Set-Valued and Variational Analysis (2023) 31:21
https://doi.org/10.1007/s11228-023-00685-0

Characteristic curves for Set-Valued Hamilton-Jacobi Equations

D. Visetti[1]

Received: 19 August 2022 / Accepted: 5 June 2023 / Published online: 6 July 2023

Abstract
The method of characteristics is extended to set-valued Hamilton-Jacobi equations. This problem arises from a calculus of variations' problem with a multicriteria Lagrangian function: through an embedding into a set-valued framework, a set-valued Hamilton-Jacobi equation is derived, where the Hamiltonian function is the Fenchel conjugate of the Lagrangian function. In this paper a method of characteristics is described and some results are given for the Fenchel conjugate.

Keywords Multicriteria calculus of variations · Multiobjective optimization · Hamilton-jacobi equation · Characteristics

Mathematics Subject Classification 49J53

1 Introduction

The method of characteristics converts a nonlinear first-order partial differential equation into a system of ordinary differential equations, both with suitable boundary conditions. In some circumstances, the solutions of the latter exist (at least locally in time) and give a solution of the first one. This happens in particular for the Hamilton-Jacobi equation both for classical solutions and for viscosity solutions (see [13]). The characteristic curves are called generalized in the second case.

This method was already suggested by Cauchy and was mentioned in Carathéodory's book [4]. In order to have the general theory and some perspective in the research that has been done, it is possible to consider [3, 7, 16].

Optimization of a multiobjective cost function is often very important and complex: the components to minimize (for example production cost and holding inventory cost) may contradict each other, in the sense that trying to minimize one of them leads to the increase of another one. Naturally, a linearization is a very strong simplification of the problem.

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

✉ D. Visetti
daniela.visetti@unimib.it

1 Department of Economics, Management and Statistics, University of Milano Bicocca, Milano, Italy

Nowadays, there are many approaches to study multiobjective variational or control problems (see for example [5, 6, 8, 9, 14]).

In this paper a Hamilton-Jacobi equation for a set-valued Hamiltonian function is presented. The Hamiltonian function takes values in a complete lattice of sets that are invariant with respect to the sum of a closed and convex cone. This problem was never studied before, to the knowledge of the author, and it is not only interesting per se, but also as a natural completion of a previous paper. In fact, in [12] the authors considered a calculus of variations' problem with a multicriteria Lagrangian function. The embedding of it into a set-valued framework was fundamental to prove a Hopf-Lax formula for the value function. Subsequently, through the Fenchel conjugate of the Lagrangian function a Hamilton-Jacobi set-valued equation was derived.

Here, like in the classical theory, we start supposing that a smooth solution of the equation is known, in order to write the system of ordinary differential equations and the characteristic curves. In this situation, a family of characteristic curves is obtained, parametrized by the elements of the dual cone. A posteriori the ordinary differential equations do not depend on the solution and can be solved independently. Under some assumptions it is so possible to write a solution of the set-valued Hamilton-Jacobi equation from the solutions of the ordinary differential equations. The solutions are global under some hypotheses.

Considering the case of the Hamiltonian function as the Fenchel conjugate of a Lagrangian function, some results are obtained. For set-valued optimization an infimizer is a set. It is proved that an infimizer for the Hopf-Lax formula proved in [12] can be constituted only by minimizers of the scalarized problem with respect to an element of the dual cone.

Moreover, some properties regarding the derivatives of the Fenchel conjugate are extended to the set-valued framework. Finally, it is proved that the characteristic curves coincide with elements of the infimizer.

This study opens a variety of new questions. First of all, it is still not clear what can be an extension of the concept of viscosity solution for a set-valued Hamilton-Jacobi equation. Then also the problem of the generalized characteristic curves should be addressed. Recently, in [15] the author considers a generalized form of characteristics. But also in the previously cited texts [3] and [16], there are several approaches that have not yet been studied in the set-valued case.

2 Preliminaries

The Minkowski sum of two non-empty sets $A, B \subseteq \mathbb{R}^d$ is $A + B = \{a + b \mid a \in A,\ b \in B\}$. It is extended to the whole power set $\mathcal{P}(\mathbb{R}^d)$ by

$$\emptyset + A = A + \emptyset = \emptyset.$$

We also use $A \oplus B := \mathrm{cl}\,(A + B)$, the 'closed sum' of two sets.

A set $C \subseteq \mathbb{R}^d$ is a cone if $sC \subseteq C$ for all $s > 0$, and it is a convex cone if additionally $C + C \subseteq C$. Let C be a closed and convex cone in $\mathbb{R}^d$ different from the empty set and the whole $\mathbb{R}^d$. The dual of a cone C is defined as

$$C^+ = \{\zeta \in \mathbb{R}^d \mid \forall z \in C,\ \zeta \cdot z \geq 0\},$$

where $\zeta \cdot z$ denotes the usual scalar product. If there is an element $\hat{z} \in C$ such that $\zeta \cdot \hat{z} > 0$ for all $\zeta \in C^+\backslash\{0\}$ (in particular, if int $C \neq \emptyset$), then the set

$$B^+(\hat{z}) = \{\zeta \in C^+ \mid \zeta \cdot \hat{z} = 1\} \tag{1}$$

is a (closed and convex) base of C^+, i.e., for each element $\xi \in C^+\backslash\{0\}$ there are unique $\zeta \in B^+(\hat{z})$ and $s > 0$ such that $\xi = s\zeta$.

We consider the following subset of the power set $\mathcal{P}(\mathbb{R}^d)$ (see for instance [10]):

$$\mathcal{G}(\mathbb{R}^d, C) = \{A \in \mathcal{P}(\mathbb{R}^d) \mid A = \mathrm{cl}\,\mathrm{co}\,(A + C)\}$$

where cl and co are the closure and the convex hull, respectively.

The pair $(\mathcal{G}(\mathbb{R}^d, C), \supseteq)$ is a complete lattice. If $\mathcal{A} \subseteq \mathcal{G}(\mathbb{R}^d, C)$, then the infimum and the supremum of $\mathcal{A}$ are given by

$$\inf \mathcal{A} = \mathrm{cl}\,\mathrm{co} \bigcup_{A \in \mathcal{A}} A \qquad \sup \mathcal{A} = \bigcap_{A \in \mathcal{A}} A. \tag{2}$$

An element $A_0 \in \mathcal{A}$ is called minimal for $\mathcal{A}$ if

$$A \in \mathcal{A},\ A \supseteq A_0 \implies A = A_0\,.$$

Let $\zeta \in C^+\backslash\{0\}$ and let

$$H^+(\zeta) = \{z \in \mathbb{R}^d \mid \zeta \cdot z \geq 0\}.$$

For two sets $A, B \in \mathcal{G}(\mathbb{R}^d, C)$, the set

$$A -_\zeta B = \{z \in \mathbb{R}^d \mid z + B \subseteq A \oplus H^+(\zeta)\}$$

is called the ζ-difference of A and B. It is either $\emptyset$, $\mathbb{R}^d$ or a closed (shifted) half-space. It is possible to see that

$$A -_\zeta B = \{z \in \mathbb{R}^d \mid \zeta \cdot z + \inf_{b \in B} \zeta \cdot b \geq \inf_{a \in A} \zeta \cdot a\}, \tag{3}$$

where it is meant that $\inf_{y \in \emptyset} \zeta \cdot y = +\infty$ and $r + (-\infty) = -\infty$ as well as $r + (+\infty) = +\infty$ for $r \in \mathbb{R}$. For more details about this set difference see [10] (where there is a slightly different definition) and [11].

Let $\{A_n\}_{n \in \mathbb{N}}$ be a sequence of sets in $\mathcal{G}(\mathbb{R}^d, C)$, we denote by $\lim_{n\to\infty} A_n$ the following set:

$$\lim_{n\to\infty} A_n = \left\{z \in \mathbb{R}^d \mid \forall n \in \mathbb{N}, \exists z_n \in A_n,\ \lim_{n\to\infty} z_n = z\right\}.$$

This definition of limit coincides with the upper limit of Painlevé-Kuratowski ($\mathrm{Liminf}_{n\to\infty} A_n = \{z \in Z \mid \lim_{n\to\infty} d(z, A_n) = 0\}$, see [1]).

Let $\{A_s\}_{s \in S}$ with $S \subseteq \mathbb{R}$ be a family of sets in $\mathcal{G}(\mathbb{R}^d, C)$ and $\bar{s} \in \mathbb{R}$. We denote by $\lim_{s\to\bar{s}} A_s$ the set which satisfies that for any sequence $\{s_n\}_{n\in\mathbb{N}} \subseteq S$ with $s_n \to \bar{s}$ one has

$$\lim_{s\to\bar{s}} A_s = \lim_{n\to\infty} A_{s_n}.$$

Let f be a function $f : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$. The graph of f is

$$\mathrm{graph}\, f = \{(x, z) \in \mathbb{R}^n \times \mathbb{R}^d \mid z \in f(x)\}\,.$$

The domain of f is

$$\mathrm{dom}\, f = \{x \in \mathbb{R}^n \mid f(x) \neq \emptyset\}$$

The function is convex if and only if graph f is a convex subset of $\mathbb{R}^n \times \mathbb{R}^d$. This is equivalent to the following condition: for any $\lambda \in (0, 1)$, $x_1, x_2 \in \mathbb{R}^n$

$$f(\lambda x_1 + (1-\lambda)x_2) \supseteq \lambda f(x_1) + (1-\lambda) f(x_2)\,.$$

Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a set-valued map from X into the closed nonempty subsets of $\mathbb{R}^d$.

The set of the integrable selections of f is:

$$\mathcal{F} = \{\varphi \in L^1(X, \mathbb{R}^d) \mid \varphi(x) \in f(x) \text{ a.e. in } X\}$$

The Aumann integral of f on $\mathbb{R}^n$ is the set of integrals of the integrable selections of f:

$$\int_X f\, d\mu = \left\{ \int_{\mathbb{R}^n} \varphi\, d\mu \mid \varphi \in \mathcal{F} \right\}$$

Let X be a non-empty set, $f : X \to \mathcal{G}(\mathbb{R}^d, C)$ a function and $f[X] = \{f(x) \mid x \in X\}$.

A family of extended real-valued functions $\varphi_{f,\zeta} : X \to \overline{\mathbb{R}}$ with $\zeta \in C^+$ is defined by

$$\varphi_{f,\zeta}(x) = \inf_{z \in f(x)} \zeta \cdot z. \tag{4}$$

A point $\bar{x} \in X$ is called a ζ-minimizer of f if

$$\forall x \in X\colon \varphi_{f,\zeta}(\bar{x}) \le \varphi_{f,\zeta}(x).$$

(a) A set $M \subset X$ is called an *infimizer for* f if

$$\inf f[M] = \inf f[X]\,.$$

(b) An element $x_0 \in X$ is called a *minimizer for* f if $f(x_0)$ is minimal for $f[X]$.
(c) A set $M \subset X$ is called a *solution* of the problem *minimize* $f(x)$ *subject to* $x \in X$ if M is an infimizer for f and each $x_0 \in M$ is a minimizer for f. It is called a *full solution* if the set $f[M]$ includes all minimal elements of $f[X]$.
(d) A set $M \subseteq X$ is called a *scalarization solution* of the problem *minimize* $f(x)$ *subject to* $x \in X$ if it is an infimizer and only includes ζ-minimizers.

The solution concept in (d) has been considered first in [11].

Let $\eta \in \mathbb{R}^n$ and $\zeta \in C^+$ be given. We recall the definition of the function $S_{(\eta,\zeta)} : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$:

$$S_{(\eta,\zeta)}(x) = \{z \in \mathbb{R}^d \mid \zeta \cdot z \ge \eta \cdot x\}\,.$$

Such a function is additive and positively homogeneous, i.e., for all $x \in \mathbb{R}^n$, $\lambda > 0$

$$S_{(\eta,\zeta)}(\lambda x) = \lambda S_{(\eta,\zeta)}(x)$$

and for all $x_1,\ x_2 \in \mathbb{R}^n$

$$S_{(\eta,\zeta)}(x_1 + x_2) = S_{(\eta,\zeta)}(x_1) + S_{(\eta,\zeta)}(x_2)\,.$$

Let $\hat{z} \in \mathbb{R}^d$ be such that $\zeta \cdot \hat{z} = 1$. Then for any $x \in \mathbb{R}^n$

$$S_{(\eta,\zeta)}(x) = (\eta \cdot x)\hat{z} + H^+(\zeta) \tag{5}$$

(see [10]).

The derivative concept that will be used in this paper is as follows. The derivatives of a function $f : \mathbb{R} \times \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ with respect to an element ζ in the dual cone of C, if they exist, will be defined in the following way:

$$\begin{aligned} D_{\zeta,t} f(t,x) &= \lim_{h\to 0^+} \frac{1}{h}[f(t+h,x) -_\zeta f(t,x)], \\ D_{\zeta,x} f(t,x)(q) &= \lim_{h\to 0^+} \frac{1}{h}[f(t,x+hq) -_\zeta f(t,x)]. \end{aligned} \tag{6}$$

See [11] and [12] for a motivation and many features including a discussion of the 'improper' function values $\mathbb{R}^d$ and $\emptyset$.

The Fenchel conjugate of the function $f : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ is defined as the function

$$\begin{aligned} f^* : \mathbb{R}^n \times C^+\backslash\{0\} &\to \mathcal{G}(\mathbb{R}^d, C) \\ (\eta, \zeta) &\mapsto \sup_{x\in\mathbb{R}^n} S_{(\eta,\zeta)}(x) -_\zeta f(x) \end{aligned} \tag{7}$$

3 Hamilton-Jacobi Equation and Characteristic Curves

Let C be a closed and convex cone in $\mathbb{R}^d$ different from the empty set and the whole $\mathbb{R}^d$ and let $\hat{z}$ be a given element in the cone such that $B^+(\hat{z})$ is a base of C^+ (see (1)). The cone and the element $\hat{z}$ are fixed from now on.

Let us consider

$$0 < T < +\infty, \quad Q_T = [0,T] \times \mathbb{R}^n, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n,$$

a set-valued Hamiltonian function $\mathcal{H} : \mathbb{R}^n \times C^+\backslash\{0\} \to \mathcal{G}(\mathbb{R}^d, C)$ and $U_0 : \mathbb{R}^n \to \mathbb{R}^d$ a function of class C^2. The Hamiltonian function was chosen in this way due to the fact that the main example for it is the Fenchel conjugate (7) of a Lagrangian function $\mathcal{H}(p,\zeta) = L^*(p,\zeta)$, where $L : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ (see [12]).

We suppose that for any $\zeta \in C^+\backslash\{0\}$ the function $\mathcal{H}_\zeta : \mathbb{R}^n \to \mathbb{R}$, defined as

$$\mathcal{H}_\zeta(p) = \inf_{z\in\mathcal{H}(p,\zeta)} \zeta \cdot z,$$

is of class C^2.

The simplest example can be when there exists $\mathcal{H}_0 : \mathbb{R}^n \to \mathbb{R}^d$ of class C^2 and $\mathcal{H}$ is the inf-extension of $\mathcal{H}_0$, $\mathcal{H}(p,\zeta) = \mathcal{H}_0(p) + C$ (see [10]), then $\mathcal{H}_\zeta(p)$ coincide with $\mathcal{H}_{0,\zeta}(p) = \mathcal{H}_0(p) \cdot \zeta$ and are automatically of class C^2.

Given $\zeta \in B^+(\hat{z})$, we say that $U : Q_T \to \mathcal{G}(\mathbb{R}^d, C)$ satisfies the ζ-property if the scalarization along ζ is finite and we define

$$u_\zeta(t,x) = \varphi_{U,\zeta}(t,x), \tag{8}$$

where $\varphi_{U,\zeta}$ is defined in (4). This property is additive and positively homogeneous with respect to the function U.

Proposition 3.1 *Given $\zeta \in B^+(\hat{z})$, the function $U : Q_T \to \mathcal{G}(\mathbb{R}^d, C)$ satisfies the ζ-property if and only if*

$$U(t,x) + H^+(\zeta) = u_\zeta(t,x)\hat{z} + H^+(\zeta), \tag{9}$$

with $u_\zeta : Q_T \to \mathbb{R}$ as in (8).

Proof It is immediate to see that if (9) holds, then

$$\begin{aligned}\inf\{z\cdot\zeta \mid z\in U(t,x)\} &= \inf\{z\cdot\zeta \mid z\in U(t,x)+H^+(\zeta)\}\\ &= \inf\{z\cdot\zeta \mid z\in u_\zeta(t,x)\hat{z}+H^+(\zeta)\} = u_\zeta(t,x).\end{aligned}$$

Vice versa, let (8) hold. For every $z\in U(t,x)$ the vector $h_\zeta = z-u_\zeta(t,x)\hat{z}$ is such that

$$h_\zeta\cdot\zeta = z\cdot\zeta - u_\zeta(t,x)\geq 0,$$

so $h_\zeta \in H^+(\zeta)$ and $U(t,x)\subset u_\zeta(t,x)\hat{z}+H^+(\zeta)$. Finally, we prove that $u_\zeta(t,x)\hat{z}\in U(t,x)+H^+(\zeta)$. Let $z_0\in U(t,x)$ be such that $z_0\cdot\zeta = u_\zeta(t,x)$. As we have already seen $z_0-u_\zeta(t,x)\hat{z} = h_\zeta \in H^+(\zeta)$, but in this case h_ζ is perpendicular to ζ. Then $u_\zeta(t,x)\hat{z} = z_0-h_\zeta\in U(t,x)+H^+(\zeta)$. □

It must be stressed that the choice of the point $\hat{z}$ is arbitrary (for example it can be any point in the interior of C), but equation (9) holds always for $\zeta\in B^+(\hat{z})$.

Proposition 3.2 *Given $\zeta\in B^+(\hat{z})$, if the function $U : Q_T\to\mathcal{G}(\mathbb{R}^d, C)$ satisfies the ζ-property and the function $u_\zeta(t,x)$ is C^1, then*

$$\begin{aligned}D_{\zeta,t}U(t,x) &= S_{\left(\frac{\partial u_\zeta}{\partial t}(t,x),\zeta\right)}(1)\,,\\ D_{\zeta,x}U(t,x)(q) &= S_{(Du_\zeta(t,x),\zeta)}(q)\,.\end{aligned}$$

Proof Using (6) and (3), we can write for $h>0$

$$\begin{aligned}&\frac{1}{h}\big(U(t+h,x) -_\zeta U(t,x)\big)\\ &\quad=\frac{1}{h}\left\{z\in\mathbb{R}^d\;\middle|\; z\cdot\zeta+\inf_{z_1\in U(t,x)} z_1\cdot\zeta \geq \inf_{z_1\in U(t+h,x)} z_1\cdot\zeta\right\}\\ &\quad=\left\{z\in\mathbb{R}^d\;\middle|\; z\cdot\zeta\geq\frac{1}{h}(u_\zeta(t+h,x)-u_\zeta(t,x))\right\}\end{aligned}$$

and analogously

$$\frac{1}{h}\big(U(t,x+hq) -_\zeta U(t,x)\big) = \left\{z\in\mathbb{R}^d\;\middle|\; z\cdot\zeta\geq\frac{1}{h}(u_\zeta(t,x+hq)-u_\zeta(t,x))\right\}.$$

Taking the limits in the two equations, it is obtained

$$\begin{aligned}D_{\zeta,t}U(t,x) &= \left\{z\in\mathbb{R}^d\;\middle|\; z\cdot\zeta\geq\frac{\partial u_\zeta}{\partial t}(t,x)\right\} = S_{\left(\frac{\partial u_\zeta}{\partial t}(t,x),\zeta\right)}(1),\\ D_{\zeta,x}U(t,x)(q) &= \left\{z\in\mathbb{R}^d\;\middle|\; z\cdot\zeta\geq Du_\zeta(t,z)\cdot q\right\} = S_{(Du_\zeta(t,x),\zeta)}(q).\end{aligned}$$

□

We define now a family of Hamilton-Jacobi equations. For $\zeta\in B^+(\hat{z})$, $U : Q_T\to\mathcal{G}(\mathbb{R}^d, C)$ is a solution of the ζ-Hamilton-Jacobi equation if it satisfies the ζ-property and is a solution of

$$\begin{cases} D_{\zeta,t}U(t,x)+\mathcal{H}\big(Du_\zeta(t,x),\zeta\big) = H^+(\zeta)\\ U(0,x) = U_0(x)+C\end{cases}\tag{10}$$

where u_ζ is of class C^2. Recalling that the supremum in $\mathcal{G}(\mathbb{R}^d, C)$ is defined in (2) and using the fact that $\bigcap_{\zeta\in B^+(\hat{z})} H^+(\zeta) = \bigcap_{\zeta\in C^+} = C$, if $U(t,x)$ is a solution of (10) for every $\zeta \in B^+(\hat{z})$, then it is also a solution of

$$\begin{cases} \sup\limits_{\zeta\in B^+(\hat{z})} \left[D_{\zeta,t}U(t,x) + \mathcal{H}\left(Du_\zeta(t,x),\zeta\right)\right] = C \\ U(0,x) = U_0(x) + C \end{cases} \tag{11}$$

We say that $U(t,x)$, satisfying property (9), is of class C^2 if all the u_ζ are C^2 for $\zeta \in B^+(\hat{z})$. The first equation in (10) gives

$$\inf\left\{\zeta\cdot z \mid z \in \left[D_{\zeta,t}U(t,x) + \mathcal{H}\left(Du_\zeta(t,x),\zeta\right)\right]\right\} = \inf\{\zeta\cdot z \mid z\in H^+(\zeta)\}$$

that can be written

$$\frac{\partial u_\zeta}{\partial t}(t,x) + \mathcal{H}_\zeta\left(Du_\zeta(t,x)\right) = 0\,.$$

So u_ζ is a solution of

$$\begin{cases} \frac{\partial \upsilon_\zeta}{\partial t}(t,x) + \mathcal{H}_\zeta\left(D\upsilon_\zeta(t,x)\right) = 0 \\ \upsilon_\zeta(0,x) = U_{0,\zeta}(x) \end{cases} \tag{12}$$

where $U_{0,\zeta}(x) = \zeta\cdot U_0(x)$ is a real-valued function.

We suppose that problem (12) has a solution υ_ζ of class C^2. For fixed $x\in\mathbb{R}^n$, we denote by $X_\zeta(t,x)$ the solution of the ordinary differential equation

$$\dot{X}_\zeta = D\mathcal{H}_\zeta\left(D\upsilon_\zeta(t,X_\zeta)\right), \qquad X_\zeta(0,x) = x\,. \tag{13}$$

Such a solution is defined on an interval $[0, T_{\zeta,x})$. The curve $(t, X_\zeta(t,x))$ is the characteristic curve associated to U with respect to ζ.

We define now

$$V_\zeta(t,x) = \upsilon_\zeta(t, X_\zeta(t,x))\,, \qquad P_\zeta(t,x) = D\upsilon_\zeta(t, X_\zeta(t,x))\,. \tag{14}$$

Using equation (12), we obtain

$$\begin{aligned} \dot{V}_\zeta &= \frac{\partial \upsilon_\zeta}{\partial t}(t,X_\zeta) + D\upsilon_\zeta(t,X_\zeta)\cdot\dot{X}_\zeta = -\mathcal{H}_\zeta\left(P_\zeta\right) + D\mathcal{H}_\zeta\left(P_\zeta\right)\cdot P_\zeta\,, \\ \dot{P}_\zeta &= \frac{\partial D\upsilon_\zeta}{\partial t}(t,X_\zeta) + D^2\upsilon_\zeta(t,X_\zeta)\dot{X}_\zeta \\ &= D\left(\frac{\partial \upsilon_\zeta}{\partial t}(t,X_\zeta) + \mathcal{H}_\zeta\left(D\upsilon_\zeta(t,X_\zeta)\right)\right) = 0\,. \end{aligned}$$

As a consequence, P_ζ is constant in time:

$$P_\zeta(t,x) \equiv DU_{0,\zeta}(x)\,.$$

Then also $\dot{X}_\zeta$ is constant in time: $\dot{X}_\zeta = D\mathcal{H}_\zeta\left(P_\zeta\right) = D\mathcal{H}_\zeta\left(DU_{0,\zeta}(x)\right)$. The solutions of the ODEs are then:

$$\begin{cases} X_\zeta(t,x) = x + tD\mathcal{H}_\zeta\left(DU_{0,\zeta}(x)\right) \\ V_\zeta(t,x) = U_{0,\zeta}(x) + t\left(-\mathcal{H}_\zeta\left(DU_{0,\zeta}(x)\right) + D\mathcal{H}_\zeta\left(DU_{0,\zeta}(x)\right)\cdot DU_{0,\zeta}(x)\right) \\ P_\zeta(t,x) = DU_{0,\zeta}(x) \end{cases} \tag{15}$$

The next step is to consider the system of ODEs

$$\begin{cases} \dot{X}_\zeta = D\mathcal{H}_\zeta\left(P_\zeta\right) \\ \dot{P}_\zeta = 0 \\ \dot{V}_\zeta = -\mathcal{H}_\zeta\left(P_\zeta\right) + D\mathcal{H}_\zeta\left(P_\zeta\right)\cdot P_\zeta \end{cases} \tag{16}$$

in order to build a solution of the Hamilton-Jacobi equation. The classical result is the following *local existence theorem* (see for example [3]):

Let U_0 be in $C^2(\mathbb{R}^n;\mathbb{R}^d)$ and $DU_{0,\zeta}$, $D^2U_{0,\zeta}$ be bounded for any $\zeta \in B^+(\hat{z})$. Let $\mathcal{H}_\zeta$ be of class C^2 for any $\zeta \in B^+(\hat{z})$. Denoting

$$T^*_\zeta = \sup\{t > 0 \mid I + tD^2\mathcal{H}_\zeta(DU_{0,\zeta}(x))D^2U_{0,\zeta}(x) \text{ is invertible} \forall x \in \mathbb{R}^n\}, \tag{17}$$

problem (12) has a unique solution $\upsilon_\zeta \in C^2([0,T^*_\zeta)\times\mathbb{R}^n)$.

By the previous hypotheses, for any $T < T^*_\zeta$, there exists $Z_\zeta : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$ of class C^1 such that

$$X_\zeta(t, Z_\zeta(t,x)) = x$$

for any $(t,x) \in [0,T]\times\mathbb{R}^n$. The solution of the previous theorem is given by

$$\upsilon_\zeta(t,x) = V_\zeta(t, Z_\zeta(t,x)), \qquad \text{for any } (t,x) \in [0,T]\times\mathbb{R}^n. \tag{18}$$

We want now to find set-valued functions $U(t,x)$ that are solutions of the set-valued Hamilton-Jacobi equations.

Theorem 3.3 *Let U_0 be in $C^2(\mathbb{R}^n;\mathbb{R}^d)$ and $DU_{0,\zeta}$, $D^2U_{0,\zeta}$ be bounded for any $\zeta \in B^+(\hat{z})$. Let $\mathcal{H}_\zeta$ be in $C^2(\mathbb{R}^n)$ for any $\zeta \in B^+(\hat{z})$. Denoting*

$$T^* = \inf_{\zeta\in B^+(\hat{z})} T^*_\zeta \tag{19}$$

where T^*_ζ is as in (17), let us assume that $T^* > 0$. For $T < T^*$ the map $U_\zeta : [0,T]\times\mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ defined as

$$U_\zeta(t,x) = \upsilon_\zeta(t,x)\hat{z} + H^+(\zeta)$$

is a solution of

$$\begin{cases} D_{\zeta,t}U(t,x) + \mathcal{H}\left(D\upsilon_\zeta(t,x),\zeta\right) = H^+(\zeta) \\ U(0,x) = U_0(x) + H^+(\zeta). \end{cases} \tag{20}$$

Moreover, let U be the map $U : [0,T]\times\mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ defined as

$$U(t,x) = \sup_{\zeta\in B^+(\hat{z})} U_\zeta(t,x). \tag{21}$$

If for any $\zeta \in B^+(\hat{z})$ and $(t,x) \in [0,T]\times\mathbb{R}^n$, there holds

$$u_\zeta(t,x) = \inf_{z\in U(t,x)} \zeta\cdot z = \inf_{z\in U_\zeta(t,x)} \zeta\cdot z = \upsilon_\zeta(t,x), \tag{22}$$

then U is a solution of (11).

Remark 3.4 *In the hypotheses of the local existence theorem (that coincide with the hypotheses of the previous theorem) for any $\zeta \in B^+(\hat{z})$ there exist $T_\zeta > 0$ and $U_\zeta(t,x)$ defined on $[0,T_\zeta)\times\mathbb{R}^n$. In some cases it is possible to deduce that $T^* > 0$. More precisely, we recall the following results linked to the existence of the characterisics of the scalarized problems (see Corollary 1.5.5 in [3]):*

1. *Set*

$$M_0 = \sup_{\zeta \in B^+(\hat{z})} \sup_{x \in \mathbb{R}^n} \|DU_{0,\zeta}(x)\|,$$

$$M_1 = \sup_{\zeta \in B^+(\hat{z})} \sup_{x \in \mathbb{R}^n} \|D^2U_{0,\zeta}(x)\|,$$

$$M_2 = \sup_{\zeta \in B^+(\hat{z})} \sup_{x \in \mathbb{R}^n,\ \|x\| \le M_0} \|D^2\mathcal{H}_\zeta(x)\|.$$

   Then problem (11) *has a* C^2 *solution at least for the time* $t \in [0, T^*)$*, where* $T^* = \frac{1}{M_1 M_2}$.
2. *If* $U_{0,\zeta}$ *and* $\mathcal{H}_\zeta$ *are convex, then problem* (11) *has a* C^2 *solution for all positive times (so* $T^* = +\infty$*).*
3. *If* $U_0(x) = Ax + b$ *with* A *a matrix of dimension* d *times* n *and* $b \in \mathbb{R}^d$*, then problem* (11) *has a* C^2 *solution for all positive times (so* $T^* = +\infty$*).*

Proof It is easy to see that $U_\zeta(t,x)$ is a solution of (20).

The map $U(t,x)$ has the property (9) thanks to hypothesis (22).

The function U satisfies the first equation in (11), because $D_{\zeta,t}U(t,x) = D_{\zeta,t}U_\zeta(t,x)$. In fact, for any $A, B \in \mathcal{G}(\mathbb{R}^d, C)$

$$A -_\zeta B = (A + H^+(\zeta)) -_\zeta (B + H^+(\zeta))$$

and so also for the derivative

$$\begin{aligned} D_{\zeta,t}U(t,x) &= D_{\zeta,t}[U(t,x) + H^+(\zeta)] = u_\zeta(t,x)\hat{z} + H^+(\zeta) \\ &= \upsilon_\zeta(t,x)\hat{z} + H^+(\zeta) = D_{\zeta,t}U_\zeta(t,x). \end{aligned}$$

For the initial condition, we have that

$$U(0,x) = \sup_{\zeta \in B^+(\hat{z})} U_\zeta(0,x) = \sup_{\zeta \in B^+(\hat{z})} [U_0(x) + H^+(\zeta)] = U_0(x) + C\,.$$

□

In the following proposition the solution is written as a set-valued version of the characteristic method. In order to do that, we define a set-valued correspondent to $V_\zeta(t,x)$ for $\zeta \in B^+(\hat{z})$. More precisely, we denote $\mathcal{V}_\zeta : [0,T] \times \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ the map

$$\begin{aligned} \mathcal{V}_\zeta(t,x) =& U_0(x) + t\left[(H^+(\zeta) -_\zeta \mathcal{H}\left(DU_{0,\zeta}(x), \zeta\right)) \right. \\ & \left. + D_{\zeta,p}\mathcal{H}\left(DU_{0,\zeta}(x), \zeta\right)(DU_{0,\zeta}(x))\right], \end{aligned} \tag{23}$$

where the derivative D_ζ denotes the derivative of the set-valued function $\mathcal{H}$.

Proposition 3.5 *Let* U_0 *be in* $C^2(\mathbb{R}^n; \mathbb{R}^d)$ *and* $DU_{0,\zeta}$, $D^2U_{0,\zeta}$ *be bounded for any* $\zeta \in B^+(\hat{z})$. *Let* $\mathcal{H}_\zeta$ *be in* $C^2(\mathbb{R}^n)$ *for any* $\zeta \in B^+(\hat{z})$. *If* (22) *holds, for* $T^* > 0$ *as in Theorem 3.3 the solution* $U : [0,T] \times \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ *with* $T < T^*$ *defined in* (21) *can be written*

$$U(t,x) = \sup_{\zeta \in B^+(\hat{z})} \mathcal{V}_\zeta(t, Z_\zeta(t,x)). \tag{24}$$

Proof Both $U_\zeta(t,x)$ and $\mathcal{V}_\zeta(t, Z_\zeta(t,x))$ are half-spaces in the positive direction of ζ. We have that

$$\inf_{z \in H^+(\zeta) -_\zeta \mathcal{H}(DU_{0,\zeta}(x),\zeta)} \zeta \cdot z = -\mathcal{H}_\zeta\left(DU_{0,\zeta}(x)\right),$$

$$\inf_{z \in D_{\zeta,p}\mathcal{H}(DU_{0,\zeta}(x),\zeta)(DU_{0,\zeta}(x))} \zeta \cdot z = D\mathcal{H}_\zeta\left(DU_{0,\zeta}(x)\right) \cdot DU_{0,\zeta}(x).$$

It is then immediate that $U_\zeta(t,x) = \mathcal{V}_\zeta(t, Z_\zeta(t,x))$. □

Next corollary presents a special case in which condition (22) in Theorem 3.3 is satisfied.

Corollary 3.6 *Let U_0 be in $C^2(\mathbb{R}^n;\mathbb{R}^d)$ and $DU_{0,\zeta}$, $D^2U_{0,\zeta}$ be bounded for any $\zeta \in B^+(\hat{z})$. Let $\mathcal{H}(\cdot,\zeta)=\mathcal{H}_0(\cdot)+C$ for any $\zeta \in B^+(\hat{z})$ and $\mathcal{H}_0$ be in $C^2(\mathbb{R}^n)$. For $T^*>0$ as in Theorem 3.3 and $T<T^*$, if for any $\zeta,\xi \in B^+(\hat{z})$*

$$\begin{aligned} &U_{0,\zeta}(Z_\xi)+t\left[-\mathcal{H}_{0,\zeta}(DU_{0,\xi}(Z_\xi))+D\mathcal{H}_0(DU_{0,\xi}(Z_\xi))DU_{0,\xi}(Z_\xi)\right] \\ &\qquad \geq U_{0,\zeta}(Z_\zeta)+t\left[-\mathcal{H}_{0,\zeta}(DU_{0,\zeta}(Z_\zeta))+D\mathcal{H}_0(DU_{0,\zeta}(Z_\zeta))DU_{0,\zeta}(Z_\zeta)\right], \end{aligned} \tag{25}$$

where $Z_\zeta = Z_\zeta(t,x)$, $Z_\xi = Z_\xi(t,x)$ and $D\mathcal{H}_0$ denotes the Jacobian matrix, then the function $U(t,x)$ defined in (24) is a solution of (11).

Proof We observe that

$$\mathcal{V}_\zeta(t,x)=U_0(x)+t\left[-\mathcal{H}_0(DU_{0,\zeta}(x))+D\mathcal{H}_0(DU_{0,\zeta}(x))DU_{0,\zeta}(x)\right]+H^+(\zeta).$$

Equation (25) implies that the vectors

$$U_0(Z_\zeta)+t\left[-\mathcal{H}_0(DU_{0,\zeta}(Z_\zeta))+D\mathcal{H}_0(DU_{0,\zeta}(Z_\zeta))DU_{0,\zeta}(Z_\zeta)\right]\in U(t,x)$$

and that hypothesis (22) is satisfied. □

Example 3.7 *We choose $\mathbb{R}^n=\mathbb{R}^d=\mathbb{R}^2$, $C=\mathbb{R}^2_+$ and the following Hamiltonian and initial condition*

$$\mathcal{H}_0(p)=\begin{pmatrix}\frac{1}{2}\|p\|^2\\ \frac{1}{4}\|p\|^4\end{pmatrix}, \qquad \mathcal{H}(p,\zeta)=\mathcal{H}_0(p)+C, \qquad U_0(x)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}x.$$

If we consider $\hat{z}=\binom{1}{1}$, $B^+(\hat{z})$ is given by all $\zeta=\binom{\zeta_1}{1-\zeta_1}$ for $0\leq\zeta_1\leq 1$. Since

$$\begin{aligned} \mathcal{H}_\zeta(p)&=\frac{\zeta_1}{2}\|p\|^2+\frac{1-\zeta_1}{4}\|p\|^4,\\ D\mathcal{H}_\zeta(p)&=\zeta_1 p+(1-\zeta_1)\|p\|^2 p,\\ U_{0,\zeta}(x)&=\zeta\cdot Ax,\\ DU_{0,\zeta}(x)&=A^T\zeta, \end{aligned}$$

where A^T denotes the transpose of the matrix A. Solving the system of ODEs (16), we obtain

$$\begin{cases} X_\zeta(t,x)=x+t\left[\zeta_1+(1-\zeta_1)\|A^T\zeta\|^2\right]A^T\zeta\\ V_\zeta(t,x)=\zeta\cdot Ax+t\left[\frac{1}{2}\zeta_1\|A^T\zeta\|^2+\frac{3}{4}(1-\zeta_1)\|A^T\zeta\|^4\right]\\ P_\zeta(t,x)=A^T\zeta \end{cases}$$

We want to calculate $Z_\zeta(t,x)$ such that

$$X_\zeta(t,Z_\zeta(t,x))=x,$$

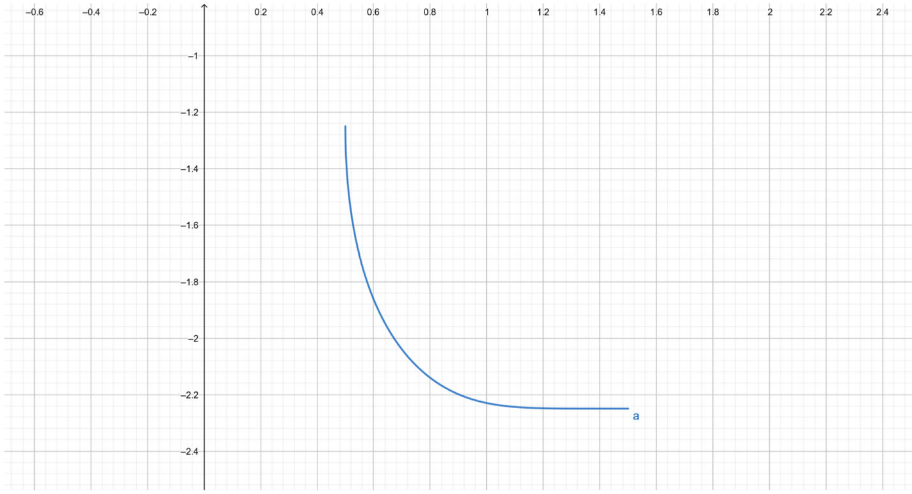
which gives

$$Z_\zeta(t,x)=x-t\left[\zeta_1+(1-\zeta_1)\|A^T\zeta\|^2\right]A^T\zeta.$$

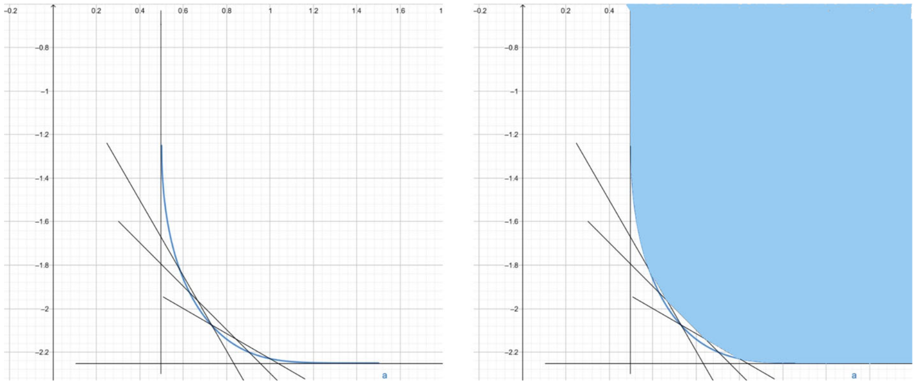
Now we can calculate

$$U_\zeta(t,x)=Ax-t[\zeta_1+(1-\zeta_1)\|A^T\zeta\|^2]AA^T\zeta+t\begin{pmatrix}\frac{1}{2}\|A^T\zeta\|^2\\ \frac{3}{4}\|A^T\zeta\|^4\end{pmatrix}+H^+(\zeta).$$

In particular, for $x_0 = \binom{1}{2}$ *and* $t = 1$ *the curve* $\gamma_\zeta(1, x_0) = Ax_0 - [\zeta_1 + (1 - \zeta_1)\|A^T\zeta\|^2]AA^T\zeta + \binom{\frac{1}{2}\|A^T\zeta\|^2}{\frac{3}{4}\|A^T\zeta\|^4}$ *is plotted in the following figure:*



while in the next two figures there are some half-spaces (corresponding to $\zeta = \binom{1}{0}$, $\binom{3/(3+\sqrt{3})}{\sqrt{3}/(3+\sqrt{3})}$, $\binom{1/2}{1/2}$, $\binom{1/(1+\sqrt{3})}{\sqrt{3}/(1+\sqrt{3})}$, $\binom{0}{1}$*) and their intersection, which is an approximation of the corresponding solution* $U(1, x_0)$.



It is possible to see that the hypothesis (22) holds and $U(t, x)$ *is a solution of the Hamilton-Jacobi equation.*

In the following example hypothesis (22) does not hold.

Example 3.8 *Like before, we choose* $\mathbb{R}^n = \mathbb{R}^d = \mathbb{R}^2$ *and* $C = \mathbb{R}^2_+$. *The Hamiltonian and the initial condition are*

$$\mathcal{H}_0(p) = \begin{pmatrix} \frac{1}{2}\|p\|^2 \\ \frac{1}{2}\|p + p_0\|^2 \end{pmatrix} + C, \quad \mathcal{H}(p, \zeta) = \mathcal{H}_0(p) + C, \quad U_0(x) = \begin{pmatrix} \frac{1}{2}\|x\|^2 \\ \frac{1}{2}\|x\|^2 \end{pmatrix}.$$

For $\zeta = \binom{\zeta_1}{1-\zeta_1} \in B^+(\hat{z})$, where $\hat{z}$ is the same as in the previous example, we have

$$\begin{aligned}\mathcal{H}_\zeta(p) &= \frac{\zeta_1}{2}\|p\|^2 + \frac{1-\zeta_1}{2}\|p+p_0\|^2,\\ D\mathcal{H}_\zeta(p) &= p + (1-\zeta_1)p_0,\\ U_{0,\zeta}(x) &= \frac{1}{2}\|x\|^2,\\ DU_{0,\zeta}(x) &= x.\end{aligned}$$

The solutions (15) *are*

$$\begin{cases} X_\zeta(t,x) = (1+t)x + t(1-\zeta_1)p_0 \\ V_\zeta(t,x) = \frac{1}{2}(1+t)\|x\|^2 - \frac{1-\zeta_1}{2}t\|p_0\|^2 \\ P_\zeta(t,x) = x \end{cases}$$

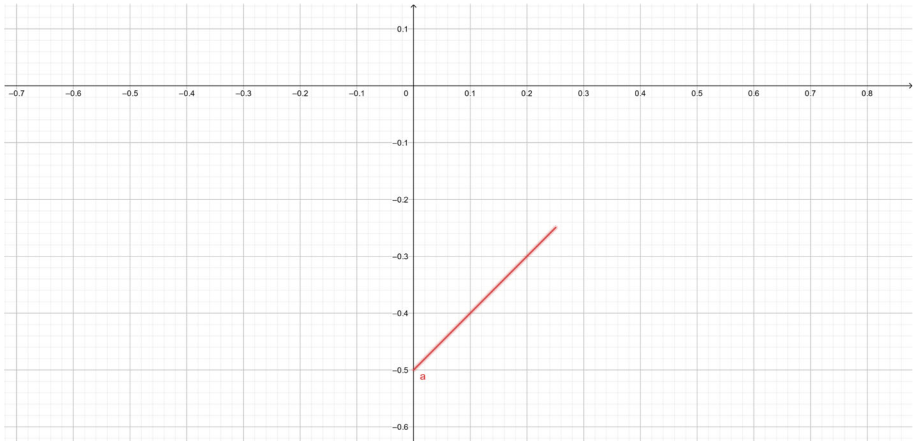
We find that $Z_\zeta(t,x)$ is well defined for any nonnegative t and

$$Z_\zeta(t,x) = \frac{x - t(1-\zeta_1)p_0}{1+t}.$$

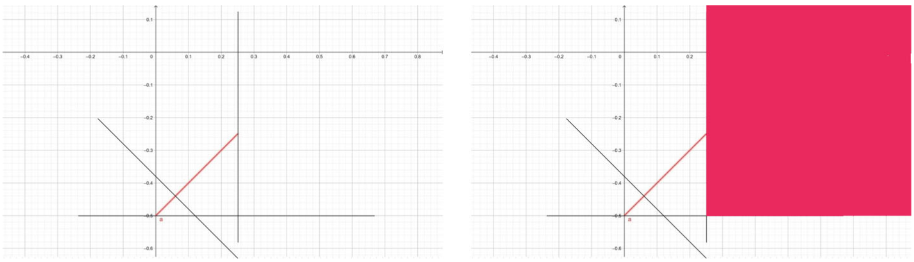
The solutions $u_\zeta(t,x)$ of (12) *are global for any ζ and any $x \in \mathbb{R}^n$. Correspondingly, we obtain*

$$U_\zeta(t,x) = \begin{pmatrix} \frac{1}{2(1+t)}\|x - t(1-\zeta_1)p_0\|^2 \\ \frac{1}{2(1+t)}\|x - t(1-\zeta_1)p_0\|^2 - \frac{1}{2}t\|p_0\|^2 \end{pmatrix} + H^+(\zeta).$$

To check the property (22)*, the curve that describes $U_\zeta(t,x)$ is plotted for $\zeta \in B^+(\hat{z})$, $x = p_0 = \binom{1}{0}$ and $t = 1$:*



In the following figures the half-spaces corresponding to $\zeta = \binom{1}{0}$, $\binom{1/2}{1/2}$ and $\binom{0}{1}$ are drawn. It is possible to observe that in the second figure the half-space corresponding to $\zeta = \binom{1/2}{1/2}$ is not on the border of the intersection of the other two half-spaces, so hypothesis (22) *is not fulfilled.*



4 A Scalarization Solution for a Multiobjective Calculus of Variations Problem

Let us consider the continuous lower bounded functions

$$L : \mathbb{R}^n \to \mathbb{R}^d , \qquad U_0 : \mathbb{R}^n \to \mathbb{R}^d$$

where L is the running cost or Lagrangian and U_0 is the initial cost.

For any $(t, x) \in [0, +\infty) \times \mathbb{R}^n$, define the set of admissible arcs:

$$Y(t, x) = \{y \in W^{1,1}([0, t], \mathbb{R}^n) \mid y(t) = x\} .$$

In [12] the problem of 'minimizing' the cost functional $J_t : W^{1,1}([0, t], \mathbb{R}^n) \to \mathbb{R}^d$

$$J_t[y] = \int_0^t L(s, y(s), \dot{y}(s))\, ds + U_0(y(0))$$

with respect to $y \in Y(t, x)$ was considered.

In order to precise the meaning of the previous minimization, we consider the functions:

$$\begin{aligned} \overline{L} &: \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C) \\ \overline{J}_t &: W^{1,1}([0, t], \mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^d, C) \end{aligned}$$

defined by the inf-extension $\overline{L}(s, y, z) = L(s, y, z) + C$ and

$$\overline{J}_t[y] = \int_0^t \overline{L}(s, y(s), \dot{y}(s))\, ds + U_0(y(0)),$$

where the integral is in the Aumann sense (see [2] or [1]).

Now the problem can be written:

$$\text{minimize } \overline{J}_t[y] \text{ over all arcs } y \in Y(t, x) . \tag{26}$$

Since the functional $\overline{J}_t$ maps into the complete lattice $\mathcal{G}(\mathbb{R}^d, C)$, the value function is well defined:

$$U(t, x) = \inf_{y \in Y(t,x)} \overline{J}_t[y]. \tag{27}$$

Let $\overline{L} : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ be a convex function. For any $\zeta \in B^+(\hat{z})$ let $L_\zeta(p) = L(p) \cdot \zeta$ be such that

$$\lim_{\|p\| \to +\infty} \frac{L_\zeta(p)}{\|p\|} = +\infty \tag{28}$$

and let $U_{0,\zeta}(x) = U_0(x) \cdot \zeta$ be Lipschitz continuous.

For any $\zeta \in B^+(\hat{z})$ there exists w_ζ such that

$$\inf_{w\in\mathbb{R}^n}\left[tL_\zeta\left(\frac{x-w}{t}\right)+U_{0,\zeta}(w)\right]=\left[tL_\zeta\left(\frac{x-w_\zeta}{t}\right)+U_{0,\zeta}(w_\zeta)\right]. \tag{29}$$

The element w_ζ is a ζ-minimizer.

The value function $U(t,x)$ was proved to be obtained as an infimum over $\mathbb{R}^n$ through the Hopf-Lax formula:

$$U(t,x)=\inf_{w\in\mathbb{R}^n}\left[t\overline{L}\left(\frac{x-w}{t}\right)+U_0(w)\right]. \tag{30}$$

We prove now that it is sufficient to take the infimum over a smaller set, instead of all $\mathbb{R}^n$. More precisely, one can consider only the set of the ζ-minimizers, for $\zeta \in B^+(\hat{z})$.

Theorem 4.1 *Let $L : \mathbb{R}^n \to \mathbb{R}^d$ and $U_0 : \mathbb{R}^n \to \mathbb{R}^d$ be continuous functions, $\overline{L} : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ be a convex function and satisfy (28). Then the value function U with values in $\mathcal{G}(\mathbb{R}^d, C)$ is given by the formula*

$$U(t,x)=\inf_{\zeta\in B^+(\hat{z})}\left[t\overline{L}\left(\frac{x-w_\zeta}{t}\right)+U_0(w_\zeta)\right].$$

Proof If we denote

$$V(t,x)=\inf_{\zeta\in B^+(\hat{z})}\left[t\overline{L}\left(\frac{x-w_\zeta}{t}\right)+U_0(w_\zeta)\right],$$

it is immediate that $V(t,x) \subseteq U(t,x)$, where U is defined by (30). Let $z_0 \in U(t,x)\backslash V(t,x)$. Since $V(t,x)$ is closed and convex, by the separation theorem there exists a nonzero $\xi \in \mathbb{R}^d$ and $K \in \mathbb{R}$ such that

$$\xi \cdot z_0 < K < \xi \cdot z \tag{31}$$

for any $z \in V(t,x)$. We want to show that $\xi \in C^+$. In fact, if not there exists $c \in C$ with $\xi \cdot c < 0$. Now, if $z \in V(t,x)$, also $z+\lambda c \in V(t,x)$ for any $\lambda \geq 0$. The following limit holds

$$\lim_{\lambda\to+\infty} \xi\cdot(z+\lambda c) = -\infty,$$

but this contradicts inequality (31). It is always possible to consider ξ in $B^+(\hat{z})$. Now (31) implies that

$$\xi \cdot z_0 < tL_\xi\left(\frac{x-w_\xi}{t}\right)+U_{0,\xi}(w_\xi)$$

and this is not possible. □

The previous theorem clarifies also that the set of all the linear arcs

$$y_\zeta(s)=w_\zeta+\frac{s}{t}(x-w_\zeta)$$

for $\zeta \in B^+(\hat{z})$ forms an infimizer for problem (26) and more precisely a scalarization solution:

Corollary 4.2 *The set*

$$M=\{y_\zeta \in Y(t,x) \mid \zeta \in B^+(\hat{z})\} \tag{32}$$

is a scalarization solution for problem (26).

5 Properties of the Set-Valued Fenchel Conjugate

In the following lemma and theorem some properties of the Fenchel conjugate are stated. In the lemma the link between the set-valued and the scalarized Fenchel conjugate is studied.

Lemma 1 *or any $\zeta \in B^+(\hat{z})$ the following equalities hold:*

$$\overline{L}^*(p,\zeta) = S_{(1,\zeta)}(L^*_\zeta(p)) \tag{33}$$

$$\inf_{z\in\overline{L}^*(p,\zeta)} \zeta\cdot z = L^*_\zeta(p). \tag{34}$$

Proof The Fenchel conjugate $\overline{L}^*(p,\zeta)$ is defined as the supremum over $\mathbb{R}^n$ of $S_{(p,\zeta)}(x) -_\zeta \overline{L}(x)$. Each of the half-spaces can be written as

$$S_{(p,\zeta)}(x) -_\zeta \overline{L}(x) = (p\cdot x - L_\zeta(x))\hat{z} + H^+(\zeta).$$

Since the half-spaces are parallel, we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^n}\left[S_{(p,\zeta)}(x) -_\zeta \overline{L}(x)\right] &= \bigcap_{x\in\mathbb{R}^n}\left[S_{(p,\zeta)}(x) -_\zeta \overline{L}(x)\right]\\ &= \left[\sup_{x\in\mathbb{R}^n}(p\cdot x - L_\zeta(x))\right]\hat{z} + H^+(\zeta)\\ &= L^*_\zeta(p)\hat{z} + H^+(\zeta).\end{aligned}$$

This proves (33) and (34). □

In the assumption that L_ζ is C^2, coercive (hypothesis (28)) and strictly convex, some well-known properties of the Fenchel conjugate (see for example [3]) hold:

$$\begin{aligned}DL^*_\zeta(p_0) &= (DL_\zeta)^{-1}(p_0),\\ D^2L^*_\zeta(p_0) &= \left[D^2L_\zeta(DL^*_\zeta(p_0))\right]^{-1},\\ L^*_\zeta(p_0) &= p_0\cdot DL^*_\zeta(p_0) - L_\zeta(DL^*_\zeta(p_0)).\end{aligned} \tag{35}$$

In the next theorem the previous properties are extended to the set-valued case.

Theorem 5.1 *Given $\zeta \in B^+(\hat{z})$, suppose that L is of class C^2, satisfies (28) and $L_\zeta(p)$ is strictly convex. Then $\overline{L}^*(p,\zeta)$ is twice differentiable in p with respect to ζ and*

$$D_{\zeta,p}\overline{L}^*(p_0,\zeta)(p) = S_{((DL_\zeta)^{-1}(p_0),\zeta)}(p), \tag{36}$$

$$D^2_{\zeta,p}\overline{L}^*(p_0,\zeta)(p_1,p_2) = S_{\left(p_1^T[D^2L_\zeta(DL^*_\zeta(p_0))]^{-1},\zeta\right)}(p_2) \tag{37}$$

$$\overline{L}^*(p_0,\zeta) = S_{\left(DL^*_\zeta(p_0),\zeta\right)}(p_0) -_\zeta \overline{L}\left(DL^*_\zeta(p_0)\right), \tag{38}$$

where p_1^T is the transpose of the vector p_1.

Proof In order to calculate the first derivative of $\overline{L}^*(\cdot,\zeta)$ at $p_0 \in \mathbb{R}^n$ in the direction $p \in \mathbb{R}^n$ with respect to ζ, we must study the limit

$$\lim_{h\to 0^+}\frac{1}{h}\left[\overline{L}^*(p_0+hp,\zeta) -_\zeta \overline{L}^*(p_0,\zeta)\right].$$

Using the previous lemma, we obtain

$$\begin{aligned}&\frac{1}{h}\left[\overline{L}^*(p_0+hp,\zeta)-_\zeta \overline{L}^*(p_0,\zeta)\right]\\&\quad=\left\{z\in\mathbb{R}^d \mid \zeta\cdot z\geq \frac{1}{h}\left[\inf_{z_1\in\overline{L}^*(p_0+hp,\zeta)}\zeta\cdot z_1-\inf_{z_2\in\overline{L}^*(p_0,\zeta)}\zeta\cdot z_2\right]\right\}\\&\quad=\left\{z\in\mathbb{R}^d \mid \zeta\cdot z\geq \frac{1}{h}\left[L_\zeta^*(p_0+hp)-L_\zeta^*(p_0)\right]\right\}\end{aligned}$$

and it is possible to calculate the limit

$$D_{\zeta,p}\overline{L}^*(p_0,\zeta)(p)=\left\{z\in\mathbb{R}^d \mid \zeta\cdot z\geq DL_\zeta^*(p_0)\cdot p\right\}.$$

The first equation in (35) completes the proof of (36).

In order to study the second derivative, we calculate

$$\begin{aligned}&\frac{1}{h}\left[D_{\zeta,p}\overline{L}^*(p_0+hp_2,\zeta)(p_1)-_\zeta D_{\zeta,p}\overline{L}^*(p_0,\zeta)(p_1)\right]\\&\quad=\frac{1}{h}\left[S_{((DL_\zeta)^{-1}(p_0+hp_2),\zeta)}(p_1)-_\zeta S_{((DL_\zeta)^{-1}(p_0),\zeta)}(p_1)\right].\end{aligned}$$

Using the first equation in (35), we obtain

$$\begin{aligned}&\frac{1}{h}\left[S_{((DL_\zeta)^{-1}(p_0+hp_2),\zeta)}(p_1)-_\zeta S_{((DL_\zeta)^{-1}(p_0),\zeta)}(p_1)\right]\\&\quad=\frac{1}{h}\left[S_{(DL_\zeta^*(p_0+hp_2),\zeta)}(p_1)-_\zeta S_{(DL_\zeta^*(p_0),\zeta)}(p_1)\right]\\&\quad=\left\{z\in\mathbb{R}^d \mid \zeta\cdot z\geq \frac{1}{h}\left[DL_\zeta^*(p_0+hp_2)-DL_\zeta^*(p_0)\right]\cdot p_1\right\}\end{aligned}$$

and, taking the limit,

$$D^2_{\zeta,p}\overline{L}(p_0)(p_1,p_2)=\left\{z\in\mathbb{R}^d \mid \zeta\cdot z\geq p_1^T D^2L_\zeta^*(p_0)p_2\right\}$$

and this with the second equation in (35) completes the proof of (37).

By (33), we have

$$\overline{L}^*(p_0,\zeta)=S_{(1,\zeta)}(L_\zeta^*(p_0)).$$

Using the third property in (35), we may write

$$\begin{aligned}\overline{L}^*(p_0,\zeta)&=S_{(1,\zeta)}(p_0\cdot DL_\zeta^*(p_0)-L_\zeta(DL_\zeta^*(p_0)))\\&=S_{(DL_\zeta^*(p_0),\zeta)}(p_0)-_\zeta \overline{L}(DL_\zeta^*(p_0)).\end{aligned}$$

□

Remark 5.2 *If we apply the Fenchel conjugate twice*

$$\overline{L}^{**}(p,\zeta)=\sup_{q\in\mathbb{R}^n}[S_{(p,\zeta)}(q)-_\zeta \overline{L}^*(q,\zeta)]$$

for $\zeta\in B^+(\hat{z})$ and if $\overline{L}$ is convex, it is easy to see that

$$\bigcap_{\zeta\in B^+(\hat{z})}\overline{L}^{**}(p,\zeta)=\overline{L}(p).$$

See for example [10] for a generalization of the Fenchel-Moreau theorem.

6 The Scalarization Solution and The Characteristic Curves

Let $L : \mathbb{R}^n \to \mathbb{R}^d$ and $U_0 : \mathbb{R}^n \to \mathbb{R}^d$ be continuous functions, $\overline{L} : \mathbb{R}^n \to \mathcal{G}(\mathbb{R}^d, C)$ be a convex function and satisfy (28). The value function (27) of the minimization problem (26) has the property (9), as it is stated in the following lemma.

Lemma 2 *he value function* (27) *of the minimization problem* (26) *is such that for any* $\zeta \in B^+(\hat{z})$

$$U(t, x) + H^+(\zeta) = u_\zeta(t, x)\hat{z} + H^+(\zeta),$$

where

$$u_\zeta(t, x) = tL_\zeta\left(\frac{x - w_\zeta}{t}\right) + U_{0,\zeta}(w_\zeta),$$

for w_ζ *as in* (29).

Proof Recalling the Hopf-Lax formula, one has that

$$\begin{aligned} u_\zeta(t, x) &= \inf_{z \in U(t,x)} \zeta \cdot z = \inf_{w \in \mathbb{R}^n} \left[tL_\zeta\left(\frac{x - w}{t}\right) + U_{0,\zeta}(w) \right] \\ &= tL_\zeta\left(\frac{x - w_\zeta}{t}\right) + U_{0,\zeta}(w_\zeta). \end{aligned}$$

□

The value function was proved in [12] to satisfy a Hamilton-Jacobi equation and we report here the result:

Theorem 6.1 *Let* $(t, x) \in [0, +\infty) \times \mathbb{R}^n$, $\zeta \in C^+$, $\|\zeta\| = 1$. *Let* $\overline{L}$ *be convex,* (28) *be satisfied,* $U_{0,\zeta}$ *be Lipschitz on* $\mathbb{R}^n$ *and* L, U_0 *be of class* C^2. *If* w_ζ *is as in* (29), *let the sum of the hessian matrices*

$$\frac{1}{t}H_{L_\zeta}\left(\frac{x - w_\zeta}{t}\right) + H_{U_{0,\zeta}}(w_\zeta)$$

be non-singular.

Then the value function $U(t, x)$ *is a solution of the Hamilton-Jacobi equation*

$$U_{t,\zeta}(t, x) + \overline{L}^*(Du_\zeta(t, x), \zeta) = H^+(\zeta)$$

The following proposition shows the link between the scalarization solution of the calculus of variations problem (26) described in (32) and the characteristic curves for the corresponding Hamilton-Jacobi equation.

Proposition 6.2 *For any* $\zeta \in B^+(\hat{z})$ *suppose that* L *is* $C^2(\mathbb{R}^n; \mathbb{R}^d)$, *satisfies* (28) *and* $L_\zeta(p)$ *is strictly convex. Suppose also that the functions* $U_{0,\zeta}$ *are Lipschitz and of class* C^1. *The elements* y_ζ *of the set* M *defined in* (32) *are such that*

$$y_\zeta(s) = X_\zeta(s, w_\zeta), \tag{39}$$

with w_ζ *as in* (29).

This results shows that there is a scalarization solution which is formed by characteristic curves.

Finally, in the following theorem it is proved that the value function obtained through the Hopf-Lax formula in (30) and the solution obtained by the characteristic method of Theorem 3.3 coincide:

Theorem 6.3 *Let the hypotheses of Theorem 3.3 and Theorem 6.1 be fulfilled. Then the value function obtained through the Hopf-Lax formula* (30) *and the solution of the Hamilton-Jacobi equation obtained through the characterics method* (24) *coincide.*

Proof First, we apply the characteristic method when $\mathcal{H}(p,\zeta)=\overline{L}^*(p,\zeta)$, obtaining, since $\mathcal{H}_\zeta(p)=L_\zeta^*(p)$, that

$$X_\zeta(t,x)=x+tDL_\zeta^*(DU_{0,\zeta}(x))$$

and that $Z_\zeta(t,x)$ is such that

$$DL_\zeta^*(DU_{0,\zeta}(Z_\zeta(t,x)))=\frac{x-Z_\zeta(t,x)}{t}. \tag{40}$$

By definition (23) and using (38), we have

$$\begin{aligned}\mathcal{V}_\zeta(t,x)=&U_0(x)+t\Big[\Big(H^+(\zeta)-_\zeta\overline{L}^*(DU_{0,\zeta}(x),\zeta)\Big)\\&+D_{\zeta,p}\overline{L}^*(DU_{0,\zeta}(x),\zeta)(DU_{0,\zeta}(x))\Big]\\=&U_0(x)+t\Big[\Big(H^+(\zeta)-_\zeta\Big(S_{\left(DL_\zeta^*(DU_{0,\zeta}(x)),\zeta\right)}(DU_{0,\zeta}(x))\\&-_\zeta\overline{L}(DL_\zeta^*(DU_{0,\zeta}(x)))\Big)\Big)+D_{\zeta,p}\overline{L}^*(DU_{0,\zeta}(x),\zeta)(DU_{0,\zeta}(x))\Big].\end{aligned}$$

Since for any $A,B\in\mathcal{G}(\mathbb{R}^d,C)$, not both equal to $\mathbb{R}^d$ (the infimum in the direction ζ is not $-\infty$) and not both equal to $\emptyset$, the equation $H^+(\zeta)-_\zeta(A-_\zeta B)=B-_\zeta A$ holds, one obtains, using (35) and (36),

$$\begin{aligned}\mathcal{V}_\zeta(t,x)=&U_0(x)+t\Big[\Big(\overline{L}(DL_\zeta^*(DU_{0,\zeta}(x))-_\zeta S_{\left(DL_\zeta^*(DU_{0,\zeta}(x)),\zeta\right)}(DU_{0,\zeta}(x))\Big)\\&+D_{\zeta,p}\overline{L}^*(DU_{0,\zeta}(x),\zeta)(DU_{0,\zeta}(x))\Big]\\=&U_0(x)+t\Big[\Big(\overline{L}(DL_\zeta^*(DU_{0,\zeta}(x))-_\zeta D_{\zeta,p}\overline{L}^*(DU_{0,\zeta}(x),\zeta)(DU_{0,\zeta}(x))\Big)\\&+D_{\zeta,p}\overline{L}^*(DU_{0,\zeta}(x),\zeta)(DU_{0,\zeta}(x))\Big].\end{aligned}$$

For any $A,B\in\mathcal{G}(\mathbb{R}^d,C)$, with $\inf_{z\in B}\zeta\cdot z\neq\pm\infty$, there holds $(A-_\zeta B)+B=A+H^+(\zeta)$. This means that

$$\mathcal{V}_\zeta(t,x)=U_0(x)+t\left[\overline{L}(DL_\zeta^*(DU_{0,\zeta}(x))+H^+(\zeta)\right].$$

Using (40), we obtain for the characteristic solution

$$\begin{aligned}U_\zeta^{ch}(t,x)&=\mathcal{V}_\zeta(t,Z_\zeta(t,x))\\&=U_0(Z_\zeta(t,x))+t\left[\overline{L}(DL_\zeta^*(DU_{0,\zeta}(Z_\zeta(t,x)))+H^+(\zeta)\right]\\&=U_0(Z_\zeta(t,x))+t\left[\overline{L}\left(\frac{x-Z_\zeta(t,x)}{t}\right)+H^+(\zeta)\right]\end{aligned}$$

and

$$U^{ch}(t,x)=\sup_{\zeta\in B^+(\hat{z})}U_\zeta^{ch}(t,x).$$

Let $U(t,x)$ be as in Theorem 4.1. From (39) it derives that $x=X_\zeta(t,w_\zeta)$ and consequently that $w_\zeta=Z_\zeta(t,x)$. Now, if $z\in U(t,x)$, for any $\zeta\in B^+(\hat{z})$ the inequality

$\zeta \cdot z \geq t L_\zeta \left(\frac{x-w_\zeta}{t}\right) + U_{0,\zeta}(w_\zeta)$ implies that $z \in U^{ch}(t,x)$. Using a separation argument like in Theorem 4.1, it is possible to conclude that $U^{ch}(t,x) = U(t,x)$. □

7 Conclusions and Open Problems

In this paper the problem of the characteristics for set-valued Hamilton-Jacobi equations is studied. This problem comes from the minimization of a multiobjective Lagrangian, studied in [12]. The results that are presented here are in agreement with the solutions found in the other paper.

The set-valued approach allows to solve the Hamilton-Jacobi equations keeping all its complexity, while a scalarization approach forgets about the other dimensions. The results obtained describe very much the set-valued nature of the problem considered, without simplifying it.

In any case, the technique uses the scalarizations and this gives rise to the following interesting open problems. First, it is still not clear whether there is a different approach that permits to avoid hypothesis (22) and solve the cases like Example 3.8, where the situation is very simple in all the scalarizations, but the set-valued approach is not applicable. Secondly, what is the choice of Hamiltonian function and/or initial data that triggers the failure of hypothesis (22)? It may be important to use different kinds of scalarizations.

Funding Open access funding provided by Universitá degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

References

1. Aubin, J.P., Frankowska, H.: Set-Valued Analysis. Birkhäuser, Boston-Basel-Berlin (1990)
2. Aumann, R.J.: Integrals of set-valued functions. J Math Anal Appl **12**, 1–12 (1965)
3. Cannarsa, P., Sinestrari, C.: Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control. Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser (2004)
4. Carathéodory, C.: Calculus of Variations and Partial Differential Equations of the First Order. Teubner, Berlin (1935)
5. Christof, C., Müller, G.: Multiobjective optimal control of a non-smooth semilinear elliptic partial differential equation, ESAIM: Control. Optim Calc Var **27**, S13 (2021)
6. Désilles, A., Zidani, H.: Pareto front characterization for multiobjective optimal control problems using Hamilton–Jacobi approach, SIAM Journal on Control and Optimization **57**(6) (2019)
7. Evans, L.C.: Partial Differential Equations, Graduate Studies in Mathematics?: Providence, p. 19. American Mathematical Society, RI (2010)
8. Guigue, A.: Set-valued return function and generalized solutions for multiobjective optimal control problems (MOC). SIAM J Control Optim **51**(3), 2379–2405 (2013)
9. Gunantara, N.: A review of multi-objective optimization: Methods and its applications. Cogent Engineering **5**(1) (2018)

10. A.H. Hamel, A.H., Heyde, F., Löhne, A., Rudloff, B., Schrage, C.:Set optimization – A rather short introduction. In: Hamel, A.H, Heyde, F., Löhne, A., Rudloff, B., Schrage, C. (Eds.), Set Optimization and Applications – The State of the Art, 65–141(2015). Springer
11. Hamel, A.H., Schrage, C.: Directional derivatives, subdifferentials and optimality conditions for set-valued convex functions. Pac J Optim **10**(4), 667–689 (2014)
12. Hamel, A.H., Visetti, D.: The value functions approach and Hopf-Lax formula for multiobjective costs via set optimization. J Math Anal Appl **483**(1), 123605 (2020)
13. Lions, P.L.:Generalized Solutions of Hamilton-Jacobi Equations, Pitman Research Notes, Math.Ser. **69**, (1982). Pitman, Boston
14. Lovison, A., Cardin, F.: A Pareto-Pontryagin maximum principle for optimal control. Symmetry **14**, 1169 (2022)
15. Hoang, N.: Hopf-Lax formula and generalized characteristics. Appl Anal **96**(2), 261–277 (2017)
16. Subbotina, N.N.: The method of characteristics for Hamilton-Jacobi equations and applications to dynamical optimization. J Math Sci **135**, 2955–3091 (2006)