On the rank of the flat unitary summand of the Hodge bundle

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Abstract

Let $f\colon S \to B$ be a non-isotrivial fibred surface. We prove that the genus g, the rank u_f of the unitary summand of the Hodge bundle $f_*\omega_f$ and the Clifford index c_f satisfy the inequality $u_f \leq g - c_f$. Moreover, we prove that if the general fibre is a plane curve of degree ≥ 5 then the stronger bound $u_f \leq g - c_f - 1$ holds. In particular, this provides a strengthening of the bounds of [BGAN18] and of [FNP18]. The strongholds of our arguments are the deformation techniques developed by the first author in [GA16] and by the third author and Pirola in [PT19], which display here naturally their power and depth.

1 Motivation and statement of the results

Let $f\colon S \to B$ be a fibred surface, i.e. a surjective morphism with connected fibres from a smooth projective surface S to a smooth projective curve B. Call g the arithmetic genus of the fibres, and suppose that $g \geq 2$. Let $q = h^0(S, \Omega^1_S)$ be the irregularity of the surface and $b = h^0(B, \omega_B)$ the genus of B. Let $\omega_f = \omega_S \otimes (f^*\omega_B)^\vee$ be the relative dualizing sheaf. The *Hodge bundle* of f is the rank g vector bundle $E_f = f_*\omega_f$ on B. The recent papers [CD17, CD16, CD14] have given new attention on the so-called Fujita decompositions of E_f (see Theorem 2.1)

$$E_f = \mathcal{O}_B^{\oplus q_f} \oplus \mathcal{F} = \mathcal{U} \oplus \mathcal{A}, \tag{1.1}$$

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where $q_f = q - b$ is the relative irregularity of f, $\mathcal{F}$ is nef with $h^1(B, \mathcal{F} \otimes \omega_B) = 0$, $\mathcal{U}$ is flat and unitary, and $\mathcal{A}$ is ample. We denote by u_f the rank of the unitary summand $\mathcal{U}$. Notice that $\mathcal{O}_B^{\oplus q_f} \subseteq \mathcal{U}$, hence $u_f \geq q_f$.

A natural issue arising in the study of the geometry of f is to find numerical relations between q_f and u_f and the other invariants (such as g, b, the self-intersection of the relative canonical divisor K_f^2, the relative Euler characteristic χ_f, the relative topological characteristic $e_f = 12\chi_f - K_f^2$).

In particular, the relations between the relative irregularity q_f and the genus g have been analyzed intensively since the first results and conjecture of Xiao in the '80's [Xia87b]. Let us summarize the main known results.

(Q1) (Beauville [Deb82, Appendix]) $q_f \leq g$, and equality holds if and only if f is trivial, i.e. S is birational to $B \times F$ and f corresponds to the first projection.

(Q2) (Serrano [Ser96]) If f is isotrivial (i.e. its smooth fibres are all mutually isomorphic) but not trivial, then $q_f \leq \frac{g+1}{2}$.

(Q3) (Xiao [Xia87b], [Xia87a]) If f is non-isotrivial and $B \cong \mathbb{P}^1$, then $q_f = q \leq \frac{g+1}{2}$. For non-isotrivial fibrations with arbitrary B the bound $q_f \leq \frac{5g+1}{6}$ holds.

(Q4) (Cai [Cai98]) If f is non-isotrivial and the general fibre is either hyperelliptic or bielliptic, the same bound $q_f \leq \frac{g+1}{2}$ holds.

(Q5) Pirola in [Pir92] constructs for non-isotrivial fibrations a higher Abel-Jacobi map whose vanishing implies $q_f \leq \frac{g+1}{2}$.

(Q6) (Barja-González-Naranjo [BGAN18]) If f is non-isotrivial, then $q_f \leq g - c_f$, where c_f is the Clifford index of the general fibre of f. Favale-Naranjo-Pirola [FNP18] have recently proven the stronger inequality $q_f \leq g - c_f - 1$ for families of plane curves of degree $d \geq 5$.

In (Q6) another important invariant enters the picture, namely the Clifford index of the general fibre c_f (see Section 3 for the definition). This quantity c_f, which is closely related to the gonality of the general fibres (see Section 3), has a deep and not yet completely understood influence on the invariants of fibred surfaces, see e.g. [BS08].

Xiao's original conjecture [Xia87b] that the first bound of (Q3) holds for any non-trivial fibration has been disproved by Pirola in [Pir92] (more counterexamples have been found later by Albano and Pirola in [AP16]) and was modified in [BGAN18] as follows.

Conjecture 1.1 (Modified Xiao's conjecture for the relative irregularity)**.** *For any non-isotrivial fibred surface $f\colon S \to B$ of genus $g \geq 2$ it holds*

$$q_f \leq \left\lceil \frac{g+1}{2} \right\rceil. \tag{1.2}$$

The results of [BGAN18] imply the conjecture in the (general) case of maximal Clifford index $c_f = \left\lfloor \frac{g-1}{2} \right\rfloor$. All counterexamples to the original conjecture found by Albano and Pirola satisfy equality for the modified one.

In this paper we address the same question for the unitary rank u_f: what are the relations between u_f and g? Let us discuss the possible counterparts of the results described above.

(U1) Clearly $u_f \leq \operatorname{rk} E_f = g$ and equality holds if and only if f is locally trivial (i.e. f is a holomorphic fibre bundle): this follows immediately observing that $u_f = g$ if and only if $\chi_f = \deg E_f = 0$.

(U2) In the case of isotrivial but non-trivial fibrations, there is no better bound for u_f, analgous to Serrano's bound for q_f. Indeed, it is easy to construct such fibred surfaces with $u_f = g$ as appropriate quotients of products of curves. We do not know wether isotrivial non-locally trivial fibrations satisfy some bound or u_f can be arbitrarily close to g.

(U3) If $B \cong \mathbb{P}^1$, then clearly the flat unitary summand is trivial, so $u_f = q_f = q \leq \frac{g+1}{2}$ for the non-isotrivial case. Moreover, as observed in [CLZ16, Lemma 3.3.1], Xiao's argument can be extended to bound the rank of the whole unitary summand, hence $u_f \leq \frac{5g+1}{6}$ for f non-isotrivial (the authors only consider semistable fibrations, but Xiao's proof works identically for any fibration).

(U4) Lu and Zuo in [LZ14, Theorem 4.7] prove that for a non-isotrivial hyperelliptic fibration there is a finite base change such that the flat summand of the new Hodge bundle is trivial, and moreover it is known that u_f is non-decreasing by base change (Remark 2.3), and so one can apply the bound for the relative irregularity proved by Cai and get the bound $u_f \leq \frac{g+1}{2}$.

(U5) Pirola's construction is specific to the case of the trivial part of E_f, it does not apply directly to the flat unitary summand.

In case (4) the bound follows by trivializing the unitary flat part by base change. However, the examples of Catanese and Dettweiler precisely show that this is not always possible: indeed they prove that the flat summand can be trivialized via a base change if and only if the image of the monodromy map associated to the unitary

summand is finite (see also [Bar00]). Catanese and Dettweiler provide examples where the monodromy image is not finite.

Let us remark that after [CD17] the modified Xiao conjecture implies:

Conjecture 1.2. *For any non-isotrivial fibred surface $f\colon S \to B$ of genus $g \geq 2$ such that the flat unitary summand has finite monodromy, it holds*

$$u_f \leq \left\lceil \frac{g+1}{2} \right\rceil. \tag{1.3}$$

Indeed, given a non-isotrivial fibred surface $f\colon S \to B$ of genus $g \geq 2$ such that $\mathcal{U}$ has finite monodromy, we can perform an (étale) base change $\pi\colon B' \to B$ obtaining a new fibred surface $f'\colon S' \to B'$ such that $\pi^*\mathcal{U}$ is trivial, hence in particular $u_f = \operatorname{rk} \pi^*\mathcal{U} \leq q_{f'}$. So modified Xiao for $q_{f'}$ implies equation (1.3).

In the case of infinite monodromy the bound (1.3) is false: this follows from Remark 38 of [CD16] and from the construction of Lu [Lu18]. In the first version of the paper this bound was stated as a conjecture. See Section 4.

Hence, we see a serious discrepancy between the behavior of the relative irregularuity and the one of unitary rank. It is therefore extremely interesting to see that the bounds of (Q6) for q_f can be extended to u_f. The main result of this paper answers positively to this question.

Theorem 1.3. *Let $f : S \to B$ be a non-isotrivial fibration of genus g, flat unitary rank u_f and Clifford index c_f. Then*

$$u_f \leq g - c_f. \tag{1.4}$$

Moreover, if the general fibre is a plane curve of degree $d \geq 5$, then

$$u_f \leq g - c_f - 1. \tag{1.5}$$

Remark 1.4. The bound (1.4) is sharp in case of general (maximal) Clifford index $c_f = \left\lfloor \frac{g-1}{2} \right\rfloor$, as proven in Section 4. What about the non-general case? Modified Xiao's conjecture would imply that the first bound of (Q.6) for the relative irregularity is not the optimal one. For the unitary rank, it is an open and extremely interesting question to understand the case of non general Clifford index. If one believes in modified Xiao's conjecture (Conjecture 1.2), extremal examples have to be searched for in the cases where the unitary summand has inifinite monodromy. Unfortunately, the infinite series of examples constructed in [CD16] (she so-called Standard Case, which are the same used in [Lu18]) all satisfy $c_f \leq 1$ (see Example 4.2).

Let us spend a couple of words about the proof of Theorem 1.3. The proof of the first inequality (1.4) follows the lines of the original argument of [BGAN18], but in this more general setting new results are needed. The key points of our arguments are the deformation thecnhniques developed by the first author in [GA16] and by the third author and Pirola in [PT19], together with an ad-hoc Castelnuovo-de Franchis theorem for tubular surfaces.

As in [BGAN18], we start by constructing a *supporting divisor* for the family (see Definitions 2.7 and 2.11), which is, rougly speaking, a divisor $\mathcal{D}$ on S whose restriction on the general fibre supports the first-order deformation induced by f. This divisor is constructed thanks to the results of the first named author in [GA16]. The proof of [BGAN18] then proceeds by treating separatedly the cases when $\mathcal{D}$ is relatively rigid or movable.

The main difference between the present case and [BGAN18] stems from the fact that sections of $\mathcal{U}$ do not correspond to global differential 1-forms on S, while sections of the trivial part do. Instead, after the recent work of Pirola and Torelli [PT19], local *flat* sections of $\mathcal{U}$ can be identified with *closed* holomorphic 1-forms on "tubes" $f^{-1}(\Delta) \subset S$ around smooth fibres (see Lemma 2.4). This local liftability of sections of $\mathcal{U}$ is enough to deal with the movable case.

To treat the rigid case, we develop a tubular version of the classical Castelnuovo-de Franchis theorem, that we believe to be interesting on its own.

Theorem 1.5. *Let $f : S \to \Delta$ be a family of smooth compact/projective curves over a disk. Let $\omega_1, \dots, \omega_k \in H^0(S, \Omega^1_S)$ ($k \geq 2$) be closed holomorphic 1-forms such that $\omega_i \wedge \omega_j = 0$ for every i, j, and whose restrictions to a general fibre F are linearly independent. Then there exist a projective curve C of genus $g(C) \geq k$ and a morphism $\phi : S \to C$ such that $\omega_i \in \phi^* H^0(C, \omega_C)$ for every i (possibly after shrinking Δ).*

This Theorem can be applied to conclude our argument, precisely because of the abovementioned closedness of 1-forms corresponding to sections of $\mathcal{U}$.

The inequality (1.5) for fibrations whose general fibres are plane curves uses the results of [FNP18]: as these regard infinitesimal deformations, the very same arguments as above apply to extend Favale-Naranjo-Pirola's inequality to hold for u_f.

In the last Section we discuss the known examples with large unitary summand.

We end the introduction by giving an application of our result to the framework of the Coleman-Oort conjecture on Shimura varieties in the Torelli locus. This is a straightforward application of the theory developed in [LZ14] and [CLZ16], just plugging in inequality (1.4), as in the proof of [CLZ16, Theorem 1.2.2]. We refer to this article for the definitions and notations. Here we just recall the following facts.

Let $\mathcal{M}_g$ be the moduli space of smooth projective curves of genus $g \geq 2$ and $\mathcal{A}_g$ the moduli space of g-dimensional principally polarized abelian varieties; consider both spaces with a full level ℓ structure, in order to be fine moduli spaces. Let $j^\circ \colon \mathcal{M}_g \to \mathcal{A}_g$ be the Torelli morphism. The Torelli locus $\mathcal{T}_g$ is the closure in $\mathcal{A}_g$ of the image of j°. Let $\mathcal{M}_g(c)$ be the locus of curves in $\mathcal{M}_g$ of Clifford index c. Call $\mathcal{T}_g^\circ(c)$ the image of $\mathcal{M}_g(c)$ in $\mathcal{A}_g$ via j°. Let $\mathcal{T}_g(c)$ be the closure of $\mathcal{T}_g^\circ(c)$. Recall that for maximal Clifford index $c = \left\lfloor \frac{g-1}{2} \right\rfloor$, $\mathcal{T}_g(c) = \mathcal{T}_g$. We say that a subvariety $Z \subset \mathcal{A}_g$ is contained generically in $\mathcal{T}_g(c)$ if $Z \subseteq \mathcal{T}_g(c)$ and $Z \cap \mathcal{T}_g^\circ(c) \neq \emptyset$.

Recall that, given $C \subset \mathcal{A}_g$ a smooth closed curve, the canonical Higgs bundle $\mathcal{E}_C$ on C is the Hodge bundle given by the universal family of abelian varieties restricted over C. The bundle $\mathcal{E}_C$ decomposes as $\mathcal{E}_C = \mathcal{F}_C \oplus \mathcal{U}_C$ where $\mathcal{U}_C$ is the maximal unitary Higgs subbundle corresponding to the maximal subrepresentation on which $\pi_1(C)$ acts through a compact unitary group. The bundles $\mathcal{E}_C$, $\mathcal{F}_C$ and $\mathcal{U}_C$ have an Hodge decomposition into a $(-1,0)$ and $(0,-1)$ part.

Corollary 1.6. *Let c be an integer between 1 and $\left\lfloor \frac{g-1}{2} \right\rfloor$. Let $C \subset \mathcal{A}_g$ be a curve with Higgs bundle decomposition $\mathcal{E}_C = \mathcal{F}_C \oplus \mathcal{U}_C$. If*

$$\mathrm{rk}\,\mathcal{U}_C^{-1,0} > g - c$$

then C is not contained generically in $\mathcal{T}_g(c)$.

Remark 1.7. Hence in particular if $\mathrm{rk}\,\mathcal{U}_C^{-1,0} > \left\lceil \frac{g+1}{2} \right\rceil$ then, if C is contained generically in the Torelli locus $\mathcal{T}_g$, the corresponding family of curves has not maximal Clifford index, which corresponds to not having general gonality.

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Basic assumptions and notation: Throughout the whole article, all varieties are assumed to be smooth and defined over $\mathbb{C}$. Unless otherwise is explicitly said, $f : S \to B$ will be a fibration (a surjective morphism with connected fibres) from a compact surface S to a compact curve B. The *genus* of f, defined as the genus of any smooth fibre, will be denoted by g, and assumed to be at least 2.

2 Some results on the Hodge bundle

2.1 Fujita decompositions of the Hodge bundle

Although we are primarily interested in the case of fibred surfaces, in this section we can assume that $f : X \to B$ is a fibration over a (smooth compact) curve B of a compact Kähler manifold X of arbitrary dimension $n+1 \geq 2$. We denote by $\omega_f = \omega_X \otimes (f^*\omega_B)^\vee$ the relative dualizing sheaf, and by $E_f = f_*\omega_f$ the *Hodge bundle* of f, which is a vector bundle on B of rank equal to the geometric genus of a general fibre of f. Fujita investigated the structure of E_f and obtained the following results. The complete proof of the second result is done by Catanese and Dettweiler in [CD17].

Theorem 2.1 (Fujita decompositions [Fuj78a, Fuj78b, CD17])**.** *Let $f : X \to B$ be a fibration as above and $E_f = f_*\omega_f$ be its Hodge bundle. Then E_f admits the following decompositions as sums of vector bundles*

$$E_f = \mathcal{O}_B^{\oplus h} \oplus \mathcal{F} = \mathcal{U} \oplus \mathcal{A}, \tag{2.1}$$

where $h = h^1(B, f_*\omega_X)$, $\mathcal{F}$ is nef with $h^1(B, \mathcal{F} \otimes \omega_B) = 0$, $\mathcal{U}$ is flat and unitary, and $\mathcal{A}$ is ample.

The decompositions (2.1) are the *first and second Fujita decompositions* of E_f. They are compatible, in the sense that $\mathcal{O}_B^{\oplus h} \subseteq \mathcal{U}$ and $\mathcal{A} \subseteq \mathcal{F}$. Moreover they can be combined into a finer decomposition

$$E_f = \mathcal{O}_B^{\oplus h} \oplus \mathcal{U}' \oplus \mathcal{A}, \tag{2.2}$$

where $\mathcal{U}'$ is flat unitary and has no sections, $\mathcal{U} = \mathcal{O}_B^{\oplus h} \oplus \mathcal{U}'$ and $\mathcal{F} = \mathcal{U}' \oplus \mathcal{A}$. In the case $X = S$ is a surface, $h = q_f = q(S) - g(B)$ is the *relative irregularity* of f.

Definition 2.2. *The* flat unitary summand *(of f) is the vector bundle $\mathcal{U}$ in the second Fujita decomposition. We denote by $\mathbb{U}$ the associated local system, and by $\nabla = \nabla_{\mathcal{U}} : \mathcal{U} \to \mathcal{U} \otimes \omega_B$ the corresponding connection. The* flat unitary rank *of f is $u_f := \mathrm{rk}\,\mathcal{U}$.*

Remark 2.3. It is worth observing that the unitary rank u_f is not necessarily constant under finite base change; however it is non decreasing. That is, if $\pi : B' \to B$ is a finite map, X' is a desingularization of the fibre product $X \times_B B'$ and $f' : X' \to B'$ is the induced fibration, then we have that $u_{f'} \geq u_f$. A similar phenomenon happens also for the relative irregularity. Indeed, the Hodge bundle itself is not stable under base change, but becomes "smaller": $E_{f'}$ is a subsheaf of π^*E_f and the quotient is a skyscraper sheaf supported on the π-branch points $b \in B$

such that $f^{-1}(b)$ is not semistable. Call $\mathcal{A}$ (resp. $\mathcal{A}'$) and $\mathcal{U}$ (resp. $\mathcal{U}'$) the ample and unitary summands of E_f (resp. $E_{f'}$); observe that the composite morphism $\mathcal{U}' \subseteq \pi^*\mathcal{U} \oplus \pi^*\mathcal{A} \to \pi^*\mathcal{U}$ is necessarily surjective (because the quotient $\pi^*E_f/E_{f'}$ is skyscraper), hence $u_{f'} = \operatorname{rk}\mathcal{U}' \geq \operatorname{rk}\pi^*\mathcal{U} = u_f$, as claimed. This fact makes it possible that somehow, using a suitable base change π, part of the ample summand of E_f becomes flat in $E_{f'}$, hence $u_{f'} > u_f$.

We now sketch the construction of the unitary summand $\mathcal{U}$. For more details, please consult [CD17]. Let $B^o \subseteq B$ denote the Zariski-open subset of non-critical values, and $f^o : f^{-1}(B^o) \to B^o$ the restriction of f to the smooth fibres. Over B^o, E_f can be interpreted as part of a variation of Hodge structures

$$H^0\left(f^{-1}(b), \Omega^n_{f^{-1}(b)}\right) \subset H^n\left(f^{-1}(b), \mathbb{C}\right).$$

Indeed, it is a holomorphic vector subbundle of $\mathcal{H} = R^n f^o_* \mathbb{C} \otimes \mathcal{O}_{B^o}$, which is equipped with the Gauß-Manin connection $\nabla_{\mathcal{H}}$ associated to the local system $\mathbb{H} = R^n f^o_* \mathbb{C}$. The basic idea is to define $\mathcal{U}_{|B^o}$ as the vector subbundle of E_f spanned by $\nabla_{\mathcal{H}}$-flat sections, which is flat and unitary with respect to the restriction of $\nabla_{\mathcal{H}}$. This construction can be extended to the whole B only if the monodromy of $\mathbb{H}$ around the singular fibres is unipotent. This holds if and only if the fibres are semistable (recalling that we are assuming the fibration to be relatively minimal with smooth total space). Hence, in the semistable case, the monodromy of $\mathcal{U}$ around the singular fibres becomes trivial (being unipotent and unitary). In case there is non-unipotent monodromy around a singular (non-semistable) fibre, $\mathcal{U}_{|B^o}$ can be a strictly smaller subbundle, being the subbundle spanned by $\nabla_{\mathcal{H}}$-flat sections that are also invariant by the local monodromy around the singular fibres. From this perspecive we also can see the fact that the unitary rank is non-decreasing under finite base change, as observed in Remark 2.3; moreover we can observe that in particular, choosing a π such that f' is semistable we maximize the unitary rank.

2.2 Flat sections of $f_*\omega_f$ and 1-forms on S

We will now more explicitly recall the interplay between the Hodge bundle and extensions of differential forms on the fibres in the case of a family of curves $f : S \to B$, i.e. a fibration of relative dimension $n = 1$ over a possibly non-compact base curve B. In order to simplify some statements we will assume that the family is semistable, which is enough to our purposes (see the end the previous section). Let g be the (arithmetic) genus of the fibres of f, and for any $b \in B$ we denote by $C_b = f^{-1}(b)$ the corresponding fibre (with the induced subscheme structure).

Consider now the sheaf of relative differentials $\Omega^1_{S/B}$ defined by the short exact sequence

$$0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \Omega^1_{S/B} \longrightarrow 0. \tag{2.3}$$

The restriction of (2.3) to any nodal C_b gives the Kodaira-Spencer class $\xi_b \in \mathrm{Ext}^1\left(\Omega^1_{C_b}, \mathcal{O}_{C_b}\right) \otimes T^\vee_{B,b}$ associated to the first-order infinitesimal deformation of C_b induced by f.

There is a natural map $\Omega^1_{S/B} \to \omega_f$, which is an isomorphism outside the singular fibres. Moreover, it is an injection if and only if every fibre of f is reduced, and in particular if f is semistable. In this case, $\Omega^1_{S/B}$ is torsion-free, and therefore also $f_*\Omega^1_{S/B}$ is torsion-free, hence a vector bundle on B. Furthermore, $f_*\Omega^1_{S/B}$ is a subsheaf of the Hodge bundle $E_f = f_*\omega_f$, and there is a connecting homomorphism

$$f_*\Omega^1_{S/B} \xrightarrow{\partial} R^1 f_* \mathcal{O}_S \otimes \omega_B$$

which at a regular value $b \in B$ is nothing but the cup product with the corresponding Kodaira-Spencer class $\cup\xi_b : H^0\left(C_b, \omega_{C_b}\right) \to H^1\left(C_b, \mathcal{O}_{C_b}\right) \otimes T^\vee_{B,b}$. Denote by $\mathcal{K}$ the kernel of ∂, which is a vector bundle on B sitting into the short exact sequence

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_S \longrightarrow \mathcal{K} \longrightarrow 0. \tag{2.4}$$

Sections of $\mathcal{K}$ correspond thus to families of forms on the fibres that locally glue to give a form on a "tubular" open neighbourhood of any fibre. Note that the pull-back of a form on B is trivial along the fibres, which also explains the exact sequence (2.4).

Suppose now that S and B are compact. Since over the regular values B^o, the Hodge bundle E_f coincides with $f_*\Omega^1_{S/B}$, it makes sense to compare $\mathcal{K}$ with the Fujita decompositions of E_f. Indeed, the trivial summand $\mathcal{O}_B^{\oplus q_f}$ of E_f corresponds to the holomorphic forms on the fibres that extend to the whole surface, that is, to $H^0\left(S, \Omega^1_S\right) / f^* H^0\left(B, \Omega^1_B\right)$. As for the flat unitary summand $\mathcal{U}$, we have the following result of Pirola and the third named author.

Lemma 2.4 (Lifting Lemma in [PT19])**.** *Let $f\colon S \to B$ be a semistable fibration, $\mathbb{U}$ the local system associated to the flat unitary summand, and $\Omega^1_{S,d}$ the sheaf of closed holomorphic 1-forms. Then there is a short exact sequence of sheaves (of abelian groups) on B*

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_{S,d} \longrightarrow \mathbb{U} \longrightarrow 0. \tag{2.5}$$

In particular, if $V \subset B$ is a disk, then there is an exact sequence of vector spaces:

$$0 \longrightarrow H^0\left(V, \omega_B\right) \longrightarrow H^0\left(f^{-1}\left(V\right), \Omega^1_{S,d}\right) \longrightarrow H^0\left(V, \mathbb{U}\right) \longrightarrow 0. \tag{2.6}$$

This means that the *flat* sections of $\mathcal{U}$ correspond to 1-forms on the fibres that extend to *closed* 1-forms on tubular neighbourhoods, and thus arbitrary sections of $\mathcal{U}$ over $V \subseteq B$ correspond to $\mathcal{O}_B\left(V\right)$-linear combinations of closed 1-forms on $f^{-1}\left(V\right)$. Note that here there is a crucial difference between the local system $\mathbb{U}$ and the vector bundle $\mathcal{U}$.

Then one gets immediately the following:

Lemma 2.5 ([PT19])**.** *Under the above assumptions, there is an injection of sheaves* $i\colon \mathbb{U} \hookrightarrow \mathcal{K}$*, that sits in the following commutative diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \omega_B & \longrightarrow & f_*\Omega^1_{S,d} & \longrightarrow & \mathbb{U} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle i} & & \\ 0 & \longrightarrow & \omega_B & \longrightarrow & f_*\Omega^1_S & \longrightarrow & \mathcal{K} & \longrightarrow & 0 \end{array}$$

Remark 2.6. The inclusion of Lemma 2.5 is actually a quite natural fact. Indeed, over the regular vaules B^o, the connecting homomorphism ∂ coincides with the composition

$$E_{f^o} = f^o_*\omega_{f^o} \xrightarrow{\nabla_{\mathcal{H}}} \mathcal{H} \otimes \omega_{B^o} \longrightarrow (\mathcal{H}/E_{f^o}) \otimes \omega_{B^o} \cong R^1 f^o_* \mathcal{O}_{S^o} \otimes \omega_{B^o}$$

of the Gauß-Manin connection (restricted to E_{f^o}) and the natural projection. Thus the kernel of the first map (which spans $\mathcal{U}$) has to be contained in the kernel $\mathcal{K}$ of ∂. This inclusion can very well be strict, as shown in the recent work [GAT18] of the first and third named authors.

2.3 Supporting divisors

We recall now some definitions and results concerning infinitesimal deformations of curves already introduced in [CP95, GA16, BGAN18]. Let C be a smooth curve and $\xi \in H^1(C, T_C)$ the Kodaira-Spencer class of a first-order infinitesimal deformation $C \hookrightarrow \mathcal{C}$, which corresponds to the exact sequence of vector bundles on C

$$0 \to N^\vee_{C/\mathcal{C}} \cong \mathcal{O}_C \to \Omega^1_{\mathcal{C}|C} \to \omega_C \to 0.$$

Definition 2.7. *We say that* ξ *is* supported on *an effective divisor* D *if and only if*

$$\xi \in \ker\left(H^1(C, T_C) \longrightarrow H^1(C, T_C(D))\right).$$

Furthermore, ξ *is* minimally *supported on* D *if it is not supported on any effective strict subdivisor* $D' < D$*.*

Equivalently, ξ is supported on D if and only if the subsheaf $\omega_C(-D)$ lifts to $\Omega^1_{\mathcal{C}|C}$, i.e. the inclusion $\omega_C(-D) \hookrightarrow \omega_C$ factors through $\Omega^1_{\mathcal{C}|C} \to \omega_C$.

Recall that the connecting homomorphism in the long exact sequence of cohomology associated to ξ is given by cup-product with ξ

$$\partial_\xi : H^0(C, \omega_C) \xrightarrow{\cup\xi} H^1(C, \mathcal{O}_C).$$

Definition 2.8. *The* rank *of ξ is defined as*

$$\operatorname{rk}\xi = \operatorname{rk}\partial_\xi = \operatorname{rk}(\cup\xi).$$

The rank of ξ is the most important numerical invariant for our purposes, and is related to supporting divisors by the following

Theorem 2.9 (Lemma 2.3 and Thm 2.4 in [BGAN18])**.** *Suppose ξ is supported on D. Then $H^0(C, \omega_C(-D)) \subseteq \ker\partial_\xi$. In particular,*

$$\operatorname{rk}\xi \leq \deg D - r(D).$$

If moreover ξ is minimally supported on D, then

$$\operatorname{rk}\xi \geq \deg D - 2r(D) = \operatorname{Cliff}(D).$$

Note that $H^1(C, T_C(D)) = 0$ if D is ample enough (e.g. $\deg D > 2g(C) - 2$), hence such D supports any deformation ξ. However, these divisors are uninteresting, since for example the inequalities provided by Theorem 2.9 are trivial. We are therefore interested in the existence of small supporting divisors, which can be constructed using the adjoint theorem (see [CP95]). In [GA16] the following numerical sufficient condition for the existence of supporting divisors is obtained.

Proposition 2.10 ([GA16] Corollary 3.1)**.** *If $\dim\ker(\partial_\xi) > \frac{g(C)+1}{2}$, then there is a two-dimensional subspace $W \subseteq \ker(\partial_\xi) \subset H^0(C, \omega_C)$ whose base divisor D supports ξ.*

We extend now the previous definitions to the case of one-dimensional families of curves as done in [GA16]. In order to simplify the exposition, we will assume that the family has no singular members, though the theroy can be developed in more general cases.

Let $f : S \to B$ be a family of smooth curves, i.e. a *smooth* proper surjective morphism with connected fibres from a (non-necessarily compact) surface S to a (non-necessarily compact) curve B. Denote by $C_b = f^{-1}(b)$ the fibre over a point $b \in B$, which is smooth of genus g, and by $\xi_b \in H^1(C_b, T_{C_b})$ the Kodaira-Spencer class of the first-order deformation of C_b induced by f (well-defined up to non-zero scalar multiplication).

Definition 2.11. *We say that an effective divisor $\mathcal{D} \subset S$ supports f if the restrictions $D_b = \mathcal{D}_{|C_b}$ support ξ_b for general $b \in B$.*

Note that Definition 2.11 only considers general fibres, hence we can always assume that a divisor $\mathcal{D}$ supporting a family f contains no fibre.

The following result is a simplification of some results in [GA16, Sec. 2], which is enough for the setting of this work.

Lemma 2.12. *Suppose that B is a disk, $f : S \to B$ has no singular fibres, and $\mathcal{D} \subset S$ is a divisor supporting f such that $\mathcal{D} \cdot F < 2g-2$ for any fibre F. Then the inclusion $\omega_f(-\mathcal{D}) \hookrightarrow \omega_f$ factors uniquely as*

$$\omega_f(-\mathcal{D}) \overset{\iota}{\hookrightarrow} \Omega^1_S \twoheadrightarrow \omega_f.$$

Proof. Note first that $\deg\left(\omega_f(-\mathcal{D})_{|F}\right) = 2g-2-\mathcal{D}\cdot F > 0$ for any fibre F. Hence $f_*\left(\omega_f(-\mathcal{D})^\vee\right) = 0$ and

$$\operatorname{Hom}\left(\omega_f(-\mathcal{D}), f^*\omega_B\right) = H^0\left(B, f_*\left(\omega_f(-\mathcal{D})\right)^\vee \otimes \omega_B\right) = 0.$$

The (left-exact) functor $\operatorname{Hom}\left(\omega_f(-\mathcal{D}), -\right)$ applied to the exact sequence

$$\xi: \quad 0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_S \longrightarrow \omega_f \longrightarrow 0$$

gives

$$0 \longrightarrow \operatorname{Hom}\left(\omega_f(-\mathcal{D}), \Omega^1_S\right) \longrightarrow \operatorname{Hom}\left(\omega_f(-\mathcal{D}), \omega_f\right) \xrightarrow{\mu} \operatorname{Ext}^1\left(\omega_f(-\mathcal{D}), f^*\omega_B\right).$$

The image of a morphism of sheaves $\phi : \omega_f(-\mathcal{D}) \to \omega_f$ by the map μ is $\phi^*\xi$, the class of the short exact sequence obtained from ξ by pull-back in the last term and completing the diagram. Thus the natural inclusion ι factors through Ω^1_S if and only if $\mu(\iota) = 0$.

By [GA16, Lemma 2.4], $\operatorname{Ext}^1\left(\omega_f(-\mathcal{D}), f^*\omega_B\right) \cong H^0(B, \mathcal{E})$, where $\mathcal{E} = \mathcal{E}\mathrm{xt}^1_f\left(\omega_f(-\mathcal{D}), \mathcal{O}_S\right) \otimes \omega_B$ is a vector bundle whose fibre over a point $b \in B$ is

$$\operatorname{Ext}^1\left(\omega_{F_b}\left(-\mathcal{D}_{|F_b}\right), T^\vee_{B,b}\right) \cong H^1\left(F_b, T_{F_b}\left(\mathcal{D}_{|F_b}\right)\right) \otimes T^\vee_{B,b}$$

(cf. [GA16, Lemma 2.3, Proposition 2.1 and Appendix]). Moreover, under this isomorphism $\mu(\iota)$ corresponds to a section σ of $\mathcal{E}$, whose value $\sigma(b)$ at any $b \in B$ is the class of the restriction to F_b of the pull-back $\iota^*\xi$. Since this pull-back and the restriction clearly conmute, $\sigma(b)$ coincides with the image of $\xi_b \in H^1(F_b, T_{F_b})$ in $H^1\left(F_b, T_{F_b}\left(\mathcal{D}_{|F_b}\right)\right)$, which is generically zero (hence identically zero, $\mathcal{E}$ being torsion-free) because the family f is supported on $\mathcal{D}$. This shows that $\mu(\iota) = 0$ and completes the proof. □

Note that, in particular, if $\mathcal{D}$ supports f, then we have the following injection

$$f_*\omega_f(-\mathcal{D}) \subseteq \mathcal{K} = \ker\left(f_*\omega_f \to R^1 f_*\mathcal{O}_S \otimes \omega_B\right).$$

Proposition 2.13. *Suppose* $\operatorname{rk}\mathcal{K} > \frac{g+1}{2}$. *Then there is an open disk $V \subseteq B$ and a divisor $\mathcal{D} \subset f^{-1}(V)$* minimally *supporting the restriction of f and such that $h^0\left(C_b, \omega_{C_b}\left(-\mathcal{D}_{|C_b}\right)\right) \geq 2$ and $H^0\left(C_b, \omega_{C_b}\left(-\mathcal{D}_{|C_b}\right)\right) \subseteq \mathcal{K}_b$ for any fibre C_b with $b \in V$.*

Proof. The proof is analogous to that of Theorem 3.3 and Corollary 3.2 in [GA16]. Roughly speaking, applying Proposition 2.10 pointwise we can locally (over any disk $V \subseteq B$) find a rank-2 vector subbundle $\mathcal{W} \subseteq \mathcal{K}$ such that the divisorial base locus $\mathcal{D}$ of the relative evaluation map

$$f^*\mathcal{W} \longrightarrow f^*f_*\omega_f \longrightarrow \omega_f$$

supports f. Up to shrinking V we may assume that $\mathcal{D}$ consists of disjoint sections of f over V. If $\mathcal{D}$ is not minimal supporting f, we can remove some of the components (or reduce their multiplicities) until obtaining a minimal one. □

2.4 A Castelnuovo-de Franchis theorem for tubular surfaces

We now give the proof of Theorem 1.5. Our argument follows the one of Beauville [Bea96, Proposition X.6]. Since all the 1-forms $\omega_1, \ldots, \omega_k$ are pointwise proportional, there are meromorphic functions $g_2, \ldots, g_k$ on S such that $\omega_i = g_i\omega_1$. Differentiating these equalities and using that $d\omega_i = 0$, we obtain $0 = dg_i \wedge \omega_1$ for $i = 2, \ldots, k$. In particular, there is another meromorphic function g_1 such that $\omega_1 = g_1 dg_2$, and also $0 = dg_1 \wedge dg_2$. This means that the meromorphic differentials $dg_2, \ldots, dg_k$ are pointwise proportional to ω_1 (wherever they are defined) and also to dg_1, hence $dg_i \wedge dg_j = 0$ for any i, j.

These functions define a meromorphic map $\psi : S \dashrightarrow \mathbb{P}^k$ as

$$\phi(p) = (1 : g_1(p) : g_2(p) : \ldots : g_k(p)) .$$

Let $\epsilon : \hat{S} \to S$ be a resolution of the indeterminacy locus of ψ, and let $\hat{\psi} : \hat{S} \to \mathbb{P}^k$ be the corresponding holomorphic map. Note that this resolution might not exist on the original surface due to the possible existence of infinitely many indeterminacy points. However, over a smaller disk there are only finitely many such points, and each of them can be resolved after finitely many blow-ups. If $(x_1, \ldots, x_k)$ are affine coordinates on $\{X_0 \neq 0\} \subset \mathbb{P}^k$, then by construction $\epsilon^*\omega_1 = \hat{\psi}^* (x_1 dx_2)$ and $\epsilon^*\omega_i = \hat{\psi}^* (x_1 x_i dx_2)$ for any $i \geq 2$. Furthermore $\hat{\psi}^* (dx_i \wedge dx_j) = dg_i \wedge dg_j = 0$, which implies that the image of $\hat{\psi}$ is locally an analytic curve $\hat{C} \subset \mathbb{P}^k$. Since $\hat{S}$ is smooth, $\hat{\psi}$ factors throught the normalization $\nu : C \to \hat{C}$, giving a holomorphic map $\hat{\phi} : \hat{S} \to C$.

The meromorphic 1-forms on C defined as $\alpha_1 = \nu^* (x_1 dx_2)$ and $\alpha_i = \nu^* (x_1 x_i dx_2)$ (for $i \geq 2$) verify that $\hat{\phi}^*\alpha_i = \epsilon^*\omega_i$ are holomorphic. A straightforward computation in local coordinates shows that

$$\operatorname{div}\left(\hat{\phi}^*\alpha_i\right) = \hat{\phi}^* \operatorname{div}(\alpha_i) + \sum (n_j - 1) E_j,$$

where the E_j are the irreducible components of the fibres of $\hat{\phi}$, and n_j is the corresponding multiplicity. Since the $\hat{\phi}^*\alpha_i$ are holomorphic, these divisors are effective,

but any contribution of a pole of α_i to E_j would be smaller or equal than $-n_j$. Therefore the α_i are holomorphic 1-forms on C.

To conclude, let $F \subset \hat{S}$ be a general fibre of $\epsilon \circ f$, and let $\pi : F \to C$ be the map induced by $\hat{\phi}$. Since the $\pi^*\alpha_i = (\epsilon^*\omega_i)_{|F}$ are linearly independent by hypothesis, in particular they are not zero and π is hence surjective. This implies that C is indeed a compact curve of genus $g(C) \geq k$.

Note that, *a fortiori*, the resolution of indeterminacy ϵ is not necessary, since every ϵ-exceptional divisor is contracted by $\hat{\phi}$ because $g(C) \geq k \geq 2$.

3 Bounding u_f: proof of Theorem 1.3

Recall that the *Clifford index* (cf. [Mar82]) of a smooth projective curve C is defined as

$$\mathrm{Cliff}(C) = \min\left\{\mathrm{Cliff}(D) = \deg D - 2r(D) \mid h^0(C, \mathcal{O}_C(D)), h^1(C, \mathcal{O}_C(D)) \geq 2\right\}.$$

The Clifford index of a curve is related to its gonality $\mathrm{gon}(C)$ (i.e. the minimum degree of surjective morphisms from C to $\mathbb{P}^1$) in the following way: $\mathrm{gon}(C) - 3 \leq \mathrm{Cliff}(C) \leq \mathrm{gon}(C) - 2$, and for a general k-gonal curve C it holds $k = \mathrm{gon}(C) = \mathrm{Cliff}(C) - 2$ (see [Bal86]). In particular, we can deduce that $0 \leq \mathrm{Cliff}(C) \leq \left\lfloor \frac{g-1}{2} \right\rfloor$, and the second is an equality for a general $C \in \mathcal{M}_g$.

Given a fibration $f : S \to B$, the *Clifford index* of f is c_f, the maximal Clifford index of the smooth fibres, which is attained over a Zariski-open subset of B.

First of all, since the flat unitary rank u_f does not decrease by finite base change, we can assume that the fibration is semistable by applying the semistable reduction. Consider now an open disk $V \subseteq B$ such that:

- V contains only regular values and the corresponding smooth fibres have Clifford index c_f, and
- the connecting homomorphism $\partial : f_*\Omega^1_{S/B} \to R^1 f_* \mathcal{O}_S \otimes \omega_B$ has constant rank on V.

From now on, let $f : S \to B$ denote only the restriction to $f^{-1}(V) \to V$. Thus f is a smooth fibration, $\Omega^1_{S/B} \cong \omega_f$, and $\mathcal{K} = \ker \partial$ is a vector bundle whose fibre K_b over any $b \in B$ is exactly $\ker(\cup\xi_b : H^0(C_b, \omega_{C_b}) \to H^1(C_b, \mathcal{O}_{C_b}))$.

3.1 Proof of inequality (1.4)

Suppose by contradiction that $u_f > g - c_f$. Since $c_f \leq \left\lfloor \frac{g-1}{2} \right\rfloor$, in particular $u_f > \frac{g+1}{2}$. Therefore, by Proposition 2.13 there is (up to shrinking B) a divisor $\mathcal{D} \subset S$

minimally supporting f such that

$$h^1\left(C_b, \mathcal{O}_{C_b}\left(\mathcal{D}_{|C_b}\right)\right) = h^0\left(C_b, \omega_{C_b}\left(-\mathcal{D}_{|C_b}\right)\right) \geq 2$$

for general $b \in B$.

As in the proof of [BGAN18, Theorem 1.2], we consider now two cases:

Case 1: The divisor $\mathcal{D}$ is relatively rigid, that is $h^0\left(C_b, \mathcal{O}_{C_b}\left(\mathcal{D}_{|C_b}\right)\right) = 1$ for a general $b \in B$. Then Theorem 2.9, together with Riemann-Roch, implies that

$$H^0\left(C_b, \omega_{C_b}\left(-\mathcal{D}_{|C_b}\right)\right) = K_b,$$

and hence $\mathcal{K} = f_*\omega_f\left(-\mathcal{D}\right)$.

Let now $\omega_f\left(-\mathcal{D}\right) \hookrightarrow \Omega^1_S$ be the map provided by Lemma 2.12, which gives a splitting $\mathcal{K} \hookrightarrow f_*\Omega^1_S$. Thus every section of $\mathcal{K}$ corresponds to a 1-form on S, and in particular every $w \in \ker \cup \xi_b \subseteq H^0\left(C_b, \omega_{C_b}\right)$ can be extended to S. Moreover, all these extensions are sections of the same line bundle $\omega_f\left(-\mathcal{D}\right)$, and therefore any two such extensions wedge to 0.

Let now $\eta_1, \ldots, \eta_{u_f}$ be a basis of flat sections of $\mathcal{U} \subseteq \mathcal{K}$, i.e. a basis of $H^0\left(V, \mathbb{U}\right)$, and let $\omega_1, \ldots, \omega_{u_f} \in H^0\left(S, \Omega^1_S\right)$ be the extensions provided by the splitting above. By the previous discussion, any two of these forms wedge to zero. We are thus in the situation of Theorem 1.5, which gives a new fibration $\phi : S \to C$ over a compact curve of genus $g\left(C\right) = u_f$. Let now C_b be any fibre of f, and $\pi : C_b \to C$ be the restriction of ϕ to the smooth fibre C_b. Applying Riemann-Hurwitz we obtain

$$2g - 2 \geq \deg \pi \left(2u_f - 2\right).$$

At the beginning of the proof we obtained that $u_f > \frac{g+1}{2}$, so that $2u_f - 2 > g - 1$, and thus

$$2(g-1) > \deg \pi \left(g - 1\right).$$

It follows that $\deg \pi = 1$, so every smooth fibre is isomorphic to C and hence f is isotrivial.

Case 2: The divisor $\mathcal{D}$ moves on any smooth fibre, i.e. $h^0\left(C_b, \mathcal{O}_{C_b}\left(\mathcal{D}_{|C_b}\right)\right) \geq 2$ for every regular value $b \in B$. We use Theorem 2.9 to obtain

$$\operatorname{rk} \xi_b \geq \operatorname{Cliff}\left(\mathcal{D}_{|C_b}\right) \geq c_f. \tag{3.1}$$

But $\mathcal{U}_b \subseteq \ker \partial_{\xi_b} = K_{\xi_b}$, so that $\operatorname{rk} \xi_b = g - \dim K_{\xi_b} \leq g - u_f$, and the inequality (3.1) implies that

$$g - u_f \geq c_f,$$

contradicting our very first hypothesis.

3.2 Proof of inequality (1.5)

Let us now assume that the general fibre of f is a plane curve of degree $d \geq 5$, and note that the Clifford index is thus $c_f = d - 4$. If C is a general fibre and $0 \neq \xi \in H^1(C, T_C)$ is the induced infinitesimal deformation, [FNP18][Theorem 1.3] gives directly $\operatorname{rk} \xi \geq d - 3 = c_f + 1$, and hence

$$\operatorname{rk} \mathcal{U} \leq \dim \ker \xi = g - \operatorname{rk} \xi \leq g - c_f - 1.$$

4 Examples

In this last section we analyze some examples satisfying the equality $u_f = \lceil \frac{g+1}{2} \rceil$. The examples of Pirola and Albano-Pirola satisfy $u_f = \lceil \frac{g+1}{2} \rceil$, and moreover they are extremal with respect to (1.2), as we verify in Example 4.1. The examples of Catanese and Dettweiler (in the standard case in the notations of [CD16]) satisfy the above equality, but are not extremal with respect to (1.4), as they are not of general gonality, as proved here: in fact the gonality is less than or equal to three. Note that these examples have $q_f = 0$.

It remains open to find examples with both c_f and u_f big, but q_f small. The first step is to investigate the non-standard examples of Catanese and Dettweiler [CD16].

Example 4.1 (cf. [AP16])**.** We first consider some non-isotrivial fibrations $f : S \to B$ constructed by Albano and Pirola in [AP16] with invariants $(q_f, g) = (4, 6), (6, 10)$. We will now show that these fibrations satisfy $q_f = u_f$.

The general fibres C_b of f are simultaneously double covers of a fixed curve D and étale cyclic covers (of prime order p) of hyperelliptic curves E_b with simple Jacobian variety. Moreover, the Jacobian of C_b is isogenous to $J(D) \times J(D) \times J(E_b)$, and the simplicity of $J(E_b)$ implies that the relative irregularity of f is $q_f = 2g(D)$ and does not grow under base change.

The curves E_b's form an hyperelliptic fibration $h : T \to B$ and (possibly after finite base change) there is a compatible étale cyclic cover $S \to T$ of order p. Thus the Hodge bundle of h is a direct summand of the Hodge bundle of f, and is complementary to the trivial summand $\mathcal{O}_B^{\oplus q_f} = \mathcal{O}_B^{\oplus 2g(D)}$ that corresponds to the constant factors $J(D) \times J(D)$ of the Jacobians (up to isogeny). Thus the non-trivial flat unitary summand $\mathcal{U}'$ of E_f (see equation (2.2)) coincides with the flat unitary summand of E_h.

Suppose that $u_f > q_f$, that is, $\operatorname{rk} \mathcal{U}' > 0$. Since h is an hyperelliptic fibration, by [LZ14, Theorem Theorem 4.7] the monodromy representation of $\pi_1(B)$ corresponding to $\mathcal{U}'$ has finite non-trivial image. But this would provide a finite étale covering

$\pi : \widetilde{B} \to B$ such that $\pi^*\mathcal{U}'$ is a trivial summand of the Hodge bundle of base-change of h, contradicting the simplicity of the Jacobian of the general fibres of h.

Example 4.2 (cf. [CD14] and [CD16])**.** Catanese and Dettweiler construct infinite families of non-isotrivial fibrations $f\colon S \to B$ with big values of u_f. We will now show that the Clifford index of these fibrations is $c_f = 1 < \left\lfloor \frac{g-1}{2} \right\rfloor$, hence it is not extremal for Theorem 1.3.

We only recall the (few) elements of the construction of the so-called standard case ([CD16, Definition 4.1]) needed for our purpose. Let n be any integer such that $GCD(n, 6) = 1$. The smooth fibres of f are cyclic coverings $\pi : C \to \mathbb{P}^1$ of degree n branched on 4 points. These curves can be realized as the normalizations of the plane curves with equation (we use the same notations as in [CD16]):

$$C_x : z_1^n - y_0 y_1 (y_1 - y_0)(y_1 - x y_0)^{n-3} = 0,$$

with $x \in \mathbb{C} \setminus \{0, 1\}$ and coordinates $(y_0 : y_1 : z_1) \in \mathbb{P}^2$. Now, we can just observe that the point $(1 : x : 0)$ has multiplicity $n - 3$ in the curve C_x, so by projecting from this point we get a 3 to 1 map to $\mathbb{P}^1$, hence a g^1_3 on the normalization of C_x.

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