Phi(w^{so}) - (|\mathcal{K}| - 1)C(w^{so}) \\ &\leq \Phi(w^*) - (|\mathcal{K}| - 1)C(w^{so}) \\ &= S(w^*) + (|\mathcal{K}| - 1)C(w^*) - (|\mathcal{K}| - 1)C(w^{so}) \\ &= S(w^*) + (|\mathcal{K}| - 1) \sum_{(i,j)\in\mathcal{R}} \frac{\alpha_{ij}}{2} \left([y_{ij}(w^*)]^2 - [y_{ij}(w^{so})]^2 \right) \\ &\leq S(w^*) + (|\mathcal{K}| - 1) \sum_{(i,j)\in\mathcal{R}} \frac{\alpha_{ij}}{2} \left(\frac{\rho^2 c_j^2}{c_i^2} - |\mathcal{K}|^2\varepsilon^2 \right), \end{aligned} \tag{40}$$

where the first and second equality follow from (36), the first inequality from Theorem 1, and the second inequality from (39). □

4. An Application to an Airport Network

In this section, we show an application of the GNEP model described in Section 3.2 to a network of airports. In particular, we consider a multigraph (V,E), where the set of nodes V is a set of airports, the set of edges E is a set of air connections between airports, and the set of classes K is the set of airlines operating these connections. In this context, the weight w_{ij}^k associated with each edge (i,j,k) represents the number of passengers who, in a certain period of time (e.g., one year), travel from airport i to airport j using flights operated by airline k. In this way, the elements of the adjacency matrix $A(w)$ represent the passenger flows between the airports considered, and the eigenvector centrality of the airports can be calculated accordingly. Assuming that an air traffic regulatory authority imposes a predetermined value on the centrality of each airport, then the GNEP described in Section 3.2 consists of a game in which airlines aim to maximize their utility functions

u_k, choosing the weights w_{ij}^k so that the overall centrality of the airports is equal to the fixed values.

For the numerical experiments, we considered the multigraph (V, E), where V is the set of major Italian airports, $\mathcal{K}$ is the set of major airlines operating flights between Italian airports, and each edge $(i, j, k) \in \mathcal{E}$ represents a flight from airport i to airport j operated by airline k. The data was downloaded from the repository (Peixoto, 2020), which collected data from the Openflights airport network (Openflights Website, 2026). Specifically, since the latter network's data concerns connections between airports around the world, only connections between Italian airports were selected and connections operated by airlines providing fewer than 10 connections were eliminated. In this way, the resulting multigraph has $n = 34$ nodes, $m = 414$ edges and $|\mathcal{K}| = 10$ classes, and is strongly connected. Figure 2 shows the underlying graph corresponding to the multigraph (V, E). The list of airports and airlines considered are shown in Tables 1 and 2, respectively.

Table 1. List of Italian airports considered in the numerical experiments and their eigenvector centrality indices.

#	Name	City	Eigenvector Centrality
1	Bari Karol Wojtyla Airport	Bari	4.83
2	Pescara International Airport	Pescara	0.23
3	Brindisi & Salento Airport	Brindisi	2.72
4	Lamezia Terme Airport	Lamezia	2.26
5	Comiso Airport	Comiso	0.15
6	Catania–Fontanarossa Airport	Catania	11.61
7	Lampedusa Airport	Lampedusa	0.32
8	Pantelleria Airport	Pantelleria	0.17
9	Falcone & Borsellino Airport	Palermo	9.42
10	Reggio Calabria Airport	Reggio Calabria	0.41
11	Vincenzo Florio Airport Trapani-Birgi	Trapani	0.27
12	Alghero–Fertilia Airport	Alghero	1.73
13	Cagliari Elmas Airport	Cagliari	4.55
14	Olbia Costa Smeralda Airport	Olbia	3.02
15	Malpensa International Airport	Milan Malpensa	6.11
16	Il Caravaggio International Airport	Bergamo	3.74
17	Turin Airport	Turin	3.54
18	Genoa Cristoforo Colombo Airport	Genoa	1.58
19	Milano Linate Airport	Milan Linate	9.31
20	Parma Airport	Parma	0.09
21	Cuneo International Airport	Cuneo	0.09
22	Bolzano Airport	Bolzano	0.19
23	Bologna Guglielmo Marconi Airport	Bologna	2.75
24	Treviso–Sant'Angelo Airport	Treviso	0.49
25	Trieste, Friuli Venezia Giulia Airport	Trieste	1.29
26	Verona Villafranca Airport	Verona	2.06
27	Ancona Falconara Airport	Ancona	0.37
28	Venice Marco Polo Airport	Venice	3.49
29	Ciampino, G. B. Pastine International Airport	Rome Ciampino	0.19
30	Leonardo da Vinci, Fiumicino Airport	Rome Fiumicino	14.59
31	Naples International Airport	Naples	5.42
32	Pisa International Airport	Pisa	1.69
33	Peretola Airport	Florence	1.26
34	Umbria International Airport	Perugia	0.06

To calculate the eigenvector centrality of airports, we used data from the top 100 connections between Italian airports by passenger numbers in 2024, which are available on the Italian Civil Aviation Authority website (Enac Website, 2026). For each connection for which data was not available, we estimated the average number of passengers equal to 30,000. The resulting adjacency matrix $A(w)$ has been divided by 1000 so that each element

$A_{ij}(w)$ represents the thousands of passengers from i to j. The spectral radius ρ is equal to 2261 and the values for the eigenvector centrality of the airports are shown in Table 1. For ease of reading, the sum of all centrality values has been set equal to 100.

Table 2. List of airlines considered in the numerical experiments and number of edges associated with each of them.

Code	Airline	No. Edges
FR	Ryanair Ltd.	128
AZ	Italia Trasporto Aereo S.p.A. dba ITA S.p.A.	100
IG	Global Aviation dba GEA	42
V7	Volotea, S.L.	40
AP	Alba Star, S.A. dba Alba Star	30
U2	EasyJet UK Limited	28
BV	Toki Air Co., Ltd.	16
VY	Vueling Airline SA	10
IB	Iberia Lineas Aereas de Espana S.A. Operadora	10
F7	I Fly Ltd.	10

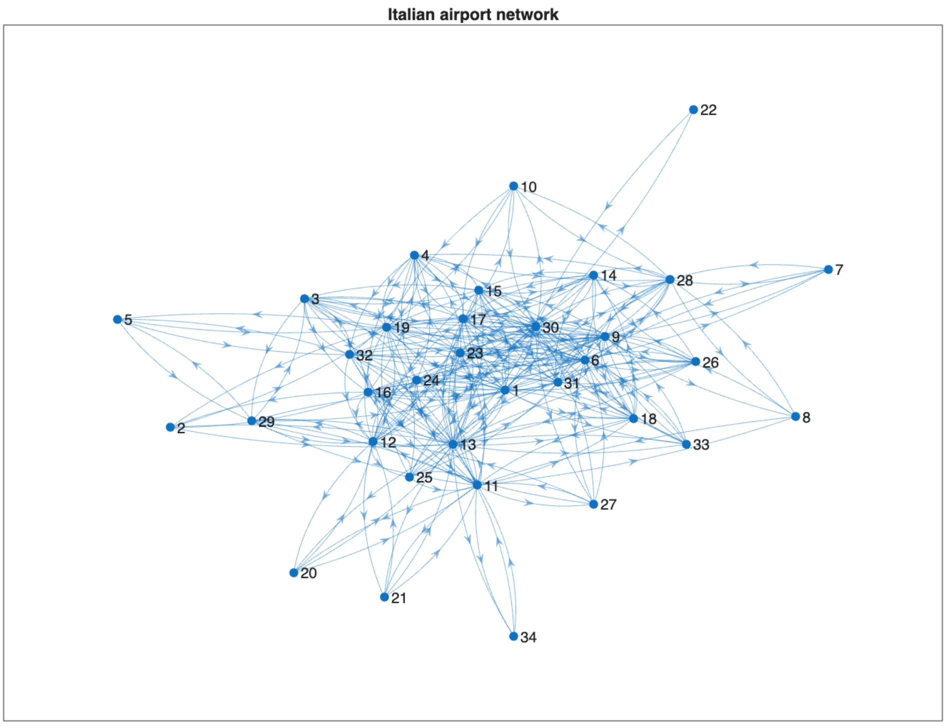

Figure 2. The Italian airport network. The node's numbers represent the airports reported in Table 1.

The parameters of the utility functions were set as follows: r^k_{ij} varies between 150 and 300 and is proportional to the number of passengers on connection (i, j); $\mu^k_{ij} = 1$ for each $(i, j, k) \in E$ and $\alpha_{ij} = 0.01$ for each $(i, j) \in \mathcal{R}$. The numerical experiments were performed with MATLAB R2025b. The variational equilibrium of the GNEP and the social optimum were computed by solving the corresponding quadratic programs through the `quadprog` function from the MATLAB optimization toolbox. In the feasible region $W_\varepsilon(c, \rho)$, the parameters c and ρ were set equal to the original values described above, while ε was set equal to 10.

Table 3 shows the utility of each airline computed at the variational equilibrium of the GNEP and at the social optimum. The results show that, for each airline, the utility values in the two cases are fairly similar, although for some airlines the value is higher at equilibrium than at the social optimum, while for others the opposite is true. Interestingly, airlines with a larger number of routes (e.g., Ryanair and ITA) exhibit modest deviations between the equilibrium and social optimum, suggesting that strategic decentralization does not significantly penalize the major carriers. On the other hand, smaller airlines experience relatively larger percentage variations.

Table 3. Utility of each airline computed at the variational equilibrium of the GNEP and at the social optimum.

Code	Name	Utility at Var. Equilibrium	Utility at Social Optimum
FR	Ryanair Ltd.	1,341,501	1,334,657
AZ	Italia Trasporto Aereo S.p.A. dba ITA S.p.A.	1,184,238	1,196,011
IG	Global Aviation dba GEA	494,426	493,875
V7	Volotea, S.L.	337,449	338,176
AP	Alba Star, S.A. dba Alba Star	341,287	336,674
U2	EasyJet UK Limited	403,598	410,399
BV	Toki Air Co., Ltd.	167,235	176,643
VY	Vueling Airline SA	73,078	76,194
IB	Iberia Lineas Aereas de Espana S.A. Operadora	73,078	76,194
F7	I Fly Ltd.	14,480	19,585

In the following, we also report a sensitivity analysis to investigate how much the variational equilibrium varies as a function of the variation in the centrality index of a given node in the network. Specifically, we changed the centrality index of a given node i by adding a constant value δ, with $\delta \in [-2, 2]$, and we computed the following relative sensitivity index:

$$RSI(\delta, c) = \frac{\dfrac{\|w^\star(c + \delta e_i) - w^\star(c)\|_s}{\|w^\star(c)\|_s}}{\dfrac{|\delta|}{\|c\|_s}}, \tag{41}$$

where c is the original eigenvector centrality vector and $w^\star(c)$ the corresponding variational equilibrium; $c + \delta e_i$ is the modified centrality vector, with e_i the i-th vector of the canonical basis of $\mathbb{R}^n$, and $w^\star(c + \delta e_i)$ the corresponding variational equilibrium; $\| \cdot \|_s$ is the s-norm. To analyze these variations from different points of view, we chose to use three types of norms: $s = 1$, $s = 2$ and $s = \infty$.

The relative sensitivity index $RSI(\delta, c)$ measures the proportional variation of the equilibrium weight vector with respect to a proportional perturbation of the target centrality. In other words, it quantifies the elasticity of the equilibrium configuration w with respect to changes in the prescribed centrality profile. Values of RSI close to zero indicate that the equilibrium weights are robust with respect to small variations in the centrality target, while larger values signal a higher structural dependence of the passenger flows on the imposed centrality level. The normalization allows for a scale-free comparison across different nodes and norms.

Figure 3 shows the graph of the RSI(δ, c) function for the four Italian airports with the highest centrality index (Rome Fiumicino, Catania, Palermo, and Milan Linate) for each type of norm.

The four sub-figures show similar results. All three sensitivity indices, computed using the 1-, 2-, and infinity norms, exhibit a regular and monotonically decreasing trend as the parameter δ increases over the considered interval. In particular, the values decrease gradually without showing any reversals in trend. Moreover, the relative ordering among the three curves remains unchanged throughout the entire interval. For Rome Fiumicino and Catania airports, the index computed using the 1-norm is largest, followed by the infinity norm. For Palermo and Milan Linate airports, however, the opposite is true. The 2-norm attains the smallest values for all four airports. No intersections between the curves are observed, and the relative differences among the three indices remain substantially stable. Finally, all curves appear smooth and continuous, with no visible jumps or discontinuities. This indicates that, within the analyzed perturbation range, the dependence of the sensitivity index on the parameter δ evolves regularly, without abrupt changes in the behavior of the solution. The sensitivity indices remain within a bounded range over the entire perturbation interval, indicating that the equilibrium weights vary in a controlled manner as the centrality target is perturbed. In the context of transportation networks, this analysis quantifies how much passenger flows must be redistributed to achieve a prescribed change in the centrality of a given airport. This could provide useful information for network design and policy planning.

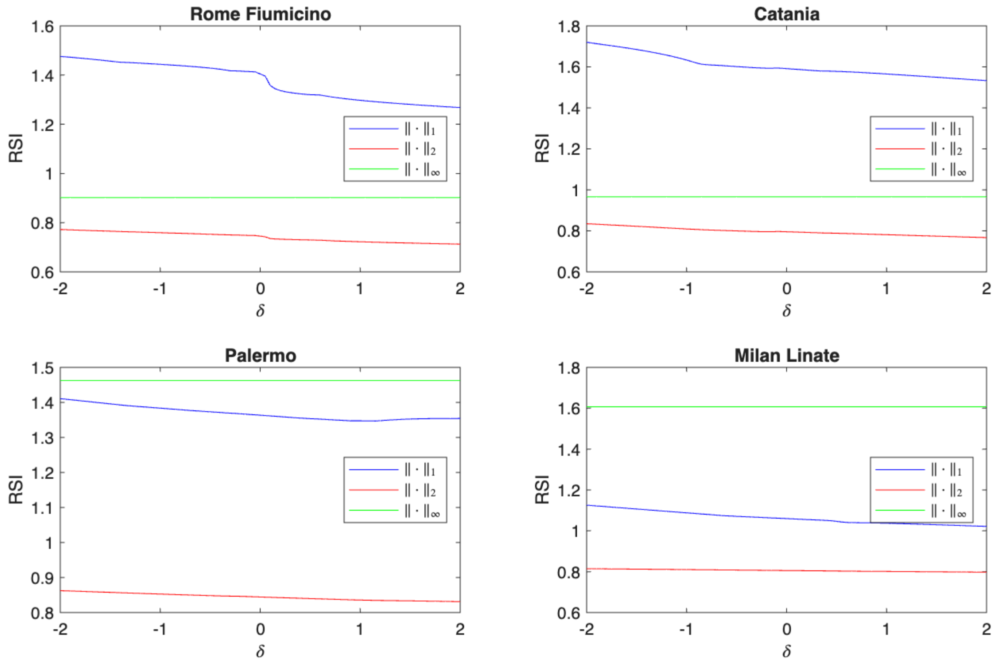

Figure 3. The relative sensitivity index with respect to the variation δ in the centrality index of one of the four Italian airports with the highest centrality.

Next, we analyzed the sensitivity of the variational equilibrium to the congestion parameters. We modified the parameter α_{ij} for a given connection (i, j) by adding a constant value δ, with $\delta \in [-0.01, 0.01]$, and we computed the following relative sensitivity index:

$$RSI(\delta, \alpha) = \frac{\dfrac{\|w^\star(\alpha + \delta e_{ij}) - w^\star(\alpha)\|_s}{\|w^\star(\alpha)\|_s}}{\dfrac{|\delta|}{\|\alpha\|_s}}, \tag{42}$$

where α is the original vector of congestion parameters and $w^\star(\alpha)$ the corresponding variational equilibrium; $\alpha + \delta e_{ij}$ is the parameter vector modified in component (i, j) only and $w^\star(\alpha + \delta e_{ij})$ the corresponding variational equilibrium.

Figure 4 shows the graph of the RSI(δ, α) function for the top four connections between Italian airports by passenger numbers (Catania–Rome Fiumicino, Rome Fiumicino–Catania, Rome Fiumicino–Palermo, and Palermo–Rome Fiumicino) for each type of norm.

We can notice that moderate changes in the congestion intensity on a single connection produce limited global adjustments in the equilibrium configuration. In economic terms, although congestion affects the strategic interaction among airlines operating on the same route, its local variation does not drastically alter the overall passenger distribution required to maintain the prescribed centrality profile.

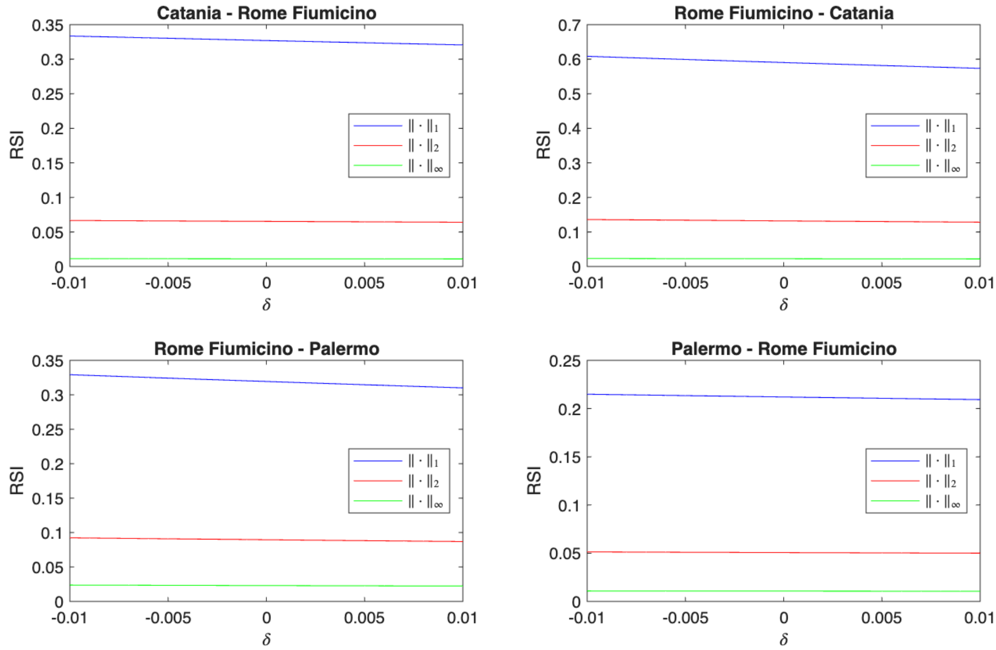

Figure 4. The relative sensitivity index with respect to the variation δ in the congestion parameter of one of the top four connections between Italian airports by passenger numbers.

Finally, we analyzed the sensitivity of the variational equilibrium to the cost parameters. We modified the parameters μ_{ij}^k for all edges corresponding to a given connection (i,j) by adding a constant value δ, with $\delta \in [-0.5, 0.5]$, and we computed the following relative sensitivity index:

$$RSI(\Delta,\mu) = \frac{\dfrac{\|w^\star(\mu+\Delta) - w^\star(\mu)\|_s}{\|w^\star(\mu)\|_s}}{\dfrac{\|\Delta\|_s}{\|\mu\|_s}}, \tag{43}$$

where μ is the original vector of cost parameters and $w^\star(\mu)$ the corresponding variational equilibrium; $\mu + \Delta$ is the cost parameter vector modified in edges corresponding to the connection (i,j), with

$$\Delta_{h\ell}^k = \begin{cases} \delta & \text{if } (h,\ell) = (i,j), \\ 0 & \text{otherwise,} \end{cases} \tag{44}$$

and $w^\star(\mu+\Delta)$ the corresponding variational equilibrium.

Figure 5 shows the graph of the RSI(δ,μ) function for the top four connections between Italian airports by passenger numbers (Catania–Rome Fiumicino, Rome Fiumicino–Catania, Rome Fiumicino–Palermo, and Palermo–Rome Fiumicino) for each type of norm.

The sensitivity analysis with respect to the cost parameters μ shows a stronger reaction of the equilibrium configuration compared to the congestion case. In particular, the RSI values are significantly larger, especially for the busiest connections, indicating that variations in marginal operating costs have a more pronounced impact on the equilibrium passenger flows. This behavior is economically intuitive: while congestion affects all airlines symmetrically through the aggregate flow, the parameters μ_{ij}^k directly modify the individual quadratic cost of each airline, thereby altering their marginal incentives. Nevertheless, even in this case the response remains smooth and monotonic, confirming the structural stability of the variational equilibrium with respect to parametric perturbations.

From an economic and policy perspective, the sensitivity analysis provides insights into the network's vulnerability and controllability. The relative sensitivity index (RSI) can be interpreted as an indicator of how strongly equilibrium passenger flows react to marginal interventions. Higher RSI values identify airports or connections where small changes in policy targets or parameters induce relatively large reallocations of traffic. These elements can therefore be regarded as more "sensitive" nodes or routes, where regulatory interventions may have amplified system-wide effects. The results also show that the equilibrium response to variations in centrality targets is smooth and bounded, indicating a significant degree of structural stability. This suggests that regulators can adjust the relative importance of airports without generating abrupt or unpredictable changes in traffic distribution. Regarding model parameters, the analysis highlights a clear difference between congestion and cost effects. Variations in congestion parameters produce relatively limited and localized adjustments, whereas changes in cost parameters have a much stronger impact on equilibrium flows. This implies that pricing policies or cost-based interventions represent more effective tools for influencing the global structure of the network. Overall, these findings suggest that the proposed framework can support policy design by identifying which nodes are more sensitive and which policy levers are more effective in shaping transportation networks.

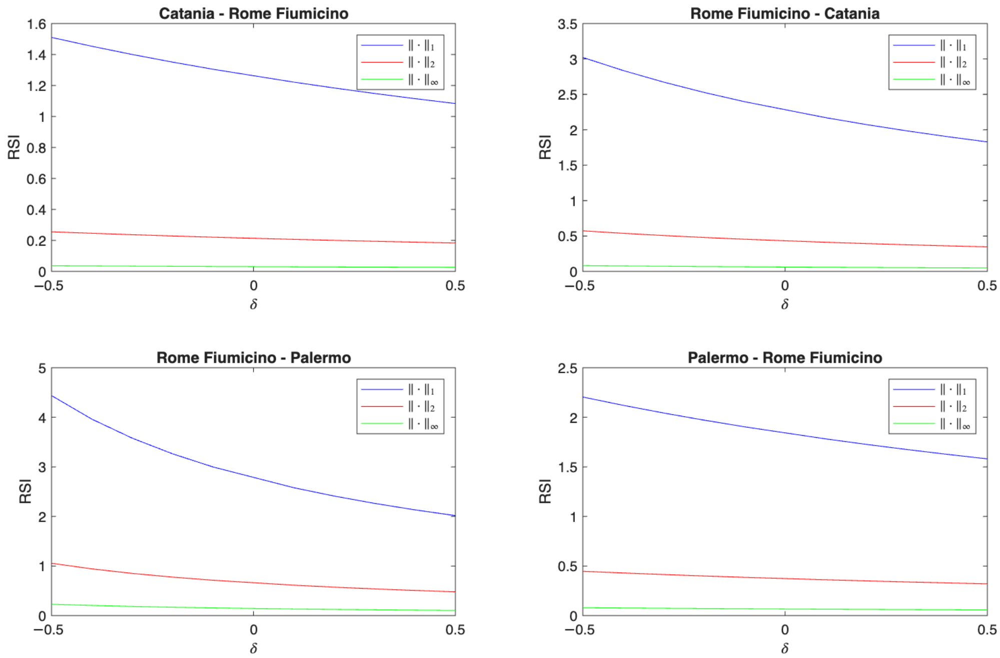

Figure 5. The relative sensitivity index with respect to the variation δ in the cost parameters of edges corresponding to one of the top four connections between Italian airports by passenger numbers.

5. Conclusions

This paper introduces a game-theoretic framework for the inverse eigenvector centrality problem on directed multigraphs. To the best of our knowledge, this is the first contribution that models the inverse centrality problem as a strategic game, interpreting the admissible set of centrality-compatible weights as the shared feasible region of a generalized Nash equilibrium problem. The resulting game admits an exact potential structure, ensuring the existence and uniqueness of the variational equilibrium. The application to the Italian airport network shows that the equilibrium configuration is economically interpretable and robust with respect to perturbations of centrality targets and model parameters. Future research may extend the approach to alternative centrality measures, dynamic settings, or regulatory design problems.

Author Contributions: Conceptualization, M.P. and F.R.; methodology, M.P. and F.R.; software, M.P. and F.R.; validation, M.P. and F.R.; formal analysis, M.P. and F.R.; investigation, M.P. and F.R.; resources, M.P. and F.R.; data curation, M.P. and F.R.; writing—original draft preparation, M.P. and F.R.; writing—review and editing, M.P. and F.R.; visualization, M.P. and F.R.; supervision, M.P. and F.R.; project administration, M.P. and F.R.; funding acquisition, M.P. and F.R. All authors have read and agreed to the published version of the manuscript.

Funding: The research was supported by the MUR research program PRIN2022 founded by the European Union—Next Generation EU (Project "ACHILLES, eco-sustAinable effiCient tecHdrIven Last miLE logiStics", CUP: E53D23005640006). It was also partially supported by the research project "Programma ricerca di Ateneo UNICT 2024-26 NOVA—Network Optimization and Vulnerability Assessment" of the University of Catania.

Data Availability Statement: The data regarding the application have been downloaded by Enac website, 2024 Dati di traffico https://www.enac.gov.it/app/uploads/2025/07/Dati-di-traffico-2024-v.0.7_cover_c.pdf (accessed on 10 February 2026) and The Netzschleuder network catalogue and repository https://networks.skewed.de/net/openflights (accessed on 10 February 2026).

Acknowledgments: The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA—National Group for Mathematical Analysis, Probability and their Applications) of the Istituto Nazionale di Alta Matematica (INdAM—National Institute of Higher Mathematics).

Conflicts of Interest: The authors declare no conflicts of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript; or in the decision to publish the results.

Abbreviation

The following abbreviation is used in this manuscript:

GNEP Generalized Nash Equilibrium Problem

Appendix A. An Overview of GNEPs with Shared Constraints and the Variational Inequality Approach to Their Solution

The material of this appendix is based on the paper (Nabetani et al., 2011). In GNEPs each player's strategy set may depend on the strategies of the other players. We consider here the following framework: each player $p = 1, \ldots, N$ controls the variable $x_p \in \mathbb{R}^{n_p}$ and, denoting by $n := \sum_{p=1}^{N} n_p$, we are given a function $g : \mathbb{R}^n \to \mathbb{R}^m$ describing the shared constraints. Moreover, for each player p, we are given a function $h_p : \mathbb{R}^{n_p} \to \mathbb{R}^{k_p}$ describing the individual constraints. The strategy set of player p is then defined as

$$K_p(x_{-p}) = \{x_p \in \mathbb{R}^{n_p} : \ g(x_p, x_{-p}) \le 0, \quad h_p(x_p) \le 0\}.$$

Thus, players share a common constraint g and have additional individual constraints given by h_p. With these ingredients, the GNEP is the problem of finding $x^* = (x_1^*, \ldots, x_N^*) \in \mathbb{R}^n$ such that, for any $p \in \{1, \ldots, N\}$, $x_p^* \in K_p(x_{-p}^*)$ and

$$u_p(x_p^*, x_{-p}^*) \ge u_p(x_p, x_{-p}^*), \qquad \forall\, x_p \in K_p(x_{-p}^*). \tag{A1}$$

We will work under the common (although not minimal) assumptions that, for each fixed x_{-p}, the function $u_p(\cdot, x_{-p})$ is concave and continuously differentiable, and that the components of g and h_i are convex and continuously differentiable. As a consequence, a necessary and sufficient condition for $x_p^* \in K_p(x_{-p}^*)$ to satisfy (A1) is

$$-\nabla_{x_p} u_p(x_p^*, x_{-p}^*)^\top (x_p - x_p^*) \ge 0, \qquad \forall\, x_p \in K_p(x_{-p}^*). \tag{A2}$$

Thus, defining

$$F(x) := -\begin{pmatrix} \nabla_{x_1} u_1(x) \\ \vdots \\ \nabla_{x_N} u_N(x) \end{pmatrix} \in \mathbb{R}^n, \tag{A3}$$

and

$$K(x) = K_1(x_{-1}) \times \cdots \times K_N(x_{-N}), \tag{A4}$$

it follows that x^* is a GNE if and only if $x^* \in K(x^*)$ and

$$F(x^*)^\top (x - x^*) \geq 0, \qquad \forall\, x \in K(x^*). \tag{A5}$$

The latter problem, where in addition the feasible set depends on the solution, is called a quasi-variational inequality and its solution is as difficult as the original GNEP.

Assume now that x^* is a solution of the GNEP. Hence, for each $p \in \{1, \ldots, N\}$, x_p^* solves the maximization problem

$$\max_{x_p} \Big\{ u_p(x_p, x_{-p}^*) :\ g(x_p, x_{-p}^*) \leq 0, \quad h_p(x_p) \leq 0 \Big\}. \tag{A6}$$

Under some standard constraint qualification, we can then write the KKT conditions for each maximization problem. We introduce the non-negative Lagrange multiplier $\lambda^p \in \mathbb{R}^m$ associated with the shared constraint $g(x_p, x_{-p}^*) \leq 0$ and the non-negative multiplier $\nu^p \in \mathbb{R}^{k_p}$ associated with the individual constraint $h_p(x_p) \leq 0$. The Lagrangian function for each player p reads as follows:

$$L_p(x_p, x_{-p}^*, \lambda^p, \nu^p) = u_p(x_p, x_{-p}^*) - [g(x_p, x_{-p}^*)]^\top \lambda^p - [h_p(x_p)]^\top \nu^p. \tag{A7}$$

The KKT conditions for all players are given by the following:

$$\nabla_{x_p} L_p(x_p^*, x_{-p}^*, \lambda^{p*}, \nu^{p*}) = 0, \qquad p = 1, \ldots, N, \tag{A8}$$

$$\lambda_j^{p*}\, g_j(x^*) = 0, \quad \lambda_j^{p*} \geq 0, \quad g_j(x^*) \leq 0, \qquad p = 1, \ldots, N,\ j = 1, \ldots, m, \tag{A9}$$

$$\nu_\ell^{p*}\, h_{p,\ell}(x_p^*) = 0, \quad \nu_\ell^{p*} \geq 0, \quad h_{p,\ell}(x_p^*) \leq 0, \qquad p = 1, \ldots, N,\ \ell = 1, \ldots, k_p. \tag{A10}$$

Conversely, under the assumptions made, if (x^*, λ^*, ν^*), where $\lambda^* = (\lambda^{1*}, \ldots, \lambda^{N*})$ and $\nu^* = (\nu^{1*}, \ldots, \nu^{N*})$, satisfy the KKT system (A8)–(A10), then x^* is a GNE.

Definition A1. *Let x^* be a GNE which, together with the Lagrange multipliers $\lambda^* = (\lambda^{1*}, \ldots, \lambda^{N*})$ and $\nu^* = (\nu^{1*}, \ldots, \nu^{N*})$, satisfies the KKT system of all players. Then, x^* is called a* normalized equilibrium *if there exist a vector $r \in \mathbb{R}_{++}^N$ and a vector $\bar{\lambda} \in \mathbb{R}_+^m$ such that*

$$\lambda^{p*} = \frac{\bar{\lambda}}{r_p}, \qquad \forall\, p = 1, \ldots, N. \tag{A11}$$

This means that, at a normalized equilibrium, the multipliers associated with the constraints shared by all players are proportional to a common multiplier. In the special case $r_p = 1$ for any $p = 1, \ldots, N$, i.e., when the multipliers associated with the shared constraints coincide for all players, p^* is called a variational equilibrium *(VE). Rosen proved that if the feasible set, which in our case is*

$$K = \big\{ x = (x_1, \ldots, x_N) \in \mathbb{R}^n :\ g(x) \leq 0, \quad h_p(x_p) \leq 0, \ \ p = 1, \ldots, N \big\}, \tag{A12}$$

is compact and convex, then there exists a normalized equilibrium for each $r \in \mathbb{R}_{++}^n$ (Rosen, 1965).

Now, for each $r \in \mathbb{R}^N_{++}$, let us define the vector function $F^r : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$F^r(x) := -\begin{pmatrix} r_1 \nabla_{x_1} u_1(x) \\ \vdots \\ r_N \nabla_{x_N} u_N(x) \end{pmatrix}. \tag{A13}$$

The variational inequality approach for finding normalized equilibria of the GNEP is expressed by the following theorem.

Theorem A1. *Suppose that, for each player $p = 1, \dots, N$, the function $u_p(\cdot, x_{-p})$ is concave and continuously differentiable on $\mathbb{R}^{n_p}$ for every fixed x_{-p}, and that the components of $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h_p : \mathbb{R}^{n_p} \to \mathbb{R}^{k_p}$ are convex and continuously differentiable. Let $r \in \mathbb{R}^n_{++}$ and assume that the feasible set defined in (A12) is nonempty. Assume moreover that, for each p and x_{-p}, a suitable constraint qualification (e.g., Slater condition) holds for the set $K_p(x_{-p})$.*

1. *If x^* is a solution of $VI(F^r, K)$ and $\bar{\lambda} \in \mathbb{R}^m_+$ and $\bar{\nu}^p \in \mathbb{R}^{k_p}_+$, with $p = 1, \dots, N$, are the associated Lagrange multipliers for the constraints $g(x) \leq 0$ and $h_p(x_p) \leq 0$, with $p = 1, \dots, N$, respectively, then x^* is a normalized equilibrium and the multipliers (λ^{p*}, ν^{p*}) of each player p satisfy the relations*

$$\lambda^{p*} = \frac{\bar{\lambda}}{r_p}, \qquad \nu^{p*} = \frac{\bar{\nu}^p}{r_p}, \qquad \forall\, p = 1, \dots, N. \tag{A14}$$

2. *If x^* is a normalized equilibrium such that the multipliers (λ^{p*}, ν^{p*}) of each player p satisfy*

$$\lambda^{p*} = \frac{\bar{\lambda}}{r_p}, \qquad \forall\, p = 1, \dots, N, \tag{A15}$$

for some $\bar{\lambda} \in \mathbb{R}^m_+$ and $r \in \mathbb{R}^N_{++}$, then x^* is a solution of $VI(F^r, K)$, and $\left(\bar{\lambda},\ r_1\nu^{1*}, \dots, r_N\nu^{N*}\right)$ are the corresponding multipliers associated with the constraints $g(x) \leq 0$ and $h_p(x_p) \leq 0$, $p = 1, \dots, N$.

References

Bonacich, P. (1972). Factoring and weighting approaches to status scores and clique identification. *Journal of Mathematical Sociology*, *2*(1), 113–120. [CrossRef]

Bonacich, P. (1987). Power and centrality: A family of measures. *American Journal of Sociology*, *92*, 1170–1182. [CrossRef]

Brin, S., & Page, L. (1998). The anatomy of a large-scale hypertextual Web search engine. *Computer Networks and ISDN Systems*, *30*(1–7), 107–117. [CrossRef]

Enac Website. (2026). *2024 Dati di Traffico*. Available online: https://www.enac.gov.it/app/uploads/2025/07/Dati-di-traffico-2024-v.0.7_cover_c.pdf (accessed on 10 February 2026).

Facchinei, F., Fischer, A., & Piccialli, V. (2007). On generalized Nash games and variational inequalities. *Operations Research Letters*, *35*, 159–164. [CrossRef]

Freeman, L. C. (1979). Centrality in social networks: Conceptual clarification. *Social Networks*, *1*(3), 215–239. [CrossRef]

Horn, R. A., & Johnson, C. R. (2013). *Matrix analysis*. Cambridge University Press.

Konnov, I. V. (2007). *Equilibrium models and variational inequalities*. Elsevier.

Minc, H. (1988). *Nonnegative matrices*. John Wiley & Sons.

Monderer, D., & Shapley, L. S. (1996). Potential games. *Games and Economic Behavior*, *14*, 124–143. [CrossRef]

Nabetani, K., Tseng, P., & Fukushima, M. (2011). Parametrized variational inequality approaches to generalized Nash equilibrium problems with shared constraints. *Computational Optimization and Applications*, *48*, 423–452. [CrossRef]

Nagurney, A. (1999). *Network economics: A variational inequality approach*. Springer.

Newman, M. E. J. (2010). *Networks: An introduction*. Oxford University Press.

Nicosia, V., Criado, R., Romance, M., Russo, G., & Latora, V. (2012). Controlling centrality in complex networks. *Scientific Reports*, *2*, 218. [CrossRef] [PubMed]

Openflights Website. (2026). Available online: https://openflights.org (accessed on 10 February 2026).

Passacantando, M., & Raciti, F. (2021). A note on generalized Nash games played on networks. In T. M. Rassias (Ed.), *Nonlinear analysis, differential equations, and applications* (Vol. 173, pp. 365–380). (Springer Optimization and Its Applications). Springer.

Passacantando, M., & Raciti, F. (2026). *Optimizing edge weights in the inverse eigenvector centrality problem* [Manuscript submitted for publication].

Peixoto, T. P. (2020). *The Netzschleuder network catalogue and repository*. Available online: https://networks.skewed.de/net/openflights (accessed on 10 February 2026).

Rosen, J. B. (1965). Existence and uniqueness of equilibrium points for concave N-person games. *Econometrica*, *33*(3), 520–534. [CrossRef]