A note on the NLS equation on Cartan-Hadamard manifolds with unbounded and vanishing potentials

April 6, 2023 3:42pm +02:00

Abstract

We study the semilinear equation $-\Delta_g u + V(\sigma)u = f(u)$ on a Cartan-Hadamard manifold $\mathcal{M}$ of dimension $N \geq 3$, and we prove the existence of a nontrivial solution under suitable assumptions on the potential function $V \in C(\mathcal{M})$. In particular, the decay of V at infinity is allowed, with some restrictions related to the geometry of $\mathcal{M}$. We generalize some results proved in $\mathbb{R}^N$ by Alves *et al.*, see [1].

1 Introduction

The study of Nonlinear Schrödinger Equations of the form

$$-\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N \tag{1}$$

is a classical research topic in the theory of Partial Differential Equations, and the lack of compactness introduced by the non-compact space $\mathbb{R}^N$ is an obstruction to the application of basic tools of Nonlinear Analysis. For example, Variational Methods typically require the validity of some *compactness condition* for the so-called Palais-Smale sequences associated to the previous equation. At this point, suitable conditions on the nonlinearity f and on the potential function $V : \mathbb{R}^N \to \mathbb{R}$ must be imposed in order to find solutions.
The basic case in which V coincides with a positive constant was studied in [11], while P.H. Rabinowitz considered in [23] the case of coercive potentials. i.e.

$$0 < \inf_{x \in \mathbb{R}^N} V(x) < \liminf_{|x| \to +\infty} V(x).$$

While standard embedding theorems for weighted Sobolev spaces ensure the existence of solutions under the strong assumption

$$\lim_{|x| \to +\infty} V(x) = +\infty,$$

a milder condition was introduced in [9]: for every $M > 0$, the Lebesgue measure of the set

$$\left\{x \in \mathbb{R}^N \mid V(x) \leq M\right\}$$

must be finite. In all these cases, the potential V may be unbounded from above, but it has to be bounded away from zero on the whole $\mathbb{R}^N$. Mathematically, this implies that equation (1) can be successfully set in the Sobolev space $H^1(\mathbb{R}^N)$.
On the other hand, the presence of a vanishing potential V, in the sense that $\inf_{x\in\mathbb{R}^N} V(x) = 0$, introduces additional difficulties, starting from the fact that solutions need not lie in $L^2(\mathbb{R}^N)$. We refer to [3, 4, 10] for some recent existence results, which have been extended in several directions later. The interested readers can consult also [2], [8], [13], [22] and the reference therein for a complete summary on the already existing results. We point out that in all the cited papers the NLS equation (1) is set in the standard Euclidean space $\mathbb{R}^N$.

Although the Schrödinger equation in $\mathbb{R}^N$ has been extensively studied, there is a surprising lack of understanding when it comes to looking for solutions for the equation on non-Euclidean spaces such as Riemannian Manifolds. One of the first contributions in this direction is given in the papers [15] and [16] where the authors proved the existence of solutions for the Schrödinger equation or for the Schrödinger-Maxwell system requiring suitable bounds on the Ricci or sectional curvature. More recently, Appolloni, Molica Bisci and Secchi proved respectively in [6] and [7] the existence of three solutions for the Schrödinger equation on a manifold with asymptotically non-negative Ricci curvature with a coercive potential and the existence of infinitely many solutions on a Cartan-Hadamard manifold with a constant potential and an oscillatory nonlinearity. We also refer to [20, Part III, Chapter 8] for recent results about nonlinear equations on Cartan-Hadamard manifolds.
In this note we deal with a semilinear elliptic problem of the form

$$\begin{cases} -\Delta_g u + V(\sigma)u = f(u) & \text{on } \mathcal{M} \\ u > 0 & \text{on } \mathcal{M}, \end{cases} \tag{2}$$

where $\mathcal{M}$ is a *non-compact* manifold of dimension $N \geq 3$, V is a real-valued continuous potential function on $\mathcal{M}$, and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Inspired by the paper [1], we prove the existence of a positive solution under suitable assumptions, for which we refer to Section 2. To the best of our knowledge, our result is new in the framework of Cartan-Hadamard manifolds.

It should be noted that the Riemannian setting may differ considerably from the Euclidean one for several reasons. First of all, the behavior of the potential V *at infinity* requires a good replacement for the basic condition $\|x\| \to +\infty$ in $\mathbb{R}^N$. The Riemannian distance from a *fixed* point is the first attempt, but of course the condition at infinity must be independent of the choice of local charts. On the other hand, the topological approach which consists in saying that a sequence diverges to infinity if and only if it escapes every compact subset, may be too weak for a quantitative analysis. A good compromise is the use of *manifolds with a pole*, see [19] and the references therein. By definition, a Riemannian manifold has a pole o if and only if the exponential map at o induces a global diffeomorphism. This allows us to replace the Euclidean norm of $\mathbb{R}^N$ with the distance from the pole o, $d_g(\cdot, o)$.

But even under very specific assumptions on the Riemaniann metric, the geometry of the manifold puts in jeopardy some standard tricks of Nonlinear Analysis, like the use of cut-off functions. See Remark 6 below.

2 Quick survey of model manifolds and main result

We will work on a Cartan-Hadamard manifold $\mathcal{M}$ of dimension $N \geq 3$, i.e. a simply-connected complete non-compact manifold with non-positive sectional curvature. As is well know, the cut locus of any point of $\mathcal{M}$ is empty, hence $\mathcal{M}$ is a manifold with a pole. We collect some basic information about Riemannian geometry in our setting. We follow [21], and we refer to [17] for more details.

- We fix a pole $o \in \mathcal{M}$, which we will consider as the *origin*. For every $\sigma \in \mathcal{M}$, $\sigma \neq o$, we may define polar coordinates as follows: we let $r = d_g(\sigma, o) > 0$ and θ be an angle such that the shortest geodesic from o to σ starts with direction θ in the tangent space $T_\sigma\mathcal{M}$. Since $T_\sigma\mathcal{M}$ can be identified with $\mathbb{R}^N$, the angle θ may be seen as an element of the sphere $\mathbb{S}^{N-1}$.

- The Riemannian metric of $\mathcal{M}$ is expressed in polar coordinates as

$$g = \mathrm{d}r^2 + A_{ij}(r,\theta)\mathrm{d}\theta^i\,\mathrm{d}\theta^j$$

  for some positive-definite matrix $[A_{ij}]$. Here $(\theta^1, \ldots, \theta^{N-1})$ are local coordinates on $\mathbb{S}^{N-1}$.

- The Laplace-Beltrami operator is then written in the form

$$\Delta_g = \frac{\partial^2}{\partial r^2} + \mathcal{F}(r,\theta)\frac{\partial}{\partial r} + \Delta_{S_r},$$

  where

$$\mathcal{F}(r,\theta) = \frac{\partial}{\partial r}\log\sqrt{\det A_{ij}(r,\theta)},$$

  and Δ_{S_r} is the Laplace-Beltrami operator on the sphere $S_r = \partial B_r(o)$.

- A model manifold is a manifold with a pole such that the Riemannian metric has the form

$$g = \mathrm{d}r^2 + h(r)^2\,\mathrm{d}\theta^2,$$

  where $\mathrm{d}\theta^2 = \beta_{ij}\,\mathrm{d}\theta^i\,\mathrm{d}\theta^j$ is the standard metric on $\mathbb{S}^{N-1}$ and h is a smooth function such that $h(0) = 0$, $h'(0) = 1$ and $h(r) > 0$ for $r > 0$.

- On a model manifold the Laplace-Beltrami operator is expressed as

$$\Delta_g = \frac{\partial^2}{\partial r^2} + (N-1)\frac{h'(r)}{h(r)}\frac{\partial}{\partial r} + \frac{1}{h(r)^2}\Delta_{\mathbb{S}^{N-1}}. \tag{3}$$

- It follows from (3) that harmonic functions on a model manifold $\mathcal{M}$ endowed with the metric $g = dr^2 + h(r)\mathrm{d}\theta^2$ must satisfy the equation

$$\frac{\partial^2 w}{\partial r^2} + (N-1)\frac{h'(r)}{h(r)}\frac{\partial w}{\partial r} + \frac{1}{h(r)^2}\Delta_{\mathbb{S}^{N-1}} w = 0.$$

  If we look for *radially symmetric* harmonic functions $w = w(r)$, i.e. harmonic functions depending only on the variable $r = d_g(\sigma, o)$, the previous equation reduces to

$$\frac{\partial^2 w}{\partial r^2} + (N-1)\frac{h'(r)}{h(r)}\frac{\partial w}{\partial r} = 0.$$

  Under mild assumptions on the function h, this equation can be solved to find the two-parameter family of harmonic functions

$$w(r) = c_1 + c_2 \int_1^r \frac{\mathrm{d}t}{h(t)^{N-1}},$$

  for any $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}$. We will assume that

  (h) h is a positive smooth function such that the improper integral

$$\int_1^{+\infty} \frac{\mathrm{d}t}{h(t)^{N-1}}$$

  is finite.

- If (h) holds, we may introduce the function

$$\mathbf{H}(r) = \int_r^{+\infty} \frac{\mathrm{d}t}{h(t)^{N-1}},$$

  and it is easy to check that $\mathbf{H}$ is a positive harmonic function on $\mathcal{M}$ (possibly singular at $r = d_g(\sigma, o) = 0$, i.e. at the pole o) which satisfies

$$\lim_{d_g(\sigma,o)\to+\infty} \mathbf{H}(\sigma) = 0.$$

We can now state our main existence result for problem (2).

Theorem 1. *Let $\mathcal{M}$ be an N-dimensional model manifold endowed with the metric*

$$g = \mathrm{d}r^2 + h(r)\,\mathrm{d}\theta^2,$$

where h is a smooth positive function on $[0, +\infty)$ satisfying (h). Suppose that

(V_0) *$V(\sigma) \geq 0$ for every $\sigma \in \mathcal{M}$;*

(V_2) *there exists a constant $V_\infty > 0$ such that $V(\sigma) \leq V_\infty$ for every $\sigma \in B$;*

(f_1) $\limsup_{t\to 0^+} \frac{f(t)}{t^{2^*-1}} < +\infty$, *where* $2^* = 2N/(N-2)$;

(f_2) *there exists* $2 < p < 2^*$ *such that*

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{p-1}} = 0;$$

(f_3) *there exists* $\mu > 2$ *such that* $\mu F(t) \le t f(t)$ *for all* $t > 0$, *where* $F(t) := \int_0^t f(s)\, ds$.

Suppose moreover that either

(V_1) $\inf_{\mathcal{M}} V > 0$

or

(Vol) *The function*

$$r \mapsto \frac{\left(\int_r^{2r} h(t)^{N-1}\,\mathrm{d}t\right)^{1/N}}{r}$$

is bounded from above on $[0, +\infty)$.

Under these assumptions, there exist $\Lambda > 0$ *and* $R > 1$ *such that, if*

$$\mathbf{H}(r) = \int_r^{+\infty} \frac{\mathrm{d}t}{h(t)^{N-1}},$$

and

$$\mathbf{H}(R)^{\frac{4}{N-2}} \inf_{d_g(\sigma,o)\ge R} \frac{V(\sigma)}{\mathbf{H}(d_g(\sigma,o))^{\frac{4}{N-2}}} \ge \Lambda,$$

then (2) *possesses at least a nontrivial positive solution.*

Before proceeding to the proof of Theorem 1, we observe that V may be unbounded at infinity (though locally bounded by condition (V2)).

Remark 2. Since $p < 2^*$, assumptions (f_1) and (f_2) imply the existence of a constant $c_0 > 0$ such that

$$|sf(s)| \le c_0|s|^{2^*}, \quad |sf(s)| \le c_0|s|^p \tag{4}$$

for every $s \in \mathbb{R}$.

3 The auxiliary problem

Since we are looking for *positive* solutions to (2), we will suppose that $f(t) = 0$ for every $t < 0$. The main idea behind our approach is based on a suitable modification of the nonlinearity f, in such a way that the Palais-Smale condition can be recovered. To complete the proof, we need to check that the solution of the modified equation is actually a solution of equation (2). This technique goes back to the paper [14].

We introduce the Sobolev space[1] X defined as

$$X = \left\{ u \in D^{1,2}(\mathcal{M}) : \int_{\mathcal{M}} V|u|^2 \, \mathrm{d}v_g < +\infty \right\}.$$

Remark 3. Assumption (V_1) guarantees that X is continuously embedded into $L^2(\mathcal{M})$. In this case, X may be considered as a subspace of $H^1_0(\mathcal{M})$.

We introduce the functional $I\colon X \to \mathbb{R}$ as

$$I(u) := \frac{1}{2}\|u\|^2 - \int_{\mathcal{M}} F(u(\sigma)) \, \mathrm{d}v_g.$$

The functional I is of class C^1 on $\mathcal{M}$ as a standard consequence of assumptions (f_1)–(f_3). Moreover, critical points of I on $\mathcal{M}$ correspond to weak solutions of problem (2). Let us set

$$k := \frac{2\mu}{\mu - 2} > 2,$$

and consider a number $R > 1$. We define $\tilde{f}\colon \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{f}(\sigma, t) := \begin{cases} f(t) & \text{if } kf(t) \leq V(\sigma)t \\ \frac{V(\sigma)}{k}t & \text{if } kf(t) > V(\sigma)t, \end{cases}$$

and $g\colon \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ by

$$g(\sigma, t) := \begin{cases} f(t) & \text{if } d_g(\sigma, o) \leq R \\ \tilde{f}(\sigma, t) & \text{if } d_g(\sigma, o) > R. \end{cases}$$

We collect the main estimates for the auxiliary functions $\tilde{f}$ and g. The proof is standard and therefore omitted.

Lemma 4. *The following relations hold:*

- (i_1) $\tilde{f}(\sigma, t) \leq f(t)$ *for every* $\sigma \in \mathcal{M}$ *and* $t \in \mathbb{R}$*;*
- (i_2) $g(\sigma, t) \leq \frac{V(\sigma)}{k}t$ *for every* $\sigma \in \mathcal{M}$ *and* $t \in \mathbb{R}$ *such that* $d_g(\sigma, o) \geq R$*;*
- (i_3) $G(\sigma, t) = F(t)$ *for every* $\sigma \in \mathcal{M}$ *and* $t \in \mathbb{R}$ *such that* $d_g(\sigma, o) \leq R$*;*
- (i_4) $G(\sigma, t) \leq \frac{V(\sigma)}{2k}t^2$ *for every* $\sigma \in \mathcal{M}$ *and* $t \in \mathbb{R}$ *such that* $d_g(\sigma, o) > R$*.*

Here $G(\sigma, t) := \int_0^t g(\sigma, s) \, ds$*.*

[1]We refer the reader to [18] for a discussion of Sobolev spaces on Riemannian manifolds.

We can now introduce a functional $J \colon X \to \mathbb{R}$ such that

$$J(u) := \frac{1}{2}\|u\|^2 - \int_{\mathcal{M}} G(\sigma, u(\sigma))\, \mathrm{d}v_g.$$

It is immediate to check that $J \in C^1(\mathcal{M})$ and that the Gâteaux derivative of J is given by

$$J'(u)[v] = \int_{\mathcal{M}} (\nabla_g u \cdot \nabla_g v + V(\sigma)uv)\, \mathrm{d}v_g - \int_{\mathcal{M}} g(\sigma, u)v\, \mathrm{d}v_g.$$

Therefore, critical points of J correspond to weak solutions of the equation

$$-\Delta_g u + V(\sigma)u = g(\sigma, u) \quad \text{in } \mathcal{M}. \tag{5}$$

Let $I_0 \colon H_0^1(B) \to \mathbb{R}$ be the functional defined by

$$I_0(u) := \frac{1}{2}\int_B \left(|\nabla_g u|^2 + V_\infty |u|^2\right)\, \mathrm{d}v_g - \int_B F(u(\sigma))\, \mathrm{d}v_g.$$

We define the *mountain pass level* of I_0 as

$$d := \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} I_0(\gamma(t)),$$

where

$$\Gamma := \left\{\gamma \in C([0,1], H_0^1(B)) \mid \gamma(0) = 0,\ \gamma(1) = e\right\},$$

and $e \in H_0^1(B)$ is such that $I_0(e) < 0$. By (3) of Lemma 4 and (V_2) we deduce that

$$J(u) \leq I_0(u) \quad \text{for every } u \in H_0^1(B).$$

It follows immediately that

$$c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} J(\gamma(t)) \leq d$$

The functional J gains some topological strength from the modified nonlinearity g.

Proposition 5. *Suppose that either assumption (V_1) or assumption (Vol) holds. Then the functional J satisfies the Palais-Smale condition on X.*

Proof. Let $\{u_n\}_n$ be a Palais-Smale sequence for J in X, i.e. the sequence $\{J(u_n)\}_n$ is bounded and $J'(u_n) \to 0$ strongly in X^*. We compute first

$$J(u_n) - \frac{1}{\mu} J'(u_n)[u_n] \geq \frac{\mu - 2}{4\mu}\|u_n\|^2 = \frac{1}{2k}\|u_n\|^2. \tag{6}$$

The left-hand side of (6) is eventually bounded by $M + \|u_n\|$ for some constant M, and we conclude that

$$\|u_n\|^2 \leq 2k \left(M + \|u_n\|\right)$$

for $n \gg 1$. Thus the sequence $\{u_n\}_n$ is bounded in X. We may assume that (up to a subsequence) u_n converges weakly to some u in X. Fix $\varepsilon > 0$, and choose a number $r > R$ such that

$$\left(\int_{A(r,2r)} |u|^{2^*} \,\mathrm{d}v_g \right)^{1/2^*} < \varepsilon, \tag{7}$$

where we have set for $s \geq 0$, $t \geq 0$,

$$A(s,t) := \{\sigma \in \mathcal{M} \mid s \leq d_g(\sigma, o) \leq t\}$$

Recalling the discussion at the and of the proof of [24, Proposition 4.1], there exists a smooth cut-off function $\eta = \eta_r$ such that $\operatorname{supp} \eta \subset \mathcal{M} \setminus B_r(o)$, $\eta = 1$ on $\mathcal{M} \setminus B_{2r}(o)$, $0 \leq \eta \leq 1$ and $|\nabla_g \eta| \leq 2/r$ on $\mathcal{M}$. The boundedness of the sequence $\{\eta u_n\}_n$ yields

$$\int_{\mathcal{M}} \left(\nabla_g u_n \cdot \nabla_g (\eta u_n) + \eta V(\sigma) |u_n|^2 \right) \,\mathrm{d}v_g = \int_{\mathcal{M}} \eta g(\sigma, u_n) u_n \,\mathrm{d}v_g + o(1).$$

Combining with (2) of Lemma 4 we see that

$$\begin{aligned} &\int_{d_g(\sigma,o) \geq r} \eta \left(|\nabla_g u_n|^2 + V(\sigma) |u_n|^2 \right) \mathrm{d}v_g \\ &\qquad \leq \frac{1}{k} \int_{d_g(\sigma,o) \geq r} \eta V(\sigma) |u_n|^2 \,\mathrm{d}v_g - \int_{d_g(\sigma,o) \geq r} u_n \nabla_g u_n \cdot \nabla_g \eta \,\mathrm{d}v_g + o(1). \end{aligned}$$

As a consequence,

$$\begin{aligned} &\left(1 - \frac{1}{k}\right) \int_{d_g(\sigma,o) \geq r} \eta \left(|\nabla_g u_n|^2 + V(\sigma) |u_n|^2 \right) \mathrm{d}v_g \\ &\qquad \leq \frac{2}{r} \int_{A(r,2r)} |u_n| \, |\nabla_g u_n| \,\mathrm{d}v_g + o(1). \end{aligned}$$

On the bounded set $A(r, 2r) = B_{2r}(o) \setminus B_r(o)$ the Sobolev embedding theorem ensures that $u_n \to u$ in the sense of L^2; the Hölder inequality and the boundedness of $\{u_n\}_n$ yield

$$\limsup_{n \to +\infty} \int_{A(r,2r)} |u_n| \, |\nabla_g u_n| \,\mathrm{d}v_g \leq C \left(\int_{A(r,2r)} |u|^2 \,\mathrm{d}v_g \right)^{1/2}, \tag{8}$$

where $C > 0$ is a suitable constant. If assumption (V_1) holds, then $u \in L^2(\mathcal{M})$ and therefore

$$\lim_{r \to +\infty} \int_{A(r,2r)} |u|^2 \,\mathrm{d}v_g = 0.$$

On the other hand, if (Vol) holds, then

$$\left(\int_{A(r,2r)} |u|^2 \,\mathrm{d}v_g\right)^{1/2} \leq \left(\int_{A(r,2r)} |u|^{2^*} \,\mathrm{d}v_g\right)^{1/2^*} \mathrm{Vol}(A(r,2r))^{1/N}.$$

In order to estimate the last volume, we recall that

$$\mathrm{Vol}(B_\varrho(o)) = \omega_N \int_0^\varrho |h(s)|^{N-1} \, ds,$$

where ω_N is the $(N-1)$-volume of the sphere $\mathbb{S}^{N-1}$. We may finally write

$$\begin{aligned}
\limsup_{n\to+\infty} \int_{d_g(\sigma,o)\geq r} &|u_n| \, |\nabla_g u_n| \,\mathrm{d}v_g \\
&\leq 2C\omega_N^{1/N} \|u\| \left(\int_{A(r,2r)} |u|^{2^*} \,\mathrm{d}v_g\right)^{1/2^*} \frac{\left(\int_r^{2r} |h(s)|^{N-1} \, ds\right)^{1/N}}{r} \\
&\leq 2C\omega_N^{1/N} \|u\| \left(\int_{A(r,2r)} |u|^{2^*} \,\mathrm{d}v_g\right)^{1/2^*}
\end{aligned}$$

In any case we get from (7) that

$$\limsup_{n\to+\infty} \int_{\mathcal{M}\setminus B_r(o)} \left(|\nabla_g u_n|^2 + V(\sigma)|u_n|^2\right) \,\mathrm{d}v_g < C\varepsilon,$$

which in turn yields the convergence of $\{u_n\}_n$ to u. □

Remark 6. Before proceeding further, we would like to discuss the role of the assumptions (V_1) and (Vol). The analysis of Palais-Smale sequences $\{u_n\}_n$ for the modified function J shows that the compactness condition

$$\limsup_{r\to+\infty} \int_{d_g(x,o)\geq r} \left(|\nabla_g u_n|^2 + V(\sigma)|u_n|^2\right) \mathrm{d}v_g = 0$$

depends on two competing ingredients: the decay of the *gradient* of the cut-off function η with respect to r, and the growth of the function $r \mapsto \mathrm{Vol}\, A(r, 2r)$.
In the familiar setting $\mathcal{M} = \mathbb{R}^N$, $\mathrm{Vol}\, A(r, 2r) \lesssim r^N$, which exactly balances the decay $|\nabla\eta| \lesssim 1/r$ in integration. In our setting, however, the volume of the annulus $A(r, 2r)$ is determined by the growth of the function h, while the decay of η remains the same as in the Euclidean setting. From a technical viewpoint, assumption (V_1) states that every element of X lies in $L^2(\mathcal{M})$, so that we can directly bound the integral

$$\int_{d_g(x,o)\geq r} |u_n| \, |\nabla_g u_n| \,\mathrm{d}v_g$$

by the Cauchy-Schwarz inequality and the size of $\operatorname{Vol} A(r,2r)$ becomes irrelevant. Of course the potential V is no longer allowed to decay to zero at infinity.
On the other hand, assumption (Vol) allows V to decay at infinity, but clearly puts a rather strong restriction on the geometry of the manifold. The survey [12] contains several sufficient conditions for the Riemannian volume of the ball to grow like in the Euclidean case. In our setting we draw the reader's attention to the following result, see [12, Theorem G]; $\rho(\sigma)$ is defined to be the lowest eigenvalue of the Ricci tensor at σ.

Theorem 7. *Consider $\mathbb{R}^N$ endowed with the metric*

$$\mathrm{d}r^2 + h(r)^2\,\mathrm{d}\theta^2,$$

where h is smooth and satisfies $h(0)=0$, $h'(0)=1$. If for some $\lambda \geq (N-1)/4$ the Schrödinger operator $-\Delta + \lambda\rho$ is non-negative, then

$$\operatorname{Vol} B(0,R) \leq c(N,\lambda) R^N.$$

Once the Palais-Smale condition holds for the functional J, the Mountain Pass Theorem [5] guarantees the existence of a critical point $u \in X$ of the functional J such that $J(u) = c > 0$.

Remark 8. If u is a nontrivial critical point of J, it follows from the relation $c \leq d$ and the estimate

$$\|u\|^2 \leq 2k\left(J(u) + J'(u)[u]\right)$$

coming from (6) that $\|u\| \leq \sqrt{2kd}$, and this upper bound does not depend on the parameter $R > 1$.

The next regularity result is based on an argument reminiscent of the De Giorgi-Nash-Moser iteration technique.

Proposition 9. *Suppose $a \in L^q(\mathcal{M})$ for some $q > N/2$ and that $v \in X$ is a weak solution of the equation*

$$-\Delta_g v + b(\sigma)v = A(\sigma, v) \quad \text{in } \mathcal{M},$$

where $A \colon \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

$$|A(\sigma,t)| \leq a(x)t \quad \text{for all } t > 0$$

and $b \geq 0$ is a (measurable) function on $\mathcal{M}$. Then there exists a constant $M > 0$, depending on q and on $\|a\|_{L^q}$ only, such that

$$\|v\|_{L^\infty} \leq M\,\|v\|_{L^{2^*}}.$$

Proof. Since the proof is standard, we only give an outline and we refer to [1] for further details. Fix $\beta > 1$ and $m \in \mathbb{N}$. Set

$$w_m := \begin{cases} v|v|^{\beta-1} & \text{on } A_m \\ mv & \text{elsewhere} \end{cases} \quad \text{where} \quad A_m := \left\{\sigma \in \mathcal{M} \mid |v|^{\beta-1} \leq m\right\}.$$

Utilizing the definition of weak solution, the Sobolev inequality and the assumption on the potential it is possible to derive the estimate

$$\left[\int_{A_m} |w_m|^{2^*}\,\mathrm{d}v_g\right]^{\frac{N-2}{N}} \le S\beta^2 \int_{\mathcal{M}} a(\sigma) w_m^2\,\mathrm{d}v_g$$

where $S > 0$ denotes the optimal Sobolev constant. From this, applying the Hölder inequality with $1/q_1 + 1/q = 1$ and letting $m \to \infty$, we get

$$\|v\|_{2^*\beta} \le \beta^{\frac{1}{\beta}} \left(S\|a\|_q\right)^{\frac{1}{2\beta}} \|v\|_{2\beta q_1}. \tag{9}$$

At this point, since $N/(N-2) > q_1$, we set $\gamma := N/q_1(N-2) > 1$. If $\beta = \gamma$, (9) becomes

$$\|v\|_{2^*\gamma} \le \beta^{\frac{1}{\gamma}} \left(S\|a\|_q\right)^{\frac{1}{2\gamma}} \|v\|_{2^*}, \tag{10}$$

while if $\beta = \gamma^2$, taking into account $2\beta q_1 = 2^*\gamma$, we obtain

$$\|v\|_{2^*\gamma^2} \le \gamma^{\frac{2}{\gamma}} \left(S\|a\|_q\right)^{\frac{1}{2\gamma^2}} \|v\|_{2^*\gamma}. \tag{11}$$

Coupling (10) and (11), we get

$$\|v\|_{2^*\gamma^2} \le \gamma^{\frac{1}{\gamma}+\frac{2}{\gamma^2}} \left(S\|a\|_q\right)^{\frac{1}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma^2}\right)} \|v\|_{2^*}.$$

Iterating this procedure j times with $\beta = \gamma^j$, we have

$$\|v\|_{2^*\gamma^j} \le \gamma^{\frac{1}{\gamma}+\ldots+\frac{j}{\gamma^j}} \left(S\|a\|_q\right)^{\frac{1}{2}\left(\frac{1}{\gamma}+\ldots+\frac{1}{\gamma^j}\right)} \|v\|_{2^*}.$$

Now, recalling that

$$\sum_{j=1}^{\infty} \frac{j}{\gamma^j} = \frac{\gamma}{(\gamma-1)^2} \quad \text{and} \quad \sum_{j=1}^{\infty} \frac{1}{\gamma^j} = \frac{1}{\gamma-1}$$

and that

$$\lim_{j\to\infty} \|v\|_{2^*\gamma^j} = \|v\|_\infty,$$

the proposition is proved selecting

$$M = \gamma^{\frac{\gamma}{(\gamma-1)^2}} \left(S\|a\|_q\right)^{\frac{1}{2}\frac{1}{\gamma-1}}.$$

The proof is now complete. □

Corollary 10. *Any positive ground state of* (5) *satisfies the estimate*

$$\|u\|_{L^\infty} \le M\sqrt{2Skd}, \tag{12}$$

where S is the best constant for the Sobolev embedding $D^{1,2}(\mathcal{M}) \subset L^{2^*}(\mathcal{M})$.

Proof. Indeed, we consider the functions

$$A(\sigma,t) := \begin{cases} f(t) & \text{if } d_g(\sigma,o) < R \text{ or } f(t) \le \frac{V(\sigma)}{k}t, \\ 0 & \text{if } d_g(\sigma,o) \ge R \text{ or } f(t) > \frac{V(\sigma)}{k}t \end{cases}$$

and

$$b(\sigma) := \begin{cases} V(\sigma) & \text{if } d_g(\sigma,o) < R \text{ or } f(t) \le \frac{V(\sigma)}{k}t, \\ \left(1-\frac{1}{k}\right)V(\sigma) & \text{if } d_g(\sigma,o) \ge R \text{ or } f(t) > \frac{V(\sigma)}{k}t. \end{cases}$$

Any positive solution u to (5) satisfies the equation

$$-\Delta_g u + b(\sigma)u = A(\sigma,u) \quad \text{in } \mathcal{M}.$$

Our assumptions on f yield that $|A(\sigma,t)| \le f(t) \le c_0|t|^{p-1}$, hence

$$|A(\sigma,t)| \le a(\sigma)|t| \quad \text{with } a(\sigma) = c_0|u(\sigma)|^{p-2}.$$

For $q = 2^*/(p-2)$ it is immediate to check that $a \in L^q(\mathcal{M})$ and

$$\|a\|_{L^q} \le c_0\,(2Skd)^{\frac{p-2}{2}}.$$

The conclusion follows from Proposition 9. ☐

4 Proof of Theorem 1

Proposition 11. *If u is a positive ground state solution to (5), then*

$$u(\sigma) \le \frac{\mathbf{H}(d_g(\sigma,o))}{\mathbf{H}(R)}\|u\|_{L^\infty} \le \frac{\mathbf{H}(d_g(\sigma,o))}{\mathbf{H}(R)}M\sqrt{2Skd} \tag{13}$$

whenever $d_g(\sigma,o) \ge R$.

Proof. Indeed, we know that the function $\mathbf{H}$ is harmonic on $\mathcal{M}$, and so is the function

$$v\colon \sigma \mapsto M\sqrt{2Skd}\,\frac{\mathbf{H}(d_g(\sigma,o))}{\mathbf{H}(R)}.$$

Since $u \le v$ whenever $d_g(\sigma,o) \ge R$ by Corollary 10, we are allowed to define $\omega \in D^{1,2}(\mathcal{M})$ as

$$\omega(\sigma) := \begin{cases} (u-v)^+ & \text{if } d_g(\sigma,o) \ge R \\ 0 & \text{otherwise.} \end{cases}$$

We then see that

$$
\begin{aligned}
&\int_{\mathcal{M}} |\nabla_g \omega|^2 \, \mathrm{d}v_g \\
&\qquad = \int_{\mathcal{M}} \nabla_g (u - v) \cdot \nabla_g \omega \, \mathrm{d}v_g = \int_{d_g(\sigma,o) \geq R} (g(\sigma, u)\omega - V(\sigma) u\omega) \, \mathrm{d}v_g \\
&\qquad\qquad\qquad\qquad\qquad\qquad \leq \left(\frac{1}{k} - 1 \right) \int_{\mathcal{M}} V(\sigma) u\omega \, \mathrm{d}v_g \leq 0.
\end{aligned}
$$

It follows that $\omega = 0$ on $\mathcal{M}$, and $u \leq v$ whenever $d_g(\sigma, o) \geq R$. □

Proof of Theorem 1. Let $u \in X$ be a positive ground state solution of (5). For every $\sigma \in \mathcal{M}$ such that $d_g(\sigma, o) \geq R$, we have for any

$$
\Lambda > k c_0 M^{\frac{4}{N-2}} (2Skd)^{\frac{2}{N-2}},
$$

the estimate

$$
\begin{aligned}
\frac{f(u)}{u} &\leq c_0 |u|^{\frac{4}{N-2}} \leq c_0 M^{\frac{4}{N-2}} (2Skd)^{\frac{2}{N-2}} \left(\frac{\mathbf{H}(d_g(\sigma, o))}{\mathbf{H}(R)} \right)^{\frac{4}{N-2}} \\
&\leq \frac{V(\sigma)}{k}.
\end{aligned}
$$

It now follows that u solves (2), and the proof is complete. □

References

[1] Claudianor O. Alves and Marco A.S. Souto, *Existence of solutions for a class of elliptic equations in $\mathbb{R}^n$ with vanishing potentials*, Journal of Differential Equations **252** (2012), no. 10, 5555–5568.

[2] A. Ambrosetti, A. Malchiodi, and D. Ruiz, *Bound states of nonlinear Schrödinger equations with potentials vanishing at infinity*, J. Anal. Math. **98** (2006), 317–348. MR 2254489

[3] A. Ambrosetti and Z.-Q. Wang, *Nonlinear Schrödinger equations with vanishing and decaying potentials.*, Differ. Integral Equ. **18** (2005), no. 12, 1321–1332 (English).

[4] Antonio Ambrosetti, Veronica Felli, and Andrea Malchiodi, *Ground states of nonlinear Schrödinger equations with potentials vanishing at infinity*, Journal of the European Mathematical Society (2005), 117–144.

[5] Antonio Ambrosetti and Paul H Rabinowitz, *Dual variational methods in critical point theory and applications*, Journal of Functional Analysis **14** (1973), no. 4, 349–381.

[6] Luigi Appolloni, Giovanni Molica Bisci, and Simone Secchi, *Multiple solutions for Schrödinger equations on Riemannian manifolds via ∇-theorems*, Ann. Global Anal. Geom. **63** (2023), no. 1, Paper No. 11, 22. MR 4540772

[7] ———, *Schrödinger equation on Cartan-Hadamard manifolds with oscillating nonlinearities*, J. Math. Anal. Appl. **519** (2023), no. 2, Paper No. 126853, 20. MR 4511374

[8] Luigi Appolloni and Simone Secchi, *Normalized solutions for the fractional NLS with mass supercritical nonlinearity*, J. Differential Equations **286** (2021), 248–283. MR 4232664

[9] Thomas Bartsch and Zhi Qiang Wang, *Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^n$*, Communications in Partial Differential Equations **20** (1995), no. 9-10, 1725–1741.

[10] Vieri Benci, Carlo R. Grisanti, and Anna Maria Micheletti, *Existence and non existence of the ground state solution for the nonlinear Schrödinger equations with $V(\infty) = 0$*, Topological Methods in Nonlinear Analysis **26** (2005), no. 2, 203.

[11] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, I existence of a ground state*, Archive for Rational Mechanics and Analysis **82** (1983), no. 4, 313–345.

[12] Gilles Carron, *Euclidean volume growth for complete riemannian manifolds*, Milan Journal of Mathematics **88** (2020), no. 2, 455–478.

[13] Thierry Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003. MR 2002047

[14] Manuel A. del Pino and Patricio L. Felmer, *Local mountain passes for semilinear elliptic problems in unbounded domains*, Calc. Var. Partial Differ. Equ. **4** (1996), no. 2, 121–137 (English).

[15] Francesca Faraci and Csaba Farkas, *A characterization related to Schrödinger equations on Riemannian manifolds*, Commun. Contemp. Math. **21** (2019), no. 8, 1850060, 24. MR 4020749

[16] Csaba Farkas and Alexandru Kristály, *Schrödinger-Maxwell systems on non-compact Riemannian manifolds*, Nonlinear Anal. Real World Appl. **31** (2016), 473–491. MR 3490853

[17] Alexander Grigor'yan, *Analytic and geometric background of recurrence and non-explosion of the Brownian motion on Riemannian manifolds*, Bulletin of the American Mathematical Society **36** (1999), no. 2, 135–249.

[18] Emmanuel Hebey, *Nonlinear analysis on manifolds: Sobolev spaces and inequalities*, Courant Lect. Notes Math., vol. 5, Providence, RI: American Mathematical Society (AMS); New York, NY: Courant Institute of Mathematical Sciences, New York Univ., 2000 (English).

[19] Atsushi Kasue, *On Riemannian manifolds with a pole*, Osaka J. Math. **18** (1981), 109–113.

[20] Giovanni Molica Bisci and Patrizia Pucci, *Nonlinear problems with lack of compactness*, de Gruyter, 2021.

[21] Dario D Monticelli, Fabio Punzo, and Marco Squassina, *Nonexistence for hyperbolic problems on Riemannian manifolds*, Asymptotic Analysis **120** (2020), no. 1-2, 87–101.

[22] Markus Poppenberg, Klaus Schmitt, and Zhi-Qiang Wang, *On the existence of soliton solutions to quasilinear Schrödinger equations*, Calc. Var. Partial Differential Equations **14** (2002), no. 3, 329–344. MR 1899450

[23] Paul H. Rabinowitz, *On a class of nonlinear schrödinger equations*, ZAMP Zeitschrift für angewandte Mathematik und Physik **43** (1992), no. 2, 270–291.

[24] Mikhail Shubin, *Essential self-adjointness for semi-bounded magnetic Schrödinger operators on non-compact manifolds*, Journal of Functional Analysis **186** (2001), no. 1, 92–116.