# Congestion control and optimal maintenance of communication networks with stochastic cost functions: a variational formulation*

Mauro Passacantando and Fabio Raciti

**Abstract** We consider a game-theory model of congestion control in communication networks, where each player is a user who wishes to maximize his/her flow over a path in the network. We allow for stochastic fluctuations of the cost function of each player, which consists of two parts: a pricing and a utility term. The solution concept we look for is the mean value of the (unique) variational Nash equilibrium of the game. Furthermore, we assume that it is possible to invest a certain amount of money to improve the network by enhancing the capacity of its links and, because of limited financial resources, an optimal choice of the links to improve has to be made. We model the investment problem as a nonlinear knapsack problem with generalized Nash equilibrium constraints in probabilistic Lebesgue spaces and solve it numerically for some examples.

**Key words:** Generalized Nash equilibrium; stochastic cost function; congestion control; investment optimization.

## 1 Introduction

In this paper we first model a congestion control problem in communication networks within a game theory approach which permits to treat stochastic costs func-

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tions and then consider the problem of improving the overall network performance in an optimal way, by investing a given amount of money. The cost function of each player is the difference of a pricing term, which promotes congestion control, and a utility term which describes the user's profit.

Game theoretical models for network equilibrium problems are very popular and, in the case of communication networks, an interesting approach has been developed in the papers [1, 2, 19, 22]. Our starting point is the model introduced in [1], where the players are network users who compete to send their flow from a given origin to a certain destination node along a route that has been computed previously by a routing algorithm. The fact that some links in the network are used by more than one player implies that the strategy space of each player also depends on the variables of other players. As a result, the game under consideration falls in the class of Generalized Nash Equilibrium Problems (GNEPs) with shared constrained, introduced by Rosen a long time ago [21], and further developed recently by using the theory of variational inequalities (see, e.g., [3, 4, 16, 18]). Indeed, it is well known (see e.g. [6]) that GNEPs are equivalent to quasi-variational inequalities which are considered very difficult problems (for an $L^2$ approach to quasi-variational inequalities see, for instance, [20]). In our approach we allow for stochastic fluctuations of the players' cost function and apply the theory of variational inequalities in probabilistic Lebesgue spaces (see, e.g., [5, 7, 8, 9, 10, 11, 12]), to find the unique variational equilibrium of the game, which is considered the most desirable from an economic point of view among the multiple equilibria [3]. Let us also mention that the theory of variational inequalities in probabilistic Lebesgue spaces has been recently applied to study the efficiency of road traffic networks [13, 17] and that a different approach to stochastic variational inequalities has been applied to communication networks in [15].

In our model the bandwidth is the most important characteristic of the network and, in this respect, a system manager may wish to improve the network by investing financial resources to enhance the capacity of the links. In a typical real situation the investment cannot cover all the links and a choice has to be made to decide which links to improve. The system manager makes his/her decision with the help of a network cost function associated to each set of improvements, which has the role of maximizing the aggregate utility, while minimizing the total delay at the links. Since this system function depends on the stochastic price and utility functions, the quantity of interest is its mean value. Thus, once the sets of variational equilibria, for all feasible improvements, is computed, we have to solve a knapsack-type problem which, for instances of reasonable dimensions can be solved by direct inspection, that is by ordering all the solutions according to their corresponding relative variation of the above mentioned system function.

The paper is organized in four sections and an appendix. In Section 2, we introduce some notations and the congestion control model proposed in [1], which we modify to include stochastic fluctuations of the cost functions; we also introduce the variational inequality whose (unique) solution gives the desired Nash equilibrium of the game, and define a system function which describes a global property of the network. Section 3 is devoted to describe the optimal investment strategy. In

Section 4, we apply our model to some small-size problems which are solved numerically. The appendix has the role of providing the frame of stochastic variational inequalities in probabilistic Lebesgue space, but for the details of the numerical approximation scheme the interested reader can refer to the references mentioned in this introduction.

## 2 The congestion control model and its stochastic variational inequality formulation

Throughout the paper, vectors of $\mathbb{R}^n$ are thought of as rows, but in matrix operations they will be considered as columns and the superscript $^\top$ will denote transposition. The scalar product between two Euclidean vectors $a$ and $b$ will be denoted by $a^\top b$, while the scalar product between two square integrable functions $f$ and $g$ will be denoted in compact form by $\langle f, g\rangle_{L^2}$. The notation $E_P[f]$ will be used to denote the mean value of a random function $f$ with respect to the probability measure $P$. The network topology consists of a set of links $\mathscr{L} = \{1,\dots,L\}$ connecting the nodes in the set $\mathscr{N} = \{n_1,\dots,n_N\}$. The users of the network belong to the set $\mathscr{G} = \{g_1,\dots,g_M\}$. A route $R$ in the network is a set of consecutive links and each user $g_i$ wishes to send a flow $x_i$ between a given pair $O_i - D_i$ of origin-destination nodes; $x \in \mathbb{R}^M$ is the (route) flow of the network; the notation $x = (x_i, x_{-i})$, common in game theory, will be used in the sequel when we need to distinguish the flow component of player $g_i$ from all the others. We assume that the routing problem has already been solved and that there is only one route $R_i$ assigned to user $g_i$. Each link $l$ has a fixed capacity $C_l > 0$, so that user $i$ cannot send a flow greater than the capacity of every link of his/her route, and we group these capacities into a vector $C \in \mathbb{R}^L$. To describe the link structure of each route, it is useful to introduce the link-route matrix whose entries are given by:

$$A_{li} = \begin{cases} 1, & \text{if link } l \text{ belongs to route } R_i, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

Using the link-route matrix the set of feasible flows can be written in compact form as

$$X := \left\{ x \in \mathbb{R}^M : \; x \geq 0, \;\; Ax \leq C \right\}. \tag{2}$$

In order to better specify the feasible set of each player, we write by components the conservation of flow in $X$ as

$$\sum_{i=1}^{M} A_{li} x_i \leq C_l, \qquad \forall \, l \in \mathscr{L}.$$

Therefore, because users share some links, the possible amount of flow $x_i$ depends on the flows sent by the other users and is bounded from above by the quantity

$$m_i(x_{-i}) = \min_{l \in R_i} \left\{ C_l - \sum_{j=1,\ j \neq i}^{M} A_{lj} x_j \right\} \geq 0.$$

Now, let $(\Omega, \mathscr{A}, P)$ be a probability space, and define the cost function $J_i : \Omega \times \mathbb{R}^M \to \mathbb{R}$ of player $g_i$ as

$$J_i(\omega, x) = P_i(\omega, x) - U_i(\omega, x_i), \tag{3}$$

where $U_i$ represents the utility function of player $g_i$, which only depends on the flow that he/she sends through the network, while $P_i$ is a pricing term which represents some kind of toll that $g_i$ pays to exploit the network resources and depends on the flows of the players with common links to $g_i$. Stochastic fluctuations of both $P_i$ and $U_i$ are described by the random parameter $\omega \in \Omega$. Players compete in a non-cooperative manner, as it is assumed that they do not communicate, and act selfishly to increase their flow. Because the conservation law in $X$ implies that users share the constraints, the solution concept adopted is the equilibrium introduced by Rosen in his seminal paper [21], which in the modern literature is known as generalized Nash equilibrium (with coupled constraints). Due to the presence of $\omega \in \Omega$, the Nash equilibrium is a random vector, according to the following definition:

$$\begin{gathered} x^* = (x_i^*(\omega), x_{-i}^*(\omega)) : \Omega \to \mathbb{R}^M \text{ is a generalized Nash equilibrium if} \\ \text{for each } i \in \{1, \dots, M\} \text{ and } P\text{-a.s.:} \\ J_i(\omega, x_i^*(\omega), x_{-i}^*(\omega)) = \min_{x_i \in X_i(x_{-i}^*(\omega))} J_i(\omega, x_i, x_{-i}^*(\omega)), \end{gathered} \tag{4}$$

where

$$X_i(x_{-i}^*(\omega)) := \left\{ x_i \in \mathbb{R} :\ (x_i, x_{-i}^*(\omega)) \in X \right\} = \{ x_i \in \mathbb{R} :\ 0 \leq x_i \leq m_i(x_{-i}^*(\omega)) \}$$

and

$$m_i(x_{-i}^*(\omega)) = \min_{l \in R_i} \left\{ C_l - \sum_{j=1,\ j \neq i}^{M} A_{lj} x_j^*(\omega) \right\}.$$

For each fixed $\omega$, it is well known (see, e.g., [6]) that, under standard differentiability and convexity assumptions, the above problem is equivalent to a quasi-variational inequality and that a particular subset of solutions (called variational equilibria) can be found by solving the variational inequality $VI(F, X)$, where $X$ is the feasible set defined in (2) and $F$ is the so-called *pseudogradient* of the game, defined by

$$F(\omega, x) = \left( \frac{\partial J_1(\omega, x)}{\partial x_1}, \dots, \frac{\partial J_M(\omega, x)}{\partial x_M} \right). \tag{5}$$

More precisely, the variational inequality under consideration is the problem of finding, for each $\omega \in \Omega$, a vector $x^*(\omega) \in X$ such that:

$$F(\omega, x^*(\omega))^\top (x - x^*(\omega)) \geq 0, \ \ \forall x \in X, P- \text{ a. s.} \tag{6}$$

In (4) and (6) the solution is a random vector, i.e., a vector function which is merely measurable with respect to the probability measure $P$ on $\Omega$. Since we wish to compute statistical quantities associated with the solution, it is natural to require that $x^*$ has finite first and second order moments. Following [7], we provide an $L^2$ formulation of both (4) and (6).

We will posit the following assumptions on the cost functions $J_i$, for each $i \in \{1,\dots,M\}$:

A) $J_i(\cdot,x)$ is a random variable for each $x \in \mathbb{R}^M$, and $J_i(\omega,\cdot) \in C^1(\mathbb{R}^M)$, $P$-a.s.;
B) $J_i(\omega,0) \in L^1(\Omega,P)$;
C) $J_i(\omega,\cdot,x_{-i})$ is convex $P$-a.s. and $\forall\, x_{-i} \in \mathbb{R}^{M-1}$;
D) $|\nabla_x J_i(\omega,x)| \le c(1+|x|)$, $\forall x \in \mathbb{R}^M$, $P-$ a.s. .

Let us now introduce, for each $i \in \{1,\dots,M\}$, the mapping $T_i : L^2(\Omega,P,\mathbb{R}^M) \to \mathbb{R}$ defined by

$$T_i(u_i,u_{-i}) = \int_\Omega J_i(\omega,u_i(\omega),u_{-i}(\omega))\,dP_\omega. \tag{7}$$

The following Lemma specifies some fundamental properties of $T_i$ and can be proved along the same lines as in [6].

**Lemma 1** *Let us assume that, for each $i \in \{1,\dots,M\}$, $J_i$ satisfies assumptions A) – D). Then, for each $i \in \{1,\dots,M\}$, $T_i$ is well defined in $L^2(\Omega,P,\mathbb{R}^M)$, $T_i(\cdot,u_{-i})$ is convex and Gateaux-differentiable in $L^2(\Omega,P)$, for each $u_{-i}$, and its derivative is given by*

$$D_iT_i(u_i,u_{-i})(v_i) = \int_\Omega \frac{\partial}{\partial x_i}[J_i(\omega,u_i(\omega),u_{-i}(\omega)]\, v_i(\omega)\,dP_\omega, \quad \forall\, v_i \in L^2(\Omega,P). \tag{8}$$

In order to provide the $L^2$-formulation of (4) and (6), we need to introduce the following sets:

$$K := \left\{u \in L^2(\Omega,P,\mathbb{R}^M):\; u(\omega) \ge 0,\, Au(\omega) \le C,\; P-\text{ a.s.}\right\}$$

and

$$K_i(u_-i) := \left\{u_i \in L^2(\Omega,P):\; (u_i(\omega),u_{-i}(\omega)) \in K,\; P-\text{ a. s.}\right\}.$$

Thus, a vector $u^* = (u_i^*,u_{-i}^*) \in L^2(\Omega,P,\mathbb{R}^M)$ is a generalized Nash equilibrium iff, for each $i \in \{1,\dots,M\}$:

$$\int_\Omega J_i(\omega,u_i^*(\omega),u_{-i}^*(\omega))\,dP_\omega = \min_{u_i \in K_i(u_{-i}^*)} \int_\Omega J_i(\omega,u_i(\omega),u_{-i}^*(\omega))\,dP_\omega. \tag{9}$$

The variational solutions of (9) can be obtained by solving the following variational inequality $VI(\Gamma,K)$: find $u^* \in K$ such that

$$\int_\Omega \sum_{i=1}^M \left[\frac{\partial}{\partial x_i} J_i(\omega,u^*(\omega))\right](v_i(\omega)-u_i^*(\omega))dP_\omega \ge 0, \qquad \forall\, v \in K, \tag{10}$$

where $\Gamma : L^2(\Omega, P, \mathbb{R}^M) \to L^2(\Omega, P, \mathbb{R}^M)$ is given by:

$$\Gamma(u) = (\Gamma_1(u), \ldots, \Gamma_M(u)) = \left( \frac{\partial}{\partial x_1} J_1(\omega, u(\omega)), \ldots, \frac{\partial}{\partial x_M} J_M(\omega, u(\omega)) \right). \quad (11)$$

It can be useful to write (10) in compact form by using the following notation:

$$\langle \Gamma(u), v - u \rangle_{L^2} \geq 0, \qquad \forall\, v \in L^2(\Omega, P, \mathbb{R}^M).$$

Problem (10) is a random (or stochastic) variational inequality in $L^2$ and the interested reader can refer to the articles mentioned in the introduction for a comprehensive treatment of this relatively new methodology as well as for several applications. In order to be self-consistent, we give in the appendix a short outline of the topic, in the general $L^p$ setting ($p \geq 2$).

In what follows, we consider the specific functional form of $P_i$ and $U_i$ treated in [1], with a slight modification, and allowing for stochastic fluctuations. Furthermore, we show the existence of a unique variational equilibrium of the game. Specifically, the utility function $U_i$ of player $g_i$ is given by

$$U_i(\omega, x_i) = a_i(\omega) \log(x_i + 1), \quad (12)$$

where $a_i \in L^\infty(\Omega, P)$ and is bounded away from zero from below for each $i \in \{1, \ldots, m\}$. The route price function $P_i$ of player $g_i$ is the sum of the price functions of the links associated to route $R_i$:

$$P_i(\omega, x) = \sum_{l \in R_i} P_l \left( \omega, \sum_{j=1}^{M} A_{lj} x_j \right). \quad (13)$$

Let us notice that $P_l$ is modeled so as to only depend on the variables of players who share the link $l$, namely:

$$P_l \left( \omega, \sum_{j=1}^{M} A_{lj} x_j \right) = \frac{k(\omega)}{C_l - \sum_{j=1}^{M} A_{lj} x_j + e}, \quad (14)$$

where $k \in L^\infty(\Omega, P)$ is a network function, bounded away from zero from below, and $e$ is a small positive number which we introduce to allow capacity saturation, while obtaining a well behaved function. The price function of $g_i$ is thus given by

$$P_i(\omega, x) = \sum_{l \in R_i} \frac{k(\omega)}{C_l - \sum_{j=1}^{M} A_{lj} x_j + e}, \quad (15)$$

and the resulting expression of the cost for $g_i$ is:

$$J_i(\omega, x) = \sum_{l \in R_i} \frac{k(\omega)}{C_l - \sum_{j=1}^{M} A_{lj} x_j + e} - a_i(\omega) \log(x_i + 1). \quad (16)$$

The following properties of the above functions are easy to check, for each fixed $\omega$: (i) $U_i(\omega,\cdot)$ is twice continuously differentiable, non-decreasing and strongly concave on any compact interval $[0,b]$ (the last condition means that there exists $\tau > 0$ such that $\partial^2 U_i(\omega,x_i)/\partial x_i^2 \leq -\tau$ for any $x_i \in [0,b]$); (ii) $P_i(\omega,\cdot)$ is twice continuously differentiable, convex and $P_i(\omega,\cdot,x_{-i})$ is non-decreasing. These properties of $U_i$ and $P_i$ entail an important monotonicity property of the pseudogradient $F$ defined in (5), as the following theorem shows.

**Theorem 1** *Let $U_i$ and $P_i$ be given as in* (12) *and* (15)*, then $F$ is strongly monotone on $X$, uniformly with respect to $\omega \in \Omega$, i.e., there exists $\alpha > 0$ such that*

$$(F(\omega,x) - F(\omega,y))^\top (x-y) \geq \alpha \|x-y\|^2, \qquad \forall\, x,y \in X, \forall\, \omega \in \Omega.$$

***Proof*** Similarly to [1], it can be shown that the Jacobian matrix of $F$ is positive definite on $X$, uniformly with respect to $x$. Moreover, since the random parameters $k$ and $a_i$ are bounded, the Jacobian is positive definite, uniformly with respect to $\omega$. Thus, $F$ is strongly monotone on $X$, uniformly with respect to $\omega$. □

The unique solvability of $VI(\Gamma,K)$ is based on standard arguments, as the following theorem shows.

**Theorem 2** *There exists a unique variational equilibrium of the GNEP* (9)*.*

***Proof*** The variational equilibria of (9) are the solutions of (10), i.e., of $VI(\Gamma,K)$. Under assumptions $A)-D)$, the operator $\Gamma$ generated by $F$ maps $L^2$ in $L^2$ and is norm-continuous, being $P$ a probability measure. Moreover, the uniform strong monotonicity of $F$ implies the uniform strong monotonicity of $\Gamma$. At last, the set $K$ is a closed and convex subset of $L^2(\Omega,P,\mathbb{R}^M)$ and is norm-bounded, hence weakly compact. Then, applying monotone operator theory we get that (10) admits a unique solution (see e.g. [14]), which is the unique variational equilibrium of (9). □

We now introduce a function $f$ which describes a global property of the game:

$$f(\omega,x) = \sum_{l\in\mathscr{L}} P_l\left(\omega, \sum_{j=1}^{M} A_{lj}x_j\right) - \sum_{i=1}^{M} U_i(\omega,x_i), \tag{17}$$

which represents the aggregate delay at the links minus the sum of the utilities of all players.

The Carathéodory function $f$ generates a functional $\Pi : L^2(\Omega,P,\mathbb{R}^M) \to \mathbb{R}$ through the position:

$$\Pi(u(\omega)) := E_P[f] = \int_\Omega f(\omega,u(\omega))dP_\omega, \quad \forall\, u \in K \subset L^2(\Omega,P,\mathbb{R}^M). \tag{18}$$

The theorem which follows shows that $\Pi$ plays the role of a potential for the game described by (9).

**Theorem 3** *The unique variational equilibrium of the GNEP* (9) *coincides with the optimal solution of the system problem* $\min\limits_{u\in K} \Pi(u)$*.*

***Proof*** Since both $\Pi$ and $K$ are convex, $\bar{u}$ is a minimizer of $\Pi$ on $K$ if and only if

$$\langle D\Pi(\bar{u}), v - \bar{u} \rangle_{L^2} \geq 0, \quad \forall\, v \in K,$$

where $D\Pi(\bar{u})$ stands for the Gateaux derivative of $\Pi$ in $\bar{u}$. Since $D\Pi = \Gamma$, the expression above is nothing else that the variational inequality $VI(\Gamma, K)$, whose solution gives the variational equilibrium of (9). □

To study our model from a numerical point of view, we need to pass from the abstract probability space $(\Omega, \mathscr{A}, P)$ to the probability space generated by the random variables under consideration: $(k, a_1, \ldots, a_M)$. The new probability space is then $(\mathbb{R}^{M+1}, \mathscr{B}, \mathbb{P})$, where $\mathscr{B}$ represents the Borel $\sigma$-algebra on $\mathbb{R}^{M+1}$ and $\mathbb{P} = P_k \otimes P_{a_1} \otimes, \ldots, \otimes P_{a_M} = P_k \otimes P_a$, where we assumed independence of all the random variables involved. In what follow, with a slight abuse of notation, we will continue to denote with $K$ and $K_i(u_{-i})$ the sets previously defined but expressed now with new variables $(k, a)$. The cost functions are thus expressed as $J_i(k, a, x)$, and problem (9) now reads as

$$\begin{aligned} &\int_{\mathbb{R}^{M+1}} J_i(k, a, u_i^*(k,a), u_{-i}^*(k,a))\, dP_k\, dP_a \\ &= \min_{u_i \in K_i(u_{-i}^*)} \int_{\mathbb{R}^{M+1}} J_i(k, a, u_i(k,a), u_{-i}^*(k,a))\, dP_k\, dP_a, \end{aligned} \tag{19}$$

while the variational solutions of (19) can be obtained by solving the following variational inequality: find $u^* \in K$ such that

$$\int_{\mathbb{R}^{M+1}} \sum_{i=1}^{M} \left[ \frac{\partial}{\partial x_i} J_i(k, a, u^*(\omega)) \right] (v_i(k,a) - u_i^*(k,a))\, dP_k\, dP_a \geq 0, \quad \forall\, v \in K. \tag{20}$$

Analogously, the system function, as a function of the random variables, reads as

$$f(k, a, x) = \sum_{l \in \mathscr{L}} P_l \left( k, a, \sum_{j=1}^{M} A_{lj} x_j \right) - \sum_{i=1}^{M} U_i(k, a, x_i), \tag{21}$$

and its mean value is expressed by

$$E_{\mathbb{P}}[f] = \Pi(u(k,a)) := \int_{\mathbb{R}^{M+1}} f(k, a, u(k,a))\, dP_k\, dP_a. \tag{22}$$

Thus, the quantities of interest in our model are $E_{\mathbb{P}}[u^*] = \int_{\mathbb{R}^{M+1}} u^*(k,a) dP_k\, dP_a$ and $\Pi(u^*(k,a))$.

## 3 The optimal network improvement model

We now suppose that the network system manager has a budget $B$ available to improve the network performance. He/she can only increase the capacity of a subset $\widetilde{\mathscr{L}} \subseteq \mathscr{L}$ of links and knows that $I_l$ is the investment required to enhance the capacity of link $l$ by a given ratio $\gamma_l$. Since the available budget is generally not sufficient to enhance the capacities of all the links of $\widetilde{\mathscr{L}}$, he/she has to decide which subset of links to invest in, in order to improve as much as possible the system cost $\Pi$ computed at the variational equilibrium of the game with new link capacities, while satisfying the budget constraint. This problem can be formulated as an integer nonlinear program.

To this end, we define a binary variable $y_l$, for any $l \in \widetilde{\mathscr{L}}$, which takes on the value 1 if the investment is actually carried out on link $l$, and 0 otherwise. A vector $y = (y_l)_{l\in\widetilde{\mathscr{L}}}$ is feasible if the budget constraint $\sum_{l\in\widetilde{\mathscr{L}}} I_l y_l \leq B$ is satisfied. Given a feasible vector $y$, the new capacity of each link $l \in \widetilde{\mathscr{L}}$ is equal to

$$C'_l(y) := \gamma_l C_l y_l + (1 - y_l)C_l,$$

i.e., $C'_l(y) = \gamma_l C_l$ if $y_l = 1$ and $C'_l(y) = C_l$ if $y_l = 0$. The network manager aims to maximize the percentage relative variation of the system cost defined as

$$\varphi(y) := 100 \cdot \frac{\Pi(u_0^*(k,a)) - \Pi(u_y^*(k,a))}{\Pi(u_0^*(k,a))},$$

where $u_0^*(k,a)$ is the variational equilibrium of the GNEP before the investment, while $u_y^*(k,a)$ is the variational equilibrium of the GNEP on the improved network according to $y$. Therefore, the proposed optimization model is

$$\begin{aligned} \max \quad & \varphi(y) \\ \text{subject to} \quad & \sum_{l\in\widetilde{\mathscr{L}}} I_l y_l \leq B, \\ & y_l \in \{0,1\} \quad l \in \widetilde{\mathscr{L}}. \end{aligned} \tag{23}$$

The above model can be considered a generalized knapsack problem because the computation of the nonlinear function $\varphi$ at a given $y$ requires to find the variational equilibrium of the GNEP both for the original and the improved network. Notice that, since the variational equilibrium of the GNEP is the minimizer of $\Pi$ (see Theorem 3), the optimization problem (23) can be reformulated as the following mixed integer nonlinear program:

$$\min \sum_{l\in\widetilde{\mathscr{L}}} \int_{\mathbb{R}^{M+1}} \frac{k}{\gamma_l C_l y_l + (1-y_l)C_l - \sum\limits_{i=1}^{M} A_{li} u_i(k,a) + e} dP_k \, dP_a$$

$$
\begin{aligned}
&+\sum_{l\in\mathscr{L}\setminus\widetilde{\mathscr{L}}}\int_{\mathbb{R}^{M+1}}\frac{k}{C_l-\sum\limits_{i=1}^{M}A_{li}u_i(k,a)+e}\,dP_k\,dP_a\\
&-\sum_{i=1}^{M}\int_{\mathbb{R}^{M+1}}a_i\log(u_i(k,a)+1)\,dP_k\,dP_a\\
\text{subject to}\quad &\sum_{i=1}^{M}A_{li}u_i(k,a)\le\gamma_l C_l y_l+(1-y_l)C_l \qquad l\in\widetilde{\mathscr{L}},\\
&\sum_{i=1}^{M}A_{li}u_i(k,a)\le C_l \qquad l\in\mathscr{L}\setminus\widetilde{\mathscr{L}},\ \mathbb{P}-\text{ a.s.}\\
&\sum_{l\in\widetilde{\mathscr{L}}}I_l y_l\le B,\\
&u_i(k,a)\ge 0,\qquad i=1,\dots,M,\ \mathbb{P}-\text{ a.s.}\\
&y_l\in\{0,1\}\qquad l\in\widetilde{\mathscr{L}}.
\end{aligned}
$$

## 4 Numerical experiments

In this section, we show some preliminary numerical experiments on two test networks for the stochastic formulation of the congestion control problem and the optimal network improvement problem. The numerical approximation of random variational equilibria was performed by implementing in Matlab 2020a the discretization procedure described in [9, 10] and exploiting the Matlab Optimization toolbox. The nonlinear knapsack problem (23) has been solved evaluating the objective function at all the feasible solutions.

**Example 1.** We consider the network shown in Fig. 1 (see also [1]) with nine nodes and nine links. The origin-destination pairs of the users and their routes are de-

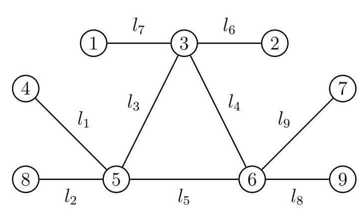


**Fig. 1** Network topology of Example 1.

scribed in Table 1. We set $e = 0.01$ and $C_l = 10$ for any $l \in \mathscr{L}$. Moreover, we assume that the random parameter $k$ is equal to $k = 100+\delta_k$, where $\delta_k$ is a random variable which varies in the interval $[-90, 90]$ with either uniform distribution or

**Table 1** Origin-Destination pairs and routes (sequence of links) of the users in Example 1.

| User | Origin | Destination | Route |
|---|---|---|---|
| 1 | 8 | 2 | $l_2, l_3, l_6$ |
| 2 | 8 | 7 | $l_2, l_5, l_9$ |
| 3 | 4 | 7 | $l_1, l_5, l_9$ |
| 4 | 2 | 7 | $l_6, l_4, l_9$ |
| 5 | 9 | 7 | $l_8, l_9$ |

truncated normal distribution with mean 0 and standard deviation 9. Moreover, for any $i \in \{1,\dots,m\}$, the random parameters $a_i$ are equal to $a_i = 100+\delta_a$, where $\delta_a$ is a random variable which varies in the interval $[-90,90]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 9. Both intervals $[-90,90]$ have been partitioned into $N^d$ subintervals in the approximation procedure. Tables 2–5 show the convergence of the approximated mean values of the variational equilibrium $u^*$ for different values of $N^d$ by using the four different combinations of probability densities.

**Table 2** Convergence of the approximated mean values of the variational equilibrium of Example 1; $\delta_k$ and $\delta_a$ vary in the interval $[-90,90]$ with uniform distribution.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 4.4478 | 4.4440 | 4.4434 | 4.4433 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.6458 | 1.6433 | 1.6429 | 1.6428 |
| $(E_{\mathbb{P}}[u^*])_3$ | 2.1001 | 2.0970 | 2.0965 | 2.0964 |
| $(E_{\mathbb{P}}[u^*])_4$ | 1.7110 | 1.7081 | 1.7077 | 1.7076 |
| $(E_{\mathbb{P}}[u^*])_5$ | 2.3924 | 2.3891 | 2.3886 | 2.3884 |

**Table 3** Convergence of the approximated mean values of the variational equilibrium of Example 1; $\delta_k$ varies in the interval $[-90,90]$ with uniform distribution, $\delta_a$ varies in the interval $[-90,90]$ with truncated normal distribution with mean 0 and standard deviation 9.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 4.7310 | 4.7326 | 4.7328 | 4.7328 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.7183 | 1.7183 | 1.7183 | 1.7183 |
| $(E_{\mathbb{P}}[u^*])_3$ | 2.2025 | 2.2026 | 2.2025 | 2.2025 |
| $(E_{\mathbb{P}}[u^*])_4$ | 1.7811 | 1.7811 | 1.7811 | 1.7811 |
| $(E_{\mathbb{P}}[u^*])_5$ | 2.4633 | 2.4634 | 2.4634 | 2.4634 |

**Table 4** Convergence of the approximated mean values of the variational equilibrium of Example 1; $\delta_k$ varies in the interval $[-90,90]$ with truncated normal distribution with mean 0 and standard deviation 9, $\delta_a$ varies in the interval $[-90,90]$ with uniform distribution.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 4.2698 | 4.2636 | 4.2627 | 4.2624 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.6233 | 1.6203 | 1.6198 | 1.6196 |
| $(E_{\mathbb{P}}[u^*])_3$ | 2.0642 | 2.0604 | 2.0598 | 2.0596 |
| $(E_{\mathbb{P}}[u^*])_4$ | 1.6945 | 1.6912 | 1.6906 | 1.6905 |
| $(E_{\mathbb{P}}[u^*])_5$ | 2.3775 | 2.3736 | 2.3730 | 2.3728 |

**Table 5** Convergence of the approximated mean values of the variational equilibrium of Example 1; $\delta_k$ and $\delta_a$ vary in the interval $[-90,90]$ with truncated normal distribution with mean 0 and standard deviation 9.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 4.5685 | 4.5680 | 4.5679 | 4.5679 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.6995 | 1.6993 | 1.6993 | 1.6993 |
| $(E_{\mathbb{P}}[u^*])_3$ | 2.1726 | 2.1723 | 2.1722 | 2.1722 |
| $(E_{\mathbb{P}}[u^*])_4$ | 1.7679 | 1.7677 | 1.7677 | 1.7677 |
| $(E_{\mathbb{P}}[u^*])_5$ | 2.4504 | 2.4502 | 2.4502 | 2.4502 |

We now consider the optimal network improvement problem. We set $e = 0.01$ and $C_l = 10$ for any $l \in \mathscr{L}$. The random parameters are of the form: $k = 100 + \delta_k$ and $a_i = 100 + \delta_a$, where $\delta_k$ and $\delta_a$ vary in the interval $[-90,90]$ with uniform distribution. Each interval $[-90,90]$ has been been partitioned into 25 subintervals in the approximation procedure. We assume that the available budget is $B = 20$ $k$€, the set of links to be improved is $\widetilde{\mathscr{L}} = \mathscr{L}$, while the values of $\gamma_l$ and $I_l$ are shown in Table 6.

**Table 6** Capacity enhancement factors and investments for links of Example 1.

| Links | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ | $l_6$ | $l_7$ | $l_8$ | $l_9$ |
|---|---|---|---|---|---|---|---|---|---|
| $\gamma_l$ | 1.2 | 1.5 | 1.1 | 1.6 | 1.3 | 1.4 | 1.1 | 1.7 | 1.3 |
| $I_l$ ($k$€) | 3 | 8 | 2 | 10 | 4 | 5 | 2 | 12 | 4 |

Table 7 shows the ten best feasible solutions together with the percentage of total cost improvement $\varphi(y)$ and the corresponding investment $I(y) = \sum_{l \in \widetilde{\mathscr{L}}} I_l y_l$.

**Table 7** The ten best feasible solutions for the optimal network improvement model in Example 1.

| Ranking | $y$ | $\varphi(y)$ | $I(y)$ |
|---|---|---|---|
| 1 | (0,1,1,0,0,1,0,0,1) | 28.0619 | 19 |
| 2 | (1,1,0,0,0,1,0,0,1) | 27.2448 | 20 |
| 3 | (0,1,0,0,0,1,1,0,1) | 26.8262 | 19 |
| 4 | (0,1,0,0,0,1,0,0,1) | 26.6207 | 17 |
| 5 | (1,0,1,0,1,1,1,0,1) | 23.2752 | 20 |
| 6 | (0,1,1,0,1,0,1,0,1) | 23.0862 | 20 |
| 7 | (1,0,1,0,1,1,0,0,1) | 23.0697 | 18 |
| 8 | (0,1,1,0,1,0,0,0,1) | 22.8807 | 18 |
| 9 | (1,1,0,0,1,0,0,0,1) | 22.8453 | 19 |
| 10 | (0,0,1,0,1,1,1,0,1) | 22.5908 | 17 |

**Example 2.** We now consider the network shown in Fig. 2 with 10 nodes and 13 links. The O-D pairs of the ten users and their routes are described in Table 8. We report numerical experiments similar to Example 1. First, we show the convergence of the approximated mean values of the variational equilibrium with respect to different probability distributions of the random parameters $k$ and $a$. Then, the solution of the optimal network improvement problem is reported.

We set $e = 0.01$ and $C_l = 10$ for any $l \in \mathscr{L}$. Moreover, we assume that the random parameter $k$ is equal to $k = 10 + \delta_k$, where $\delta_k$ is a random variable which varies in the interval $[-9,9]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 0.9. Moreover, for any $i \in \{1,\ldots,m\}$, the random parameters $a_i$ are equal to $a_i = 10 + \delta_a$, where $\delta_a$ is a random variable which varies in the interval $[-9,9]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 0.9. Both intervals $[-9,9]$ have been partitioned into $N^d$ subintervals in the approximation procedure. Tables 9–12 show the convergence of the approximated mean values of the variational equilibrium for

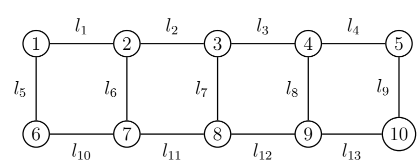


**Fig. 2** Network topology of Example 2.

**Table 8** Origin-Destination pairs and routes (sequence of links) of the users in Example 2.

| User | Origin | Destination | Route | User | Origin | Destination | Route |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 5 | $l_1, l_2, l_3, l_4$ | 6 | 5 | 1 | $l_4, l_3, l_2, l_1$ |
| 2 | 6 | 10 | $l_{10}, l_{11}, l_{12}, l_{13}$ | 7 | 10 | 6 | $l_{13}, l_{12}, l_{11}, l_{10}$ |
| 3 | 2 | 10 | $l_6, l_{11}, l_{12}, l_{13}$ | 8 | 5 | 8 | $l_9, l_{13}, l_{12}$ |
| 4 | 8 | 5 | $l_7, l_3, l_4$ | 9 | 4 | 6 | $l_8, l_{12}, l_{11}, l_{10}$ |
| 5 | 6 | 5 | $l_5, l_1, l_2, l_3, l_4$ | 10 | 8 | 1 | $l_7, l_2, l_1$ |

different values of $N^d$ by using the four different combinations of probability densities.

**Table 9** Convergence of the approximated mean values of the variational equilibrium of Example 2; $\delta_k$ and $\delta_a$ vary in the interval $[-9,9]$ with uniform distribution.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 1.3364 | 1.3342 | 1.3338 | 1.3337 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.3470 | 1.3450 | 1.3447 | 1.3446 |
| $(E_{\mathbb{P}}[u^*])_3$ | 1.4515 | 1.4491 | 1.4487 | 1.4486 |
| $(E_{\mathbb{P}}[u^*])_4$ | 2.7526 | 2.7499 | 2.7495 | 2.7493 |
| $(E_{\mathbb{P}}[u^*])_5$ | 1.2546 | 1.2523 | 1.2519 | 1.2518 |
| $(E_{\mathbb{P}}[u^*])_6$ | 1.3364 | 1.3342 | 1.3338 | 1.3337 |
| $(E_{\mathbb{P}}[u^*])_7$ | 1.3470 | 1.3450 | 1.3447 | 1.3446 |
| $(E_{\mathbb{P}}[u^*])_8$ | 1.8810 | 1.8773 | 1.8768 | 1.8766 |
| $(E_{\mathbb{P}}[u^*])_9$ | 1.6685 | 1.6652 | 1.6646 | 1.6645 |
| $(E_{\mathbb{P}}[u^*])_{10}$ | 2.7526 | 2.7499 | 2.7495 | 2.7493 |

**Table 10** Convergence of the approximated mean values of the variational equilibrium of Example 2; $\delta_k$ varies in the interval $[-9,9]$ with uniform distribution, $\delta_a$ varies in the interval $[-9,9]$ with truncated normal distribution with mean 0 and standard deviation 0.9.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 1.4076 | 1.4076 | 1.4076 | 1.4076 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.4204 | 1.4206 | 1.4206 | 1.4206 |
| $(E_{\mathbb{P}}[u^*])_3$ | 1.5275 | 1.5276 | 1.5276 | 1.5276 |
| $(E_{\mathbb{P}}[u^*])_4$ | 2.8993 | 2.9003 | 2.9004 | 2.9004 |
| $(E_{\mathbb{P}}[u^*])_5$ | 1.3380 | 1.3380 | 1.3380 | 1.3380 |
| $(E_{\mathbb{P}}[u^*])_6$ | 1.4076 | 1.4076 | 1.4076 | 1.4076 |
| $(E_{\mathbb{P}}[u^*])_7$ | 1.4204 | 1.4206 | 1.4206 | 1.4206 |
| $(E_{\mathbb{P}}[u^*])_8$ | 1.9536 | 1.9532 | 1.9532 | 1.9531 |
| $(E_{\mathbb{P}}[u^*])_9$ | 1.7588 | 1.7585 | 1.7584 | 1.7584 |
| $(E_{\mathbb{P}}[u^*])_{10}$ | 2.8993 | 2.9003 | 2.9004 | 2.9004 |

Let us consider the optimal network improvement problem. We set $e = 0.01$ and $C_l = 10$ for any $l \in \mathscr{L}$. The random parameters are $k = 10 + \delta_k$ and $a_i = 10 + \delta_a$, where $\delta_k$ and $\delta_a$ vary in the interval $[-9,9]$ with uniform distribution. Each interval $[-90,90]$ has been been partitioned into 25 subintervals in the approximation procedure. We assume that the available budget is $B = 20$ $k$€, the set of links to be improved is $\widetilde{\mathscr{L}} = \mathscr{L}$, while the values of $\gamma_l$ and $I_l$ are shown in Table 13.

**Table 11** Convergence of the approximated mean values of the variational equilibrium of Example 2; $\delta_k$ varies in the interval $[-9,9]$ with truncated normal distribution with mean 0 and standard deviation 0.9, $\delta_a$ varies in the interval $[-9,9]$ with uniform distribution.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 1.3135 | 1.3109 | 1.3104 | 1.3103 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.3123 | 1.3098 | 1.3094 | 1.3093 |
| $(E_{\mathbb{P}}[u^*])_3$ | 1.4258 | 1.4229 | 1.4224 | 1.4223 |
| $(E_{\mathbb{P}}[u^*])_4$ | 2.6674 | 2.6632 | 2.6625 | 2.6624 |
| $(E_{\mathbb{P}}[u^*])_5$ | 1.2263 | 1.2235 | 1.2231 | 1.2230 |
| $(E_{\mathbb{P}}[u^*])_6$ | 1.3135 | 1.3109 | 1.3104 | 1.3103 |
| $(E_{\mathbb{P}}[u^*])_7$ | 1.3123 | 1.3098 | 1.3094 | 1.3093 |
| $(E_{\mathbb{P}}[u^*])_8$ | 1.8871 | 1.8831 | 1.8824 | 1.8823 |
| $(E_{\mathbb{P}}[u^*])_9$ | 1.6590 | 1.6552 | 1.6546 | 1.6545 |
| $(E_{\mathbb{P}}[u^*])_{10}$ | 2.6674 | 2.6632 | 2.6625 | 2.6624 |

**Table 12** Convergence of the approximated mean values of the variational equilibrium of Example 2; $\delta_k$ and $\delta_a$ vary in the interval $[-9,9]$ with truncated normal distribution with mean 0 and standard deviation 0.9.

| $E_{\mathbb{P}}[u^*]$ | $N^d = 10$ | $N^d = 25$ | $N^d = 50$ | $N^d = 100$ |
|---|---|---|---|---|
| $(E_{\mathbb{P}}[u^*])_1$ | 1.3892 | 1.3890 | 1.3889 | 1.3889 |
| $(E_{\mathbb{P}}[u^*])_2$ | 1.3888 | 1.3887 | 1.3886 | 1.3886 |
| $(E_{\mathbb{P}}[u^*])_3$ | 1.5059 | 1.5057 | 1.5056 | 1.5056 |
| $(E_{\mathbb{P}}[u^*])_4$ | 2.8207 | 2.8205 | 2.8204 | 2.8204 |
| $(E_{\mathbb{P}}[u^*])_5$ | 1.3157 | 1.3155 | 1.3155 | 1.3155 |
| $(E_{\mathbb{P}}[u^*])_6$ | 1.3892 | 1.3890 | 1.3889 | 1.3889 |
| $(E_{\mathbb{P}}[u^*])_7$ | 1.3888 | 1.3887 | 1.3886 | 1.3886 |
| $(E_{\mathbb{P}}[u^*])_8$ | 1.9665 | 1.9662 | 1.9661 | 1.9661 |
| $(E_{\mathbb{P}}[u^*])_9$ | 1.7584 | 1.7580 | 1.7579 | 1.7579 |
| $(E_{\mathbb{P}}[u^*])_{10}$ | 2.8207 | 2.8205 | 2.8204 | 2.8204 |

**Table 13** Capacity enhancement factors and investments for links of Example 2.

| Links | $l_1$ | $l_2$ | $l_3$ | $l_4$ | $l_5$ | $l_6$ | $l_7$ | $l_8$ | $l_9$ | $l_{10}$ | $l_{11}$ | $l_{12}$ | $l_{13}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\gamma_l$ | 1.2 | 1.5 | 1.1 | 1.6 | 1.3 | 1.4 | 1.1 | 1.7 | 1.3 | 1.5 | 1.1 | 1.8 | 1.3 |
| $I_l$ ($k$€) | 3 | 8 | 2 | 10 | 4 | 5 | 2 | 12 | 4 | 8 | 2 | 13 | 4 |

Table 14 shows the ten best feasible solutions together with the value of $\varphi$ and the corresponding investment.

**Table 14** The ten best feasible solutions for the optimal network improvement model in Example 2.

| Ranking | $y$ | $\varphi(y)$ | $I(y)$ |
|---|---|---|---|
| 1 | (0,0,0,0,0,0,0,0,0,0,1,1,1) | 19.3768 | 19 |
| 2 | (1,0,0,0,0,0,0,0,0,0,0,1,1) | 18.9122 | 20 |
| 3 | (0,0,1,0,0,0,0,0,0,0,0,1,1) | 18.1916 | 19 |
| 4 | (0,0,0,0,0,0,1,0,0,0,0,1,1) | 17.6307 | 19 |
| 5 | (0,0,0,0,0,0,0,0,0,0,0,1,1) | 17.0483 | 17 |
| 6 | (1,0,1,0,0,0,0,0,0,0,1,1,0) | 15.8179 | 20 |
| 7 | (1,0,0,0,0,0,1,0,0,0,1,1,0) | 15.2526 | 20 |
| 8 | (1,0,0,0,0,0,0,0,0,0,1,1,0) | 14.6397 | 18 |
| 9 | (1,0,1,0,0,0,1,0,0,0,0,1,0) | 14.5318 | 20 |
| 10 | (0,0,1,0,0,0,1,0,0,0,1,1,0) | 14.5210 | 19 |

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## Appendix

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $A, B : \mathbb{R}^k \to \mathbb{R}^k$ two given mappings, and $b, c \in \mathbb{R}^k$ two given vectors in $\mathbb{R}^k$. Moreover, let $R$ and $S$ be two real-valued random variables defined on $\Omega$, $D$ a random vector in $\mathbb{R}^m$, and $G \in \mathbb{R}^{m \times k}$ a given matrix. For $\omega \in \Omega$, we define a random set

$$M(\omega) := \left\{ x \in \mathbb{R}^k : \; Gx \leq D(\omega) \right\}.$$

Consider the following stochastic variational inequality: for almost every $\omega \in \Omega$, find $\hat{x} := \hat{x}(\omega) \in M(\omega)$ such that

$$(S(\omega)A(\hat{x}) + B(\hat{x}))^\top (z - \hat{x}) \geq (R(\omega)\, c + b)^\top (z - \hat{x}), \qquad \forall\, z \in M(\omega). \tag{24}$$

To facilitate the foregoing discussion, we set $T(\omega, x) := S(\omega)A(x) + B(x)$. We assume that $A, B$ and $S$ are such that the map $T : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ is a Carathéodory function. We also assume that $T(\omega, \cdot)$ is monotone for every $\omega \in \Omega$. Since we are only interested in solutions with finite first- and second-order moments, our approach is to consider an integral variational inequality instead of the parametric variational inequality (24).

Thus, for a fixed $p \geq 2$, consider the Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors $V$ from $\Omega$ to $\mathbb{R}^k$ such that the expectation ($p$-moment) is given by

$$E^P(\|V\|^p) = \int_\Omega \|V(\omega)\|^p dP(\omega) < \infty.$$

For subsequent developments, we need the following growth condition

$$\|T(\omega,z)\| \leq \alpha(\omega)+\beta(\omega)\|z\|^{p-1}, \qquad \forall\, z \in \mathbb{R}^k, \tag{25}$$

where $\alpha \in L^q(\Omega,P)$ and $\beta \in L^\infty(\Omega,P)$. Due to the above growth condition, the Nemytskii operator $\hat{T}$ associated to $T$, acts from $L^p(\Omega,P,\mathbb{R}^k)$ to $L^q(\Omega,P,\mathbb{R}^k)$, where $p^{-1}+q^{-1}=1$, and is defined by $\hat{T}(V)(\omega) := T(\omega,V(\omega))$, for any $\omega \in \Omega$. Assuming $D \in L^p_m(\Omega) := L^p(\Omega,P,\mathbb{R}^m)$, we introduce the following nonempty, closed and convex subset of $L^p_k(\Omega)$:

$$M^P := \left\{V \in L^p_k(\Omega) :\; GV(\omega) \leq D(\omega),\; P-a.s.\right\}.$$

Let $S(\omega) \in L^\infty$, $0 < \underline{s} < S(\omega) < \overline{s}$, and $R(\omega) \in L^q$. Equipped with these notations, we consider the following $L^p$ formulation of (24): find $\hat{U} \in M^P$ such that for every $V \in M^P$, we have

$$\begin{aligned}\int_\Omega (S(\omega)A[\hat{U}(\omega)] + B[\hat{U}(\omega))]^\top (V(\omega)-\hat{U}(\omega))\,dP(\omega)\\ \geq \int_\Omega (b+R(\omega)\,c)^\top (V(\omega)-\hat{U}(\omega))dP(\omega).\end{aligned} \tag{26}$$

Classical theorems for the solvability of (26) can be found in [14].

To get rid of the abstract sample space $\Omega$, we consider the joint distribution $\mathbb{P}$ of the random vector $(R,S,D)$ and work with the special probability space $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d),\mathbb{P})$, where $d := 2+m$ and $\mathscr{B}$ is the Borel $\sigma$-algebra on $\mathbb{R}^d$ . For simplicity, we assume that $R$, $S$ and $D$ are independent random vectors. We set

$$r = R(\omega), \quad s = S(\omega), \quad t = D(\omega), \quad y = (r,s,t).$$

For each $y \in \mathbb{R}^d$, we define the set

$$M(y) := \left\{x \in \mathbb{R}^k :\; Gx \leq t\right\}.$$

Consider the space $L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$ and introduce the closed and convex set

$$M_{\mathbb{P}} := \{v \in L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) :\; Gv(r,s,t) \leq t,\; \mathbb{P}-a.s.\}.$$

Without any loss of generality, we assume that $R \in L^q(\Omega,P)$ and $D \in L^p(\Omega,P,\mathbb{R}^m)$ are non-negative. Moreover, we assume that the support (i.e., the set of possible outcomes) of $S \in L^\infty(\Omega,P)$ is the interval $[\underline{s},\overline{s}[\subset (0,\infty)$. With these ingredients, we consider the variational inequality problem of finding $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$ we have

$$\begin{aligned}\int_0^\infty\int_{\underline{s}}^{\overline{s}}\int_{\mathbb{R}^m_+} (sA[\hat{u}(y)] + B[\hat{u}(y)])^\top (v(y)-\hat{u}(y))\,d\mathbb{P}(y)\\ \geq \int_0^\infty\int_{\underline{s}}^{\overline{s}}\int_{\mathbb{R}^m_+} (b+r\,c)^\top (v(y)-\hat{u}(y))\,d\mathbb{P}(y).\end{aligned} \tag{27}$$

For the details on the numerical approximation of the solution $\hat{u}$ the interested reader can refer to the references in the introduction. Here, we only recall that the set $M_{\mathbb{P}}$ can be approximated by a sequence $\{M^n_{\mathbb{P}}\}$ of finite dimensional sets, and the functions $r$ and $s$ can be approximated by the sequences $\{\rho_n\}$ and $\{\sigma_n\}$ of step functions, with $\rho_n \to \rho$ in $L^p$ and $\sigma_n \to \sigma$ in $L^\infty$, respectively, where $\rho(r,s,t) = r$ and $\sigma(r,s,t) = s$. When the solution of (27) is unique, we can compute a sequence of step functions $\hat{u}_n$ which converges strongly to $\hat{u}$, under suitable hypotheses, by solving, for $n \in \mathbb{N}$, the following discretized variational inequality: find $\hat{u}_n := \hat{u}_n(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\begin{aligned}\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (\sigma_n(y) A[\hat{u}_n(y)] + B[\hat{u}_n(y)])^\top (v_n(y) - \hat{u}_n(y))\, d\mathbb{P}(y) \\ \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (b + \rho_n(y)\, c)^\top (v_n(y) - \hat{u}_n(y))\, d\mathbb{P}(y).\end{aligned} \tag{28}$$

In absence of strict monotonicity, the solution of (26) and (27) can be not unique and the previous approximation procedure must be coupled with a regularization scheme as follows. We choose a sequence $\{\varepsilon_n\}$ of regularization parameters and choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k) \to L^q(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. We assume that $\varepsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\varepsilon_n \downarrow 0$ as $n \to \infty$.

We can then consider the following regularized stochastic variational inequality: for $n \in \mathbb{N}$, find $w_n = w_n^{\varepsilon_n}(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\begin{aligned}\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} \big(\sigma_n(y) A[w_n(y)] + B[w_n(y)] + \varepsilon_n J(w_n(y))\big)^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y) \\ \geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^m_+} (b + \rho_n(y)\, c)^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y).\end{aligned} \tag{29}$$

As usual, the solution $w_n$ will be referred to as the regularized solution. Weak and strong convergence of $w_n$ to the minimal-norm solution of (27) can be proved under suitable hypotheses (see, e.g., [10]).

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