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A note on the compactness properties of discontinuous Galerkin time discretizations of nonlinear evolution problems

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ABSTRACT

This work extends the discrete compactness results of Walkington (SIAM J. Numer. Anal., 47(6):4680–4710, 2010) for high-order discontinuous Galerkin time discretizations of parabolic problems to more general function space settings. In particular, we show a discrete version of the Aubin–Lions–Simon lemma that holds for general Banach spaces X, B, and Y satisfying $X \hookrightarrow B$ compactly and $B \hookrightarrow Y$ continuously. Our proofs rely on the properties of a time reconstruction operator and remove the need for quasi-uniform time partitions assumed in previous works. Thus, we provide a useful and flexible tool for the analysis of high-order discontinuous Galerkin time discretizations of complex nonlinear partial differential equations.

1. Introduction

The aim of this work is to present some discrete compactness results for sequences of (possibly discontinuous) piecewise polynomial functions in time. In particular, we show an extension of [1, Thm. 3.1] to a more general variational framework. Moreover, our proofs are simpler and avoid the requirement of (global) quasi-uniformity of the time partitions, as they rely on continuous-in-time reconstructions that allow us to use compactness results from the continuous setting. Our main result (Theorem 2.1) can also be seen as a high-order generalization of [2, Thm. 1], which addressed the case of piecewise-constant functions in time.

Previous works. Using discrete compactness for sequences of piecewise-constant functions in time is a standard tool not only to show the existence of weak solutions to nonlinear evolution problems, but also to prove convergence of numerical schemes; see, for instance, applications in flow dynamics [3], nonlinear cross-diffusion systems [4–6], and hyperbolic–parabolic problems [7]. Such a technique is particularly useful when deriving *a priori* error estimates is too challenging, as well as to show convergence in mesh-independent norms under minimal regularity assumptions.

In contrast, the literature on discrete compactness results for higher-order piecewise polynomial functions in time is scarcer. To the best of our knowledge, Walkington's result in [1, Thm. 3.1] was the first of this type. Subsequently, a variation used for the miscible displacement problem was presented in [8, Thm. 3.6], and an extension to nonconforming approximations in space was established in [9, Thm. 3.2]. The main idea of these results was to show that discontinuous Galerkin (DG) approximations in time are uniformly equicontinuous (see [1, Lemma 3.3]), thus circumventing the lack of differentiability of such discrete functions, at

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the expense of assuming quasi-uniform time partitions. The above results have been used for high-order DG time discretizations of reaction–diffusion [10,11], incompressible fluid dynamics [12,13], porous media [8,9,14], and optimal control [15,16] problems.

Recently, in [12, Thm. A.1], it was shown that some of the assumptions in [1, Thm. 3.1] and [9, Thm. 3.2] can be replaced by conditions on the time derivative of the continuous-in-time reconstructions introduced in [17, §2.1]. Following this approach, we show that the equicontinuity argument can be avoided, as discrete compactness can instead be established by only exploiting the properties of the reconstruction operator.

Compactness results for continuous functions in time. We first recall some standard results for the continuous setting, which we use in the proof of Theorem 2.1 below. The Aubin–Lions lemma establishes some sufficient conditions to prove that a set of functions $\mathcal{U}$ is relatively compact in $L^p(0,T;B)$, where $p \in [1,\infty)$, $T>0$, and B is a Banach space. Under these conditions, any bounded sequence in $\mathcal{U}$ has a subsequence that converges strongly in $L^p(0,T;B)$. The following version from [18, Cor. 4] is called the Aubin–Lions–Simon lemma, as Simon established the result without the assumption in [19, Thm. 1] and[20, Thm. I.5.1] that the Banach spaces involved are reflexive.

Theorem 1.1 (Aubin–Lions–Simon). *Let X, B, and Y be Banach spaces with*

$$X \hookrightarrow B \text{ compactly} \quad \text{and} \quad B \hookrightarrow Y \text{ continuously}.$$

Let also $p, r \in [1,\infty]$, and $\mathcal{U} \subset L^p(0,T;X)$ be a set of functions such that

$$\mathcal{U} \text{ is bounded in } L^p(0,T;X), \tag{1.1a}$$

$$\partial_t \mathcal{U} := \left\{\partial_t u \ : \ u \in \mathcal{U}\right\} \text{ is bounded in } L^r(0,T;Y). \tag{1.1b}$$

- *If $1 \le p < \infty$, then $\mathcal{U}$ is relatively compact in $L^p(0,T;B)$.*
- *If $p=\infty$ and $r>1$, then $\mathcal{U}$ is relatively compact in $C^0([0,T];B)$.*

In [18, Thm. 5], Simon showed that condition (1.1b) on $\mathcal{U}$ can be weakened to

$$\lim_{\delta\to 0^+} \|\sigma_\delta u - u\|_{L^r(0,T-\delta;Y)} = 0, \quad \text{uniformly for } u \in \mathcal{U}, \tag{1.2}$$

where the shift operator $\sigma_\delta u(t) := u(t+\delta)$, for $t \in [0, T-\delta]$. This version with weaker assumptions is particularly useful when dealing with (possibly discontinuous) piecewise polynomial functions in time, as they are not differentiable and hence condition (1.1b) cannot hold. However, verifying (1.2) for all time shifts $\delta>0$ can be cumbersome and conditions that can be deduced directly from the bounds on the discrete solutions are more convenient (see [2]).

Next theorem concerns relative compactness of higher regularity under some slightly more restrictive assumptions; see [21, Thm. 1.1].

Theorem 1.2 (Amann). *Let X, B, and Y be Banach spaces such that*

$$X \hookrightarrow B \text{ and } B \hookrightarrow Y \text{ continuously}, \tag{1.3a}$$

$$X \hookrightarrow Y \text{ compactly}, \tag{1.3b}$$

and, for some $\theta \in (0,1)$, there exists $C_\theta > 0$ such that

$$\|u\|_B \le C_\theta \|u\|_X^\theta \|u\|_Y^{1-\theta} \qquad \text{for all } u \in X. \tag{1.3c}$$

Let $\mathcal{U} \subset L^p(0,T;X)$ be a set of functions satisfying condition (1.1a) and either

- *condition (1.1b) (setting $s=1$), or*
- *for $p \in [1,\infty)$, $r \le p$, and some $0<s<1$, there is $C_s>0$ such that*

$$\|\sigma_\delta u - u\|_{L^r(0,T-\delta;Y)} \le C_s \delta^s \quad \text{for all } \delta>0 \text{ and } u \in \mathcal{U}. \tag{1.4}$$

Then $\mathcal{U}$ is relatively compact in $L^q(0,T;B)$ for all $1 \le q < rp/((1-\theta)(1-sr)p+\theta r)$ provided that $(sr-1)p/r \le \theta/(1-\theta)$.

Remark 1.1. If X is a Banach space, B is a Hilbert space, and we have a Gelfand triplet $X \hookrightarrow B \hookrightarrow X'$ with dense compact embeddings, then condition (1.3c) holds with $Y = X'$, $\theta = 1/2$, and $C_\theta = 1$. In this setting, if condition (1.4) holds with $r=1$ for all $0<s<1$, then Theorem 1.2 implies that $\mathcal{U}$ is relatively compact in $L^q(0,T;B)$ for $1 \le q < 2p$. The result in [1, Thm. 3.1(1)] for the high-order DG time stepping can be seen as a discrete counterpart of this particular situation. ∎

2. Discontinuous Galerkin time discretizations

In Section 2.1, we introduce some DG notation and define spaces of piecewise polynomial functions in time. In Section 2.2, we present the definition and main properties of the time reconstruction operator used in the proof of our discrete compactness result in Section 2.3. A brief discussion of the application of Theorem 2.1 to parabolic problems is presented in Section 2.4.

2.1. DG notation and piecewise polynomial spaces

Let X be a Banach space, and let $\{\mathcal{T}_\tau\}_{\tau>0}$ be a family of partitions of the time interval $(0,T)$, where each $\mathcal{T}_\tau$ is determined by some nodes $0 =: t_0 < \ldots < t_{N_\tau} := T$. For $n = 1, \ldots, N_\tau$, we define the time interval $I_n := (t_{n-1}, t_n)$ and the time step $\tau_n := t_n - t_{n-1}$. The subscript τ stands for the maximum time step, i.e., $\tau := \max_{n=1,\ldots,N_\tau} \tau_n$. We make the following assumption on $\{\mathcal{T}_\tau\}_{\tau>0}$, which is less restrictive than the (global) quasi-uniformity assumption in [1, Thm. 3.1], [8, Thm. 3.6], and [9, Thm. 3.2].

Assumption 2.1 (Time steps ratio). There exists a positive constant $C_\star$ such that, for all $\tau > 0$ and $n = 2, \ldots, N_\tau$,

$$\tau_n/\tau_{n-1} \le C_\star.$$

In particular, this condition holds for families of geometrically refined partitions in time, which can be used to resolve temporal singularities (see, e.g., [22,23]).

For each $\tau > 0$, let X_τ be a subspace of X. Given $\ell \in \mathbb{N}$, we define the following spaces:

$$\mathcal{P}_\tau^\ell(X_\tau) := \left\{ v \in L^1(0,T;X_\tau) \ : \ v_{|_{I_n}} \in \mathcal{P}^\ell(I_n) \otimes X_\tau, \text{ for } n = 1, \ldots, N_\tau \right\},$$
$$\mathcal{P}_\tau^{\ell,\mathrm{cont}}(X_\tau) := \mathcal{P}_\tau^\ell(X_\tau) \cap C^0([0,T];X_\tau),$$

where $\mathcal{P}^\ell(I_n)$ is the space of polynomials of degree at most ℓ defined on I_n, and $\otimes$ denotes the algebraic product for vector spaces.

We also define the time jumps ($[\![\cdot]\!]_n$) for piecewise smooth scalar functions in time (v) as follows:

$$[\![v]\!]_n := v(t_n^+) - v(t_n^-), \qquad \text{for } n = 1, \ldots, N_\tau,$$

where

$$v(t_n^+) := \lim_{\varepsilon \to 0^+} v(t_n + \varepsilon) \quad \text{ and } \quad v(t_n^-) := \lim_{\varepsilon \to 0^+} v(t_n - \varepsilon).$$

2.2. Time reconstruction operator

We now recall the definition and some properties of the time reconstruction operator introduced in [17, §2.1]. To do so, we resort to the pointwise-in-time definition in [24, Eq. (33)] (see also [24, Lemmas 6 and 7]).

For any Banach space Z and $\ell \in \mathbb{N}$, we define the lifting operator $\mathcal{L}_\tau : Z \to \mathcal{P}_\tau^\ell(Z)$ and the time reconstruction operator $\mathcal{R}_\tau : \mathcal{P}_\tau^\ell(Z) \to \mathcal{P}_\tau^{\ell+1,\mathrm{cont}}(Z)$ as follows: for $n = 1, \ldots, N_\tau$,

$$\mathcal{L}_\tau z(t) := \frac{z}{\tau_n} \sum_{i=0}^{\ell} (-1)^i (2i+1) L_{n,i}(t) \qquad \text{for all } z \in Z \text{ and } t \in I_n,$$
$$\mathcal{R}_\tau v_\tau(t) := v_\tau(t) - [\![v_\tau]\!]_{n-1} + \int_{t_{n-1}}^{t} \mathcal{L}_\tau([\![v_\tau]\!]_{n-1})(s)\,\mathrm{d}s \qquad \text{for all } v_\tau \in \mathcal{P}_\tau^\ell(Z) \text{ and } t \in \overline{I_n},$$

where $L_{n,i}$ is the ith *mapped Legendre polynomial* defined on the interval I_n, and we have set $[\![v_\tau]\!]_0 := v_\tau(0) - v_0$, for some "initial datum" $v_0 \in Z$. The explicit dependence of $\mathcal{R}_\tau v_\tau$ on v_0 will be neglected. The continuity in time of $\mathcal{R}_\tau$ was proven in [17, Lemma 2.1], and the following result from [17, Lemma 2.2] provides an estimate of the error between a piecewise polynomial function in time and its continuous-in-time reconstruction.

Lemma 2.1 (Properties of $\mathcal{R}_\tau$). *For any Banach space Z, $\ell \in \mathbb{N}$, and $p \in [1,\infty]$, there exists a positive constant $C_{\mathcal{R}}$ depending only on p and ℓ such that*

$$\|\mathcal{R}_\tau v_\tau - v_\tau\|_{L^p(0,T;Z)} \le C_{\mathcal{R}} \Big(\sum_{n=1}^{N_\tau} \tau_n \|[\![v_\tau]\!]_{n-1}\|_Z^p \Big)^{1/p} \qquad \textit{for all } v_\tau \in \mathcal{P}_\tau^\ell(Z),$$

with the usual notational conventions for the case $p = \infty$.

2.3. Discrete compactness

Henceforth, we assume that the degree of approximation $\ell \in \mathbb{N}$, the final time $T > 0$, and the "initial datum" $u_0 \in X$ are fixed. We are now in a position to prove our discrete compactness result. Some extensions are discussed in the arXiv version of this manuscript; see [25, Remarks 2.3 and 2.4].

Theorem 2.1 (Discrete Aubin–Lions–Simon compactness). *Let X, B, and Y be Banach spaces such that the embedding $X \hookrightarrow B$ is compact, and the embedding $B \hookrightarrow Y$ is continuous. Let also $u_0 \in X$, and $\{\mathcal{T}_\tau\}_{\tau>0}$ be a family of partitions of $(0,T)$ with $\tau \to 0^+$ satisfying Assumption 2.1. For each τ, let $u_\tau \in \mathcal{P}_\tau^\ell(X_\tau)$ for some subspace X_τ of X.*

Assume that

- *(h1) $\{u_\tau\}_{\tau>0}$ is uniformly bounded in $L^p(0,T;X)$ for some $p \in [1,\infty]$.*
- *(h2) $\{\partial_t \mathcal{R}_\tau u_\tau\}_{\tau>0}$ is uniformly bounded in $L^r(0,T;Y)$ for some $r \in [1,\infty]$.*
- *(h3) There exists a positive constant C_J independent of τ such that $\sum_{n=1}^{N_\tau} \|[\![u_\tau]\!]_{n-1}\|_B^2 \le C_J$ for all $\tau > 0$.*

The following hold:

(i) If $1 \le p < \infty$, then $\{u_\tau\}_{\tau>0}$ is relatively compact in $L^p(0,T;B)$.
(ii) If $p = \infty$ and $r > 1$, then there exists a subsequence of $\{u_\tau\}_{\tau>0}$ that converges strongly in $L^q(0,T;B)$ for all $1 \le q < \infty$ to a function belonging to $C^0([0,T];B)$.
(iii) If conditions (1.3a)–(1.3c) hold, then $\{u_\tau\}_{\tau>0}$ is relatively compact in $L^q(0,T;B)$ for all $1 \le q < rp/((1-\theta)(1-r)p+\theta r)$ provided that $(r-1)p/r \le \theta/(1-\theta)$.

Proof. We split the proof into three parts.

Part I) Uniform boundedness of $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$. We first show that the time reconstruction $\mathcal{R}_\tau u_\tau$ inherits the uniform boundedness of u_τ in $L^p(0,T;X)$. Below, we focus on the case $1 \le p < \infty$, but the case $p = \infty$ follows analogously.

Using the triangle inequality and Lemma 2.1, we obtain

$$\begin{aligned}\|\mathcal{R}_\tau u_\tau\|_{L^p(0,T;X)} &\le \|u_\tau\|_{L^p(0,T;X)} + \|\mathcal{R}_\tau u_\tau - u_\tau\|_{L^p(0,T;X)} \\ &\le \|u_\tau\|_{L^p(0,T;X)} + C_{\mathcal{R}}\Big(\sum_{n=1}^{N_\tau} \tau_n \|[\![u_\tau]\!]_{n-1}\|_X^p\Big)^{1/p}. \end{aligned} \tag{2.1}$$

By a standard inverse-trace inequality (see, e.g., [26, Lemma 12.8]) extended to Bochner spaces, there exists a positive constant $C_{\rm tr}$ independent of $\mathcal{T}_\tau$ such that

$$\begin{aligned}\|[\![u_\tau]\!]_{n-1}\|_X &\le C_{\rm tr}\big(\tau_n^{-1/p}\|u_\tau\|_{L^p(I_n;X)} + \tau_{n-1}^{-1/p}\|u_\tau\|_{L^p(I_{n-1};X)}\big), \qquad \text{for } n = 2,\ldots,N_\tau, \\ \|[\![u_\tau]\!]_0\|_X &\le C_{\rm tr}\tau_1^{-1/p}\|u_\tau\|_{L^p(I_1;X)} + \|u_0\|_X,\end{aligned}$$

which, together with the triangle inequality and Assumption 2.1, implies

$$\begin{aligned}\|\mathcal{R}_\tau u_\tau - u_\tau\|_{L^p(0,T;X)} &\le C_{\mathcal{R}}\Bigg[\sum_{n=2}^{N_\tau}\big(C_{\rm tr}\|u_\tau\|_{L^p(I_n;X)} + C_{\rm tr}(\tau_n/\tau_{n-1})\|u_\tau\|_{L^p(I_{n-1};X)}\big)^p + \big(C_{\rm tr}\|u_\tau\|_{L^p(I_1;X)} + \tau_1\|u_0\|_X\big)^p\Bigg]^{1/p} \\ &\le C_{\mathcal{R}}\big(C_{\rm tr}(1+C_\star)\|u_\tau\|_{L^p(0,T;X)} + \tau_1^{1/p}\|u_0\|_X\big).\end{aligned} \tag{2.2}$$

The uniform boundedness of $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$ in $L^p(0,T;X)$ then follows from hypothesis (h1) by combining inequalities (2.1) and (2.2).

Part II) Relative compactness of $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$. Since $\{\partial_t \mathcal{R}_\tau u_\tau\}_{\tau>0}$ is uniformly bounded in $L^r(0,T;Y)$ by hypothesis (h2), and $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$ is uniformly bounded in $L^p(0,T;X)$ by Part I) of this proof, we can use Theorems 1.1 and 1.2 to deduce that the relative compactness results (i) and (iii) hold replacing $\{u_\tau\}_{\tau>0}$ by $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$. Moreover, if $p = \infty$ and $r > 1$, the sequence $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$ is relatively compact in $C^0([0,T];B)$ by Theorem 1.1.

Part III) Relative compactness of $\{u_\tau\}_{\tau>0}$. We focus on cases (i) and (ii), as case (iii) can be proven analogously.

Proof of case (i). Assume that $p \in [1,\infty)$ and $r \in [1,\infty]$. From Part II) of this proof, there exists a subsequence, still denoted by $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$, that converges strongly in $L^p(0,T;B)$ to some function $u^* \in L^p(0,T;B)$. Using the triangle inequality and Lemma 2.1, we have

$$\|u_\tau - u^*\|_{L^p(0,T;B)} \le \|u_\tau - \mathcal{R}_\tau u_\tau\|_{L^p(0,T;B)} + \|\mathcal{R}_\tau u_\tau - u^*\|_{L^p(0,T;B)} \le C_{\mathcal{R}}\Big(\sum_{n=1}^{N_\tau}\tau_n\|[\![u_\tau]\!]_{n-1}\|_B^p\Big)^{1/p} + \|\mathcal{R}_\tau u_\tau - u^*\|_{L^p(0,T;B)}. \tag{2.3}$$

Therefore, it only remains to show that the first term on the right-hand side of (2.3) converges to 0 as $\tau \to 0^+$.

For $p \ge 2$, the following bound can be obtained from hypothesis (h3) and the standard inequality for vector norms $\|\boldsymbol{x}\|_p \le \|\boldsymbol{x}\|_2$:

$$C_{\mathcal{R}}\Big(\sum_{n=1}^{N_\tau}\tau_n\|[\![u_\tau]\!]_{n-1}\|_B^p\Big)^{1/p} \le C_{\mathcal{R}}\sqrt{C_J}\,\tau^{1/p} \to 0 \quad \text{as } \tau \to 0^+.$$

As for the case $1 \le p < 2$, we use the Hölder inequality (with $\gamma = 2/(2-p)$ and $\gamma' = 2/p$) to get

$$\begin{aligned}C_{\mathcal{R}}\Big(\sum_{n=1}^{N_\tau}\tau_n\|[\![u_\tau]\!]_{n-1}\|_B^p\Big)^{1/p} &\le C_{\mathcal{R}}\Bigg[\Big(\sum_{n=1}^{N_\tau}\tau_n^{\gamma/\gamma}\Big)^{1/\gamma}\Big(\sum_{n=1}^{N_\tau}\tau_n^{(1-\frac{1}{\gamma})\gamma'}\|[\![u_\tau]\!]_{n-1}\|_B^{\gamma' p}\Big)^{1/\gamma'}\Bigg]^{1/p} \\ &\le C_{\mathcal{R}}T^{\frac{2-p}{2p}}\tau^{1/2}\Big(\sum_{n=1}^{N_\tau}\|[\![u_\tau]\!]_{n-1}\|_B^2\Big)^{1/2} \le C_{\mathcal{R}}T^{\frac{2-p}{2p}}\sqrt{C_J}\,\tau^{1/2} \to 0 \quad \text{as } \tau \to 0^+.\end{aligned}$$

This completes the proof of case (i).

Proof of case (ii). For this case, we proceed similarly as in [2, Thm. 1].

Assume that $p = \infty$ and $r > 1$. From Part II) of this proof, there exists a subsequence, still denoted by $\{\mathcal{R}_\tau u_\tau\}_{\tau>0}$, that converges strongly in $C^0([0,T];B)$ to some function $u^\star \in C^0([0,T];B)$. Moreover, using case (i) with $p = q \in [1,\infty)$ instead of $p = \infty$, there exists another subsequence, still denoted by $\{u_\tau\}_{\tau>0}$, which converges strongly in $L^q(0,T;B)$ to some function $\hat{u} \in L^q(0,T;B)$. Using the triangle inequality, for all $\tau > 0$ in the intersection of the two subsequences, we have

$$\begin{aligned}\|u^\star - \hat{u}\|_{L^q(0,T;B)} &\leq \|u^\star - \mathcal{R}_\tau u_\tau\|_{L^q(0,T;B)} + \|\mathcal{R}_\tau u_\tau - u_\tau\|_{L^q(0,T;B)} + \|u_\tau - \hat{u}\|_{L^q(0,T;B)} \\ &\leq T^{1/q}\|u^\star - \mathcal{R}_\tau u_\tau\|_{C^0([0,T];B)} + \|\mathcal{R}_\tau u_\tau - u_\tau\|_{L^q(0,T;B)} + \|u_\tau - \hat{u}\|_{L^q(0,T;B)}. \end{aligned} \tag{2.4}$$

All the terms on the right-hand side of (2.4) converge to 0 as $\tau \to 0^+$, which implies that $u^\star(t) = \hat{u}(t)$ in B for a.e. $t \in (0,T)$. □

2.4. Insights into parabolic problems

We now give a brief account of how the assumptions of Theorem 2.1 can be verified for the DG time discretization of parabolic problems. Assume that $X \hookrightarrow B \hookrightarrow X'$ is a Gelfand triplet, where X is a separable reflexive Banach space with dense compact embedding $X \hookrightarrow B$, and B is a separable Hilbert space with inner product $(\cdot,\cdot)_B$ and norm $\|\cdot\|_B$.

Given $p \in (1,\infty)$ with $p' := p/(p-1)$, a (non)linear operator $A : X \to X'$, and a nonlinear source term $f : X \to B$, we consider the following first-order evolution equation: find $u \in W^p(0,T;X) := \{v \in L^p(0,T;X) \,:\, \partial_t v \in L^{p'}(0,T;X')\}$ such that

$$\int_0^T \langle \partial_t u, v\rangle_{X'\times X}\, dt + \int_0^T \langle A(u), v\rangle_{X'\times X}\, dt = \int_0^T \big(f(u), v\big)_B\, dt \qquad \text{for all } v \in L^p(0,T;X), \tag{2.5a}$$

$$u(0) = u_0 \qquad \text{in } B, \tag{2.5b}$$

where $\langle\cdot,\cdot\rangle_{X'\times X}$ denotes the duality product between X' and X, and the continuous embedding $W^p(0,T;X) \hookrightarrow C^0([0,T];B)$ (see, e.g., [27, Lemma 7.1]) ensures that (2.5b) makes sense.

For a given discrete subspace X_τ of X, and a degree of approximation $\ell \in \mathbb{N}$, the DG time discretization of the abstract evolution problem (2.5) reads: find $u_\tau \in \mathcal{P}^\ell_\tau(X_\tau)$ such that, for all $v_\tau \in \mathcal{P}^\ell_\tau(X_\tau)$,

$$\sum_{n=1}^{N_\tau} \Big(\int_{I_n} (\partial_t u_\tau, v_\tau)_B\, dt + \big([\![u_\tau]\!]_{n-1}, v_\tau(t^+_{n-1}) \big)_B \Big) + \int_0^T a(u_\tau, v_\tau)\, dt = \int_0^T (f(u_\tau), v_\tau)_B\, dt, \tag{2.6}$$

where $a : X \times X \to \mathbb{R}$ is defined by $a(u,v) := \langle A(u), v\rangle_{X'\times X}$ for all $u, v \in X$.

If there is a positive constant C_a such that $a(u,u) \geq C_a\|u\|^p_X$ for all $u \in L^p(0,T;X)$, and $f(\cdot)$ satisfies an appropriate continuity or growth condition, then one can obtain a stability bound of the form

$$\frac{1}{2}\Big(\|u_\tau(T)\|^2_B + \sum_{n=1}^{N_\tau} \|[\![u_\tau]\!]_{n-1}\|^2_B \Big) + C_a\|u_\tau\|^p_{L^p(0,T;X)} \leq C(T, u_0, f),$$

thus verifying hypotheses (h1) and (h3). To verify hypothesis (h2), we use the equivalent definition of $\mathcal{R}_\tau u_\tau$ given in [17, Eq. (13)] and rewrite (2.6) as follows: find $u_\tau \in \mathcal{P}^\ell_\tau(X_\tau)$ such that, for all $v_\tau \in \mathcal{P}^\ell_\tau(X_\tau)$,

$$\int_0^T (\partial_t \mathcal{R}_\tau u_\tau, v_\tau)_B\, dt = \sum_{n=1}^{N_\tau} \Big(\int_{I_n} (\partial_t u_\tau, v_\tau)_B\, dt + \big([\![u_\tau]\!]_{n-1}, v_\tau(t^+_{n-1}) \big)_B \Big) = \int_0^T \langle \mathcal{F}_\tau, v_\tau\rangle_{X'_\tau\times X_\tau}\, dt, \tag{2.7}$$

where $\langle\cdot,\cdot\rangle_{X'_\tau\times X_\tau}$ denotes the duality product between X'_τ and X_τ, and the functionals $\{\mathcal{F}_\tau\}_{\tau>0}$, arising from the terms involving $a(\cdot,\cdot)$ and $f(\cdot)$, are assumed to satisfy that the norms $\{\|\mathcal{F}_\tau\|_{L^{p'}(0,T;X'_\tau)}\}_{\tau>0}$ are uniformly bounded.

We denote by $\Pi_{X_\tau} : B \to X_\tau$ the B-orthogonal projection onto X_τ, and assume that its restriction to X is stable in the X norm (i.e., $\|\Pi_{X_\tau}\phi\|_X \lesssim \|\phi\|_X$ for all $\phi \in X$). We also denote by $\pi_t : L^1(0,T) \to \mathcal{P}^\ell_\tau(\mathcal{T}_\tau)$ the $L^2(0,T)$-orthogonal projection onto $\mathcal{P}^\ell_\tau(\mathcal{T}_\tau)$, and define the mixed projection $\Pi^t_{X_\tau} := \pi_t \circ \Pi_{X_\tau}$. Then, since $\partial_t \mathcal{R}_\tau u_\tau \in \mathcal{P}^\ell_\tau(X_\tau)$, it follows from Eq. (2.7) and the orthogonality properties of π_t and Π_{X_τ} that, for any $\phi \in L^p(0,T;X)$,

$$\int_0^T (\partial_t \mathcal{R}_\tau u_\tau, \phi)_B\, dt = \int_0^T (\partial_t \mathcal{R}_\tau u_\tau, \Pi^t_{X_\tau}\phi)_B\, dt = \int_0^T \langle \mathcal{F}_\tau, \Pi^t_{X_\tau}\phi\rangle_{X'_\tau\times X_\tau}\, dt \leq \|\mathcal{F}_\tau\|_{L^{p'}(0,T;X'_\tau)} \|\Pi^t_{X_\tau}\phi\|_{L^p(0,T;X)}. \tag{2.8}$$

Combining this estimate with the stability of π_t (see, e.g., [26, Thm. 18.16(ii)]) and the stability of Π_{X_τ} in the X norm, we obtain that the sequence $\{\partial_t \mathcal{R}_\tau u_\tau\}_{\tau>0}$ is uniformly bounded in $L^{p'}(0,T;X')$, thus verifying (h2) with $Y = X'$ and $r = p'$. Therefore, if $u_0 \in X$, the sequence $\{u_\tau\}_{\tau>0}$ is relatively compact in $L^q(0,T;B)$ for all $1 \leq q < \infty$ by Theorem 2.1 case (iii) with $r = p'$ and $\theta = 1/2$; cf. [1, Thm. 3.1(2)].

3. Concluding remarks

We have presented a compactness result for high-order piecewise polynomial functions in time, which serves as a discrete counterpart of the Aubin–Lions–Simon lemma and of Amann's theorem on compactness of higher regularity. By exploiting the properties of a time reconstruction operator, we have circumvented the need for equicontinuity arguments and the assumption of quasi-uniform time partitions required in earlier works. Our approach is expected to contribute to the analysis of discontinuous Galerkin time discretizations of complex nonlinear evolution problems.

Conflict of interest

The author declares no competing interests.

Data availability

No data was used for the research described in the article.

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