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Compact Anti de Sitter manifolds with Spin-cone singularities

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Introduction

A pseudo-Riemannian structure consists of a manifold with a $\mathscr{C}^\infty$ field of non degenerate symmetric bilinear forms on its tangent bundle. The most studied case, the so-called Riemannian structure, has signature of the form $(n, 0)$. In the following, we will consider structures with signature of the form $(n-1, 1)$, i.e., Lorentzian structures. As in the Riemannian setting, we can study Lorentzian manifolds using classical objects of differential geometry, like geodesics, connections and curvature; despite this, the two cases have a lot of differences. The most remarkable one concerns completeness: for example, compact Riemannian manifolds are geodesically complete, instead in a compact Lorentzian manifold this is guaranteed only if the sectional curvature is constant, as proved by Carrière [Car89] and Klingler [Kli96].

A space-form is a complete pseudo-Riemannian manifold of dimension greater than 2 with constant curvature [KR85]. A Lorentzian manifold of constant sectional curvature positive, zero, negative is locally isometric to the so-called de-Sitter (dS), Minkowski, Anti-de Sitter (AdS) space-form, respectively. By the completeness theorems, it follows that every compact manifold of constant sectional curvature is isometric to the quotient of the universal cover of one of the *models* by a discrete subgroup of the isometry group.

In this thesis, we deal with three-dimensional AdS geometry, mostly with

AdS 3-manifolds which admits a foliation by closed timelike geodesics. The three-dimensional Anti-de Sitter space $\mathbb{A}\mathrm{d}\mathbb{S}^3$ identifies with $\mathrm{PSL}_2(\mathbb{R})$ endowed with the Lorentzian structure induced by the Killing form of its Lie algebra. The identity component of the isometry group of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is naturally isomorphic to $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, acting on $\mathrm{PSL}_2(\mathbb{R}) \simeq \mathbb{A}\mathrm{d}\mathbb{S}^3$ by right and left multiplication.

Kulkarni and Raymond proved that every compact $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold is a Seifert bundle over a surface [KR85]. The fibration can be chosen so that the fibers are geodesics. Each of these manifolds can be obtained as quotient of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ by a discrete subgroup of the group of isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, acting properly discontinuously, up to finite cover.

Moreover, the subgroup has the form $(j \times \rho)(\Gamma)$ where, up to switching the two factors of $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$, the group Γ is isomorphic to the fundamental group of an orientable closed surface and $j, \rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$ are representations, with j Fuchsian. We recall that $\mathrm{PSL}_2(\mathbb{R})$ is the group of orientation preserving isometries of the hyperbolic plane. If j is discrete and faithful, then $j(\Gamma)$ acts properly discontinuously on $\mathbb{H}^2$. Thanks to the work of Gueritaud, Kassel, Wolff [GKW15], ρ is "strictly dominated" by j, i.e., there exists a (j, ρ)-equivariant map from $\mathbb{H}^2$ to $\mathbb{H}^2$ which is a contraction and this property is strictly related to the existence of a geodesic foliation.

In Chapter 3, after giving a background on Anti-de Sitter geometry, we extend the results of [GKW15] to the non complete case. In particular, we study *AdS* manifolds which admit foliations by timelike geodesics of length π. More precisely, given a (even non complete) hyperbolic surface (S, g), a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ and an equivariant contractive map $f : \widetilde{S} \to \mathbb{H}^2$ with respect to ρ, there exists an *AdS* spacetime $\mathscr{M}(S, \rho, f)$ with a fiber bundle $\tau : \mathscr{M}(S, \rho, f) \to S$ such that the fiber $\tau^{-1}(\cdot)$ is a timelike geodesic of length π.

Then we show that this construction is in some sense universal.

Theorem A. Up to switching the time-orientation, any spacetime $\mathscr{M}$ foliated by timelike geodesics of length π is of the form $\mathscr{M}(S, \rho, f)$.

In order to be more accurate, we prove that if X is a unit future-directed timelike vector field generating a foliation $\mathscr{F}$ on the manifold $\mathscr{M}$ and ω is the volume form on $\mathscr{M}$, then the quadratic form $q(v) = \omega(v, X, \nabla_v X)$ is either positive or negative for almost all spacelike vectors v. Geometrically, this means that if $\gamma : [0,1] \to \mathscr{M}$ is a spacelike geodesic and $\ell(t)$ is the leaf in $\mathscr{F}$ passing through $\gamma(t)$, then $\ell(t)$ rotates always in the same direction. This allows us to distinguish two classes for the foliations: $\mathscr{F}$ is *right-handed* if $q(v) > 0$, *left-handed* otherwise. The way we label the two classes depends on the time-orientation.

We find that a foliation for $\mathscr{M}(S, \rho, f)$ is always left-handed and in the following we stipulate to fix the orientation so that all the foliations are left-handed.

The second goal of the thesis is to analyze the singularities that arises when S is the regular part of a surface with cone singularities. To this aim, we introduce the notion of *generalized spin-cone singularity.* The idea is to mimick the definition given by Barbot and Meusburger [BM12] in the flat case, notably, flat 3-dimensional Lorentzian manifolds with singularities coming from Euclidean surfaces with cone singularities and closed 1-forms on these surfaces. They considered a wedge in the Minkowski space quotiented by isometries which are composition of a rotation and a translation in the axis direction. Other exampels of Anti-de Sitter manifolds with cone singularities, along timelike lines and called *particles*, can be found in previous works of Benedetti, Barbot, Bonsante and Schlenker [BBS11, BB09, BS09].

In our case, the singularity is constructed around a timelike geodesic of the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-space, that is a closed geodesic. However, since the tube around a geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is not simply connected, we need to lift it to the universal cover. More precisely, we denote with $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ the complement of a geodesic and we reproduce something similar to the flat case in the universal cover of $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$. Indeed, a local model for this kind of singularities can be obtained by taking into account the quotient of the universal cover $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ by a lattice Λ of the group $\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})$. We notice that $\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) = \widetilde{\mathrm{Stab}(i)} \times \widetilde{\mathrm{Stab}(i)} \simeq \mathbb{R}^2$.

In this thesis, we are mainly interested in manifolds that admit timelike foliations. For this purpose, we prove:

Theorem B. The quotient $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \diagup \Lambda$ admits a foliation in timelike geodesics of length π if and only if $F_0 = (0, 2\pi)$ is a primitive element of the lattice Λ.

Let $\Lambda = \langle F_0, G_0 \rangle$ be a discrete lattice in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ with F_0 as in Theorem B. The element G_0 is not uniquely determined, in fact it is defined up to integer multiples of F_0. We prove that $G_0 = (\theta_0, \eta_0) \in \mathbb{R}_+ \times \mathbb{R} \mod (0, 2\pi)$ classifies all the lattices Λ such that $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \diagup \Lambda$ admits a timelike foliation.

In Chapter 2, we provide the *generalized spin-cone model for* the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-*spacetime.*

Definition. Let $\Lambda = \langle (0, 2\pi), (\theta_0, \eta_0) \rangle$ be a discrete lattice in $\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3})$ with $(\theta_0, \eta_0) \in \mathbb{R}_+ \times \mathbb{R} \mod (0, 2\pi)$. We define *model for a generalized spin-cone singularity associated to* (θ_0, η_0) the quotient manifold:

$$\mathbb{A}\mathrm{d}\mathbb{S}^3_{(\theta_0, \eta_0)} := \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \diagup \Lambda.$$

One can define an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold $\mathscr{M}$ with generalized spin-cone singularities as an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold in the complement of singular sets which are locally modelled on $\mathbb{A}\mathrm{d}\mathbb{S}^3_{(\theta_0, \eta_0)}$. Let us remark that θ_0 is a well-defined positive number, while $\eta_0 \in \mathbb{R} \diagup 2\pi\mathbb{Z}$.

In Chapter 3 we prove the following result:

Theorem C. A manifold $\mathscr{M}(S, \rho, f)$ has generalized spin-cone singularities if and only if S is a hyperbolic surface with cone singularities. Moreover, chosing a contraction map f with equivariance ρ is equivalent to provide a (left-handed) foliation.

Cone points on the surface correspond to spin-cone singularities on the manifold. Moreover, if (θ_0, η_0) is the pair of invariants determining a spin-cone singularity on $\mathscr{M}(S, \rho, f)$, θ_0 is the conical angle of the corresponding cone point in S.

As we said, a compact Anti-de Sitter manifold is, up to a finite cover, a quotient of $\mathrm{PSL}_2(\mathbb{R})$ by a discrete group of the form $j, \rho \in \mathrm{Hom}(\Gamma, \mathrm{PSL}_2(\mathbb{R}))$. Tholozan in [Tho15] proved that the volume of such a quotient is proportional to the sum of the Euler classes of the representations j and ρ.

In Chapter 4, we compute the volume of a manifold $\mathscr{M}(S, \rho, f)$ that allows us to state a result similar to Tholozan's theorem for compact manifolds.

Theorem D. Let $\mathscr{M} = \mathscr{M}(h, f, \rho)$ be a bundle over an oriented hyperbolic surface S. Then the volume of $\mathscr{M}$ is determined as function of the maps $f, d\colon \widetilde{S} \to \mathbb{H}$:

$$\mathrm{Vol}\,(\mathscr{M}) = \pi\Big[\,\mathrm{Area}(f) + \mathrm{Area}(d)\Big].$$

Outline of the thesis. In Chapter 1 we give a brief background of pseudo-riemannian geometry and we introduce the formalism of (G, X)-structures. Furthermore, we lay out the models for $\mathbb{A}\mathrm{d}\mathbb{S}^3$, then we study timelike geodesics and foliations for this space. In Chapter 2, after reviewing the notion of conical singularities for the hyperbolic plane, we define the model for the generalized spin-cone singularity in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. In Chapter 3, we provide the construction of an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold as fiber bundle over a hyperbolic surface and we show the way to obtain $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold with generalized spin-cone singularities. In Chapter 4 we compute the volume of such manifolds.

CHAPTER 1

Background on Geometric structures and Lorentzian Geometry

The intent of this chapter is to provide basic elements of geometric structures on manifolds and to present the formalism of (G, X)-strucures.

Endowing a manifold M with a geometry means to locally identify M with a homogeneous space X, namely a space equipped with a group G acting transitively on it. The idea of a *model* geometry starts with Klein and Lie. After them, Ehresmann initiated a general study of geometric structures [Ehr36].

1.1 The formalism of geometric structures

1.1.1 Definition of geometric structures on a manifold

We introduce in this section the notion of (G, X)-structure and the basic theory [BBS11]. For more details on the topic of geometric structures we refer to the surveys [Gol10],[Gol19], [Gol88]. Informally, we can say that a geometric structure is a way to equip a manifold with a geometry.

Definition 1. A *geometry* is a pair (G, X) where X is an analytical manifold and G is a Lie group acting on X by diffeomorphism. We will require that the action is faithful, transitive and analytic.

Very basic examples of geometries are given by:

- *Euclidean geometry*: $(G, X) = (\mathrm{Isom}(R^n), \mathbb{R}^n)$;
- *affine geometry* $(G, X) = (\mathrm{Aff}(\mathbb{R}^n), \mathbb{R}^n)$.

If we want instead an example which is not a geometry in the above sense, we can consider the pair $(\mathrm{Diffeo}(\mathbb{R}^n), \mathbb{R}^n)$: the action of the group of diffeomorphisms satisfies the first two conditions, but not the third one.

Definition 2. Let (G, X) be a geometry and M a compact manifold having the same dimension as X.

A (G, X)-*atlas* on M is an atlas $\mathscr{A} = \{(U_i, \varphi_i)\}$ where $\{U_i\}$ is an open covering of M and

$$\varphi_i : U_i \to X$$

are injective local diffeomorphisms such that for all $i, j \in I$, over each connected component of $U_i \cap U_j$, there exists $g \in G$ such that

$$\varphi_j = g \circ \varphi_i.$$

The element $g \in G$ is said *transition function*.

Since the set of (G, X)-*atlas* is preordered by inclusion, we can give the following definition.

Definition 3. A (G, X)-*structure* is a maximal (G, X)-*atlas*.

Remark 1. Every (G,X)-*atlas* is contained in a unique (G,X)-*structure* formed by all the charts $\varphi : U \to X$ that are $\mathscr{A}$-*compatible*, i.e., such that $\mathscr{A} \cup \{(U,\varphi)\}$ is still a (G,X)-*atlas*.

If M is equipped with a (G,X)-structure, we will say that M is *locally modeled* on X or that M is a (G,X)-*manifold*.

As first examples, we can get (G,X)-structures on a manifold M from a local diffeomorphism φ of M in X. As charts we take the immersion restricted to open sets small enough so that the restriction is a diffeomorphism onto the image. In this case the atlas is given by $\{U, \varphi_{|U}\}$, where U is chosen so that $\varphi_{|U}$ is injective. Notice that transition maps in this atlas are trivial.

An other example is given in low dimensional topology.

Example 1.1.1. Let S be a closed surface. It admits a (G,X)-structure, where

$$X = \begin{cases} O(3)/O(2) = \mathbb{S}^2 & \text{if } \chi(S) > 0 \\ O(2) \ltimes \mathbb{R}^2/O(2) = \mathbb{E}^2 & \text{if } \chi(S) = 0 \\ PGL(\mathbb{R})/O(2) = \mathbb{H}^2 & \text{if } \chi(S) < 0 \end{cases}$$

where $\chi(S)$ is the Euler characteristic of the surface S.

Definition 4. Let $f : M \to N$ a local diffeomorphism and $\mathscr{A} = \{(V_j, \psi_j)\}$ (G,X)-stucture on N. We define the *pull-back* $f^*(\mathscr{A})$ through f as the (G,X)-structure on M given by $f^*(\mathscr{A}) = \{(U_i, f \circ \psi_j)\}$.

Proposition 1.1.2. If M is simply connected, any (G,X)-structure on M is induced by a local diffeomorphism dev: $M \to X$.

Definition 5. Let M be a (G,X)-manifold and f a diffeomorphism of M. We will say that f *preserves* the (G,X)-structure or that is a (G,X)-*structure automorphism* if the (G,X)-structure is equal to its pull-back through f.

More generally, we will say that a local diffeomorphism $f: M \to N$ is a *morphism of* (G, X)*-manifolds* if the (G, X)-structure on M is equal to the pull-back by f of the (G, X)-structure on N.

Proposition 1.1.3. Let M be a (G, X)-manifold and H a group of diffeomorphisms of M acting freely and properly discontinuously on M and preserving the (G, X)-structure. Then, there exists a unique (G, X)-structure on the quotient M/H such that the covering map from M to M/H is a morphism of (G, X)-structures.

The above proposition allows us to construct more (G, X)-structures. Let M be a manifold of the same dimension as X and $\widetilde{M}$ its universal covering, where the fundamental group $\pi_1(M)$ is identified with the group $\mathrm{Aut}(\widetilde{M} \to M)$ of the automorphisms of the covering. Let $\rho: \pi_1(M) \to G$ be a representation. Now let us assume that there exists a ρ-equivariant diffeomorphism $\mathrm{dev}: \widetilde{M} \to X$ and equip $\widetilde{M}$ with the pull-back (G, X)-structure by dev. Thanks to the equivariance of dev, the action of $\pi_1(M)$ preserves the (G, X)-structure and so we obtain a (G, X)-structure on M.

A more specific way to construct (G, X)-manifolds is to consider a discrete subgroup Γ of G acting freely and properly discontinuously. Then, we have that the quotient X/Γ is a (G, X)-manifold.

Actually, by the following proposition [Ehr36] we can say that all the (G, X)-structures are constructed in this way.

Proposition 1.1.4. A (G, X)-structure on M determines a pair (dev, ρ), where $\mathrm{dev}: \widetilde{M} \to X$ is a local diffeomorphism, called the *developing map*, and $\rho: \pi_1(M) \to G$ is a homomorphism, called *holonomy* representation. Moreover, the following equivariance property holds:

$$\mathrm{dev}(\gamma \tilde{x}) = \rho(\gamma) \, \mathrm{dev}(\tilde{x})$$

for every $\tilde{x} \in \widetilde{M}$.

The maps dev and ρ are unique up to the action of G, which means that two pairs (dev, ρ) and (dev', ρ') induce the same (G, X)-structure on M if and only if there exists an element $g \in G$ such that

$$\mathrm{dev}' = g \circ \mathrm{dev}$$

and

$$\rho' = g\rho g^{-1}$$

that is dev' is a translate of dev itself and $g\rho g^{-1}$ in a conjugate of ρ.

1.1.2 Completeness and geodesic completeness

In this section, we will treat the fundamental topic of completeness, transcribed in the language of (G, X)-structures.

Definition 6. A (G, X)-manifold M is *complete* if $\mathrm{dev} : \widetilde{M} \to X$ is a covering map.

Let us note that if X is simply connected, the notion of completeness is the same as saying that the developing map is a diffeomorphism. In other words, the developing map diffeomorphically identifies $\widetilde{M}$ with X. In that case, the holonomy representation identifies the fundamental group with a discrete subgroup of G acting as deck transformations on X. Therefore, complete (G, X)-manifold are the quotients $\Gamma \backslash X$ where Γ is a discrete subgroup of G acting freely and properly discontinuous on X.

In order to motivate Definition 6 we state the following proposition.

Proposition 1.1.5. Let X be a G-homogeneous space. Let us suppose that G preserves an affine connection on X that is geodesically complete. Let M be a manifold equipped with a (G, X)-structure and an induced connection.

Then, M is geodesically complete if and only if M is isomomorphic, as (G, X)-structure, to a quotient of X by a discrete subgroup of G acting freely and properly discontinuously on X.

In the case of compact manifolds, we can give a further characterization for the completeness.

Proposition 1.1.6. Let X be a simply connected G-homogeneous space, M a compact (G, X)-manifold and (dev, ρ) a pair associated to the (G, X)-structure of M. Then, M is a complete (G, X)-manifold if and only if $\rho(\pi_1(M))$ acts freely and properly discontinuously on X.

1.1.3 Basics on Lorentzian geometry

Definition 7. A *Lorentzian metric* on a manifold of dimension $n+1$ is a non-degenerate symmetric 2-tensor g of signature $(n, 1)$. A *Lorentzian manifold* is a connected manifold M equipped with a Lorentzian metric g.

Definition 8. In a Lorentzian manifold M we say that a non-zero vector $v \in TM$ is

- *spacelike* if $g(v, v) > 0$;
- *timelike* if $g(v, v) < 0$;
- *lightlike* if $g(v, v) = 0$.

Thus, we will refer to linear subspaces $V \subset T_x M$ as *spacelike*, *timelike*, *lightlike* if the restriction of g_x to V is respectively positive definite, degenerate or indefinite. A differentiable curve is spacelike, lightlike, timelike if its tangent vector is respectively spacelike, lightlike, timelike at every point. It is *causal* if the tangent vector is either timelike or lightlike.

1.2 Anti-de Sitter geometry

There are several models of Anti-de Sitter geometry. In the following we will introduce some of them.

1.2.1 The quadric model

Let $\mathbb{R}^{n,2}$ be the vector space $\mathbb{R}^{n+2}$ endowed with the non-degenerate bilinear symmetric form $\langle x,y\rangle_{n,2} \coloneqq \sum_{i=1}^{n} x_i y_i - x_{n+1}y_{n+1} - x_{n+2}y_{n+2}$ with signature $(n,2)$. We define the $(n+1)$-dimensional Anti de Sitter space as

$$\mathbb{H}^{n,1} \coloneqq \{x \in \mathbb{R}^{n+2} \;\mid\; \langle x,x\rangle = -1\}.$$

For $x \in \mathbb{H}^{n,1}$, the tangent space $T_x\mathbb{H}^{n,1}$ is $x^{\perp}$ and $g \coloneqq \langle\ ,\ \rangle_{|T\mathbb{H}^{n,1}}$ is a bilinear form of signature $(n,1)$. As a consequence, $(\mathbb{H}^{n,1}, g)$ is a Lorentzian manifold and it can be verified that it has constant curvature -1 with respect to the lorentzian metric g.

Furthermore, the pair $(\mathrm{Isom}(\mathbb{H}^{n,1}), \mathbb{H}^{n,1})$ defines a (G,X)-structure and every Lorentzian manifold of constant curvature -1 is locally modelled on $\mathbb{H}^{n,1}$.

Let us observe that $\mathbb{H}^{n,1}$ is diffeomorphic to $\mathbb{H}^n \times S^1$. Let $\mathbb{H}^n = \{z = (z_1,...,z_{n+1}) \in \mathbb{R}^{n+1} \,|\, z_1^2 + ... + z_n^2 - z_{n+1}^2 = -1, z_{n+1} > 0\}$ the hyperboloid model for the hyperbolic space. Then, the map $\varphi\colon \mathbb{H}^n \times \mathbb{R} \to \mathbb{H}^{n,1}$ defined by $\varphi(z,t) = (z_1,...,z_n, z_{n+1}\cos t, z_{n+1}\sin t)$ is a covering for the Anti-de Sitter space with deck transformations of the form $(z,t) \mapsto (z, t+2k\pi)$ for every $k \in \mathbb{Z}$. Hence, it induces the diffeomorphism $\bar{\varphi} : \mathbb{H}^n \times {}^{\mathbb{R}}\!/_{2\pi} \to \mathbb{H}^{n,1}$ [BS10].

1.2.2 Klein model

Under the canonical projection $\pi : \mathbb{R}^{n,2}\setminus\{0\} \to \mathbb{R}P^{n+1}$, the Anti-de Sitter space of dimension $n+1$ identifies with the subset $\Omega = \{[x] \,|\, q_{n,2} < 0\}$. The

Klein model, that we denote with $\mathbb{A}\mathrm{d}\mathbb{S}^{n+1}$, is a domain in the projective space $\mathbb{R}P^{n+1}$ whose boundary is the quadric of signature $(n,2)$. In order to be more specific,

$$\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n+1} = \{[x] \in \mathbb{R}P^{n+1} \,|\, q_{n,2}(x) = 0\},$$

that is the projectivization of the set of lightlike vectors in $\mathbb{R}^{n,2}$.

Isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^{n+1}$ are projective transformations which preserve $\partial\mathbb{A}\mathrm{d}\mathbb{S}^{n+1}$ [BS20].

For instance, in the case $n = 2$, if we consider the affine chart $x_4 \neq 0$ of $\mathbb{R}P^3$, the intersection of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ with the affine chart is mapped to the interior of the one-sheeted hyperboloid given by the equation $\{x_1^2 + x_2^2 - x_3^2 = 1\}$.

In this model, geodesics are given by straight lines. In particular, spacelike geodesics intersect with the boundary in two points, timelike geodesics do not have any intersection with the boundary and lightlike geodesics are tangent to the boundary [Tou16].

A hyperboloid of one sheet is foliated by two families of straight lines, which we refer to as the *right* family and the *left* family. The group $\mathrm{Isom}_+(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ of space and time-orientation preserving isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ preserves each family of the foliation. Fixing a spacelike plane Π in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, its intersection with the boundary $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ is a spacelike circle. This provides an identification between $\partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ and $\mathbb{R}P^1 \times \mathbb{R}P^1$. Indeed, given a point $x \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$, there exists a unique line in the right family and a unique line in the left one which pass through x. It follows that $x \in \partial\mathbb{A}\mathrm{d}\mathbb{S}^3$ gives a point in $\mathbb{R}P^1 \times \mathbb{R}P^1$. If we take another spacelike plane, the pair of points in $\mathbb{R}P^1$ is conjugated by an element of $\mathrm{PSL}_2(\mathbb{R})$.

Notice that the model $\mathbb{H}^{n,1}$ is a 2 : 1 covering over the Klein model $\mathbb{A}\mathrm{d}\mathbb{S}^{n+1}$.

1.3 Anti-de Sitter space in dimension three

1.3.1 $\mathrm{PSL}_2(\mathbb{R})$-model

In this thesis we are interested in 3-dimensional Anti-de Sitter manifolds, so here we provide the model we will use mostly in this dimension [BS20]. Notice that $\mathbb{R}^4$ equipped with the quadratic form $q(x,x) := \langle x,x\rangle_{2,2}$, i.e., $\mathbb{R}^{2,2}$, is isometric to the space of 2×2 matrices $M_2(\mathbb{R})$ with the quadratic form of signature $(2,2)$ given by $-det$. In this way, $\mathbb{H}^{2,1}$ is identified with the Lie group $\mathrm{SL}_2(\mathbb{R})$.

Moreover, it is possible to identify $\mathrm{PSL}_2(\mathbb{R}) = {}^{\mathrm{SL}_2(\mathbb{R})}\diagup_{\pm\mathrm{Id}}$ to $\mathrm{Isom}_+(\mathbb{H}^2)$, i.e., the group of orientation-preserving isometries of the hyperbolic plane, in the upper-half plane model

$$\mathbb{H}^2 := \left(\{z\in\mathbb{C}\,|\,\mathfrak{I}(z)>0\}, \frac{|dz|^2}{\mathfrak{I}(z)^2}\right)$$

by associating to an isometry of $\mathbb{H}^2$ its extension to the visual boundary $\partial_\infty\mathbb{H}^2 = \mathbb{R}P^1$, which is a projective transformation.

As a consequence, we can see the model $\mathbb{A}\mathrm{d}\mathbb{S}^3$ as the group of isometries of the hyperbolic plane.

1.3.2 The isometry group

Using the isomorphism between $\mathbb{R}^{2,2}$ and $M_2(\mathbb{R})$ we can also provide a description of the isometry group of $\mathbb{H}^{n,1}$ through the group $\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})$. The left action of this group on $M_2(\mathbb{R})$ is given by left and right multiplications, that is

$$(A,B)\cdot T = ATB^{-1}.$$

In particular, thanks to the Binet formula, this action preserves the bilinear form induced by $-det$, so we have a group representation

$$\mathrm{SL}_2(\mathbb{R})\times\mathrm{SL}_2(\mathbb{R})\to O(M_2(\mathbb{R}),-det),$$

where $O(M_2(\mathbb{R}), -det) \simeq O(2,2)$ identifies with the group $\mathrm{Isom}\,\mathbb{H}^{2,1}$. Taking the connected component of identity of the target, since the groups have same dimension, the representation is surjective. Moreover, its kernel is $\{(Id, Id), (-Id, -Id)\}$, so by the *First Isomorphism Theorem* we obtain that

$$\mathrm{Isom}_0(\mathbb{H}^{2,1}) = {\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})}\big/{\{(Id, Id), (-Id, -Id)\}},$$

so the isometry group of $\mathbb{H}^{2,1}$ is identified (up to a finite covering) with $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acting on $\mathrm{SL}_2(\mathbb{R})$ by left and right multiplication, while $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is identified with the group $\mathrm{PSL}_2(\mathbb{R})$ with isometry group $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ acting by left and right multiplication.

1.3.3 Timelike geodesics

Let us consider the set given by

$$\begin{aligned}\ell_{i,i} :=& \mathrm{Stab}_{\mathrm{PSL}_2(\mathbb{R})} = \mathrm{PSO}_2(\mathbb{R})\\ =&\{T \in \mathrm{PSL}_2(\mathbb{R}) \mid T(i) = i\}.\end{aligned}$$

Notice that $\ell_{i,i}$ is a projective line in $\mathbb{P}(M_4(\mathbb{R}))$ entirely contained in $\mathrm{PSL}_2(\mathbb{R})$ so it is a timelike line. More precisely

$$\gamma(\theta) = R_i^\theta = \left\{\begin{bmatrix}\cos\frac{\theta}{2} & -\sin\frac{\theta}{2}\\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{bmatrix} \mid \theta \in [0, 2\pi)\right\}$$

provides a parametrization of $\ell_{i,i}$. Moreover, we notice that $\left\langle \dot{\gamma}(\theta), \dot{\gamma}(\theta)\right\rangle = -1$, hence $\ell_{i,i}$ has length π. Notice that R_i^θ geometrically is the rotation of $\mathbb{H}^2$ around i of angle θ.

Remark 2. As already mentioned, there is a $2:1$ covering map between $\mathbb{H}^{2,1}$ and $\mathbb{A}\mathrm{d}\mathbb{S}^3$ so that geodesics in the first model have length 2π.

The line $\ell_{i,i}$ is a closed timelike geodesic. In particular $\ell_{i,i,}$ passes through the identity. We can get every timelike geodesic by the action of (the identity component of) the isometry group $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ on $\mathrm{PSL}_2(\mathbb{R})$.

Proposition 1.3.1. Every timelike geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ has the form:

$$\ell_{x,y} := \{T \in \mathrm{PSL}_2(\mathbb{R}) \mid T(y) = x\}$$

for $x, y \in \mathbb{H}^2$.

Lemma 1.3.2. $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ acts on $\ell_{x,y}$ in a way that

$$(A, B) \cdot \ell_{x,y} = \ell_{A(x),B(y)},$$

for every $(A, B) \in \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$.

Proof. Take first $F \in (A, B) \cdot \ell_{x,y}$, so

$$F = ATB^{-1} \qquad \text{for some} \qquad T \in \mathrm{SL}_2(\mathbb{R}) \;\; \text{s.t.} \qquad T(y) = x.$$

In particular,

$$FB(y) = AT(y) = A(x),$$

so $F \in \ell_{A(x),B(y)}$.

On the other hand, if $F \in \ell_{A(x),B(y)}$, we have

$$F(B(y)) = A(x),$$

so that $A^{-1}FB \in \ell_{x,y}$ and $F \in A\ell_{x,y}B^{-1}$.

□

Remark 3. For every $x, y \in \mathbb{H}^2$ there exist $A, B \in \mathrm{Isom}_+(\mathbb{H}^2) \simeq \mathrm{PSL}_2(\mathbb{R})$ such that $A(i) = x$ and $B(y) = i$. Then $(A, B) \cdot \ell_{i,i} = \ell_{x,y}$ is a geodesic, since $(A, B) \in \mathrm{Isom}_+(\mathrm{PSL}_2(\mathbb{R}))$.

Proof of Proposition 1.3.1. First of all, let us observe that $\ell_{x,y}$ is a geodesic. For every $x, y \in \mathbb{H}^2$ there exist $A, B \in \mathrm{Isom}_+(\mathbb{H}^2) \simeq \mathrm{PSL}_2(\mathbb{R})$ such that $A(i) = x$ and $B(y) = i$. By the Lemma 1.3.2 we can write $\ell_{x,y} = (A, B) \cdot \ell_{i,i}$ with $(A, B) \in \mathrm{Isom}_+(\mathrm{PSL}_2(\mathbb{R}))$. Thus it is well-defined the map

$$\begin{aligned} \Psi : \mathbb{H}^2 \times \mathbb{H}^2 &\to \{\text{timelike geodesics in } \mathrm{PSL}_2(\mathbb{R})\} \\ (x, y) &\mapsto \ell_{x,y} \end{aligned}$$

that is $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$-equivariant. The group $\mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ acts on the set of timelike geodesic transitively, so the map Ψ is a bijection.

□

Proposition 1.3.3. The following equivalence holds:

$$\ell_{x_1,y_1} \cap \ell_{x_2,y_2} = \varnothing \iff \mathrm{dist}_{\mathbb{H}^2}(x_1, x_2) \neq \mathrm{dist}_{\mathbb{H}^2}(y_1, y_2).$$

Proof. Let us assume that $\ell_{x_1,y_1} \cap \ell_{x_2,y_2} \neq \varnothing$. There exists $A \in \ell_{x_1,y_1} \cap \ell_{x_2,y_2}$ such that

$$\begin{cases} T(y_1) = x_1 \\ T(y_2) = x_2 \end{cases}$$

and then we have that

$$\mathrm{dist}_{\mathbb{H}^2}(x_1, x_2) = \mathrm{dist}_{\mathbb{H}^2}(T(y_1), T(y_2)) = \mathrm{dist}_{\mathbb{H}^2}(y_1, y_2),$$

where the second equality holds because T is an isometry of the hyperbolic plane.

Conversely, if $\mathrm{dist}_{\mathbb{H}^2}(x_1, x_2) = \mathrm{dist}_{\mathbb{H}^2}(y_1, y_2)$, then there exists $T \in \mathrm{Isom}(\mathbb{H}^2)$ sending the geodesic arc between y_1 and y_2 onto that between x_1 and x_2. Thus, $T \in \ell_{x_1,y_1} \cap \ell_{x_2,y_2}$.

□

Corollary 1.3.4. If $f : \mathbb{H}^2 \to \mathbb{H}^2$ is weak a contraction, for every $x \neq y \in \mathbb{H}^2$,

$$\ell_{x,f(x)} \cap \ell_{y,f(y)} = \varnothing.$$

Proof. Since f is a weak contraction,

$$\mathrm{dist}_{\mathbb{H}^2}(f(x), f(y)) < \mathrm{dist}_{\mathbb{H}^2}(x, y). \tag{1.1}$$

In particular, $\mathrm{dist}_{\mathbb{H}^2}(f(x), f(y)) \neq \mathrm{dist}_{\mathbb{H}^2}(x, y)$ for every $x \neq y$, therefore, by Proposition 1.3.3, we attain the desired result. □

Thanks to Corollary 1.3.4, given $f : \mathbb{H}^2 \to \mathbb{H}^2$ contraction, the set $\{\ell_{x,f(x)} \,|\, x \in \mathbb{H}^2\}$ is a foliation for $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Indeed, if $A \in \mathrm{PSL}_2(\mathbb{R})$ namely $A \in \ell_{x,f(x)}$ where $x = \mathrm{Fix}(A \circ f)$ is the fix point of the map $A \circ f$ which exists and it is unique.

1.3.4 Orientation choice

Let us observe that if $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$ then $\mathrm{tr}(A) = 0$. If we also require that $\det(A) > 0$ we have necessarily that $bc < 0$. The two possibilities, either $b < 0$ and $c > 0$ or $b > 0$ and $c < 0$ determines the choice of the time orientation: respectively future or past directed. Let us introduce

$$\mathbf{H}^+ \coloneqq \{A = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right] \in T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3 \,|\, \det(A) = 1, b < 0, c > 0\}. \tag{1.2}$$

Then, it is well-defined the map

$$\begin{aligned} \Phi : \mathbf{H}^+ &\to \mathbb{H}^2 \\ A &\mapsto \mathrm{Fix}(\exp tA). \end{aligned} \tag{1.3}$$

Observation 1.3.5. If $A = \left[\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right]$ is an element in $\mathbf{H}^+$, the fixed point of $\exp(tA) \in \mathrm{PSL}_2(\mathbb{R})$ is $\mathrm{Fix}(\exp tA) = \frac{a+i}{c}$.

Proof. Let $z \in \mathbb{H}^2$ the point such that $\exp(tA)(z) = z$ for every t. Then z is the zero of the vector field $V_A = \frac{d}{dt}\exp(tA)_{|t=0}$.

We can write $\exp(tA) = \left[\begin{smallmatrix} 1+ta & tb \\ tc & 1-ta \end{smallmatrix}\right] + o(t)$, thus

$$\begin{aligned} \exp(tA)(z) &= \frac{(1 + ta + o(t)z + (tb + o(t)))}{(tc + o(t))z + (1 - ta + o(t))} \\ &= (z + t(az + b) + o(t))(1 + t(a - cz) + o(t)) \\ &= z + t(-cz^2 + 2az + b) + o(t). \end{aligned}$$

Therefore $V_A(z) = -cz^2 + 2az + b$. The zeros on $\mathbb{C}P^1$ are $z_1 = \frac{a+\sqrt{a^2+bc}}{c}$ and $z_2 = \frac{a-\sqrt{a^2+bc}}{c}$. Since $a^2 + bc = -\det A = -1$, we have $c \neq 0$ and $bc < 0$. Then $z_1 = \frac{a+i}{c}$ and $z_2 = \frac{a-i}{c}$. As consequence we have $\mathrm{Fix}(\exp tA) = \frac{a+i}{c}$ with $c > 0$.

□

Proposition 1.3.6. The map $\Phi : \mathbf{H}^{+} \to \mathbb{H}^2$ is an isometry with inverse map $\Phi^{-1} : \mathbb{H}^2 \to \mathbf{H}^{+}$ such that associates to every $x \in \mathbb{H}^2$ the future directed vector $X \in T^1_{\mathrm{Id}}\,\mathrm{Stab}(x)$.

Remark 4. Recall that $\mathrm{Stab}(x) = \ell_{x,x}$, so the unitary vector tangent X is a generator of $\ell_{x,x}$.

Observation 1.3.7. If $B \in T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $\det B < 0$, then $\exp(tB)$ is hyperbolic.

Proof. The conditions $\mathrm{tr}(B) = 0$ and $\det(B) < 0$ imply that the matrix B diagonalize on $\mathbb{R}$. If $v_1 = (x_1, y_1)$ and $v_2 = (x_2, y_2)$ are real eigenvectors for B, then they are also real eigenvectors for $\exp(tB)$ and $[x_1 : y_1]$, $[x_2 : y_2]$ are fixed point in $\mathbb{R}P^1$ for $\exp(tB)$. □

Lemma 1.3.8. $B \in T_A H^{+}$ if and only if $\mathrm{Fix}(\exp tA) \subseteq \mathrm{Axis}(\exp tB)$.

Proof. ($\Rightarrow$) Let us observe that there exists $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$\mathrm{Ad}(\gamma)A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \mathrm{Ad}(\gamma)B = \lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{1.4}$$

for some $\lambda \in \mathbb{R}$. Indeed, choosing λ such that $\lambda^2 = \langle B, B \rangle$ it easy to check that

$$\langle A, A \rangle = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\rangle, \tag{1.5}$$

$$\langle A, B \rangle = 0 = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \right\rangle, \tag{1.6}$$

$$\langle B, B \rangle = \lambda^2 = \left\langle \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix} \right\rangle. \tag{1.7}$$

So there exists $T \in SO^+(T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3)$ such that $T(A) = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $T(B) = \lambda\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Since $\mathrm{Ad} : \mathrm{PSL}_2(\mathbb{R}) \to SO^+(T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3)$ is an isomorphism, we prove the existence of γ satisfying (1.4).

Notice that $\mathrm{Fix}\left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\right) = i$ and $\mathrm{Axis}\left(\left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]\right) = i\mathbb{R}_+$, thus we have

$$\begin{aligned}\mathrm{Fix}\exp(tA) &= \mathrm{Fix}(\exp(t\gamma^{-1}\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\gamma)) \\ &= \mathrm{Fix}(\gamma^{-1}\exp t\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\gamma)) = \gamma^{-1}i\end{aligned} \tag{1.8}$$

and

$$\mathrm{Axis}(\exp tB) = \gamma^{-1}\,\mathrm{Axis}(\exp t\left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right]) = \gamma^{-1}(i\mathbb{R}_+). \tag{1.9}$$

Since $i \in i\mathbb{R}_+$, then $\gamma^{-1}i \in \gamma^{-1}(i\mathbb{R}_+)$.

($\Leftarrow$) Similarly, there exists $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ such that $\gamma(\mathrm{Fix}(\exp(tA))) = i$ and $\gamma(\mathrm{Axis}(\exp tB)) = i\mathbb{R}_+$. Therefore,

- from $\mathrm{Fix}(\exp t(\gamma A\gamma^{-1})) = i$

$$\begin{aligned}&\Rightarrow \exp t(\gamma A\gamma^{-1}) = \mathrm{Stab}\, i \\ &\Rightarrow \gamma A\gamma^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\end{aligned}$$

- from $\mathrm{Axis}(\exp t(\gamma B\gamma^{-1})) = i\mathbb{R}_+$

$$\begin{aligned}&\Rightarrow \exp t(\gamma B\gamma^{-1}) = \exp t\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \mathrm{Stab}(i\mathbb{R}_+) \\ &\Rightarrow \gamma B\gamma^{-1} = \lambda\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.\end{aligned}$$

In conclusion, $\langle A, B\rangle = \langle \gamma A\gamma^{-1}, \gamma B\gamma^{-1}\rangle = \left\langle \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \lambda\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right\rangle = 0.$

□

Lemma 1.3.9. Let $\Phi\colon H^+ \to \mathbb{H}^2$ as defined before. For every $A \in H^+$ and $B \in T_A H^+$, $B \neq 0$, the vector $i\, d_A\Phi(B) \in T_{\Phi(A)}\,\mathbb{H}^2$ has the same direction and orientation of $\mathrm{Axis}(\exp tB)$.

Proof. Firstly let us observe that if the statement is true for the pair (A, B), it is true for $(\gamma A\gamma^{-1}, \gamma B\gamma^{-1})$ and viceversa.

Since $\Phi(\gamma A\gamma^{-1}) = \gamma\Phi(A)$, we have that $B \in T_A H^+$ implies $\gamma B\gamma^{-1} \in T_{\gamma A\gamma^{-1}} H^+$ and as a consequence

$$d_{\gamma A\gamma^{-1}}\Phi(\gamma B\gamma^{-1}) = \frac{d}{dt}\Phi(\gamma A(t)\gamma^{-1})|_{t=0} = d\gamma\Big(\frac{d}{dt}\Phi(A(t))\Big) = d\gamma \circ d_A\Phi(B).$$

Besides, $\mathrm{Axis}(\exp(t(\gamma B\gamma^{-1}))) = \gamma\,\mathrm{Axis}(\exp tB)$ as oriented geodesics.

This allows us to consider just the matrices

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}.$$

Notice that in this case:

- $d_A\Phi(B) = \lambda d_A\Phi\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$;
- $\mathrm{Axis}(\exp tB) = \mathrm{sign}(\lambda)\,\mathrm{Axis}(\exp t\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right])$.

This leads us to prove the lemma in the simpler case with $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$. Let us consider a path $A(t) = \left[\begin{smallmatrix} \sinh t & -\cosh t \\ \cosh t & -\sinh t \end{smallmatrix}\right]$ in $\mathbf{H}^+$ such that $A(0) = A$ and $\dot{A}(0) = B$.

Then

$$d_A\Phi(B) = \frac{d}{dt}\Phi(A(t))|_{t=0} = \frac{d}{dt}\frac{\sinh t + i}{\cosh t}|_{t=0} = 1.$$

Since $T_i\,\mathrm{Axis}(\exp t\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]) = i\mathbb{R}$ and i is positively oriented, the thesis is proved. □

Proof of the Proposition 1.3.6. We need to prove that for every $A \in \mathbf{H}^+$ and $B \in T_A\mathbf{H}^+$, if $\|B\| = 1$ then $\|d_A\Phi(B)\|_{\mathbb{H}^2} = 1$. Using the equivariance rule of Φ for the action of $\mathrm{PSL}_2(\mathbb{R})$, we can lead the proof to the case $A = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$

and $B = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$. In that case, $\Phi(A) = i$, $d_A\Phi(B) = 1$ and $\|d_A\Phi(B)\|_{\mathbb{H}^2} = \frac{1}{\Im(\Phi(A))^2}\|d_A\Phi(B)\|_{\mathbb{E}^2} = 1$. □

Definition 9. Let $[\ ,\ ]$ be the Lie bracket on $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$. We define the 3-form $\omega(x, y, z) = -\big\langle [x, y], z\big\rangle$.

Proposition 1.3.10. The 3-form ω is

- ω is alternating;
- ω is non-zero.

Since ω is a non-zero 3-form, it gives an orientation over $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Definition 10. A basis (v_1, v_2, v_3) for $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$ is *positive* if $\omega(v_1, v_2, v_3) > 0$. It is *negative* otherwise.

Remark 5. For every $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, $\omega(\mathrm{Ad}(\gamma)(x), \mathrm{Ad}(\gamma)(y), \mathrm{Ad}(\gamma)(z)) = \omega(x, y, z)$.

Proposition 1.3.11. Let (A, B_1, B_2) be an orthonormal future directed basis for $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$. Then (A, B_1, B_2) is positive if and only if $(d_A\Phi(B_1), d_A\Phi(B_2))$ is a positive basis for $T_{\Phi(A)}\,\mathbb{H}^2$.

Proof. Let us prove the thesis for the case $A = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, $B_1 = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$, $B_2 = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. In this case $\omega(A, B_1, B_2) = 4$, thus it is a positive basis. Indeed, $T_i\,\mathrm{Axis}(\exp tB_1) = \mathbb{R}$ and $T_i\,\mathrm{Axis}(\exp tB_2) = i\mathbb{R}$. By the previous Lemma $d\Phi(B_1) \in \mathbb{R}_+ \cdot (-i)$ and $d\Phi(B_2) \in \mathbb{R}_+ \cdot (1)$, therefore $d\Phi(B_1)$ and $d\Phi(B_2)$ form a positive basis for $T_i\,\mathbb{H}^2$.

Now let us consider the general case and define the map $L\colon T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3 \to T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$ such that $L(A) = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$, $L(B_1) = \left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$, $L(B_2) = \left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$.

The map $L \in O^+(T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3)$ because it preserves orthonormal basis and future directed vectors. If A, B_1, B_2 is a positive basis, then $L \in SO^+(T_I\mathbb{A}\mathrm{d}\mathbb{S}^3)$ and so there exists $\gamma \in \mathrm{PSL}_2(\mathbb{R})$ such that $L = Ad\gamma^{-1}$. Since $A = \mathrm{Ad}(\gamma)\left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right]$,

$B_1 = \mathrm{Ad}(\gamma)\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, $B_2 = \mathrm{Ad}(\gamma)\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, then $d_A\Phi(B_1) = d\gamma d_{\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]}\Phi(\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right])$ and $d_A\Phi(B_2) = d\gamma d_{\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]}\Phi(\left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right])$. Noting that $d\gamma$ preserves the orientation, also $(d_A\Phi(B_1), d_A\Phi(B_2))$ is positive. Moreover, if A, B_1, B_2 is a negative basis, we observe that $A, B_1, -B_2$ is positive, then $d_A\Phi(B_1), -d_A\Phi(B_2)$ is positive too and in conclusion $d_A\Phi(B_1), d_A\Phi(B_2)$ is negative.

□

1.3.5 Pure rotation and translation in AdS-space

Given a basis of $T_{\mathrm{Id}}\,\mathrm{SL}_2(\mathbb{R})$ composed by the matrices

$$J_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\ J_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

we can write $\ell_{i,i} = \{\exp_{\mathrm{Id}}(tJ_0)\}$, as J_0 is a vector tangent to the geodesic $\ell_{i,i}$.

If R_i^θ is a rotation of angle 2θ as isometry of $\mathbb{H}^2$ fixing the imaginary unit, $\mathscr{S} := (R_i^{\theta_1}, R_i^{\theta_2})$ is an isometry of $\mathrm{PSL}_2(\mathbb{R})$ that fixes the geodesic $\ell_{i,i}$. In particular, this kind of isometries translate a point on $\ell_{i,i}$ of a parameter $\theta_1 - \theta_2$ and rotate tangent vectors in $T\,\mathrm{PSL}_2(\mathbb{R})$ orthogonal to the geodesic of an angle $\theta_1 + \theta_2$.

Let us explain this claim more in detail. As for the first fact, let us consider $\eta : [0,\pi] \to \mathrm{PSL}_2(\mathbb{R})$ parametrization of $\ell_{i,i}$ such that $\eta(0) = \eta(\pi) = \mathrm{Id}$, then there exists $k \in \mathbb{R}$ arc length such that $\mathscr{S}\eta(t) = \eta(t+k)$. For $t = 0$, $(R_i^{\theta_1}, R_i^{\theta_2})\eta(0) = R_i^{\theta_1 - \theta_2} = \eta(k)$, from which follows that the translation is $k = \theta_1 - \theta_2$.

As for the second assertion, if $J \in T_I\mathbb{A}\mathrm{d}\mathbb{S}^3$ with $J \perp T_{\mathrm{Id}}\ell_{i,i,}$, then $J \in \mathrm{Span}\{J_1, J_2\}$, the angle between $\boldsymbol{P}_0^t(J)$ and $d_{\mathrm{Id}}\mathscr{S}(J)$ is $\theta_1 + \theta_2$, where $\boldsymbol{P}_0^t(J)$ is the parallel transport of J along $\ell_{i,i}$ (for $t = \theta_1 - \theta_2$) and $d_{\mathrm{Id}}\mathscr{S}(J)$ is its image through the differential map.

This angle can be easily calculated noting that, in our case, for every

$t \in \mathbb{R}$:

$$\boldsymbol{P}_0^t(J) = J,$$

under the identification of $T_{\eta(t)}\mathbb{A}\mathrm{d}\mathbb{S}^3 \simeq \eta(t)^\perp$ with a subspace of $M_2(\mathbb{R})$.

Moreover, using that $R_i^\theta J = J R_i^\theta$ for every $J \in \mathrm{Span}\{J_1, J_2\}$, we can check that

$$d_{\mathrm{Id}}\mathscr{S}(J) = R_i^{\theta_1+\theta_2} J$$

so we have

$$\langle \boldsymbol{P}_0^t(J), d_{\mathrm{Id}}\mathscr{S}(J)\rangle_{\mathscr{S}(\mathrm{Id})} = \cos(\theta_1 + \theta_2).$$

It follows that the desired angle is $\theta_1 + \theta_2$.

Let us define $\mathscr{R}^\theta = (R_i^{\theta/2}, R_i^{\theta/2})$ and $\mathscr{T}^\varphi = (R_i^{\varphi/2}, R_i^{-\varphi/2})$ two specific isometries of $\mathrm{PSL}_2(\mathbb{R})$ fixing $\ell_{i,i}$. Note that the first isometry has no translation parameter along $\ell_{i,i}$, but it just rotates the plane orthogonal to the tangent vector J_0 by an angle θ. The second has translation φ and no rotation around $\ell_{i,i}$.

Notation 1.3.12. From here on, we will denote

$$\mathbb{A}\mathrm{d}\mathbb{S}^3_* := \mathbb{A}\mathrm{d}\mathbb{S}^3 \backslash \ell_{i,i}$$

that is the Anti-de Sitter space identified with $\mathrm{PSL}_2(\mathbb{R})$ without the geodesic $\ell_{i,i,} \subset \mathrm{PSL}_2(\mathbb{R})$.

1.4 Timelike geodesics in $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$

In this section we study $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, i.e, the universal cover of the Lie group $\mathrm{PSL}_2(\mathbb{R})$.

Let us observe that the center of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ is a cyclic subgroup generated by $\tau_0 \in \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, where $\tau_0 = \tilde{\gamma}(1)$ for $\gamma\colon [0,1] \to \mathrm{PSL}_2(\mathbb{R})$ timelike geodesic with $\gamma(0) = \gamma(1) = \mathrm{Id}$ and $\tilde{\gamma}$ is lifting such that $\tilde{\gamma}(0) = \mathrm{Id}$.

Therefore, for every closed timelike geodesic $\alpha : [0,1] \to \mathrm{PSL}_2(\mathbb{R})$ it is $\tilde{\alpha}(1) = \tau_0\tilde{\alpha}(0)$. All the geodesics of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ are lifting of closed geodesics in $\mathrm{PSL}_2(\mathbb{R})$, thus $\tau_0(\tilde{\ell}) = \tilde{\ell}$ for every timelike geodesic $\tilde{\ell} \subset \widetilde{\mathrm{PSL}_2(\mathbb{R})}$.

Let $\pi : \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R})$ be the canonical projection and $T \in \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$. Both π and $T \circ \pi$ can be regarded as developing maps for $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, hence there exists $\pi_*(T) \in \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$ such that $\pi \circ T = \pi_*(T) \circ \pi$. As a consequence, we can define a homomorphism

$$\pi_* : \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})}) \to \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$$

such that the following diagram

$$\begin{tikzcd}
\widetilde{\mathrm{PSL}_2(\mathbb{R})} \arrow[r, "T"] \arrow[d, "\pi"] & \widetilde{\mathrm{PSL}_2(\mathbb{R})} \arrow[d, "\pi"] \\
\mathrm{PSL}_2(\mathbb{R}) \arrow[r, "\pi_*(T)"] & \mathrm{PSL}_2(\mathbb{R})
\end{tikzcd} \tag{1.10}$$

is commutative.

Given $(\widetilde{\alpha}, \widetilde{\beta}) \in \widetilde{\mathrm{PSL}_2(\mathbb{R})} \times \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, we can define a diffeomorphism $\widetilde{\Theta}(\widetilde{\alpha}, \widetilde{\beta})$ such that for every $\widetilde{\gamma} \in \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, $\widetilde{\Theta}(\widetilde{\alpha}, \widetilde{\beta})\gamma = \widetilde{\alpha}\widetilde{\gamma}\widetilde{\beta}^{-1}$. There exists $\Theta(\alpha, \beta)$ with $\pi(\widetilde{\alpha}) = \alpha$, $\pi(\widetilde{\beta}) = \beta$, such that

$$\begin{tikzcd}
\widetilde{\mathrm{PSL}_2(\mathbb{R})} \arrow[r, "\widetilde{\Theta}(\widetilde{\alpha},\widetilde{\beta})"] \arrow[d, "\pi"] & \widetilde{\mathrm{PSL}_2(\mathbb{R})} \arrow[d, "\pi"] \\
\mathrm{PSL}_2(\mathbb{R}) \arrow[r, "\Theta(\alpha,\beta)"] & \mathrm{PSL}_2(\mathbb{R})
\end{tikzcd}$$

is commutative. Moreover, since π is a group homomorphism, $\Theta(\alpha, \beta)\gamma = \alpha\gamma\beta^{-1}$ for every $\gamma \in \mathrm{PSL}_2(\mathbb{R})$.

In particular, $\widetilde{\Theta} : \widetilde{\mathrm{PSL}_2(\mathbb{R})} \times \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$ and, by 1.10, $\pi_*(\widetilde{\Theta}(\widetilde{\alpha}, \widetilde{\beta})) = \Theta(\alpha, \beta)$. Furthermore, notice that

$$\ker \pi_* = \widetilde{\Theta}(Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})}) \times Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})})),$$

in particular $\ker \pi_*$ is a cyclic subgroup generated by $T_0 \coloneqq \widetilde{\Theta}(\tau_0, \mathrm{Id}) = \widetilde{\Theta}(\mathrm{Id}, \tau_0)$.

Lemma 1.4.1. Every isometry of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ is induced by left and right multiplication.

Proof. There is a short exact sequence

$$1 \to \Delta \to \widetilde{\mathrm{PSL}_2(\mathbb{R})} \times \widetilde{\mathrm{PSL}_2(\mathbb{R})} \xrightarrow{\widetilde{\Theta}} \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})}) \to 1,$$

where $\Delta = \{(\tau, \tau) \,|\, \tau \in Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})})\}$.

It is easy to show that $\ker \widetilde{\Theta} = \Delta < Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})}) \times Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$. If $(\alpha, \beta) \in \ker \widetilde{\Theta}$, then $\alpha = \beta$ and $\alpha \in Z(\mathrm{PSL}_2(\mathbb{R}))$. Indeed, if $\widetilde{\Theta}(\alpha, \beta) \cdot \gamma = \gamma$ for every $\gamma \in \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, in particular for $\gamma = \mathrm{Id}$ we have $\alpha = \beta$. Imposing that $\widetilde{\Theta}(\alpha, \alpha)\gamma = \gamma$ for every $\gamma \in \widetilde{\mathrm{PSL}_2(\mathbb{R})}$, we obtain $\alpha\gamma\alpha^{-1} = \gamma$ for every γ, thus $\alpha \in Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$. On the othe hand, let $(\alpha, \alpha) \in \Delta$. Then $\widetilde{\Theta}(\alpha, \alpha)\gamma = \alpha\gamma\alpha^{-1} = \gamma$ for every $\gamma \in \mathrm{PSL}_2(\mathbb{R})$, that is $\widetilde{\Theta}(\alpha, \alpha) = \mathrm{Id}$.

Let us see now that $\widetilde{\Theta}$ is surjective. If $T \in \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$, then there exists $(\alpha, \beta) \in \mathrm{PSL}_2(\mathbb{R})$ such that $\pi_*(T) = \Theta(\alpha, \beta)$. Fix $\widetilde{\alpha}$, $\widetilde{\beta}$ such that $\pi(\widetilde{\alpha}) = \alpha$, $\pi(\widetilde{\beta}) = \beta$. Then $\pi_*(\widetilde{\Theta}(\widetilde{\alpha}, \widetilde{\beta})) = \pi_*(T)$ and so $\pi_*(T^{-1}\widetilde{\Theta}(\widetilde{\alpha}, \widetilde{\beta})) = 1$. In conclusion $T^{-1}\widetilde{\Theta}(\widetilde{\alpha}, \widetilde{\beta}) = \widetilde{\Theta}(\widetilde{\gamma_1}, \widetilde{\gamma_2})$ for $\widetilde{\gamma_1}$ and $\widetilde{\gamma_2} \in Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$, thus $T = \widetilde{\Theta}(\widetilde{\gamma_1}^{-1}\widetilde{\alpha}, \widetilde{\gamma_2}^{-1}\widetilde{\beta})$.

□

Notation 1.4.2. We write Stab instead for the stabilizer of a point or a geodesic in the isometry group of $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$.

Proposition 1.4.3. Let $\ell_1 = \ell_{a_1,b_1}$ and $\ell_2 = \ell_{a_2,b_2}$ a pair of disjoint geodesics in $\mathrm{PSL}_2(\mathbb{R})$ and $\widetilde{\ell_1}$, $\widetilde{\ell_2}$ their lifts in $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$.

(i) If $a_1 \neq a_2$ and $b_1 \neq b_2$, then $\mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2}) = \mathbb{Z} \cdot T_0 = (\tau_0, 1) = (1, \tau_0)$;

(ii) If $a_1 \neq a_2$ and $b_1 = b_2 = b$, then $\mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2}) = \widetilde{\Theta}(\{1\} \times \widetilde{\ell}_{b,b})$.

In both cases, if $T \in \mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2})$:

- for all $p \in \widetilde{\ell_1}$ and $q \in \widetilde{\ell_2}$, $\mathrm{dist}(p, Tp) = \mathrm{dist}(q, Tq) = \xi$;

- $\xi = \pi$ if and only if $T = T_0$.

Remark 6. In the former case $\xi \in \pi\mathbb{Z}$, while in the later case $\xi \in \mathbb{R}$.

Proof of Proposition 1.4.3. Let us note that $T \in \mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2})$ if and only if $\pi_*(T) \in Stab(\ell_1) \cap \mathrm{Stab}(\ell_2)$. Therefore,

$$\mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2}) = \pi_*^{-1}(\mathrm{Stab}(\ell_1) \cap \mathrm{Stab}(\ell_2)).$$

In the first case, $\mathrm{Stab}(\ell_1) \cap \mathrm{Stab}(\ell_2) = \{\mathrm{Id}\}$. Thus,

$$\mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2}) = \pi_*^{-1}(\mathrm{Id}) = \ker \pi_* = \mathbb{Z}T_0.$$

In the second case, $\mathrm{Stab}(\ell_1) \cap \mathrm{Stab}(\ell_2) = \{\mathrm{Id}\} \times \mathrm{Stab}(b_1)$, where $\mathrm{Stab}(b_1) = \ell_{b_1,b_1}$. Therefore,

$$\begin{aligned}
\mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2}) &= \pi_*^{-1}(\mathrm{Id} \times \mathrm{Stab}(b_1)) \\
&= \widetilde{\Theta}((\pi,\pi)^{-1}(\{\mathrm{Id}\} \times \ell_{b_1,b_1})) \\
&= \widetilde{\Theta}(Z(\widetilde{\mathrm{PSL}_2(\mathbb{R})}) \times \widetilde{\ell_{b_1,b_1}}) \\
&= \widetilde{\Theta}(\{1\} \times \widetilde{\ell_{b_1,b_1}}).
\end{aligned}$$

Let us notice that $\{\mathrm{Id}\} \times \widetilde{\ell_{b_1,b_1}} \cap \Delta = \{\mathrm{Id}\}$. Thus,

$$\widetilde{\Theta} : \{1\} \times \widetilde{\ell_{b_1,b_1}} \to \mathrm{Stab}(\widetilde{\ell_1}) \cap \mathrm{Stab}(\widetilde{\ell_2}).$$

□

1.5 Timelike foliations for $\mathbb{A}\mathrm{d}\mathbb{S}^3$

Definition 11. A *time-tube* in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is an open subset $\mathscr{U} \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$ with a foliation $\{\ell_i\}_{i\in\mathscr{I}}$ such that

i) for every $i \in \mathscr{I}$, ℓ_i is a closed timelike geodesic;

ii) $\mathscr{U} = \cup \ell_i$;

iii) $\ell_i \cap \ell_j = \varnothing$ if $i \neq j$: in fact, for every $p \in \mathscr{U}$ there exists unique $i(p) \in \mathscr{I}$ such that $p \in \ell_{i(p)}$;

iv) the unitary tangent field $X \in (\Gamma(\mathscr{U}), T\mathbb{A}\mathrm{d}\mathbb{S}^3)$ such that

- $X(p) \in T_p \ell_{i(p)}$,
- $X(p)$ is future oriented,

is $\mathscr{C}^\infty$.

Let us recall that every geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ is in the form

$$\ell_{a,b} = \{\gamma \in \mathbb{A}\mathrm{d}\mathbb{S}^3 \simeq \mathrm{PSL}_2(\mathbb{R}) \,|\, \gamma(b) = a\}.$$

For every connected time-tube $\mathscr{U}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ foliated by $\mathscr{F} = \{\ell_{a(i),b(i)}\}$, with $\{a(i) \,|\, i \in \mathscr{I}\}$ and $\{b(i) \,|\, i \in \mathscr{I}\}$ subsets in $\mathbb{H}^2$, one of the following inequalities holds:

(i) $\mathrm{dist}_{\mathbb{H}^2}(a(i), a(j)) < \mathrm{dist}_{\mathbb{H}^2}(b(i), b(j))$;

(ii) $\mathrm{dist}_{\mathbb{H}^2}(a(i), a(j)) > \mathrm{dist}_{\mathbb{H}^2}(b(i), b(j))$.

Moreover, one between $\{a(i) \,|\, i \in \mathscr{I}\}$ and $\{b(i) \,|\, i \in \mathscr{I}\}$ is an open subset.

Let us define the following subsets in $\mathbb{A}\mathrm{d}\mathbb{S}^3$:

(i) $\mathscr{U}_L \coloneqq \{(p,q) \,|\, \mathrm{dist}_{\mathbb{H}^2}(\bar{a}(p), \bar{a}(q)) < \mathrm{dist}_{\mathbb{H}^2}(\bar{b}(p), \bar{b}(q))\}$;

(ii) $\mathscr{U}_R \coloneqq \{(p,q) \,|\, \mathrm{dist}_{\mathbb{H}^2}(\bar{a}(p), \bar{a}(q)) > \mathrm{dist}_{\mathbb{H}^2}(\bar{b}(p), \bar{b}(q))\}$.

Theorem 1.5.1. There exists the following dichotomy:

$$\textit{or} \quad \mathscr{U} = \mathscr{U}_L \quad \textit{and} \quad \mathscr{U}_R = \varnothing \qquad \textit{or} \quad \mathscr{U} = \mathscr{U}_R \quad \textit{and} \quad \mathscr{U}_L = \varnothing.$$

Lemma 1.5.2. Let us define the maps $\bar{a}, \bar{b} : \mathscr{U} \to \mathbb{H}^2$ such that $\bar{a}(p) := a(i_p)$ and $\bar{b}(p) := b(i_p)$ with $p \in \ell_{i_p} := \ell_{a(i_p),b(i_p)}$. Then, they are $\mathscr{C}^\infty$.

Proof. Let ρ_p: $\mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ be a map such that $\rho_p(q) = qp^{-1}$. As consequence we have that $\rho_p \ell_{\bar{a}(p),\bar{b}(p)} = \ell_{\bar{a}(p),\bar{a}(p)}$. It holds that $\bar{a} = \Phi(d_p(\rho_p)X(p))$ and $\bar{b} = \Phi(d_p(\lambda_p)X(p))$, where $\lambda_p(q) = p^{-1}q$ and Φ is the isometry defined in 1.3.6. □

Proof of Theorem 1.5.1. As consequence of the Lemma 1.5.2, the subsets $\mathscr{U}_L$ and $\mathscr{U}_R$ are open subsets in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Since $\mathscr{U}$ is connected, and $\mathscr{U} = \mathscr{U}_L \cup \mathscr{U}_R$ and $\mathscr{U}_L \cap \mathscr{U}_R = \varnothing$, we obtain the thesis. □

Proposition 1.5.3. Let us assume $\mathscr{U} = \mathscr{U}_L$. The following properties hold:

(i) $\bar{b}(p) = \bar{b}(q)$ if and only if $i(p) = i(q)$, that is p and q are in the same leaf;

(ii) $\Omega = \{\bar{b}(p) \,|\, p \in \mathscr{U}\}$ is open in $\mathbb{H}^2$;

(iii) $f : \Omega \to \mathbb{H}^2$ such that $f(\bar{b}(p)) = \bar{a}(p)$ is well defined and it is a distance decreasing map.

Proof. (i) Let us observe that if $\bar{b}(p) = \bar{b}(q)$, being $\mathrm{dist}(\bar{a}(p), \bar{a}(q)) < \mathrm{dist}(\bar{b}(p), \bar{b}(q))$ when $i(p) \neq i(q)$, it follows necessarily that $i(p) = i(q)$.

(ii) Let Π be a spacelike plane. Then $\mathscr{D} := \mathscr{U} \cap \Pi$ is an open subset in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. In particular $\mathscr{D} \subset \mathscr{U}$ it is a totally geodesic spacelike disk. Since $\mathscr{D}$ intersects every leaf of the foliation at most once, we can define the injective map $\mathscr{D} \to \mathbb{H}^2$ such that $p \mapsto \bar{b}(p)$. This map is continuous (Lemma 1.5.2), so as a consequence of the Invariance of Domain Theorem $\Omega = \bar{b}(\mathscr{D})$ is an open subset in $\mathbb{H}^2$.

(iii) Since $\Omega = \{\bar{b}(p) \,|\, p \in \mathscr{U}_L\}$, by definition of $\mathscr{U}_L$ it follows that $\mathrm{dist}_{\mathbb{H}^2}(f(\bar{b}(q), f(\bar{b}(q)))) = \mathrm{dist}_{\mathbb{H}^2}(\bar{a}(p), \bar{a}(q)) < \mathrm{dist}_{\mathbb{H}^2}(\bar{b}(p), \bar{b}(q))$.

□

Before considering a foliation, let us study the mutual position of two timelike geodesics. Let ℓ_1 and ℓ_2 be disjoint timelike geodesics in $\mathrm{PSL}_2(\mathbb{R})$. Let us fix a spacelike geodesic γ joining a point $p \in \ell_1$ and a point $q \in \ell_2$. Let $A_1, A_2 \in \mathrm{SL}_2(\mathbb{R})$ such that $p = [A_1]$ and $q = [A_2]$ and $\tilde{\gamma}(0) = A_1$, $\tilde{\gamma}(1) = A_2$. Let $B_1 \in T_{A_1}\mathrm{SL}_2(\mathbb{R})$ and $B_2 \in T_{A_2}\mathrm{SL}_2(\mathbb{R})$ be future oriented unitary vectors and denote $X_1 \coloneqq d_{A_1}\widetilde{\pi}(B_1)$ and $X_2 \coloneqq d_{A_2}\widetilde{\pi}(B_2)$, where $\widetilde{\pi} : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{PSL}_2(\mathbb{R})$ is the projection map. Notice that:

- $\langle A_1, A_1 \rangle = -1, \langle B_1, B_1 \rangle = -1, \langle A_1, B_1 \rangle = 0$, thus
$$\ell_1 = \widetilde{\pi}(\mathrm{Span}(A_1, B_1)) \cap \mathrm{PSL}_2(\mathbb{R}) = \widetilde{\pi}(\mathrm{Span}(A_1, B_1));$$
- $\langle A_2, A_2 \rangle = -1, \langle B_2, B_2 \rangle = -1, \langle A_2, B_2 \rangle = 0$, thus
$$\ell_2 = \widetilde{\pi}(\mathrm{Span}(A_2, B_2)) \cap \mathrm{PSL}_2(\mathbb{R}) = \widetilde{\pi}(\mathrm{Span}(A_2, B_2)).$$

This chain of equivalences holds:

$$\begin{aligned} \ell_1 \cap \ell_2 = \varnothing &\iff \mathrm{Span}(A_1, B_1) \cap \mathrm{Span}(A_2, B_2) = \varnothing \\ &\iff \mathbb{R}^{2,2} = \mathrm{Span}(A_1, B_1) \oplus \mathrm{Span}(A_2, B_2) \\ &\iff \{A_1, B_1, A_2, B_2\} \quad \text{is a basis for} \quad \mathbb{R}^{2,2}. \end{aligned}$$

Proposition 1.5.4. With the previous notations, let Y be the parallel extension of X_1 along the spacelike geodesic γ such that $\gamma(0) = p$ and $\gamma(1) = q$. Then $\{\dot{\gamma}(1), X_2, Y(q)\}$ is a basis for $T_q\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Proof. Let $B(t) \in T_{\tilde{\gamma}(t)}SL_2(\mathbb{R})$ be such that $d\widetilde{\pi}(B(t)) = Y(t)$. $B(t)$ is parallel along $\tilde{\gamma}(t)$. Let us recall that

$$\frac{DB(t)}{dt} = (B'(t))^T = B'(t) + \langle B'(t), \tilde{\gamma}(t) \rangle \tilde{\gamma}(t) = 0.$$

Therefore, $B'(t) \in \mathrm{Span}\,\tilde{\gamma}(t) \subset \mathrm{Span}(A_1, A_2)$ and thus $B(t) \in B + \mathrm{Span}(A_1, A_2)$, where $B_1 \in T_{A_1}\mathrm{SL}_2(\mathbb{R})$ such that $d_{A_1}\widetilde{\pi}(B_1) = X_1$. In particular, $B(1) \in B + \mathrm{Span}(A_1, A_2) \subset \mathrm{Span}(B_1, A_1, A_2)$, from which $\mathrm{Span}(B_1, A_1, A_2) = \mathrm{Span}(B(1), A_1, A_2) = \mathrm{Span}(B(1), \dot{\tilde{\gamma}}(1), A_2)$. Let us consider B_2 such that $d_{\widetilde{\pi}}(B_2) = X_2$. It is $B_2 \notin \mathrm{Span}(B_1, A_1, A_2)$, hence $B_2 \notin \mathrm{Span}(B(1), \dot{\tilde{\gamma}}(1))$, so $\{B_2, B(1), \dot{\tilde{\gamma}}(1)\}$ is actually a basis for $T_{A_1} SL_2(\mathbb{R})$. Since $\dot{\tilde{\gamma}}(1)$ and $B(1)$ are indipendent vectors (in particular, one is space-like the other one is timelike), it follows that $\{\dot{\tilde{\gamma}}, B(1), B_2\}$ form a basis for $T_{A_2} SL_2(\mathbb{R})$. Moreover, since $d_{A_2}\widetilde{\pi}(\dot{\tilde{\gamma}}) = \dot{\gamma}$, $d_{A_2}\widetilde{\pi}(B(1)) = Y(1)$ and $d_{A_2}\widetilde{\pi}(B_2) = X_2$ with $d_{A_2}\widetilde{\pi} : T_{A_2} SL_2(\mathbb{R}) \to T_q \,\mathrm{PSL}_2(\mathbb{R})$ isomorphism, we can conclude that $\{\dot{\gamma}(1), X_2, Y(q)\}$ is a basis. $\square$

Definition 12. Let $\ell_1 = \exp_p(tX_1)$ and $\ell_1 = \exp_q(tX_2)$ be disjoint timelike geodesics. Given a space-like geodesic γ joining $p \in \ell_1$ and $q \in \ell_2$, we will say that ℓ_2 is *right-rotated* (respectively *left-rotated*) with respect to ℓ_1 along γ if and only if $\{\dot{\gamma}(1), X_2, Y(q))\}$ is a positive (resp. negative) basis for $T_q\mathbb{A}\mathrm{d}\mathbb{S}^3$, where Y is the parallel extension of X_1 along γ.

Lemma 1.5.5. Let $\ell = \ell_{a,b}$ and $\ell' = \ell_{a',b'}$ be disjoint geodesics. Let γ a spacelike geodesic orthogonal to ℓ joining $p \in \ell$ and $q \in \ell'$, then ℓ' right-rotated with respect to ℓ along γ if and only if $d(a, a') < d(b, b')$.

Proof. Up to isometry of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, we can suppose $p = \mathrm{Id}$, $q = \left[\begin{smallmatrix} e^{\delta} & 0 \\ 0 & e^{-\delta}\end{smallmatrix}\right]$ with $\delta > 0$ and $\ell = \ell_{i,i}$. Let us call $X = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right]$ and $\gamma(t) = \cosh t \,\mathrm{Id} + \sinh t \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right]$. Notice that $\left\langle \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} e^{t} & 0 \\ 0 & e^{-t}\end{smallmatrix}\right] \right\rangle = 0$, thus $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right] \in T_{\gamma(t)}\mathbb{A}\mathrm{d}\mathbb{S}^3$. As a consequence, $X(t) = X$ is the parallel field along γ.

We have

$$\begin{aligned} T_q\mathbb{A}\mathrm{d}\mathbb{S}^3 &= \mathrm{Span}\{\dot{\gamma}(1), \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\} \\ &= \mathrm{Span}\{\begin{bmatrix} e^{\delta} & 0 \\ 0 & -e^{-\delta} \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\} \end{aligned}$$

and we can check that $\{\left[\begin{smallmatrix} e^{\delta} & 0 \\ 0 & -e^{-\delta}\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right]\}$ is a positive basis for $T_q\mathbb{A}\mathrm{d}\mathbb{S}^3$.

Thus, writing

$$Y = \alpha \begin{bmatrix} e^{\delta} & 0 \\ 0 & -e^{-\delta} \end{bmatrix} + \beta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \xi \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

with $\alpha^2 + \xi^2 - \beta^2 = -1$, ℓ' is right-rotated if and only if $\xi > 0$.

Now, if $\ell = \ell_{a',b'}$, we have that $q^{-1}\ell' = \mathrm{Stab}(b)$ and $\ell' q^{-1} = \mathrm{Stab}(a)$, then $\Phi(q^{-1}Y) = b$, $\Phi(Yq^{-1}) = a$. The map Φ is an isometry, thus

$$\begin{aligned} \mathrm{dist}_{\mathbb{H}^2}(a,a') &= \mathrm{dist}_{\mathbb{H}^2}(\Phi(X),\Phi(Yq^{-1})) \\ &= \mathrm{dist}_{\mathbb{H}^2}(X,Yq^{-1}) \\ &= \mathrm{arcosh}(-\langle X, Yq^{-1}\rangle), \end{aligned}$$

$$\begin{aligned} \mathrm{dist}_{\mathbb{H}^2}(b,b') &= \mathrm{dist}_{\mathbb{H}^2}(\Phi(X),\Phi(Yq^{-1})) \\ &= \mathrm{dist}_{\mathbb{H}^2}(X,Yq^{-1}) \\ &= \mathrm{arcosh}(-\langle X, q^{-1}Y\rangle). \end{aligned}$$

Since $q^{-1}Y = \alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \beta \begin{bmatrix} 0 & -e^{-\delta} \\ e^{\delta} & 0 \end{bmatrix} + \xi \begin{bmatrix} 0 & e^{-\delta} \\ e^{\delta} & 0 \end{bmatrix}$ and $Yq^{-1} = \alpha \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} +$ $\beta \begin{bmatrix} 0 & -e^{\delta} \\ e^{-\delta} & 0 \end{bmatrix} + \xi \begin{bmatrix} 0 & e^{\delta} \\ e^{-\delta} & 0 \end{bmatrix}$ we have that $\langle X, Yq^{-1}\rangle = -(\beta\cosh\delta - \gamma\sinh\delta)$ and $\langle X, q^{-1}Y\rangle = -(\beta\cosh\delta + \xi\sinh\delta)$ from which

$$\begin{cases} \cosh\mathrm{dist}_{\mathbb{H}^2}(a,a') = [\beta\cosh\delta - \xi\sinh\delta], \\ \cosh\mathrm{dist}_{\mathbb{H}^2}(b,b') = [\beta\cosh\delta + \xi\sinh\delta], \end{cases} \tag{1.11}$$

therefore $\cosh\mathrm{dist}_{\mathbb{H}^2}(b,b') - \cosh\mathrm{dist}_{\mathbb{H}^2}(a,a') = 2\sinh\delta\xi$. It follows $\xi > 0$ if and only if $\mathrm{dist}_{\mathbb{H}^2}(b,b') > \mathrm{dist}_{\mathbb{H}^2}(a,a')$. □

Definition 13. Let $\mathscr{F} = \{\ell_i\}_{i\in\mathscr{I}}$ a timelike foliation of an open subset U of $\mathbb{A}\mathrm{d}\mathbb{S}^3$. We will say that $\mathscr{F}$ is *right-handed* if $\mathrm{dist}_{\mathbb{H}^2_*}(a(i),a(j)) < \mathrm{dist}_{Hyp^2_*}(b(i),b(j))$ for every $i,j \in \mathscr{I}$, $i \neq j$. Equally, $\mathscr{F}$ is *right-handed* if ℓ_j is right-rotated with respect to ℓ_i for every i,j. In the other way, we will say that $\mathscr{F}$ is *left-handed*.

Lemma 1.5.6. If X is the unitary vector field generating a right-handed foliation $\mathscr{F}$ for the open subset $\mathscr{U}$ of $\mathbb{A}\mathrm{d}\mathbb{S}^3$, then for every $p \in \mathscr{U}$:

$$\omega(v, X, \nabla_v X) > 0$$

for almost all $v \in T_p\mathscr{U}$ spacelike vector.

Proof. Let $w \in T_p\mathbb{A}\mathrm{d}\mathbb{S}^3$ be a vector such that $\{v, X(p), w\}$ is a positive basis. Let $e_1(t)$ and $e_2(t)$ be respectively the parallel transport of $X(p)$ and w along $\gamma = \exp(tv)$. Let us notice that

$$X(t) = \lambda(t)\dot{\gamma}(t) + \mu(t)e_1(t) + \nu(t)e_2(t)$$

with $\nu(t) > 0$ if $t > 0$. Then

$$\nabla_v X = \frac{DX}{dt}_{|t=0} = \dot{\lambda}(0)v + \dot{\mu}(0)X + \dot{\nu}(0)w.$$

In particular, since $\nu(t) > 0$ for $t > 0$, $\nu(0) = 0$, thus $\dot{\nu}(0) \geq 0$. If ω is the positive volume form on $\mathbb{A}\mathrm{d}\mathbb{S}^3$, $\omega(v, X, \nabla_v X) = \dot{\nu}(0)\omega(v, X, w) \geq 0$. $\square$

Remark 7. In general we cannot expect $\omega(v, X, \nabla_v X) > 0$.

Let us assume now that $v \perp X(p)$. Let us fix $a(t), b(t) \in \mathbb{H}^2$ such that $\ell(\gamma(t)) = \ell_{a(t),b(t)}$. Let us notice that from (1.11) it follows

$$\begin{cases} \cosh \mathrm{dist}_{\mathbb{H}^2}(a(0), a(t)) = (\mu(t)\cosh t - \nu(t)\sinh t), \\ \cosh \mathrm{dist}_{\mathbb{H}^2}(b(0), b(t)) = (\mu(t)\cosh t + \nu(t)\sinh t). \end{cases}$$

Let us set $\mathrm{dist}_{\mathbb{H}^2}(a(0), a(t)) = At + o(t)$ and $\mathrm{dist}_{\mathbb{H}^2}(b(0), b(t)) = Bt + o(t)$, expanding at the second order both the members:

$$\begin{cases} 1 + \frac{A^2}{2}t^2 + o(t^2) = (1 + \dot{\mu}t)(1 + \frac{t^2}{2}) - \dot{\nu}t^2 + o(t^2), \\ 1 + \frac{B^2}{2}t^2 + o(t^2) = (1 + \dot{\mu}t)(1 + \frac{t^2}{2}) + \dot{\nu}t^2 + o(t^2), \end{cases} \tag{1.12}$$

from which $\dot{\mu} = 0$, $\frac{A^2}{2} = 1 - \frac{\dot{\nu}}{2}$, $\frac{B^2}{2} = 1 + \frac{\dot{\mu}}{2}$. It follows that $B^2 - A^2 = 4\dot{\nu}$ and $(1 - \frac{A^2}{B^2}) = \frac{4\dot{\nu}}{B^2} = \frac{2\dot{\nu}}{1+\dot{\nu}^2}$. Therefore $\frac{A^2}{B^2} = 1 - \frac{2\dot{\nu}}{1+\dot{\nu}} = \frac{1-\dot{\nu}}{1+\dot{\nu}}$. Let us observe that if

$$b(t) = \exp_{b(0)} u(t), \qquad B = \|\dot{u}(0)\| = \|\dot{b}(0)\|, \tag{1.13}$$

$$a(t) = \exp_{a(0)} w(t), \qquad A = \|\dot{w}(0)\| = \|\dot{a}(0)\|. \tag{1.14}$$

Setting $a(t) = f(b(t))$ then $\dot{a}(0) = d_{b(0)} f \dot{b}(0)$, then $\frac{A}{B} = \frac{\|d_{b(0)} f(\dot{b}(0))\|}{\|\dot{b}(0)\|}$.

For every $b \in \bar{b}(\mathscr{U})$, $\mathscr{L}(b) = \sup_{u \in T_{b(0)}\mathbb{H}^2} \frac{\|d_{b(0)} f(u)\|}{\|u\|}$, then $\mathscr{L}(b) \le 1$ because f decreseas the distances. Fixing $p \in \ell = \ell_{a,b}$

$$\mathscr{L}(b)^2 = \sup_{v \perp \ell,\, v \in T_p \mathbb{A}\mathrm{d}\mathbb{S}^3} \frac{1 - \dot{\nu}}{1 + \dot{\nu}},$$

where we recall that $\dot{\nu} = \omega(v, X(p), \nabla_v X)$.

Corollary 1.5.7. The following are equivalent:

(i) f is a contraction;

(ii) there exists ν such that for every $v \perp X$, $\omega(v, X, \nabla_v X) \ge \nu \|v\|^2$.

Moreover, if $\nu_0 > 0$ is the biggest value realizing the inequlity then $c^2 = \frac{1-\nu_0^2}{1+\nu_0^2}$ and c is the contraction constant for f.

Proposition 1.5.8. Let $\mathscr{U}$ a time-tube with a right-handed foliation $\mathscr{F}$. Then the map $\bar{b} \colon \mathscr{U} \to \mathbb{H}^2$ is a summersion with $\ker d_p \bar{b} = \mathrm{Span}(X(p))$.

Proof. Since $B(\exp_p tX(p)) = b(p)$, it holds that $d_p b(X(p)) = 0$. Thus, we need to show that if $v \perp X(p)$, with $\|v\| = 1$, then $d_p b(v) \neq 0$. If $b(t) = b(\exp_p tv)$, then $d(b(0), b(t)) = (2 + \dot{\nu})t + o(t)$ with $\dot{\nu} = \omega(X, v, \nabla_v X) \ge 0$. It follows that $\dot{\bar{b}} \neq 0$. □

CHAPTER 2

AdS-manifolds with singularities

In this Chapter, we recall the definition of a conical singularity on a hyperbolic surface and we construct the *model for generalized spin-cone singularities* as the quotient of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ by a specific lattice of the group $\mathrm{Isom}\,\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$. Then we provide a way to compute the length of a timelike geodesic in this kind of quotient.

2.1 Conical singularities on the hyperbolic plane

In this section we recall in detail the hyperbolic case. In particular, we will introduce the notion of conical singularity on hyperbolic surfaces.

Fix a point $p \in \mathbb{H}^2$ and consider the space $\mathbb{H}^2_* = \mathbb{H}^2 \setminus \{p\}$ and its universal covering $\widetilde{\mathbb{H}^2_*} \simeq \mathbb{R}_+ \times \mathbb{R}$ given by

$$\widetilde{\mathbb{H}^2_*} \simeq \mathbb{R}_+ \times \mathbb{R} \to \mathbb{H}^2 \setminus \{p\}$$
$$(r, \theta) \mapsto R^\theta \gamma(t)$$

where $\gamma\colon (0, +\infty) \to \mathbb{H}^2$ is a unit speed geodesic with $\gamma(0) = p$ and R^θ a rotation of angle θ around p. In this model of $\widetilde{\mathbb{H}^2_*}$, we have that $\mathrm{Isom}^+\left(\widetilde{\mathbb{H}^2_*}\right) \simeq$

$\mathbb{R}$ and the group acts in this way:

$$\tau \cdot (t, \theta) = (t, \theta + \tau).$$

The fundamental group $\pi_1(\mathbb{H}^2_*)$ is identified with the subgroup $2\pi\mathbb{Z}$ in $\mathbb{R}$. In particular, it acts properly discontinuously and freely on $\widetilde{\mathbb{H}^2_*}$ by translations. Given a parameter $\theta_0 \in \mathbb{R}$, we define the *model for the cone singularity of angle* θ_0 as the quotient

$$\mathbb{H}^2_{\theta_0} = \widetilde{\mathbb{H}^2_*} \big/ \langle F_{\theta_0} \rangle,$$

where F_{θ_0} is the translation such that

$$F_{\theta_0}(t, \theta) = (t, \theta + \theta_0).$$

Let us notice that $\mathbb{H}^2_{\theta_0}$ is homeomorphic to a punctured disk, so we will denote the induced metric on this space with g_{θ_0}.

Definition 14. Let S be a hyperbolic surface and $\widetilde{S}$ its universal cover. For $\theta_0 \in \mathbb{R}$, an isometric embedding $\Phi : \widetilde{S} \to \widetilde{\mathbb{H}^2_*}$ is said θ_0-*equivariant isometry* if

$$\Phi(\gamma \cdot \tilde{x}) = F_{\theta_0} \circ \Phi(\tilde{x})$$

with $\tilde{x} \in \widetilde{S}$ and $\gamma \in \pi_1(S)$.

Definition 15 (Hyperbolic surface with cone singularities)**.** A hyperbolic surface S with cone singularities is an oriented surface with a set of points $\mathbf{p} = \{p_1, ...p_n\}$ associated to angles $\{\theta_1, ..., \theta_n\} \subset [0, 2\pi]^n$. The surface $S_* = S - \mathbf{p}$ is endowed with the standard hyperbolic metric and for every p_i there exists a neighbourhood U_i, an $\epsilon > 0$ and a θ_0-equivariant isometry $\Delta : \widetilde{U_i \backslash \{p_i\}} \to \widetilde{\mathbb{H}^2_*}$ such that

$$\Delta \circ \pi^{-1}(U_i) \supseteq \{(r, \theta) \,|\, r < \epsilon\}$$

where here $\pi : \widetilde{S} \to S$ is the universal covering of S and (r, θ) are the coordinates of $\widetilde{\mathbb{H}^2_*}$.

Remark 8. If $d : \widetilde{S} \to \mathbb{H}^2$ is a developing map for S, for every neighbourhood U_j of the singular point p_j, the holonomy is given by a representation $h : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ such that, for every peripheral $\gamma \in \pi_1(S_*)$

$$h(\gamma) = R_q^{\theta_j}$$

with $q = d(p_j) \in \mathbb{H}^2$.

Moreover, for every connected component $\widetilde{U_j} \subset \pi^{-1}(U_j)$, there exists an open subset $V \ni q$ of the hyperbolic plane such that

$$V \backslash \{q\} \subseteq d(\widetilde{U_j} \backslash \{\tilde{p_j}\}),$$

where $\widetilde{p_j}$ is the point in $\widetilde{U_j}$ such that $\pi(\widetilde{p_j}) = p_j$.

2.2 The generalized spin-cone model

For the 3-dimensional Anti-de Sitter space, we shall define a more general type of singularity. In this section we introduce the model that is the *generalized spin-cone singularities.* This model is based on a classification of quotients of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_{**}} = \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3 \backslash \ell_{i,i}}$ by $\mathbb{Z}^2$-lattices of isometries, which admit a foliation by timelike geodesics of length π.

2.2.1 Universal cover of $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$

Let us start by studying the universal cover of $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ and its group of isometry.

Definition 16. For $z \in \mathbb{H}^2$ we denote by L_z the unique orientation-preserving isometry fixing the geodesic through i and z and such that $L_z(i) = z$.

Let be $\Lambda_\eta = (R_i^\eta, \mathrm{Id})$, $P_\xi = (\mathrm{Id}, R_i^\xi)$. Then, the map

$$\begin{aligned} \Pi : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} &\to \mathbb{A}\mathrm{d}\mathbb{S}^3_* \\ (r, \eta, \xi) &\mapsto P_\xi \Lambda_\eta (L_{ie^r}) = R_i^\eta L_{ie^r} R_i^{-\xi} \end{aligned}$$

is the universal covering of $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$.

Notation 2.2.1. We will denote the covering space with $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$.

Remark 9. The map Π coincides with the developing map of $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$.

Proposition 2.2.2. The group of isometry is $\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) \simeq \mathbb{R}^2$. The action on $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ is given by

$$(\eta_0, \xi_0) \cdot (r, \eta, \xi) = (r, \eta + \eta_0, \xi + \xi_0).$$

Remark 10. Under this identification, the automorphisms group for Π is given by $\mathrm{Aut}(\Pi) = \langle (2\pi, 0), (0, 2\pi) \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Proof of Proposition 2.2.2 . Let us observe that the isometries of $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ are the isometries of the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-space preserving the geodesic $\ell_{i,i}$, moreover

$$\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3_*) \simeq \mathrm{Stab}(i) \times \mathrm{Stab}(i).$$

Now, given $\varphi \in \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3_*)$ and $X = (r, \eta, \xi) \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$, the image through the isometry is $\varphi(X) = (r', \eta', \xi') \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ and its projection is $\Pi(\varphi(X)) = P_{\xi'} \Lambda_{\eta'}(L_{ie^{r'}}) \in \mathbb{A}\mathrm{d}\mathbb{S}^3_*$. On the other hand $\Pi(X) = P_\xi \Lambda_\eta(L_{ie^r}) \in \mathbb{A}\mathrm{d}\mathbb{S}^3_*$. Both Π and $\Pi \circ \varphi$ can be regarded as developing maps, so there exists $\Pi^*(\varphi) \in \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3_*)$ such that the following diagram is commutative:

$$\begin{tikzcd}
\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \arrow[r, "\varphi"] \arrow[d, "\Pi"] & \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \arrow[d, "\Pi"] \\
\mathbb{A}\mathrm{d}\mathbb{S}^3_* \arrow[r, "\Pi_*(\varphi)"] & \mathbb{A}\mathrm{d}\mathbb{S}^3_*
\end{tikzcd} .$$

The isometry $\Pi_*(\varphi)$ has to preserve $\mathrm{Im}(\Pi) = \mathbb{A}\mathrm{d}\mathbb{S}^3_*$. In particular, $\Pi_*(\varphi)$ preserves the singular geodesic, that means

$$\Pi_*(\varphi) = (R_i^\theta, R_i^\lambda),$$

where $\theta, \lambda \in [0, 2\pi]$.

Accordingly, there exists a pair $(\tilde{\theta}, \tilde{\lambda}) \in \mathbb{R}^2$ defining an isometry $u \in \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})$ and such that $\Pi_*(u) \in \mathrm{Stab}(i) \times \mathrm{Stab}(i)$ and $\Pi_*(f) = \Pi_*(u)$. It follows $\Pi_*(f \circ u^{-1}) = id$. Thus $f \circ u^{-1} \in \Gamma = \mathrm{Aut}(\Pi)$. Since Γ is a subgroup of $\mathbb{R}^2$ and $\varphi = (\varphi \circ u^{-1}) \circ u$, if $u \in \Gamma$, then $\varphi \in \Gamma$ and we can also conclude $\varphi \in \mathbb{R}^2$. $\square$

2.2.2 $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ and $\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ as coverings of $\mathrm{PSL}_2(\mathbb{R})$

Since the lifting of the map $p : \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \to \mathrm{PSL}_2(\mathbb{R})$ to $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \to \widetilde{\mathrm{PSL}_2(\mathbb{R})}$ can be regarded as a developing map, there exists a homomorphism $\widetilde{p}_* : \mathrm{Isom}\,\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \to \mathrm{Isom}\,\widetilde{\mathrm{PSL}_2(\mathbb{R})}$, then for every $\varphi \in \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})$ there exists a map $\widetilde{p}_*(\varphi) \in \mathrm{Isom}\,\widetilde{\mathrm{PSL}_2(\mathbb{R})}$ such that the diagram

$$\begin{tikzcd}
\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \arrow[r, "\varphi"] \arrow[d, "p"] & \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \arrow[d, "p"] \\
\widetilde{\mathrm{PSL}_2(\mathbb{R})} \arrow[r, "\widetilde{p}_*(\varphi)"] & \widetilde{\mathrm{PSL}_2(\mathbb{R})}
\end{tikzcd}$$

is commutative. Similarly, it is the map $p_* : \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) \to \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$ such that $p_*(\varphi) \circ p = p \circ \varphi$ for every $\varphi \in \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})$.

Proposition 2.2.3. Let $\varphi_{x,y} \in \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})$ be the isometry such that $\varphi_{x,y}(r, a, b) = (r, a + x, b + y)$. Then,

- $p_*(\varphi) = (R^x, R^y) \in \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$
- $\widetilde{p}_*(\varphi_{x,y}) = \widetilde{\Theta}(\widetilde{R}^x, \widetilde{R}^y) \in \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$.

Proof. Let us call $\pi : \widetilde{\mathrm{PSL}_2(\mathbb{R})} \to \mathrm{PSL}_2(\mathbb{R})$ the covering map, then there exists $\pi_* : \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})}) \to \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$.

We have $\pi_*(\widetilde{p}_*(\varphi_{x,y})) = \pi_*(\widetilde{\Theta}(\widetilde{R}^x, \widetilde{R}^y))$, hence for every $(x, y) \in \mathbb{R}^2$ there exists $k(x, y) \in \mathbb{Z}$ such that

$$\widetilde{p}_*(x, y) = \widetilde{\Theta}(\widetilde{R}^{x+2\pi k}, \widetilde{R}^y).$$

The map $k \mapsto k(x,y)$ is continuous, so it is constant. Since $\varphi_{0,0} = (0,0) = \mathrm{Id}$ and $\widetilde{p}_*(\varphi_{0,0}) = \mathrm{Id} = \widetilde{\Theta}(\widetilde{R}^0, \widetilde{R}^0)$, we conclude that $k(0,0) = 0$ and so $k(x,y) = 0$ for every $(x,y) \in \mathbb{R}^2$. $\square$

2.2.3 Homotopy class of a timelike geodesic

Let us consider now the homotopy class of a timelike geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3_* = \mathbb{A}\mathrm{d}\mathbb{S}^3 \setminus \ell_{i,i}$.

First of all we remark that $\pi_1(\mathbb{A}\mathrm{d}\mathbb{S}^3_*, x_0)$ is Abelian so it can be canonically identified with the free homotopic classes $[S^1, \mathbb{A}\mathrm{d}\mathbb{S}^3_*] = \mathbb{Z} \oplus \mathbb{Z}$. We will always implicity use this identification getting rid of the base point.

Now, any closed timelike geodesic contained in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ is of the form $\ell_{a,b}$ for some $a, b \in \mathbb{H}^2$. Notice that since $\ell_{a,b} \cap \ell_{i,i} = \varnothing$, the distances in $\mathbb{H}^2$ are $\mathrm{dist}_{\mathbb{H}^2}(a,i) \neq \mathrm{dist}_{\mathbb{H}^2}(b,i)$ so there are two possibilities: $\mathrm{dist}_{\mathbb{H}^2}(a,i) < \mathrm{dist}_{\mathbb{H}^2}(b,i)$ or $\mathrm{dist}_{\mathbb{H}^2}(a,i) > \mathrm{dist}_{\mathbb{H}^2}(b,i)$. This fact allows to distinguish two classes of timelike geodesics in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$. Those two classes correspond to two different homotopy classes:

$$[\ell_{a,b}] = \begin{cases} \alpha_r & \text{if } \ d(a,i) < d(b,i), \\ \alpha_l & \text{if } \ d(a,i) > d(b,i). \end{cases}$$

We want to prove that every geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ is homotopic either to $\ell_{i,b}$ and $\ell_{a,i}$ and that they represent the unique homotopy classes in $\pi_1(\mathbb{A}\mathrm{d}\mathbb{S}^3_*)$. Firstly we have to ensure that $\ell_{i,b}$ does not represent a class for every $b \in \mathbb{H}^2 \setminus \{p\}$ (and the same for $\ell_{a,i}$).

Lemma 2.2.4. The geodesic $\ell_{i,b}$ and $\ell_{i,b'}$ are in the same homotopy class for every $b, b' \in \mathbb{H}^2 \setminus \{p\}$, where p is singular point.

Proof. It follows by the observation that $\mathbb{H}^2 \setminus \{i\}$ is connected. $\square$

Proposition 2.2.5. The geodesic $\ell_{a,b}$ is homotopic to $\ell_{i,b}$ if $d(a,i) < d(b,i)$ while it is homotopic to $\ell_{a,i}$ otherwise.

Proof. It is clear that $\ell_{a,b} \cap \ell_{i,i} \neq \varnothing$ if and only if $d(a,i) = d(i,b)$. So, let us consider the cases where $\ell_{a,b}$ doesn't intersect the singular geodesic $\ell_{i,i,}$, i.e., $d(a,i) < d(b,i)$ or $d(b,i) < d(a,i)$.

The two cases are analogous. Let us consider the case $0 < \mathrm{dist}(a,i) < \mathrm{dist}(b,i)$: there exists a path $a_t \subset \mathbb{H}^2$ with $a_0 = i$ and $a_1 = a$, such that for every t

$$\mathrm{dist}(a_t, i) < \mathrm{dist}(b,i) \qquad \text{and} \qquad \ell_{a_t,b} \subset \mathbb{A}\mathrm{d}\mathbb{S}^3_*,$$

that is, $\ell_{a_t,b} \cap \ell_{i,i} \neq \varnothing$ for every $t \in [0,1]$. Notice that is not possible to construct a similar path between $\ell_{a,b}$ and $\ell_{a,i}$ without meeting $\ell_{i,i}$. □

These results and in particular the proposition 2.2.5 show that $\ell_{i,b}$ and $\ell a, i$ are representatives respectively for the classes α_r and α_r.

Remark 11. The classes α_r and α_r form a basis of $\pi_1(\mathbb{A}\mathrm{d}\mathbb{S}^3_*)$.

At this point, it is useful to notice that we can write

- $\ell_{i,b} = \{R_i^\theta (L_b)^{-1} \mid \theta \in [0, 2\pi]\}$,
- $\ell_{a,i} = \{(L_a) R_i^\theta \mid \theta \in [0, 2\pi]\}$,

for every $a, b \in \mathbb{H}^2$.

Proposition 2.2.6. Let $\tilde{\ell}$ be a complete timelike geodesic in the universal cover $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ and $\Pi(\tilde{\ell}) = \ell$ its image in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$.
Then,

$$\mathrm{Stab}(\tilde{\ell}) = \begin{cases} \mathbb{R} \times \{0\} & \text{if} \quad \ell = \ell_{i,b}, \\ \{0\} \times \mathbb{R} & \text{if} \quad \ell = \ell_{a,i}, \\ \mathbb{Z} \cdot (2\pi, 0) & \text{if} \quad [\ell] = \alpha_r, \\ \mathbb{Z} \cdot (0, 2\pi) & \text{if} \quad [\ell] = \alpha_l. \end{cases}$$

Proof. We already know that every geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ has the form $\ell_{a,b}$ with $a, b \in \mathbb{H}^2$ distinct points and we can distinguish the cases $0 < d(a,i) < d(i,b)$ and $0 < d(i,b) < d(a,i)$ so that every timelike geodesic will be homotopic either to the special timelike geodesic $\ell_{i,b}$ or $\ell_{a,i}$.

Let us first consider the case $\ell = \ell_{i,b}$ and $\tilde{\ell} \in \Pi^{-1}(\ell_{i,b})$. Since

$$\Pi^{-1}(\ell_{i,b}) = \{(d(i,b), x_0 + x, y_0 + 2\pi k), x \in \mathbb{R}, k \in \mathbb{Z}\}$$

with $x_0, y_0 \in \mathbb{R}$ fixed, knowing that the lift $\tilde{\ell}$ is just an element in the preimage of $\ell_{i,b}$ so

$$\tilde{\ell} = \{(d(i,b), x_0 + x, y_0), x \in \mathbb{R}\},$$

where we chose $k_0 = 0$.

If $\varphi \in \mathrm{Stab}(\tilde{\ell})$, by definition $\varphi(\tilde{\ell}) = \tilde{\ell}$ and, because of the commutative diagram, it is also $H(\varphi)(\ell) \subseteq \ell$.

Recalling that $\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) \simeq \mathbb{R}^2$ (Proposition 2.2.2) acts by translation on the last two components of $\tilde{\ell}$, it is clear that

$$\mathrm{Stab}(\tilde{\ell}) = \mathbb{R} \times \{0\}.$$

It is the same for the symmetric case with $\ell = \ell_{a,i}$.

Let us consider the case for $0 < d(a,i) < d(i,b)$ so that ℓ is homotopic to $\ell_{i,b}$ and let $\tilde{\ell}_0$ be a lift of $\ell_{i,b}$. Thus, we want to show that

$$\mathrm{Stab}(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell}_0),$$

where $\Gamma = \mathrm{Aut}(\Pi) = 2\pi\mathbb{Z} \times 2\pi\mathbb{Z}$.

First we see that $\mathrm{Stab}_\Gamma(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell}_0)$ for every $\tilde{\ell}, \tilde{\ell}_0 \in \Pi^{-1}(\ell)$: in general the stabilizers of the lifting of the free homotopic curves are conjugates in Γ. In our case $\mathrm{Aut}(\Pi) \simeq \pi_1(\mathbb{A}\mathrm{d}\mathbb{S}^3_*)$ is Abelian, so the stabilizers in Γ are exactly the same. By the previous discussion $\mathrm{Stab}_\Gamma(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell}_0) = \mathrm{Stab}_\Gamma(\tilde{\ell}_0) \cap 2\pi\mathbb{Z} \times 2\pi\mathbb{Z} = 2\pi\mathbb{Z} \times \{0\}$.

Now let us prove $\mathrm{Stab}(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell})$. Recall the map $p_* : \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) \to \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$ seen in Section 2.2.2 and notice that $\ker p_* = \Gamma$. Let $\varphi \in \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})$ preserving $\tilde{\ell}$ that is not in Γ. If $g \in \mathrm{Stab}(\tilde{\ell})$, then $p_*(g) \in \mathrm{Stab}(\ell_{a,b}) \cap \mathrm{Stab}(\ell_{i,i})$, being $\ell_{a,b} \in \mathrm{PSL}_2(\mathbb{R})$ with $i \notin \{a,b\}$ geodesic corresponding to $\tilde{\ell}$. Notice that

$$\mathrm{Stab}(\ell_{a,b}) \cap \mathrm{Stab}(\ell_{i,i}) = \big(\mathrm{Stab}(a) \times \mathrm{Stab}(b)\big) \cap \big(\mathrm{Stab}(i) \times \mathrm{Stab}(i)\big) = (\mathrm{Id}, \mathrm{Id}),$$

so that $p_*(g) = (\mathrm{Id}, \mathrm{Id})$ and $g \in \Gamma$. We conclude that $\mathrm{Stab}(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell}) = \mathrm{Stab}_\Gamma(\tilde{\ell}_0) = (2\pi, 0) \cdot \mathbb{Z}$.

□

Proposition 2.2.7. Let $\widetilde{\alpha}$ be a complete geodesic in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ such that $\Pi(\widetilde{\alpha})$ is in the homotopy class α_l and let $T = (0, y)$ an isometry of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ preserving $\widetilde{\alpha}$. Then, T acts on $\widetilde{\alpha}$ as a translation of $\frac{x}{2}$.

Proof. Let us distinguish two cases:

(i) $x \in 2\pi\mathbb{Z}$,

(ii) $x \notin 2\pi\mathbb{Z}$.

Let us see the case (i). Here $\Pi(\widetilde{\alpha})$ is homotopic to $\ell_{a,i}$. The map $\widetilde{p} : \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \to \widetilde{\mathrm{PSL}_2(\mathbb{R})}$ is such that $\widetilde{p}_{|\tilde{\ell}}$ is injective and realizes an isometry between $\tilde{\ell}$ and $\widetilde{p}(\tilde{\ell})$. If $\widetilde{p}_* : \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) \to \mathrm{Isom}(\widetilde{\mathrm{PSL}_2(\mathbb{R})})$ is the induced map over the isometry groups, then by Proposition 2.2.3:

$$\widetilde{p}_*(0, 2\pi) = \widetilde{\Theta}(\widetilde{R}^0, \widetilde{R}^{2\pi}) = \widetilde{\Theta}(\mathrm{Id}, \widetilde{R}^{2\pi}) = T_0.$$

The following diagram

$$\begin{array}{ccc} \tilde{\ell} & \xrightarrow{\widetilde{p}} & \widetilde{p}(\tilde{\ell}) \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle T_0^{k_0}} \\ \tilde{\ell} & \xrightarrow{\widetilde{p}} & \widetilde{p}(\tilde{\ell}) \end{array}$$

where $T_0^{k_0}$ is a translation of πk_0 (Proposition 1.4.3).

Let us see now the case (ii). Here $\Pi(\widetilde{\alpha}) = \ell_{a,i}$. Then, we can give the following parametrization to $\widetilde{\alpha}$.

$$\widetilde{\alpha}(t) = (r_0, x_0, t).$$

Then, $\widetilde{\alpha}'(t) = \dfrac{\partial}{\partial y}$. To compute the speed of $\Pi\left(\widetilde{\alpha}(t)\right)$, we need to compute first the norm of the derivative in y:

$$\begin{aligned}
&\left\| \frac{d}{dh}\Pi\left(\gamma_L(t+h)\right)_{|h=0} \right\| = \\
&\left\| \frac{d}{dh}\left(R_i^{x_0} L_{e^{ir_0}} R_i^{-(t+h)}\right)_{|h=0} \right\| = \\
&\left\| \frac{d}{dh}\left(\left(R_i^{x_0} L_{e^{ir_0}} R_i^{-t}\right) R_i^{-h}\right)_{|h=0} \right\| = \\
&\frac{1}{2}\left\| \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\| = -\frac{1}{2},
\end{aligned}$$

where we recall that $\|A\| := \langle A, A\rangle = -det(A)$.

Therefore

$$\begin{aligned}
\text{length}\left(\Pi\left(\widetilde{\alpha}(t)\right)\right) &= \left| \int_0^{2\pi} \frac{d}{dt} \|\Pi\left(\widetilde{\alpha}(t)\right)\| dt \right| \\
&= \left| \int_0^{2\pi} -\frac{1}{2} dt \right| = 2\pi \cdot \frac{1}{2} = \pi.
\end{aligned}$$

□

We are studying quotients of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ by $\mathbb{Z}^2$-lattices with foliation $\mathscr{F}$ in timelike geodesics of length π and here we provide a condition for the manifold to have such a foliation. Remember that two types of foliation are possible in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$, right-handed and left-handed. We prove the following result first.

Proposition 2.2.8. Let Λ be a lattice and $\mathscr{F}$ be a foliation in timelike geodesics for $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ which is Λ-invariant. Then, $\mathscr{F}$ is left-handed if and only if every $\ell \in \mathscr{F}$ is in the hotopopy class α_l.

Proof. Let $\gamma(t) = (t, \theta_0, \varphi_0)$ be a spacelike geodesic in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ and $\ell(t)$ the leaf through $\gamma(t)$, with projection $\Pi(\ell(t)) = \ell_{a(t),b(t)}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$. Using the fact that $\ell_{a(t),b(t)}$ rotates always on the left along $\Pi(\gamma(t))$ (Lemma 1.5.5), we see that there exists $\lim_{t\to 0} \ell_{a(t),b(t)} = \ell_{a,b}$ with $\ell_{a,b} \cap \ell_{i,i} \neq \varnothing$, since $\Pi(\gamma(t))$ converges to a point in $\ell_{i,i}$.

Denote by $\alpha(t)$ the point on $\ell(t)$ in the future of $\Pi(\gamma(t))$ at distance ϵ. Notice that $\alpha(t)$ extends to a continuous path $\alpha : [0, +\infty) \to \mathbb{A}\mathrm{d}\mathbb{S}^3_*$ since $\alpha(0) \in \ell_{a,b} \backslash \ell_{i,i}$. So there exists $\widetilde{\alpha} : [0, +\infty) \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ such that $\widetilde{\alpha}(0) \in \ell(0)$. It follows that the leaf through $\widetilde{\alpha}(t) = \ell(t)$, hence $\lim_{t\to 0} \ell(t) = \ell_0$ is a complete geodesic in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$. Thus, $\Pi(\ell_0)$ is a geodesic contained in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$, $\Pi(\ell_0) \neq \ell_{a,b}$. But this contradicts the assumption $\Pi(\ell(t)) = \ell_{a(t),b(t)} \to \ell_{a,b}$.

Since $\mathscr{F}$ is left-handed, then $\mathrm{dist}_{\mathbb{H}^2}(a_1, a(t)) \geq \mathrm{dist}_{\mathbb{H}^2}(b_1, b(t))$, so passing to the limit $\mathrm{dist}_{\mathbb{H}^2}(a_1, i) \geq \mathrm{dist}_{\mathbb{H}^2}(b_1, i)$. The geodesics ℓ_{a_1,b_1} and $\ell_{i,i}$ are disjoint, thus

$$\mathrm{dist}_{\mathbb{H}^2}(a_1, i) > \mathrm{dist}_{\mathbb{H}^2}(b_1, i).$$

□

Proposition 2.2.9. Let Λ be a lattice for $\mathrm{Isom}\,\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ such that $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} / \Lambda$ admits a foliation in timelike geodesics of length π, then $F_0 = (0, 2\pi)$ is a generator for Λ if and only if the foliation is left-handed.

Proof. Since $\mathrm{Stab}(\tilde{\ell}) \simeq \mathbb{Z} \cdot \gamma$, using Proposition 2.2.6 we know that $\gamma \in \{0\} \times \mathbb{R}$ with length π, then $\gamma = (0, 2\pi)$. □

Remark 12. If $\langle F_0, G_0 \rangle$ and $\langle F_0, G_1 \rangle$ are two basis of Λ, then there exists $k \in \mathbb{Z}$ such that $G_0 + G_1 = kF_0$. Indeed, we can write as linear combination $G_1 = kF_0 + hG_0$ with $k, h \in \mathbb{Z}$. Since $\langle F_0, G_1 \rangle$ is a basis if and only if

$$\left| \det \begin{bmatrix} 1 & \alpha \\ 0 & \beta \end{bmatrix} \right| = 1,$$

then $\beta = \pm 1$. Thus, $G_1 = kF_0 \pm G_0$.

Since we are interested in lattices Λ such that ${\widetilde{\mathbb{AdS}^3_*}}\diagup_{\Lambda}$ admits a foliation in timelike geodesics of length π, by the previous results, we need to take a lattice $\Lambda = \langle F_0, G_0\rangle$ with F_0 irreducible element in the lattice such that $F_0 = (2\pi, 0)$ or $F_0 = (0, 2\pi)$. Indeed, the transformations in $\Lambda = \langle F_0, G_0\rangle$ permute the leaves of the foliation and F_0 preserves every leaf acting on each by a translation of π. Moreover we can say also something about the generator G_0.

Corollary 2.2.10. Let $\{F_0, G_0\}$ and $\{F_0, G_1\}$ be $\mathbb{Z}$-basis of Λ, with $F_0 = (2\pi, 0)$. Then if $G_0 = (\theta_0, \eta_0)$ and $G_1 = (\theta_1, \eta_1)$, there exists $k \in \mathbb{Z}$ such thatd $\theta_1 = \theta_0$ an $\eta_1 = \eta_0 + 2\pi k$.

Proof. By Remark 12, there exists $k \in \mathbb{Z}$ such that $G_1 = kF_0 + G_0$, thus

$$\begin{cases} \theta_1 = \theta_0, \\ \eta_1 = 2\pi k + \eta_0. \end{cases}$$

□

Definition 17. Let $\Lambda = \langle (0, 2\pi), (\theta_0, \eta_0)\rangle$ be a lattice in $\mathrm{Isom}(\widetilde{\mathbb{AdS}^3})$. We define *model for a generalized cone-spin singularity with* $(\theta_0, \eta_0) \in \mathbb{R} \times {\mathbb{R}}\diagup_{2\pi\mathbb{Z}}$ the quotient manifold:

$$\mathbb{AdS}^3_{(\theta_0,\eta_0)} := {\widetilde{\mathbb{AdS}^3_*}}\diagup_{\Lambda}$$

Remark 13. By Proposition 2.2.10, generalized spin-cone singularities are classified by $\theta_0 \in \mathbb{R}$ and $\eta_0 \in {\mathbb{R}}\diagup_{2\pi\mathbb{Z}}$.

Remark 14. $\mathbb{AdS}^3_{(\theta_0,\eta_0)}$ with $\theta_0 \in \mathbb{R}$ and $\eta_0 \in {\mathbb{R}}\diagup_{2\pi\mathbb{Z}}$ has a left-handed foliation in timelike geodesics of length π.

CHAPTER 3

Foliated Anti-de Sitter manifolds

In this Chapter, we introduce the construction of a three dimensional Anti-de Sitter manifold as fiber bundle over a hyperbolic surface. The main purpose is to show that all the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold with generalized spin-cone singularities are fibrations over a surface with conical singularities.

3.1 Quotients of the 3-dimensional Anti-de Sitter space

By the works of Kulkarni and Raymond [KR85], Klingler [Kli96] and Kassel [GKW15], the three dimensional compact Anti-de Sitter manifolds are in the form

$$\mathrm{PSL}_2(\mathbb{R}) \big/ j \times \rho(\pi_1(S))$$

where S is an oriented compact surface, $j : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian representation and $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is a representation strictly dominated by j. It follows that a compact Anti-de Sitter manifold admits a finite covering which is a circle bundle over a compact surface of genus greater than 2, and the surface is a Seifert bundle.

Thus, it is possible to describe the deformation space of Anti-de Sitter structures on a circle bundle of Euler class k over an oriented compact surface S of genus $g \geq 2$.

In the further section, we provide a construction for compact $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifolds as fiber bundle over a compact hyperbolic surface.

3.2 AdS-manifold as circle bundle over a surface

3.2.1 First result: construction of the circle bundle

Let (S, g) be an oriented hyperbolic surface, not necessarily complete, with developing map d and holonomy j. We consider also an equivariant map $f : \widetilde{S} \to \mathbb{H}^2$, that is

$$f(\gamma\tilde{x}) = \rho(\gamma) f(\tilde{x})$$

with $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ with monodromy representation.

Definition 18. We say that $f : \widetilde{S} \to \mathbb{H}^2$ equivariant map is *a weak contraction* if and only if

$$\mathrm{dist}_{\mathbb{H}^2}(f(x), f(y)) < \mathrm{dist}_{\widetilde{S}}(x, y) \tag{3.1}$$

for every distinct points $x, y \in \widetilde{S}$ sufficently close.

In the following, we will associate any tern (S, ρ, f) such that f is a weak contraction with a three dimensional Anti-de Sitter manifold $\mathscr{M} := \mathscr{M}(S, \rho, f)$ so that:

- the holonomy map of $\mathscr{M}$ is given by the pair (j, ρ), that is

  $$\mathscr{D} \circ \gamma = (j(\gamma), \rho(\gamma)) \cdot \mathscr{D},$$

  with $\mathscr{D}$ a developing map for $\mathscr{M}$;

- there exists a fibration $\tau : \mathscr{M} \to S$ whose fiber $\tau^{-1}(x)$ is a closed geodesic in $\mathscr{M}$ for every $x \in S$.

We will construct explicitly the manifold $\mathscr{M}$ as fiber bundle over the surface. Let us fix $\{U_\alpha\}_{\alpha \in \mathscr{A}}$ a finite cover of S with each U_α geodesically convex set for the metric g, and let us denote by $\{\tilde{U}_{\tilde{\alpha}}\}_{\tilde{\alpha} \in \tilde{\mathscr{A}}}$ the open cover of $\widetilde{S}$, given by lifting the open sets in $\{U_\alpha\}$ and such that $d_{|\widetilde{U}_{\tilde{\alpha}}}$ is a diffeomorphism onto its image. If $\pi : \widetilde{S} \to S$ is the covering map, we define an induced map $\pi_* : \widetilde{\mathscr{A}} \to \mathscr{A}$ over the indices such that

$$\forall \tilde{\alpha} \in \widetilde{\mathscr{A}} \qquad \pi(\widetilde{U}_{\tilde{\alpha}}) = U_{\pi_*(\tilde{\alpha})}.$$

For every $x \in S$, the action of the fundamental group $\pi_1(S)$ on $\pi^{-1}(x)$ is free and transitive, therefore we can define an action of $\pi_1(S)$ on $\widetilde{\mathscr{A}}$ in the following way:

$$\text{for every } \gamma \in \pi_1(S) \qquad \gamma \cdot \tilde{\alpha} = \tilde{\beta}$$
$$\text{where } \tilde{\beta} \in \widetilde{\mathscr{A}} \text{ is the index such that } \gamma \cdot \widetilde{U}_{\tilde{\alpha}} = \widetilde{U}_{\tilde{\beta}}.$$

Given a pair of maps (d, f) as defined before, for every $\tilde{x} \in \widetilde{S}$ we can look at the timelike geodesic:

$$\ell_{\tilde{x}} \coloneqq \ell_{d(\tilde{x}), f(\tilde{x})} = \{T \in \mathrm{PSL}_2(\mathbb{R}) \mid T(f(\tilde{x})) = d(\tilde{x})\}.$$

Since $d_{|\widetilde{U}_{\tilde{\alpha}}}$ is injective, for every $\tilde{x} \in \widetilde{U}_{\tilde{\alpha}}$ the geodesic $\ell(\tilde{x})$ is the same as $\ell_{z, f \circ (d_{|\widetilde{U}_{\tilde{\alpha}}})^{-1}(z)}$, where $z = d(\tilde{x}) \in \mathbb{H}^2$. Moreover, as f is a weak contraction, the composition $f \circ d^{-1}{}_{|\widetilde{V}_{\tilde{\alpha}}}$ is a contraction for each $\widetilde{V}_{\tilde{\alpha}} = d(\widetilde{U}_{\tilde{\alpha}})$.

Therefore, thanks to the Corollary 1.3.4, for every $\tilde{x} \neq \tilde{y} \in \widetilde{U}_\alpha$

$$\ell_{\tilde{x}} \cap \ell_{\tilde{y}} \neq \varnothing.$$

For this reason, the set

$$M_{\tilde{\alpha}} = \bigcup_{\tilde{x} \in \widetilde{U}_{\tilde{\alpha}}} \ell_{\tilde{x}} = \{T \in \mathrm{PSL}_2(\mathbb{R}) \mid T(f(\tilde{x})) = d(\tilde{x}) \text{ for some } \tilde{x} \in \widetilde{U}_{\tilde{\alpha}}\} \tag{3.2}$$

is indeed foliated by timelike geodesics, and we can rewrite it as

$$M_{\tilde{\alpha}} = \{T \in \mathrm{PSL}_2(\mathbb{R}) \mid \exists!\, \tilde{x} \in \widetilde{U}_{\tilde{\alpha}} \text{ such that } T(f(\tilde{x})) = d(\tilde{x})\}. \tag{3.3}$$

Proposition 3.2.1. The set $\mathscr{U}_{\tilde{\alpha}}$ is a time-tube.

Proof.
- $M_{\tilde{\alpha}}$ is foliated by closed timelike geodesics. Let us call ν the unitary tangent field generating the foliation.
- $M_{\tilde{\alpha}}$ is an open subset in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. Given Π spacelike plane, the following map is well-defined:

$$\begin{aligned} F\colon \widetilde{U}_{\tilde{\alpha}} &\to \Pi \subset \mathrm{PSL}_2(\mathbb{R}) \\ \tilde{x} &\mapsto \ell_{\tilde{x}} \cap \Pi. \end{aligned} \tag{3.4}$$

  Moreover it is injective, because of the fact that the timelike geodesics form a foliation. For the Invariance of Domain Theorem, $F(\widetilde{U}_{\tilde{\alpha}})$ is open in Π.

  We define now the injective map:

$$\begin{aligned} \Theta : \widetilde{U}_{\tilde{\alpha}} \times [0,\pi] &\to \mathrm{PSL}_2(\mathbb{R}) \\ (\tilde{x}, t) &\mapsto exp_{F(\tilde{x})}(t\nu(\tilde{x})). \end{aligned} \tag{3.5}$$

  Therefore, again for the Invariance of Domain Theorem $\Theta(\widetilde{U}_{\tilde{\alpha}} \times [0,\pi]) = M_{\tilde{\alpha}}$ is an open subset.

□

Then, we set

$$M \coloneqq \bigsqcup_{\tilde{\alpha} \in \widetilde{\mathscr{A}}} M_{\tilde{\alpha}}. \tag{3.6}$$

Let $\iota : M \to \mathrm{PSL}_2(\mathbb{R})$ be a map such that, for every $X \in M_{\tilde{\alpha}}$,

$$\iota_{|M_{\tilde{\alpha}}} = \iota_{\tilde{\alpha}},$$

where $\iota_{\tilde{\alpha}} : M_{\tilde{\alpha}} \to \mathrm{PSL}_2(\mathbb{R})$ is the inclusion map.

Furthermore, thanks to (3.3), there are well-defined projections $\tau_{\tilde{\alpha}} : M_{\tilde{\alpha}} \to \widetilde{U}_{\tilde{\alpha}}$ such that for every $\tilde{x} \in U_{\tilde{\alpha}}$,

$$\tau_{\tilde{\alpha}}^{-1}(\tilde{x}) = \ell_{\tilde{x}}.$$

Remark 15. Using a similar argument as the one seen in Proposition 3.2.1, we can say that $\tau_{\tilde{\alpha}}$ is a continuous map for every $\widetilde{\alpha}$.

Therefore, we can define $\tau : M \to \widetilde{S}$ as the map such that

$$\tau_{|M_{\tilde{\alpha}}} = \tau_{\tilde{\alpha}}. \tag{3.7}$$

Remark 16. As we already noted, two geodesics in $M_{\tilde{\alpha}}$ have no intersection, so the map ι is injective if restricted to $M_{\tilde{\alpha}}$, but for $X \in M_{\tilde{\alpha}}$ and $Y \in M_{\tilde{\beta}}$ it can happen that they correspond to the same element in $\mathrm{PSL}_2(\mathbb{R})$. As a consequence, we can just say that the map ι is locally injective.

For every $\tilde{x} \in \widetilde{U}_{\tilde{\alpha}} \cap \widetilde{U}_{\tilde{\beta}} \neq \varnothing$, the images of $\tau_{\tilde{\alpha}}^{-1}(\tilde{x})$ and $\tau_{\tilde{\beta}}^{-1}(\tilde{x})$ through ι represent the same timelike geodesics in $\mathrm{PSL}_2(\mathbb{R})$. This proves that τ is not globally injective.

Taking into account the previous remarks, in order to identify two points in M, we introduce on this domain the following equivalence relation:

$$X \sim Y \iff \tau_{\tilde{\alpha}}(X) = \tau_{\tilde{\beta}}(Y) \quad \text{and} \quad \iota_{\tilde{\alpha}}(X) = \iota_{\tilde{\beta}}(Y), \tag{3.8}$$

for $X \in M_\alpha$ and $Y \in M_\beta$. This allows us to consider the quotient space

$$\widehat{\mathscr{M}} := {}^{M}\!/\!_{\sim} \tag{3.9}$$

together with the projection map $p : M \to {}^{M}\!/\!_{\sim}$.

Remark 17. The projection p has the following properties:

- $p_{|M_{\tilde{\alpha}}}$ is injective;

- $p_{|M_{\tilde{\alpha}}}\colon M_{\tilde{\alpha}} \to p(M_{\tilde{\alpha}})$ is a homeomorphism onto its image;
- $p(M_{\tilde{\alpha}})$ is an open set in $\widehat{\mathscr{M}}$.

For every class $\mathscr{C}$ in $\widehat{\mathscr{M}}$ we have $\mathscr{C} = [X]$ with $X \in M_{\tilde{\alpha}}$ for some $\tilde{\alpha} \in \widetilde{\mathscr{A}}$, so we can define the map $\widetilde{\tau} : \widehat{\mathscr{M}} \to \widetilde{S}$ by

$$\widetilde{\tau}([X]) = \tau_{\tilde{\alpha}}(X). \tag{3.10}$$

In particular, $\widetilde{\tau}$ does not depend on the choice of the representative, thanks to the definition on $\widehat{\mathscr{M}}$.

Moreover $\widetilde{\tau}$ is a circle bundle over $\widetilde{S}$: for every $\tilde{x} \in \widetilde{S}$ there exists a neighbourhood $\widetilde{U}_{\tilde{\alpha}}$ such that

$$\widetilde{\tau}^{-1}(\widetilde{U}_{\tilde{\alpha}}) = M_{\tilde{\alpha}} \simeq \Theta(\widetilde{U}_{\tilde{\alpha}} \times [0,\pi]) \simeq \widetilde{U}_{\tilde{\alpha}} \times S^1, \tag{3.11}$$

where Θ is the map defined in 3.5. Notice that $\Theta(\widetilde{U}_{\tilde{\alpha}} \times [0,\pi])$ is foliated by timelike geodesics in $\mathrm{PSL}_2(\mathbb{R})$ and every timelike geodesic is homeomorphic to a circle of lenght π.

Lemma 3.2.2. The map $\widetilde{\tau}$ is continuous.

Proof. More generally, $f : {}^{X}\!/\!\sim\, \to Y$ is continuous if and only if $f \circ p : X \to Y$ is continuous with $p : X \to {}^{X}\!/\!\sim$ projection. In our case, $\widetilde{\tau} \circ p(X) = \widetilde{\tau}([X]) = \tau_{\tilde{\alpha}}$. Furthermore, we know that $\tau_{\tilde{\alpha}}$ is continuous, so it is $\widetilde{\tau}$. □

Proposition 3.2.3. The quotient space $\widehat{\mathscr{M}}$ is an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold.

Proof. First of all, the map $\widetilde{\tau}$ allows us to say that $\widehat{\mathscr{M}}$ is a Hausdorff space:

- since $\widetilde{\tau}$ is a continuous map, $\widetilde{\tau}^{-1}(\widetilde{U}_{\tilde{\alpha}})$ is an open set for the induced topology on $\widehat{\mathscr{M}}$, as $\widetilde{U}_{\tilde{\alpha}}$ is an open set in $\widetilde{S}$. In particular, $\tau_{\alpha}^{-1}(\widetilde{U}_{\tilde{\alpha}}) \simeq M_{\tilde{\alpha}}$ is Hausdorff;
- let us consider $\mathscr{C}_1$ and $\mathscr{C}_2$ distinct points in $\widehat{\mathscr{M}}$. For every $X \in \mathscr{C}_1$ and every $Y \in \mathscr{C}_2$ we can distinguish two cases:

(i) $\widetilde{\tau}(X) \neq \tau(Y)$, so there exist $V, U \subset \widetilde{S}$ neighbourhoods of $\widetilde{\tau}(X), \widetilde{\tau}(Y)$ such that $U \cap V = \varnothing$ and $\mathscr{C}_1 \in \widetilde{\tau}^{-1}(V)$, $\mathscr{C}_2 \in \widetilde{\tau}^{-1}(U)$ with $\widetilde{\tau}^{-1}(V) \cap \in \widetilde{\tau}^{-1}(U) = \varnothing$;

(ii) $\widetilde{\tau}(X) = \widetilde{\tau}(Y) = \tilde{x} \in \widetilde{U}\widetilde{\alpha}$ and $\iota(X) \neq \iota(Y)$ hence X and Y are distinct points in $i_{\widetilde{\alpha}}(M_{\widetilde{\alpha}})$ so they can be separated.

Then, we can define for $\widehat{\mathscr{M}}$ the atlas $\{(p(M_{\widetilde{\alpha}}), \iota_{\tilde{\alpha}} \circ {p_{|M_{\widetilde{\alpha}}}}^{-1})\}$. $\square$

3.2.2 Action of the fundamental group on a timelike foliation of an AdS-domain

In this section we introduce a natural action of $\pi_1(S)$ on $\widehat{\mathscr{M}}$ in order to prove:

Theorem 3.2.4. $\mathscr{M} \coloneqq \widehat{\mathscr{M}} / \sim$ is a circle bundle over the surface S with geodesic fibers.

To this aim, firstly we consider the homomorphism $H : \pi_1(S) \to \mathrm{Isom}(\mathrm{PSL}_2(\mathbb{R}))$ as $H(\gamma) \coloneqq (j(\gamma), \rho(\gamma))$ for every $\gamma \in \pi_1(S)$. In particular, we show the action of $\gamma \in \pi_1(S)$ on the leaves of a foliated open subset in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and the action of $H(\gamma)$ on the sets $M_{\tilde{\alpha}}$ immersed in $\mathrm{PSL}_2(\mathbb{R})$. By the action on M we obtain an induced action on the quotient $\widetilde{\mathscr{M}}$.

Considering that d and f are $\pi_1(S)$-equivariant with holonomies respectively j and ρ, for every $\gamma \in \pi_1(S)$ and $\tilde{x} \in \tilde{S}$:

$$\begin{aligned} \ell_{\gamma\tilde{x}} = \ell_{d(\gamma\tilde{x}), f(\gamma\tilde{x})} &= \ell_{j(\gamma)d(\tilde{x}), \rho(\gamma)f(\tilde{x})} \\ &= (j(\gamma), \rho(\gamma))\ell_{d(\tilde{x}), f(\tilde{x})} = H(\gamma) \cdot \ell_{\tilde{x}}, \end{aligned}$$

where $H(\gamma) \coloneqq (j(\gamma), \rho(\gamma))$ is an isometry of $\mathrm{PSL}_2(\mathbb{R})$. Therefore, for every $\tilde{\alpha} \in \widetilde{\mathscr{A}}$ and for every $\gamma \in \pi_1(S)$ we have

$$\begin{aligned} H(\gamma) \cdot \iota_{\tilde{\alpha}}(M_{\tilde{\alpha}}) &= j(\gamma) \Bigg(\bigcup_{\tilde{x} \in \widetilde{U}_{\tilde{\alpha}}} \ell_{\tilde{x}} \Bigg) \rho(\gamma)^{-1} = \bigcup_{\tilde{x} \in \widetilde{U}_{\tilde{\alpha}}} j(\gamma) \ell_{\tilde{x}} \rho(\gamma)^{-1} \\ &= \bigcup_{\tilde{x} \in \widetilde{U}_{\tilde{\alpha}}} H(\gamma) \cdot \ell_{\tilde{x}} = \bigcup_{\tilde{x} \in \widetilde{U}_{\tilde{\alpha}}} \ell_{\gamma\tilde{x}} = \iota_{\gamma \cdot \tilde{\alpha}}(M_{\gamma \cdot \tilde{\alpha}}). \end{aligned}$$

As a consequence, we can define an action on M in the following way: for every $\gamma \in \pi_1(S)$ there exists a map $\widehat{H(\gamma)}: M \to M$ such that

$$\iota(\widehat{H(\gamma)}(M_{\tilde{\alpha}})) = H(\gamma)(\iota_{\tilde{\alpha}}(M_{\tilde{\alpha}})). \tag{3.12}$$

We set $\widehat{H(\gamma)}_{|M_{\tilde{\alpha}}} \coloneqq \iota^{-1}_{\gamma\cdot\alpha} \cdot H(\gamma) \cdot \iota_{\alpha}$.

Lemma 3.2.5 (Equivariance of τ and ι)**.** For every $\gamma \in \pi_1(S)$ the map $\widehat{H(\gamma)}$ is such that

$$\begin{cases} \iota \circ \widehat{H(\gamma)} & = H(\gamma) \circ \iota, \\ \tau \circ \widehat{H(\gamma)} & = \gamma \cdot \tau. \end{cases} \tag{3.13}$$

Proof. The first equivariance relation holds by the definition. Let us now consider the second relation. If $X \in M_{\tilde{\alpha}}$, then $X \in \ell_{\tilde{x}}$. Now, $\widehat{H(\gamma)} \cdot X \in M_{\gamma\cdot\tilde{\alpha}}$, indeed $\widehat{H(\gamma)} \cdot X \in \widehat{H(\gamma)} \cdot \ell_{\tilde{x}} = H(\gamma)\ell_{\tilde{x}} = \ell_{\gamma\tilde{x}}$. As a consequence,

$$\tau(\widehat{H(\gamma)}(X)) = \gamma\tilde{x}.$$

□

Notation 3.2.6. Taking into account the equivariance of the maps ι and τ with respect to the action of $\pi_1(S)$, for simplicity, we will denote the action by $\gamma\cdot$ for both the cases $\mathscr{M}$ and $\widetilde{S}$.

From the lemma 3.2.5, the action of $\pi_1(S)$ on M induces an action on the quotient $\widehat{\mathscr{M}} = {}^{M}\!/\!_{\sim}$. We have to verify that it is well-defined, i.e., if $Y \in [X]$ then $[\gamma \cdot Y] = [\gamma \cdot X]$ for every $\gamma \in \pi_1(S)$. If $X \in M_{\tilde{\alpha}}$ and $Y \in M_{\tilde{\beta}}$:

$$[\gamma \cdot Y] = [\gamma \cdot X] \iff \begin{cases} \tau_{\gamma\cdot\tilde{\beta}}(\gamma \cdot Y) = \tau_{\gamma\cdot\tilde{\alpha}}(\gamma \cdot X) \\ \iota_{\gamma\cdot\tilde{\beta}}(\gamma \cdot Y) = \iota_{\gamma\cdot\tilde{\alpha}}(\gamma \cdot X) \end{cases}$$

by definition (3.8). Considering that the action of $\pi_1(S)$ commutes with τ and ι, we have:

$$\begin{cases} \tau_{\gamma\cdot\tilde{\beta}}(\gamma \cdot Y) = \tau_{\gamma\cdot\tilde{\alpha}}(\gamma \cdot X) \\ \iota_{\gamma\cdot\tilde{\beta}}(\gamma \cdot Y) = \iota_{\gamma\cdot\tilde{\alpha}}(\gamma \cdot X) \end{cases} \iff \begin{cases} \gamma \cdot \tau_{\tilde{\beta}}(Y) = \gamma \cdot \tau_{\tilde{\alpha}}(X) \\ \gamma \cdot \iota_{\tilde{\beta}}(Y) = \gamma \cdot \iota_{\tilde{\alpha}}(X) \end{cases}$$

that is $[Y] = [X]$.

Lemma 3.2.7. The action of $\pi_1(S)$ on $\widehat{\mathscr{M}}$ is properly discontinuous and free.

- Properly discontinuous: if $K \subset \widehat{\mathscr{M}}$ is a compact set, also $\tau(K) \subset \widetilde{S}$ is compact; since $\pi_1(S)$ acts on $\widetilde{S}$ properly discontinuously,

  $$\gamma \cdot \tau(K) \cap \tau(K) = \varnothing \text{ for almost every } \gamma \in \pi_1(S);$$

  since $\gamma \circ \tau = \tau \circ \gamma$, we have

  $$\tau(\gamma K \cap K) \subset \tau(\gamma K) \cap \tau(K) = \gamma \cdot \tau(K) \cap \tau(K) = \varnothing,$$

  and so

  $$\gamma K \cap K = \varnothing \text{ for almost every } \gamma \in \pi_1(S).$$

- The action is free because $\pi_1(S)$ is torsion free and the action is properly discontinuous.

Therefore, thanks to the previous lemma, we can conclude that the quotient space $\mathscr{M} \coloneqq \widehat{\mathscr{M}} \big/ \pi_1(S)$ is a circle bundle over S.

3.2.3 Developing map of the AdS-circle bundle

Let us notice that the map $p_{|M_{\tilde{\alpha}}} : M_{\tilde{\alpha}} \to \widehat{\mathscr{M}}$ is an injective isometry and let us define $W_{\tilde{\alpha}} = p(M_{\tilde{\alpha}})$. Then we can define a local isometry $\mathscr{D} \colon \widehat{\mathscr{M}} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ such that, for every $\tilde{\alpha}$

$$\mathscr{D}_{|W_{\tilde{\alpha}}} = \iota_{\tilde{\alpha}} \circ (p_{|M_{\tilde{\alpha}}})^{-1}.$$

This map is well-defined, indeed for every $W_{\tilde{\alpha}} \cap W_{\tilde{\beta}} \neq \varnothing$, it occurs that

$$\iota_{\tilde{\alpha}} \circ (p_{|M_{\tilde{\alpha}}})^{-1}_{|W_{\tilde{\alpha}} \cap W_{\tilde{\beta}}} = \iota_{\tilde{\beta}} \circ (p_{|M_{\tilde{\beta}}})^{-1}_{|W_{\tilde{\alpha}} \cap W_{\tilde{\beta}}}.$$

Moreover, for $[X] \in W_{\tilde{\alpha}}$ and $\gamma \in \pi_1(S)$:

$$\begin{aligned} \mathscr{D}_{|W_{\gamma \cdot \tilde{\alpha}}}(\gamma \cdot [X]) &= \mathscr{D}_{|W_{\tilde{\alpha}}}([\gamma \cdot X]) = \iota_{\gamma \cdot \tilde{\alpha}}(\gamma \cdot X) \\ &= H(\gamma) \cdot \iota_{\tilde{\alpha}}(X) = H(\gamma) \cdot \mathscr{D}_{|W_{\tilde{\alpha}}}([X]) \end{aligned}$$

where $H : \pi_1(S) \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$.

3.2.4 A different interpretation

Let us denote $\tau := (\widetilde{\tau}, \iota) : \widehat{\mathscr{M}} \to \widetilde{S} \times \mathbb{A}\mathrm{d}\mathbb{S}^3$. This map is injective by definition and identifies $\widehat{\mathscr{M}}$ with the subspace of $\widetilde{S} \times \mathbb{A}\mathrm{d}\mathbb{S}^3$ given by

$$\{(\tilde{x}, T) \,|\, T(f(\tilde{x})) = d(\tilde{x})\}.$$

Notice that τ is π_1-equivariant, where the action on $\widetilde{S} \times \mathbb{A}\mathrm{d}\mathbb{S}^3$ is the product action. Moreover, from this perspective, the map $\mathscr{D}$ coincides with the projection on the second factor, while the circle bundle map is the projection on the first fact.

3.3 Functoriality of the construction

We can consider the category whose objects are the terns (S, ρ, f), where S is a hyperbolic surface with weak contraction f equivariant with respect to the representation ρ. To every (S, ρ, f) we associate the $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold given by $\mathscr{M}(S, \rho, f)$ as constructed in the above section. We say that the map $i : (S_1, \rho_1, f_1) \to (S_2, \rho_2, f_2)$ is a morphism in the category if

- $i : S_1 \to S_2$ is local isometry,
- $f_1 = f_2 \circ i$,
- $\rho_1 = \rho_2 \circ i_*$.

By the naturality of the construction is not difficult to check that any map i is associated to a map $\mathscr{I} : \mathscr{M}(S_1, \rho_1, f_1) \to \mathscr{M}(S_2, \rho_2, f_2)$ which is a local isometry. Then, the following diagram

$$\begin{tikzcd} \mathscr{M}(S_1, \rho_1, f_1) \arrow[r] \arrow[d, "\tau_1"] & \mathscr{M}(S_2, \rho_2, f_2) \arrow[d, "\tau_2"] \\ S_1 \arrow[r] & S_2 \end{tikzcd}$$

is commutative.

3.4 AdS-manifold with singularities as fibration on a surface with cone singularities

In the previous section, we have seen how to obtain an Anti-de Sitter manifold as circle bundle over a hyperbolic surface. In this section we analyze more closely the case where S is a compact surface with cone singularities.

Let S_* be the complement of the set of singularities $\{p_j\}$ in S, with $d : \widetilde{S} \to \mathbb{H}^2$ developing map for S and $f : \widetilde{S} \to \mathbb{H}^2$ weak contraction. For the construction seen in the Section 3.2, there exists an Anti de Sitter 3-manifold $\mathscr{M}_* := \mathscr{M}_*(S_*, \rho, f)$ that is a circle bundle over S_*. The purpose of this section is to show that for every point on the fiber of a singular point on S_*, the manifold $\mathscr{M}_*$ has a generalized cone-spin singularity.

Now, we look at one singular point p_j. Let us define $U_* = U \backslash \{p_j\}$. Given the set $\widetilde{U_*}$ as connected component of $\pi^{-1}(U_*)$, it results that

- the map $d_{|\widetilde{U}_*}$ avoids the point $p \in \mathbb{H}^2$;
- for every $\tilde{x} \in \widetilde{U}$ and for every $\gamma \in \pi_1(S)$:
$$d(\gamma\tilde{x}) = R_p^{\theta_1} d(\tilde{x})$$
with $\theta_1 \in \mathbb{R}$.

Proposition 3.4.1. Let $f : \widetilde{S} \to \mathbb{H}^2$ be equivariant with respect to the representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$. For every $\gamma \in \pi_1(S)$ the isometry $\rho(\gamma)$ is elliptic, that is
$$f(\gamma\tilde{x}) = R_q^{\theta_2} f(\tilde{x}),$$
with $\theta_2 \in \mathbb{R}$.

Proof. Let us consider a sequence $\{x_n\}$ in $\widetilde{U}$ that converges to $\widetilde{p_j}$ lifting of a singular point $\{p_j\} \in S$. Thanks to the equivariance of f, it holds
$$\mathrm{dist}_{\mathbb{H}^2}(\rho(\gamma) f(x_n), f(x_n)) = \mathrm{dist}_{\mathbb{H}^2}(f(\gamma x_n), \mathbf{f}(x_n)).$$

Because f is weak contraction,

$$\mathrm{dist}_{\mathbb{H}^2}(f(\gamma x_n), f(x_n)) < \mathrm{dist}_{\widetilde{S}}(\gamma x_n, x_n) < \theta \sinh(\mathrm{dist}_{\widetilde{S}}(x_n), \widetilde{p_j}) \to 0$$

where θ is the conical angle in p_j. Thus

$$\mathrm{dist}_{\widetilde{S}}(\gamma x_n, x_n) \leq \theta \sinh(\mathrm{dist}_{\widetilde{S}}(x_n), \widetilde{p_j}) \to 0.$$

Therefore we can conclude that $\mathrm{dist}_{\mathbb{H}^2}(\rho(\gamma) f(x_n), f(x_n))$ converges to zero, so $\{f(x_n)\}$ converges to $f(\widetilde{p_j})$ which is the fixed point for $\rho(\gamma)$. $\square$

Remark 18. If $\ell_{p,q} = \{A \in PSL_2(\mathbb{R}) \mid A(q) = p\}$ is a generic timelike geodesic in $\mathbb{A}\mathrm{d}\mathbb{S}^3$, identified by $\mathrm{PSL}_2(\mathbb{R})$, and $(R_p^{\theta_1}, R_q^{\theta_2}) \in \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-isometry fixing $\ell_{p,q}$, then $(R_p^{\theta_1}, R_q^{\theta_2})$ is conjugated to the pair $(R_i^{\theta_1}, R_i^{\theta_2})$ fixing the geodesic $\ell_{i,i}$.

Proposition 3.4.2. For every point $\tilde{x} \in \widetilde{U}_* = \widetilde{U \setminus \{p_j\}}$ the timelike geodesic $\ell(\tilde{x})$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ avoids the timelike geodesic $\ell(\tilde{p}_j) = \ell_{p,q}$.

Proof. Thanks to the previous remark, we can show the proposition in the case $p = q = i$, so that $\ell(\tilde{p}_j) = \ell_{i,i}$. If $\ell(\tilde{x}) \cap \ell(\tilde{p}_j)$, then there exists $A \in \mathrm{PSL}_2(\mathbb{R})$ such that

$$A(f(\tilde{x})) = d(\tilde{x}) \quad \text{and} \quad A(i) = i.$$

As a consequence, we have

$$\mathrm{dist}_{\mathbb{H}^2}(f(\tilde{x}), i) = \mathrm{dist}_{\mathbb{H}^2}(Af(\tilde{x}), Ai) = \mathrm{dist}_{\mathbb{H}^2}(d(\tilde{x}), i).$$

Being $d : \widetilde{S} \to \mathbb{H}^2$ a local isometry, this contradicts the fact that f is a a weak contraction, indeed if $\tilde{y}$ is a point on the ray between $\tilde{x}$ and $\tilde{p}_j$, we have

$$\begin{aligned}\mathrm{dist}_{\mathbb{H}^2}(f(\tilde{x}), i) &\leq \mathrm{dist}_{\mathbb{H}^2}(f(\tilde{x}), i) + \mathrm{dist}_{\mathbb{H}^2}(f(\tilde{x}), f(\tilde{y})) \\ &< \mathrm{dist}_{\mathbb{H}^2}(d(\tilde{x}), i) + \mathrm{dist}_{\mathbb{H}^2}(d(\tilde{x}), d(\tilde{y})) = \mathrm{dist}_{\mathbb{H}^2_*}(d(\tilde{x}), i).\end{aligned}$$

$\square$

Since $j(\gamma) = R_p^{\theta_1}$ and $\rho(\gamma) = R_q^{\theta_2}$ fix respectively p and q in $\mathbb{H}^2$, for a given $\gamma \in \pi_1(S)$ we have

$$(j(\gamma), \rho(\gamma)) \cdot \ell_{p,q} = \ell_{p,q}.$$

Thanks to the previous considerations, we can study the manifold $\mathscr{M}$ around the fiber of a singular point on S just looking at the neighbourhood of the geodesic $\ell_{i,i}$. Thanks to the functoriality of the construction, for simplicity, we can consider as base the disk $(\mathbb{D}, g_\theta)$ where g_θ os the metric such that $\mathbb{D}$ is a hyperbolic surface with one conical singularity at the point x_0.

In this case, we have the maps $d, f \colon \widetilde{\mathbb{D}_*} \to \mathbb{H}^2$ such that, for every $\tilde{x_0} \in \pi^{-1}(x_0)$, $d(\tilde{x_0}) = f(\tilde{x_0}) = i$. Their holonomies will be given by $j(\gamma) = R_i^{\theta_1}$ and $\rho(\gamma) = R_i^{\theta_2}$, while the closed curve $\ell(\tilde{x_0})$ corresponds to the geodesic $\ell_{i,i}$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ fixed by the pair $(R_i^{\theta_1}, R_i^{\theta_2})$. We will refer to $\mathscr{M} \coloneqq \mathscr{M}(\theta, \rho, f)$ as the manifold constructed on $\mathbb{D}$.

Notation 3.4.3. Taking account of the Remark 18, we will consider just the rotation R_i^θ that fixes $i \in \mathbb{H}^2$. As a consequence, from now on, we will often indicate the rotations of $\mathbb{H}^2$ fixing the point i just with R^θ, for any $\theta \in \mathbb{R}$. Moreover, we notice that $R^\theta = R^{\theta+2\pi}$ and that we can think of θ as an element of S^1.

3.4.1 From the disk to $\mathbb{H}^2_\theta$

In the previous section we reduced the construction passing from a closed surface to a disk $\mathbb{D}$. However, dealing with a disk we should discuss the construction over the boundary. To overcome this critical issue we can think in a more general way extending the construction to the case where the base surface for the fibration is $\mathbb{H}^2_* \coloneqq \mathbb{H}^2 \setminus \{i\}$.

In this respect, we notice that every equivariant weak contraction $f : \widetilde{\mathbb{D}_*} \to \mathbb{H}^2$ extends to an equivariant contraction $\overline{f} : \widetilde{\mathbb{H}^2_*} \to \mathbb{H}^2$ in the following way:

$$\overline{f}(t,\theta) = \begin{cases} f(t,\theta) & \text{if } t \le m_0 \\ f(m_0,\theta) & \text{if } t > m_0. \end{cases}$$

This map is a contraction too, because we are composing f with the 1-Lipschitz map $F : \widetilde{\mathbb{H}^2_*} \to \widetilde{\mathbb{D}_*}$ such that

$$F(t,\theta) = \begin{cases} (t,\theta) & t \le m_0 \\ (m_0,\theta) & t > m_0. \end{cases}$$

Then, we have a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathbb{D}_*} & \xrightarrow{d} & \mathbb{H}^2_* \\ \downarrow & & \downarrow \\ \widetilde{\mathbb{H}^2_*} & \longrightarrow & \mathbb{H}^2_\theta\,. \end{array}$$

where $\mathbb{H}^2_\theta$ is the model of the cone singularity on the hyperbolic plane as defined in Section 2.1. We can also construct an Anti-de Sitter manifold as fibration over $\mathbb{H}^2_\theta$, depending in this case on the functions $\overline{f}$ and π.

Therefore, we obtain an other commutative diagram:

$$\begin{array}{ccc} \mathscr{M}(\theta,\rho,f) & \xrightarrow{\mathscr{I}} & \mathscr{M}(\mathbb{H}^2_\theta,\rho,\overline{f}) \\ \downarrow & & \downarrow \\ \mathbb{D}_* & \xrightarrow{i} & \mathbb{H}^2_\theta, \end{array}$$

and $\mathscr{I}$ and i are isometric inclusions.

3.5 Fibration over $\mathbb{H}^2_\theta$

Definition 19. Let us define *admissible data* the tern (θ,η,f) where θ is the conical angle for the singularity in $\mathbb{H}^2_\theta$ and $f : \widetilde{\mathbb{H}^2_\theta} \to \mathbb{H}^2$ a contractive map equivariant with respect to R^η_i.

Given an admissible data (θ,η,f), as we have seen in the previous section, we can construct an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold $\mathscr{M}(\theta,\eta,f)$ fibered over $\mathbb{H}^2_\theta$.

Proposition 3.5.1. The map $\Psi : \mathbb{R}_+ \times S^1 \times S^1 \to \mathbb{A}\mathrm{d}\mathbb{S}^3_*$ given by the map

$$\Psi(r, e^{i\theta}, e^{i\varphi}) = (R^\theta, R^\varphi) \cdot L_{ir}$$

is a diffeomorphism.

Proof. It is enough to show that the map is injective and proper to conclude that the map is a diffeomorphism.

The map is injective. Let us remember that $(R^\theta, R^\varphi) \cdot L_{ir} = R^\theta L_{ir} R^{-\varphi}$, then we want to show that $R^\theta L_{ir} R^{-\varphi} = R^{\theta'} L'_{ir} R^{-\varphi'}$ imply $(r, e^{i\theta}, e^{i\varphi}) = (r', e^{i\theta'}, e^{i\varphi'})$. First we notice that $\mathrm{dist}_{\mathbb{H}^2}(R^\theta L_{ir} R^{-\varphi}(i), i) = r$, so if $R^\theta L_{ir} R^{-\varphi} = R^{\theta'} L'_{ir} R^{-\varphi'}$ certainly $r = r'$. Then we have $R^\theta L_{ir} R^{-\varphi}(i) = R^{\theta'} L_{ir} R^{-\varphi'}(i)$ and so $R^\theta L_{ir}(i) = R^{\theta'} L_{ir}(i)$, that imply $e^{i\theta} = e^{i\theta'}$, because it it $L_{ir}(i) \neq i$. Now we have $R^\theta L_{ir} R^{-\varphi} = R^\theta L_{ir} R^{-\varphi'}$. Repeating the same argument on the inverses we can conclude that also $e^{i\varphi} = e^{i\varphi'}$.

The map is also proper. Remember that $R^\theta L_{ir} R^{-\varphi}(i) \neq i$ and $R^\varphi(i) = i$, so $L_{ir} R^\varphi(i) = L_{ir}(i)$. Since L_{ir} is the traslation of the geodesic through i, we have $\mathrm{dist}(L_{ir}(i), i) = r$. Then,

$$\begin{aligned}
\mathrm{dist}_{\mathbb{H}^2}(R^\theta L_{ir} R^{-\varphi}(i)), i) &= \mathrm{dist}_{\mathbb{H}^2}(R^\theta L_{ir}(i), i) \\
&= \mathrm{dist}_{\mathbb{H}^2}(R^\theta L_{ir}(i)), R^\theta(i)) \\
&= \mathrm{dist}_{\mathbb{H}^2}(L_{ir}(i), i) = r > 0.
\end{aligned}$$

A sequence $(r_n, \theta_n \varphi_n)$ is divergent in $\mathbb{R}_+ \times S^1 \times S^1$ if $r_n \to 0$ or $r_n \to \infty$.

Let us denote $R^{\theta_n} L_{r_n} R^{-\varphi_n} = \gamma_n$. The sequence γ_n diverges in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$, since

$$\mathrm{dist}_{\mathbb{H}^2}(\gamma_n(i), i) = r_n,$$

then if $r_n \to \infty$, γ_n can not converge, while if $r_n \to 0$, γ_n converges to something fixing i, that is an element in ℓ_i. $\square$

Let us notice also that $S^1 \times S^1$ acts on $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ as pair of rotations and the diffeomorphism is equivariant, that is

$$\Psi(r, \theta + \theta', \varphi + \varphi') = (R^{\theta'}, R^{\varphi'}) \cdot \Psi(r, \theta, \varphi).$$

We recall that in $\mathbb{A}\mathrm{d}\mathbb{S}^3$ the pair $(R^{\alpha/2}, R^{\alpha/2})$ is a *pure rotation* while $(R^{\alpha/2}, R^{-\alpha/2})$ is a *pure traslation* along the geodesic $\ell_{i,i}$. In $\mathbb{R}_+ \times S^1 \times S^1$ these correspond respectively to the isometries $(r,\theta,\varphi) \mapsto (r, \theta + \alpha/2, \varphi - \alpha/2)$ and $(r,\theta,\varphi) \mapsto (r, \theta + \alpha/2, \varphi + \alpha/2)$.

Let us consider $\widehat{\mathscr{M}} = {\bigsqcup M_{\tilde{\alpha}}} \big/ {\sim}$ and the following diagram

$$\begin{tikzcd} \widetilde{\mathscr{M}} \arrow[r, "\widetilde{\mathrm{Dev}}"] \arrow[d, "p"] & \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \arrow[d, "\Pi"] \\ \widehat{\mathscr{M}} \arrow[r, "\mathrm{Dev}"] & \mathbb{A}\mathrm{d}\mathbb{S}^3_* \end{tikzcd}$$

where $\widetilde{\mathscr{M}}$ is the universal cover of $\mathscr{M}$ and $\widetilde{\mathrm{Dev}}$ is the lifting of Dev, that is a developing map for $\mathscr{M} = {\widehat{\mathscr{M}}} \big/ {\pi_1(\mathbb{H}^2_\theta)}$.

The map Dev has holonomy:

$$H : \pi_1(\mathscr{M}) \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3_*).$$

If $\alpha \in \pi_1(\mathscr{M})$ is the homotopy class of some curve of $\mathscr{M}$, then $H(\alpha)$ is an isometry of

$$\mathbb{A}\mathrm{d}\mathbb{S}^3_* \simeq \mathbb{R}_+ \times S^1 \times S^1$$

and the group of isometry is given by

$$\begin{aligned} \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3_*) &= \{F_{\theta_0,\varphi_0} \mid F_{\theta_0,\varphi_0}(r,\theta,\varphi) = (r, \theta_0 + \theta, \varphi_0 + \varphi) \text{ with } \theta_0, \varphi_0 \in \mathbb{R} \mod 2\pi\} \\ &\simeq {\mathbb{R}^2} \big/ {\langle (2\pi, 0), (0, 2\pi) \rangle}. \end{aligned}$$

Let $\Gamma = \langle (a,b) \rangle$ be a ciclic discrete group with $a, b \neq 0$. We define the quotient space

$$\mathbb{A}\mathrm{d}\mathbb{S}^3_{(a,b)} \coloneqq {\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}} \big/ {\Gamma},$$

with group of isometry

$$\mathrm{Isom}\big(\mathbb{A}\mathrm{d}\mathbb{S}^3_{(a,b)}\big) = {\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*})} \big/ {\langle (a,b) \rangle} \simeq {\mathbb{R}^2} \big/ {\Gamma}.$$

The map $\widetilde{\mathrm{Dev}}$, that can be considered a developing map, has holonomy

$$\widetilde{H} : \pi_1(\mathscr{M}) \to \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}) = \mathbb{R}^2$$

and clearly $\widetilde{H}(\alpha))$ is a lift of $H(\alpha)$.

3.5.1 Holonomy for $\widetilde{\mathrm{Dev}}$

Let us take a $\gamma : [0,1] \to \mathbb{H}^2_\theta$, a loop around the singularity and let $\widetilde{\gamma}$ be a lift in $\widetilde{\mathbb{H}^2_\theta}$. Notice that a lift α in $\mathscr{M}$ of γ is a non trivial loop different from the fiber and remember that γ acts as automorphism of the covering $\widetilde{\mathscr{M}} \to \mathscr{M}$. Given a lifting $\widetilde{\alpha}$ of γ in $\widetilde{\mathscr{M}}$, we denote $\widetilde{A}_t \coloneqq \widetilde{\mathrm{Dev}}(\widetilde{\alpha}(t))$ for every $t \in \mathbb{R}$. By equivariance of $\widetilde{\mathrm{Dev}}$, we have

$$\widetilde{\mathrm{Dev}}(\widetilde{\alpha}(t+1)) = \widetilde{\mathrm{Dev}}(\gamma \cdot \widetilde{\alpha}(t)) = \widetilde{H}(\gamma)\widetilde{\mathrm{Dev}}(\gamma \cdot \widetilde{\alpha}(t)),$$

with $\mathrm{Dev}(\alpha(t)) = \pi(\widetilde{A}_t) = A_t \in \mathbb{A}\mathrm{d}\mathbb{S}^3_*$.

By construction of the manifold $\mathscr{M}$, the path A_t in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ has the property that

$$A_t(f(\widetilde{\gamma}(t)) = d(\widetilde{\gamma}(t))$$

for every t. Let us denote $d_t \coloneqq d(\widetilde{\gamma}(t))$ and $f_t \coloneqq f(\widetilde{\gamma}(t))$ with $\mathrm{dist}(i, f_t) < \mathrm{dist}(i, d_t)$.

The aim of this section is to compute $\widetilde{H}(\widetilde{\alpha})$. Since $H(\alpha) = (R^\theta, R^\eta) = \Pi(\widetilde{H}(\widetilde{\alpha}))$, we already know that there exists $h, k \in \mathbb{Z}$ such that $\widetilde{H}(\widetilde{\alpha}) = (\theta + 2h\pi, \eta + 2k\pi)$. However, we will show

Proposition 3.5.2. There exists $k \in \mathbb{Z}$ such that $\widetilde{H}(\widetilde{\alpha}) = (\theta, \eta + 2k\pi)$.

To this purpose let us introduce the valuation maps

Definition 20. Let $A \in \mathbb{A}\mathrm{d}\mathbb{S}^3_*$. We introduce the projection maps $\mathrm{val}, \overline{\mathrm{val}} : \mathbb{A}\mathrm{d}\mathbb{S}^3_* \to \mathbb{H}^2 \setminus \{i\}$ defined as

$$\begin{cases} \mathrm{val}(\widetilde{A}) = A(i), \\ \overline{\mathrm{val}}(\widetilde{A}) = A^{-1}(i) \end{cases}$$

If we denote with $d\vartheta$ the angle form on $\mathbb{H}_* = \mathbb{H}^2 \setminus \{i\}$ and recalling the definition of pull-back, we can define on $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$ the 1-forms $\mathrm{val}^*(d\vartheta)$ and $\overline{\mathrm{val}}^*(d\vartheta)$.

In general, given any curve $c(t)$ in $\mathbb{A}\mathrm{d}\mathbb{S}^3_*$, a lift $\tilde{c}(t) = (r(t), x(t), y(t))$ in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ such that $\widetilde{c}(0) = (r_0, x_0, y_0)$, it can be described as

$$\tilde{c}(t) = (\rho(\alpha(t)), x_0 + \int_{c(t)} \mathrm{val}^*(d\vartheta), y_0 + \int_{c(t)} \overline{\mathrm{val}}^*(d\vartheta)), \tag{3.14}$$

where $\rho(A) := \mathrm{dist}(i, A(i))$ denotes the distance between a point $A \in \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*}$ and the geodesic $\ell_{i,i}$. In order to prove Proposition 3.5.2, we will need the following technical lemma:

Lemma 3.5.3. Let $d\vartheta$ the angle form on $\mathbb{H}^2_*$. Let $\gamma_1 : \mathbb{R} \to \mathbb{H}^2_*$ and $\gamma_2 : \mathbb{R} \to \mathbb{H}^2_*$ path in $\mathbb{H}^2_*$ with same equivariance, that is $\gamma_i(t+t_0) = R_i^\theta \gamma_i(t)$, and such that

$$\mathrm{dist}(\gamma_1(t), \gamma_2(t)) < \mathrm{dist}(\gamma_2(t), i),$$

then

$$\int_{\gamma_1} d\vartheta = \int_{\gamma_2} d\vartheta.$$

Proof. Firslty, let us observe that $i \notin \overline{\gamma_1, \gamma_2}$, i.e., the geodesic path between $\gamma_1(t)$ and $\gamma_2(t)$, because otherwise there is contradiction with the distance condition.

Secondly, there exists a homotopy $\Theta : \mathbb{R} \times [0,1] \to \mathbb{H}^2_*$ such that

$$\begin{cases} \Theta(t,0) = \gamma_1(t), \\ \Theta(t,1) = \gamma_2(t), \\ \Theta(t+t_0, s) = R_i^\theta \Theta(t,s). \end{cases}$$

Since $\exp_{\gamma_1(t)} : T_{\gamma_1(t)} \mathbb{H}^*_2 \to \mathbb{H}^2_*$ is a diffeomorphism, we can write

$$\Theta(t,s) = \exp_{\gamma_1(t)} \left(s (\exp_{\gamma_1(t)})^{-1} \gamma_2(t) \right).$$

By Stokes' Theorem and since $\Theta^*(d\vartheta)$ is a closed form:

$$\int_{\partial([0,t_0]\times[0,1])} \Theta^*(d\vartheta) = \int_{[0,t_0]\times[0,1]} d\Theta^*(d\vartheta) = 0,$$

where $d\vartheta$ is the angle form on $\mathbb{H}^2_*$. Thus:

$$\int_{\partial([0,t_0]\times\{0\})} \Theta^*(d\vartheta) + \int_{\partial([0,t_0]\times\{1\})} \Theta^*(d\vartheta) + \\ + \int_{\partial(\{0\}\times[0,1])} \Theta^*(d\vartheta) + \int_{\partial(\{t_0\}\times[0,1])} \Theta^*(d\vartheta) = 0$$

from which

$$\int_0^{t_0} {\gamma_1}^*(d\vartheta) - \int_0^{t_0} {\gamma_2}^*(d\vartheta) + \int_0^1 \Theta^*(d\vartheta)(0,s)ds - \int_0^1 \Theta^*(d\vartheta)(t_0,s)ds = 0.$$

□

Proof of Proposition 3.5.2. Using the notation at the beginning of the section, by formula 3.14, we need to show that

$$\int_{A_t} \mathrm{val}^*(d\vartheta) = \theta. \tag{3.15}$$

Now, consider the paths $\gamma_1(t) = A_t(i)$ and $\gamma_2(t) = f_t(i) = d_t$. We have

$$\int_{\gamma_2} d\vartheta = \theta, \tag{3.16}$$

by definition of cone singularity. On the other hand

$$\mathrm{dist}_{\mathbb{H}^2}(\gamma_1,\gamma_2) = \mathrm{dist}_{\mathbb{H}^2}(i, f_t) < \mathrm{dist}_{\mathbb{H}^2}(i, d_t), \tag{3.17}$$

thus the previous lemma implies that

$$\int_{\gamma_1} d\vartheta = \int_{\gamma_2} d\vartheta = \theta. \tag{3.18}$$

□

3.6 The fibration is an AdS-manifold with spin-cone singularity

The main goal of this section is to prove that a manifold $\mathscr{M}(\theta,\eta,f)$ is diffeomorphic to the quotient $\mathbb{A}\mathrm{d}\mathbb{S}^3_\Lambda$.

Proposition 3.6.1. Let (θ, η, f) be an admissible data, then $\mathscr{M} := \mathscr{M}(\theta, \eta, f)$ is diffeomorphic to

$$\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \big/ \Lambda,$$

where $\Lambda = (0, 2\pi) \cdot \mathbb{Z} \oplus (\theta, \eta) \cdot \mathbb{Z}$ is the lattice in $\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3})$.

Notation 3.6.2. In the following we will use just $\mathscr{M}$ to indicate $\mathscr{M}(\theta, \eta, f)$ and $\tau : \mathscr{M} \to \mathbb{H}^2_\theta$ is the fibration map over $\mathbb{H}^2_\theta$. The manifold $\mathscr{M}$ has develping map $\mathscr{D} : \widetilde{\mathscr{M}} \to \mathbb{A}\mathrm{d}\mathbb{S}^3_*$.

Lemma 3.6.3. Let $\varphi \in \pi_1(\mathscr{M})$ a loop homotopic to the fiber (future directed) and α any lift of the generator γ of $\pi_1(\mathbb{H}^2_\theta)$. Then

(i) $\pi_1(\mathscr{M}) = \mathbb{Z}\varphi \oplus \mathbb{Z}\alpha$,

(ii) $\widetilde{H}(\varphi) = (0, 2\pi)$ and there exists $k \in \mathbb{Z}$ such that $\widetilde{H}(\alpha) = (\theta, \eta + 2k\pi)$.

Proof. The statement (i) follows from the fact that any S^1-bundle over $\mathbb{D}^2_*$ is trivial. The second part of (ii) is true by Proposition 3.5.2. Let us compute now $\widetilde{H}(\varphi)$. Let $c : I \to \widetilde{M}$ a parametrization of the fiber, that means the homotopy class is $[c] = \varphi$. Then, we fix a lift $\hat{c} : I \to \widehat{\mathscr{M}}$ and a lift $\widetilde{c} : I \to \widetilde{\mathscr{M}}$. Notice that $\hat{c}$ is a closed path in $\widehat{\mathscr{M}}$, instead $\tilde{c}$ is not necessarily closed. Remember that Dev is the map such that $\mathrm{Dev} \circ \hat{\pi} = \mathbb{A}\mathrm{d}\mathbb{S}^3_*$ and so that the following diagram is commutative:

$$\begin{tikzcd}
\widetilde{\mathscr{M}} \arrow[r, "\mathrm{Dev}"] \arrow[d, "\hat{\pi}"] & \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3_*} \arrow[d, "\Pi"] \\
\widehat{\mathscr{M}} \arrow[r, "\mathrm{Dev}"] & \mathbb{A}\mathrm{d}\mathbb{S}^3_*,
\end{tikzcd}$$

that is $\mathrm{Dev} \circ \hat{\pi} \circ \tilde{c} = \mathrm{Dev} \circ \hat{c} = \Pi \circ \widetilde{\mathrm{Dev}} \circ \tilde{c}$. This allows to say that $\widetilde{\mathrm{Dev}} \circ \tilde{c}$ is a lift of $\mathrm{Dev} \circ \hat{c}$. By Proposition 2.2.6, we know that $\widetilde{\mathrm{Dev}} \circ \tilde{c}(1) = (2\pi, 0) \cdot \widetilde{\mathrm{Dev}} \circ \tilde{c}(0)$ or $\widetilde{\mathrm{Dev}} \circ \tilde{c}(1) = (0, 2\pi) \cdot \widetilde{\mathrm{Dev}} \circ \tilde{c}(0)$. Moreover, by the equivariance of $\widetilde{\mathrm{Dev}}$:

$$\widetilde{\mathrm{Dev}}(\varphi \cdot \tilde{c}(0)) = \widetilde{H}(\varphi) \cdot \widetilde{\mathrm{Dev}} \circ \tilde{c}(0)$$

where $\varphi \cdot \tilde{c}(0) = \tilde{c}(1)$, because $\tilde{c}$ is a lift of a parametrization of the fiber.

$\square$

In conclusion, the map $\widetilde{\mathrm{Dev}}$ induces a diffeomorphism

$$\overline{\mathrm{Dev}} : \widetilde{\mathscr{M}} \big/ \pi_1(\mathscr{M}) \longrightarrow \widetilde{\mathbb{AdS}^3_*} \big/ \Lambda.$$

that proves the initial statement of the Proposition 3.6.1.

Therefore, we have the following results.

Theorem 3.6.4. If $\tau : \mathscr{M}(\theta, \eta, f) \to \mathbb{H}^2_\theta$ is fiber bundle for an admissible data (θ, η, f), then $\tau^{-1}(U)$ is a generalized cone-spin singularity for every U neighbourhood of the singular point $x_0 \in \mathbb{H}^2_\theta$.

Corollary 3.6.5. If $\tau : \mathscr{M}(\theta, \eta, f) \to S$ fiber bundle over S surface with conical singularities, then $\tau^{-1}(U)$ is a a generalized cone-spin singularity for every U neighbourhood of a singular point in S.

Corollary 3.6.6. $\mathbb{AdS}^3_{(\theta,\eta)} \to \mathbb{H}^2_\theta$ is a fibration.

Remark 19. The fact that only the first component of $\widetilde{H}(\widetilde{\alpha})$ is determined as a real number (not only modulo 2π) is a conseguence of this more general fact. If $\mathscr{M}$ is a circle bundle over a pointed disk $\mathbb{D}_*$, that is

$$\begin{array}{ccc} S^1 & \xrightarrow{F} & \mathscr{M} \\ & & \big\downarrow \tau \\ & & \mathbb{D}_* \end{array},$$

there is a short exact sequence of this type:

$$1 \to \pi_1(S^1) \to \pi_1(\mathscr{M}) \to \pi_1(\mathbb{D}_*) \to 1.$$

that splits, determining an isomorphism $f : \pi_1(\mathscr{M}) \to \pi_1(S^1) \oplus \pi_1(\mathbb{D}_*)$.

Let φ be a generator of $\pi_1(S^1)$ and α an element of $\pi_1(\mathscr{M})$. Then

- $\{F_*(\varphi), \gamma\}$ is a basis of $\pi_1(\mathscr{M})$ if and only if $\tau_*(\gamma)$ is generator of $\pi_1(\mathbb{D}_*)$. Let $\sigma : \mathbb{D}_* \to \mathscr{M}$ be a section of $\tau : \mathscr{M} \to \mathbb{D}_*$, i.e., such that $\tau \circ \sigma = \mathrm{Id}_{\mathbb{D}_*}$ and $(\tau \circ \sigma)_* = \tau_* \circ \sigma_* = (\mathrm{Id}_{\mathbb{D}_*}) : \pi_1(\mathbb{D}_*) \to \pi_1(\mathbb{D}_*)$;

- two sections σ_1 and σ_2 are homotopic if and only if $(\sigma_1)_* = (\sigma_2)_*$, that is $(\sigma_1)_* - (\sigma_2)_* : \pi_1(\mathbb{D}_*) \to \ker \tau_* = \pi_1(\varphi)$, where $\tau : \mathscr{M} \to \mathbb{D}_*$.

Therefore $(\sigma_1)_*(\tau_*(\gamma)) = (\sigma_2)_*(\tau_*(\gamma)) + n\varphi$.

3.7 Anti-de Sitter manifolds foliated by timelike geodesics

In this section we want to show that every $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold with a foliation $\mathscr{F}$ in timelike geodesics of length π has the form $\mathscr{M}(S, \eta, f)$ for a surface S.

Notation 3.7.1. In the following we will refer to $\mathscr{M}$ as an orientable and time-orientable $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold with developing map $\widetilde{\mathrm{Dev}} : \widetilde{\mathscr{M}} \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ and holonomy $\widetilde{H} : \pi_1(\mathscr{M}) \to \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3})$. Given $\Pi_* : \mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}) \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$, we define $H = \Pi_* \circ \widetilde{H} : \pi_1(\mathscr{M}) \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \simeq \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ with H_L, H_R projections over the left and right componenst of $\mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \simeq \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$.

Proposition 3.7.2. Let $\mathscr{M}$ be an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold foliated by timelike geodesics of length π, then for every $p \in \mathscr{M}$ there exists an open subset $\mathscr{W}$ union of leaves isometric to a timetube in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

This proposition is a direct consequence of the the following topology fact:

Proposition 3.7.3. Let $\varphi : M \to N$ local diffeomorphism and $X \subset M$ compact subset such that $\varphi_{|X}$ is injective. Then there exists a neighbourhood U of X such that $\varphi_{|U}$ is injective.

Proof. Since M is locally compact, X admits a basis of neighbourhoods $\{U_n\}$ such that $U_{n+1} \subset U_n$ and $\bigcap U_n = X$ with $\overline{U_n}$ compact. Let us suppose, by contraddiction, that there exists two sequences $\{x_n\}, \{y_n\}$ such that for every n

- $x_n, y_n \in U_n$,
- $x_n \neq y_n$,
- $\varphi(x_n) = \varphi(y_n)$.

Notice that up to a subsequence $\{x_n\}$ and $\{y_n\}$ converge respectively to $x_\infty, y_\infty \in X$. Since φ is continuous $\varphi(x_\infty) = \varphi(y_\infty)$, thus by injectivity on the compact set X we conclude that $x_\infty = y_\infty$.

On the other hand, there exists a neighbourhood V of $x_\infty = y_\infty$ such that $\varphi_{|V}$ is injective. Since x_n, y_n are both convergent to $x_\infty = y_\infty$, then there exists a n_0 such that $x_n, y_n \in V$ for every $n \geq n_0$. Hence $x_n = y_n$, getting a contradiction. □

Proposition 3.7.4. Let $\mathscr{M}$ an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold that admits an oriented and timeoriented foliation $\mathscr{F}$ in closed geodesics of length α. Then, there exists an oriented surface S and a summersion $\pi : \mathscr{M} \to S$ such that the fiber $\pi^{-1}(x)$ is a leaf of the foliation for every $x \in S$.

Proof. By a theorem due to Epstein [Eps72], there exists a Seifert fibration $\pi : \mathscr{M} \to S$ such that $\gamma_x := \pi^{-1}(x)$ is a leaf in $\mathscr{F}$ and it is associated to the coprime integers (p, q). To prove that π is actually a fibration, namely $p = 1$ for every neighbourhood of a fiber, we fix a disk U_0 aroud $x_0 \in S$ and we consider $\pi^{-1}(U_0)$. Notice that $\pi_1(\pi^{-1}(U_0)) \simeq \mathbb{Z} \cdot [\gamma_{x_0}]$. Let us fix an other point $y_0 \in U_0$. In general, $[\gamma_y] = p[\gamma_{x_0}]$. We want to show that $p = 1$.

In the universal covering, let us fix $\widetilde{\gamma_{x_0}}$ and $\widetilde{\gamma_y}$ lifts of γ_{x_0} and γ_y such that $[\gamma_y] \cdot \widetilde{\gamma_y} = \widetilde{\gamma_y}$ and $[\gamma_{x_0}] \cdot \widetilde{\gamma_{x_0}} = \widetilde{\gamma_{x_0}}$. Since $[\gamma_y] = p[\gamma_{x_0}]$, we have that $[\gamma_y]$

preserves also $\widetilde{\gamma_{x_0}}$. Let us observe that $\ell_{x_0} = \mathrm{Dev}(\widetilde{\gamma_{x_0}})$ and $\ell_y = \mathrm{Dev}(\widetilde{\gamma_y})$ are geodesics. Moreover, by the equivariance:

- $\widetilde{H}([\gamma_{x_0}])(\ell_{x_0}) = \ell_{x_0}$,
- $\widetilde{H}([\gamma_y])(\ell_y) = \ell_y$,
- $\widetilde{H}([\gamma_y])(\ell_{x_0}) = \ell_{x_0}$.

We can say that the restriction of $\widetilde{H}(\gamma_{x_0})$ is a translation of lenght α, i.e. the lenght of the segment on ℓ_{x_0} with endpoints A and $\widetilde{H}([\gamma_{x_0}])A$, for every $A \in \ell_{x_0}$. Likewise, the restriction of $\widetilde{H}(\gamma_y)$ to ℓ_y is a translation of length α. The restriction of $\widetilde{H}([\gamma_y])$ to ℓ_{x_0} is also a translation of length α. Since $\widetilde{H}([\gamma_y]) = \widetilde{H}([\gamma_{x_0}])^p$, it follows that the translation lenght of $\widetilde{H}([\gamma_y])_{|\ell_{x_0}}$ is also equal to αp. We know that $\alpha \neq 0$ hence it holds that $p = 1$. □

Proof of Proposition 3.7.2. Considering the developing map $\mathrm{Dev} : \mathscr{M} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$. Let γ be the deck transformation of $\mathscr{M}$ corresponding to the homotopy class of a leaf of $\mathscr{F}$. We have that γ preserves every leaf of $\widetilde{\mathscr{F}}$ and acts on each as a translation of π.

Now, Dev sends every leaf of $\widetilde{\mathscr{F}}$ to a geodesic of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ and the restriction $\mathrm{Dev}_{|\ell_x}\, \ell_x \to \mathrm{Dev}(\ell_x)$ is a local isometry and thus a covering. Using that $\mathrm{length}(\mathrm{Dev}(\ell_x)) = \pi$ we deduce that for every $x \in \ell_x$

$$\mathrm{Dev}(\gamma X) = \mathrm{Dev}(X),$$

so $H(\gamma) = \mathrm{Id}_{\mathbb{D}_*}$.

In this way we see there exists a wel-defined local isometry

$$\overline{Dev} : \widetilde{\mathscr{M}} \big/ \langle\gamma\rangle \to \mathbb{A}\mathrm{d}\mathbb{S}^3.$$

Notice that $\widetilde{\mathscr{M}} \big/ \langle\gamma\rangle$ is still foliated by geodesics of length π. Since this foliation is sent to the foliation $\mathscr{M}$ by the covering map $\widetilde{\mathscr{M}} \big/ \langle\gamma\rangle \to \mathscr{M}$ and the length of the leaves of $\mathscr{M}$ is also π, we deduce that small neighbourhoods of leaves of $\widetilde{\mathscr{M}}\diagup_{\langle\gamma\rangle}$ are sent isometrically to neighbourhoods of leaves of $\mathscr{M}$. So it is sufficient to prove that the image of some neighbourhood of a leaf ℓ in $\widetilde{\mathscr{M}}\diagup_{\langle\gamma\rangle}$ is a timetube in $\mathbb{A}\mathrm{d}\mathbb{S}^3$. By Proposition 3.7.4, the leaves of the foliation over $\mathscr{M}$ are the leaves of a fibration on some surface, so the neighbourhoods of ℓ that are union of leaves form a fundamental system of neighbourhoods of ℓ.

Thus, it is sufficient to prove that there exists a neighbourhood of ℓ on which $\overline{\mathrm{Dev}}$ is injective. By Proposition 3.7.3 it is sufficient to check that $\overline{\mathrm{Dev}}_{|\ell}$ is injective, and this follows from the fact that the length of ℓ is π. $\square$

Proposition 3.7.5. Let $\mathscr{M}$ be an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold with a foliation $\mathscr{F}$ in closed timelike geodesics. Let X be a generator for the foliation and ω the volume form on $\mathscr{M}$. Given $\gamma : I \to \mathscr{M}$ space-like geodesic and $Y(t)$ parallel transport of $X(\gamma(0))$ along γ, one of the two holds:

$$\textit{or}\quad \omega(\dot{\gamma}, Y(t), X(\gamma(t))) > 0 \qquad\qquad \textit{or}\quad \omega(\dot{\gamma}, Y(t), X(\gamma(t)) < 0.$$

almost everywhere.

Proof. The result is true for a timetube by Proposition 1.5.6, so it holds on $\mathscr{M}$ as the manifold is union of open subsets isometric to timetubes. $\square$

Theorem 3.7.6. Let $\mathscr{M}$ be an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold admitting a foliation $\mathscr{F}$ by timelike geodesics of length π. Then, there exists a hyperbolic surface S, a representation $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ and a ρ-equivariant weak contraction $f : \widetilde{S} \to \mathbb{H}^2$ such that $\mathscr{M}$ is diffeomorphic to $\mathscr{M}(S, \rho, f)$ so that leaves of $\mathscr{F}$ correspond to fibers of the fibration $\tau : \mathscr{M}(S, \rho, f) \to S$.

Remark 20. Notice that in the case of left-handed timelike foliations the same result can be stated by reversing the maps (f, d) and (a, b)

To prove this theorem we need the following lemma.

Lemma 3.7.7. Let $\pi : \mathscr{M} \to S$ be a circle fibration and $\widetilde{\pi} : \widetilde{\mathscr{M}} \to \widetilde{S}$ its lift to the universal cover. For all $\tilde{p} \in \widetilde{S}$ there exists a neighbourhood U of p such that $\mathrm{Dev}_{|\widetilde{\pi}^{-1}(\widetilde{U})}$ is injective.

Proof. By Proposition 3.7.2, there exists a neighbourhood U such that $\pi^{-1}(U)$ is isometric to a timetube. Denote by $\delta : \pi^{-1}(U) \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ the isometric embedding. Notice that δ is injective and π_1-injective, so $\widetilde{\delta} : \widetilde{\pi}^{-1}(\widetilde{U}) \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^3_*$ is injective. Now, $\widetilde{\delta}$ differ by $\mathrm{Dev}_{|\widetilde{\pi}^{-1}(\widetilde{U})}$ post-composing by an isometry of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$, so we conclude. $\square$

Proof of Theorem 3.7.6. By Lemma 3.7.7, there exists U neighbourhood of $p \in S$ and a lift $\widetilde{U} \subset \widetilde{S}$ such that $\mathrm{Dev} : \widetilde{\pi}^{-1}(\widetilde{U}) \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}}^3$ is injective. Let us observe that $\widetilde{\mathscr{U}} = \mathrm{Dev}(\widetilde{\pi}^{-1}(\widetilde{U}))$ is an open subset in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ foliated by geodesics of the form $\{\tilde{\ell}_{\tilde{p}} = \mathrm{Dev}(\widetilde{\pi}^{-1}(\tilde{p}))\}_{\tilde{p} \in \widetilde{U}}$. Let T_0 be the generator of $Z(\mathrm{Isom}(\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}))$ such that $T_0 \widetilde{\mathscr{U}} = \widetilde{\mathscr{U}}$. Thus, $\mathscr{U} = \widetilde{\mathscr{U}} / T_0 \subseteq \mathbb{A}\mathrm{d}\mathbb{S}^3$ is a foliated open subset in $\mathbb{A}\mathrm{d}\mathbb{S}^3$.

For every $\tilde{p} \in \widetilde{S}$, $\mathrm{Dev}(\ell_{\tilde{p}})$ is a geodesic of $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ hence it is of the form $\widetilde{\ell}_{\mathbf{d}(\tilde{p}),\mathbf{f}(\tilde{p})}$ for some $\mathbf{f} : \widetilde{S} \to \mathbb{H}^2$ and $\mathbf{d} : \widetilde{S} \to \mathbb{H}^2$.

We will show that $\mathbf{d}$ is local diffeomorphism and $\mathbf{f}$ a contractive map. Let $T^{1,+}\mathbb{A}\mathrm{d}\mathbb{S}^3$ be the tangent space of $\mathbb{A}\mathrm{d}\mathbb{S}^3$ of future directed vectors and let us define $(G_l, G_r) : T^{1,+}\mathbb{A}\mathrm{d}\mathbb{S}^3 \to \mathbb{H}^2 \times \mathbb{H}^2$ as the map such that $\{\exp(tv)\} = \ell_{G_l(p,v),G_r(p,v)} \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$. Notice that $G_l(p,v) = \Phi(vp^{-1})$ and $G_r(p,v) = \Phi(p^{-1}v)$, where Φ is the map introduced in subsection 1.3.4. Recalling that $\Lambda : \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ is the covering map, we can consider

$$(\widetilde{G_l}, \widetilde{G_r}) : T^{1,+}_{\mathrm{Id}}\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3} \xrightarrow{d\Lambda} T^{1,+}\mathbb{A}\mathrm{d}\mathbb{S}^3 \xrightarrow{(G_l,G_r)} \mathbb{H}^2 \times \mathbb{H}^2$$

such that $\{\exp(t\tilde{v})\} = \tilde{\ell}_{\widetilde{G_l}(\tilde{p},\tilde{v}),\widetilde{G_r}(\tilde{p},\tilde{v})}$.

Let U be an open subset of S such that $\mathrm{Dev} : \widetilde{\pi}^{-1}(\widetilde{U}) \to \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ is injective, for some lift $\widetilde{U}$. Therefore we can redefine

$$\widetilde{\mathscr{U}} = \bigsqcup\{\widetilde{\ell}_{\mathbf{d}(\tilde{p}),\mathbf{f}(\tilde{p})} \mid \tilde{p} \in \widetilde{U}\} \qquad \text{and} \qquad \mathscr{U} = \bigsqcup\{\ell_{d(p),f(p)} \mid p \in U\}.$$

Let $\widetilde{X}$ be the field generating the foliation on $\widetilde{\pi}^{-1}(\widetilde{U})$. Then

$$d(p) = \widetilde{G_l}(\mathrm{Dev}_{|\widetilde{\pi}^{-1}(p)}\,\sigma(p), \widetilde{X}(\sigma(p))),$$
$$f(p) = \widetilde{G_r}(\mathrm{Dev}_{|\widetilde{\pi}^{-1}(p)}\,\sigma(p), \widetilde{X}(\sigma(p))),$$

where $\sigma : \widetilde{U} \to \widetilde{\pi}^{-1}(\widetilde{U})$ is any section of $\widetilde{\pi}$ on $\widetilde{U}$.

Let us recall the summersion $\overline{a} : \mathscr{U} \to \mathbb{H}^2$ defined in Lemma 1.5.2 with $\ker d_p\overline{a} = \operatorname{Span} X(p)$ (Proposition 1.5.8), where in this case X is the unitary tangent vector field generating the foliation of $\mathscr{U}$. Notice that $d(p) = \overline{b}(\Lambda \circ \mathrm{Dev}_{|\widetilde{\pi}^{-1}(\tilde{p})}(\sigma(\tilde{p}))$. Since σ is a section of the fibration, it shall not be tangent to the fibers and consequently $\ker d_p d = \{\mathrm{Id}\}$. Let us say $\tilde{h} = d^*(g_{\mathbb{H}^2})$. It follows that if $\gamma \in \pi_1(M)$ generates the leaf, then $\widetilde{H}(\gamma) = (\tilde{g}, 1)$. In particular, $H_R(\gamma) = \mathrm{Id}$.

It is well defined the map

$$j : \pi_1(S) = {}^{\pi_1(M)}\!\big/_{\mathbb{Z}\cdot[\gamma]} \to \mathrm{PSL}_2(\mathbb{R})$$

such that $H_R(\alpha) = j(\pi_*(\alpha))$. Let us observe that from

$$\widetilde{\ell}_{d(\gamma x), f(\gamma x)} = \widetilde{H}(\gamma)\widetilde{\ell}_{d(x), f(x)}$$

it follows that $d(\gamma x) = j(\gamma) d(x)$, thus j is holonomy for the metric $\tilde{h}$ which descends to a hyperbolic metric h on S.

The map f is a distance decreasing map, in fact $f(p) = \overline{a}(\Lambda \circ \mathrm{Dev}_{|\widetilde{\pi}^{-1}(\tilde{p})}(\sigma(\tilde{p}))$.

□

We can conclude at this point that every $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold foliated by time-like geodesic is a fibration over a surface. In particular, it is diffeomorphic to an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold of the form $\mathscr{M}(S, \rho, f)$ for a surface S with hyperbolic metric g_θ and a ρ-equivariant contractive map f.

Let us notice that for $\theta \neq 0$ the manifold $\mathscr{M}(\theta, \eta, f)$ is a fibration over a surface with conical singularities, hence it has generalized spin-cone singularities.

Definition 21. Let $\mathscr{M}$ be an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold. We will say that a foliation $\mathscr{F}$ on $\mathscr{M}$ is *standard* if the map $f : \widetilde{S} \to \mathbb{H}^2$ is constant. In this case, the foliation on $\widetilde{\mathscr{M}}$ is the pull-back through the developing map of the foliation on $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ given by $\{\tilde{\ell}_{i,x} | x \in \mathbb{H}^2\}$.

Proposition 3.7.8. Let $\mathscr{M}$ be an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-manifold with holonomy $\widetilde{H}$ such that $\pi : \mathscr{M} \to S$ fibration in circles and $\mathscr{F}$ foliation with timelike geodesics of length α.

- If $\alpha = k\pi$ with $k \in \mathbb{Z}$, then

  (1) there exists $\mathscr{M}_0$ covered by $\mathscr{M}$ such that

  - $\hat{\pi}\colon \mathscr{M} \to \mathscr{M}_0$ sends leaves in leaves and it is locally isometric;
  - the leaves of $\mathscr{M}_0$ have length π;
  - $\hat{\pi}$ is a normal cover;

  (2) there exists $\widehat{H} = (\rho, j) : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R}) \times \mathrm{PSL}_2(\mathbb{R})$ such that

  - $\Pi_* \circ \widetilde{H} = \widehat{H} \circ \pi_*$
  - f is ρ-equivariant.

- If $\alpha \notin \mathbb{Z}\pi$, the foliation is standard.

Proof. We show first the case with $\alpha \notin \mathbb{Z}\pi$. Let us consider U as in the Lemma 3.7.7. Then $\mathrm{Dev}(\widetilde{U})$ is an open subset in $\widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3}$ foliated by geodesics $\ell_{\mathbf{d}(p),\mathbf{f}(p)}$. If $\alpha \notin \mathbb{Z}$, then $H(\gamma) \notin \mathbb{Z}T_0$ and so we are in the case (ii) of the Proposition 1.4.3, i.e., there exists $b \in \mathbb{H}^2$ such that $f(x) = b$ for every x in an open subset of $\widetilde{U}$. Since f is constant on an open subset $\widetilde{U}$ and $\widetilde{S}$ is connected, f is constant everywhere.

Let us see now the case with $\alpha = k\pi$. For the statement (1), by Theorem 3.7.6 we can say that $\widehat{H} = (\rho, j)$ and S has hyperbolic metric h. Let be $\mathscr{M}_0 = \mathscr{M}(h, \rho, j)$ so by construction there exists a map $\pi_0 : \mathscr{M}_0 \to S$ such that $\pi_0^{-1}(\cdot)$ is a geodesic of lenght π. Then, there exists a covering map $p : \mathscr{M} \to \mathscr{M}_0$ of order k such that the following diagram

$$\begin{tikzcd} \mathscr{M} \arrow[d, "p"] \arrow[dr, "\pi"] & \\ \mathscr{M}_0 \arrow[r, "\pi_0"] & S \end{tikzcd}$$

is commutative. Let us remember that if $\mathbf{d} : \widetilde{S} \to \mathbb{H}^2$ is developing map for the metric h, we have $\widehat{\mathscr{M}_0} = \{(x, A) \in \widetilde{S} \times \mathbb{A}\mathrm{d}\mathbb{S}^3 \,|\, A \in \ell_{f(x),d(x)}\}$ with $\mathrm{Dev}_0 : \widehat{\mathscr{M}_0} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ such that $\mathrm{Dev}_0(x, A) = A$ and $\widetilde{\pi}_0(x, A) = x$. Therefore, it is well-defined a map $\tilde{p} : \widetilde{\mathscr{M}} \to \widehat{\mathscr{M}_0}$ such that $\tilde{p}(\xi) = (\tilde{\pi}(\xi), \mathbf{P} \circ \mathrm{Dev}\,\xi)$, where $\mathbf{P} \colon \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3} \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ is a projection map. Let us notice that this map satisfies the equivariance rule $\tilde{p}(\alpha\xi) = \pi_*(\alpha)\tilde{p}(\xi)$, where $\pi_1(S)$ acts on $\widehat{\mathscr{M}_0}$ as $\beta \cdot (x, A) = (\beta(x), \widehat{H}(\beta)A)$. Thus, $\tilde{p}$ descends to a map $p : \mathscr{M} \to \mathscr{M}_0 = \widehat{\mathscr{M}_0} / \pi_1(S)$. The map $\tilde{p}$ is a local isometry, thus p is local isometry too. Since $\tilde{\pi}_0 \circ \tilde{p} = \tilde{\pi}$, then $\pi_0 \circ p = \pi$ and it follows that also the diagram

$$\begin{tikzcd} \pi_1(\mathscr{M}) \arrow[d, "p_*"] \arrow[dr, "\pi_*"] & \\ \pi_1(\mathscr{M}_0) \arrow[r, "\pi_{0*}"] & \pi_1(S) \end{tikzcd}$$

is commutative. Then $p_*(\ker(\pi_*)) \subseteq \ker(\pi_{0*})$

For the statement (2), let be $\gamma \in \pi_1(\mathscr{M})$ generating the fiber. We have $\widetilde{H}(\gamma) = T_0^k$, thus $\Pi_* \circ \widetilde{H}(\gamma) = \mathrm{Id}$. Since $\ker(\pi_* : \pi_1(\mathscr{M}) \to \pi_1(S)) = \langle\gamma\rangle$, for the *First Isomorphism Theorem*, it follows that there exists a map $\widehat{H} : \pi_1(S) \to \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3)$ such that the following diagram is commutative:

$$\begin{tikzcd} \pi_1(\mathscr{M}) \arrow[r, "\Pi_* \circ \widetilde{H}"] \arrow[d, "\pi_*"] & \mathrm{Isom}(\mathbb{A}\mathrm{d}\mathbb{S}^3) \\ \pi_1(S) & \end{tikzcd} .$$

that is $\Pi_* \circ \widetilde{H} = \widehat{H} \circ \pi_*$. In conclusion, f is $\rho \coloneqq H_L$-equivariant. $\square$

Remark 21. $\widetilde{\mathscr{M}} = \{(x, \tilde{\xi}) \in \widetilde{S} \times \widetilde{\mathbb{A}\mathrm{d}\mathbb{S}^3} \,|\, \tilde{\xi} \in \tilde{\ell}_{f(x),d(x)}\}$.

We have the following sequence:

$$\mathrm{Id} \to \langle \gamma \rangle \to \pi_1(\mathscr{M}) \to \pi_1(S) \to \mathrm{id}\,.$$

The action of $\pi_1(S)$ on $\widehat{\mathscr{M}}$ is $\alpha \cdot (x, \xi) = (\alpha(x), (\rho(\alpha), j(\alpha)) \circ \xi)$.

CHAPTER 4

Volume of AdS compact manifold with singularities

4.1 Computation of the volume of $\mathscr{M}$

In this section, we want to conclude the description of the three dimensional Anti-de Sitter manifold $\mathscr{M}(\theta, \eta, f)$ by calculating their volume.

Definition 22. Let $d\,\mathrm{Vol}_{\mathbb{H}^2}$ be the area form on $\mathbb{H}^2$. Then $\mathbf{f}^*(d\,\mathrm{Vol}_{\mathbb{H}^2})$ is π_1-invariant. Moreover, it exists a 2-form dA on the surface S such that

$$\pi^*(dA) = f^*(d\,\mathrm{Vol}_{\mathbb{H}^2}).$$

Then we can define

$$\mathrm{Area}(\mathbf{f}) = \int_S dA.$$

Theorem 4.1.1. Let $\mathscr{M} = \mathscr{M}(\theta, \eta, \mathbf{f})$ be a three dimensional Anti-de Sitter manifold as bundle over a oriented hyperbolic surface (S, g_θ). Then the volume of $\mathscr{M}$ is determined as function of the maps $\mathbf{f}, \mathbf{d}\colon \widetilde{S} \to \mathbb{H}^2$:

$$\mathrm{Vol}\,(\mathscr{M}) = \pi\big[\,\mathrm{Area}(f) + \mathrm{Area}(d)\big].$$

Let $\mathscr{U}$ be an $\mathbb{A}\mathrm{d}\mathbb{S}^3$-domain foliated by timelike geodesics and let X be the vector field generating the foliation. On this foliation we can define a 1-form dt such that, for every $A \in \mathscr{U}$:

$$\begin{cases} dt(X(A)) = 1, \\ \quad dt(V) = 0 \quad \text{if } V \perp X(A). \end{cases}$$

Let us recall now the timelike unit tangent space of Anti-de Sitter at Id:

$$H^+ := T^1_{Id}\mathbb{A}\mathrm{d}\mathbb{S} = \{J \in M_2(\mathbb{R}) \mid \operatorname{tr} J = 0, \, \det J = 1\},$$

which is a hypersurface in the tangent space $T_{\mathrm{Id}}\mathbb{A}\mathrm{d}\mathbb{S}^3$ and it is a copy of $\mathbb{H}$.

Thanks to the Caley-Hamilton theorem we can write:

$$H^+ = \{J \in M_2(\mathbb{R}) \mid J^2 = -1\}.$$

Therefore, the tangent of H^+ can be seen in two ways:

$$\begin{aligned} T_J H^+ &= \{J \in M_2(\mathbb{R}) \mid \operatorname{tr} \dot{J} = 0, \det J\dot{J} = 0\} \\ &= \{J \in M_2(\mathbb{R}) \mid J\dot{J} + \dot{J}J = 0.\} \end{aligned}$$

Let ω be the area form on H^+. We want to show that ω is given by

$$\omega := \omega_J(\dot{J}_1, \dot{J}_2) = -\langle \dot{J}_1, J\dot{J}_2 \rangle.$$

We need to define the map $\mu_J: T_J H \to T_J H^+$ such that $\mu_J(\dot{J}) = J\dot{J}$. If $\dot{J} \in T_J H^+$, then $J\dot{J} \in T_J H^+$, because

$$\begin{cases} \operatorname{tr} J\dot{J} = 0 \\ \operatorname{tr} J(J\dot{J}) = -tr(\dot{J}) = 0, \end{cases}$$

so the map μ_J is well-defined.

Moreover, we have

$$\langle \mu_J \dot{J}_1, \mu_J \dot{J}_2 \rangle = \langle \dot{J}_1, \dot{J}_2 \rangle$$

and since $J^2 = -1$ we obtain that

$$\langle \mu_J \dot{J}_1, \dot{J}_2 \rangle = \langle \mu_J \mu_J \dot{J}_1, \mu_J \dot{J}_2 \rangle = -\langle \dot{J}_1, \mu_J \dot{J}_2 \rangle.$$

In particular

$$\langle \mu_J \dot{J}_1, \dot{J}_1 \rangle, = 0$$

and so we can say that $\{\dot{J}_1, J\dot{J}_1\}$ is an alternating form and $\{\dot{J}_1, J\dot{J}_1\}$ is an orthonormal basis, because if $\dot{J}_1$ is a unit vector in $T_J\mathbb{H}$, then $J\dot{J}_1$ is still a unit vector orthogonal to $\dot{J}_1$.

Therefore,

$$\omega_J(\dot{J}_1, J\dot{J}_1) = -\langle \dot{J}_1, JJ\dot{J}_1 \rangle$$

and so the alternating 2-forms ω_J and $-\langle \cdot, J\cdot \rangle$ are coincident on the orthonormal basis $\{\dot{J}_1, J\dot{J}_1\}$, then they are the same.

Now, let us set the maps $\lambda, \rho : \mathscr{W} \to H^+$ such that

- $\lambda(A) = A^{-1}X(A)$;
- $\rho(A) = X(A)A^{-1}$,

where X is the generator of the foliation with future orientation. Then, fix $\dot{A} \in T_A\mathbb{A}\mathrm{d}\mathbb{S}^3$ and set $\lambda(A) = J$, $\rho(A) = J'$. We have

$$\begin{aligned}(d_A\lambda)(\dot{A}) &= -A^{-1}\dot{A}A^{-1}X(A) + A^{-1}d_AX(\dot{A}) \\ &= J(A^{-1}\dot{A}) + A^{-1}d_AX(\dot{A}),\end{aligned}$$

so, for $\dot{A}_1, \dot{A}_2 \in T_A\mathbb{A}\mathrm{d}\mathbb{S}^3$ orthogonal to X,

$$\begin{aligned}(\lambda^*\omega)(\dot{A}_1, \dot{A}_2) &= -\langle d_A\lambda\dot{A}_1, Jd\lambda\dot{A}_2 \rangle \\ &= \langle J(A^{-1}\dot{A}_1) + A^{-1}d_AX(\dot{A}_1), -A^{-1}\dot{A}_2 + JAd_AX(\dot{A}_2) \rangle \\ &= \langle J(A^{-1}\dot{A}_1), A^{-1}\dot{A}_2 \rangle + \langle d_AX(\dot{A}_1, \dot{A}_2) \rangle \\ &- \langle \dot{A}_1, d_AX(\dot{A}_2) \rangle - \langle A^{-1}d_AX(\dot{A}_1), JA^{-1}d_AX(\dot{A}_2) \rangle\end{aligned}$$

while

$$d_A\rho(\dot{A}) = d_A X(\dot{A})A^{-1} - XA^{-1}\dot{A}A^{-1} = d_A X(\dot{A})A^{-1} - J'\dot{A}A'$$

so that

$$\begin{aligned}\rho^*\omega(\dot{A}_1, \dot{A}_2) &= -\big\langle d_A\rho(\dot{A}_1), J' d_A\rho(\dot{A}_2)\big\rangle \\ &= \big\langle d_A X(\dot{A})A^{-1} - J'\dot{A}A', J' d_A X(\dot{A})A^{-1} + \dot{A}_2 A^{-1}\big\rangle \\ &= \big\langle J'\dot{A}_1 A^{-1}, \dot{A}_2 A^{-1}\big\rangle + \big\langle \dot{A}_1 A^{-1}, d_A X(\dot{A}_2)A^{-1}\big\rangle \\ &- \big\langle d_A X(\dot{A}_1), \dot{A}_2\big\rangle - \big\langle d_A X(\dot{A}_1), J' d_A X(\dot{A}_2)\big\rangle.\end{aligned}$$

Using the fact that $J' = X(A)A^{-1}$ and $J = A^{-1}X(A)$, we have $J' = AJA^{-1}$. Given that, we obtain

$$\begin{aligned}\big\langle J'\dot{A}_1 A^{-1}, \dot{A}_2 A^{-1}\big\rangle &= \big\langle J'\dot{A}_1, \dot{A}_2\big\rangle \\ &= \big\langle AJA^{-1}\dot{A}_1, \dot{A}_2\big\rangle = \big\langle JA^{-1}\dot{A}, A^{-1}\dot{A}_2\big\rangle.\end{aligned}$$

Analogously,

$$\big\langle d_A X(\dot{A}_1), J' d_A X(\dot{A}_2)\big\rangle = \big\langle A^{-1} d_A X(\dot{A}_1), JA^{-1} d_A X(\dot{A}_2)\big\rangle.$$

We can conclude

$$\begin{aligned}\big[(\lambda^*\omega) + (\rho^*\omega)\big](\dot{A}_1, \dot{A}_2) &= 2\big[\big\langle JA^{-1}\dot{A}_1, A^{-1}\dot{A}_2\big\rangle + \big\langle A^{-1} d_A X(\dot{A}_1), JA^{-1} d_A X(\dot{A}_2)\big\rangle\big] \\ &= 2\big[\psi_1(\dot{A}_1, \dot{A}_2) + \psi_2(\dot{A}_1, \dot{A}_2)\big],\end{aligned}$$

where

- $\psi_1(\dot{A}_1, \dot{A}_2) = \big\langle JA^{-1}\dot{A}_1, A^{-1}\dot{A}_2\big\rangle$;
- $\psi_2(\dot{A}_1, \dot{A}_2) = \big\langle A^{-1} d_A X(\dot{A}_1), JA^{-1} d_A X(\dot{A}_2).\big\rangle$.

Remark 22. We want to evaluate $[(\lambda^*\omega) + (\rho^*\omega)] \wedge \mathrm{dt}$ on the orthogonal basis $\{\dot{A}_1, \dot{A}_2, X(A)\}$. Since $\mathrm{dt}(\dot{A}_1) = \mathrm{dt}(\dot{A}_2) = 0$ and $\mathrm{dt}(X(A)) = 1$, we obtain

$$[(\lambda^*\omega) + (\rho^*\omega)] \wedge \mathrm{dt}(\dot{A}_1, \dot{A}_2, X(A)) = [(\lambda^*\omega) + (\rho^*\omega)](\dot{A}_1, \dot{A}_2).$$

Let us now consider $\{\dot{A}_1, \dot{A}_2, X(A)\}$: if this is a positive orthonormal basis of $T_A\mathbb{A}\mathrm{d}\mathbb{S}^3$, then $\{A^{-1}\dot{A}_1, A^{-1}\dot{A}_2, J\}$ is a positive orthonormal basis too. So that

$$\psi_1 \wedge dt\big[\dot{A}_1, \dot{A}_2, X(A)\big] = \psi_1(\dot{A}_1, \dot{A}_2) = 1$$

and so $\psi_1 \wedge dt = dVol_{\mathbb{A}\mathrm{d}\mathbb{S}^3}$.

For ψ_2 we need some more consideration. Chosen a vector field $X\colon W \to \mathbb{R}^{2,2}$ such that $X(A) \perp A$, then $\nabla_{\dot{A}}X$ is the tangent part of $d_AX(\dot{A})$ while the normal part is

$$\big[d_AX(\dot{A})\big]^N = -\big\langle dX(\dot{A}), A\big\rangle A.$$

Differentiating we obtain:

$$d\big\langle X(A), A\big\rangle = \big\langle dX(\dot{A}), A\big\rangle + \big\langle X, \dot{A}\big\rangle,$$

from which

$$d_AX(\dot{A}) = \nabla_{\dot{A}}X + \big\langle X, \dot{A}\big\rangle A.$$

Thus the calculation here becomes

$$\begin{aligned}
\psi_2(\dot{A}_1, \dot{A}_2) &= \big\langle A^{-1}\nabla_{\dot{A}_1}X + \big\langle X, \dot{A}_1\big\rangle Id, JA^{-1}\nabla_{\dot{A}_2}X + \big\langle X, \dot{A}_2\big\rangle J\big\rangle \\
&= \big\langle A^{-1}\nabla_{\dot{A}_1}X, JA^{-1}\nabla_{\dot{A}_2}X\big\rangle + \big\langle X, \dot{A}_1\big\rangle\big\langle Id, JA^{-1}\nabla_{\dot{A}_2}X\big\rangle + \\
&\quad + \big\langle A^{-1}\nabla_{\dot{A}_1}X, J\big\rangle\big\langle X, \dot{A}_2\big\rangle + \big\langle X, \dot{A}_1\big\rangle\big\langle X, \dot{A}_2\big\rangle\big\langle Id, J\big\rangle.
\end{aligned}$$

Let us remember that $\big\langle Id, J\big\rangle = 0$ because $J \in T_{Id}\mathbb{A}\mathrm{d}\mathbb{S}^3$ and, since $\big\langle X, X\big\rangle = -1$, both $\big\langle \mathrm{Id}, JA^{-1}\nabla_{\dot{A}_2}X\big\rangle = 0$ and $\big\langle A^{-1}\nabla_{\dot{A}_1}X, J\big\rangle = 0$. Indeed,

- $\big\langle Id, JA^{-1}\nabla_{\dot{A}_2}X\big\rangle = 0$ because $\big\langle Id, JA^{-1}\nabla_{\dot{A}_2}X\big\rangle = \big\langle J, A^{-1}\nabla_{\dot{A}_2}X\big\rangle = \big\langle A^{-1}X, A^{-1}\nabla_{\dot{A}_2}X\big\rangle = \big\langle X, \nabla_{\dot{A}_2}X\big\rangle = 0$;
- $\big\langle A^{-1}\nabla_{\dot{A}_1}X, J\big\rangle = \big\langle A^{-1}\nabla_{\dot{A}_1}X, A^{-1}X\big\rangle = \big\langle \nabla_{\dot{A}_1}X, X\big\rangle = 0.$

In conclusion, we can say

$$\psi_2(\dot{A}_1, \dot{A}_2) = \big\langle A^{-1}\nabla_{\dot{A}_1}X, JA^{-1}\nabla_{\dot{A}_2}X\big\rangle.$$

We want to calculate the integral

$$\int_U \psi_2 \wedge \mathrm{dt}\,.$$

Let us assume, up to shrink the domain $\mathscr{W}$, that there exists a surface parametrized by

$$\mathbf{S} : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to \mathbb{A}\mathrm{d}\mathbb{S}$$

that intersect all the leaves in the foliation of $\mathscr{W}$ once and only once.

Then, we can parametrize the whole domain $\mathscr{W}$ using the map

$$\hat{\mathbf{S}} : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{A}\mathrm{d}\mathbb{S}$$

given by

$$\hat{\mathbf{S}}(x, y, t) = \varphi_t(\mathbf{S}(x, y))$$

where φ_t is the flow of X. With this choice, we have $\frac{\partial}{\partial t} = X$. Note that $\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} = 0$ because $\frac{\partial}{\partial t} = \frac{\partial \hat{S}}{\partial t}$ is the velocity field of geodesics. In these coordinates

$$\big[\psi_2 \wedge dt\big]\big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial t}\big) = \psi_2\big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\big)$$

so that

$$\int_W \psi_2 \wedge dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \psi_2(\frac{\partial}{\partial x}, \frac{\partial}{\partial y})\,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dt}\,.$$

Since

$$\nabla_{\frac{\partial}{\partial x}} x = \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x},$$

then

$$\begin{aligned}
\psi_2(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) &= \big\langle \nabla_{\frac{\partial}{\partial x}} X, AJA^{-1} \nabla_{\frac{\partial}{\partial y}} X \big\rangle = \\
&= \big\langle \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial x}, AJA^{-1} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial t} \big\rangle = \\
&= \frac{\partial}{\partial t} \big\langle \frac{\partial}{\partial x}, AJA^{-1} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial t} \big\rangle - \big\langle \frac{\partial}{\partial x}, \frac{\nabla}{\partial t}\big[AJA^{-1} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial t}\big] \big\rangle.
\end{aligned}$$

From these relations

$$\begin{cases} A(t) = \cos t A(0) + \sin t X(0) = A(0)(\cos t \,\mathrm{Id} + \sin t J(0)), \\ X(t) = -\sin t A(0) + \cos t X(0) = A(0)(-\sin t\,\mathrm{Id} + \cos t J(0)), \end{cases} \tag{4.1}$$

we have

$$\begin{aligned} AJA^{-1} &= AA^{-1}XA^{-1} = XA^{-1} \\ &= A(0)\big[\big(-\sin t\cos t(1 - J(0)^2) + (\sin^2 t + \cos^2 t)J(0)\big)\big]A(0)^{-1}. \end{aligned}$$

Thus we can conclude that $\frac{\partial}{\partial t}AJA^{-1} = 0$ and so AJA^{-1} is constant on the geodesics tangent to $\frac{\partial}{\partial t}$.

Therefore,

$$\frac{D}{\partial t}\big(AJA^{-1}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial t}\big) = AJA^{-1}\nabla_{\frac{\partial}{\partial y}}\nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} + AJA^{-1}R\big(\frac{\partial}{\partial t}, \frac{\partial \hat{S}}{\partial y}\big)\frac{\partial}{\partial t}.$$

Now, using the fact that the Anti-de Sitter space has curvature -1,

$$\begin{aligned} R\big(\frac{\partial}{\partial t}, \frac{\partial \hat{S}}{\partial y}\big)\frac{\partial}{\partial t} &= -\big\langle\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\big\rangle\frac{\partial \hat{S}}{\partial y} + \big\langle\frac{\partial \hat{S}}{\partial y}, \frac{\partial}{\partial t}\big\rangle\frac{\partial}{\partial t} = \\ &= \frac{\partial \hat{S}}{\partial y} + \big\langle\frac{\partial \hat{S}}{\partial y}, \frac{\partial}{\partial t}\big\rangle\frac{\partial}{\partial t}X. \end{aligned}$$

Since

$$AJA^{-1}X = AJA^{-1}AJ = AJ^2 = -A,$$

then

$$\begin{aligned} \psi_2\big(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\big) &= \frac{\partial}{\partial t}\big[\big\langle\frac{\partial \hat{S}}{\partial x}, AJA^{-1}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial t}\big\rangle\big] + \\ &\quad - \big\langle\frac{\partial \hat{S}}{\partial x}, AJA^{-1}\frac{\partial}{\partial y}\big\rangle + \big\langle\frac{\partial \hat{S}}{\partial y}, \frac{\partial}{\partial t}\big\rangle\big\langle\frac{\partial \hat{S}}{\partial x}, A\big\rangle. \end{aligned}$$

The last part of the equation in zero because $\frac{\partial \hat{\mathbf{S}}}{\partial x} \in T_A\mathbb{A}\mathrm{d}\mathbb{S}^3$, where $A = \hat{\mathbf{S}}(x, y, t)$.

We observe that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\partial}{\partial t}\big\langle\frac{\partial \hat{\mathbf{S}}}{\partial x}, AJA^{-1}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial t}\big\rangle\,\mathrm{dt} = \big\langle\frac{\partial \hat{\mathbf{S}}}{\partial x}, AJA^{-1}\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial t}\big\rangle\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

and $\hat{\mathbf{S}}$ is π-periodic, so the integral is zero. It remains

$$\begin{aligned}\int_{\mathscr{W}} \psi_2 \wedge \mathrm{dt} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \big\langle \frac{\partial \hat{\mathbf{S}}}{\partial x}, AJA^{-1} \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial t} \big\rangle \,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dt} = \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} \psi_1 \big(\frac{\partial \hat{\mathbf{S}}}{\partial x}, \frac{\partial \hat{\mathbf{S}}}{\partial y} \big) \,\mathrm{dx}\,\mathrm{dy}\,\mathrm{dt} = \\ &= \int_{\mathscr{W}} d\,\mathrm{vol}\,.\end{aligned}$$

Definitely

$$\int_{\mathscr{W}} \big(\lambda^* \omega + \rho^* \omega \big) \wedge \mathrm{dt} = 4 \int_{\mathscr{W}} d\,\mathrm{vol} = 4\,\mathrm{Vol}(\mathscr{W})$$

with

$$\int_{\mathscr{W}} \lambda^* \omega \wedge \mathrm{dt} = \pi \int_{\lambda(\mathscr{W})} \omega = \pi\,\mathrm{Area}(\lambda(\mathscr{W}))$$

and

$$\int_{\mathscr{W}} \rho^* \omega \wedge \mathrm{dt} = \int_{\rho(\mathscr{W})} \omega = \pi\,\mathrm{Area}(\rho(\mathscr{W})).$$

In conclusion

$$\mathrm{Vol}(\mathscr{W}) = \pi \big[\,\mathrm{Area}(\lambda(\mathscr{W}) + \mathrm{Area}(\rho(\mathscr{W})) \big].$$

Let $\Phi : H^+ \to \mathbb{H}^2$ be the isometry defined in 1.3.6.

Through the map Φ, the open subset $\lambda(\mathscr{W}) \subset \mathbb{A}\mathrm{d}\mathbb{S}^3$ identifies an open set in the hyperbolic plane. We want now to express the geodesics in the foliation of $\mathscr{W}$ using the map Φ.

First, we look at the geodesic ℓ_0 passing through the identity. Let $V \in T_{\mathrm{Id}}\ell_0$ be a unit tangent vector. We can write $\ell_0 = \exp tV$ and we know that it fix a point $a \in \mathbb{H}^2$, that is $\ell_0 = \ell_{a,a}$. Then,

$$\Phi(V) = \mathrm{Fix}(\exp tV) = a.$$

Let us take ℓ a timelike geodesic in the foliation of $\mathscr{W}$, so that $\ell = \ell_{a,b}$ with $a, b \in \mathbb{H}^2$, $a \neq b$, and let $A \in \ell$ be a point in the geodesic.

We can write $\ell_{a,b} = (A, \mathrm{Id}) \cdot \ell_{b,b}$ and $\ell_{a,b} = (\mathrm{Id}, A^{-1})\ell_{a,a}$ from which we have $\ell_{b,b} = A^{-1}\ell_{a,b}$ and $\ell_{a,a} = \ell_{a,b}A^{-1}$.

Therefore, each element is obtained respectively as left and right translation by A^{-1}. In particular, we can write

$$\mathrm{Id} = L_{A^{-1}}(A) = R_{A^{-1}}(A),$$

where $L_{A^{-1}}, R_{A^{-1}} \colon \mathbb{A}\mathrm{d}\mathbb{S}^3 \to \mathbb{A}\mathrm{d}\mathbb{S}^3$ are such that $L_{A^{-1}}(X) = A^{-1}X$ and $R_{A^{-1}}(X) = XA^{-1}$.

We remind that $V \in T_A\ell_{a,b}$, then

- $d_A L_{A^{-1}}(V) = A^{-1}V \in T_{\mathrm{Id}}\ell_{b,b}$;
- $d_A R_{A^{-1}}(V) = VA^{-1} \in T_{\mathrm{Id}}\ell_{a,a}$,

and so we have $\ell_{b,b} = \exp(tA^{-1}V)$ and $\ell_{a,a} = \exp(tVA^{-1})$. Then,

- $\Phi(A^{-1}V) = b$;
- $\Phi(VA^{-1}) = a$.

Remark 23. Let $U \subset S$ and let $\widetilde{U}$ be an open subset in the lift of U in the universal cover of S. As constructed in Chapter 3, $\tau^{-1}(\widetilde{U})$ is a domain in the anti-de Sitter manifold $\widetilde{\mathscr{M}}$ foliated by timelike geodesics of the form $\ell_{\mathbf{f}(\tilde{x}),\mathbf{d}(\tilde{x})}$ with $\tilde{x} \in \widetilde{U}$. Thanks to what we have seen before,

$$\begin{aligned}\mathrm{Vol}(\tau^{-1}(U)) &= \pi\Big[\,\mathrm{Area}(\lambda(\tau^{-1}(\widetilde{U}))) + \mathrm{Area}(\rho(\tau^{-1}(\widetilde{U})))\Big]\\ &= \pi\Big[\,\mathrm{Area}(\mathbf{d}(\widetilde{U})) + \mathrm{Area}(\mathbf{f}(\widetilde{U}))\Big].\end{aligned}$$

Let $\{U_\alpha\}$ be a finite cover of the surface S with lifting $\{\widetilde{U}_{\tilde{\alpha}}\}$ in the universal cover of S such that $\mathbf{d}|_{\widetilde{U}_{\tilde{\alpha}}}$ is a diffeomorphism onto its image.

Then, we can compute the volume of the Anti-de Sitter manifold $\widetilde{\mathscr{M}}$ as volume of the union over $\widetilde{\mathscr{A}}$:

$$\begin{aligned}\mathrm{Vol}\left(\bigcup_{\tilde{\alpha}\in\widetilde{\mathscr{A}}} \widetilde{U}_{\tilde{\alpha}}\right) &= \sum_{i=1}^{n}(-1)^{i+1}\sum_{1\le\tilde{\alpha}_1<\ldots<\tilde{\alpha}_i\le n}\pi\left[\mathrm{Area}\left(\mathbf{d}\left(\bigcap_{k=1}^{i}\widetilde{U}_{\tilde{\alpha}}\right)\right) + \mathrm{Area}\left(\mathbf{f}\left(\bigcap_{k=1}^{i}\widetilde{U}_{\tilde{\alpha}}\right)\right)\right]\\ &= \pi\Big[\,\mathrm{Area}(d) + \mathrm{Area}(f)\Big].\end{aligned}$$

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