Birational properties of tangent to the identity germs without non-degenerate singular directions

SAMUELE MONGODI AND MATTEO RUGGIERO

Abstract. We provide a family of isolated tangent to the identity germs $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ which possess only degenerate characteristic directions, and for which the lift of f to any modification (with suitable properties) has only degenerate characteristic directions. This is in sharp contrast with the situation in dimension 2, where any isolated tangent to the identity germ f admits a modification where the lift of f has a non-degenerate characteristic direction. We compare this situation with the resolution of singularities of the infinitesimal generator of f, showing that this phenomenon is not related to the non-existence of complex separatrices for vector fields of Gomez-Mont and Luengo. Finally, we describe the set of formal f-invariant curves, and the associated parabolic manifolds, using the techniques recently developed by López-Hernanz, Raissy, Ribón, Sanz Sánchez, Vivas.

Introduction

In this paper, we investigate birational properties of tangent to the identity germs in $\mathbb{C}^3$, in relation with the construction of (strong) separatrices and parabolic manifolds.

A holomorphic germ $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ is tangent to the identity when its differential at 0 is the identity. In the one-dimensional case, Leau-Fatou's flower theorem [Lea97a, Lea97b, Fat19] ensures the existence of simply connected invariant domains (petals) containing the origin at their boundary, where f is conjugated to a translation. Petals for f and f^{-1} cover a pointed neighborhood of the origin, and allow a precise description of the local dynamics of these germs. The properties of tangent to the identity germs and their petals are fundamental both in the global (see e.g. the monography [Mil06]) and in the local (see e.g. the topological classification of tangent to the identity germs [Cam78]) aspects of the theory of holomorphic dynamical systems in dimension 1, as well as for understanding bifurcations via parabolic implosion (see e.g. the survey [Shi00]).

In higher dimensions, it is not possible to give, in general, such a precise description of the dynamics near the origin, but one can still aim at describing higher dimensional analogues of the petals, called *parabolic manifolds*.

They are attached to complex tangent directions at 0, called *characteristic directions*. Characteristic directions can be described as either fixed points (*non-degenerate case*) or indeterminacy points (*degenerate case*) for the action induced by $f - \mathrm{id}$ on the exceptional divisor of the blow-up of the origin (or equivalently, of the action of the homogeneous part H of smallest degree of $f - \mathrm{id}$ on $\mathbb{P}^{d-1}$).

A fundamental result by Hakim [Hak98] shows the existence of parabolic curves tangent to non-degenerate characteristic directions (in any dimension).

Later, Abate [Aba01] shows the existence of parabolic curves for isolated tangent to the identity germs in dimension $d = 2$. In analogy with Camacho-Sad's construction of complex separatrices for holomorphic foliations in dimension 2 [CS82], the proof consists in showing that, after a finite number of blow-ups along characteristic (and in fact singular, see Definition 1.2) directions, one can always find a regular modification (i.e., a composition of blow-ups) where the lift of f has at least one non-degenerate characteristic direction. This allows to apply Hakim's result to get a parabolic curve for the lifted germs, transversal to the exceptional divisor of the modification, so that it descends to a parabolic curve for f. Several authors addressed the problem of finding stable manifolds for (possibly non-isolated) 2-dimensional tangent to the identity germs, and the picture is quite complete

Date: December 10, 2022.
2010 *Mathematics Subject Classification.* 32H50, 37F99, 32S65.

now, see e.g. [Éca85, Wei98, Hak98, Aba01, ABT04, BMCLH08, Mol09, Viv12, Ron15, LHSS18, LHRRSS19, LHR20]. The description of parabolic manifolds has been recently instrumental for the construction of examples of wandering domains, see [ABD+16, ABTP21]. (Semi-)parabolic implosion in dimension 2 (or higher) and applications to bifurcation theory can also be found in the literature (see e.g. [BSU17, DL15, Bia19]), and mainly rely on a careful study of the dynamics on parabolic curves.

We briefly expose here some of the reasons why the study of tangent to the identity germs and their parabolic manifolds is much harder in higher dimensions. Firstly, the homogeneous part H introduced above acts on $\mathbb{P}^{d-1}$: for $d = 2$ all indeterminacy points can be avoided by saturation, while they persist when $d \geq 3$. Since 2-dimensional modifications are composition of point blow-ups, most of the phenomenon are combinatorial. In higher dimensions, we can blow-up higher dimensional centers, and their geometry needs to be taken into account. Moreover, we only have a weak factorization theorem (see [AKMW02, Bon02]). Resolution theorems for vector fields are available in dimension 2 (see [Sei68]), and recently dimension 3 (see [Pan06, MP13]): here we need in general to introduce singularities on the ambient space, by considering weighted blow-ups and orbifolds. Finally, the *infinitesimal generator* of a tangent to the identity germ may not admit complex separatrices when $d \geq 3$, as showed by Gomez-Mont and Luengo [GML92]. Adapting their construction to tangent to the identity germs, Abate and Tovena [AT03] give examples of tangent to the identity germs in dimension 3 that do not admit *robust* parabolic curves, i.e., parabolic curves attached to invariant formal curves, the analogue of (formal) complex separatrices in this setting. In their examples, all characteristic directions are non-degenerate, and (non-robust) parabolic curves exist thanks to Hakim's theorem.

In this paper, we investigate the existence of parabolic manifolds attached to degenerate characteristic directions in dimension 3, by studying the following family of tangent to the identity germs:

$$f(x, y, z) = (x + yz(y - z) + P, y + x(x^2 - z^2) + Q, z + xz(y - z) + R). \tag{1}$$

Here P, Q, R are holomorphic germs with order at least 4 at the origin. The coefficients of the formal power series expansion of P, Q, R are considered as parameters of the family. We say that a certain property holds for a generic element of the family if it holds for an open dense subset of the parameters with respect to the Zariski topology over $\mathbb{C}$.

Since characteristic directions are determined only by the homogeneous part of smallest degree of $f - \mathrm{id}$, all these maps share the same characteristic directions: there are five of them, which we label v_1, v_2, v_3, v_4, v_5, all of them degenerate. We denote by p_1, p_2, p_3, p_4, p_5 the corresponding points on the exceptional divisor of the blow-up of the origin. Other examples are easy to construct, building on the examples of rational maps in $\mathbb{P}^2$ with no (holomorphic) fixed points given by [Iva11].

For the maps described by (1), we investigate two possible strategies to find parabolic manifolds. The first strategy, following [Aba01], consists in looking for a suitable birational model, where we can find non-degenerate characteristic directions (that are non-exceptional, i.e., transverse to the exceptional divisor). Since non-degenerate characteristic directions correspond to eigenvectors of the linear part of the saturated infinitesimal generator $\hat{\chi}$ of f, it is natural to start our study from a birational model $\pi_0 : X_{\pi_0} \to (\mathbb{C}^3, 0)$, which provides a resolution of the singularities of the infinitesimal generator.

Our example shows that, unlike dimension 2, this first strategy may fail in higher dimensions.

Theorem A. *A generic element $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ of the family (1) satisfies the following property:*

> *For any regular modification $\pi : X \to (\mathbb{C}^3, 0)$ strongly adapted to f and dominating π_0, and for any point $p \in \pi^{-1}(0)$ in the exceptional divisor, the lift $\tilde{f} : X \to X$ of f at p has only degenerate characteristic directions.*

Here "regular" means that we only allow sequences of blow-ups of smooth centers, while "adapted to f" means that we only allow blow-up of centers that are invariants by the saturated infinitesimal generator $\hat{\chi}$, and with "strongly adapted" we only allow to blow-up points or curves belonging to the singular locus of $\hat{\chi}$.

We can actually say a little more about this family: one cannot find any non-degenerate characteristic direction also for any point modification (see Subsection 5.4.2), nor along the curves C_1 and C_2 (see below) for regular modifications (not necessarily strongly) adapted to f above the points p_1 and p_2 (see Subsection 5.4.1).

The second strategy is in line with the recent works [LHRRSS19, LHR20, LHRSSV]. It consists in looking for complex separatrices for the dynamics, and study parabolic manifolds attached to them. While we know that this second strategy may fail in general by [GML92, AT03], it proves quite fruitful in this case. We are able to find formal invariant curves tangent to the directions $v_1, \dots, v_4$, and deduce the existence of parabolic manifolds by [LHRSSV].

Theorem B. *For generic elements $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ of the family (1), there exists formal invariant curves $C_1, \dots, C_4$ tangent to $v_1, \dots, v_4$ respectively. These curves are smooth, and they are the only formal invariant curves tangent to any direction (but possibly v_5). Finally, there are 3 (resp., 5) parabolic manifolds asymptotic to C_1 and C_2 (resp., C_3 and C_4), of dimension either 1 or 2.*

For a generic choice of parameters, the parabolic manifolds asymptotic to C_1 and C_2 are of dimension 2, as well as either 2 or 3 out of the 5 asymptotic to C_3 and C_4, the others being of dimension 1 (this is a consequence of the computations done in the proof of Corollary 5.12).

The dynamics above p_5 remains more complicated to describe. We are able to exclude the existence of formal invariant curves that are transverse to the exceptional divisor of the model X_{π_0}, while we are able to find a formal invariant surface S tangent to v_5.

Besides being only formal, the surface S is also singular, and [LHR20] cannot be applied to $f|_S$ even if S were convergent. When working on the model X_{π_0}, the strict transform $\widetilde{S}$ of S is smooth and invariant by the lift $\widetilde{f}$ of f. A direct computation shows that $\widetilde{f}|_{\widetilde{S}}$ has only two characteristic directions, corresponding to the tangent space of the exceptional divisor $\pi_0^{-1}(0)$. In general $\pi_0^{-1}(0)$ could provide the only separatrices of $\widetilde{f}$, and constructing parabolic manifolds would require other techniques (similar to [LHR20]).

The techniques used to prove Theorem A are mainly combinatorial. In particular, we identify three new classes of tangent to the identity germs, namely *degenerate spikes*, *spinning corners* and *half corners*, and show that all singularities in a suitable model dominating X_{π_0} belong to one of these classes (or *simple corners* introduced in [AT03]). Then we show that these classes are invariant by (strongly) adapted regular modifications, and they do not admit non-degenerate non-exceptional characteristic directions.

To prove Theorem B, we use the combinatorial knowledge achieved in the previous step, and some computations using normal forms, to describe the set of formal invariant curves attached to the classes introduced above. Moreover, we compute the reduction to Ramis-Sibuya normal form, and apply the results in [LHRSSV] to deduce the existence of parabolic manifolds attached to these formal invariant curves.

In both results, the genericity conditions are explicit and easy to check. They are not essential to the results: they are taken to simplify the birational study and the exposition of the dynamical properties of germs of the form (1).

Besides giving an explicit way to find formal invariant curves and parabolic manifolds in a non-trivial example, the identification of classes invariant by (adapted) modifications provide ideal candidates to replace the final reduced forms $\star 1$ and $\star 2$ of [Aba01]. The reduction to these classes would be a fundamental step towards proving in general the existence of parabolic manifolds in higher dimensions.

The paper is organized as follows. In Section 1, we recall some basics about tangent to the identity germs, vector fields, birational geometry and construction of formal curves, as well as the theory of Ramis-Sibuya normal forms and the construction of parabolic manifolds in the case of tangent to the identity germs.

In Section 2 we introduce the family of maps (1), study characteristic directions, and exhibit the resolution $\pi_0 : X_{\pi_0} \to (\mathbb{C}^3, 0)$ of the infinitesimal generator.

In Section 3 we recall the definition of simple corners, and introduce the three new classes. We then study their combinatorics in terms of point blow-ups.

In Section 4 we study the behaviour of these classes under regular modifications strongly adapted to the dynamics, and conclude the proof of Theorem A.

Finally, in Section 5 we use the combinatorial picture portrayed in the previous section to construct formal invariant curves, compute Ramis-Sibuya normal forms, and conclude the proof of Theorem B. We end this section by some remarks on not strongly adapted modifications, point modifications, and on the dynamical picture above p_5.

Acknowledgements. This project has been developed during a collaboration with Leandro Arosio. The authors thank Leandro for the insightful discussions. The interest for the phenomena described in this paper arose after a fruitful discussion with Filippo Bracci, whom we thank as well. The authors would also like to thank the anonymous referee for their useful comments that improved the presentation of this paper.

Part of this work has been developed while the first named author was visiting the University of Paris Diderot, supported by the LIA LYSM AMU CNRS ECM INdAM funding. The second named author has been partially supported by the ANR grant Fatou ANR-17-CE40-0002-01.

1. Background

1.1. Modifications

We start by some terminology about sequences of blow-ups.

Definition 1.1. A modification of $(\mathbb{C}^d, 0)$ is a proper bimeromorphic map $\pi : X_\pi \to (\mathbb{C}^d, 0)$ which is a biholomorphism outside the exceptional divisor $\pi^{-1}(0)$. A modification is called *smooth* if X_π is smooth, *regular* if X_π is obtained as a composition of blow-ups of smooth centers.

If $d = 2$, any smooth modification is obtained as a finite composition of point blow-ups. In general, building blocks of modifications are still given by blow-ups, whose centers have codimension at least 2. In particular for $d = 3$, we can blow-up both points and curves. The study of the birational geometry of tangent to the identity germs needs to take into account the geometry of such curves, and not only the combinatorial data of blow-ups.

Moreover, it is not anymore true that smooth modifications are given by composition of blow-ups (see [AKMW02, Bon02]), which gives a further technical difficulty to deal with generic modifications.

Most of the modifications we will consider will be *point modifications*, i.e., composition of point blow-ups, since they are more directly related to characteristic directions.

When doing so, we will perform local computations on suitable charts.

Let $\pi : X_\pi \to (\mathbb{C}^d, 0)$ be the blow-up of the origin, and fix local coordinates $(x_1, \dots, x_d)$ at $0 \in \mathbb{C}^d$. The total space X_π of the blow-up is covered by d charts U_j, with $j = 1, \dots, d$, corresponding to the complementary in X_π of the strict transform of the hyperplane $\{x_j = 0\}$. With abuse of notation, we denote by $(x_1, \dots, x_d)$ also the coordinates in U_j, for which the map π takes the form

$$\pi(x_1, \dots, x_d) = (x_1 x_j, \dots, x_{j-1} x_j, x_j, x_j x_{j+1}, \dots, x_j x_d).$$

In this case, we will say that we work *in the x_j-chart*. A point p corresponding to a direction $v = [a_1 : \ldots : a_d]$ belongs to U_j if and only if $a_j \neq 0$. If this is the case, p has coordinates $\left(\frac{a_1}{a_j}, \ldots, \frac{a_{j-1}}{a_j}, 0, \frac{a_{j+1}}{a_j}, \ldots, \frac{a_d}{a_j}\right)$ in the x_j-chart.

1.2. **Characteristic directions**

We introduce here some terminology about characteristic directions for tangent to the identity germs in $(\mathbb{C}^d, 0)$.

Definition 1.2. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ. Denote by H the homogeneous part of smallest degree of $f - \mathrm{id}$, by ℓ the greatest common divisor of the d coordinates of $f - \mathrm{id}$ (defined up to units), and let H_ℓ be the homogeneous part of smallest degree of $\ell^{-1}(f - \mathrm{id})$. A tangent direction $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$ is called

- *characteristic* if there exists $\lambda \in \mathbb{C}$ so that $H(v) = \lambda v$;
- *singular* if there exists $\lambda \in \mathbb{C}$ so that $H_\ell(v) = \lambda v$.

In both cases, v is called *non-degenerate* if $\lambda \neq 0$, and *degenerate* if $\lambda = 0$.

The degree of H is called the *order* of f, while the degree of H_ℓ is called the *pure order* of f.

Remark 1.3. The value λ in the previous definition is sometimes called *multiplier* of the characteristic direction. Notice that such value is not well defined up to change of coordinates, but its vanishing is.

Notice also that if v is a singular direction, then it is a characteristic direction. In fact, if L is the homogeneous part of ℓ of smallest degree, then $H = LH_\ell$ and $H(v) = L(v)H_\ell(v) = \lambda L(v)v$. We also infer that any characteristic direction that is tangent to $\{L = 0\}$ (or equivalently to $\{\ell = 0\}$) is automatically degenerate (as a characteristic direction).

Remark 1.4. Borrowing some terminology from algebraic geometry, one can see characteristic and singular directions as the same object.

Let X be a smooth manifold, $p \in X$, $f : (X, p) \to (X, p)$ be a tangent to the identity germ, and $D = \{\psi = 0\}$ be an effective divisor. Assume that its support is contained in $\mathrm{Fix}(f)$. Then locally at p we can write

$$f(x) = x + \psi(x) \cdot (H_\psi(x) + \text{h.o.t.}),$$

where h.o.t. stands for "higher other terms". In this situation, a *D-characteristic direction* (or a *singular direction with respect to D*) is an element $v \in \mathbb{P}(T_pX)$ such that $H_\psi(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. Characteristic directions are obtained when $D = 0$ (or equivalently $\psi = 1$), while singular directions are obtained when $\psi = \ell$ as above (in this case the support of $D = \mathrm{div}(\ell)$ is the pure $(d-1)$-dimensional part of $\mathrm{Fix}(f)$).

We need some terminology to describe the interaction between the exceptional divisor of a given modification and characteristic and singular directions.

Definition 1.5. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ, and $\pi : X_\pi \to (\mathbb{C}^d, 0)$ be a smooth modification. Denote by E the exceptional divisor of π, and let $f_\pi : X_\pi \dashrightarrow X_\pi$ be the lift of f in X_π. Let $p \in E$ be a point in the exceptional divisor so that the germ of f_π at p defines a tangent to the identity germ. We say that a characteristic direction of f_π is *exceptional* if it belongs to the projectivization of the tangent space of E at p.

In other terms, we can consider the blow-up of p, getting another modification $\pi' : X_{\pi'} \to (\mathbb{C}^d, 0)$ dominating π: $\pi' = \pi \circ \eta$ with η the blow-up at p. Then a characteristic direction v of f_π is exceptional if the corresponding point p_v in $\eta^{-1}(0)$ belongs to the strict transform of the exceptional divisor E. If v is a characteristic (resp., singular) direction for f_π, we will call the corresponding point p_v a *characteristic* (resp., *singular*) *point*.

Clearly, the set of singular points describe an algebraic subvariety of $\mathbb{CP}^{d-1}$. If the maximal dimension of the irreducible components of this subvariety is k, we say that the germ f is *k-dicritical.* Notice that 0-dicritical germs have only finitely many singular directions. When f is $(d-1)$-dicritical, the set of singular points coincide with $\mathbb{CP}^{d-1}$, and we simply say that f is *dicritical.*

Remark 1.6. Assume that $f:(\mathbb{C}^d,0)\to(\mathbb{C}^d,0)$ is a tangent to the identity germ, with an isolated fixed point. Let $\pi: X_\pi\to(\mathbb{C}^d,0)$ be any modification strongly adapted to f. Since π defines a local isomorphism outside the exceptional divisor, the lift f_π of f at X_π satisfies $\mathrm{Fix}(f_\pi)=\pi^{-1}(0)$. More generally if $\mathrm{Fix}(f)$ has no divisorial components, then $\mathrm{Fix}(f_\pi)$ has no divisorial components outside the exceptional divisor $E=\pi^{-1}(0)$.

In the families we will study in the next chapters, we will often consider exceptional the directions tangent to the divisor F of fixed points of f, since these families arise when studying the lift of the maps of (1) with respect to some modification.

1.3. Infinitesimal generators

To any tangent to the identity germ $f\colon(\mathbb{C}^d,0)\to(\mathbb{C}^d,0)$ is associated a unique (formal, possibly non-convergent) vector field χ, that has multiplicity at 0 at least 2, and satisfying $f=\exp\chi$ (see e.g. [BMCLH08] for the construction in dimension 2). We recall that if ϕ is a local coordinate (hence defining a germ of smooth hypersurface $\{\phi=0\}$) at 0, we have

$$\phi\circ\exp\chi=\sum_{n=0}^{\infty}\frac{\chi^n(\phi)}{n!},$$

where χ^n denotes the derivation χ applied n times. The vector field χ is called the *infinitesimal generator* of f, and denoted by $\chi=\log f$.

Remark 1.7. Notice that the homogeneous part of degree k of χ contributes to the factor $\chi^n(\phi)$ only starting from order $(n-1)(\mathrm{ord}_0\chi-1)+k$.

In particular, if f is given by the example (1), then $\phi\circ(f-\mathrm{id})$ and $\chi(\phi)$ coincide up to order 4.

Consider now a smooth manifold X, a compact (smooth) submanifold $Z\subset X$ of codimension at least 2, and $\pi: X_\pi\to X$ the blow-up of X along Z. If χ is a vector field on X, then χ lifts to a vector field χ_π on X_π, satisfying $\chi_q=(d\pi)_p(\chi_\pi)_p$ for any $q=f(p)$, $p\in X_\pi$, as far as χ is tangent to Z. When Z is reduced to a point $\{p\}$ this happens exactly when p is a singular point of χ.

Applying this situation to the infinitesimal generator χ of a tangent to the identity germ f, we get:

Proposition 1.8. *Let X be a smooth manifold, $Z\subset X$ be a compact smooth submanifold of codimension at least 2, and $\pi: X_\pi\to X$ the blow-up of X along Z. Let $f: X\to X$ be a holomorphic map fixing Z pointwise, and such that the germ of f at any point of Z is tangent to the identity; denote by χ the infinitesimal generator of f. Finally, denote by $f_\pi: X_\pi\to X_\pi$ the lift of f, and by χ_π the lift of χ.*

Then $f_\pi=\exp\chi_\pi$.

Proof. Assume for the moment that χ is analytic. Let Θ be the flow of χ, so that $f(z)=\Theta(z,1)$. Set $z=\pi(x)$ and $\Omega(x,t)=\pi^{-1}\circ\Theta(z,t)$ for any $x\notin E$. As Ω is analytic and bounded in a neighborhood of E, it extends holomorphically to E. Now, let us consider $x\in X_\pi\setminus E$. On the one hand, we get

$$\pi\circ f_\pi(x)=f(z)=\Theta(z,1)=\pi\circ\Omega(x,1)\ .$$

On the other hand, we get

$$\chi_{\Theta(z,t)}=\Theta'(z,t)=d\pi_{\pi^{-1}(\Theta(z,t))}\Omega'(x,t)=d\pi_{\Omega(x,t)}(\chi_\pi)_{\Omega(x,t)},$$

and Ω is the flow of χ_π. As this holds outside E and all the maps involved extend holomorphically to E, we obtain the desired result for χ analytic.

The result for χ formal follows, by applying the previous calculation to truncations, and by Remark 1.7. ∎

1.4. **Vector fields and characteristic directions**

In more abstract terms, Proposition 1.8 says that the operator associating to a tangent to the identity germ its infinitesimal generator is functorial (with respect to regular modifications adapted to f).

In order to explicit the link between characteristic/singular directions, and singularities of the infinitesimal generator, we need to introduce partial saturations.

Definition 1.9. Let X be a complex manifold, $Z \subset X$ a compact submanifold of X, and $f : (X, Z) \to (X, Z)$ a germ of holomorphic map fixing Z pointwise, and for which f is a tangent to the identity germ at any $p \in Z$. Let $\pi : X_\pi \to (X, Z)$ be a modification over Z, adapted to f. Denote by E the exceptional divisor of π, and by f_π the lift of f at X_π. Finally, let χ_π be the infinitesimal generator of f_π.

The *partial saturation* of χ_π with respect to $\pi^{-1}(D)$ at a point $q \in E$ is the vector field

$$(x_1^{h-1}\ell \circ \pi(z))^{-1}\chi_\pi,$$

where $D = \{\ell = 0\}$ locally at $p = \pi(q)$, $E = \{x_1 = 0\}$ locally at q and $h = \mathrm{ord}_p\big(\ell^{-1}(f - \mathrm{id})\big)$.

Remark 1.10. In the setting of Definition 1.9, denote by F the divisorial part of the fixed locus of f, and by F_π the divisorial part of the fixed locus of f_π the lift of f. We let $E = \pi^{-1}(0)$ be the exceptional divisor of π; we also set $D = F = \{\ell = 0\}$, and denote again by h the order at 0 of $\ell^{-1}(f - \mathrm{id})$.

When f is non-dicritical, then we have that $F_\pi = \pi^{-1}F + (h - 1)E$, and the partial saturation $\hat{\chi}_\pi$ of χ_π corresponds to the saturation of a vector field in the usual sense.

When f is dicritical, then we have that $F_\pi \geq \pi^{-1}F + hE > \pi^{-1}F + (h - 1)E$. In this case, in the partial saturation we only simplify the factor due to $\pi^{-1}F + (h-1)E$, and not the one due to F_π, as the saturation in the usual sense would require.

Notice that by Remark 1.7, the set of singular points of a tangent to the identity germ f coincides with the set of singular points of the (partially) saturated infinitesimal generator $\hat{\chi}_\pi$.

Proposition 1.11. *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ, and $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$ be a tangent direction at 0. Denote by $\pi : X_\pi \to (\mathbb{C}^d, 0)$ the blow-up of the origin, by f_π the lift of f at X_π, and by χ_π the infinitesimal generator of f_π. Let D be an effective divisor whose support is contained in* $\mathrm{Fix}(f)$*. Then v is a D-characteristic direction for f if and only if the partial saturation of χ_π with respect to $\pi^{-1}D$ is singular at the corresponding point $p_v \in \pi^{-1}(0)$.*

Proof. Fix coordinates $x = (x_1, \dots, x_d)$ on $(\mathbb{C}^d, 0)$ so that $v = [1 : 0 : \cdots : 0]$. Write $D = \{\ell = 0\}$. Then we can write f as:

$$f(x) = x + \ell(x)\big(H(x) + \mathfrak{m}^{h+1}\big),$$

where $\mathfrak{m}$ is the maximal ideal at 0, and $H := H_\ell$ is a non-vanishing homogeneous polynomial of degree $h \geq 0$. Then v is D-characteristic if and only if $H_\ell(v) = \lambda v$ for some $\lambda \in \mathbb{C}$. We work in the x_1-chart. The lift f_π of f satisfies

$$x_1 \circ f_\pi = x_1 + \ell \circ \pi(x)\Big(x_1^h H_1(1, x_2, \dots, x_d) + \langle x_1^{h+1}\rangle\Big),$$

$$x_j \circ f_\pi = \frac{x_j + \ell \circ \pi(x)\Big(x_1^{h-1} H_j(1, x_2, \dots, x_d) + \langle x_1^h\rangle\Big)}{1 + \ell \circ \pi(x)\Big(x_1^{h-1} H_1(1, x_2, \dots, x_d) + \langle x_1^h\rangle\Big)},$$

where $H = (H_1, \dots, H_d)$ and $j = 2, \dots, d$. Notice that $\mathrm{ord}_0(\ell) + h \geq 2$, hence f_π leaves $\pi^{-1}(0)$ fixed. By Proposition 1.8, the infinitesimal generator χ_π has the following form when developed near p_v (corresponding to the origin in the coordinates $(x_1, \dots, x_d)$):

$$\chi_\pi = \Big(x_1^{h-1}\ell \circ \pi(x)\Big)\left(\sum_{j=2}^{d}(H_j - x_j H_1)(1, x_2, \dots, x_d)\partial_j + x_1\xi\right).$$

where ξ is a suitable vector field. The partial saturation of χ_π with respect to $\pi^{-1}(D)$ is, by definition, given by $\hat{\chi}_\pi = \left(x_1^{h-1}\ell\circ\pi(x)\right)^{-1}\chi_\pi$, which coincides with

$$\sum_{j=2}^{d}(H_j - x_j H_1)(1, x_2, \ldots, x_d)\partial_j$$

on $\pi^{-1}(0) = \{x_1 = 0\}$.

Then, $v = [1 : 0 : \cdots : 0]$ is D-characteristic if and only if $H_j(1, 0, \ldots, 0) = 0$ for all $j = 2, \ldots, d$. But this happens if and only if $\hat{\chi}_\pi$ has a singularity at the origin. □

We extend the notion of singular points, using the interpretation in terms of saturated infinitesimal generator, for models not obtained as point modifications.

Definition 1.12. Let X be a complex manifold, $Z \subset X$ a compact submanifold of X, and $f : (X, Z) \to (X, Z)$ a germ of holomorphic map fixing Z pointwise, and for which f is a tangent to the identity germ at any $p \in Z$. Let $\pi : X_\pi \to (X, Z)$ be a modification over Z, adapted to f. Denote by f_π the lift of f at X_π, and by $\hat{\chi}_\pi$ its saturated infinitesimal generator (with respect to $\pi^{-1}(Z)$). Then we say that f_π is *singular* at $p \in \pi^{-1}(Z)$ if p is a singularity of $\hat{\chi}_\pi$.

1.5. **Resolution of singularities of vector fields**

In [Pan06], the author provides an algorithm to resolve singularities for analytic vector fields locally defined at the origin of $\mathbb{R}^3$. He shows that, up to a finite sequence of weighted blow-ups, any real analytic vector field can be assumed to have *elementary* singularities. Up to further blow-ups, one can get even better final normal forms, called *strongly elementary*.

In [MP13, Theorem p.281], these results have been transported to the complex-analytic case. In this case the singularities are classified, following the minimal model problem for algebraic varieties, according to positivity properties of the canonical bundle of the associated foliation. Elementary singularities are called here *log-canonical* (see [MP13, I.ii.1 Definition]). Again, a further improvement can be achieved, obtaining *canonical* singularities.

One of the major difficulties in this setting is that weighted blow-ups don't preserve the class of smooth manifolds: one has to consider some mild singularities, namely, cyclic quotients, which correspond to working with orbifolds.

When studying our example given by (1), we will only need smooth models (see Proposition 2.3). We recall here the definition of log-canonical singularities in this setting.

Definition 1.13. Let X be a smooth 3-fold, D a simple normal crossings (SNC) divisor on X, and χ a vector field locally defined at a point $p \in D$. Then χ is called *log-canonical* if its D-saturation is tangent to D, and either regular, or singular at p with a non-nilpotent linear part.

In general, when working with a cyclic quotient singularity (X, p), we can see it as the quotient of $(\mathbb{C}^3, 0)$ by the action of some finite group Γ. Then a log-canonical foliation on (X, p) is induced by a log-canonical Γ-invariant foliation on $(\mathbb{C}^3, 0)$ (see [MP13, I.ii.5 Fact/Definition]).

We also need to recall the definition of (isolated) canonical singularities, (see [MP13, III.i.2 Definition and III.i.3 Fact]).

Definition 1.14. Let X be a smooth 3-fold, D a SNC divisor on X, and χ a saturated vector field at X with an isolated singularity at $p \in D$. Then χ is called (D-)*radial* if it is tangent to D and its linear part has eigenvalues $(\lambda_1, \lambda_2, \lambda_3) \in \lambda(\mathbb{N}^*)^3$ for some $\lambda \neq 0$.

A vector field χ as above is called (D-)*canonical* if it is (D-)log-canonical, but not (D-)radial.

The reduction of singularities for vector fields can be then stated as follows.

Theorem 1.15 ([MP13, Theorem p.281])**.** *Let $(X, \mathcal{F})$ be a holomorphic foliation by curves on a 3-manifold X. Then there exists a sequence of weighted blow-ups $\pi : (X_\pi, D_\pi, \mathcal{F}_\pi) \to (X, \mathcal{F})$ so that $\mathcal{F}_\pi$ has only log-canonical singularities.*

Moreover, "log-canonical" in the previous statement can be replaced with "canonical" by [MP13, III.ii.2 Resolution]. In the present paper, both log-canonical and canonical singularities are considered (without further mention) with respect to the exceptional divisor D whose support is $\pi^{-1}(0)$.

1.6. **Parabolic manifolds**

Definition 1.16. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ. A *parabolic manifold* for f is a connected complex manifold $\Delta \subseteq \mathbb{C}^d$ of positive dimension such that

- $0 \in \partial\Delta$;
- Δ is f-invariant, and $f^n(z) \to 0$ for all $z \in \Delta$ as $n \to +\infty$, uniformly on compact subsets of Δ.

When moreover it has dimension 1 (resp., dimension d), it is called a *parabolic curve* (resp., *parabolic domain*).

Remark 1.17. Sometimes parabolic manifolds are also asked to be simply connected, and not simply connected parabolic manifolds are sometimes called stable manifolds. To avoid confusion with respect to the classical stable/unstable manifolds, we will stick with the terminology of "parabolic manifolds", and specify if they are simply connected if necessary.

Parabolic manifolds are often attached to complex directions, in the following sense.

Definition 1.18. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ. Denote by $[\cdot]$ the canonical projection from $\mathbb{C}^d \setminus \{0\}$ to $\mathbb{P}^{d-1}_{\mathbb{C}}$, and let $v \in \mathbb{P}^{d-1}_{\mathbb{C}}$ be a tangent direction at 0. Let p be a point in $\mathbb{C}^d$. We say that its orbit *converges to the origin tangent to v* if $f^n(p) \to 0$ and $[f^n(p)] \to v$ when $n \to +\infty$.

We say that a parabolic manifold Δ for f is tangent to v if the orbit of every point $p \in \Delta$ converges to the origin tangent to v.

Proposition 1.19 ([Hak98, Proposition 2.3])**.** *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ. If the orbit of a point converges to the origin tangent to a direction v then v is a characteristic direction.*

The following result is a geometric reformulation of [AT03, Proposition 3.1].

Proposition 1.20. *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ. Suppose that there exists an effective divisor D with simple normal crossings at 0 and supported in* $\operatorname{Fix} f$*, so that the D-saturated infinitesimal generator $\hat{\chi}$ of f is regular at 0, and tangent to D. Then no infinite orbit for f can stay arbitrarily close to 0.*

Proof. In what follows, $x^a = x_1^{a_1} \cdots x_d^{a_d}$. By our assumptions, we can find local coordinates at 0 so that $D = \{x^a = 0\}$ for some $a \in \mathbb{N}^d$, and

$$f(x) = (x + x^a g(x)).$$

Here $g : (\mathbb{C}^d, 0) \to \mathbb{C}^d$ is a holomorphic map, with homogeneous part of smallest degree denoted by G. The multiplication $x^a g(x)$ is meant as the product of a scalar x^a and a vector $g(x)$.

The saturated infinitesimal generator $\hat{\chi}$ is tangent to D if and only if $x_k | x_k \circ g$ for all k satisfying $a_k > 0$. It is regular if and only if there exists k so that $a_k = 0$ and $x_k \circ g(0) \neq 0$, where $x_k \circ g$ is the k-th coordinate of g.

The result follows from [AT03, Proposition 3.1]. □

Suppose we have a tangent to the identity germ $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$. We apply the previous proposition to the lift of f to the blow-up of 0, obtaining the following.

Corollary 1.21. *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ, and let D be a (possibly trivial) SNC divisor with support contained in* $\mathrm{Fix}(f)$*. Suppose that f is not dicritical, and the saturation of the infinitesimal generator $\hat{\chi}$ of f is tangent to D.*

If an orbit converges to 0 tangent to a (characteristic) direction v, then v is singular (with respect to D).

Proof. Let $\pi : X_\pi \to (\mathbb{C}^d, 0)$ be the blow-up of the origin, and let f_π be the lift of f on X_π. The condition on the non-dicriticity of f corresponds to the fact that the saturation $\hat{\chi}_\pi$ of the infinitesimal generator of f_π is tangent to the exceptional divisor $E = \pi^{-1}(0)$. Together with the hypothesis of tangency to D, we get that $\hat{\chi}_\pi$ is tangent to $\pi^{-1}(D) \cup E$.

Finally, a direction v is singular with respect to D if and only if $\hat{\chi}_\pi$ is singular at the associated point $p \in E$.

We conclude by Proposition 1.20. □

Corollary 1.21 can be restated in terms of point blow-ups. Under the same assumptions (and using the same notations as in the proof), if a orbit converges to a point $p \in \pi^{-1}(0)$, then p is a singular point for $\hat{\chi}_\pi$.

In general, Proposition 1.20 forces $\hat{\chi}_\pi$ to be either singular at p, or regular at p and transverse to the exceptional divisor. The latter case is excluded thanks to the non-dicriticity hypothesis on f.

When working with blow-up of curves, we lack the correspondence between characteristic directions of f and singular points of f_π, so we apply directly Proposition 1.20 in this case.

Verifying these conditions during the proof of Theorem A is straightforward and left to the reader.

1.7. Invariant curves and point modifications

Point modifications allow to study (formal) curves. We first introduce some terminology.

Definition 1.22. An *increasing sequence of infinitely near points* (above the origin) is a sequence $\mathfrak{p} = (p_n)_{n\in\mathbb{N}}$ of infinitely near points, which starts with $p_0 = 0 \in \mathbb{C}^d$ and satisfying the following property: for any $n \in \mathbb{N}$, p_{n+1} is a point in the exceptional divisor of the blow-up $\pi_n : X_{n+1} \to X_n$ of p_n (where $X_0 = \mathbb{C}^d$). We set $\hat{\pi}_n = \pi_0 \circ \ldots \circ \pi_{n-1} : X_n \to X_0$.

Proposition 1.23. *Let $\mathfrak{p} = (p_n)_n$ be an increasing sequence of infinitely near points. Suppose that for any n, p_n is a smooth point of $\hat{\pi}_n^{-1}(0)$, i.e., it belongs to $\pi_{n-1}^{-1}(p_{n-1})$ but not to the strict transform of $\hat{\pi}_{n-1}^{-1}(0)$.*

Then there exists a unique (possibly non-convergent) smooth curve $C = C_{\mathfrak{p}}$, with the property that the strict transform C_n of C with respect to $\hat{\pi}_n$ passes through p_n.

Proof. This can be done explicitly as follows. Without losing generality, we may assume that p_1 is the point associated to the direction $[a_1^{(1)} : \cdots : a_{d-1}^{(1)} : 1]$ for some $a^{(1)} = (a_1^{(1)}, \ldots, a_{d-1}^{(1)}) \in \mathbb{C}^{d-1}$. This allows us to make computations in the x_d-chart, and write $\pi_1(x_1, \ldots, x_d) = (x_1 x_d, \ldots, x_{d-1} x_d, x_d)$. We now take the local coordinates $(x_1 - a_1^{(1)}, \ldots, x_{d-1} - a_{d-1}^{(1)})$ at p_1. The smoothness hypothesis ensures that p_2 is associated to a point of the form $[a^{(2)} : 1]$ with $a^{(2)} \in \mathbb{C}^{d-1}$. By induction we obtain that p_n is associated to a point of the form $(a^{(n)} : 1)$ for some $a^{(n)} \in \mathbb{C}^{d-1}$, when all computations for π_n are made in the x_d-chart (after having translated coordinates as shown above).

We consider the curve C, parametrized by $\big(x_1(t), \ldots, x_{d-1}(t), t\big)$, where

$$x_k(t) = \sum_{n=1}^{\infty} a_k^{(n)} t^n.$$

It is a simple computation to show that C satisfies the statement. Moreover, a curve C tangent to a vector of the form $(a : 1)$ is parametrized uniquely as $(x_1(t), \ldots, x_{d-1}(t), t)$ for some formal power series $x_k(t) \in \mathbb{C}[\![t]\!]$ for $k = 1, \ldots, d-1$, whose linear terms are uniquely determined by $a \in \mathbb{C}^{d-1}$, from which we infer the uniqueness of C. □

Remark 1.24. Notice also that if an increasing sequence of infinitely near points does not satisfy the condition of Proposition 1.23, at least starting from a certain n_0, then it does not identify a curve (not even singular). In fact, any truncation $(p_n)_{n\leq m}$ identifies a set $\mathcal{C}_m$ of curves tangent to them. If p_m is a singular point of $\hat{\pi}_m^{-1}(0)$, and p_{m+1} is a smooth point of $\hat{\pi}_{m+1}^{-1}(0)$, then the minimal multiplicity of the curves in $\mathcal{C}_{m+1}$ is strictly larger than the analogous quantity for $\mathcal{C}_m$. Since curves are desingularized by blowing-up points (for irreducible curves, the intersection of the strict transform of the curve with the exceptional divisor, see e.g. [CC05, Section 3.2]), and smooth curves are characterized by sequences of smooth infinitely near points, the condition in Proposition 1.23 is also necessary.

Given an irreducible curve C, we denote by $\mathfrak{p} = \mathfrak{p}(C)$ the increasing sequence of infinitely near points attached to it, starting with $p_0 = 0 \in \mathbb{C}^d$.

We want to apply Proposition 1.23 to increasing sequences of infinitely near points which are singular points for the lifts of a tangent to the identity germ.

Proposition 1.25. *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ, and $\mathfrak{p} = (p_n)_n$ be an increasing sequence of infinitely near points satisfying the hypothesis of Proposition 1.23. Let $C = C_{\mathfrak{p}}$ be the formal curve associated to $\mathfrak{p}$. If p_n are singular points for the lift of f on X_n for all $n \in \mathbb{N}$, then C is f-invariant.*

Proof. Denote by $f_n : X_n \dashrightarrow X_n$ the lifts of f with respect to $\hat{\pi}_n$. Being p_n a singular point for f_n, we have in particular that $f_n(p_n) = p_n$. It follows that $f(C)$ is an irreducible curve whose strict transform with respect to $\hat{\pi}_n$ passes through p_n. By Proposition 1.23, this is exactly the curve $C_{\mathfrak{p}}$. □

The invariant curves constructed here are sometimes called *(strict) separatrices* for the tangent to the identity germ f (see [LHR20] for the analogous in dimension 2). They are in fact the analogous of separatrices for the (reduced) infinitesimal generator (see [BMCLH08]).

Remark 1.26. Notice that Proposition 1.23 and Proposition 1.25 do not hold if we replace point modifications with sequences of blow-ups of centers with positive dimension. The main reason is that curves are not anymore uniquely determined by the sequence of points of intersection of their strict transform with the exceptional divisor.

As an example, consider the blow-up of the line $\{x = z = 0\}$ in $\mathbb{C}^3$, and coordinates in the blown-up space so that the projection takes the form $\pi(x, y, z) = (xz, y, z)$. Reiterate the process, so to construct a sequence $\pi_n : X_n \to (\mathbb{C}^3, 0)$, where each element consists in the blow-up of n lines. In this case, for any curve C parametrized by $(0, y(z), z)$ (with $y \in z\mathbb{C}[\![z]\!]$ a formal power series with vanishing constant term), its strict transform C_n would intersect $\pi_n^{-1}(0)$ at the origin p_n of the corresponding chart. In particular, if the points p_n are singular for the lifts f of a tangent to the identity germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ we could only infer that $f(C)$ is another curve lying in the plane $\{x = 0\}$.

We conclude with a version of Corollary 1.21 for invariant curves, which gives a partial converse to Proposition 1.25.

Proposition 1.27. *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ, and let D be a (possibly trivial) SNC divisor with support contained in* $\mathrm{Fix}(f)$. *Suppose that f is not dicritical, and the saturation of the infinitesimal generator $\hat{\chi}$ of f is tangent to D.*

If C is a (formal) f-invariant curve, then C is tangent to a singular direction of f.

Proof. Let $\pi : X_\pi \to (\mathbb{C}^d, 0)$ be the blow-up of the origin, and let f_π be the lift of f on X_π. The condition on the non-dicriticity of f corresponds to the fact that the saturation $\hat{\chi}_\pi$ with respect to $\pi^{-1}(D)$ of the infinitesimal generator of f_π is tangent to the exceptional divisor $E = \pi^{-1}(0)$. But then invariant curves for $\hat{\chi}_\pi$ at non-singular points must be contained in E. □

1.8. Parabolic manifolds asymptotic to invariant curves

We have seen how orbits of points converging to the origin must be tangent to a characteristic direction. One could be interested in controlling higher orders of tangency. This corresponds to imposing conditions on lifts to other birational models. We need here some terminology to deal with these conditions, which are expressed in terms of asymptoticity to (formal invariant) curves (see [LHRRSS19, LHRSSV]).

Definition 1.28. Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ. Let $\mathfrak{p} = (p_n)$ be an increasing sequence of infinitely near points above the origin. Denote by $\hat{\pi}_n : X_n \to (\mathbb{C}^d, 0)$ the composition of the blow-ups of the points $p_0, \ldots, p_{n-1}$, and by $f_n : X_n \dashrightarrow X_n$ the lift of f to X_n. We say that the orbit of a point $p \in \mathbb{C}^d \setminus \{0\}$ converges to the origin *asymptotic to* $\mathfrak{p}$ if for any $n \in \mathbb{N}$, the limit of the f_n-orbit of $\hat{\pi}_n^{-1}(p)$ is exactly p_n.

If $\mathfrak{p} = \mathfrak{p}(C)$ for some irreducible curve C, we say that the orbit converges to the origin *asymptotic to* C.

We say that a stable manifold Δ is *asymptotic to* $\mathfrak{p}$ (resp., *to* C)if the orbit of p is asymptotic to $\mathfrak{p}$ (resp., to $\mathfrak{p}(C)$) for any $p \in \Delta$.

We now recall [LHRSSV, Theorem 1], which allows to construct parabolic manifolds from formal invariant curves.

Theorem 1.29 ([LHRSSV, Theorem 1])**.** *Let $f : (\mathbb{C}^d, 0) \to (\mathbb{C}^d, 0)$ be a tangent to the identity germ, and let C be a formal invariant curve for f. Then either C is contained in* $\mathrm{Fix}(f)$*, or there exist finitely many parabolic manifolds asymptotic to C.*

1.9. Ramis-Sibuya normal forms

To describe precisely the number and dimension of the parabolic manifolds produced by Theorem 1.29, we need to introduce some terminology.

We first introduce Ramis-Sibuya normal forms, for tangent to the identity germs in dimension 3.

Definition 1.30. Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a tangent to the identity germ, and let C be a smooth f-invariant formal curve. We say that the couple (f, C) is in *Ramis-Sibuya normal form* with respect to local coordinates (x, y, z) at the origin, if C is transverse to $\{z = 0\}$, and f takes the form

$$f(x,y,z) = \begin{pmatrix} \exp(d_1(z))\big(x(1 + c_{11}z^r) + c_{12}yz^r\big) + \langle z^{r+1}\rangle \\ \exp(d_2(z))\big(y(1 + c_{22}z^r) + c_{21}xz^r\big) + \langle z^{r+1}\rangle \\ z - z^{r+1} + bz^{2r+1} + \langle z^{2r+2}\rangle \end{pmatrix}, \tag{2}$$

where d_1 and d_2 are polynomials of degree at most $r - 1$ vanishing at the origin, and $c_{12} = c_{21} = 0$ unless $d_1 \equiv d_2$.

Notice that on the z-coordinate, assuming that C has sufficiently high tangency with $\{x = y = 0\}$, we find the formal normal form of the action of $f|_C$. In particular the existence of such a normal form implies that $f|_C$ defines a parabolic 1-dimensional germ of multiplicity $r + 1$.

Theorem 1.31 ([LHRSSV, Theorem 5.11])**.** *Let $f : (\mathbb{C}^3, 0)$ be a tangent to the identity germ, admitting an (irreducible) f-invariant formal curve C. Suppose that $f|_C \neq \mathrm{id}$. Then there exists a sequence of weighted blow-ups $\pi : X_\pi \to (\mathbb{C}^3, 0)$ so that the strict transform C_π of C is smooth, and, if $f_\pi : (X_\pi, p) \to (X_\pi, p)$ denotes the lift of f at $p = C_\pi \cap \pi^{-1}(0)$, then (f_π, C_π) is in Ramis-Sibuya normal form.*

In [LHRSSV], the authors show the reduction (up to taking iterates) to Ramis-Sibuya normal form in the more general setting of automorphisms admitting an f-invariant formal curve where $f|_C$ has multiplier 1 (in particular, the linear part of f does not need to be the identity).

The reduction process consists in three steps. The first consists in an embedded resolution of C. In the second step, one applies Theorem 1.15 to solve the singularities of the infinitesimal generator of f. The third step reduces the pair (f_π, C_π) to the desired normal form, by performing further blow-ups. Notice that both the second and third steps may require weighted blow-ups.

Once the couple (f, C) is reduced in Ramis-Sibuya normal form, one can describe explicitly the number and dimension of the parabolic manifolds provided by Theorem 1.29.

With the notations of (2), write d_j for $j = 1, 2$ as

$$d_j(z) = \sum_{k=1}^{r-1} d_k^{(j)} z^k.$$

Given an attracting direction ξ for $f|_C$ (i.e., any complex r-th root of 1, see [LHRSSV, Section 6]), we set

$$R_j(\xi) = \big(\mathrm{Re}(d_1^{(j)}\xi), \dots, \mathrm{Re}(d_{r-1}^{(j)}\xi^{r-1})\big). \tag{3}$$

Definition 1.32. We say that ξ is a *node direction* for the variable x (resp., y) if $R_1(\xi) < 0$ (resp., $R_2(\xi) < 0$), and a *saddle direction* otherwise, where $<$ denotes the lexicographic order.

Theorem 1.33 ([LHRSSV, Theorem 6.1])**.** *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a tangent to the identity germ, and let C be an f-invariant formal curve. Suppose that (f, C) is in Ramis-Sibuya normal form. For any attracting direction ξ for $f|_\Gamma$, let $s = s(\xi) \in \{0, 1, 2\}$ be the number of variables for which ξ is a node direction. Then there exists a parabolic manifold $\Delta(\xi)$ asymptotic to C, of dimension $s(\xi) + 1$, which is connected, simply connected, and which is a fundamental domain for the set of points whose orbit converges to 0 asymptotic to C and tangent to ξ.*

2. The example

2.1. Rational maps with no holomorphic fixed points

We want to start with a tangent to the identity germ f which has a finite number of characteristic directions, all degenerate.

Recall that if $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is a tangent to the identity germ, its characteristic directions can be reinterpreted in terms of the action induced by the homogeneous part of smallest degree of $f - \mathrm{id}$ on $\mathbb{P}^2$: non-degenerate characteristic directions correspond to holomorphic fixed points, while degenerate characteristic directions correspond to indeterminacy points.

In order to work with a germ with only degenerate characteristic directions, we start with a rational map in $\mathbb{P}^2$ which has no holomorphic fixed points:

$$H([x : y : z]) = \big[yz(y - z) : x(x^2 - z^2) : xz(y - z)\big]. \tag{4}$$

Hence we focus on germs of the form:

$$f(x, y, z) = \begin{pmatrix} x + yz(y - z) + P \\ y + x(x^2 - z^2) + Q \\ z + xz(y - z) + R \end{pmatrix}, \tag{1}$$

with P, Q, R of order at least 4. Notice that, by abuse of notation, we denote by H both the homogeneous part of smallest degree of $f - \mathrm{id}$, and the action on $\mathbb{P}^2$ induced by it.

Remark 2.1. The choice of H as in equation (4) is inspired by [Iva11, Example 2.1]. In op. cit., the author provides examples of rational maps of $\mathbb{P}^2$ without (*holomorphic*) fixed points, of any given degree. The word *holomorphic* is used to distinguish the case of *meromorphic* fixed points, i.e., fixed points p with the additional condition $H(p) \ni p$. In other terms, there exists a sequence of non-indeterminacy points p_n converging to p whose image $H(p_n)$ converges to p. The original example [Iva11, Example 2.1] is a rational maps of $\mathbb{P}^1 \times \mathbb{P}^1$ without holomorphic fixed points, that we consider in the birationally equivalent model $\mathbb{P}^2$.

The map H acting on $\mathbb{P}^2$ still has no holomorphic fixed points, while it has exactly 5 indeterminacy points $p_1 = [0 : 0 : 1]$, $p_2 = [0 : 1 : 1]$, $p_3 = [1 : 1 : 1]$, $p_4 = [-1 : 1 : 1]$, and $p_5 = [0 : 1 : 0]$. Out of these indeterminacy points, one can check that only p_3, p_4 and p_5 are meromorphic fixed points.

When we need to develop P, Q, R in formal power series, we will use the following notations:

$$P = \sum_{i,j,k} P_{ijk} x^i y^j z^k, \quad Q = \sum_{i,j,k} Q_{ijk} x^i y^j z^k, \quad R = \sum_{i,j,k} R_{ijk} x^i y^j z^k,$$

where the indices i, j, k vary in $\mathbb{N}$ with $i + j + k \geq 4$. We will also denote by $P^{(h)}$ (resp., $Q^{(h)}$, $R^{(h)}$) the homogeneous part of degree h of P (resp., Q, R).

2.2. Characteristic directions

As a consequence of Remark 2.1, f has exactly 5 characteristic directions, given by $v_1 = [0 : 0 : 1], v_2 = [0 : 1 : 1], v_3 = [1 : 1 : 1], v_4 = [-1 : 1 : 1], v_5 = [0 : 1 : 0]$. All these directions are degenerate, of multiplicities $1, 1, 3, 3, 5$ respectively. We denote by p_1, p_2, p_3, p_4, p_5 the corresponding characteristic points.

For a definition the multiplicity $\mu_f(v)$ of a characteristic direction v, see [AT03, p. 278]. For the reader's convenience, we show here how to compute the multiplicity of v_5. The multiplicity $\mu_f([0 : 1 : 0])$ is the local intersection multiplicity at $[0 : 1 : 0]$ of $y(x \circ f^{(3)}) - x(y \circ f^{(3)})$ and $y(z \circ f^{(3)}) - z(y \circ f^{(3)})$ in $\mathbb{P}^2$. We denote by $\langle \phi, \psi \rangle_p$ the local intersection multiplicity at p of $\{\phi = 0\}$ and $\{\psi = 0\}$. In this case, computing the intersection in the chart $\{y = 1\}$, we obtain

$$\begin{aligned}
\mu_f([0 : 1 : 0]) &= \langle z(1 - z) - x^2(x^2 - z^2), xz(1 - z) - xz(x^2 - z^2) \rangle_0 \\
&= \langle z - z^2 - x^4 + x^2 z^2, xz(1 - z - x^2 + z^2) \rangle_0 \\
&= \langle z - z^2 - x^4 + x^2 z^2, x \rangle_0 + \langle z - z^2 - x^4 + x^2 z^2, z \rangle_0 \\
&= 1 + 4 = 5.
\end{aligned}$$

Computations for the other multiplicities are similar and left to the reader.

Remark 2.2. Consider the local diffeomorphism $\sigma(x, y, z) = (-\mathrm{i}x, \mathrm{i}y, \mathrm{i}z)$. Then we get

$$\sigma^{-1} \circ f \circ \sigma(x, y, z) = \begin{pmatrix} x + yz(y - z) + \mathrm{i}P \circ \sigma \\ y + x(x^2 - z^2) - \mathrm{i}Q \circ \sigma \\ z + xz(y - z) - \mathrm{i}R \circ \sigma \end{pmatrix}.$$

In particular, the 3-jet $f^{(\leq 3)}$ is invariant by this conjugacy. The action of σ on the characteristic directions $v_1, \ldots, v_5$ is a bijection that fixes v_1, v_2, v_5 while it exchanges v_3 and v_4.

It follows that one can recover the birational study of the lifts of f above the point p_4 associated to v_4 from the behaviour of the lifts of f above p_3.

2.3. Resolution of singularities of the infinitesimal generator

In this section, we provide a resolution of the infinitesimal generator χ of a tangent to the identity germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ of the form (1), in the sense of [MP13] (see Proposition 2.3).

In Section 3, we will show that after further blow-up (see Proposition 3.14), all the singularities will be isolated and belonging to one of three classes (simple corners, degenerate spikes and spinning corners). We will then study the behaviour of these families under point modifications (introducing a fourth family, half corners).

We will finally study the behaviour of these families under general admissible modifications (strongly adapted to the dynamics) in Section 4.

By Remark 1.7, the infinitesimal generator χ of f takes the form

$$\chi = \big(yz(y-z) + P^{(4)}\big)\partial_x + \big(x(x^2-z^2) + Q^{(4)}\big)\partial_y + \big(xz(y-z) + R^{(4)}\big)\partial_z + \xi,$$

where ξ is a (possibly formal) vector field of multiplicity at least 5.

To study the resolution of χ, we will perform computations from the point of view of maps instead of vector fields, relying on Proposition 1.8.

2.3.1. *First blow-up.*

To resolve the singularities of χ, we first blow-up the origin. We write the blow-up $\pi_1 : X_1 \to (\mathbb{C}^3, 0)$, and compute the lift f_1 of f with respect to π_1. By Proposition 1.11, the singularities of the saturated infinitesimal generator $\hat{\chi}_1$ of f_1 are isolated, given by the points p_1, p_2, p_3, p_4, p_5.

We first work in the z-chart, for which the map π_1 is written as $\pi_1(x, y, z) = (xz, yz, z)$. We obtain:

$$f_1(x,y,z) = \begin{pmatrix} \dfrac{x - z^2y(1-y) + z^{-1}P\circ\pi_1}{1 - z^2x(1-y) + z^{-1}R\circ\pi_1} \\[2ex] \dfrac{y - z^2x(1-x^2) + z^{-1}Q\circ\pi_1}{1 - z^2x(1-y) + z^{-1}R\circ\pi_1} \\[2ex] z\big(1 - z^2x(1-y) + z^{-1}R\circ\pi_1\big) \end{pmatrix}. \tag{5}$$

We rewrite f_1 developing around a point in $\{z = 0\}$, obtaining

$$f_1(x,y,z) = \begin{pmatrix} x + z^2(-y + x^2 + y^2 - x^2y) + z^3(P^{(4)} - xR^{(4)})(x,y,1) + \langle z^4\rangle \\ y + z^2x(-1 + y + x^2 - y^2) + z^3(Q^{(4)} - yR^{(4)})(x,y,1) + \langle z^4\rangle \\ z + z^3x(-1+y) + z^4R^{(4)}(x,y,1) + \langle z^5\rangle \end{pmatrix}. \tag{6}$$

We study f_1 around the characteristic points $p_1, \ldots, p_4$. The point p_1 corresponds to the origin in this chart. In this case the linear part of the reduced infinitesimal generator is:

$$(-y + P_{004}z)\partial_x + (-x + Q_{004}z)\partial_y.$$

Hence $\hat{\chi}_1$ has a canonical singularity at p_1, with eigenvalues of the linear part given by 1, -1 and 0.

Similarly, the point p_2 corresponds to $(0, 1, 0)$ in this chart. By setting $y = 1 + v$, we get

$$f_1(x,v,z) = \begin{pmatrix} x + z^2v(1 + v - x^2) + z^3(P^{(4)} - xR^{(4)})(x, 1+v, 1) + \langle z^4\rangle \\ v + z^2x(-1 - v + x^2 - v^2) + z^3(Q^{(4)} - (1+v)R^{(4)})(x, 1+v, 1) + \langle z^4\rangle \\ z + z^3xv + z^4R^{(4)}(x, 1+v, 1) + \langle z^5\rangle \end{pmatrix}. \tag{7}$$

We get again a canonical singularity, with eigenvalues of the linear part $\mathfrak{i}$, $-\mathfrak{i}$ and 0.

The points p_3 and p_4 have coordinates $(1, 1, 0)$ and $(-1, 1, 0)$ respectively in the z-chart. We treat p_3, the case of p_4 being completely analogous by Remark 2.2. By setting $x = 1+u$, we get

$$f_1(u,v,z) = \begin{pmatrix} u + z^2v(-2u + v - u^2) + z^3(P^{(4)} - (1+u)R^{(4)})(1+u, 1+v, 1) + \langle z^4\rangle \\ v + z^2(1+u)(2u - v + u^2 - v^2) + z^3(Q^{(4)} - (1+v)R^{(4)})(1+u, 1+v, 1) + \langle z^4\rangle \\ z + z^3(1+u)v + z^4R^{(4)}(1+u, 1+v, 1) + \langle z^5\rangle \end{pmatrix}. \tag{8}$$

In this case the linear part of $\hat{\chi}_1$ is

$$z\big(P^{(4)}-R^{(4)}\big)(1,1,1)\partial_u+\Big(2u-v+z\big(Q^{(4)}-R^{(4)}\big)(1,1,1)\Big)\partial_v,$$

which gives an isolated canonical singularity with eigenvalues -1, and 0 (of multiplicity 2).

It remains to study the characteristic direction $v_5=[0:1:0]$. In this case we work in the y-chart, and write $\pi_1(x,y,z)=(xy,y,yz)$. The lift f_1 of f takes the form

$$f_1(x,y,z)=\begin{pmatrix}\dfrac{x+y^2z(1-z)+y^{-1}P\circ\pi_1}{1+y^2x(x^2-z^2)+y^{-1}Q\circ\pi_1}\\[2ex] y\big(1+y^2x(x^2-z^2)+y^{-1}Q\circ\pi_1\big)\\[1ex] \dfrac{z+y^2xz(1-z)+y^{-1}R\circ\pi_1}{1+y^2x(x^2-z^2)+y^{-1}Q\circ\pi_1}\end{pmatrix}. \tag{9}$$

The Taylor expansion at the origin gives the following expression for $f_1(x,y,z)$:

$$\begin{pmatrix}x+y^2\Big(z-z^2-x^4+x^2z^2+P_{040}y+(P_{130}-Q_{040})xy+P_{031}yz+P_{050}y^2+y\mathfrak{m}^2\Big)\\ y+y^3\Big(x^3-xz^2+Q_{040}y+y\mathfrak{m}\Big)\\ z+y^2\Big(xz-xz^2-x^3z+xz^3+R_{040}y+R_{130}xy+(R_{031}-Q_{040})yz+R_{050}y^2+y\mathfrak{m}^2\Big)\end{pmatrix}. \tag{10}$$

The linear part of the reduced infinitesimal generator is:

$$(P_{040}y+z)\partial_x+R_{040}y\partial_z.$$

We get a nilpotent linear part (of rank 1 if $R_{040}=0$, and of rank 2 otherwise). In this case the singularity is not log-canonical, and we need to keep blowing-up.

2.3.2. *Second blow-up.*

For simplicity, we will assume $R_{040}\neq 0$. In this case, f_1 has only one singular direction $v_{5,1}=[1:0:0]$. Consider the blow-up $\pi_2\colon X_2\to X_1$ of the point p_5. In the x-chart we have $\pi_2(x,y,z)=(x,xy,xz)$. Set $\hat{\pi}_2(x,y,z)=\pi_1\circ\pi_2(x,y,z)=(x^2y,xy,x^2yz)$. The lift f_2 of f in X_2 is given by

$$f_2(x,y,z)=\begin{pmatrix}x\dfrac{1+x^2y^2z(1-xz)+x^{-2}y^{-1}P\circ\hat{\pi}_2}{1+x^5y^2(1-z^2)+x^{-1}y^{-1}Q\circ\hat{\pi}_2}\\[2ex] y\dfrac{\big(1+x^5y^2(1-z^2)+x^{-1}y^{-1}Q\circ\hat{\pi}_2\big)^2}{1+x^2y^2z(1-xz)+x^{-2}y^{-1}P\circ\hat{\pi}_2}\\[2ex] \dfrac{z+x^3y^2z(1-xz)+x^{-2}y^{-1}R\circ\hat{\pi}_2}{1+x^2y^2z(1-xz)+x^{-2}y^{-1}P\circ\hat{\pi}_2}\end{pmatrix}. \tag{11}$$

We rewrite f_2 developing around the origin, obtaining

$$f_2(x,y,z)=\begin{pmatrix}x+x^3y^2\Big(P_{040}y+z+\big(P_{130}-Q_{040}\big)xy-x^3-xz^2+x^3z^2+\langle xy\rangle\mathfrak{m}\Big)\\ y+x^2y^3\Big(-P_{040}y-z+\big(2Q_{040}-P_{130}\big)xy+2x^3+xz^2-2x^3z^2+\langle xy\rangle\mathfrak{m}\Big)\\ z+x^2y^2\Big(R_{040}y+R_{130}xy+xz-z^2-P_{040}yz-x^2z^2+xz^3+\langle xy\rangle\mathfrak{m}\Big)\end{pmatrix}. \tag{12}$$

2.3.3. *Third blow-up.*

Let $\pi_3 : X_3 \to X_2$ be the blow-up of the point $p_{5,1}$ corresponding to the origin in the last coordinate chart we considered. To study the singular points associated to f_2, we will need to consider two different charts.

First, in the x-chart we get

(13)
$$f_3(x,y,z) = \begin{pmatrix} x + x^6y^2\Big(P_{040}y + z - x^2 + \big(P_{130} - Q_{040}\big)xy - x^2z^2 + x^4z^2 + \langle x^2y\rangle\Big) \\ y + x^5y^3\Big(-2P_{040}y - 2z + 3x^2 + \big(3Q_{040} - 2P_{130}\big)xy + 2x^2z^2 - 3x^4z^2 + \langle x^2y\rangle\Big) \\ z + x^4y^2\Big(R_{040}y + R_{130}xy + xz - 2P_{040}xyz - 2xz^2 + x^2\langle x,y\rangle\Big) \end{pmatrix}.$$

For any $z_0 \in \mathbb{C}$, the saturated infinitesimal generator $\hat{\chi}_3$ of f_3 has a singularity at $(0,0,z_0)$, with nilpotent linear part of rank 2. In this case the singular directions of f_3 form the line $[pR_{040} : pz_0(2z_0-1) : r]$ with $[p:r]$ varying in $\mathbb{P}^1_{\mathbb{C}}$.

We now work in the z-chart, so that $\pi_3(x,y,z) = (xz, yz, z)$, and get

$$(14)\quad f_3(x,y,z) = \begin{pmatrix} x + x^3y^2z^4\Big(-R_{040}y + 2z - xz + 2P_{040}yz - R_{130}xyz + z^2\langle y,z\rangle\Big) \\ y + x^2y^3z^4\Big(-R_{040}y - xz - R_{130}xyz + z^2\langle y,z\rangle\Big) \\ z + x^2y^2z^5\Big(R_{040}y - z + xz - P_{040}yz + R_{130}xyz + z^2\langle y,z\rangle\Big) \end{pmatrix}.$$

In this case $\hat{\chi}_3$ has a singularity of order 2 at the origin (and of order 1 with nilpotent linear part at $(x_0,0,0)$, with $x_0 \neq 0$, that we already know about from the previous computation).

2.3.4. *Fourth blow-up.*

Finally, we consider the blow-up $\pi_4 : X_4 \to X_3$ along the line L of singular points of $\hat{\chi}_3$.

The line L is covered by two charts in X_3, the one where the exceptional divisor is $\{x=0\}$ and the line is given by $L = \{x = y = 0\}$, and the one where the exceptional divisor is $\{z=0\}$ and the line is given by $L = \{y = z = 0\}$. This gives a total of four charts to be considered on X_4, to cover the exceptional divisor $\pi_4^{-1}(L)$.

We first consider the chart in X_3 that gives (13), so that $L = \{x = y = 0\}$.

We put ourselves in the chart of X_4 not intersecting the strict transform of the exceptional divisor $E_3 = \{x = 0\}$ of π_3, obtaining $\pi_4(x,y,z) = (x, xy, z)$. Computing the lift of f_3, we get

$$(15)\qquad f_4(x,y,z) = \begin{pmatrix} x + x^8y^2\Big(z + P_{040}xy + \langle x^2\rangle\Big) \\ y + x^7y^3\Big(-3z - 3P_{040}xy + \langle x^2\rangle\Big) \\ z + x^7y^2\Big(R_{040}y + z - 2z^2 + R_{130}xy - 2P_{040}xyz + \langle x^2\rangle\Big) \end{pmatrix}.$$

The saturation $\hat{\chi}_4^x$ of the infinitesimal generator of f_4 with respect to $\{x=0\}$ takes the form

$$\hat{\chi}_4^x = xy^2z\partial_x + y^3\big(-3z - 3P_{040}xy\big)\partial_y + y^2\big(R_{040}y + z - 2z^2 + R_{130}xy - 2P_{040}xyz\big)\partial_z + x^2\xi,$$

where ξ is a suitable vector field. We study this vector field on the point $(0, y_0, z_0)$.

If $y_0 \neq 0$, we have that $(0, y_0, z_0)$ is singular if and only if

$$\begin{cases} -3y_0z_0 = 0, \\ R_{040}y_0 + z_0 - 2z_0^2 = 0. \end{cases}$$

Since $R_{040} \neq 0$, this system does not have solutions.

Suppose now $y_0 = 0$. Then the saturation $\hat{\chi}_4$ with respect to the exceptional divisor, locally given by $\{xy = 0\}$, gives

$$\hat{\chi}_4 = xz\partial_x - 3yz\partial_y + \big(R_{040}y + z - 2z^2\big)\partial_z + \xi',$$

where ξ' is a vector field whose coefficients belong to $x\langle x, y\rangle$. First notice that $\hat{\chi}_4$ is regular unless $z_0(1 - 2z_0) = 0$.

At the point q_1 corresponding to the value $z_0 = 0$, $\hat{\chi}_4$ has a linear part with a non-vanishing eigenvalue (of eigenspace generated by ∂_z): hence we get an isolated canonical singularity. Similarly, at the point q_2 corresponding to $z_0 = \frac{1}{2}$, $\hat{\chi}_4$ has an isolated canonical singularity, with linear part with eigenvalues $\frac{1}{2}(1, -3, -2)$.

With respect to suitable coordinates in a chart intersecting E_3, we get the form $\pi_4(x, y, z) = (xy, y, z)$. For the lift of f_3, we get

$$f_4(x,y,z) = \begin{pmatrix} x + x^6y^7\Big(3P_{040}y + 3z + \langle y^2\rangle\Big) \\ y + x^5y^8\Big(-2P_{040}y - 2z + \langle y^2\rangle\Big) \\ z + x^4y^7\Big(R_{040} + x(z - 2z^2) + \langle y\rangle\Big) \end{pmatrix}. \tag{16}$$

The saturation $\hat{\chi}_4^y$ of the infinitesimal generator of f_4 with respect to $\{y = 0\}$ takes the form

$$\hat{\chi}_4^y = 3x^6z\partial_x + x^4\big(R_{040} + x(z - 2z^2)\big)\partial_z + y\xi,$$

where ξ is a suitable vector field.

We study $\hat{\chi}_4$ at points $(x_0, 0, z_0)$. The case $x_0 \neq 0$ corresponds to previous computations, and we have no singularities here.

When $x_0 = 0$, again we get regular points, hence no singularities arise in this chart.

We finally consider the chart in X_3 giving (14), so that $L = \{y = z = 0\}$.

We pick the coordinate chart of X_4 not intersecting the strict transform of the exceptional divisor $E_3 = \{z = 0\}$ of π_3, obtaining $\pi_4(x, y, z) = (x, yz, z)$. For the lift of f_3, we get

$$f_4(x,y,z) = \begin{pmatrix} x + x^3y^2z^7\Big(2 - x - R_{040}y + 2P_{040}yz - R_{130}xyz + \langle z^2\rangle\Big) \\ y + x^2y^3z^7\Big(1 - 2x - 2R_{040}y + P_{040}yz - 2R_{130}xyz + \langle z^2\rangle\Big) \\ z + x^2y^2z^8\Big(-1 + x + R_{040}y - P_{040}yz + R_{130}xyz + \langle z^2\rangle\Big) \end{pmatrix}. \tag{17}$$

As usual, we denote by $\hat{\chi}_4$ the saturated infinitesimal generator of f_4, and study its germ at points $(x_0, y_0, 0)$. At the point q_3 corresponding to the origin, we get an isolated canonical singularity, whose linear part has eigenvalues $(2, 1, -1)$.

When $x_0 = 0$ and $y_0 \neq 0$, we have a singularity if and only if $1 - 2R_{040}y_0 = 0$, i.e., $y_0 = \frac{1}{2R_{040}}$. At the corresponding point q_4, consider local coordinates (x, v, z) with $y = y_0 + v$. In these coordinates, the linear part of $\hat{\chi}_4$ takes the form (up to renormalization of a factor y_0^2):

$$\frac{3}{2}x\partial_x + (-2y_0x - y + y_0^2P_{040}z)\partial_y - \frac{1}{2}z\partial_z, \tag{18}$$

hence we get another isolated canonical singularity.

The case $x_0 \neq 0$ corresponds to the study carried on above: we get again a singularity when $x_0 = 2$ and $y_0 = 0$, which corresponds to q_2.

To finish our study, we consider a chart of X_4 intersecting the strict transform of E_3, getting $\pi_4(x,y,z) = (x,y,yz)$. For the lift of f_3, we get

$$f_4(x,y,z) = \begin{pmatrix} x + x^3y^7z^4\Big(-R_{040} + 2z - xz + \langle y\rangle\Big) \\ y + x^2y^8z^4\Big(-R_{040} - xz + \langle y\rangle\Big) \\ z + x^2y^7z^5\Big(2R_{040} - z + 2xz + \langle y\rangle\Big) \end{pmatrix}. \tag{19}$$

The only point q_5 that remains to be studied corresponds to the origin in this chart, and $\hat{\chi}_4$ has an isolated canonical singularity there, with linear part having eigenvalues $R_{040}(-1,-1,2)$.

To sum up, we proved the following result.

Proposition 2.3. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a germ of the form (1) with $R_{040} \neq 0$. Let $\pi_0 : X_{\pi_0} \to (\mathbb{C}^3, 0)$ be the regular modification obtained as the composition $\pi_0 = \pi_1 \circ \pi_2 \circ \pi_3 \circ \pi_4$ described above (hence $X_{\pi_0} = X_4$).*

Then the reduced infinitesimal generator $\hat{\chi}_{\pi_0}$ of the lift f_{π_0} of f at X_{π_0} has only isolated canonical singularities, namely $p_1, \ldots, p_4, q_1, \ldots, q_5 \in X_{\pi_0}$.

Notice the abuse of notation, where we denote by $p_1, \ldots, p_4$ both the points in X_1, and their unique preimages through $\pi_2 \circ \pi_3 \circ \pi_4$ in X_4.

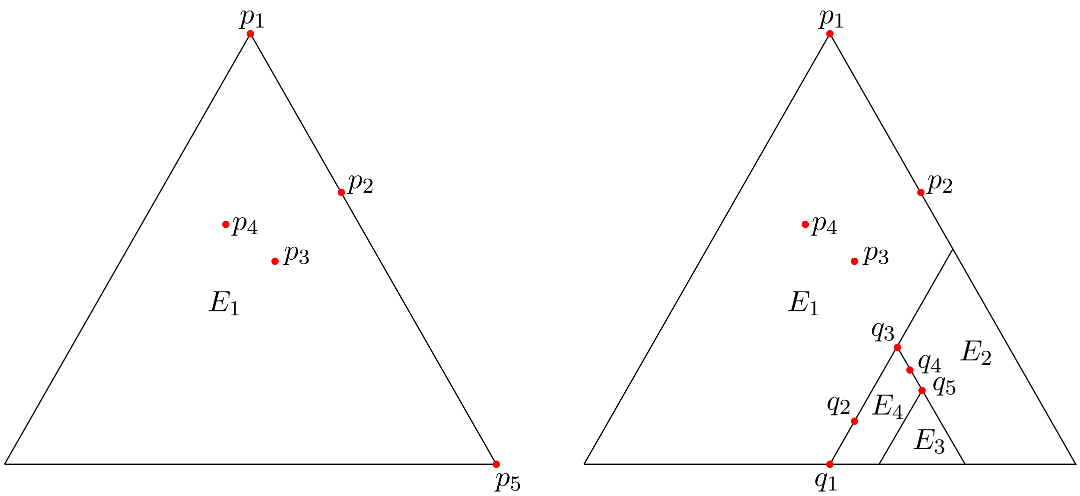


FIGURE 1. Singular points of the saturated infinitesimal generator at X_1 (on the left) and $X_4 = X_{\pi_0}$ (on the right).

Remark 2.4. One can check that Panazzolo's algorithm [Pan07] would perform two weighted blow-ups to solve χ: the first is the blow-up $\pi_1 : X_1 \to (\mathbb{C}^3, 0)$ of the origin, and the second is the blow-up $\pi_\tau : X_\tau \to X_1$ of the point p_5, with respect to the weight $\omega = (1,3,2)$. The weighted blow-up π_τ produces a divisor that is birationally equivalent to E_4 (meaning that there exists a birational map from X_4 to X_τ sending E_4 to $\pi_\tau^{-1}(p_5)$).

To compute ω, notice that the Newton polyhedron (see [Pan07, Section 3] for a definition) associated to the saturated infinitesimal generator of f_1 given by (10) is generated by

$$(1,0,0), (0,1,-1), (-1,0,1), (-1,1,0).$$

It has two bounded faces (generated by the first and last three vertices in the list respectively), and the one generated by the first three vertices has ω as normal vector.

3. Birational study above the resolution

In this section, we study the dynamics of f and its behaviour under point modifications, starting from the model π_0 given by Proposition 2.3.

3.1. Special families

First, we introduce some special families of tangent to the identity germs that will appear in the birational models. We describe here a few notations that we will use all long the rest of the paper. Any family f of germs will be introduced by giving a name and a code. For example *simple corners* [R_0]. The code will be used in all the diagrams below. The letter R in the codes refers to the saturated infinitesimal generator of these families being reduced (see also Subsection 5.4.2). As for the names, they were inspired by the geometric characteristic of the germs. For example, degenerate spikes possess only one degenerate characteristic direction, pointing out from the exceptional divisor.

Any family is described in some special coordinates, and there will be some formal power series P, Q, R, belonging to suitable ideals (that will be explicited according to cases). We will always develop, without further mention, P, Q, R in formal power series, as:

$$P = \sum_{i,j,k} a_{ijk} x^i y^j z^k, \quad Q = \sum_{i,j,k} b_{ijk} x^i y^j z^k, \quad R = \sum_{i,j,k} c_{ijk} x^i y^j z^k.$$

Unless otherwise specified, we will also replace a_{100} with a_x, a_{010} with a_y and a_{001} with a_z, and analogously for Q and R. Finally, we will often replace a_{000}, b_{000} and c_{000} with a_0, b_0, c_0, or with α, β, γ, according to the situation.

Recall also that P, Q, R denote also the parts of degree 4 of higher or the maps f of the form (1) that we are studying. In this case we will keep developing them with coefficients P_{ijk}, Q_{ijk}, R_{ijk}, to avoid confusion.

3.1.1. *Simple corners*

We start from *simple corners*, introduced for vector fields in [GML92] and adapted to tangent to the identity germs in [AT03].

Definition 3.1 ([AT03, p. 288])**.** A tangent to the identity germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is a *simple corner* [R_0] if there are $a, b \in \mathbb{N}^*$, $c \in \mathbb{N}$, $\lambda \in \mathbb{C}^*$, $\mu \in \mathbb{C} \setminus (\lambda \mathbb{Q}_{>0})$, and local coordinates (x, y, z) so that

$$f(x,y,z) = \begin{pmatrix} x + (x^a y^b z^c) x \big(\lambda + P\big) \\ y + (x^a y^b z^c) y \big(\mu + Q\big) \\ z + (x^a y^b z^c) R \end{pmatrix}, \tag{20}$$

with $P, Q, R \in \mathfrak{m}$, and $z | R$ if $c > 0$.

Remark 3.2. We will discuss singular and exceptional directions with respect to the divisor $D = \{x^a y^b z^c = 0\}$, whose support is the union or two or three coordinates planes, depending on the vanishing of c.

The saturated infinitesimal generator $\hat{\chi}$ of f has the following properties:

(a) $\hat{\chi}$ is tangent to D;
(b) $\hat{\chi}$ is a canonical singularity,

where we recall that a canonical singularity is a non-radial log-canonical singularity (the non-radial behavior is ensured by the condition on the eigenvalues λ, μ). These properties completely characterize simple corners when $c \geq 1$ (among tangent to the identity germs fixing pointwise a set of the form $\{xyz = 0\}$).

In order to characterize simple corners with $c = 0$, one would have to ask for additional properties on the foliation induced on the normal bundle of the curve $\{x = y = 0\}$ obtained intersecting the two irreducible components of D.

3.1.2. *Degenerate spikes*

Definition 3.3. A tangent to the identity germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is a *degenerate spike* [R_1] (of *Siegel* type) if there are $c \in \mathbb{N}^*$, and local coordinates (x, y, z) so that

$$f(x,y,z) = \begin{pmatrix} x + z^c \big(\lambda x + P\big) \\ y + z^c \big(\mu y + Q\big) \\ z + z^{c+1} R \end{pmatrix}, \tag{21}$$

where $\lambda \in \mathbb{C}^*$, $\mu \in \lambda\mathbb{R}_{<0}$, while $P, Q \in \mathfrak{m}^2$ and $R \in \mathfrak{m}$.

Remark 3.4. Notice that any germ of the form

$$\begin{pmatrix} x + z^c\big(a_y y + a_z z + P\big) \\ y + z^c\big(b_x x + b_z z + Q\big) \\ z + z^{c+1}R \end{pmatrix}, \tag{22}$$

with $a_y b_x \neq 0$, $a_z, b_z \in \mathbb{C}$, $P, Q \in \mathfrak{m}^2$, and $R \in \mathfrak{m}$ is a degenerate spike, associated to $\lambda = -\mu = \sqrt{a_y b_x}$.

Remark 3.5. Degenerate spikes will appear at points belonging to a unique irreducible component of the exceptional divisor $D = \{z = 0\}$ on blown-up models.

In terms of the saturated infinitesimal generator $\hat{\chi}$ of f, degenerate spikes are characterized by the following properties:

(a) $\hat{\chi}$ is tangent to D;
(b) the induced foliation on D has a Siegel singularity;
(c) the linear part of $\hat{\chi}$ has exactly one vanishing eigenvalue.

In particular, by the classical Briot-Bouquet's theorem, $\hat{\chi}|_D$ has no invariant curves passing through p but for the two complex separatrices, tangent to $\{x = 0\}$ and $\{y = 0\}$ when f is given by (21).

Clearly the Siegel type refers to condition (b) above. In general we could ask only for the non-resonance condition $\mu/\lambda \notin \mathbb{Q}_{\geq 0}$, i.e., the induced foliation on D is canonical (and with invertible linear part). In this paper, without further mention, all degenerate spikes are of Siegel type.

3.1.3. *Spinning corners*

Definition 3.6. A tangent to the identity germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is a *spinning corner* [R$_2$] if there are local coordinates (x, y, z) such that f can be written as

$$f(x,y,z) = \begin{pmatrix} x + y^b z^c(x + P) \\ y + y^{b+1} z^c Q \\ z + y^b z^{c+1} R \end{pmatrix} \tag{23}$$

where $b,\ c \in \mathbb{N}^*$, $Q,\ R \in \mathfrak{m}$ and $P \in \mathfrak{m}^2$.

Remark 3.7. A germ which can be written, in local coordinates (x, y, z), as

$$f(x,y,z) = \begin{pmatrix} x + y^b z^c(a_x x + a_y y + a_z z + P) \\ y + y^{b+1} z^c Q \\ z + y^b z^{c+1} R \end{pmatrix} \tag{24}$$

with $a_x \in \mathbb{C}^*$, $a_y,\ a_z \in \mathbb{C}$, and $b,\ c,\ P,\ Q,\ R$ as above, is in fact a spinning corner. Indeed, by a linear change of coordinates $(x, y, z) \mapsto (x, \mu y, \nu z)$ with μ, ν satisfying $\mu^b \nu^c = a_x$, one may assume $a_x = 1$. Moreover, in new coordinates $u = a_x x + a_y y + a_z z$, y and z, we get

$$f(u,y,z) = \begin{pmatrix} u + y^b z^c(a_x u + \widetilde{P} \circ \phi^{-1}) \\ y + y^{b+1} z^c Q \circ \phi^{-1} \\ z + y^b z^{c+1} R \circ \phi^{-1} \end{pmatrix},$$

with $\widetilde{P} = a_x P + a_y y Q + a_z z R$ and $\phi^{-1}(u, y, z) = \big(a_x^{-1}(u - a_y y - a_z z), y, z\big)$.

If we denote $Q = b_x x + b_y y + b_z z + \mathfrak{m}$ and $R = c_x x + c_y y + c_z z + \mathfrak{m}$ for the germ f, and an analogous expression with tildes over the coefficients for the same germ in the new coordinates, we have that $\widetilde{b}_y = b_y - \frac{a_y}{a_x} b_x$, and analogously for $\widetilde{b}_z$, $\widetilde{c}_y$ and $\widetilde{c}_x$.

Remark 3.8. Spinning corners will appear in the intersection of two irreducible components D_1 and D_2 of the exceptional divisor D, given in local coordinates by $\{yz = 0\}$.

In terms of the reduced infinitesimal generator $\hat{\chi}$ of f, degenerate spikes are characterized by the following properties:

(a) $\hat{\chi}$ is tangent to D;
(b) the linear part of $\hat{\chi}$ has rank 1, with the eigenspace of non-zero eigenvalue tangent to $D_1 \cap D_2$.

In fact, if $D = \{yz = 0\}$ then we have $\phi \circ (f - \mathrm{id}) = y^{b_\phi} z^{c_\phi} A_\phi$, with $b_\phi, c_\phi \in \mathbb{N}$ and A_ϕ a holomorphic germ that is not a multiple of y or z, with $\phi \in \{x, y, z\}$. The tangency condition on $\{y = 0\}$ says that $b_y > \min\{b_x, b_z\}$, while the one on $\{z = 0\}$ gives $c_z > \min\{c_x, c_y\}$. The existence of an eigenvalue tangent to $D_1 \cap D_2$ says that $x \circ (f - \mathrm{id}) = y^{b_x} z^{c_x}(\alpha x + \beta y + \gamma z + P)$ with $\alpha \neq 0$ and $P \in \mathfrak{m}^2$, and $b_x \leq b_z$ and $c_x \leq c_y$. By setting $b = b_x$, $c = c_x$, $Q = y^{b_y - b - 1} z^{c_y - c} A_y$, $R = y^{b_z - b} z^{c_z - c - 1} A_z$, and checking the linear part of $\hat{\chi}$ in extreme cases for the parameters (i.e., if $b_y = b + 1$ and $c_y = c$, or $b_z = b$ and $c_z = c + 1$), we get a germ of the form (24).

3.1.4. *Half corners*

Definition 3.9. A tangent to the identity germ $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ is a *half corner* [R3] if there are local coordinates (x, y, z) such that f can be written as

$$f(x,y,z) = \begin{pmatrix} x + z^c(x + P) \\ y + z^{c+1}(\beta + Q) \\ z + z^{c+2} R \end{pmatrix} \tag{25}$$

where $c \in \mathbb{N}^*$, $\beta \in \mathbb{C}$, $P \in \mathfrak{m}^2$, $Q \in \mathfrak{m}$ and $R \in \mathcal{O}$ with $\gamma = R(0,0,0) \in \mathbb{C}$.

Remark 3.10. As for the case of spinning corners, one can show that any germ of the form

$$f(x,y,z) = \begin{pmatrix} x + z^c(a_x x + a_y y + a_z z + P) \\ y + z^{c+1}(\beta + Q) \\ z + z^{c+2} R \end{pmatrix} \tag{26}$$

with $a_x \neq 0$, $a_y, a_z \in \mathbb{C}$ and all other entries as above is indeed a half corner. In fact, we may assume $a_x = 1$ by a linear change of coordinates $(x, y, z) \mapsto (x, y, \nu z)$ with $\nu^c = a_x$. Then, one can assume $a_y = 0$ by performing the change of coordinates $u = x + a_y y$ (which changes the value of a_z to $a_z' = a_z + \beta a_y$), and finally we can set $u' = x + a_z' z$ and get a germ of the form (25).

Notice also that when $\beta \neq 0$, we may assume it equals 1, by performing the change of coordinates $(x, y, z) \mapsto (x, \beta y, z)$.

The value of β (its vanishing) will be important in the sequel. We will say that a half corner is *simple* if $\beta \neq 0$, *non-simple* otherwise.

In fact, we can independently normalize (by conjugating by linear diagonal maps) both the second and third coordinates, for example by assuming that $\beta \in \{0, 1\}$ and $\gamma := R(0,0,0) \in \{0, 1\}$.

Remark 3.11. Once in form (25) we still have some freedom up to linear change of coordinates. Assume $\beta = 0$. In this case we can conjugate by a map of the form $(x, y, z) \mapsto (\lambda x, \mu y, \nu z)$ with $\nu^c = 1$. In this case we get $\widetilde{b}_y = \nu b_y$, $\widetilde{\gamma} = \nu \gamma$. In particular their ratio is well defined up to homotheties (and it is in fact an invariant of conjugacy for half corners in form (25)).

Remark 3.12. Half corners will appear in points contained in a unique irreducible component D of the exceptional divisor, which we will assume having local equation $\{z = 0\}$.

One can characterize half corners in terms of their infinitesimal generator also in this case, but the description is more intricated. We just remark that again the saturated infinitesimal generator is tangent to D. Moreover, its linear part has a non-zero eigenvalue (whose eigenspace is tangent to D), and:

- either a Jordan block associated to the zero eigenvalue in the simple case, with the kernel being tangent to D; or
- a kernel of dimension 2 in the non-simple case.

3.2. From the resolution to special families

We show here how, possibly up to further blow-up, the singularities appearing in the model $\pi_0 : X_{\pi_0} \to (\mathbb{C}^3, 0)$ given by Proposition 2.3, belong to one of the families described in Section 3.1

In fact, from the study done in Section 2.3, the lift $f_{\pi_0} : X_{\pi_0} \to X_{\pi_0}$ satisfies the following properties.

- At the singularity p_1, f_{π_0} takes the form (6), which is a degenerate spike of the form (22) with respect to the coordinates (x, y, z), with parameters $c = 2$, $\alpha = \beta = -1$.
- At the singularity p_2, f_{π_0} takes the form (7), which is a degenerate spike of the form (22) with respect to the coordinates (x, y, z), with parameters $c = 2$, $\alpha = 1$, $\beta = -1$.
- At the singularity q_1, f_{π_0} takes the form (15), which is a spinning corner of the form (24) with respect to coordinates (z, y, x), with parameters $a_x = 1$, $a_y = R_{040}$, $a_z = 0$, $b = 2$, $c = 7$.
- At the singularity q_2, f_{π_0} is a simple corner of the form (20) with respect to coordinates (x, y, w) with $w = z - \frac{1}{2}$ (notations of (15)), with parameters $a = 7$, $b = 2$, $c = 0$, $\lambda = \frac{1}{2}$ and $\mu = -\frac{3}{2}$.
- At the singularity q_3, f_{π_0} takes the form (17), which is a simple corner of the form (20) with respect to coordinates (z, y, x), with parameters $a = 7$, $b = 2$, $c = 2$, $\lambda = -1$ and $\mu = 1$.
- At the singularity q_4, f_{π_0} is a simple corner of the form (20) with respect to coordinates (x, z, v) with $v = y - \frac{1}{2R_{040}}$ (notations of (17)), with parameters $a = 2$, $b = 7$, $c = 0$, $\lambda = \frac{3}{2}$ and $\mu = -\frac{1}{2}$ (up to common factors, see (18)).
- At the singularity q_5, f_{π_0} takes the form (19), which is a simple corner of the form (20) with respect to coordinates (z, y, x), with parameters $a = 4$, $b = 7$, $c = 2$, $\lambda = 1$ and $\mu = -1$ (up to a common factor R_{040}).

The only singularities not falling in one of the families described in Section 3.1 are p_3 and p_4. By symmetry (see Remark 2.2), we will only deal with p_3, the case of p_4 being completely analogous.

On suitable coordinates (u, v, z) centered at p_3, the germ f_{π_0} takes the form:

(8)
$$f_{\pi_0}(u, v, z) = \begin{pmatrix} u + z^2 v(-2u + v - u^2) + z^3 (P^{(4)} - (1+u) R^{(4)})(1+u, 1+v, 1) + \langle z^4 \rangle \\ v + z^2 (1+u)(2u - v + u^2 - v^2) + z^3 (Q^{(4)} - (1+v) R^{(4)})(1+u, 1+v, 1) + \langle z^4 \rangle \\ z + z^3 (1+u) v + z^4 R^{(4)}(1+u, 1+v, 1) + \langle z^5 \rangle \end{pmatrix}.$$

In this case, the homogeneous part of smallest degree of $z^{-2}(f_{\pi_0} - \mathrm{id})$ is linear, with associated matrix

$$\begin{pmatrix} 0 & 0 & \alpha \\ 2 & -1 & \beta \\ 0 & 0 & 0 \end{pmatrix},$$

where $\alpha = (P^{(4)} - R^{(4)})(1, 1, 1)$ and $\beta = (Q^{(4)} - R^{(4)})(1, 1, 1)$.

The computation of singular directions depend on weather α vanishes or not. In both cases, $v_{3,1} = [0 : 1 : 0]$ is a singular direction (associated to the eigenvalue -1), as is $v_{3,2} = [1 : 2 : 0]$ (with multiplier 0). If $\alpha \neq 0$, the generalized eigenspace associated to the eigenvalue 0 is associated to a Jordan block of size 2. It follows that $v_{3,1}$ and $v_{3,2}$ are the only singular directions (which are both exceptional). If $\alpha = 0$, the kernel has rank 2, which gives a line of degenerate directions, generated by $[1 : 2 : 0]$ and $[0 : \beta : 1]$.

For simplicity, we will assume that $\alpha = P^{(4)}(1, 1, 1) - R^{(4)}(1, 1, 1) \neq 0$.

Blow-up of p_3.

We consider $\widetilde{\pi}_1 : X_{\widetilde{\pi}_1} \to X_{\pi_0}$ the blow-up of p_3 in X_{π_0}. We consider the chart in $X_{\widetilde{\pi}_1}$ so that $\widetilde{\pi}_1(x,y,z) = (xy, y, yz)$. The lift $\widetilde{f}_1$ of f_{π_0} is given by:

$$\widetilde{f}_1(x,y,z) = \begin{pmatrix} x + y^2z^2\Big(x - 2x^2 + y(-3x^3 + x^2 - x + 1) + z(\alpha - \beta x) + y\langle y, z\rangle\Big) \\ y + y^3z^2\Big(-1 + 2x + y(3x^2 - x - 1) + \beta z + y\langle y, z\rangle\Big) \\ z + y^2z^3\Big(1 - 2x + y(2 + x - 3x^2) - \beta z + y\langle y, z\rangle\Big) \end{pmatrix}.$$

This is clearly a simple corner at $p_{3,1}$ (which corresponds to the origin in this chart). It is with respect to coordinates (z, y, x), with $a = b = 2$ and $c = 0$, $\lambda = 1$ and $\mu = -1$.

The point $p_{3,2}$ corresponds in this chart to $(\frac{1}{2}, 0, 0)$. By setting $x = \frac{1}{2} + u$, we get

$$\widetilde{f}_1(u,y,z) = \begin{pmatrix} u + y^2z^2\Big(-u + \frac{3}{8}y + (\alpha - \frac{1}{2}\beta)z + \mathfrak{m}^2\Big) \\ y + y^3z^2\Big(2u - \frac{3}{4}y + \beta z + \mathfrak{m}^2\Big) \\ z + y^2z^3\Big(-2u + \frac{7}{4}y - \beta z + \mathfrak{m}^2\Big) \end{pmatrix}. \tag{27}$$

This is a spinning corner of the form (24) with respect to the coordinates (u, y, z), with parameters $b = 2$, $c = 2$.

Remark 3.13. Thanks to Remark 3.7, we can perform the change of coordinate $x = a_u u + a_y y + a_z z$, where $a_u = -1$, $a_y = \frac{3}{8}$, and $a_z = (\alpha - \frac{1}{2}\beta)$, in order to conjugate the germ $\widetilde{f}_1$ given by (27) to an analogous germ with the coefficients $\widetilde{a}_y = \widetilde{a}_z = 0$ (coefficients with a tilde correspond to the new variables). In this case, we get $\widetilde{b}_y = 0$, $\widetilde{c}_y = 1$, $\widetilde{b}_z = 2\alpha$ and $\widetilde{c}_z = -2\alpha$.

We proved the following:

Proposition 3.14. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3), 0)$ be a germ of the form (1) with $R_{040} \neq 0$ and $P^{(4)}(\pm 1, 1, 1) \neq \pm R^{(4)}(\pm 1, 1, 1)$. Let $\widetilde{\pi}_0 : X_{\widetilde{\pi}_0} \to (\mathbb{C}^3, 0)$ be the regular modification obtained as the composition $\widetilde{\pi}_0 = \pi_0 \circ \widetilde{\pi}_1 \circ \widetilde{\pi}_2$, where $\widetilde{\pi}_1$ is the blow-up of p_3 and $\widetilde{\pi}_2$ is the blow-up of p_4.*

Then the lift $f_{\widetilde{\pi}_0}$ of f to $X_{\widetilde{\pi}_0}$ has finitely many singular points, where it is either a simple corner, a degenerate spike, or a spinning corner.

3.3. Birational study

Here we describe the behaviour of the families introduced in Section 3.1 under point blow-up.

3.3.1. *Simple corners*

The situation for simple corners is already known, we summarize here their behaviour under point blow-up.

Proposition 3.15 ([AT03, Proposition 4.1])**.** *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a simple corner, and denote by $\widetilde{f}$ the blow-up of f at 0. Then*

(i) 0 is never 2-dicritical;

(ii) the singular directions of f are always simple corners of $\widetilde{f}$.

We will need the behavior of simple corners with respect to any admissible blow-up, and to do so we need to be more explicit on the geometry of the singular directions of a simple corner.

Proposition 3.16. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a simple corner of the form (20), write $R = \alpha x + \beta y + \gamma z + \mathfrak{m}^2$, with $\alpha = \beta = 0$ if $c > 0$. Then we get the following singular directions:*

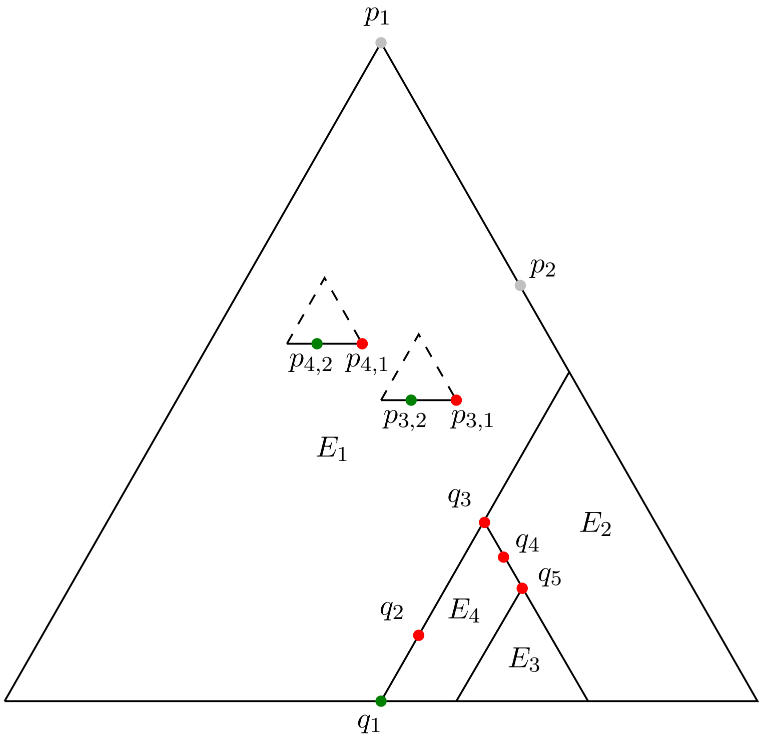


FIGURE 2. Singular points of the saturated infinitesimal generator at $X_{\widetilde{\pi}_0}$. We have degenerate spikes at p_1 and p_2, spinning corners at $p_{3,2}$, $p_{4,2}$ and q_1, and simple corners at the other marked points.

- $[\lambda - \gamma : 0 : \alpha]$ *if* α *and* $\lambda - \gamma$ *are not both vanishing;*
- $[p : 0 : r]$ *for all* $[p : r] \in \mathbb{P}^1_{\mathbb{C}}$*, if* $\alpha = \lambda - \gamma = 0$*;*
- $[0 : \mu - \gamma : \beta]$ *if* β *and* $\mu - \gamma$ *are not both vanishing;*
- $[0 : q : r]$ *for all* $[q : r] \in \mathbb{P}^1_{\mathbb{C}}$*, if* $\beta = \mu - \gamma = 0$*;*
- $[0 : 0 : 1]$.

All directions are exceptional, and simple corners.

Proof. The computation of singular directions is straightforward, since it corresponds on determining the eigenspaces of the linear map represented by

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ \alpha & \beta & \gamma \end{pmatrix}.$$

Since we will need this computation later, we verify that the singularities arising are again simple corners (property we already know from Proposition 3.15), at least for the case of non-isolated singular directions.

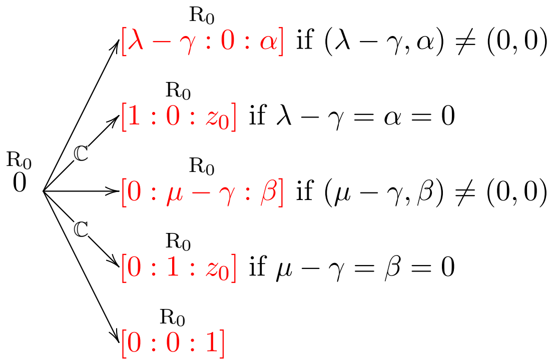
By working on the z-chart, and developing in formal power series, the lift $\widetilde{f}$ of f takes the form

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x\Big(1 + x^a y^b z^s \big(\lambda - \gamma - \alpha x - \beta y + \langle z \rangle\big)\Big) \\ y\Big(1 + x^a y^b z^s \big(\mu - \gamma - \alpha x - \beta y + \langle z \rangle\big)\Big) \\ z\Big(1 + x^a y^b z^s \big(\gamma + \alpha x + \beta y + \langle z \rangle\big)\Big) \end{pmatrix}, \tag{28}$$

where $s = a+b+c$. At the origin, corresponding to the direction $[0 : 0 : 1]$, we get a simple corner. If $\gamma = 0$, we get a simple corner again with respect to (x, y, z); if $\gamma \neq 0$, note that, if $(\lambda - \gamma)/\gamma$, $(\mu - \gamma)/\gamma \in \mathbb{Q}_{>0}$, then $\mu/\lambda \in \mathbb{Q}_{>0}$, which is impossible, so we obtain a simple corner with respect to (z, x, y) or (z, y, x).

If $\lambda = \gamma$ and $\alpha = 0$, we get singularities at all points $(x_0, 0, 0)$. By replacing $x = x_0 + u$, we get simple corners of the form (20) with respect to coordinates (y, z, u). The other cases are analogous and left to the reader. $\square$

We depict the situation in the next diagram. Exceptional directions are depicted in red, while non-exceptional (degenerate) directions will be depicted in blue (there are none for simple corners). We also indicate the type of tangent to the identity germ we get at each characteristic point (in this case, all simple corners). Finally, we indicate the geometry of singular points in case they come in a family (in this case, with a parameter $z_0 \in \mathbb{C}$).



Remark 3.17. We say that a simple corner is *resonant* if either $\lambda - \gamma = \alpha = 0$ or $\mu - \gamma = \beta = 0$, where we used the notations of Proposition 3.16. Notice that both istances cannot happen at the same time, since we would have $\lambda = \mu$, which is not allowed.

If we are in the first case $\lambda - \gamma = \alpha = 0$, and we apply to (28) the change of coordinates $(x, y, z) \mapsto (z, y, x - x_0) =: (x', y', z')$, then we get a germ of the form (20), with $R \in \langle x', y' \rangle$. In particular, the linear part $\alpha' x' + \beta' y' + \gamma' z'$ of R satisfies $\gamma' = 0$, hence all these simple corners are not resonant. A similar computation holds in the second case.

Remark 3.18. Suppose C is a formal invariant curve for a simple corner and let $\mathfrak{p}(C) = (p_n)$ be the increasing sequence of infinitely near points associated to C. By Proposition 1.27 (and with the notation of Proposition 1.25), we have that p_n is a singular direction for f_{n-1}; by Proposition 3.16, singular directions of a simple corner are again simple corners and exceptional, i.e. contained in the exceptional divisor. Inductively, we have that all infinitely near points associated to C are contained in the exceptional divisor of the corresponding blow-up, therefore C is contained in the exceptional divisor of the simple corner.

3.3.2. *Degenerate spikes*

Lemma 3.19. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a degenerate spike of the form (21). Then f has three singular directions, given by:*

- $\overrightarrow{v} = [0 : 0 : 1]$*, which is non-exceptional and degenerate;*
- $\overrightarrow{w_1} = [1 : 0 : 0]$*, and* $\overrightarrow{w_2} = [0 : 1 : 0]$*, which are exceptional, with multipliers λ and μ (seen as singular directions).*

Proof. The proof is a direct computation, left to the reader. $\square$

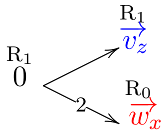
Remark 3.20. For maps of the form (22), we have $\overrightarrow{v} = \left[-\frac{b_z}{b_x} : -\frac{a_z}{a_y} : 1\right]$, and $\overrightarrow{w_j} = \left[\sqrt{a_y} : (-1)^j \sqrt{b_x} : 0\right]$ for $j = 1, 2$ (for some determinations of the square roots of a_y and b_x).

Proposition 3.21. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a degenerate spike of the form (21). Let $\pi : X \to (\mathbb{C}^3, 0)$ be the blow-up at the origin in $\mathbb{C}^3$. For the lift $\widetilde{f}$ of f to X, we have that:*

- $\overrightarrow{v} = [0 : 0 : 1]$ *is a degenerate spike;*

- $\overrightarrow{w_1} = [1:0:0]$ *and* $\overrightarrow{w_2} = [0:1:0]$ *are simple corners.*

The following diagram portrays the situation for degenerate spikes. We recall that exceptional directions are depicted in red and non-exceptional degenerate directions are depicted in blue. To help the reader, we also indicate with a subscript the chart in which we make the computations, i.e., the equation of the exceptional divisor obtained with the last blow-up.



Proof.

$[0:0:1]$ We make computations in the z-chart, so that $\pi(x,y,z) = (xz, yz, z)$. For the lift $\widetilde{f}$ of f, we obtain

$$\widetilde{f}(x,y,z) = \begin{pmatrix} \dfrac{x + z^c\big(\lambda x + z^{-1} P \circ \pi\big)}{1 + z^c R \circ \pi} \\[2ex] \dfrac{y + z^c\big(\mu y + z^{-1} Q \circ \pi\big)}{1 + z^c R \circ \pi} \\[2ex] z\big(1 + z^c R \circ \pi\big) \end{pmatrix}. \tag{29}$$

By developing in formal power series, we get

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x + z^c\big(\lambda x + a_{002} z + \mathfrak{m}^2\big) \\ y + z^c\big(\mu y + b_{002} z + \mathfrak{m}^2\big) \\ z + z^{c+1}\mathfrak{m} \end{pmatrix},$$

which is again a degenerate spike.

$[1:0:0], [0:1:0]$ We study $[1:0:0]$, the case $[0:1:0]$ being obtained by exchanging the role of x and y. We make computations in the x-chart, so that $\pi(x,y,z) = (x, xy, xz)$, and get

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x\Big(1 + x^c z^c\big(\lambda + x^{-1} P \circ \pi\big)\Big) \\[1ex] \dfrac{y + x^c z^c\big(\mu y + x^{-1} Q \circ \pi\big)}{1 + x^c z^c\big(\lambda + x^{-1} P \circ \pi\big)} \\[2ex] z\dfrac{1 + x^c z^c R \circ \pi}{1 + x^c z^c\big(\lambda + x^{-1} P \circ \pi\big)} \end{pmatrix}.$$

By developing in formal power series, we get

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x + x^{c+1} z^c\big(\lambda + \langle x \rangle\big) \\ y + x^c z^c\big((\mu - \lambda) y + \langle x \rangle\big) \\ z + x^c z^{c+1}\big(-\lambda + \langle x \rangle\big) \end{pmatrix},$$

which is a simple corner with respect to coordinates (x, z, y).

□

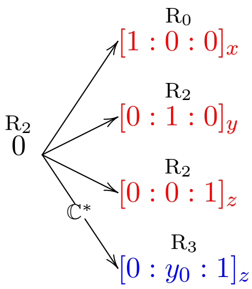
3.3.3. *Spinning corners*

Proposition 3.22. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a spinning corner of the form (23). The singular directions of f are $[1 : 0 : 0]$ (non-degenerate) and the points of the line $[0 : p : q]$, with $[p : q] \in \mathbb{P}^1_{\mathbb{C}}$ (all degenerate).*

Let $\pi : X \to (\mathbb{C}^3, 0)$ be the blow-up at the origin. For the lift $\widetilde{f}$ of f to X, we have that:

- *(i)* $[1 : 0 : 0]$ *is a simple corner;*
- *(ii)* $[0 : 1 : 0]$ *and* $[0 : 0 : 1]$ *are spinning corners;*
- *(iii)* $[0 : p : q]$ *are half corners for any p, q with $pq \neq 0$.*

We sum up the situation for spinning corners.



Proof. The list of singular directions is easily obtained by the fact that the homogeneous part of smallest degree of $f - \mathrm{id}$ is given by $y^b z^c \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$.

$\boxed{[1 : 0 : 0]}$ We make computations in the x-chart, and we obtain

$$\widetilde{f}(x, y, z) = \begin{pmatrix} x\Big(1 + x^s y^b z^c \big(1 + x^{-1} P \circ \pi\big)\Big) \\[2ex] y \dfrac{1 + x^s y^b z^c Q \circ \pi}{1 + x^s y^b z^c \big(1 + x^{-1} P \circ \pi\big)} \\[2ex] z \dfrac{1 + x^s y^b z^c R \circ \pi}{1 + x^s y^b z^c \big(1 + x^{-1} P \circ \pi\big)} \end{pmatrix},$$

where $s = b + c$. This gives a simple corner.

$\boxed{[0 : 0 : 1], [0 : 1 : 0]}$ We study $[0 : 0 : 1]$, the case $[0 : 1 : 0]$ being obtained by exchanging the role of y and z. Making computations in the z-chart, we get

$$\widetilde{f}(x, y, z) = \begin{pmatrix} \dfrac{x + y^b z^s (x + z^{-1} P \circ \pi)}{1 + y^b z^s R \circ \pi} \\[2ex] y \dfrac{1 + y^b z^s Q \circ \pi}{1 + y^b z^s R \circ \pi} \\[2ex] z\big(1 + y^b z^s R \circ \pi\big) \end{pmatrix}.$$

We develop in formal power series, obtaining

(30)

$$\widetilde{f}(x, y, z) = \begin{pmatrix} x + y^b z^s \Big(x + z\big(P^{(2)} - x R^{(1)}\big)(x, y, 1) + \langle z^2 \rangle\Big) \\ y + y^{b+1} z^{s+1} \Big(\big(Q^{(1)} - R^{(1)}\big)(x, y, 1) + z\big(Q^{(2)} - R^{(2)}\big)(x, y, 1) + \langle z^2 \rangle\Big) \\ z + y^b z^{s+2} \Big(R^{(1)}(x, y, 1) + z R^{(2)}(x, y, 1) + \langle z^2 \rangle\Big) \end{pmatrix},$$

where for any $k \in \mathbb{N}^*$, $P^{(k)}$ denotes the homogeneous part of degree k of P (and analogously for Q and R).

In particular, $\widetilde{f}$ is a spinning corner of the form (24) with respect to coordinates (x, y, z).

$[0:y_0:1]$ It remains to study the germ of $\widetilde{f}$ at points of the form $[0:y_0:1]$, with $y_0 \in \mathbb{C}^*$. We write $Q = b_x x + b_y y + b_z z + Q'$, and $R = c_x x + c_y y + c_z z + R'$ with $Q',\ R' \in \mathfrak{m}^2$. We center coordinates at $[0:y_0:1]$ by setting $y = y_0 + v$, and from (30) we get:

(31)

$$\begin{pmatrix} x + y_0^b z^s\Big(x + zP^{(2)}(0,y_0,1) + \langle x,z\rangle\mathfrak{m}\Big) \\ v + y_0^{b+1}z^{s+1}\Big(\widetilde{\beta} + x(b_x - c_x) + v\big((b+1)y_0^{-1}\widetilde{\beta} + (b_y - c_y)\big) + z\big(Q-R\big)^{(2)}(0,y_0,1) + \mathfrak{m}^2\Big) \\ z + y_0^b z^{s+2}\Big(\widetilde{\gamma} + xc_x + v\big(by_0^{-1}\widetilde{\gamma} + c_y\big) + zR^{(2)}(0,y_0,1) + \mathfrak{m}^2\Big) \end{pmatrix},$$

where $\widetilde{\beta} = \widetilde{\beta}(y_0) = b_z - c_z + y_0(b_y - c_y)$ and $\widetilde{\gamma} = \widetilde{\gamma}(y_0) = c_z + y_0 c_y$. This is a half corner, non-simple or simple depending on the vanishing of $\widetilde{\beta}(y_0)$.

□

Remark 3.23. In what follows, we will be interested in the existence of non-simple half corners, hence in the vanishing of the coefficient $\widetilde{\beta}(y_0)$.Three situations can occur:

- if $b_y = c_y$ and $b_z = c_z$, then all half corners are non-simple;
- if exactly one of the two equalities above hold, then all half corners are simple;
- if none of the two equalities above hold, then there exists a unique y_0 at which $\widetilde{f}$ is non-simple, and all the others produce simple half corners.

Notice that the value of $\widetilde{\beta}(y_0)$ has the same formula for spinning corners of the form (24) with $a_y = a_z = 0$ (i.e., where we allow a_x to be different from 1).

3.3.4. *Half corners*

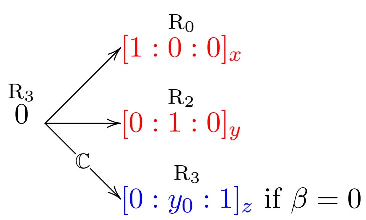
Proposition 3.24. *Let $f : (\mathbb{C}^3,0) \to (\mathbb{C}^3,0)$ be a half corner of the form (25). Its singular directions are given by:*

- $[1:0:0]$, *exceptional;*
- $[0:1:0]$, *exceptional;*
- $[0:y_0:1]$ *for all $y_0 \in \mathbb{C}$, non-exceptional degenerate (when f is non-simple).*

Let $\widetilde{f}$ be the lift of f to the blow-up of the origin in $\mathbb{C}^3$. Then

- *$\widetilde{f}$ is a simple corner at $[1:0:0]$,*
- *$\widetilde{f}$ is a spinning corner at $[0:1:0]$,*
- *if f is non-simple, then $\widetilde{f}$ is a half corner at $[0:y_0:1]$ for all $y_0 \in \mathbb{C}$.*

Here is a depiction of the situation for half corners.



Proof. The list of singular directions is easily obtained by the fact that the homogeneous part of smallest degree of $f - \mathrm{id}$ is given by $z^c \begin{pmatrix} x \\ \beta z \\ 0 \end{pmatrix}$.

$\boxed{[1:0:0]}$ We make computations in the x-chart, and get

$$\widetilde{f}(x,y,z)=\begin{pmatrix} x\big(1+x^cz^c(1+x^{-1}P\circ\pi)\big) \\ \dfrac{y+x^cz^{c+1}(\beta+Q\circ\pi)}{1+x^cz^c(1+x^{-1}P\circ\pi)} \\ z\dfrac{1+x^{c+1}z^{c+1}R\circ\pi}{1+x^cz^c(1+x^{-1}P\circ\pi)} \end{pmatrix}.$$

Since $x^2 \mid P\circ\pi$, by direct computation we get

$$\widetilde{f}(x,y,z)=\begin{pmatrix} x+x^{c+1}z^c(1+\mathfrak{m})) \\ y+x^cz^c\mathfrak{m} \\ z+x^cz^{c+1}(-1+\mathfrak{m}) \end{pmatrix},$$

which is a simple corner with respect to coordinates (x,z,y).

$\boxed{[0:1:0]}$ In the y-chart, we get

$$\widetilde{f}(x,y,z)=\begin{pmatrix} \dfrac{x+y^cz^c(x+y^{-1}P\circ\pi)}{1+y^cz^{c+1}(\beta+Q\circ\pi)} \\ y\big(1+y^cz^{c+1}(\beta+Q\circ\pi)\big) \\ z\dfrac{1+y^{c+1}z^{c+1}R\circ\pi}{1+y^cz^{c+1}(\beta+Q\circ\pi)} \end{pmatrix}.$$

By developing in formal power series, we get

$$\widetilde{f}(x,y,z)=\begin{pmatrix} x+y^cz^c\big(x+a_{020}y+\mathfrak{m}^2\big) \\ y+y^{c+1}z^c\big(z\beta+\langle yz\rangle\big) \\ z+y^cz^{c+1}\big(-\beta z+\langle yz\rangle\big) \end{pmatrix}, \tag{32}$$

and $\widetilde{f}$ is a spinning corner at $[0:1:0]$.

$\boxed{[0:y_0:1]}$ Finally, suppose $\beta=0$. By doing computation in the z-chart we get

$$\widetilde{f}(x,y,z)=\begin{pmatrix} \dfrac{x+z^c(x+z^{-1}P\circ\pi)}{1+z^{c+1}R\circ\pi} \\ \dfrac{y+z^cQ\circ\pi}{1+z^{c+1}R\circ\pi} \\ z\big(1+z^{c+1}R\circ\pi\big) \end{pmatrix}.$$

Write $Q=b_xx+b_yy+b_zz+Q'$ and $R=\gamma+c_xx+c_yy+c_zz+R'$, with $Q',\ R'\in\mathfrak{m}^2$ and expand $\widetilde{f}$ in formal power series:

$$\widetilde{f}(x,y,z)=\begin{pmatrix} x+z^c\big(x+zP^{(2)}(0,y,1)+z\langle x,z\rangle\big) \\ y+z^{c+1}\Big(b_z+(b_y-\gamma)y+b_xx+z\big(Q^{(2)}(0,y,1)-c_zy-c_yy^2\big)+z\langle x,z\rangle\Big) \\ z+z^{c+2}\Big(\gamma+z(c_z+c_yy)+z\langle x,z\rangle\Big) \end{pmatrix}.$$

We develop at the direction $[0:y_0:1]$ for some $y_0\in\mathbb{C}$, by setting $y=y_0+v$, and we get

$$\begin{pmatrix} x+z^c\big(x+zP^{(2)}(0,y_0,1)+z\mathfrak{m}\big) \\ v+z^{c+1}\Big(b_z+(b_y-\gamma)y_0+b_xx+(b_y-\gamma)v+z\big(Q^{(2)}(0,y_0,1)-c_zy_0-c_yy_0^2\big)+z\mathfrak{m}\Big) \\ z+z^{c+2}\Big(\gamma+z(c_z+c_yy_0)+z\mathfrak{m}\Big) \end{pmatrix}. \tag{33}$$

By Remark 3.10, $\widetilde{f}$ is again a half corner, non-simple or simple according to the vanishing of $\widetilde{\beta}(y_0) = b_z + (b_y - \gamma)y_0$.

□

4. Blow-up of singular curves

We study here the behaviour of the families introduced in the previous two sections when blowing-up curves contained in the singular locus S_π of f_π the lift of f at a model X_π (i.e., the singular locus of its saturated infinitesimal generator).

4.1. Patterns

We start by describing the structure of S_π when π is a point modification (adapted to f) dominating X_{π_0}. To do so we will use the following terminology.

Definition 4.1. A (rational) *pattern* is a triple (X, C, f), where X is a smooth 3-fold, C is a smooth compact rational curve inside X, and $f : (X, C) \to (X, C)$ is a holomorphic germ at C, fixing C pointwise, and defining tangent to the identity germs at p for any $p \in C$. Moreover, if $\hat{\chi}$ is the saturated infinitesimal generator of f, we impose that its singular set S contains C. The curve C is called the *core* of the pattern.

If $\mathcal{G}$ is a family of tangent to the identity germs, we say that a pattern (X, C, f) is of *type* $\mathcal{G}$ (or a $\mathcal{G}$*-pattern*) if the germ of f at p belongs to $\mathcal{G}$ for all but finitely many $p \in C$. Any such point p is called a *generic point* of the pattern, while any point at which the germ of f does not belong to $\mathcal{G}$ is called a *special point*. The *generic locus* of the pattern is the set of generic points of C, while the *special locus* is its complement.

If we need to express the fact that special points of a $\mathcal{G}$-pattern belong to some classes $\mathcal{S}$, we will talk about $\mathcal{S}$-$\mathcal{G}$-patterns. A $\mathcal{G}$-$\mathcal{G}$-pattern is a $\mathcal{G}$-pattern without special points.

Remark 4.2. One should think of patterns as germs of dynamical systems on germs of 3-dimensional manifolds around the core. These could be also described in more algebraic geometrical terms (by using formal schemes for example).

Proposition 4.3. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a germ of the form (1) satisfying the conditions of Proposition 3.14. Let $\pi : X_\pi \to (\mathbb{C}^3, 0)$ be any point modification adapted to f and dominating X_{π_0}. Let S_π be the singular set of the saturated infinitesimal generator $\hat{\chi}_\pi$ of the lift f_π of f at X_π. Then any positive-dimensional irreducible component C_π of S_π is a rational curve, and (X_π, C_π, f_π) is either a R_0-R_0-pattern or a R_2-R_3-pattern.*

Proof. By Proposition 3.14, the model X_{π_0} has finitely many singularities, which are either simple corners, degenerate spikes or spinning corners.

By blowing-up points over such families, we either stay in such families, or we obtain half corners. Non-isolated singularities may arise only when blowing-up simple corners (and in this case we get R_0-R_0-patterns), or spinning corners and half corners (and in both cases we get R_2-R_3-patterns). To conclude, we need to control the strict transform of the cores C of such patterns, when blowing-up points p in the core.

Since the singularities above simple corners are themselves simple corners, when we blow-up points in the core of R_0-R_0-patterns we still get R_0-R_0-patterns.

For the case of R_3-patterns, we need to determine the equations of the core C at any point $p \in C$ with respect to the local coordinates at p used to describe spinning corners and half corners.

It is easy to check that for R_2-R_3-patterns coming from the blow-up of either a spinning corner or a half corner, the core is given by $C = \{x = z = 0\}$ (both at the special points where we have spinning corners, or at generic points where we have half corners), see Proposition 3.22 and Proposition 3.24.

If we blow-up any point $p \in C$, the strict transform $\widetilde{C}$ of C intersects the exceptional divisor necessarily at the spinning corner at $p = [0 : 1 : 0]$, and it is locally given by $\widetilde{C} = \{x = y = 0\}$.

This situation is stable by further blow-ups, and we are done. ☐

We now study the behaviour of these patterns under blow-up of their cores.

4.2. Blow-up of R_0-R_0-patterns

Lemma 4.4. *Let (X, C, f) be a R_0-R_0-pattern given by Proposition 4.3. Then for point $p \in C$ there exists local coordinates (x, y, z) at p so that $C = \{x = y = 0\}$ and f is of the form (20), with $R \in \langle x, y\rangle$.*

Proof. From Proposition 4.3 R_0-patterns arise when blowing up simple corners, and a direct computation shows that locally f can be written as in (20) with $C = \{x = y = 0\}$. Imposing that points in C are singular for f imply that R vanishes at all points in C, which is equivalent to asking $R \in \langle x, y\rangle$. ☐

Proposition 4.5. *Let (X, C, f) be a R_0-R_0-pattern given by Proposition 4.3, and let $\pi : \widetilde{X} \to (X, C)$ be the blow-up of C. Denote by $E = \pi^{-1}(C)$ the exceptional divisor, and by $\widetilde{S}$ the set of singularities of the lift $\widetilde{f}$ of f at $\widetilde{X}$. Then $E \cap \widetilde{S}$ consists of exactly two sections $\widetilde{C}_0$ and $\widetilde{C}_\infty$ of $\pi|_E : E \to C$, not intersecting each-other. Finally, for $t = 0$ and $t = \infty$, $(\widetilde{X}, \widetilde{C}_t, \widetilde{f})$ defines a R_0-R_0-pattern, satisfying the same conditions as in Lemma 4.4.*

Proof. To study the fiber above p, we have to consider two charts of X, where in local coordinates π acts respectively as $\pi(x, y, z) = (x, xy, z)$, and $\pi(x, y, z) = (xy, y, z)$.

In the first case, $\widetilde{f}$ takes the form

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x + (x^{a+b}y^b z^c)x\big(\lambda + \langle x, z\rangle\big) \\ y + (x^{a+b}y^b z^c)y\big(\mu - \lambda + \langle x, z\rangle\big) \\ z + (x^{a+b}y^b z^c)\langle x\rangle \end{pmatrix}, \tag{34}$$

where the rest in the latter coordinate belongs to $\langle xz\rangle$ whenever $c > 0$. We study (34) at points $(0, y_0, 0)$ with $y_0 \in \mathbb{C}$.

At $y_0 = 0$, we have a singular point and we clearly get a simple corner with the wanted properties. When $y_0 \neq 0$, we get a regular point, since $\mu - \lambda \neq 0$.

The computations on the second chart are completely analogous, and left to the reader. We get another simple corner at the point associated to the direction $[0 : 1]$. ☐

4.3. Blow-up of R_2-R_3-patterns

Lemma 4.6. *Let (X, C, f) be a R_2-R_3-pattern given by Proposition 4.3. For any point $p \in C$, there are coordinates (x, y, z) so that $C = \{x = z = 0\}$, $p = (0, y_0, 0)$ and f has the form:*

$$f(x,y,z) = \begin{pmatrix} x + y^b z^c(x + P) \\ y + y^B z^{c+1} Q \\ z + y^b z^{c+2} R \end{pmatrix}, \tag{35}$$

with $c \geq 1$ and $P \in \langle z\rangle$. Moreover either $B - 1 = b \geq 1$, or $b = B = 0$.

Proof. From Proposition 4.3, R_3-patterns arise when blowing up spinning corners and (non-simple) half corners. A direct computation shows that there one can find coordinates (x, y, z) at p so that f is of the form (23) or (25), and $C = \{x = z = 0\}$, or $C = \{x = y = 0\}$ for spinning corners. Being (23) symmetric on y, z, we may assume we are in the first case. The statement follows from rewriting (31) of Proposition 3.22 under the form (35), and from (33) of Proposition 3.24. ☐

Proposition 4.7. *Let (X, C, f) be a R_2-R_3-pattern given by Proposition 4.3, and let $\pi : \widetilde{X} \to (X, C)$ be the blow-up of C. Denote by $E = \pi^{-1}(C)$ the exceptional divisor, and by $\widetilde{S}$ the set of singularities of the lift $\widetilde{f}$ of f at $\widetilde{X}$. Then $E \cap \widetilde{S}$ consists of exactly two sections $\widetilde{C}_0$ and $\widetilde{C}_\infty$ of $\pi|_E : E \to C$, not intersecting each other. Finally,*

- *$(\widetilde{X}, \widetilde{C}_\infty, \widetilde{f})$ defines a R_0-R_0-pattern, satisfying the same conditions as in Lemma 4.4;*

- $(\widetilde{X}, \widetilde{C}_0, \widetilde{f})$ *defines a* R_2*-*R_3*-pattern, admitting local coordinates of the form* (35).

Proof. Let $p \in C$ be any point in the core, and pick (x, y, z) local coordinates so that f is written as in (35). We write $P = z\big(\alpha(y) + \langle x, z\rangle\big)$. To study the fiber above p, we have to consider two charts of X, where in local coordinates π acts respectively as $\pi(x, y, z) = (x, y, xz)$, and $\pi(x, y, z) = (xz, y, z)$.

In the first case, $\widetilde{f}$ takes the form

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x\Big(1 + x^c y^b z^c\big(1 + z\alpha(y) + \langle xz\rangle\big)\Big) \\ y + x^{c+1} y^B z^{c+1} Q \circ \pi \\ z\Big(1 + x^c y^b z^c\big(-1 - z\alpha(y) + \langle xz\rangle\big)\Big) \end{pmatrix}. \tag{36}$$

The singular points if $\widetilde{f}$ in the exceptional divisor $E = \{x = 0\}$ are of the form $(0, y_0, z_0)$ with $z_0(1 + z_0\alpha(y_0)) = 0$.

When y_0 varies, the closure of points $z_0 = 0$ define a rational curve C_∞. From (36) we deduce that $(\widetilde{X}, C_\infty, \widetilde{f})$ is a R_0-R_0-pattern satisfying the conditions of Lemma 4.4.

To study the points satisfying $z_0\alpha(y_0) = -1$, we work on the second chart. We get

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x + y^b z^c\big(x + \alpha(y) + \langle z\rangle\big)\Big) \\ y + y^B z^{c+1} Q \circ \pi \\ z + y^b z^{c+2} R \circ \pi \end{pmatrix}. \tag{37}$$

In this chart, the singularities in $E = \{z = 0\}$ have the form $q_0 = (x_0, y_0, 0)$ with $x_0 = -\alpha(y_0)$. These points form a rational curve C_0 not intersecting C_∞, for which $(\widetilde{X}, C_0, \widetilde{f})$ is a R_2-R_3-pattern. More precisely, $\widetilde{f}$ is a spinning corner at q_0 exactly when $y_0 = 0$ and $b \geq 1$, i.e., if and only if f is a spinning corner at p.

By the change of coordinates $(x, y, z) \mapsto (x + \alpha(y), y, z)$, we get an expression of the form (35). □

We sum up the study of blow ups of singular points and patterns in Figure 3.

4.4. Proof of Theorem A

Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a generic germ of the form (1) (i.e., with parameters P, Q, R satisfying the conditions of Proposition 3.14).

Any regular modification $\pi : X_\pi \to (\mathbb{C}^3, 0)$ adapted to f and dominating π_0 is either a point modification, or it dominates $\widetilde{\pi}_0$ given by Proposition 3.14.

By Proposition 4.3, in the first case the only patterns that appear are R_0-R_0-patterns or R_2-R_3-patterns. In the second case, patterns may appear from regular modifications adapted to the dynamics above simple corners or spinning corners, which are again R_0-R_0-patterns or R_2-R_3-patterns. By Proposition 4.5 and Proposition 4.7, no new patterns arise when blowing-up cores these two type of patters, and similarly the blow-up of points doesn't provide new type of special points in a pattern. Hence for any such modification π, we have only simple corners, degenerate spikes, spinning corners and half corners, which admit no non-exceptional non-degenerate singular directions.

5. Invariant curves and parabolic manifolds

5.1. Invariant curves

5.1.1. *Degenerate spikes*

Proposition 5.1. *Let* $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ *be a degenerate spike of the form* (21)*. Then there exists a unique* f*-invariant formal curve* C *not contained in* $E := \{z = 0\}$*. Moreover,* C *is smooth and transverse to* E*.*

Proof. By Proposition 3.21 (see also Figure 3), there exists a unique sequence of infinitely near points $\mathfrak{p}$ consisting of singular points for the lifts of f. By Proposition 1.25, these

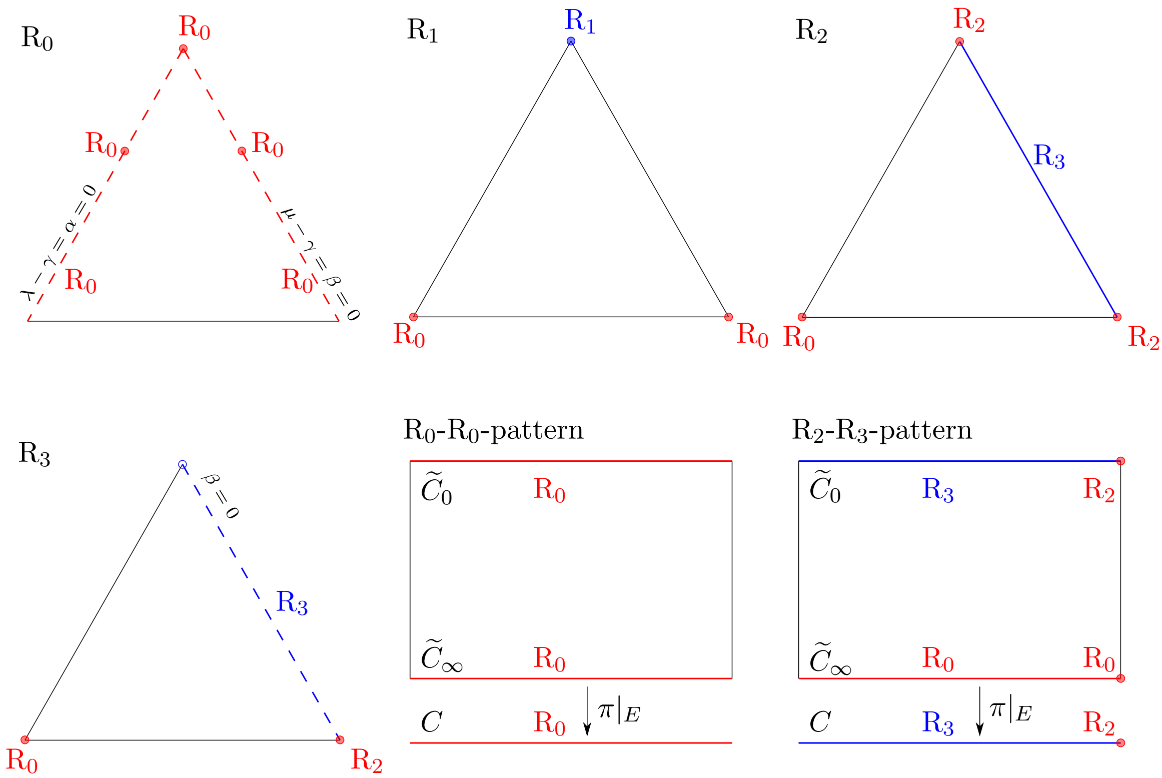


FIGURE 3. Blow-up of special families and patterns.

points induce a formal invariant curve $C_{\mathfrak{p}}$ which is f-invariant, smooth and transverse to E.

Let now C be a formal f-invariant curve. Since curves are resolved by point blow-ups, there exists a point modification $\pi : X_\pi \to (\mathbb{C}^3, 0)$ so that the curve C lifts to C_π which is smooth and transverse to the exceptional divisor E_π of π. Denote by f_π the lift of f at X_π.

Then C_π must intersect E_π transversely at a point p, and f_π must be a degenerate spike at p. In fact, by Theorem 1.29 f_π admits a parabolic manifold tangent to C_π, and by Corollary 1.21 we deduce that p must be a singular point for f_π. Since, by Remark 3.18, simple corners don't admit formal invariant curves (not lying in the exceptional divisor), we must have that p is a degenerate spike.

Since there is a unique sequence $\mathfrak{q}$ of infinitely near points consisting of singular points and satisfying the conditions of Proposition 1.23 above a degenerate spike, we deduce that $C_\pi \equiv C_{\mathfrak{q}}$, and by projecting down, we get $C \equiv C_{\mathfrak{p}}$. □

5.1.2. *Half corners*

Proposition 5.2. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a half corner of the form (25). Write $Q = b_x x + b_y y + b_z z + Q'$, $R = \gamma + R'$ with $Q' \in \mathfrak{m}^2$ and $R' \in \mathfrak{m}$. Set $E = \{z = 0\}$.*

- *If $\beta \neq 0$ (i.e., the half corner is simple), then f does not admit any formal f-invariant curve transverse to E.*
- *If $\beta = 0$ (i.e., the half corner is non-simple), and*

$$b_y \notin \gamma \mathbb{N}^*, \tag{38}$$

then f admits a unique smooth f-invariant formal curve C transverse to E.

Proof. Let us write f under the following form:

$$f(x,y,z)=\begin{pmatrix} x+z^c(x+P)\\ y+z^{c+1}(\beta+Q)\\ z+z^{c+2}R\end{pmatrix} \tag{39}$$

with $P=\sum_{k\geq 1} z^k a_k(y)+\langle xz\rangle$.

(Step 1) We want to show that up to formal conjugacy, we can suppose that $a_k\equiv 0$ for all k, hence $P\in\langle xz\rangle$.

When $k=1$, the condition $a_1\equiv 0$ corresponds to having that the R_2-R_3-pattern obtained after blowing-up $\{x=z=0\}$ has core which corresponds to the intersection of the strict transform of $\{x=0\}$ and the exceptional divisor.

Using (37) and arguing by induction, having $a_1\equiv\ldots\equiv a_k\equiv 0$ corresponds to the analogous statement for the iterated blow-up h-times, $h=1,\ldots,j$, of the cores of the R_2-R_3-patterns we meet at each step. We set $X_0=\mathbb{C}^3$ (as a germ at the origin), and X_k to be the blow-up of X_{k-1} along $\{x=z=0\}$. Since a change of coordinates of the form $x'=x+\alpha(y)$ in X_k corresponds to a change of coordinates of the form $x'=x+z^k\alpha(y)$, in X_0, these change of coordinates converge to a formal change of coordinates $x'=x+A(y,z)$.

(Step 2) By Step 1, we may assume $P\in\langle xz\rangle$. This corresponds to having the surface $S=\{x=0\}$ invariant by f.

Let C be an f-invariant curve. If $\beta\neq 0$, the only singular directions are $[1:0:0]$ and $[0:1:0]$, which are exceptional. By Proposition 1.27, C must be tangent to one of these directions, hence it cannot be transverse to E.

Suppose now that $\beta=0$ and C is transverse to E. Then, again by Proposition 1.27, C must be tangent to a direction of the form $[0:y_1:1]$ for some $y_1\in\mathbb{C}$. Let $\pi_1: X_1\to(\mathbb{C}^3,0)$ be the blow-up of the origin $p_0=0$, and let p_1 be the point corresponding to $[0:y_1:1]$. By computing the lift f_{π_1} of f with respect to the z-chart, we get another half corner of the form

$$f_{\pi_1}(x,y,z)=\begin{pmatrix} x+z^c\big(x+\langle xz\rangle\big)\\ y+z^{c+1}\big(b_z+(b_y-\gamma)y+b_xx+\langle z\rangle\big)\\ z+z^{c+2}\big(\gamma+\langle z\rangle\big)\end{pmatrix}. \tag{40}$$

The germ f_{π_1} cannot be simple at p_1, as the strict transform of C cannot be tangent to the strict transform of E, which is again given by $\{z=0\}$ in these coordinates. Hence we must have $b_z+(b_y-\gamma)y_1=0$.

By induction, we infer that C must be contained in $S=\{x=0\}$.

(Step 3) Let p_n be the sequence of infinitely near points associated to C, and let $\pi_n: X_n\to(\mathbb{C}^3,0)$ be the blow-up of the points $p_0,\ldots,p_{n-1}$. The lift f_{π_n} of f at $p_n\in X_n$ (with computations done always in the z-chart) is a half corner of the form (40), with second coordinate given by

$$y+z^{c+1}\big(b_z^{(n)}+(b_y-n\gamma)y+\langle x,z\rangle\big),$$

for some $b_z^{(n)}\in\mathbb{C}$. In particular, if $b_y\notin\gamma\mathbb{N}^*$, then there exists a unique $y_n\in\mathbb{C}$ so that $b_z^{(n)}+(b_y-n\gamma)y_n=0$. We deduce in this case the uniqueness of the f-invariant curve transverse to E. □

Remark 5.3. Some of the steps in the proof of Proposition 5.2 can be replaced by alternative arguments. For example, Step 3 can be replaced by studying directly the action of $g=f|_S: S\to S$, and its saturated infinitesimal generator $\hat{\xi}$. In fact, $\hat{\xi}$ has a singularity at the origin if and only if $\beta=0$ and in this case its linear part is $(b_yy+b_zz)\partial_y+\gamma z\partial_z$. As long as b_y and γ do not both vanish, we get a log-canonical singularity. When $b_y\notin\gamma\mathbb{Q}_{>0}$, the singularity is in fact canonical, and we have exactly two complex separatrices: one given by $E\cap S$, and the other transverse to E in S. The case $b_y\in\gamma(\mathbb{Q}_{>0}\setminus\mathbb{N}^*)$ can be also treated explicitly using normal forms for 2-dimensional vector fields.

The existence of formal invariant curves for non-simple half corners can be deduced directly from Proposition 3.24. In fact, the computations made in the proof, show that when blowing-up such a germ, we obtain half corners with parameters

$$\widetilde{\beta}(y_0) = b_z + (b_y - \gamma)y_0, \qquad \widetilde{b}_y = b_y - \gamma, \qquad \widetilde{\gamma} = \gamma,$$

where $y_0 \in \mathbb{C}$. In particular, as long as $b_y \notin \gamma\mathbb{N}^*$, we may construct an increasing sequence of infinitely near points which are non-simple half corners, which identify a formal invariant curve by Proposition 1.25.

One can also replace Step 3 of Proposition 5.2 by a direct computation, following the techniques developed in [Rug12, Rug13, Rug15]. This would correspond to parametrize a curve C transverse to E inside S as $(0, \hat{y}(t), t^e)$ for some $e \geq 1$ and formal power series $\hat{y} = \sum_{n\geq 1} y_n t^n \in \mathbb{C}[\![t]\!]$. We then impose the invariance condition

$$y \circ f(0, y(t), t^e) = \hat{y}\Big(\big(z \circ f(0, y(t), t^e)\big)^{\frac{1}{e}}\Big), \tag{41}$$

and solve this equation by expanding everything in formal power series on t.

When $\beta \neq 0$, the contradiction to the existence is obtained by checking (41) at order $e(c+1)$. When $\beta = 0$, for $e = 1$ and for any $n > c+1$, (41) contains a term of the form

$$\big(b_y + (n-c-1)\gamma\big)y_{n-c-1} = \mathrm{l.\,o.\,t.}\,,$$

where l. o. t. is a polynomial expression depending on y_h for $h < n-c-1$. We deduce from this the existence and uniqueness of $\hat{y}$ solution of (41).

5.1.3. *Spinning corners*

In the following result, we say that an irreducible curve C is transverse to $E = \{yz = 0\}$ if the strict transforms of E and C do not intersect on the exceptional divisor of the blow-up of the origin. Notice that this definition does not coincide with the common definition of transversality when C is singular.

Corollary 5.4. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a spinning corner of the form (23). Write $Q = b_x x + b_y y + b_z z + \mathfrak{m}^2$, and $R = c_x x + c_y y + c_z z + \mathfrak{m}^2$. Set $E = \{yz = 0\}$.*

- *If $b_y = c_y$ and $b_z = c_z$, and they are not all vanishing, then there exists infinitely many f-invariant formal smooth curves transverse to E.*
- *If $b_y \neq c_y$ and $b_z \neq c_z$, and*
$$(c_z - b_z)(b_y - c_y) \notin (b_y c_z - b_z c_y)\mathbb{N}^*, \tag{42}$$
then there exists a unique formal f-invariant curve smooth and transverse to E.
- *If exactly one of the two equalities $b_y = c_y$ and $b_z = c_z$ is satisfied, then there are no formal f-invariant curves transverse to E.*

Proof. Consider the blow-up of the origin. From Proposition 3.22, the points $p(y_0)$ corresponding to the directions $[0 : y_0 : 1]$ for $y_0 \in \mathbb{C}^*$ have a non-simple half corner when y_0 satisfies $b_z - c_z + y_0(b_y - c_y) = 0$. The parameters of the half corner are given (up to a factor y_0^b) by:

$$\widetilde{b}_y = y_0(b_y - c_y) = c_z - b_z, \qquad \widetilde{\gamma} = c_z + y_0 c_y,$$

see (31).

- If $b_y = c_y$ and $b_z = c_z$, then p_0 is a non-simple half corner for all values of $y_0 \in \mathbb{C}^*$, with parameters $\widetilde{b}_y = 0$ and $\widetilde{\gamma} = c_z + y_0 c_y$. As long as we do not have $c_y = c_z = 0$, then for all y_0 but at most one special value, the corresponding non-simple half corner at $p(y_0)$ satisfies the non-resonance condition (38), and there exists a unique invariant curve at $p(y_0)$ and transverse to the exceptional divisor.
- If $b_y \neq c_y$ and $b_z \neq c_z$, the only non-simple half corner is obtained at $p(y_0)$ with $y_0 = \dfrac{c_z - b_z}{b_y - c_y}$. In this case, we have $\widetilde{\gamma} = \dfrac{\delta}{b_y - c_y}$, where $\delta = b_y c_z - c_y b_z$.

If $\delta = 0$, being $\widetilde{b}_y \neq 0$, the condition (38) is satisfied. If $\delta \neq 0$, then the condition (38) gives exactly (42).

- If exactly one of the two equalities $b_y = c_y$ and $b_z = c_z$ is satisfied, then $p(y_0)$ is a simple half corner for all $y_0 \in \mathbb{C}^*$. By Proposition 5.2, we have no invariant formal curves transverse to the exceptional divisor, and hence no f-invariant formal curves transverse to E.

□

Remark 5.5. Corollary 5.4 does not deal with the existence of formal invariant curves that may be tangent to the exceptional divisor.

Given a spinning corner in the form (23), and using the notations of Corollary 5.4, we set $A = \begin{pmatrix} b_y & b_z \\ c_y & c_z \end{pmatrix}$.

Without further mention, germs or patterns that we blow-up will be considered in the special coordinates used to obtain Figure 3.

Spinning corners may arise either blowing-up other spinning corners, at the point associated to $[0:1:0]$ and $[0:0:1]$; or by blowing-up a half corner, at the point associated to $[0:1:0]$. Finally they are also obtained by blowing-up the core of a R_2-R_3-pattern.

From spinning corners — Assume f is a spinning corner, and consider the lift $\widetilde{f}$ with respect to the blow-up of the origin, at the point associated to $[0:0:1]$. At this point, $\widetilde{f}$ is a spinning corner, with associated matrix $\widetilde{A} = \begin{pmatrix} 0 & b_z - c_z \\ 0 & c_z \end{pmatrix}$. We can apply Corollary 5.4, and deduce that if $b_z \neq 2c_z$, there are no invariant curves transverse to the exceptional divisor for $\widetilde{f}$, while if $b_z = 2c_z \neq 0$, then there exists infinitely many invariant curves.

By repeating this argument, we get infinitely many invariant curves as long as $b_z/c_z \in \mathbb{N}^*$, or $c_y/b_y \in \mathbb{N}^*$ (this last condition is obtained by exchanging the role of y and z and studying the direction $[0:1:0]$).

From half corners — We need to study the direction $[0:1:0]$. In this case, we get $\widetilde{A} = \begin{pmatrix} 0 & \beta \\ 0 & -\beta \end{pmatrix}$. Hence, for simple half corners, we have $\beta \neq 0$, and no invariant curve transverse to the exceptional divisor exists. For non-simple half corners, we have $\beta = 0$, and the existence of invariant curves depend on the terms of higher degrees of f.

From R_2-R_3-patterns — In this case, it is easy to check from (37) that the spinning corner $\widetilde{f}$ above a spinning corner of the core of the pattern satisfies $\widetilde{A} = A$, and we can apply directly Corollary 5.4.

One can also use formal computation techniques (see Remark 5.3), which show again how the existence of invariant curves may depend on the higher order terms of P, Q, R.

5.2. Parabolic manifolds

In this section we describe how to get Ramis-Sibuya normal forms from the special classes of tangent to the identity germs introduced in the previous sections, and apply Theorem 1.33 in order to obtain parabolic manifolds attached to such germs.

5.2.1. *Degenerate spikes*

We start from degenerate spikes, for which we are able to describe explicitly the algorithm that brings them into Ramis-Sibuya normal form.

Lemma 5.6. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a degenerate spike. Then f is formally conjugated to a germ of the form*

$$\widetilde{f}(x,y,z) = \begin{pmatrix} x + z^c\Big(\lambda x\big(1 + P_0(z) + \langle xy \rangle\big)\Big) \\ y + z^c\Big(\mu y\big(1 + Q_0(z) + \langle xy \rangle\big)\Big) \\ z + z^{c+1}\mathfrak{m} \end{pmatrix}. \tag{43}$$

Proof. Firstly, we may assume that f is on the form (21).

The saturated infinitesimal generator $\hat{\chi}_f$ takes the form

$$\hat{\chi}_f = \big(\lambda x + p(x,y,z)\big)\partial_x + \big(\mu y + q(x,y,z)\big)\partial_y + zr(x,y,z)\partial_z,$$

where $p, q \in \mathfrak{m}^2$ and $r \in \mathfrak{m}$.

We recall that any vector field can be conjugated to a vector field in Poincaré-Dulac normal form, where only resonant monomials are allowed in the formal power expansion of the coefficients of the vector field (see, e.g., [IY08, Chapter I.4]). In our case, resonant monomials for the first, second and third coordinates respectively, are of the form $x^{i+1}y^j z^k$, $x^i y^{j+1} z^k$ and $x^i y^j z^k$ respectively, where $\lambda i + \mu j = 0$.

When $\mu/\lambda \in \mathbb{C} \setminus \mathbb{Q}$, we get $i = j = 0$, while k ranges among positive integers. When $\mu/\lambda = -n/m \in \mathbb{Q}_{<0}$ (assume m and n coprime), we get that $i = hn$ and $j = hm$ for some $h \in \mathbb{N}^*$.

Since $n, m \geq 1$, the Poincaré-Dulac normal forms can be written as

$$\lambda x\big(1 + p_0(z) + \langle xy\rangle\big)\partial_x + \mu y\big(1 + q_0(z) + \langle xy\rangle\big)\partial_y + \big(r_0(z) + \langle xy\rangle\big)\partial_z,$$

where p_0, q_0, r_0 are suitable formal power series vanishing at 0.

In general, the change of coordinates that puts $\hat{\chi}_f$ in its Poincaré-Dulac normal form could move the exceptional divisor $\{z = 0\}$; in this particular case, however, we content ourselves with putting f into a slightly less precise form with a change of coordinates adapted to the exceptional divisor. Let us consider a change of coordinates of the form $\Phi(x,y,z) = (x + \phi(x,y,z), y + \psi(x,y,z), z)$ with $\phi, \psi \in \mathfrak{m}^2$; by suitably choosing ϕ, ψ, we conjugate $\hat{\chi}_f$ to a vector field of the form

$$\hat{\chi} = \lambda x\big(1 + p_0(z) + \widetilde{p}(x,y,z)\big)\partial_x + \mu y\big(1 + q_0(z) + \widetilde{q}(x,y,z)\big)\partial_y + \big(z\widetilde{r}(x,y,z)\big)\partial_z, \tag{44}$$

with p_0 and q_0 are as above, $\widetilde{p}, \widetilde{q} \in \langle xy\rangle$, and $\widetilde{r} \in \mathfrak{m}$.

Since the exceptional divisor $\{z = 0\}$ is invariant by Φ, if we conjugate f by Φ, we obtain a tangent to the identity germ $\widetilde{f}$ of the form (43), whose associated vector field will be $\chi_{\widetilde{f}} = z^c \hat{\chi}$. □

Remark 5.7. When f has the form (43) then the unique formal invariant curve given by Proposition 5.1 is given by $C = \{x = y = 0\}$.

Notice that by replacing Φ by its truncation at order N in the proof of Lemma 5.6, we may assume that a degenerate spike germ f is analytically conjugated to a map of the form (43) up to terms in $\mathfrak{m}^M$, for M large enough. In this case we may assume that C is arbitrarily tangent to the z-axis.

Since Ramis-Sibuya normal forms depend only on the truncation at a suitable high order of a given germ, there is no loss of generality in working with germs up to formal conjugacy.

Proposition 5.8. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a degenerate spike and let C be the unique f-invariant formal curve given by Proposition 5.1. Suppose that C is not pointwise fixed by f.*

For any $n \in \mathbb{N}^*$, consider $\pi_n : X_n \to (\mathbb{C}^3, 0)$ the point modification, obtained recursively starting by π_1 the blow-up of $p_0 = 0$, and π_n obtained from π_{n-1} by blowing-up the point $p_{n-1} := \pi_{n-1}^{-1}(0) \cap C_{n-1}$, where C_{n-1} is the strict transform of C by π_{n-1}.

Denote by f_n the lift of f at X_n, as a germ at p_n. Then for $n \gg 0$, and up to an analytic change of coordinates, the pair (f_n, C_n) is in Ramis-Sibuya normal form.

Proof. By Lemma 5.6, we may assume that $C = \{x = y = 0\}$, and f is of the form (43). We can write the third coordinate of f as

$$z \circ f = z + z^{c+1} R = z + z^{c+1}\big(h(z) + \langle x, y\rangle\big).$$

Notice that $f|_C(z) = z + z^{c+1}h(z)$: up to a polynomial change of coordinates in the variable z, we may assume that

$$h(z) = -z^e + \beta z^{c+2e} + \langle z^{c+2e+1}\rangle, \tag{45}$$

with $e = \mathrm{ord}_0(h)$. Notice that performing this change of coordinates changes the values of λ and μ, but their ratio stays invariant (see Remark 5.9).

The blow-ups π_n can be computed with respect to the z-chart, and the point p_n corresponds to the origin in this chart. By direct computation we get

$$f_n(x,y,z) = \begin{pmatrix} \dfrac{x + z^c\big(x(\lambda + P_0(z)) + z^n\langle xy\rangle\big)}{\Big(1 + z^c\big(h(z) + z^n\langle x,y\rangle\big)\Big)^n} \\ \dfrac{y + z^c\big(y(\mu + Q_0(z)) + z^n\langle xy\rangle\big)}{\Big(1 + z^c\big(h(z) + z^n\langle x,y\rangle\big)\Big)^n} \\ z\Big(1 + z^c\big(h(z) + z^n\langle x,y\rangle\big)\Big) \end{pmatrix}. \tag{46}$$

Set $r = c + e \geq c + 1$, and take $n > c + 2e$. Then (46) can be rewritten as

$$f_n(x,y,z) = \begin{pmatrix} x\big(1 + z^c(\lambda + P_0(z)) + nz^r)\big) + \langle z^{r+1}\rangle \\ y\big(1 + z^c(\mu + Q_0(z)) + nz^r)\big) + \langle z^{r+1}\rangle \\ z - z^{r+1} + \beta z^{2r+1} + \langle z^{2r+2}\rangle \end{pmatrix}, \tag{47}$$

which is on the form (2). □

Remark 5.9. When we change coordinates to obtain (45), the values of λ and μ are replaced by $\lambda h_e^{-c/r}$ and $\mu h_e^{-c/r}$, where $h(z) = h_e z^e + \langle z^{e+1}\rangle$.

Suppose we have a degenerate spike of the form

$$f(x,y,z) = \begin{pmatrix} x + z^c a(x,y,z) \\ y + z^c b(x,y,z) \\ z + z^{c+1}R(x,y,z) \end{pmatrix}, \tag{48}$$

and we want to put it under the form used in the computations of Proposition 5.8. This boils down to first put the linear part of (a,b) (evaluated in $z = 0$) in diagonal form, and then perform a change of coordinates $x \mapsto x + \alpha(z)$ and $y \mapsto y + \beta(z)$ for suitable formal power series $\alpha, \beta \in z\mathbb{C}[\![z]\!]$. In particular, if we need to know the action of $f|_C$ (where C is the unique formal f-invariant curve transverse to $\{z = 0\}$) up to order $c + 1 + e$, we only need to know the values of α and β up to order e.

By Theorem 1.33, we can describe easily the parabolic manifolds attached to a degenerate spike f. The situation is particularly simple when the multiplicity $r+1$ of $(f - \mathrm{id})|_C$ at the origin is minimal, case that covers the study above the points p_1 and p_2 in the proof of Theorem B.

Corollary 5.10. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a degenerate spike (of Siegel type) of the form (21), and let C be the unique f-invariant formal curve given by Proposition 5.1. Suppose that C is not pointwise fixed by f, and let $r + 1$ be the multiplicity of $(f - \mathrm{id})|_C$ at the origin.*

Then f admits r parabolic domains Δ_k. When $e := r - c = 1$, these parabolic domains have dimension 1 or 2.

Proof. This is a direct consequence of Proposition 5.8 and Theorem 1.33. In particular, when writing (47) under the form (2) when $e = 1$, we get the values $d_1(z) = \lambda z^c$ and $d_2(z) = \mu z^c$. Being $\lambda/\mu =: -\eta \in \mathbb{R}_{<0}$, we deduce that the vectors $R_1(\xi)$ and $R_2(\xi)$ associated to the attracting direction ξ for the Ramis-Sibuya normal form (3) satisfy $R_1(\xi) + \eta R_2(\xi) = 0$. Hence either $R_j(\xi) = 0$ for $j = 1, 2$, and in this case ξ is a saddle direction for both coordinates, and the dimension of the associated parabolic manifold is 1, or $R_j(\xi) \neq 0$, and in this case ξ is a node direction for exactly one of the two coordinates, and the associated parabolic manifold has dimension 2. □

5.2.2. *Half corners*

For half corners, describing the explicit reduction to Ramis-Sibuya normal forms is more involved. We describe here the situation where the multiplicity $r+1$ of $(f-\mathrm{id})|_C$ at the origin is minimal. This case corresponds exactly to non-simple half corners of the form (25) with $\gamma \neq 0$, and it covers the study above the points p_3 and p_4 in the proof of Theorem B.

Proposition 5.11. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a non-simple half corner of the form (25), with $\gamma \neq 0$. Suppose that f admits a f-invariant smooth formal curve C transverse to $\{z = 0\}$.*

For any $n \in \mathbb{N}^*$, consider $\pi_n : X_n \to (\mathbb{C}^3, 0)$ the point modification, obtained recursively starting by π_1 the blow-up of $p_0 = 0$, and π_n obtained from π_{n-1} by blowing-up the point $p_{n-1} := \pi_{n-1}^{-1}(0) \cap C_{n-1}$, where C_{n-1} is the strict transform of C by π_{n-1}.

Denote by f_n the lift of f at X_n, as a germ at p_n. Then for $n \gg 0$, and up to an analytic change of coordinates, the pair (f_n, C_n) is in Ramis-Sibuya normal form.

Proof. Up to a formal change of coordinates, we may assume that $C = \{x = y = 0\}$, and f is of the form (25) with $P \in \langle x, y\rangle\mathfrak{m}$ and $Q \in \langle x, y\rangle$. We can write the third coordinate of f as

$$z \circ f = z + z^{c+2}R = z + z^{c+2}\big(h(z) + \langle x, y\rangle\big),$$

with $h(0) = \gamma \neq 0$

Notice that $f|_C(z) = z + z^{c+2}h(z)$: up to a polynomial change of coordinates in the variable z, we may assume that

$$h(z) = -1 + \beta z^{c+1} + \langle z^{c+2}\rangle, \tag{49}$$

which is compatible with the third coordinate of (2) when we set $r = c+1$. In this case, the first coordinate of f becomes $x + z^c(\alpha^c x + \langle x, y\rangle\mathfrak{m})$, where $\alpha^r = 1/\gamma$. To sum up, we can write f in the form:

$$f(x, y, z) = \begin{pmatrix} x + z^c(\alpha^c x + a_{101}xz + a_{011}yz + \mathfrak{r}) \\ y + z^{c+1}(b_x x + b_y y + \langle x, y\rangle\mathfrak{m}) \\ z + z^{c+2}(h(z) + \langle x, y\rangle) \end{pmatrix}, \tag{50}$$

for suitable $a_{101}, a_{011}, b_x, b_y \in \mathbb{C}$, and we set $\mathfrak{r} = \langle x, y\rangle(\langle x, y, z^2\rangle)$.

The blow-ups π_n can be computed with respect to the z-chart, and the point p_n corresponds to the origin in this chart. By direct computation we get

$$f_n(x, y, z) = \begin{pmatrix} \dfrac{x + z^c\big(\alpha^c x + a_{101}xz + a_{011}yz + z^2\langle x, y\rangle\big)}{\Big(1 + z^{c+1}\big(h(z) + z^n\langle x, y\rangle\big)\Big)^n} \\ \dfrac{y + z^{c+1}\big(b_x x + b_y y + z\langle x, y\rangle\big)}{\Big(1 + z^{c+1}\big(h(z) + z^n\langle x, y\rangle\big)\Big)^n} \\ z\Big(1 + z^{c+1}\big(h(z) + z^n\langle x, y\rangle\big)\Big) \end{pmatrix}. \tag{51}$$

Take $n > c+1$. Then (51) can be rewritten as

$$f_n(x, y, z) = \begin{pmatrix} x\big(1 + (\alpha z)^c + (a_{101} + n)z^{c+1}\big) + a_{011}yz^{c+1} + \langle z^{c+2}\rangle \\ \\ y\big(1 + (b_y + n)z^{c+1}\big) + b_x xz^{c+1} + \langle z^{c+2}\rangle \\ \\ z - z^{c+2} + \beta z^{2c+3} + \langle z^{2c+4}\rangle \end{pmatrix}, \tag{52}$$

which is on the form (2) with $r = c+1$, $d_1(z) = (\alpha z)^c$ and $d_2(z) \equiv 0$, as long as $a_{011} = b_x = 0$.

We claim that this last property can be achieved by performing a change of coordinates (before blowing up) of the form $\Phi(x, y, z) = (x + \lambda yz, y + \mu xz, z)$. Its inverse is given by

$$\Phi^{-1}(x, y, z) = \left(\frac{x - \lambda yz}{1 - \lambda\mu z^2}, \frac{y - \mu xz}{1 - \lambda\mu z^2}, z\right).$$

If f is of the form (50), then we get

$$\Phi^{-1} \circ f \circ \Phi(x,y,z) = \Phi^{-1}\begin{pmatrix} x + \lambda yz + z^c\big(\alpha^c x + a_{101}xz + (a_{011} + \alpha^c\lambda)yz + \mathfrak{r}\big) \\ y + \mu xz + z^{c+1}(b_x x + b_y y + \langle x,y\rangle\mathfrak{m}) \\ z + z^{c+2}(h(z) + \langle x,y\rangle) \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{x - \lambda\mu xz^2 + z^c\big(\alpha^c x + a_{101}xz + (a_{011} + \alpha^c\lambda)yz + \mathfrak{r}\big)}{1 - \lambda\mu z^2(1 + \langle z^{c+1}\rangle)} \\ \dfrac{y - \lambda\mu yz^2 + z^{c+1}\big((b_x - \mu\alpha^c)x + b_y y + \langle x,y\rangle\mathfrak{m}\big)}{1 - \lambda\mu z^2(1 + \langle z^{c+1}\rangle)} \\ z + z^{c+2}(h(z) + \langle x,y\rangle) \end{pmatrix}$$

$$= \begin{pmatrix} x + z^c\big(\alpha^c x + a_{101}xz + (a_{011} + \alpha^c\lambda)yz + \mathfrak{r}\big) \\ y + z^{c+1}\big((b_x - \mu\alpha^c)x + b_y y + \langle x,y\rangle\mathfrak{m}\big) \\ z + z^{c+2}(h(z) + \langle x,y\rangle) \end{pmatrix}.$$

It suffices to set $\lambda = -\alpha^{-c}a_{011}$ and $\mu = \alpha^{-c}b_x$. □

Corollary 5.12. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a non-simple half corner satisfying the same hypotheses of Proposition 5.11.*

Then f admits $r = c + 1$ parabolic manifolds Δ_k, which are of dimension 1 or 2.

Proof. By Proposition 5.8, we may assume up to point blow-ups that f is in the Ramis-Sibuya normal form (52). Denote by R_1 and R_2 the invariants associated to the Ramis-Sibuya normal form given by (3). Let $\xi = e^{2\pi ik/r}$ be a r-th root of unity. Since $d_2 = 0$, then $R_2 = 0$, and all directions are a saddle in the second coordinate. For the first coordinate, we get a node or a saddle depending on the sign of the real part of $\alpha^c\xi^c$. We conclude by Theorem 1.33. □

5.3. Proof of Theorem B

Here we apply the results of the previous sections to our example (1). From Proposition 3.14, we get a model with 11 singularities. Among those, we get two degenerate spikes at p_1 and p_2, and three spinning corners at $p_{3,2}$, $p_{4,2}$ and q_1. The others are simple corners and do not give rise to parabolic manifolds, see [AT03].

p_1 At p_1 the lift of f takes the form (6), which is a degenerate spike. To compute the parameters appearing in Proposition 5.8 and Corollary 5.10, we need some further change of coordinates (see Remark 5.9).

After performing the change of coordinates $x' = x + y$, $y' = x - y$, we get

$$f_1(x',y',z) = \begin{pmatrix} x' + z^2\big(-x' + (P_{004} + Q_{004})z + \mathfrak{m}^2\big) \\ y' + z^2\big(y' + (P_{004} - Q_{004})z + \mathfrak{m}^2\big) \\ z + z^3\big(-\frac{1}{2}x' - \frac{1}{2}y' + zR_{004} + \mathfrak{m}^2\big) \end{pmatrix}. \tag{53}$$

After the change of coordinates $x'' = x' - (P_{004} + Q_{004})z$, $y'' = y' + (P_{004} - Q_{004})z$ we finally get

$$f_1(x'',y'',z) = \begin{pmatrix} x'' + z^2\big(-x'' + \mathfrak{m}^2\big) \\ y'' + z^2\big(y'' + \mathfrak{m}^2\big) \\ z + z^3\big(-\frac{1}{2}x'' - \frac{1}{2}y'' + z(R_{004} - Q_{004}) + \mathfrak{m}^2\big) \end{pmatrix}. \tag{54}$$

The change of coordinates provided by Lemma 5.6, that reduces f_1 to the form (43), leaves the linear part of $z^{-3}(z \circ (f_1 - \mathrm{id}))$ invariant. We deduce from Corollary 5.10 that, if $R_{004} \neq Q_{004}$, the parameters of Proposition 5.8 are $c = 2$, $e = 1$, and we get $r = c + e = 3$ parabolic manifolds attached to p_1.

Here $\lambda = -1$, $\mu = 1$, $h_e = R_{004} - Q_{004}$, and by a direct check we get that all parabolic manifolds have dimension 2, unless $h_e^2 \in \mathrm{i}\mathbb{R}$, in which case one of the three parabolic manifolds has dimension 1, while the others have dimension 2.

$\boxed{p_2}$ For the degenerate spike at p_2, computations are similar and left to the reader. In this case we get again $c = 2$, $e = 1$, and $r = 3$ parabolic manifolds whenever $h_e := R^{(4)}(0,1,1) \neq 0$. They are all of dimension 2, unless $h_e^2 \in \mathbb{R}$, where one of the three parabolic manifolds has dimension 1.

$\boxed{p_{3,2}}$ At the point $p_{3,2}$ we have a spinning corner of the form (27). According to Remark 3.13, after a suitable change of coordinates we get the form

$$\widetilde{f}_1(x,y,z) = \begin{pmatrix} x + y^2z^2\Big(-x + \mathfrak{m}^2\Big) \\ y + y^3z^2\Big(-2x + 2\alpha z + \mathfrak{m}^2\Big) \\ z + y^2z^3\Big(2x + y - 2\alpha z + \mathfrak{m}^2\Big) \end{pmatrix}.$$

In particular, the parameters of Corollary 5.4 are given by $b_y - c_y = -1$, $c_z - b_z = -4\alpha$, and $\delta = b_y c_z - c_y b_z = -2\alpha$.

The ratio $(b_y - c_y)(c_z - b_z)/\delta$ of (42) equals -2, which is not a positive integer, hence the conditions of Corollary 5.4 are satisfied, and there exists a unique formal f-invariant curve smooth and transverse to the exceptional divisor.

If we blow-up the origin via the map $\pi(x,y,z) = (xz, yz, z)$, we get the lift

$$\widetilde{f}_2 = \begin{pmatrix} x + y^2z^4\Big(-x + \langle z\rangle\Big) \\ y + y^3z^5\Big(-4x - y + 4\alpha + \langle z\rangle\Big) \\ z + y^2z^6\Big(2x + y - 2\alpha + \langle z\rangle\Big) \end{pmatrix}.$$

At the point $y_0 = 4\alpha$ we get a non-simple half corner, with parameters $c = 4$, $\gamma = 2\alpha$. Being $\alpha \neq 0$, we get $e = 0$. By Corollary 5.12, we get $r = c+1+e = 5$ parabolic manifolds, which are of dimension 1 or 2.

$\boxed{p_{4,2}}$ Since the germ f_1 at p_3 is conjugated to the one at p_4 (see Remark 2.2), a similar situation arises above $p_{4,2}$.

$\boxed{q_1}$ Finally, at the point q_1 we have a spinning corner of the form (15). By conjugating by the map $\phi(x,y,z) = (z + R_{040}y, x, y)$, we get

$$f_4(x,y,z) = \begin{pmatrix} x + y^7z^2\Big(x + \mathfrak{m}^2\Big) \\ y + y^8z^2\Big(x - R_{040}z + \mathfrak{m}^2\Big) \\ z + y^7z^3\Big(-3x + 3R_{040}z + \mathfrak{m}^2\Big) \end{pmatrix}. \tag{55}$$

In this case we have $b_y = c_y = 0$ and $c_z = -3b_z \neq 0$. In particular, there are no formal f_4-invariant curves transverse to E by Corollary 5.4 (see also Remark 5.5), but we cannot exclude f_4-invariant curves tangent to E (see Subsection 5.4.3).

5.4. **Further remarks**

5.4.1. *Curve blow-ups over degenerate spikes*

When studying resolution of singularities for vector fields, it is often natural to consider (possibly weighted) blow-ups of centers that are invariant by the dynamics (and not necessarily contained in the singular locus).

In our setting, this would correspond to allowing the blow-up of curves that are invariant by the saturated infinitesimal generator $\hat{\chi}$ of f (in a given model), and contained in the exceptional divisor (obtained from previous blow-ups).

In the case of degenerate spikes (of Siegel type), the study can be easily done, since we can determine explicitly such curves. In fact, if f is a degenerate spike of the form (21), then the restriction of the saturated infinitesimal generator $\hat{\chi}$ on $E = \{z = 0\}$ gives a canonical singularity of Siegel type, which admits exactly two (strong) complex separatrices. Up to a (possibly formal, since the coordinates of $\hat{\chi}$ do not converge in general) change of coordinates, we may assume that these curves are $x = 0$ and $y = 0$. Hence we may assume that the conditions $x|P(x, y, 0)$ and $y|Q(x, y, 0)$ are satisfied. The next proposition gives the description of the lift of a degenerate spike when we blow-up one of the two complex separatrices (the other is completely analogous, we just need to interchange the role of x and y).

Proposition 5.13. *Let $f : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a degenerate spike of the form*

$$f(x, y, z) = \begin{pmatrix} x + z^c\big(\lambda x(1 + a(x, y)) + zP\big) \\ y + z^c\big(\mu y(1 + b(x, y)) + zQ\big) \\ z + z^{c+1}R \end{pmatrix}, \tag{56}$$

with $a, b \in \mathfrak{m}_2$, $P, Q, R \in \mathfrak{m}$. Let $\pi : X \to (\mathbb{C}^3, 0)$ be the blow-up of the line $\{x = z = 0\}$ in $\mathbb{C}^3$, and denote by $\widetilde{f}$ the lift of f in X.

Then the saturated infinitesimal generator of $\widetilde{f}$ has two singularities on the fiber above the origin, namely $[1 : 0]$ and $[0 : 1]$ Moreover for $\widetilde{f}$ we have that:

- $[0 : 1]$ *is a degenerate spike.*
- $[1 : 0]$ *is a simple corner.*

Proof. Computations are analogous to the ones performed in the previous sections, and left to the reader. □

Suppose now that f is a degenerate spike (of Siegel type), and let C be a formal f-invariant curve. Let $\pi : X \to (\mathbb{C}^3, 0)$ be a modification. By Remark 3.18, the strict transform of C on X cannot contain a simple corner. Since C must intersect the singular points, we infer that for any regular modification π (not necessarily strongly) adapted to f, C must intersect the (unique) degenerate spike. We infer the uniqueness of the formal f-invariant curve, and the fact that, for any such modification, the non-exceptional characteristic directions are always degenerate. This applies in particular to degenerate spikes that we find at the points p_1 and p_2.

Notice that, a priori, we cannot exclude the case of non-degenerate non-exceptional characteristic directions giving rise (via Hakim's results [Hak98]) to a non-robust parabolic curve (while we can exclude robust ones by the uniqueness of the f-invariant curve).

5.4.2. *Point modifications*

With the same techniques adopted in Section 3, it is possible to study characteristic directions on any model $\pi : X_\pi \to (\mathbb{C}^3, 0)$ obtained via *point modifications*.

If we only allow point modifications, we cannot resolve the singularities of the infinitesimal generator χ of f, and this leads to having to deal with singularities of the saturated infinitesimal generator $\widetilde{\chi}_\pi$ (on a given model X_π) that are not log-canonical.

We omit definitions and computations in this case because they would stretch the length of this paper excessively. Just to give a hint of what happens in this case, let us follow the resolution of singularity above p_5. At p_5, the map f_1 obtained as lift of f by the blow-up of the origin takes the form (10), for which the linear part of the saturated infinitesimal generator is nilpotent, of rank 2 if we assume $R_{040} \neq 0$. Let us say that this germ is a N_1-form (N stands for *nilpotent*).

After blowing-up p_5, we get a second form at the point $p_{5,1}$ corresponding to $[1 : 0 : 0]$. In this case the linear part of the infinitesimal generator has still rank 2, but the exceptional divisor locally consists of two irreducible components. Say that we get a N_2-form.

Blowing-up $p_{5,1}$, we get a line L of singularities, corresponding to the singular directions $[p : 0 : r]$ with $[p : r] \in \mathbb{P}^1_{\mathbb{C}}$. In this case, at $[1 : 0 : 0]$ we get another N_2-form. At $[0 : 0 : 1]$

we obtain a singularity for the saturated infinitesimal generator with vanishing linear part, that we call H_1-form (H stands for *higher order*). At $[p : 0 : r]$ with $p, r \neq 0$, we get another nilpotent singularity, call it N_3-form. In other terms, we got a N_3-pattern, special points N_2 and H_1, and with core L. If we blow-up L, we get the resolution of singularities π_0 described by Proposition 2.3. If we blow-up points, we need for example to deal with the blow-up of H_1-forms, which is quite intricate. Fundamental for the definition of these classes is the identification of the right non-resonant conditions, in the same spirit of the ones appearing for simple corners, as well as suitable conditions on the higher order terms of the saturated infinitesimal generator.

The birational study can be completed for point modifications, and one can show that no non-degenerate characteristic directions can appear in this case. However, the additional forms, and the appearance of several new types of patterns, make the birational study for all possible modifications (strongly) adapted to f combinatorially much more involved. We suspect that no non-degenerate characteristic directions can be found in this way either.

5.4.3. *Dynamics over q_1*

We have shown that at the point q_1 there are no formal f_4-invariant curves transverse to the exceptional divisor, while the existence of (non-transverse) f_4-invariant curves remains open in this case. Notice that we can construct a formal f_4-invariant surface S, transverse to the exceptional divisor at q_1. In fact, the saturated infinitesimal generator $\hat{\chi}_4$ of f_4 is reduced, and one can proceed as in the proof of Lemma 5.6, and find new coordinates $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ on which $\hat{\chi}_4$ is in Poincaré-Dulac normal form. One can check that the Poincaré-Dulac change of coordinates can be done so that the exceptional divisor is described by $\{\widetilde{y}\widetilde{z} = 0\}$, while the invariant surface S is described by $\{\widetilde{x} = 0\}$.

By blowing-up the point q_1, we find a R_2-R_3-pattern (denote by $\widetilde{f}_4$ the lift of f_4). If we compute the blow-up in the $\widetilde{z}$-chart, and then translate coordinates at an half corner point of the pattern, we get a germ as in Proposition 5.2 after Step 1 of the proof, and the invariant surface built in Step 1 is exactly the strict transform of S.

The map $f_4|_S$ gives a (formal) 2-dimensional tangent to the identity germ, with saturated infinitesimal generator $\hat{\xi}$ of order 2. A direct computation shows that $f_4|_S$ has exactly two characteristic directions, corresponding to the two irreducible components of the exceptional divisor. If $\hat{\xi}$ admits another complex separatrix, we can apply again the arguments of Subsection 5.2.2 to reduce f in Ramis-Sibuya normal form and find parabolic manifolds. If $\hat{\xi}$ has no other complex separatrices, we cannot apply the results [LHRSSV] in order to find parabolic manifolds attached to q_1, and one needs to study the dynamics of f_4 more in details.

Finally, notice that the f-invariant surface $S_0 := \widetilde{\pi}_0(S)$ is not smooth. Hence, even if S were convergent, we could not apply the results of [LHR20] to find paraboic curves for $f|_{S_0}$.

References

[Aba01] Marco Abate. The residual index and the dynamics of holomorphic maps tangent to the identity. *Duke Math. J.*, 107(1):173–207, 2001.

[ABD+16] Matthieu Astorg, Xavier Buff, Romain Dujardin, Han Peters, and Jasmin Raissy. A two-dimensional polynomial mapping with a wandering Fatou component. *Ann. of Math. (2)*, 184(1):263–313, 2016.

[ABT04] Marco Abate, Filippo Bracci, and Francesca Tovena. Index theorems for holomorphic self-maps. *Ann. of Math. (2)*, 159(2):819–864, 2004.

[ABTP21] Matthieu Astorg, Luka Boc Thaler, and Han Peters. Wandering domains arising from lavaurs maps with siegel disks, 2021. Preprint available at https://arxiv.org/abs/1907.04140.

[AKMW02] Dan Abramovich, Kalle Karu, Kenji Matsuki, and Jarosław Włodarczyk. Torification and factorization of birational maps. *J. Amer. Math. Soc.*, 15(3):531–572 (electronic), 2002.

[AT03] Marco Abate and Francesca Tovena. Parabolic curves in $\mathbf{C}^3$. *Abstr. Appl. Anal.*, (5):275–294, 2003.

[Bia19] Fabrizio Bianchi. Parabolic implosion for endomorphisms of $\mathbb{C}^2$. *J. Eur. Math. Soc. (JEMS)*, 21(12):3709–3737, 2019.

[BMCLH08] F. E. Brochero Martínez, F. Cano, and L. López-Hernanz. Parabolic curves for diffeomorphisms in $\mathbb{C}^2$. *Publ. Mat.*, 52(1):189–194, 2008.

[Bon02] Laurent Bonavero. Factorisation faible des applications birationnelles (d'après Abramovich, Karu, Matsuki, Włodarczyk et Morelli). *Astérisque*, (282):Exp. No. 880, vii, 1–37, 2002. Séminaire Bourbaki, Vol. 2000/2001.

[BSU17] Eric Bedford, John Smillie, and Tetsuo Ueda. Semi-parabolic bifurcations in complex dimension two. *Comm. Math. Phys.*, 350(1):1–29, 2017.

[Cam78] César Camacho. On the local structure of conformal mappings and holomorphic vector fields in $\mathbf{C}^2$. In *Journées Singulières de Dijon (Univ. Dijon, Dijon, 1978)*, volume 59 of *Astérisque*, pages 3, 83–94. Soc. Math. France, Paris, 1978.

[CC05] Antonio Campillo and Julio Castellanos. *Curve singularities. An algebraic and geometric approach*. Actualités Mathématiques. Hermann, Editeurs des sciences et des arts, 2005.

[CS82] César Camacho and Paulo Sad. Invariant varieties through singularities of holomorphic vector fields. *Ann. of Math. (2)*, 115(3):579–595, 1982.

[DL15] Romain Dujardin and Mikhail Lyubich. Stability and bifurcations for dissipative polynomial automorphisms of $\mathbb{C}^2$. *Invent. Math.*, 200(2):439–511, 2015.

[Éca85] Jean Écalle. *Les fonctions résurgentes. Tome III*, volume 85 of *Publications Mathématiques d'Orsay [Mathematical Publications of Orsay]*. Université de Paris-Sud, Département de Mathématiques, Orsay, 1985. L'équation du pont et la classification analytique des objects locaux. [The bridge equation and analytic classification of local objects].

[Fat19] Pierre Fatou. Sur les équations fonctionnelles. *Bull. Soc. Math. France*, 47:161–271, 1919.

[GML92] Xavier Gómez-Mont and Ignacio Luengo. Germs of holomorphic vector fields in $\mathbf{C}^3$ without a separatrix. *Invent. Math.*, 109(2):211–219, 1992.

[Hak98] Monique Hakim. Analytic transformations of $(\mathbf{C}^p, 0)$ tangent to the identity. *Duke Math. J.*, 92(2):403–428, 1998.

[Iva11] Sergei Ivashkovich. On fixed points of rational self-maps of complex projective plane. *Indiana Univ. Math. J.*, 60(3):803–812, 2011.

[IY08] Yulij Ilyashenko and Sergei Yakovenko. *Lectures on analytic differential equations*, volume 86 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[Lea97a] Léopold Leau. Étude sur les équations fonctionnelles à une ou à plusieurs variables. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.*, 11(2):E1–E24, 1897.

[Lea97b] Léopold Leau. Étude sur les équations fonctionnelles à une ou à plusieurs variables. *Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys.*, 11(3):E25–E110, 1897.

[LHR20] Lorena López-Hernanz and Rudy Rosas. Characteristic directions of two-dimensional biholomorphisms. *Compos. Math.*, 156(5):869–880, 2020.

[LHRRSS19] Lorena López-Hernanz, Jasmin Raissy, Javier Ribón, and Fernando Sanz-Sánchez. Stable Manifolds of Two-dimensional Biholomorphisms Asymptotic to Formal Curves. *International Mathematics Research Notices*, 09 2019. rnz143.

[LHRSSV] Lorena López-Hernanz, Javier Ribón, Fernando Sanz Sánchez, and Liz Vivas. Stable manifolds of biholomorphisms in $\mathbb{C}^n$ asymptotic to formal curves. Preprint available at https://arxiv.org/abs/2002.07102.

[LHSS18] Lorena López-Hernanz and Fernando Sanz Sánchez. Parabolic curves of diffeomorphisms asymptotic to formal invariant curves. *J. Reine Angew. Math.*, 739:277–296, 2018.

[Mil06] John Milnor. *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, third edition, 2006.

[Mol09] Laura Molino. The dynamics of maps tangent to the identity and with nonvanishing index. *Trans. Amer. Math. Soc.*, 361(3):1597–1623, 2009.

[MP13] Michael McQuillan and Daniel Panazzolo. Almost étale resolution of foliations. *J. Differential Geom.*, 95(2):279–319, 2013.

[Pan06] Daniel Panazzolo. Resolution of singularities of real-analytic vector fields in dimension three. *Acta Math.*, 197(2):167–289, 2006.

[Pan07] Daniel Panazzolo. Resolution of singularities of vector fields in dimension three. Preprint available at http://arxiv.org/abs/math/0506209, 2007.

[Ron15] Feng Rong. New invariants and attracting domains for holomorphic maps in $\mathbf{C}^2$ tangent to the identity. *Publ. Mat.*, 59(1):235–243, 2015.

[Rug12] Matteo Ruggiero. Rigidification of holomorphic germs with noninvertible differential. *Michigan Math. J.*, 61(1):161–185, 2012.

[Rug13] Matteo Ruggiero. Contracting rigid germs in higher dimensions. *Ann. Inst. Fourier (Grenoble)*, 63(5):1913–1950, 2013.

[Rug15] Matteo Ruggiero. Classification of one-dimensional superattracting germs in positive characteristic. *Ergodic Theory and Dynamical Systems*, FirstView:1–27, 1 2015.

[Sei68] Abraham Seidenberg. Reduction of singularities of the differential equation $A\,dy = B\,dx$. *Amer. J. Math.*, 90:248–269, 1968.

[Shi00] Mitsuhiro Shishikura. Bifurcation of parabolic fixed points. In *The Mandelbrot set, theme and variations*, volume 274 of *London Math. Soc. Lecture Note Ser.*, pages 325–363. Cambridge Univ. Press, Cambridge, 2000.

[Viv12] Liz Vivas. Degenerate characteristic directions for maps tangent to the identity. *Indiana Univ. Math. J.*, 61(6):2019–2040, 2012.

[Wei98] Brendan J. Weickert. Attracting basins for automorphisms of $\mathbf{C}^2$. *Invent. Math.*, 132(3):581–605, 1998.

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via Roberto Cozzi, 55, I-20125, Milano, Italy

Email address: samuele.mongodi@unimib.it

Université Paris Cité and Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75006, Paris, France

Email address: matteo.ruggiero@imj-prg.fr