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The geometry of the Hessian locus: from a theorem of Gordan and Noether to cubic fourfolds, via Gorenstein rings

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Abstract

In this thesis, we will analyze the Hessian locus associated to a projective hypersurface. Interest in this topic goes back centuries. For example, in 1876 Gordan and Noether showed that a hypersurface X defined by a homogeneous polynomial f in a projective space of dimension at most 3 is a cone if and only if the hessian polynomial of f (i.e. the determinant of the Hessian matrix of f) is identically zero. In the first chapter of this thesis, we will give a new proof of this fundamental result by showing an equivalent algebraic statement regarding the validity of Lefschetz properties for specific standard Artinian Gorenstein algebras. The techniques used in this setting, for example the construction and description of a geometric framework arising from the assumption of the failure of a specific Lefschetz property, will then be improved and exploited in Chapter 2, where we will show these properties for specific Gorenstein algebras, such as the Jacobian rings of smooth cubic hypersurfaces in projective spaces of dimension 4 and 5 (i.e. cubic threefolds and cubic fourfolds). Finally, in Chapter 3, we will analyze the Hessian hypersurface $\mathcal{H}_f$ associated to a smooth cubic hypersurface $X = V(f)$, i.e. the zero locus of the hessian polynomial of f. By exploiting properties coming from some Gorenstein algebras and by using a natural identification between quadratic forms and points of the Hessian hypersurface, we will study the singular loci of $\mathcal{H}_f$ and a natural desingularization. We will finally study the Hessian $\mathcal{H}_f$ associated to a generic smooth cubic fourfold by describing geometric properties and birational invariants of the smooth surface over which $\mathcal{H}_f$ is singular. Moreover, for such a surface, we will construct a natural connected unramified double cover, by using tools coming from representation theory.

Introduction

Hypersurfaces in projective space play a central role in algebraic geometry. Many mathematicians have studied their geometric and algebraic properties, which are also reflected in the so called Hessian locus.

Consider $\mathbb{K}$ an algebraically closed field of characteristic 0 and $X = V(f) \subset \mathbb{P}^n$, a hypersurface defined by a homogeneous polynomial $f \in \mathbb{K}[x_0, \cdots, x_n]_d$ of degree d. The Hessian matrix of f is the matrix

$$\mathrm{Hess}(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{i,j=0,\cdots,n}$$

and the Hessian locus of X is the zero locus $\mathcal{H}_f := V(\mathrm{hess}(f))$, where $\mathrm{hess}(f)$ is the determinant of $\mathrm{Hess}(f)$, which is either a homogeneous polynomial of degree $(d-2)(n+1)$ or is identically zero. Note that $\mathrm{hess}(f)$ is a homogeneous polynomial of degree $(d-2)(n+1)$ if X is a smooth hypersurface. The problem of characterizing the hypersurfaces $X = V(f)$ for which the determinant $\mathrm{hess}(f)$ is identically zero has a long history. Since the middle of the 19^{th} century, several authors have worked on this problem: in the 1850's, in both [Hes51] and [Hes59], Hesse proposed a remarkable equivalence, by claiming that a hypersurface defined by a polynomial with vanishing hessian is a cone (the converse is clearly true). This happens to be false and in a fundamental paper of 1876 ([GN76]) Gordan and Noether proved the following:

Theorem A. ***(Gordan-Noether)***
Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface defined over a field of characteristic 0 and assume that $\mathrm{hess}(f) \equiv 0$. Then, if $n \leq 3$, X is a cone.

Gordan and Noether introduced the fundamental restriction on the admissible dimension of the projective space and provided counterexamples for $n \geq 4$. The so called *Perazzo cubic 3-fold* in $\mathbb{P}^4$ (introduced in [Per00]) is the simplest of such counterexamples, which will be analyzed in Section 1.5. This theorem by Gordan and Noether still inspires many researchers (see [dBW20, CO20, DS21, GR15, Los04, Rus16]), who also have revisited the original proof in recent decades, trying to simplify it or to interpret it in a more geometric way. Much work has also been done to improve our understanding of the counterexamples for $n \geq 4$ and to provide a classification in every dimension (let us mention [Rus16, Chapter 7.4] and also [GN76, Per00, Fra54, Per57, Per64, Los04, CRS08, GR15, dB18]).

The first result presented in this thesis is a direct proof of Theorem B, which is an algebraic version of the theorem of Gordan and Noether. Before introducing Theorem B, let us present some basic notions (see Section 1.1 for details). A standard Artinian Gorenstein $\mathbb{K}-$algebra (SAGA) is an

Artinian graded $\mathbb{K}-$algebra $R = \oplus_{i=0}^{N} R^i$, such that the vector spaces R^i are of finite dimension and R is generated in degree 1 and satisfies the Poincaré-Gorenstein duality (i.e. $R^N \simeq R^0 \simeq \mathbb{K}$ and the pairing given by the multiplication map $R^i \times R^{N-i} \to R^N$ is perfect).
Examples are given by the even cohomology ring of an oriented compact variety X of even dimension which is generated in degree 2: if $\dim(X) = n$ then $R = \oplus_{i=0}^{n} H^{2i}(X, \mathbb{C})$ is a SAGA. In this setting, the following result is well known (see for example [Voi07a, Theorem 6.25]):

Theorem (*Hard Lefschetz Theorem*)**.** *If X is a compact Kähler manifold of dimension n, then the cup product of the $r-$th power of a Kähler form induces an isomorphism between $H^{n-r}(X)$ and $H^{n+r}(X)$.*

A natural question is whether a general standard Artinian Gorenstein algebra satisfies an analogous property and in the 80's, inspired by the above theorem, the so-called Lefschetz properties for an Artinian algebra were defined. The property described in the Hard Lefschetz theorem is roughly the definition of the *strong Lefschetz property* for an Artinian algebra (for details see Definition 1.1.6). Similarly, we say that an algebra R satisfies the *weak Lefschetz property* if the multiplication map $x\cdot : R^k \to R^{k+1}$ is of maximal rank for all $k \geq 0$ and $x \in R^1$ general. One can refer to Section 1.1 for rigorous definitions. For a comprehensive treatment of Lefschetz properties the interested reader can refer to [HMM+13].

In the first part of this thesis, we will discuss the two seemingly unrelated subjects of Artinian algebras and Hessian loci, which turn out to be strongly connected. In particular, we will consider standard Artinian Gorenstein algebras $R = \oplus_{i=0}^{N} R^i$ for which the strong Lefschetz property in degree 1 fails, i.e. such that the multiplication $x^{N-2}\cdot : R^1 \to R^{N-1}$ has non trivial kernel for every $x \in R^1$. To such an algebra, as in [AR19], one can associate an incidence correspondence

$$\Gamma = \{([x], [y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \mid x^{N-2}y = 0\}.$$

By studying projective and differential properties of varieties arising in this framework, we will find some constraints on their dimensions and on the dimensions of some of the graded parts of R. In particular, we will prove the following:

Theorem B.
All standard Artinian Gorenstein $\mathbb{K}-$algebras R with $\dim(R^1) \leq 4$ satisfy the strong Lefschetz property in degree 1, i.e. there exists an element $x \in R^1$ such that the multiplication map $x^{N-2}\cdot : R^1 \to R^{N-1}$ is an isomorphism.

Despite the algebraic nature of the statement, our approach to Theorem B is geometric and gives as a byproduct a new proof of Theorem A.

The interesting and, in some sense, surprising equivalence between Theorem A and Theorem B (see Section 1.4 or [HMM+13, Rus16] for example) is realized by a connection between these different settings based on Macaulay's theory of inverse systems ([HMM+13, Theorem 2.71] or the original [Mac94]), which allows to construct any standard Artinian Gorenstein algebra, from a homogeneous form in a finite number of variables.

We would also like to highlight the analogue of the famous Gordan-Noether identity (see (1.2)) in the world of Gorenstein Artinian algebras. This identity can be considered as the heart of the classical treatment of the Gordan-Noether Theorem and its proof involves some delicate manipulations. From our geometric (differential) approach, we derive what we will call the *Gorenstein-Gordan-Noether Identity* (see Corollary 1.2.4), which has a very elementary treatment and, as the original Gordan-Noether identity, is a key relation for proving Theorem B.

A natural question arising from our work is whether the methods used to prove Theorem B have more applications and, in particular, if they could be applied to study problems related to other strong or weak Lefschetz properties for Gorenstein rings. Let us stress that both the weak and the strong Lefschetz properties are known for only a few Artinian algebras, as observed in Subsection 1.1.1. With this in mind, in Chapter 2, we will treat some open cases, focusing on Jacobian rings of smooth hypersurfaces. Given $X = V(f) \subset \mathbb{P}^n$, a smooth hypersurface of degree d, one can consider the Jacobian ideal of f

$$J_f := \left(\frac{\partial f}{\partial x_0}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

and the Jacobian ring $R = S/J_f$. This is a particular example of standard Artinian Gorenstein $\mathbb{K}-$algebras. The importance of the Jacobian ring of a smooth hypersurface X lies in its geometric relation to X itself. For example, building on works of Grothendieck, in the seminal works [CG80, CGGH83, GH83, Gri83], Carlson, Griffiths, Green and Harris proved that a portion of the primitive part of the Dolbeault cohomology of X is codified in R and that R plays a crucial role in the infinitesimal variation of Hodge structure of X.
We will focus on the case of smooth cubic hypersurfaces, which has captured the interest of many mathematicians in relation to many different problems. In particular, we will deal with Jacobian rings of smooth cubic threefolds in $\mathbb{P}^4$ and smooth cubic fourfolds in $\mathbb{P}^5$ and we will prove the following:

Theorem C.
The Jacobian ring R of a smooth cubic threefold satisfies the strong Lefschetz property, i.e. if $x \in R^1$ is general the multiplication maps $x^3\cdot : R^1 \to R^4$ and $x\cdot : R^2 \to R^3$ are isomorphisms.
The Jacobian ring R of a smooth cubic fourfold satisfies the strong Lefschetz property in degree 1, i.e. given $x \in R^1$ general the multiplication map $x^4\cdot : R^1 \to R^5$ is an isomorphism.

Theorem C will follow from a more general statement for complete intersection Gorenstein algebras presented by quadrics, i.e. quotients of $\mathbb{K}[x_0, \cdots, x_n]$ by ideals generated by a regular sequence of homogeneous polynomials of degree 2 (see Definition 1.1.3). These results provide evidence for a well known conjecture which states that complete intersection Gorenstein algebras in characteristic 0 should satisfy the Lefschetz properties (see for example [HMM+13, Conjecture 3.46]).

In Section 2.4, we will extend our proof of some of the strong Lefschetz properties to complete intersection Gorenstein algebras presented by quadrics, when the dimension of R^1 is larger. In particular, we will prove the following:

Theorem D.
Let R be a complete intersection standard Artinian Gorenstein $\mathbb{K}-$algebra presented by quadrics with $\dim(R^1) = n + 1$. Given $k \in \{2, 3, 4\}$, if $n \geq k + 1$, then for x general in R^1 the multiplication map $x^k\cdot : R^1 \to R^{1+k}$ is of maximal rank.

Since the Lefschetz properties hold for a *general* complete intersection Gorenstein algebras, it is interesting to analyze special algebras exhibiting uncommon behaviors. In particular, for complete intersection Gorenstein algebras presented by quadrics (such as Jacobian rings of smooth cubic hypersurfaces), we will study what we call the *nihilpotent loci*

$$\mathcal{N}_i := \{[x] \in \mathbb{P}(R^1) \mid x^i = 0\}$$

and the *non-Lefschetz loci* (see Definition 2.5.1), the subschemes parametrizing linear forms for which the injectivity of the multiplication map fails (these were studied for example in [AR19] and [BMMRN18]). While for general Gorenstein algebras these loci are empty (at least in low degree), we will study their geometric and algebraic behaviour when this is not the case. For example, we will obtain a characterization of the cubic Fermat hypersurface, by studying the cardinality of the nihilpotent locus $\mathcal{N}_2$ in its corresponding Jacobian ring (see Corollary 2.6.2). Moreover, by assuming the non-emptiness of a suitable non-Lefschetz locus, in Section 2.5 we will derive a lifting criterion for the weak Lefschetz property. This gives a sort of converse for results which prove that Lefschetz properties are inherited by suitable quotients (see for example [HMM+13, Proposition 3.11] for the strong Lefschetz property or [Gue19] for the weak one).

In Chapter 3, we continue the study of cubic hypersurfaces from the perspective of their Hessian loci. As observed above, given a smooth cubic hypersurface $X = V(f) \subset \mathbb{P}^n$, we can consider the associated Hessian locus $\mathcal{H}_f$, which is a hypersurface of degree $n+1$.

The geometry of cubic hypersurfaces in $\mathbb{P}^n$ and their Hessians has been studied by many authors (see for example [CO20, GR15, Huy]). In particular, for $n = 3$, [DvG07] studies the classical case of the general cubic surface and of the associated Hessian quartic surface, which is singular in exactly 10 isolated points. Moreover, [AR96, Appendix IV] studies the case of cubic threefold in $\mathbb{P}^4$. In particular, the author considers the Hessian quintic threefold $\mathcal{H}$ associated to a general cubic threefold and constructs a correspondence variety over $\mathcal{H}$, which is a desingularization of the Hessian hypersurface. Adler shows that in the general case this Hessian hypersurface is singular along a curve and he also studies the geometric properties of such a curve, such as smoothness and irreducibility, and computes its degree and its genus.

With the aim of studying these Hessian loci and their singularities in higher dimension, we analyze and generalize some constructions described in [AR96, Appendix IV]. In particular, given a cubic hypersurface $V(f)$, the fact that the Hessian matrix $\mathrm{Hess}(f)$ has linear forms as coefficients allows us, via the evaluation map, to identify $\mathrm{Hess}(f)|_v$, for $[v] \in \mathbb{P}^n$ with the quadric in J_f (the Jacobian ideal of f), defined as the partial derivative $\partial_v f$. We will consider the loci

$$\mathcal{D}_k(f) = \{[x] \in \mathbb{P}^n \mid \mathrm{Rank}(\mathrm{Hess}(f)|_x) \leq k\},$$

which will be identified with the intersections $\mathcal{Q}_k \cap \mathbb{P}(J_f^2)$, where $\mathcal{Q}_k$ is the locus of quadrics in $\mathbb{P}^n$ whose rank is at most k. By using the results described in [Har95], we can then get the expected dimension and the degree (if they are non-empty) of the loci $\mathcal{D}_k$. This can be seen as a first step in the analysis of the Hessian hypersurface $\mathcal{H}$, since the loci $\mathcal{D}_k$ are strongly related with the singularities of $\mathcal{H}$. Indeed, by generalizing one of the results presented in [AR96, Appendix IV], we will prove the following:

Theorem E.
Given a smooth cubic hypersurfaces $V(f) \subset \mathbb{P}^n$, the varieties $\mathcal{D}_{n-1}(f)$ and $\mathrm{Sing}(\mathcal{H})$ coincide.

To prove the above result, we will consider a correspondence variety over $\mathcal{H}$ which can be described as

$$\Gamma_f = \{([v],[w]) \in \mathbb{P}^n \times \mathbb{P}^n \mid \partial_v \partial_w(f) = 0\}$$

which is equipped with a naturally defined involution. This variety Γ_f also has an important intrinsic geometric meaning:

Theorem F.
Given a general smooth cubic hypersurface $V(f)$, Γ_f is smooth and the natural projection $\pi_1 : \Gamma_f \to \mathcal{H}$ is a desingularization of the Hessian hypersurface.

Let us stress here that by studying a particular kind of standard Artinian Gorenstein algebra A, defined as the quotient of the ring of differential operators by the annihilator of a general cubic form f, one can observe two interesting facts. Indeed, the correspondence Γ_f just defined coincides with the incidence correspondence Γ used to study the Lefschetz properties for specific Artinian algebras. Moreover, the Hessian hypersurface $\mathcal{H}_f$ associated to a general cubic form f is exactly the non-Lefschetz locus of A.

Returning to the analysis of the loci $\mathcal{D}_k(f)$, we will also show (see Theorem 3.4.1 and Corollary 3.4.2) that for a general smooth cubic hypersurface $X = V(f)$ we have that $\mathrm{Sing}(\mathcal{D}_k(f)) = \mathcal{D}_{k-1}(f)$.

This means that for f general, the locus $\mathcal{D}_k(f)$ is smooth outside the points where the Hessian matrix has rank strictly smaller than k. Since in [RV17] the authors show that for $V(f)$ smooth and general cubic fourfold in $\mathbb{P}^5$, the locus $\mathcal{D}_3(f)$ is empty, we will get that in the general case the Hessian hypersurface associated to a smooth cubic fourfold is singular along a smooth surface. On the other hand, looking at the expected dimensions of these loci, one has that for bigger dimensions (namely for Hessians associated to cubic hypersurfaces in $\mathbb{P}^n$, with $n \geq 6$) the singular locus of the Hessian hypersurface is itself singular. It is then natural to approach the study in the case of $\mathbb{P}^5$, the first open and the last one with a smooth singular locus, for f general.

In [AR96, Appendix IV], Adler has developed the study of the curve along which the Hessian hypersurface associated to a general cubic threefold is singular, by focusing on a specific case, namely the Klein cubic threefold defined by the polynomial $f = x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_0$. He obtains a complete description of this curve, by exploiting the properties of this polynomial and of the associated hypersurface, such as the invariance under symmetric transformations. In the case of the Klein cubic in $\mathbb{P}^5$, unfortunately, the singular locus of the associated Hessian hypersurface is itself singular, with singularities arising also from the locus $\mathcal{D}_3(f)$. Thus, we have not based our analysis in the study of a specific cubic fourfold. Instead we exploit the nature and the properties of the loci $\mathcal{D}_k$. In this last part, we will fix the field $\mathbb{K} = \mathbb{C}$, since we use singular cohomology, though some of the results still hold in a more general setting. In general, given a vector bundle E and a line bundle

L over a projective variety X, one can consider a symmetric vector bundle morphism $\varphi : E \to E^* \otimes L$ and define the loci

$$\mathcal{D}'_k(\varphi) = \{x \in X \mid \operatorname{Rank}(\varphi_x) \leq k\},$$

known as degeneraci loci. In the last decades this locus has been studied in several works (for example in [FL81], [FL83], [HT84a], [HT84b], [HT90], [Laz04], [Tu86], [Tu90], [Tu89]).

By considering the symmetric vector bundle map $\varphi = \operatorname{Hess}(f)\cdot : \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{O}_{\mathbb{P}^n}^{n+1}(1)$, given by the multiplication by the Hessian matrix of a cubic hypersurface $V(f)$, the loci $\mathcal{D}_k(f)$ considered above coincide with the degeneraci loci $\mathcal{D}'_k(\operatorname{Hess}(f)\cdot)$. Analyzing these loci from this perspective and using results of [FL83, HT90, Tu89], one can show the non-emptiness and the connectedness of suitable loci $\mathcal{D}_k$ and, by using the approach presented in [HT84a], one can also calculate some Chern classes. In the case of the Hessian locus $\mathcal{H}_f$ associated to a general smooth cubic fourfold $X = V(f) \subset \mathbb{P}^5$, we will compute that the canonical divisor of the surface $Z = \mathcal{D}_4(f)$ is $K_Z = 3H_{|Z} + \eta$, where H is the hyperplane class in $\mathbb{P}^5$ and η a $2-$torsion element of $\operatorname{Pic}^0(Z)$. To compute invariants of the surface Z and to better understand the nature of η, we have constructed (see Subsection 3.5.1) an unramified double cover of Z. We have done this by seeing the elements of such a surface as rank 4 quadrics and by exploiting the existence of families of isotropic subspaces of these quadrics. Moreover, by using tools coming from representation theory (see Appendix A), we will prove that this unramified double cover is connected and that Z is a regular surface. Finally, by using some formulas of [Pra88] and the software *Magma*, we will get that the $2-$torsion element appearing in the canonical divisor is not trivial and we will obtain the description of the locus $\operatorname{Sing}(\mathcal{H}_f)$ in the case of a general smooth cubic fourfold $V(f) \subset \mathbb{P}^5$:

Theorem G.
Let $V(f)$ be a general smooth cubic fourfold. Then the singular locus of the associated Hessian hypersurface $Z := \operatorname{Sing}(\mathcal{H}_f)$ is a smooth, irreducible, and minimal surface of general type with degree 35 *and numerical invariants*

- $K_Z^2 = 315$
- *geometric genus* $p_g(Z) = 55$
- *irregularity* $q(Z) = 0$
- *(topological) Euler characteristic* $e(Z) = 357$

Moreover, its canonical divisor is $K_Z = 3H|_Z + \eta$ (where H is the hyperplane class in $\mathbb{P}^5$ and η is a non-trivial 2*-torsion element in* $\operatorname{Pic}^0(Z)$*) and Z is projectively normal.*

Outline of the thesis.
In Chapter 1, we will set the basic definitions, we will prove Theorem B and recall the relation with the Gordan-Noether Theorem. In Chapter 2, we will exploit the techniques introduced in the previous chapter and we will prove Theorem C and Theorem D. Moreover, we will study the nihilpotent and

non-Lefschetz loci arising from specific Artinian algebras. In Chapter 3, we will study in detail the Hessian locus associated to a general cubic hypersurface and prove Theorems E and F. Finally, we will focus on the case of a general smooth cubic fourfold in $\mathbb{P}^5$ to prove Theorem G.

In appendices A and B, we will present technical proofs of two results used in Chapter 3.

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Contents

Chapter 1

Gordan and Noether Theorem and Lefschetz properties

In this first chapter, we will introduce the first definitions concerning the theory around the theorem of Gordan and Noether (Theorem A) and the theory of Gorenstein algebras and their Lefschetz properties. We will also present (in Section 1.1) the most expressive examples and the first preliminary results; in Subsection 1.1.1 we will point out some known results concerning the validity of the Lefschetz properties, while in Subsection 1.1.2 we will set the preliminaries for Gordan-Noether theorem, by also giving an idea of its classical proof. In Section 1.2 we will set our framework and our construction, that will be studied and exploited to obtain the main results of this chapter and of the following one. We will prove the first bounds on the dimensions of the varieties that will be involved and we will present our strategy. In Section 1.3, we will prove Theorem B, also obtaining a new proof of Gordan and Noether theorem. In Section 1.4, we will show the equivalence between Gordan-Noether theorem and Theorem B and we will analyze the Gordan-Noether Identity with our language (in Subsection 1.4.1). Finally, in Section 1.5, we will use our techniques to study the *Perazzo cubic* in $\mathbb{P}^4$, one of the first counterexamples to Hesse's statement.

The results of this chapter appear in [BFP22] and [BF22].
In this chapter we will work on a field $\mathbb{K}$, which will be an algebraically closed field of characteristic 0.

1.1 Preliminaries and definitions

Good references for the content of this introductory section are [HMM$^+$13, Rus16, Voi07b].
First of all, let us define a *standard Artinian Gorenstein algebra*, (SAGA in short):

Definition 1.1.1. *An Artinian graded* $\mathbb{K}$*-algebra* $R = \bigoplus_{i=0}^{N} R^i$ *is a* standard Artinian Gorenstein algebra (SAGA) *if:*

- *it is* **standard**, *i.e. if it is generated, as* $\mathbb{K}$*-algebra, by* R^1*;*
- *it satisfies the* **Poincaré duality** *if* $R^N \simeq \mathbb{K}$ *and the multiplication map* $R^s \times R^{N-s} \to R^N$ *is a perfect pairing whenever* $0 \leq s \leq N$*.*

If R is a graded Artinian algebra, having the Poincaré duality is equivalent to ask that R is Gorenstein so the above duality is also called **Gorenstein duality***.*
Moreover, the **codimension** *of R is the dimension of R^1 as $\mathbb{K}$-vector space and R^N is said to be the* **socle** *of the SAGA R.*

We basically recall that a ring is Artinian if it satisfies the descending chain condition and the pairing above being perfect means that it induces an isomorphism (of $\mathbb{K}-$vector spaces) $R^s \simeq \operatorname{Hom}(R^{N-s}, R^N)$.

Let us now present two ways to construct standard Artinian Gorenstein algebras which are relevant for this work. Throughout this thesis, we will denote by $S = \mathbb{K}[x_0, \ldots, x_n] = \bigoplus_{k\geq 0} S^k$, where $S^k = H^0(\mathcal{O}_{\mathbb{P}^n}(k))$, the polynomial ring in $n+1 \geq 2$ variables with coefficient in the field $\mathbb{K}$ and by D the ring of differential operators in the variables $x_0, \ldots, x_n$, i.e. $D = \mathbb{K}[y_0, \ldots, y_n]$ where we denote for brevity $y_i = \frac{\partial}{\partial x_i}$.

Example 1.1.2. *If $e_0, \ldots, e_n \geq 1$, let us consider a regular sequence $\{g_0, \ldots, g_n\}$ in S with $g_i \in S^{e_i}$ (we can think of a regular sequence as a set of homogeneous polynomials for which the common zero locus is trivial). If we set $I = (g_0, \ldots, g_n)$, then $R = S/I$ is a standard Artinian Gorenstein algebra with socle in degree $\sum_{i=0}^n (e_i - 1)$. Particular algebras obtained via this construction are Jacobian rings associated to smooth hypersurfaces of degree $d \geq 2$ in $\mathbb{P}^n$. In this case, if $X = V(f)$, with $f \in S^d$, one takes $g_i = \partial f/\partial x_i \in S^{d-1}$: the ideal $J_f = (g_i)_{i=0,\cdots,n}$ and the quotient S/J_f are respectively the Jacobian ideal and the Jacobian ring of f, with socle in degree $N = (d-2)(n+1)$.*

Definition 1.1.3. *According to example 1.1.2, if all the degrees of the homogenous polynomials of the regular sequence we are considering are equal to an integer e, we say that the corresponding SAGA is a* **complete intersection SAGA presented by forms of degree e***.*

In particular, we have that the Jacobian ring associated to a smooth hypersurface $V(f) \subset \mathbb{P}^n$ is a complete intersection SAGA presented by forms of degree $\deg(f) - 1$.
As second example, let us consider the following:

Example 1.1.4. *If $g \in S = \mathbb{K}[x_0, \ldots, x_n]$ is any fixed homogeneous polynomial of degree $d \geq 1$, one can define the annihilator of g in $D = \mathbb{K}[y_0, \cdots, y_n]$ as the ideal*

$$\operatorname{Ann}_D(g) = \{\delta \in D \mid \delta(g) = 0\}.$$

One can see that the quotient $A = D/\operatorname{Ann}_D(g)$ is a standard Artinian Gorenstein algebra with socle in degree d.

A very important and interesting fact is that every standard Artinian Gorenstein algebra has a description as in Example 1.1.4 by an important result of Macaulay and its theory of inverse systems (see [Mac94] for a revisited reprint of original work by Macaulay of 1916). In particular, we have the following (see for example [MW09, Theorem 2.1] for a statement proposed with a modern language):

Theorem 1.1.5. *If $R = \oplus_{i=0}^d R^i \simeq \mathbb{K}[x_0, \ldots, x_n]/I$ is an Artinian standard graded $\mathbb{K}-$algebra with socle in degree d, then it is Gorenstein if and only if there exists a homogeneous polynomial $g \in \mathbb{K}[x_0, \ldots, x_n]$ of degree d such that $R \simeq D/\operatorname{Ann}_D(g)$.*

Let us now define the Lefschetz properties for a SAGA R.

Definition 1.1.6. *A SAGA $R = \bigoplus_{i=0}^{N} R^i$ is said to satisfy*

- *the weak Lefschetz property in degree k (WLP_k in short), if there exists $x \in R^1$ such that the multiplication map $x\cdot : R^k \to R^{k+1}$ has maximal rank;*
- *the strong Lefschetz property in degree k at range s ($SLP_k(s)$ in short), if there exists $x \in R^1$ such that the multiplication map $x^s\cdot : R^k \to R^{k+s}$ has maximal rank and the strong Lefschetz property in degree k (SLP_k) if $SLP_k(s)$ holds for all s.*

Then we say that R satisfies the **weak (strong) Lefschetz property** *- WLP (respectively SLP) - if it satisfies WLP_k (respectively SLP_k) for all k.*
R is also said to satisfy the **strong Lefschetz property in narrow sense**, *if $SLP_k(N-2k)$ holds for all $k \leq N/2$.*

Remark 1.1.7. *Let us stress that for SAGAs, the above two definitions of SLP and SLP in narrow sense (when satisfied for every suitable k) are equivalent (see Definition* 3.18 *and subsequent discussion in* [HMM+13]*). We will actually prove most of our results, by considering the definition in narrow sense. Moreover, let us observe that for the strong Lefschetz property in narrow sense $SLP_k(N-2k)$, for an integer $k \leq N/2$, we are looking at the multiplication map $x^{N-2k}\cdot : R^k \to R^{N-k}$, where the $\mathbb{K}-$vector spaces R^k and R^{N-k} have the same dimension, by definition of SAGA and of Gorenstein duality (see Definition 1.1.1). Hence, the property $SLP_k(N-2k)$ is satisfied if the above map $x^{N-2k}\cdot$ is an isomorphism (for x general).*

In the following, we will often deal with kernels of the multiplication maps involved in the above definitions, so it is convenient to set

$$K_\eta^i := \ker\left(\eta\cdot : R^i \to R^{i+h}\right) \qquad \text{for } \eta \in R^h.$$

(One can also write $K_\eta^i = \mathrm{Ann}_{R^i}(\eta)$.)

Finally, let us introduce the following subsets of the graded parts of a SAGA R:

$$\mathcal{N}_k^{(a)} = \{[x] \in \mathbb{P}(R^a) \mid x^k = 0\}.$$

We will refer to $\mathcal{N}_k^{(a)}$ as **nihilpotent loci of order** k of $\mathbb{P}(R^a)$. For brevity, we will set $\mathcal{N}_k^{(1)} = \mathcal{N}_k$. Let us observe that, by construction, for suitable a and k we have $\mathcal{N}_k^{(a)} \subseteq \mathcal{N}_{k+1}^{(a)}$.

Remark 1.1.8. *If R is a standard $\mathbb{K}$-algebra with socle in degree N, then $\mathcal{N}_k \subsetneq \mathbb{P}(R^1)$ for all $k \leq N$. Indeed, if $\mathcal{N}_k = \mathbb{P}(R^1)$, we have that all k-th powers of elements of R^1 are equal to 0. Since R is standard, these k-th powers of R^1 generate R^k as vector space, so $R^k = 0$. Then $k > N$.*

Let us now recall a standard result concerning particular quotient Gorenstein rings that will be used in the following of this thesis. For a simple proof one can refer to [FP21, Lemma 2.3].

Lemma 1.1.9. *Let $R = \oplus_{i=0}^{N} R^i$ be a Gorenstein ring with socle in degree N. Fix $\alpha \in R^e \setminus \{0\}$ and consider the ideal*

$$(0:\alpha) = \bigoplus_{i=0}^{N} \ker(\alpha\cdot : R^i \to R^{i+e}),$$

which is called **conductor ideal of** α.
We have that $\tilde{R} := R/(0:\alpha)$ is a Gorenstein ring with socle in degree $\tilde{N} = N - e$.

1.1.1 Lefschetz properties and state of the art

We would like to stress that the Lefschetz properties, both the strong and the weak one, are known to hold only for very few examples of SAGAs. In this subsection, we would like to partially present and summarize some known results and conjectures about such properties, which are relevant with respect to the topics of this thesis. Since this subsection can't be exhaustive, the interested reader can refer to [HMM+13] and [MN13a] for a deeper and more rigorous treatment.

One of the first results in this direction is the following theorem (see [MN13a, Theorem 1.1]), proved by different mathematicians, with different techniques:

Theorem 1.1.10. *Let $S = \mathbb{K}[x_0, \dots, x_n]$, with $\mathbb{K}$ a field of characteristic 0 and let I be an Artinian monomial complete intersection ideal, i.e. $I = (x_0^{a_0}, \dots, x_n^{a_n})$. Then the complete intersection SAGA $R = S/I$ satisfies the strong Lefschetz property.*

Let us now consider SAGAs R with low codimension, in particular such that $\dim(R^1) = 2$ or 3: in this case we have

Theorem 1.1.11. [HMNW03, Thm. 2.3, Prop. 4.4]
Let $\mathbb{K}$ be a field of characteristic 0. Any Artinian standard algebra of codimension 2 over $\mathbb{K}$ satisfies the strong Lefschetz property.
Moreover, a complete intersection SAGA of codimension 3 over $\mathbb{K}$ satisfies the weak Lefschetz property.

If we consider Jacobian rings of smooth curves of degree d in $\mathbb{P}^2$, which are particular complete intersection SAGAs of codimension 3 (see Example 1.1.2), the validity of the strong Lefschetz property is known only up to $d = 4$. Let us observe here that the strong Lefschetz property for jacobian rings of smooth cubic curves in $\mathbb{P}^2$ coincides simply with the strong Lefschetz property in degree 1. Moreover, in this case the validity of the SLP_1 follows directly from a famous theorem due to Gordan and Noether (see Theorem A and B and Section 1.4). For jacobian rings of smooth quartic curves, we have:

Theorem 1.1.12. [DGI20, Prop.2.23]
Let $\{f = 0\}$ be a smooth curve in $\mathbb{P}^2$ of even degree $d = 2d'$ and let $R = S/J_f$ be the associated jacobian ring. If $L \in R^1$ is a general element of degree 1, then the multiplication map $L^2\cdot : R^{3d'-4} \to R^{3d'-2}$ is an isomorphism. In particular, if $d = 4$, then R satisfies the strong Lefschetz property.

In particular, nothing is known up to now for the validity of the strong Lefschetz property for Jacobian rings of smooth curves with higher degree: quintic curves in $\mathbb{P}^2$ are the first open case.

If we focus on the case of codimension 4, the validity of the strong Lefschetz property for the Jacobian ring of a smooth cubic surface is again a direct consequence of Theorem A of Gordan and Noether. Moreover, regarding the validity of the weak Lefschetz property, we have the following very recent result:

Theorem 1.1.13. [BMMRN22, Prop. 5.2, Thm. 5.3, Coroll. 5.4]
Let $R = S/(g_0, g_1, g_2, g_3)$ be a complete intersection SAGA of codimension 4. If $\deg(g_i) \leq 5$ for all i, then R satisfies the weak Lefschetz property. In particular, this holds for Jacobian rings of smooth surfaces in $\mathbb{P}^3$ of degree 4, 5, and 6.

Let us now consider a complete intersection SAGA R presented by quadrics, i.e. $R = S/(g_0, \dots, g_n)$, where $\deg(g_i) = 2$ for every $i = 0, \dots, n$ (e.g. R is the jacobian ring of a smooth cubic hypersurface). In this case, the validity of the weak Lefschetz property in degree 1 is known, indeed we have:

Theorem 1.1.14. [MN13b, Prop. 4.3]
Let R be a complete intersection SAGA presented by quadrics with socle in degree $N \geq 3$ and defined over a field of characteristic 0. If L is a general element in R^1, then the multiplication map $L\cdot : R^1 \to R^2$ is injective.

Let us stress that this result holds in any codimension and that we will give a new proof of this fact in Corollary 2.1.4. In [MN13b] the authors conjectured that an Artinian Gorenstein algebra presented by quadrics satisfies the weak Lefschetz property, but in [GZ18] families (not complete intersection algebras) of counterexamples to this conjecture have been given. Anyway, let us stress that such a conjecture is still valid for complete intersection Aritinian Gorenstein algebras presented by quadrics (see for example [HMM$^+$13, Conjecture 3.46]).
If we now focus on the case of codimension 5, we have:

Theorem 1.1.15. [AR19, Theorem 1]
For a complete intersection SAGA (over the field of complex numbers $\mathbb{C}$) presented by quadrics and with codimension 5, the weak Lefschetz property holds.

In particular, we have that the jacobian ring of smooth cubic threefolds in $\mathbb{P}^4$ satisfies the WLP. Let us stress that in [AR19], the authors propose a strategy that we will present and exploit in the following sections.

Let us observe that the cases of higher codimension or of jacobian rings of hypersurfaces with higher degree are completely open up to now: in Chapter 2, we will consider complete intersection SAGAs presented by quadrics and we will prove the SLP in the case where the codimension is 5 and the SLP_1 for codimension 6.

To conclude this subsection, let us just mention some other papers, which treat problems related to Lefschetz properties for Artinian algebras, as for example [BK07, MMRN11, MMRO13, Gon17, Ila18, AAI$^+$22, DI22].

1.1.2 Hessians, cones and Gordan-Noether theorem

In this subsection, we will focus on the result proved by Gordan and Noether in 1876 (Theorem A): after a preliminary part, for completeness, we will give, very briefly, an idea of the original proof.
The principal reference for this section is the book [Rus16], by Francesco Russo. An even more geometric proof of Gordan-Noether theorem has been presented in [GR09].

Let us start with some preliminaries and, first of all, let us take a projective hypersurface $X = V(f)$, defined by a homogeneous polynomial $f \in S^d$ of degree $d \geq 1$ (without multiple factors).

Definition 1.1.16. *The* **Hessian matrix of** f*, or of* X*, is the square symmetric matrix whose entries are the second partial derivatives of* f*:*

$$\mathrm{Hess}(f) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_{0 \leq i,j \leq n}.$$

The **hessian (determinant) of** f*, or of* X*, is the determinant of the Hessian matrix of* X*:*

$$\mathrm{hess}(f) = \det(\mathrm{Hess}(f)).$$

Observe that we can suppose that the degree d of f is at least 2, since in the case where $d = 1$, and only in this case, the Hessian matrix is the zero matrix.
Let us now recall some basic facts concerning the hessian (determinant) of a homogeneous polynomial f of degree $d \geq 2$:

Remark 1.1.17.

- *Either the hessian of* f *is a homogeneous polynomial of degree* $(n+1)(d-2)$ *or it is identically zero.*
- *If for some* $i \in \{0, \cdots, n\}$ *the first partial derivative* $\frac{\partial f}{\partial x_i}$ *is zero, then* $\mathrm{hess}(f)$ *is identically zero.*
- *If the partial derivatives* $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ *are linearly dependent, then* $\mathrm{hess}(f)$ *is identically zero.*

Definition 1.1.18. *Let* $X \subset \mathbb{P}^n$ *be a closed (irreducible) subvariety.* X *is a cone if there exists a point* $p \in X$ *such that for every other point* $x \in X$ *different from* p*, the line* $\langle p, x\rangle$ *is contained in* X*.*

Let us now state a well-known characterization for a variety for being a cone:

Proposition 1.1.19. *For a variety* $X = V(f) \subset \mathbb{P}^n$*, with* $d = \deg(f) \geq 2$*, being a cone is equivalent to one of the following:*

(a) the partial derivatives of f $\frac{\partial f}{\partial x_0}, \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}$ *are linearly dependent*

(b) there exists a point p *of multiplicity* d

(c) up to a projective transformation, f *depends on at most* n *variables*

(d) X^**, the dual variety of* X*, is degenerate, i.e.* X^* *(which can be defined as the closure of the image of the Gaussian map of* X*) is contained in a proper projective linear subspace of* $(\mathbb{P}^n)^*$*.*

By the above Proposition 1.1.19, it is clear that if a variety $X = V(f)$ is a cone, then $\mathrm{hess}(f)$ is identically zero, since the partial derivatives of f are linearly dependent. Twice, both in 1851 and in 1859 ([Hes51] and [Hes59]), Hesse stated that also the converse holds. In particular, he claimed that:

Claim 1.1.20 (Hesse)**.** *If a variety* $X = V(f) \subset \mathbb{P}^n$ *is such that* $h(f) \equiv 0$*, then it is a cone.*

In other words, he stated that for the partial derivatives of a homogeneous polynomial f of degree $d \geq 2$ being algebraically dependent is equivalent to be linearly dependent.

Remark 1.1.21. *Observe that for the cases* $n = 1$ *and* $d = 2, n \geq 2$ *the claim is trivially true.*

Thus, we can suppose from now on that $n \geq 2$ and $d \geq 3$.
In 1876, Gordan and Noether fixed Hesse's statement and proved its validity for $n \leq 3$. They also showed its failure as soon as $n \geq 4$: indeed they proposed some counterexamples, namely some hypersurfaces, which are not cones, with determinant of the Hessian matrix identically zero.

Theorem 1.1.22 (Gordan and Noether, Theorem A)**.** *Let $X = V(f) \subset \mathbb{P}^n$ be a hypersurface defined over a field $\mathbb{K}$ of characteristic 0 and assume that X has vanishing hessian, i.e.* $\mathrm{hess}(f) \equiv 0$*. Then, if $n \leq 3$, X is a cone.*
Moreover, for every $n \geq 4$ and for every $d \geq 3$ there exist counterexamples to Hesse's claim.

For completeness, let us now give an idea of the proof of the first statement (we refer to the proof presented in [Rus16]).

Sketch of the proof: Let us start by considering a reduced polynomial $f \in \mathbb{K}[x_0, \cdots, x_n]_d$ of degree d. Suppose that f has vanishing hessian (i.e. $h(f) \equiv 0$) and let $X = V(f) \subset \mathbb{P}^n$ be the degree d hypersurface associated to f. Let us introduce the *polar map*

$$\nabla_f = \nabla_X : \mathbb{P}^n \dashrightarrow (\mathbb{P}^n)^* \qquad p \mapsto \nabla_f(p) = \left(\frac{\partial f}{\partial x_0}(p) : \cdots : \frac{\partial f}{\partial x_n}(p) \right).$$

Now, let $Z' = \overline{\nabla_f(\mathbb{P}^n)} \subseteq (\mathbb{P}^n)^*$ be the *polar image* of $\mathbb{P}^n$ with respect to f. By considering the restriction of the polar map to X, we get the *Gauss map* of X:

$$\mathcal{G}_X := \nabla_{f|X} : X \dashrightarrow (\mathbb{P}^n)^* \qquad X_{\mathrm{reg}} \ni p \mapsto \mathcal{G}_X(p) = [T_p(X)].$$

Hence, we have that $X^* := \overline{\mathcal{G}_X(X)} \subseteq Z' \subseteq (\mathbb{P}^n)^*$, where X^* is the dual variety of X.
One can show that $\dim(Z') = \mathrm{rk}(H(f)) - 1$ and, since $\mathrm{rk}(H(f)) = n+1$ if and only if the determinant of $H(f)$ is not the zero polynomial, we get

$$h(f) \equiv 0 \iff Z' \subsetneq (\mathbb{P}^n)^*. \tag{1.1}$$

One can also show (see [Rus16][Lemma 7.2.7]) that, in general, we have

$$\dim(X^*) \leq \dim(Z') - 1$$

(this is actually true for every irreducible component of X). Hence, with our assumption (since f has vanishing hessian) we have

$$X^* \subsetneq Z' \subsetneq (\mathbb{P}^n)^*.$$

From this we get that there exists an irreducible non-zero polynomial $g \in \mathbb{K}[y_0, \cdots, y_n]_e$ (where $y_i = \frac{\partial f}{\partial x_i}$) such that $g(\nabla_f(\mathbf{x})) = \mathbf{0}$ and, in particular, $Z' \subseteq W = V(g) \subset \mathbb{P}^n$ (with $Z' = W$ if $\mathrm{codim}(Z') = 1$). For g we can equivalently assume that either there exists an index i such that $\frac{\partial g}{\partial y_i}(\nabla_f(\mathbf{x})) \neq \mathbf{0}$ or $Z' \not\subseteq \mathrm{Sing}(W)$.

Remark 1.1.23. *One can observe that by taking for example as g a generator of minimal degree in the homogeneous ideal $I(Z')$, these assumptions are satisfied.*

Under these hypotheses the map

$$\psi_g = \nabla_g \circ \nabla_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n,$$

called the *Gordan – Noether map associated to* g, is well defined and one can also show that such a map can be written as

$$\psi_g = (h_0 : \cdots : h_n) : \mathbb{P}^n \dashrightarrow \mathbb{P}^n,$$

where the h_i's are suitable rational functions such that g.c.d.$(h_0, \dots, h_n) = 1$.
A very important step at this point of the proof is the following: by letting $F \in \mathbb{K}[x_0, \dots, x_n]_m$, $\mathbb{K}' \supseteq \mathbb{K}$ be a field extension and ψ_g be the Gordan-Noether map defined above, we have the **Gordan-Noether Identity**:

$$\sum_{i=0}^{n} \frac{\partial F}{\partial x_i}(\underline{x}) h_i(\underline{x}) = 0 \qquad \Longleftrightarrow \qquad F(\underline{x}) = F(\underline{x} + \lambda \psi_g(\underline{x})) \qquad \forall\ \lambda \in \mathbb{K}', \forall\ \underline{x} \in \mathbb{K}^{n+1}.$$

Actually, it turns out that the functions h_i's satisfy the Gordan-Noether identity and, with this key formula, one can finally prove the (probably, most) important consequence of the Gordan-Noether Identity.

$$\psi_g(\underline{x}) = \psi_g(\underline{x} + \lambda \psi_g(\underline{x})), \qquad \forall\ \lambda \in \mathbb{K},\ \forall\ \underline{x} \in \mathbb{K}^{n+1}. \tag{1.2}$$

In particular, one also gets that

$$\psi_g(\mathbb{P}^n) \subseteq V(h_0, \dots, h_n) = \mathrm{Bs}(\psi_g) \subset \mathbb{P}^n.$$

Finally, by using all this machinery, one can prove Gordan-Noether Theorem, both in the case of $\mathbb{P}^2$ and $\mathbb{P}^3$ (see [Rus16, Corollaries 7.3.8, 7.3.9]).

□

To conclude, Gordan and Noether proved that Hesse's claim is in general false, for hypersurfaces in $\mathbb{P}^n$ with $n \geq 4$, but as a consequence of their theory, and in particular of Identity (1.2), that we call again Gordan-Noether Identity, they also proved the following:

Theorem 1.1.24. [Rus16, Theorem 7.1.6] *Let* $X = V(f) \subset \mathbb{P}^n$ *be a hypersurface of degree* $d \geq 2$ *with* $\mathrm{hess}(f) \equiv 0$. *Then there exists a Cremona transformation* $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ *such that* $\Phi(X)$ *is a cone.*

In other words, we can say that Hesse's claim is birationally true, despite the condition that the determinant of the Hessian matrix of a homogeneous polynomial f is identically zero is not invariant under birational transformation.

1.2 Constructions and strategy

Let us now present the construction that will be the key framework in the whole first half of this thesis.
Let $R = \oplus_{i=0}^{N} R^i$ be a SAGA (i.e. a standard Artinian Gorenstein $\mathbb{K}$-algebra) with socle in degree N and assume it has codimension $\dim R^1 = n+1$ with $n \geq 1$.

For $j \in \{1, \cdots, N-1\}$, we define

$$\Gamma_j = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \,|\, x^j y = 0\} \subset \mathbb{P}(R^1) \times \mathbb{P}(R^1),$$

where $\mathbb{P}(R^1)$ is the projectivization of the vector space R^1. By construction, we have $\Gamma_j \subseteq \Gamma_{j+1}$.

From now on, we fix an integer k such that $1 \leq k \leq N-2$ and, denoting by p_1 and p_2 the projections of Γ_k on the two factors, we assume that the following condition holds:

$$(\star) \quad p_1 : \Gamma_k \to \mathbb{P}(R^1) \simeq \mathbb{P}^n \text{ is surjective.}$$

After proving some general results that hold for every value of k in the above range, in the following we will focus on the specific case $k = N-2$, the most relevant one for our aims, as observed in the following:

Remark 1.2.1. *Notice that $(\star)$ is equivalent to asking that the multiplication map $x^k \cdot : R^1 \to R^{k+1}$ is never injective for $x \in R^1$, i.e. that R does not satisfy SLP at range k in degree 1. If $k = N-2$ (as we will assume in the following), $(\star)$ holds if and only if R does not satisfy the strong Lefschetz property (in narrow sense) in degree 1.*
Indeed, if the first projection p_1 is surjective, it means that for every element $[x] \in \mathbb{P}(R^1)$ there exists $[y] \in \mathbb{P}(R^1)$ such that $x^k y = 0$: the multiplication map $x^k \cdot : R^1 \to R^{k+1}$ has non trivial kernel and it can not be of maximal rank (observe that since R is a SAGA we have that $\dim(R^1) \leq \dim(R^i)$ for every $1 \leq i \leq N-1$). But this fact denies the validity of the strong Lefschetz property in degree 1 at range k. In the same exact way, one can show the converse.

Since, by assumption, $p_1 : \Gamma_k \to \mathbb{P}(R^1)$ is surjective, there exists an irreducible component of Γ_k that dominates $\mathbb{P}(R^1)$ via p_1. We can easily observe that all the fibers of p_1 are irreducible (not necessarily all of the same dimension): indeed the fiber by p_1 over $[x] \in \mathbb{P}(R^1)$ is $[x] \times \mathbb{P}(K^1_{x^k})$ and so it is isomorphic to a projective space. One can then easily obtain the following:

Lemma 1.2.2. *Under assumption $(\star)$, there exists a unique irreducible component of Γ_k which dominates $\mathbb{P}(R^1)$ via first projection.*

We will denote by Θ such a unique component of Γ_k and by π_i the restriction of p_i to Θ for $i = 1, 2$. Set

$$Y := \pi_2(\Theta)$$

and $\forall\ [y] \in Y$,

$$F_y := \pi_1(\pi_2^{-1}([y])) = \{[x] \in \mathbb{P}(R^1) \,|\, x^k y = 0 \text{ and } ([x],[y]) \in \Theta\}.$$

The following diagram summarizes the framework we are going to focus on.

$$\begin{tikzcd}
F_y \times [y] \arrow[rrr] \arrow[dr, hook] \arrow[ddd, "\simeq"'] & & & {[y]} \arrow[d, hook] \\
& \Theta \arrow[rr, hook, two heads, "\pi_2"] \arrow[dr, hook] \arrow[ddr, bend right, "\pi_1"'] & & Y \arrow[d, hook] \\
& & \Gamma_k \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}(R^1) \\
F_y \arrow[rr, hook] & & \mathbb{P}(R^1) &
\end{tikzcd}$$

We stress that, in this case, since Θ is the unique irreducible component which dominates $\mathbb{P}(R^1)$ via first projection, we have

$$\pi_1^{-1}([x]) = p_1^{-1}([x]) = [x] \times \mathbb{P}(K^1_{x^k}) \quad \text{for general} \quad [x] \in \mathbb{P}(R^1).$$

On the contrary, for specific $[x] \in \mathbb{P}(R^1)$, it can happen that $\pi_1^{-1}([x]) \subsetneq p_1^{-1}([x])$ and that $\pi_1^{-1}([x])$ is not a projective subspace of $[x] \times \mathbb{P}(R^1)$.

Let us now prove the following proposition which gives a collection of equations satisfied by the points of Θ. (We remark that this principle has already been used in [FP21] for studying the Jacobian ring of a smooth plane curve in relation with the infinitesimal variation of the periods of the curve itself.)

Proposition 1.2.3 (Ker-Coker principle). *If $p = ([x],[y]) \in \Theta$ then*

$$x^i y^j = 0,$$

for all $i \geq 0$ and $j \geq 1$ such that $i + j = k + 1$.

Proof. Let us consider a general point $p = ([x],[y]) \in \Theta$, so $x^k y = 0$ by definition. We claim that p satisfies also $x^{k-1}y^2 = 0$.

For any $v \in R^1$ and $t \in \mathbb{K}$, let us take $x' = x + tv \in R^1$. By assumption $(\star)$, we have that there exists y' in $R^1 \setminus \{0\}$ such that $(x')^k y' = 0$. Then we can define $\beta(t)$ such that $\beta(0) = y$ and $(x')^k \beta(t) = 0$ for all $t \in \mathbb{K}$. We can consider the expansion of β and write this relation as

$$0 \equiv (x + tv)^k (y + tw + t^2(\cdots)) = x^k y + t(kvx^{k-1}y + wx^k) + t^2(\cdots).$$

If we multiply by y both sides of the above relation, we get that

$$kvx^{k-1}y^2 = 0 \qquad \forall v \in R^1.$$

Since the multiplication map $R^1 \times R^{k+1} \to R^{k+2}$ is non degenerate we have that $x^{k-1}y^2 = 0$ as claimed. This proves that all the points of Θ satisfy also the relation $x^{k-1}y^2 = 0$.

In the same way one shows that if all the points of Θ satisfy the relation $x^i y^j = 0$ with $i+j = k+1$ and $j \geq 1$, then they also satisfy the relation $x^{i-1}y^{i+1} = 0$. This concludes the proof. □

As a consequence of Proposition 1.2.3, we obtain the following:

Corollary 1.2.4 (Gorenstein-Gordan-Noether identity). *Let $([x],[y]) \in \Theta$. Then the following relations hold for all $t \in \mathbb{K}$ and $(\lambda : \mu) \in \mathbb{P}^1$:*

$$(x + ty)^{k+1} = x^{k+1} \in R^{k+1} \qquad \text{and} \qquad [(\lambda x + \mu y)^{k+1}] = [x^{k+1}] \in \mathbb{P}(R^{k+1}) \quad (\text{if } x^{k+1} \neq 0). \tag{1.3}$$

Proof. Let us show the first equality: the second one is simply the projective version of the first. For $t \in \mathbb{K}$ and $[x],[y] \in \mathbb{P}(R^1)$, we have

$$(x + ty)^{k+1} = x^{k+1} + \sum_{i=1}^{k+1} \binom{k+1}{i} t^i x^{k+1-i} y^i,$$

but, since $([x],[y]) \in \Theta$, then by Proposition 1.2.3, we have that all the summands but the first one (namely x^{k+1}) are zero. □

The origin of the name we have given to these equalities lies in the classical Gordan-Noether identity (1.2) presented as one of the key steps of the original proof of Gordan-Noether theorem (see also Section 1.4, where we will analyze the relation between the two identities).

For completeness, let us now present the natural generalization of the previous construction, by considering incidence varieties over the other possible pairs of graded parts of the SAGA R, defined as above, with codimension $n+1$ and socle in degree N.
For $1 \leq a, b \leq N$ we set

$$\Gamma_{i,j}^{(a,b)} = \{([x],[y]) \in \mathbb{P}(R^a) \times \mathbb{P}(R^b) \,|\, x^i y^j = 0\}.$$

As above, we will denote by p_1 and p_2 the standard projections from $\Gamma_{i,j}^{(a,b)}$ to $\mathbb{P}(R^a)$ and $\mathbb{P}(R^b)$, respectively. Notice that, by setting $a = b = j = 1$ and $i = k$, we obtain again the variety Γ_k introduced above, i.e. $\Gamma_{k,1}^{(1,1)} = \Gamma_k$. Moreover, when we consider $\Gamma_{s,1}^{(a,b)}$, we have

$$p_1^{-1}([x]) = \{([x],[y]) \,|\, x^s y = 0\} = [x] \times \mathbb{P}(K_{x^s}^b)$$

so all the fibers of p_1 are projective spaces.

In the same exact way as done before for the case of Γ_k, we can make the following considerations:

- Assume that $b \leq N/2$ and that $SLP_b(s)$ does not hold. Then, for all $[x] \in \mathbb{P}(R^1)$ we have that the multiplication map $x^s\cdot : R^b \to R^{b+s}$ is not injective. In particular, there exists $[y] \in \mathbb{P}(R^b)$ such that $x^s y = 0$ in R^{b+s}. This shows that the failure of $SLP_b(s)$ is equivalent to ask that $p_1 : \Gamma_{s,1}^{(1,b)} \to \mathbb{P}(R^1)$ is surjective.

- Assume that $p_1 : \Gamma_{s,1}^{(a,b)} \to \mathbb{P}(R^a)$ is surjective. Then, as observed above, we have that all the fibers of p_1 are projective spaces and this implies that there exists a unique irreducible component Θ of $\Gamma_{s,1}^{(a,b)}$ which dominates $\mathbb{P}(R^a)$ via p_1 and, again, we can set

$$\pi_i = p_i|_{\Theta}, \quad Y = p_2(\Theta) = \pi_2(\Theta) \qquad \text{and} \qquad F_y = \pi_1(\pi_2^{-1}([y])) \text{ for all } [y] \in Y.$$

Construction 1.2.5. *To summarize, if R is a SAGA of codimension $n+1$ and socle in degree N and we assume that $SLP_b(s)$ does not hold for R, we can construct the loci $\Gamma_{s,1}^{(1,b)}, \Theta, Y$ and F_y as above and we have the following diagram*

$$\begin{tikzcd}
F_y \times [y] \arrow[rrr, "\pi_2"] \arrow[dr, hook] \arrow[ddd, "\simeq"'] & & & {[y]} \arrow[d, hook] \\
& \Theta \arrow[rr, "\pi_2"] \arrow[dr] \arrow[ddr, bend right, "\pi_1"'] & & Y \arrow[d, hook] \\
& & \Gamma_{s,1}^{(1,b)} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}(R^b) \\
F_y \arrow[rr, hook] & & \mathbb{P}(R^1). &
\end{tikzcd} \tag{1.4}$$

We present also the obvious generalisation of Ker-Coker principle (Proposition 1.2.3):

Proposition 1.2.6. *Let T be an irreducible variety in $\mathbb{P}(R^a) \times \mathbb{P}(R^b)$ such that $p_1|_T : T \to \mathbb{P}(R^a)$ is surjective. Assume that $T \subseteq \{x^i y^j = 0\} = \Gamma_{i,j}^{(a,b)}$ with $i, j \geq 1$. Then*

1. *For all $v \in R^a$ one has $vx^{i-1}y^{j+1} = 0$;*
2. *If $a(i) + b(j+1) \leq N$, then all points of T satisfy also $x^{i-1}y^{j+1} = 0$.*

Proof. It is enough to prove the claim for a general smooth point $p = ([x], [y]) \in T$. For any $v \in R^a, t \in \mathbb{K}$ consider $x' = x + tv \in R^a$. Since $p_1 : T \to \mathbb{P}(R^a)$ is surjective by hypothesis, we have that there exists y' in $R^b \setminus \{0\}$ such that $(x')^i(y')^j = 0$. Then we can define $\beta(t)$ such that $\beta(0) = y$ and $(x + tv)^i(\beta(t))^j = 0$ for all $t \in \mathbb{K}$. We can consider the expansion of β and write this relation as

$$0 \equiv (x + tv)^i(y + tw + t^2(\cdots))^j = x^i y^j + t(ivx^{i-1}y^j + jwx^i y^{j-1}) + t^2(\cdots).$$

In particular we have $ivx^{i-1}y^j + jwx^i y^{j-1} = 0$ for all $v \in R^a$. If we multiply by y we have $ivx^{i-1}y^{j+1} = 0$ for all $v \in R^a$ which yields the first claim since $i \geq 1$ by hyphotesis.

For the second claim, consider the multiplication map $R^a \times R^{(i-1)a+(j+1)b} \to R^{ia+jb+b}$ and notice that it is non degenerate by the assumption $ia + bj + b \leq N$. Hence, if $vx^{i-1}y^{j+1} = 0$ for all $v \in R^a$, then one has also $x^{i-1}y^{j+1} = 0$ as claimed. □

Let us now go back to the first construction and the study of Γ_k: let us present the first properties of the varieties introduced so far. In particular, we now show that Y is contained in some nihilpotent locus and that the general fiber F_y is a connected cone. Moreover we prove the first bounds for the dimension of these varieties.

Proposition 1.2.7. *Let us consider the correspondence Γ_k, dominating $\mathbb{P}(R^1)$ via first projection, and also the varieties Θ, Y and F_y introduced above. Then the following properties hold:*

(a) $Y \subseteq \mathcal{N}_{k+1} = \{[y] \in \mathbb{P}(R^1) : y^{k+1} = 0\} \subsetneq \mathbb{P}(R^1)$;

(b) If $[y] \in Y$ is general, then F_y is a cone with vertex $[y]$. Moreover, the general F_y is connected;

(c) $\dim F_y + \dim Y \geq \dim(\Theta) \geq n$;

(d) $1 \leq \dim F_y \leq n - 1$ *and* $1 \leq \dim Y \leq n - 1$.

Proof. By Proposition 1.2.3 we have that all the points of Θ of the form $([x], [y])$ satisfy also the equation $y^{k+1} = 0$. Then, by definition, we have $\pi_2(\Theta) = Y \subseteq \mathcal{N}_{k+1}$. Since $1 \leq k \leq N - 2$, by Remark 1.1.8 we have $\mathcal{N}_{k+1} \neq \mathbb{P}(R^1)$: we have proved claim (a).

Before proving (b), notice the following properties. For brevity, denote by Θ^c the union of all the irreducible components of Γ_k different from Θ. For any $p = ([x], [y]) \in \Theta$ one can consider the curve $\gamma_p : \mathbb{P}^1 \to \mathbb{P}(R^1) \times \mathbb{P}(R^1)$ defined by

$$\gamma_p((\lambda : \mu)) = ([\lambda x + \mu y], [y]).$$

Since $x^i y^j = 0$ whenever $i + j = k + 1$ and $j \geq 1$ we have $(\lambda x + \mu y)^k y = 0$ so γ_p has image in Γ_k. Whenever $p = ([x], [y]) \in \Theta \setminus \Theta^c$, we have that the curve γ_p has image in Θ. In this case, the line parametrized by $\pi_1 \circ \gamma_p$ is contained in F_y and it is spanned by $[x]$ and $[y]$.

Now, we will prove (b). If $[y] \in Y$ is general, we have that $\pi_2^{-1}([y]) \cap (\Theta \setminus \Theta^c)$ is an open dense subset of $F_y \times [y]$. Let C be a connected component of F_y and consider any $p = ([x],[y]) \in (C \times [y]) \cap (\Theta \setminus \Theta^c)$, then the image of the curve γ_p is contained in Θ and pass through $([y],[y])$. So $[y] \in C$ and the line in $\mathbb{P}(R^1)$ passing through $[x]$ and $[y]$ is contained in C. Since $[x]$ is general, we have that C is a cone with vertex $[y]$. Moreover, if C' is another connected component of F_y we have $[y] \in C \cap C'$ so $C = C' = F_y$ and F_y is connected.

In order to prove (c) recall that Θ and Y are irreducible and $\pi_2 : \Theta \to Y$ is surjective. Then, for all $[y] \in Y$ we have

$$\dim(\pi_2^{-1}([y])) \geq \dim(\Theta) - \dim(Y).$$

Since $\dim(\pi_2^{-1}([y])) = \dim F_y$ by definition of F_y and since $\dim(\Theta) \geq \dim(\mathbb{P}(R^1)) = n$ by hypothesis, we get claim (c).

For the last point (d), fix $[y] \in Y$. Assume, by contradiction, that $\dim(F_y) = n$, i.e. $F_y = \mathbb{P}(R^1)$. Then, for all $x \in R^1$, we have $x^k y = 0$. Since k-th powers of elements in R^1 generates R^k (since R is a standard algebra) we have that $y \cdot R^k = 0$. But this is impossible since R is Gorenstein and $R^1 \times R^k \to R^{k+1}$ is non-degenerate. This proves that $\dim(F_y) \leq n-1$. Using (c) we also get that $\dim(Y) \geq 1$. By (a) we have $\dim(Y) \leq \dim(\mathcal{N}_{k+1}) < n$ so $\dim(Y) \leq n-1$. Then, using again (c), we obtain $\dim(F_y) \geq 1$ as claimed. □

In particular, when we consider a SAGA R as above, for which the strong Lefschetz property in degree 1 at range k ($SLP_1(k)$) does not hold, we can construct the varieties Θ, Y, F_y which satisfy the properties described in Proposition 1.2.7.

Let us now present some geometric properties involving the incidence variety Γ_k and, by assuming condition $(*)$, also the varieties Θ, Y and F_y.

Lemma 1.2.8. *Let Δ be the diagonal in $\mathbb{P}(R^1) \times \mathbb{P}(R^1)$ and let us define $\tau : \mathbb{P}(R^1) \times \mathbb{P}(R^1) \to \mathbb{P}(R^1) \times \mathbb{P}(R^1)$ as the involution $\tau(([x],[y])) = ([y],[x])$. Under assumption $(\star)$, the following properties hold:*

(a) There exists a component Θ' of Γ_k different from Θ with $\dim(\Theta') \geq \dim(\Theta)$;

(b) $\pi_i(\Gamma_k \cap \Delta) = \mathcal{N}_{k+1}$;

(c) $\pi_i(\Theta \cap \Delta) = Y$;

(d) $\tau(\Theta) \cap \Theta \subseteq Y \times Y$.

(e) Assume that $Y \subsetneq \mathcal{N}_{k+1}$ and that Γ_k has pure dimension. Then, there exists an irreducible component Λ of Γ_k with $\Theta \neq \Lambda \neq \Theta'$.

(f) For all $y \in Y$ we have $F_y \subseteq \mathbb{P}(K^1_{y^k})$.

Proof. For (a), if $p = ([x],[y]) \in \Theta$ then, by Proposition 1.2.3, $x^k y = x y^k = 0$ hence $\tau(p) \in \Gamma_k$ so there exists a component Θ' such that $\tau(\Theta) \subseteq \Theta'$. On the other hand, by construction, Θ dominates

$\mathbb{P}(R^1)$ via π_1 (by Lemma 1.2.2) and does not dominate $\mathbb{P}(R^1)$ via π_2 (since $\pi_2(\Theta) = Y$, which is a proper subvariety of $\mathbb{P}(R^1)$, by Proposition 1.2.7(a)). Since $\tau(\Theta)$ dominates $\mathbb{P}(R^1)$ via π_2 the same holds for Θ' and this implies that $\Theta \neq \Theta'$. For (b), if we take an element $[y] \in \mathcal{N}_{k+1}$, we clearly have that $y^k y = y^{k+1} = 0$, hence $([y],[y]) \in \Gamma_k \cap \Delta$. On the other hand, an element in this intersection is of the form $([y],[y])$ such that $y^k y = 0$, hence $[y] \in \mathcal{N}_{k+1}$. Point (c) follows from Proposition 1.2.7: we have $\pi_2(\Theta) = Y$ so $\pi_2(\Theta \cap \Delta) \subseteq Y$ and for $[y] \in Y$ general, $[y] \in F_y$ so $([y],[y]) \in \Theta$. (d) is trivial. For (e), let $[w] \in \mathcal{N}_{k+1} \setminus Y$. Then $w^k w = w^{k+1} = 0$ by construction so $([w],[w]) \in \Gamma_k$. Since $w \notin Y$ we have $p = ([w],[w]) \notin \Theta$. If we assume $([w],[w]) \in \tau(\Theta)$ we would also have $\pi_1([w],[w]) = [w] \in Y$ as $\tau(\Theta) = \Theta'$ (this follows since we are assuming that Γ_k has pure dimension). Then $p \in \Gamma_k \setminus (\Theta \cup \tau(\Theta))$. For (f), recall that $F_y = \{[x] \in \mathbb{P}(R^1) \,|\, ([x],[y]) \in \Theta \text{ and } x^k y = 0\}$. By Proposition 1.2.3, $[x] \in F_y$ implies that also that $xy^k = 0$ so we have the claim. $\square$

To conclude this second section, let us present the strategy we will exploit in the following:

Strategy 1.2.9. *In the next sections, we will consider a SAGA R not satisfying a specific (strong) Lefschetz property. As we have done before, this assumption allows us to construct the varieties Θ, Y, F_y. Our aim will then be the improvement of inequalities, as the ones in Proposition 1.2.7, involving the dimensions of such varieties. With stricter bounds on these dimensions, under suitable assumptions, we will be able to deny the existence of such varieties, in particular of the irreducible component Θ. But then, as specified in Remark 1.2.1, it is not possible for the first projection over $\mathbb{P}(R^1)$ to be surjective: the Lefschetz property under consideration can't fail.*

1.3 The SLP in degree 1 and the proof of Theorem B

The aim of this section is to prove Theorem B and so, as a byproduct, to give a new proof of Gordan-Noether Theorem 1.1.22.

To do this, we consider a SAGA R and let us assume that it does not satisfy the strong Lefschetz property (in narrow sense) in degree 1. According to the notation of the previous section, we then set $k = N-2$ and we consider the correspondence variety

$$\Gamma_{N-2} = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \,|\, x^j y = 0\}.$$

By assumption, the first projection $p_1 : \Gamma_{N-2} \to \mathbb{P}(R^1)$ is surjective (see Remark 1.2.1). We can then consider the varieties Θ, Y and F_y, as defined in Section 1.2. By exploiting geometric properties of these varieties, we will now follow the Strategy 1.2.9.

Let us start by constructing the dual variety of Y and by showing that $Y \subset \mathbb{P}(R^1) \simeq \mathbb{P}^n$ can't be linear and that its dimension is at most $n-2$.

First of all, let us consider the map

$$\varphi : R^1 \to R^{N-1} \quad x \mapsto x^{N-1}$$

and its projective version, that is clearly not defined on the nihilpotent locus of order $N-1$,

$$\psi : \mathbb{P}(R^1) \setminus \mathcal{N}_{N-1} \to \mathbb{P}(R^{N-1}) \quad [x] \mapsto [x^{N-1}].$$

Observe that, since R is a SAGA, we have that R^{N-1} is the dual vector space of R^1, and so $\dim(R^{N-1}) = n+1$.
By setting $Z := \overline{\psi(\mathbb{P}(R^1) \setminus \mathcal{N}_{N-1})}$, we have that this subvariety Z of $\mathbb{P}(R^{N-1}) \simeq \mathbb{P}^n$ is irreducible and non-degenerate: indeed, if Z was degenerate, and so contained in a hyperplane, we would have that the $(N-1)-$th powers of elements of R^1 do not generate the whole R^{N-1}, but this is not possible, since R is a standard algebra.

For all $[x] \in \mathbb{P}(R^1)$, let us now define $\mathcal{K}_x$ as the kernel of the differential

$$d_{[x]}\psi : T_{\mathbb{P}(R^1),[x]} \to T_{Z,[x^{N-1}]} \qquad w \mapsto (d_{[x]}\psi)(w) = (N-1)x^{N-2}w$$

i.e. $\mathcal{K}_x := \ker(d_{[x]}\psi)$.

Remark 1.3.1. *Observe that the kernel $\mathcal{K}_x$ just introduced coincides with the kernel $K^1_{x^{N-2}}$ of the multiplication map $x^{N-2}\cdot : R^1 \to R^{N-1}$. Hence, the projectivization of $\mathcal{K}_x$ is isomorphic to the fibres of the first projection $p_1 : \Gamma_{N-2} \to \mathbb{P}(R^1)$ (see Section 1.2).*

Now, let Δ be the diagonal of $\mathbb{P}(R^1) \times \mathbb{P}(R^1)$ and whenever $p = ([x],[y]) \notin \Delta$ let us set

$$L_p = \{[\lambda x + \mu y] \in \mathbb{P}(R^1) \mid (\lambda : \mu) \in \mathbb{P}^1\}, \tag{1.5}$$

the line in $\mathbb{P}(R^1)$ passing through $[x]$ and $[y]$.
We have the following:

Lemma 1.3.2. *For $p = ([x],[y]) \in \Theta$ general, the line L_p is contracted by ψ. Moreover, we have $\dim(Z) = n - \dim(\mathcal{K}_x) \le n-1$.*

Proof. Let $p = ([x],[y]) \in \Theta$ be general: we can then assume that $x^{N-1} \neq 0$, i.e. $[x] \notin \mathcal{N}_{N-1}$. Indeed, if $x^{N-1} = 0$ for $p \in \Theta$ general, then $\Theta \subseteq \mathcal{N}_{N-1} \times Y$ and thus $\mathcal{N}_{N-1} = \mathbb{P}(R^1)$ by $(\star)$. This is impossible by Proposition 1.2.7(a). In particular, we have also that $p = ([x],[y]) \notin \Delta$, indeed, while $x^{N-1} \neq 0$, by Proposition 1.2.3 we have that $y^{k+1} = 0$, where, in this case, $k = N-2$.

By using the Gorenstein-Gordan-Noether identity (see Corollary 1.2.4) we have

$$\psi([\lambda x + \mu y]) = [(\lambda x + \mu y)^{N-1}] = [\lambda^{N-1}x^{N-1}] = [x^{N-1}] = \psi([x])$$

so the line L_p is contracted by ψ (more precisely, $L_p \setminus \mathcal{N}_{N-1}$ is contracted to a point by ψ).

If we assume that $[z] := \psi([x]) = [x^{N-1}] \in Z_{smooth}$, then we have $\dim(Z) = \dim(T_{Z,[z]})$ so

$$\dim(Z) = \dim(\mathbb{P}(R^{N-1})) - \dim(\mathcal{K}_x) = n - \dim(\mathcal{K}_x). \tag{1.6}$$

Since L_p is contracted by ψ and $[x] \in L_p$ is not in $\mathcal{N}_{N-1}$ we have that $T_{L_p,[x]} = \langle y \rangle \subseteq \mathcal{K}_x$ so $\dim(\mathcal{K}_x) \ge 1$ and we have the claim. □

Recall that, as said above, via Gorenstein duality we have a linear isomorphism $R^1 \simeq (R^{N-1})^*$ which induces an isomorphism $\mathbb{P}(R^{N-1})^* \simeq \mathbb{P}((R^{N-1})^*) \simeq \mathbb{P}(R^1)$. If $H \in \mathbb{P}(R^{N-1})^*$ and $\alpha \in \mathbb{P}((R^{N-1})^*)$ correspond under the first isomorphism, we have that the hyperplane H contains a linear variety $\mathbb{P}(W) \subseteq \mathbb{P}(R^{N-1})$ if and only if α, a linear form on R^{N-1}, annihilates all the vectors in W, i.e. we have, by using the second isomorphism, $\alpha \in \mathbb{P}(\mathrm{Ann}_{R^1}(W))$.

Let X be a proper projective subvariety of $\mathbb{P}^n$ and assume that $[x] \in X_{smooth}$. We will denote with $(\mathbb{P}^n)^*$ the dual projective space of $\mathbb{P}^n$ (i.e. the projective variety parametrizing the hyperplanes of $\mathbb{P}^n$) and with $T_{[x]}(X)$ the projective tangent space to $[x]$ in X. If $\tilde{X} \subseteq \mathbb{K}^{n+1}$ is the affine cone associated to X we have $T_{[x]}(X) = \mathbb{P}(T_{\tilde{X},x})$. We recall that the dual variety of X (as subvariety of $\mathbb{P}^n$) is

$$X^* = \overline{\{H \in (\mathbb{P}^n)^* \quad | \quad \exists\, [x] \in X_{smooth} \text{ such that } T_{[x]}(X) \subseteq H\}}.$$

As one of the key results of this section, it turns out that the dual variety of such Z coincides exactly with $Y = \pi_2(\Theta)$:

Proposition 1.3.3. *We have $Y = Z^*$.*

Proof. First of all, notice that if $[z] = [x^{N-1}] = \psi([x]) \in Z_{smooth}$, we have that the tangent (projective) space to Z in $[z]$ is described as

$$T_{[z]}(Z) = \mathbb{P}(x^{N-2} \cdot R^1) = \mathbb{P}\left(\{wx^{N-2} \,|\, w \in R^1\}\right).$$

Assume that $[y] \in Y$ is a general point. We claim that $y \in Z^*$. Since $[y] \in Y$ is general, we can take $([x],[y]) \in \Theta$ such that $x^{N-1} \neq 0$ (recall that the nihilpotent loci are not the whole projective space) and $[z] = [x^{N-1}]$ is a smooth point of Z. In particular $x^{N-2}y = 0$ so $[y] \in \mathrm{Ann}_{R^1}(x^{N-2} \cdot R^1)$. Hence, by the above considerations, the hyperplane H of $\mathbb{P}(R^{N-1})$ corresponding to $[y]$ contains $T_{[z]}(Z)$ so $[y] \in Z^*$. Since $[y]$ was general in Y, we have proved $Y \subseteq Z^*$.

For the other inclusion, let H be a general element in Z^*. Let $[y] \in \mathbb{P}(R^1)$ be its corresponding point. Since $H \in Z^*$ (and H is general) we have that there exists $[z] = [x^{N-1}] \in Z_{smooth}$ such that H contains the tangent (projective) space $T_{[z]}(Z)$. Then, equivalently, y annihilates $x^{N-2} \cdot R^1$. On the other hand, since the product $R^1 \times R^{N-1} \to R^N$ is a perfect pairing, having $x^{N-2}wy = 0$ for all $w \in R^1$ implies that $x^{N-2}y = 0$ so $([x],[y]) \in \Gamma_{N-2}$. Since H was generic in Z^* and, by Lemma 1.2.2, Θ is the only component of Γ_{N-2} which dominates $\mathbb{P}(R^1)$ via π_1, we can assume that $[x]$ is outside $\overline{\pi_1(\Gamma_{N-2} \setminus \Theta)}$. Then, we have that $([x],[y]) \in \Theta$ so $[y] \in Y$ as claimed. □

Let us now recall that a variety $X \subset \mathbb{P}^n_{\mathbb{K}}$ is said to be *reflexive* if it coincides with the dual of its dual variety. One has the following (see [Wal56, Kle86]):

Theorem 1.3.4. *If $\mathbb{K}$ is a field of characteristic 0, an irreducible variety $X \subset \mathbb{P}^n_{\mathbb{K}}$ is reflexive.*

As a consequence of Proposition 1.3.3, we can then prove:

Corollary 1.3.5. *The variety $Y \subset \mathbb{P}(R^1)$ is not linear.*

Proof. Let us suppose by contradiction that Y is a proper linear subvariety of $\mathbb{P}(R^1)$. Since $\mathbb{K}$ is a field of characteristic 0 and Z is irreducible, by theorem 1.3.4, we have that $Z = Z^{**}$. From Proposition 1.3.3, we have that $Y = Z^*$ and so $Y^* = Z$. Since we are assuming that Y is linear, we have that also Y^* is linear: namely, it is the linear subspace of $\mathbb{P}(R^1)$ of the hyperplanes containing Y, which is proper. Then Z is linear and thus degenerate, but this, as we have observed above, is not possible. Then Y is not linear. □

We are now going to improve the inequalities (d) in Proposition 1.2.7, by showing that Y cannot be a hypersurface of $\mathbb{P}(R^1)$ (see Corollary 1.3.7).

Let $p = ([x],[y]) \in \Theta$ with $x \neq y$. As above, we will denote by L_p the line in $\mathbb{P}(R^1)$ joining the points $[x]$ and $[y]$, i.e. the line $L_p = \{\lambda x + \mu y \,|\, (\lambda : \mu) \in \mathbb{P}^1\}$.

Lemma 1.3.6. *Let $p = ([x],[y]) \in \Theta$ such that $x^{N-1} \neq 0$. Then*

(a) $L_p \cap Y = [y]$;

(b) if p is general, L_p is not tangent to Y at $[y]$.

Proof. Let $p = ([x],[y])$ be as in the hypothesis and let us consider the line L_p (since by assumption $x^{N-1} \neq 0$, we have that $p \notin \Delta$). Clearly, by construction, we have that $[y] \in L_p \cap Y$. To show that there are no other points in this intersection, let us recall, by Proposition 1.2.3 that a point $[y]$ in Y satisfies the condition $y^{N-1} = 0$. Then

$$[y] \in L_p \cap Y \subseteq L_p \cap \mathcal{N}_{N-1} = \{[\lambda x + \mu y] \quad | \quad (\lambda : \mu) \in \mathbb{P}^1, \, (\lambda x + \mu y)^{N-1} = 0\}$$

On the other hand, the Gorenstein-Gordan-Noether identity (see Corollary 1.2.4) yields $(\lambda x + \mu y)^{N-1} = \lambda^{N-1} x^{N-1}$. This is zero if and only if $\lambda = 0$, so $L_p \cap Y = [y]$ as claimed in (a).

To prove (b), let us take $p = ([x],[y]) \in \Theta$ general. Then, we can assume that $x^N \neq 0$ (since R is a standard $\mathbb{K}$-algebra, see Remark 1.1.8), that p is a smooth point for Θ and that the differential $d_p\pi_2 : T_{\Theta,p} \to T_{Y,[y]}$ is surjective.

Assume by contradiction that L_p meets Y non-transversely. Since the tangent in $[y]$ to L_p is spanned by x and since $d_p\pi_2$ is surjective, we have that there exists a tangent vector of the form (v, x) in $T_{\Theta,p}$. Hence, there is a curve $\gamma(t)$ in Θ, that we can write as $\gamma(t) = (\alpha(t), \beta(t))$, passing at $t = 0$ through the point $p = ([x],[y])$ and such that $\alpha'(0) = v$ and $\beta'(0) = x$. As γ has image in Θ, we have that α and β satisfy the relation $\alpha(t)^{N-2}\beta(t) = 0$. By considering the expansion of this relation, as in Proposition 1.2.3, we obtain the equation

$$(N-2)x^{N-3}vy + x^{N-1} = 0.$$

If we multiply by x, we get $x^N = 0$ which is impossible since we are assuming $x^N \neq 0$. Then L_p and Y meet transversely. $\square$

Finally, we get new bounds for the dimensions of Y and of the fibers F_y:

Corollary 1.3.7. *We have*

$$1 \leq \dim(Y) \leq n-2 \qquad \text{and} \qquad 2 \leq \dim(F_y) \leq n-1$$

for any $[y] \in Y$.

Proof. Let us show the first claim: the second one is an immediate consequence, by Proposition 1.2.7(c). Assume by contradiction that $\dim(Y) = n-1$, so Y is an irreducible hypersurface in $\mathbb{P}(R^1)$. By Lemma 1.3.6, there is a line which meets Y transversely in one point, so Y is a hyperplane. On the other hand, by Corollary 1.3.5, Y is not linear, so we have a contradiction. $\square$

Our aim is now to show the following result, that will be a key step in the proof of Theorem B. In particular, after ruling out the possibility for Y to be a hypersurface, we would like to find some necessary condition to have $\dim(Y) = 1$:

Proposition 1.3.8. *If* $\dim(Y) = 1$ *then* $n \geq 4$.

Observe, first of all, that if the general $[y] \in Y$ is such that $\dim(F_y) = n-1$, then this equality actually holds for all the fibers F_y's, since the dimension of the fibres can only increase and, on the other hand, there can not exist a fibre F_y of dimension n and so equal to $\mathbb{P}^n$.
Before proving Proposition 1.3.8, we need two technical results.

Proposition 1.3.9. *Assume that* F_y *has dimension* $n-1$ *for all* $y \in Y$*. Then*

(a) $Y \subseteq \bigcap_{y \in Y} F_y$;

(b) $\mathrm{Sec}(Y) \subseteq \mathcal{N}_{N-1}$.

Proof. Recall that Θ^c denotes the union of all the irreducible components of Γ_{N-2} different from Θ. Let us take an element $[y] \in Y$ and fix $[x] \in \mathbb{P}(R^1)$ general, satisfying the following assumptions:

$$x^{N-1} \cdot y \neq 0, \qquad [x] \in \pi_1(\Theta \setminus \Theta^c) \qquad \text{and} \qquad x^N \neq 0.$$

This can be done since R is a standard algebra and since Θ is the only component dominating $\mathbb{P}(R^1)$ via first projection.

Since π_1 is dominant, there exists $[y_1] \in Y$ (which can be assumed general as for $[x]$), such that $p_1 = ([x], [y_1]) \in \Theta \setminus \Theta^c$. In particular, we have $x^{N-2} y_1 = 0$ and $[y_1] \neq [y]$ since, otherwise, we would have that $x^{N-2} y = x^{N-2} y_1 = 0$ which gives a contradiction.

Let us now consider the line L_{p_1}, joining the points $[x]$ and $[y_1]$, i.e.

$$L_{p_1} = \{[\lambda y_1 + \mu x] \quad | \quad (\lambda : \mu) \in \mathbb{P}^1\}.$$

As in point (b) of Proposition 1.2.7, we have $L_{p_1} \subseteq F_{y_1}$ by the assumptions on $[x]$.

We claim now that $L_{p_1} \cap F_y = [y_1]$.

Since, by assumption, F_y has dimension $n-1$, the intersection $L_{p_1} \cap F_y$ cannot be empty. We will show now that $(L_{p_1} \setminus [y_1]) \cap F_y$ is empty.

Notice that $L_{p_1} \setminus [y_1]$ is the affine line parametrized by $x(t) = x + t y_1$ with $t \in \mathbb{K}$. Suppose that the intersection between F_y and this affine line is not empty, i.e. there exists $\tilde{t} \in \mathbb{K}$ such that $x + \tilde{t} y_1 \in F_y$. This means that

$$(x + \tilde{t} y_1)^{N-2} y = 0 \qquad \text{and multiplying by } x \qquad x(x + \tilde{t} y_1)^{N-2} y = 0.$$

By construction, we have that $[x] \in F_{y_1}$ (equivalently, $([x],[y_1]) \in \Theta$), and so by Proposition 1.2.3 we know that $x^i y_1^j = 0$ for $j \geq 1$ and $i + j = N - 1$. Then we get $x(x + \tilde{t}y_1)^{N-2} = x^{N-1}$ and finally, by the above, $x^{N-1}y = 0$, that is impossible by our assumptions. In conclusion, L_{p_1} and F_y meet each other at a single point, namely $[y_1]$.

We have proved that for general $[y_1] \in Y$ we have $[y_1] \in F_y$. Then, by the irreducibility of Y, we get $Y \subset F_y$. Since, this is true for every choice of $y \in Y$, we obtain claim (a).

For (b), let us consider two distinct points $[y_1], [y_2] \in Y$. From (a) we have that $[y_2] \in F_{y_1}$ and then $p = ([y_2],[y_1]) \in \Theta$. Let us now consider the projective line

$$L_p = \{[\lambda y_1 + \mu y_2] \,|\, (\lambda : \mu) \in \mathbb{P}^1\}$$

so we have $L_p \subseteq \mathrm{Sec}(Y)$. By Proposition 1.2.3 we know that $y_2^i y_1^j = 0$ for every i, j with $j \geq 1$ and $i + j = N - 1$. On the other hand, we have that $y_2 \in F_{y_2}$ so $0 = y_2^{N-2} y_2 = y_2^{N-1}$. By the above equations we get

$$(\lambda y_1 + \mu y_2)^{N-1} = 0$$

so $L_p \subseteq \mathcal{N}_{N-1}$. Since every secant line is contained in $\mathcal{N}_{N-1}$, we have claim (b). $\square$

If we assume that F_y has dimension $n - 1$ for all $y \in Y$ we can strengthen the results of Corollary 1.3.7:

Proposition 1.3.10. *Assume that F_y has dimension $n - 1$ for all $y \in Y$. Then $1 \leq \dim(Y) \leq n - 3$.*

Proof. Recall that $\{[x]\} \times \mathbb{P}(\mathcal{K}_x)$ $(= \{[x]\} \times \mathbb{P}(K^1_{x^{N-2}}))$ is the fiber of a general point $[x]$ in $\mathbb{P}(R^1)$ with respect to $\pi_1 : \Theta \to \mathbb{P}(R^1)$. Denote by $r - 1$ the dimension of the general fiber $\mathbb{P}(\mathcal{K}_x)$.

$$\begin{tikzcd}
 & \{[x]\} \times \mathbb{P}(\mathcal{K}_x) \arrow[r, hook] \arrow[dl] & \Theta \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & F_y \times \{[y]\} \arrow[l, hook'] \arrow[dr] & \\
\{[x]\} \arrow[rr, hook] & & \mathbb{P}(R^1) \quad\quad Y & & \{[y]\} \arrow[ll, hook']
\end{tikzcd}$$

Being in the above diagram $\mathbb{P}(R^1)$, Θ and Y irreducible, π_1 dominant and π_2 surjective by construction, we have

$$\dim(\Theta) = \dim(\mathbb{P}(R^1)) + \dim(\mathbb{P}(\mathcal{K}_x)) = n + r - 1 \qquad \dim(\Theta) = \dim(Y) + \dim(F_y) = \dim(Y) + n - 1$$

so $\dim(Y) = r$. By Corollary 1.3.7 we have

$$1 \leq \dim(Y) \leq n - 2$$

so it is enough to prove that $\dim(Y)$ cannot be equal to $n - 2$. This is clearly true if $n = 2$ so we can assume $n \geq 3$. By contradiction, assume that $\dim(Y) = r = n - 2$. Denote by s the dimension of $\mathrm{Sec}(Y)$. By Proposition 1.3.9 we have that $Y \subseteq \mathrm{Sec}(Y) \subseteq \mathcal{N}_{N-1} \subsetneq \mathbb{P}(R^1)$ so we have $n-2 \leq s \leq n-1$.

Notice, first of all, that s cannot be $n - 2$. Indeed, if $\dim(\mathrm{Sec}(Y)) = \dim(Y) = n - 2$, we would have that Y is linear. This is impossible by Corollary 1.3.5. Hence we can assume $s = n - 1$.

Assume first that Y is non-degenerate. We have that Y and $\mathrm{Sec}(Y)$ have codimension 2 and 1 respectively in the smallest projective space that contains Y (and $\mathrm{Sec}(Y)$). By considering the general hyperplane section $Y' = Y \cap H$ and its secant variety $\mathrm{Sec}(Y') = \mathrm{Sec}(Y) \cap H$, we preserve the above properties and Y' is as well non-degenerate (in H). We can then cut with $n-3$ general hyperplanes in order to obtain a curve C in $\mathbb{P}^3$ and its secant variety which is a surface in $\mathbb{P}^3$. This is impossible since, in this case, C would be a plane curve, and so degenerate.

The only remaining case to analyze is where Y is degenerate of dimension $n-2$, $\dim(\mathrm{Sec}(Y)) = n-1$ and the smallest projective subspace H containing Y is an hyperplane in $\mathbb{P}(R^1)$. In particular, Y is an hypersurface in $H = \mathrm{Sec}(Y)$ and its degree is at least 2 (otherwise Y would be linear).

First of all, we will prove that $H \subseteq F_y$ for $[y] \in Y$ general. Let $[y] \in Y$ be a general point. The general line L through $[y]$ in H cuts Y in at least another point $[y_1]$. By Proposition 1.3.9 (a), we have that $[y], [y_1] \in F_y$ and then, by point (b) of Proposition 1.2.7, L is contained in F_y. Since such lines cover a dense open subset of H we have that $H \subseteq F_y$. Then $H \times [y] \subset F_y \times [y]$ and then $H \times Y \subseteq \Theta$. Since they have the same dimension and they are both irreducible we have $H \times Y = \Theta$. This is impossible by $(\star)$: if $H \times Y = \Theta$ we would have $\pi_1(\Theta) = H \neq \mathbb{P}(R^1)$, which is impossible, since π_1 is dominant: $\dim(Y) \leq n-3$ as claimed. □

We can now prove Proposition 1.3.8:

Proof of Proposition 1.3.8. Assume, by contradiction, that $n \leq 3$. Since we are assuming $\dim(Y) = 1$ we have that $\dim(F_y) = n-1$ by Proposition 1.2.7. Then, by Proposition 1.3.10, we have $1 \leq \dim(Y) \leq n-3 \leq 0$, which is clearly impossible. □

To conclude this section, let us finally restate and prove Theorem B:

Theorem 1.3.11 (Theorem B)**.** *For all standard Artinian Gorenstein $\mathbb{K}$-algebras of codimension at most $n+1 = 4$ there exists $x \in R^1$ such that the multiplication map $x^{N-2}\cdot : R^1 \to R^{N-1}$ is an isomorphism, i.e. the strong Lefschetz property (in narrow sense) holds in degree* 1.

Proof. Assume, by contradiction, that for all $x \in R^1$ the multiplication map $x^{N-2}\cdot : R^1 \to R^{N-1}$ is not an isomorphism. Then we can construct the incidence variety Γ_{N-2} and by Remark 1.2.1 we know that the projection $p_1 : \Gamma_{N-2} \to \mathbb{P}(R^1)$ is surjective. Under these assumptions, we can also construct the varieties Θ, Y and the fibers F_y's, as done above. By Corollary 1.3.7 we have $1 \leq \dim(Y) \leq n-2$. Since $n \leq 3$, the only possibility is that n is equal to 3 and $\dim(Y) = 1$. But this is impossible by Proposition 1.3.8. □

1.4 Gordan-Noether and strong Lefschetz properties

In this section, for completeness, let us show the classical equivalence between Gordan-Noether theorem 1.1.22 (Theorem A), presented in Subsection 1.1.2 and Theorem 1.3.11 (Theorem B), proved in Section 1.3.

As done before, let $\mathbb{K}$ be an algebraically closed field of characteristic 0, $S = \mathbb{K}[x_0, \ldots, x_n]$ the ring of polynomials in $n+1 \geq 2$ variables and $D = \mathbb{K}[y_0, \ldots, y_n]$, with $y_i = \partial/\partial x_i$ the ring of differential operators in the x_i. Let us recall here that, by Macaulay's inverse systems, every SAGA R can be written as $D/\operatorname{Ann}_D(g)$ for some homogeneous polynomial $g \in S$ (see Theorem 1.1.5).

Remark 1.4.1. *The codimension of a SAGA $A = D/\operatorname{Ann}_D(g)$, i.e. the dimension of A^1, is at most $n+1$ and equality holds as long as $(\operatorname{Ann}_D(g))_1 = \{0\}$. This is equivalent to ask that the partial derivatives of g are linearly independent, i.e.*

$$\dim(A^1) = n+1 \iff X = V(g) \subseteq \mathbb{P}^n \textit{ is not a cone.}$$

Let us start with a strong connection between the non-vanishing of a hessian determinant and the validity of the strong Lefschetz property in degree 1 for a SAGA as in example 1.1.4. For this, we need the well known differential Euler Identity ([Rus16, Lemma 7.2.19]):

Lemma 1.4.2. *Let $g \in S^e$ be a homogeneous polynomial of degree e and let $L = a_0 \frac{\partial}{\partial x_0} + \cdots + a_n \frac{\partial}{\partial x_n}$ be an element of D^1. Then*

$$L^e(g) = e! \cdot g(a_0, \cdots, a_n).$$

Lemma 1.4.3. *Fix $g \in S^d \setminus \{0\}$ and consider the SAGA $A = D/\operatorname{Ann}_D(g)$. Then A has the strong Lefschetz property in degree 1 if and only if $\operatorname{hess}(g) \not\equiv 0$.*

Proof. For any fixed $L = \sum_{i=0}^n k_i \frac{\partial}{\partial x_i} \in A^1$ we can consider the symmetric bilinear map

$$\varphi_L : A^1 \times A^1 \to A^d \simeq \mathbb{K}$$

given by $\varphi_L(\eta, \xi) = (L^{d-2}\eta\xi)(g)$. Let $\mathcal{B} = \{y_0, \ldots, y_n\}$ be a basis of A^1. Denote with M_L the matrix associated to φ_L with respect to $\mathcal{B}$. Then we have $M_L = [\alpha_{ij}]_{0 \le i,j \le n}$ with

$$\alpha_{ij} = (L^{d-2} y_i y_j)(g) = L^{d-2}(y_i y_j(g)) = L^{d-2}(\operatorname{Hess}(g)_{ij})$$

where $\operatorname{Hess}(g)$ is the Hessian matrix of g. Since $\operatorname{Hess}(g)_{ij}$ is either 0 or has degree $d-2$, one can apply the differential Euler Identity (Lemma 1.4.2) in order to obtain

$$M_L = (d-2)! \operatorname{Hess}(g)(k_0, \ldots, k_n). \tag{1.7}$$

Hence, having $\operatorname{hess}(g)(= \det(\operatorname{Hess}(g))) \equiv 0$ is equivalent to ask that φ_L is degenerate, i.e. for all $L, z \in A^1$ there exists $y \in A^1 \setminus \{0\}$ such that $L^{d-2}yz = 0$. By Gorenstein duality, this is equivalent to $L^{d-2}y = 0$, i.e. A does not satisfy the SLP in degree 1. ☐

We can now show the equivalence between Theorem A and Theorem B:

Proposition 1.4.4. *Theorem A and Theorem B are equivalent.*

Proof. **Assume first that Theorem B holds**. Let $X = V(F)$ be a hypersurface of degree $d \ge 2$ in $\mathbb{P}^n$ with $n \le 3$ and we assume that X is not a cone. We have to show that $\operatorname{hess}(F) \neq 0$. Since X is not a cone, the partial derivatives of F are linearly independent. Hence, if we consider the SAGA $A = D/\operatorname{Ann}_D(F)$ as above, we have that A has codimention $n+1 \le 4$ and socle in degree d. By Theorem B, the SAGA A has the strong Lefschetz property in degree 1 so, by Lemma 1.4.3, $\operatorname{hess}(F) \neq 0$ as claimed.

Assume now that Theorem A holds. Let us consider a standard Artinian Gorenstein $\mathbb{K}$-algebra A of codimension $n+1$, with $n \leq 3$, and socle in degree d. This algebra can be described as

$$A = \frac{D}{\operatorname{Ann}_D(F)},$$

for a homogeneous polynomial F of degree d in the variables $x_0, \ldots, x_n$, by Macaulay's theory. We can suppose that F is such that $(\operatorname{Ann}_D(F))_1 = 0$, i.e. its partial derivatives are linearly independent. Let us now assume by contradiction that A does not satisfy the SLP in degree 1. Then, by Lemma 1.4.3, we would have $\operatorname{hess}(F) = 0$. This is impossible since, by Theorem A we would have that $V(F)$ is a cone: indeed, this would imply that the partial derivatives of F are linearly dependent, which is against our assumptions. □

1.4.1 The Gordan-Noether identity

In this subsection, let us rephrase in our setting the important consequence of the Gordan-Noether Identity, the identity (1.2) (that, for brevity, we will call again Gordan-Noether Identity) and by proving it in the framework described in the previous sections.

First of all, let us briefly recall that, given a homogeneous polynomial $f \in S^d$ with $d \geq 1$ without multiple factors, where $S = \mathbb{K}[x_0, \cdots, x_n]$, then the closure Z' of the image of ∇_f in $(\mathbb{P}^n)^*$ is easily seen to be a proper subvariety of $(\mathbb{P}^n)^*$ if and only if $\operatorname{hess}(f) \equiv 0$ (see (1.1)). In this case, for any hypersurface $W = V(g)$ containing Z', we can consider the *Gordan-Noether map associated to g*

$$\psi_g := \nabla_g \circ \nabla_f : \mathbb{P}^n \dashrightarrow \mathbb{P}^n.$$

One of the key steps in the classical proof of Gordan-Noether theorem, as we have seen (see (1.2)), is the following: if f has vanishing hessian, then the *Gordan-Noether identity*

$$\psi_g(\underline{x} + \lambda \psi_g(\underline{x})) = \psi_g(\underline{x}) \tag{1.8}$$

holds for all $\lambda \in \mathbb{K}$ and for all $\underline{x} \in \mathbb{K}^{n+1}$.

Let us now express the map ψ_g using the framework introduced in Sections 1.2, 1.3. Let R be a standard Artinian Gorenstein algebra with socle in degree N and assume that the SLP_1 does not hold. By Macaulay Theorem we can write R as $D/\operatorname{Ann}_D(F)$ for some suitable $F \in \mathbb{K}[x_0, \cdots, x_n]$ with $(\operatorname{Ann}_D(F))_1 = (0)$ and (by Lemma 1.4.3) $\operatorname{hess}(F) = 0$. By [HMM+13, Lemma 3.74], F can be taken to be the function $x \mapsto x^N$, via the isomorphism $R^N \simeq \mathbb{K}$.

By considering this function, one can observe that ∇_F is exactly the map $\psi : \mathbb{P}(R^1) \dashrightarrow \mathbb{P}(R^{N-1})$ introduced in Section 1.3, i.e. the map such that $\psi([x]) = [x^{N-1}]$ for $[x] \in \mathbb{P}(R^1) \setminus \mathcal{N}_{N-1}$, so our variety Z coincides with the variety Z' introduced above. If $W = V(g)$ is an hypersurface containing Z, then the Gordan-Noether map ψ_g defined above is the composition $\nabla_g \circ \nabla_F = \nabla_g \circ \psi$. Since the image of ∇_g lives in $\mathbb{P}(R^{N-1})^* \simeq \mathbb{P}(R^1)$ we interpret ψ_g as a (rational) map from $\mathbb{P}(R^1)$ to $\mathbb{P}(R^1)$.

Observe that the image of $\psi_g = \nabla_g \circ \psi$ is contained in Y. Indeed, if $[x] \in \mathbb{P}(R^1)$ is general, we can assume that $\psi([x]) = [z] = [x^{N-1}]$ is smooth in Z (and so for $W = V(g)$). By definition, and since W

is an hypersurface, $\nabla_g([z])$ is the point on $\mathbb{P}(R^1)$ corresponding to $T_{[z]}(W)$. Since $T_{[z]}(Z) \subseteq T_{[z]}(W)$ we have that $\nabla_g([z]) \in Y$ as claimed.

Proposition 1.4.5. *In this setting, the Gordan-Noether identity - Equation (1.8) - follows from the Gorenstein-Gordan-Noether identity - Equation (1.3).*

Proof. Let $[x] \in \mathbb{P}(R^1)$ be a general point. Then we can assume that $\psi([x]) = [x^{N-1}] = [z]$ is a smooth point for Z. Set $[y] = \psi_g([x])$ and notice that $[y] \in Y$, since we have seen that the image of ψ_g lies in Y. We claim that $([x],[y]) \in \Theta$. Indeed, $[y] \in Y \subset \mathbb{P}(R^1)$ corresponds to an hyperplane H_y of $\mathbb{P}(R^{N-1})$ tangent to $V(g)$ containing the (projective) tangent space $T_{[z]}(Z) = \mathbb{P}(x^{N-2} \cdot R^1)$ by construction. This implies that y annihilates the vector space $x^{N-2} \cdot R^1$. Since $(x^{N-2}y) \cdot R^1 = 0$, by Gorenstein duality we have $x^{N-2}y = 0$ so $([x],[y]) \in \Gamma_{N-2}$. Since $[x]$ was general and since Θ is the only component of Γ_{N-2} dominating $\mathbb{P}(R^1)$ via π_1, we have that $([x],[y]) \in \Theta$. Then, by the Gorenstein-Gordan-Noether identity (i.e. Equation (1.3)) we have

$$\psi_g([x] + \lambda\psi_g([x])) = \psi_g([x+\lambda y]) = \nabla_g(\psi([x+\lambda y])) = \nabla_g(\psi([x])) = \psi_g([x])$$

as claimed. □

1.5 The analysis of the Perazzo cubic threefold

In this last section of the first chapter, we briefly study the Perazzo cubic threefold $V(f) \subset \mathbb{P}^4$ and, in particular, the standard Artinian Gorenstein algebra defined as $A = D/\operatorname{Ann}_D(f)$. For this section we will set $\mathbb{K} = \mathbb{C}$.

The Perazzo cubic (introduced by Perazzo in [Per00]) is the cubic threefold $X = V(f)$ with

$$f = x_0x_3^2 + 2x_1x_3x_4 + x_2x_4^2 \in \mathbb{C}[x_0, x_1, x_2, x_3, x_4]$$

and it is the first counterexample to Hesse's claim 1.1.20: up to projective transformations, it is the only cubic threefold with vanishing hessian in $\mathbb{P}^4$ which is not a cone. This follows from the work of several authors which obtain a classification of the hypersurfaces in $\mathbb{P}^4$ with vanishing hessian that are not cones. A comprehensive treatment of this problem can be found in [Rus16, Chapter 7.4] whereas the original articles dealing with this classification problem (also in higher dimension) are [GN76, Per00, Fra54, Per57, Per64, Los04, CRS08, GR09].

Fix the notations as in Example 1.1.4 with $n = 4$ and let f be the above cubic form. Then

$$A = D/\operatorname{Ann}_D(f) = A^0 \oplus A^1 \oplus A^2 \oplus A^3$$

is a SAGA with codimension 5 and socle in degree $N = \deg(f) = 3$. As recalled in Section 1.4 (see Lemma 1.4.3), since X is not a cone and its hessian vanishes, A does not satisfy SLP (and WLP as well). Notice that, in this case, since A has socle in degree 3, the whole strong Lefschetz property coincides with the SLP in degree 1.

By recalling that $y_i = \frac{\partial}{\partial x_i}$, one has that

$$(\operatorname{Ann}_D(f))_2 = \langle y_0^2, y_0y_1, y_0y_2, y_0y_4, y_1^2, y_1y_2, y_2^2, y_2y_3, y_0y_3 - y_1y_4, y_1y_3 - y_2y_4\rangle \simeq \mathbb{K}^{10}$$

and, moreover, $\{y_0y_3^2, y_1y_3y_4, y_2y_4^2\}$ are the only monomials of degree 3 which are not 0 in A. More precisely, using the above relations, one has $A^3 = \langle\sigma\rangle$ where $\sigma = y_0y_3^2 = y_1y_3y_4 = y_2y_4^2$. From these relations, one has that

$$B_1 = \{b_i\}_{i=1}^5 = \{y_0, y_1, y_2, y_3, y_4\} \quad \text{and} \quad B_2 = \{c_i\}_{i=1}^5 = \{y_3^2, y_3y_4, y_4^2, y_0y_3, y_2y_4\}$$

are basis for A^1 and A^2 respectively. Moreover, it is easy to check that $b_i \cdot c_j = \delta_{ij}\sigma$ so that B_2 is the dual basis of B_1 (by choosing the isomorphism $\mathbb{K} \to A^3$ such that $1 \mapsto \sigma$). Denote by $\{w_i\}_{i=1}^5$ and by $\{z_i\}_{i=1}^5$ the coordinates induced by B_1 and B_2 on A^1 and A^2 respectively and by τ the involution $\tau([x],[y]) = ([y],[x])$, introduced in Lemma 1.2.8. With these notations, we have that $\Gamma_{N-2} = \Gamma_1 = \{([x],[y]) \,|\, xy = 0\} \subseteq \mathbb{P}(A^1) \times \mathbb{P}(A^1)$ has 3 irreducible components, $\Theta, \tau(\Theta)$ and Λ, all of dimension 4. Using coordinates w_{1i} and w_{2i} on the two factors of $\mathbb{P}(A^1) \times \mathbb{P}(A^1)$, we have

$$\Theta = V(w_{13}w_{20} + w_{14}w_{21}, w_{13}w_{21} + w_{14}w_{22}, w_{21}^2 - w_{20}w_{22}, w_{23}, w_{24}) \text{ and } \Lambda = V(w_{13}, w_{14}, w_{23}, w_{24})$$

so $Y = V(w_1^2 - w_0w_2, w_3, w_4)$ is a conic. In particular, for $[y] \in Y$ general, we have $\dim(F_y)) = 3$. The morphism $\varphi(x) = x^2$ can be written in coordinates as

$$\underline{z} = \tilde{\varphi}(\underline{w}) = (w_3^2, 2w_3w_4, w_4^2, 2(w_0w_3 + w_1w_4), 2(w_1w3 + w_2w_4))$$

so $\mathcal{N}_2 = V(w_3, w_4) \simeq \mathbb{P}^2$ is the plane containing the conic Y - here we have taken the reduced structure - and $Z = V(4z_0z_2 - z_1^2)$ is a cone over a conic with vertex the line $V(z_0, z_1, z_2)$. The polar map ∇_Z associate to Z is

$$\nabla_Z : [\underline{z}] \mapsto [\underline{w}] = [4z_2 : -2z_1 : 4z_0 : 0 : 0]$$

and has image Y. The Gordan-Noether map ψ_g associated to $g = 4z_0z_2 - z_1^2$ can be written in coordinates as

$$\psi_g(\underline{w}) = [2w_4^2 : -2w_3w_4 : 2w_4^2 : 0 : 0],$$

it is defined outside $\mathcal{N}_2$ and it defines a rational map from $\mathbb{P}^4$ to Y. Finally, one can check that $\Theta \cap \tau(\Theta) = Y \times Y$ and

$$\Lambda = \mathcal{N}_2 \times \mathcal{N}_2 \qquad \Theta \cap \Lambda = \mathcal{N}_2 \times Y \qquad \tau(\Theta) \cap \Lambda = Y \times \mathcal{N}_2$$

so $\Theta \cap \tau(\Theta) \cap \Lambda = Y \times Y$.

Chapter 2

Complete intersection SAGAs presented by quadrics

In this second chapter, we exploit the framework introduced in Chapter 1 and the construction described in Sections 1.2 and 1.3, used to prove Theorem B. Here, we specialize this setting to complete intersection SAGAs presented by homogeneous polynomials of the same degree, in particular presented by quadrics, and their Lefschetz properties. In other words, we will analyze the validity of some Lefschetz properties for a SAGA R obtained as the quotient of the polynomial ring $S = \mathbb{K}[x_0, \cdots, x_n]$ by an ideal $I = (g_0, \ldots, g_n)$ generated by a regular sequence of degree $d-1$ polynomials:

$$R = \frac{\mathbb{K}[x_0, \ldots, x_n]}{(g_0, \ldots, g_n)} = \bigoplus_{i=0}^{N} R^i,$$

where $\deg(g_i) = d - 1$ for every i and the zero locus $V(g_0, \cdots, g_n)$ is trivial.
In what follows, we will focus on the case where $d = 3$, i.e. on complete intersection SAGAs presented by quadrics. Let us observe that in this specific setting the codimension of R is $\dim(R^1) = n + 1$ and the socle is in degree $N = (n + 1)(d - 2) = n + 1$. Notice, moreover, that as a particular case of this kind of SAGAs we have the Jacobian ring of a smooth cubic hypersurface.

After presenting in Section 2.1 some results and properties that hold for any SAGAs or for complete intersection ones, in Section 2.2 we will focus on the case of complete intersection SAGAs presented by quadrics. In particular, we will prove some technical results, which will allow us to improve the bounds on the dimensions of the varieties arising in our construction, when we assume the failure of some Lefschetz properties. By using these results, in Section 2.3, we will prove Theorem C. In Section 2.4, we will generalize some of the previous results to SAGAs with higher codimension. In Section 2.5 we will prove a lifting criterion for the weak Lefschetz property of complete intersection SAGAs presented by quadrics. Finally, in Section 2.6, we will analyze, from a geometric point of view, the nihilpotent loci $\mathcal{N}_i$ introduced in Section 1.1: in the case of a Jacobian ring of a cubic hypersurface, it turns out that these loci can reflect some properties of the cubic hypersurface itself.
The results of this second chapter appear in [BFP22, BF22].
In this chapter, as in the previous one, we will work on an algebraically closed field $\mathbb{K}$ of characteristic 0.

2.1 Some preliminary results

In this first section, we study some loci arising in a natural way inside a general SAGA. In particular, we will analyze the nihilpotent loci $\mathcal{N}_i$ and the kernels K^i_η of suitable multiplication maps, introduced in Section 1.1. In the case of a complete intersection SAGA presented by forms of degree $d-1$, it is possible to obtain some bounds for the dimensions of these loci, which will play, when for example $d=3$, an important role in proving the technical results of the following sections.

First of all, let us recall that, as stated in Section 1.1, if $[\eta]$ is an element in $\mathbb{P}(R^h)$ we set

$$K^i_\eta = \ker\left(R^i \xrightarrow{\cdot\eta} R^{i+h}\right).$$

Moreover, for a suitable integer $i \geq 2$, we have defined the nihilpotent loci in degree 1 as

$$\mathcal{N}_i := \{[x] \in \mathbb{P}(R^1) \mid x^i = 0\}.$$

Let us focus on the case of a complete intersection SAGA $R = S/I = \oplus_{i=0}^N R^i$ presented by forms of degree $d-1$. For this specific kind of SAGAs, let us start by showing a result which gives a description of the kernels K^i_η when $i=1$, by also giving an upper bound for their dimensions:

Proposition 2.1.1. *Assume that $1 \leq h \leq N-1$ and let R be a complete intersection SAGA as above, presented by forms of degree $d-1$ and with socle in degree N. Then the following properties hold:*

(a) If $\eta \in R^h \setminus \{0\}$, then $h \geq (d-2)\dim(K^1_\eta)$;

(b) Let $\eta, \zeta \in R^h \setminus \{0\}$ and assume $h = (d-2)\dim(K^1_\eta) = (d-2)\dim(K^1_\zeta)$. Then $K^1_\eta = K^1_\zeta$ if and only if $[\eta] = [\zeta]$ in $\mathbb{P}(R^h)$.

Proof. Assume that $\dim(K^1_\eta) = k$ and chose $y_0, \dots, y_{k-1}$ linearly independent elements in K^1_η. We can find $g_k, \dots, g_n \in I^{d-1}$ such that $\tilde{I} = (y_0, \dots, y_{k-1}, g_k, \dots, g_n)$ is the irrelevant ideal (i.e. the set $\{y_0, \dots, y_{k-1}, g_k, \dots, g_n\}$ is a regular sequence). Then $\tilde{R} = S/\tilde{I}$ is a standard Gorenstein Artinian algebra with socle in degree $\tilde{N} = (d-2)(n+1-k)$. In particular, any element of S of degree at least $\tilde{N}+1$ belongs to $\tilde{I}$. We claim that $\eta \cdot R^{\tilde{N}+1} = 0$. Indeed, if $g \in S^{\tilde{N}+1}$ we have

$$\eta \cdot g = \eta \cdot \left(\sum_{i=0}^{k-1} \lambda_i y_i + \sum_{i=k}^{n} \mu_i g_i\right) \in I$$

since $y_i \in K^1_\eta$ and $g_i \in I$. This is possible, by Gorenstein duality, if and only if $\tilde{N}+h+1 > N$, i.e. if and only if $h \geq (d-2)k$ as claimed by (a).

For (b) assume that $\eta, \zeta \in R^h \setminus \{0\}$ are such that $K^1_\eta = K^1_\zeta$ and $h = (d-2)\dim(K^1_\eta)$. Then we can proceed as before and construct the ideal $\tilde{I}$ and the ring $\tilde{R}$ with socle in degree $\tilde{N} = N-h$. We claim that $K^{N-h}_\eta = K^{N-h}_\zeta$. Let $\tilde{\sigma}$ be a representant of the socle of $\tilde{R}$. Then we can write $S^{N-h} = \langle \tilde{\sigma}, \tilde{I}^{\tilde{N}} \rangle$. One can easily check that $\eta \cdot \tilde{I} \subseteq I$ and $\zeta \cdot \tilde{I} \subseteq I$. On the other hand, η, ζ are not zero so $\eta \cdot R^{N-h}$ and $\zeta \cdot R^{N-h}$ are not 0, i.e. $\eta \cdot \tilde{\sigma}, \zeta \cdot \tilde{\sigma} \neq 0$ in R. Hence, we have that $K^1_\eta = K^1_\zeta = \tilde{I}^{\tilde{N}}$ and then η and ζ are multiples. $\square$

Before proceeding, let us recall that if p is a smooth point of a variety X, we denote by $T_{X,p}$ the differential tangent space, while by $T_p(X)$ the embedded Zariski tangent space in the projective space where X lives; if $\tilde{X}$ is the affine cone over X, then $T_{\tilde{X},\tilde{p}}$ is the affine tangent space at the smooth point $\tilde{p}$ and $T_p(X) = \mathbb{P}(T_{\tilde{X},\tilde{p}})$.

As an easy application of the previous proposition 2.1.1, we have the following bound for the dimension of the nihilpotent locus $\mathcal{N}_i = \{[y] \in \mathbb{P}(R^1) \mid y^i = 0\}$.

Corollary 2.1.2. *We have*

$$\dim(\mathcal{N}_i) \leq \frac{i-1}{d-2} - 1.$$

Proof. Take the general point $[y]$ of any irreducible component C of $\mathcal{N}_i$ of maximal dimension which is not contained in $\mathcal{N}_{i-1}$. If such a component does not exist, set $\epsilon > 0$ to be the biggest integer such that $\mathcal{N}_i = \mathcal{N}_{i-\epsilon}$. The bound for $\dim(\mathcal{N}_{i-\epsilon})$ implies the one for the dimension of $\mathcal{N}_i$.

Let $\tilde{C}$ be the associated affine cone. We claim that $T_{\tilde{C},y} = K^1_{y^{i-1}}$. Indeed if v is a tangent vector to $\tilde{C}$ in y, we have a curve $\gamma(t) = y + tv + t^2(\cdots)$ which is contained in $\mathcal{N}_i$. Then, by expanding the relation $\gamma(t)^i = 0$, one has $vy^{i-1} = 0$ so $v \in T_{\tilde{C},y}$ if and only if $v \in K^1_{y^{i-1}}$. Then, by Proposition 2.1.1, we have

$$\dim(\mathcal{N}_i) = \dim(\tilde{C}) - 1 = \dim(K^1_{y^{i-1}}) - 1 \leq \frac{i-1}{d-2} - 1$$

as claimed. □

Moreover, with the same idea of the above proof up to minimal changes, one has a sort of generalisation of the previous result, that is valid for any SAGA (not necessarily complete intersection ones) and that presents a description of the tangent spaces for the nihilpotent loci:

Corollary 2.1.3. *Let R be any SAGA. Then for $[\eta] \in \mathcal{N}^{(a)}_k$ general we have $T_{[\eta]}(\mathcal{N}^{(a)}_k) \subseteq \mathbb{P}(K^a_{\eta^{k-1}})$. If, moreover, $\eta^{k-1} \neq 0$, we have an equality.*

As a consequence of this preliminary discussion, we have a new proof of the following result of Migliore and Nagel ([MN13b, Proposition 4.3]).

Corollary 2.1.4. *Let $R = S/I$ be a standard Artinian Gorenstein algebra with I generated by a regular sequence of polynomials of degree e with $e \geq 2$. Then R has the weak Lefschetz property in degree 1.*

Proof. The result is clear if $e \geq 3$ since, in this case, $R^1 = S^1$ and $R^2 = S^2$. If $e = 2$ one can consider the incidence variety $\Gamma_1 = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \mid xy = 0\}$ introduced in Section 1.2 and its projection p_1 on $\mathbb{P}(R^1)$. By contradiction, assume that the weak Lefschetz property does not hold in degree 1. This is equivalent to ask that p_1 is surjective. Proceeding as in Section 1.2 one has that there exists a unique irreducible component Θ of Γ_1 that dominates $\mathbb{P}(R^1)$ via first projection. Moreover we have $Y = \pi_2(\Theta) \subseteq \mathcal{N}_2$ and $\dim(Y) \geq 1$ (proceeding as in Proposition 1.2.7) so $\dim(\mathcal{N}_2) \geq 1$. On the other hand, by Corollary 2.1.2 we have $\dim(\mathcal{N}_2) \leq 0$, which gives a contradiction. □

To conclude this section, let us focus on the case that will be treated in what follows: let us assume that R is a complete intersection SAGA presented by forms of degree $d-1 = 2$ and with socle in degree

$N = n+1$. For this specific case, let us make the point on what we can say about the dimensions of nihilpotent loci and kernels presented above:

$$(1) \text{ If } \eta \in R^h \setminus \{0\}, \text{ for } 1 \leq h \leq N-1, \text{ then } \dim(K^1_\eta) \leq h \qquad (2) \dim(\mathcal{N}_i) \leq i-2. \tag{2.1}$$

2.2 Technical results: new dimension bounds

In this section we consider a SAGA R of codimension $\dim(R^1) = n+1$ and socle in degree N. Moreover, in addition to what we have done in the previous Section 2.1, we will also assume that R is a SAGA which does not satisfy the strong Lefschetz property in degree 1 at range k, i.e. $SLP_1(k)$, with $2 \leq k \leq N-2$. Equivalently, the multiplication map $x^k \cdot : R^1 \to R^{k+1}$ is never injective. Hence, we are in the situation described more generally in Section 1.2: let us recall that under the above assumptions we have

$$\begin{tikzcd}
F_y \times [y] \arrow[rrr, "\pi_2"] \arrow[dr, hook] \arrow[ddd, "\simeq"'] & & & {[y]} \arrow[d, hook] \\
& \Theta \arrow[rr, two heads, "\pi_2"] \arrow[dr, hook] \arrow[ddr, bend right, "\pi_1"'] & & Y \arrow[d, hook] \\
& & \Gamma_k \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}(R^1) \\
F_y \arrow[rr, hook] & & \mathbb{P}(R^1) &
\end{tikzcd} \tag{2.2}$$

where we have set $\Gamma_k := \Gamma^{(1,1)}_{k,1} = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \mid x^k y = 0\}$. We recall that Θ is the unique irreducible component of Γ_k that dominates $\mathbb{P}(R^1)$ via its first projection π_1, $Y = \pi_2(\Theta)$ and $F_y = \pi_1(\pi_2^{-1}([y]))$ for $[y] \in Y$.

Remark 2.2.1. *We recall that all complete intersection SAGAs presented in degree $d-1$ satisfy $SLP_1(1) = WLP_1$ (see* [MN13b, Proposition 4.3] *and 2.1.4). For this reason in this section we do not consider $k=1$ since in this case it is possible to construct Γ_1, but p_1 is never dominant (so Θ, Y and F_y cannot be constructed).*

First of all, let us show a technical result that holds for any SAGA when we deny the $SLP_1(k)$ for some suitable k. The following lemma states properties, that will be used later, concerning the nature of the tangent spaces to the varieties arising from denying some strong Lefschetz properties.

Lemma 2.2.2. *With notations as above, if $p = ([x],[y]) \in \Theta$ is a general point, we have:*

(a) $[y] \in T_{F_y,[x]}$ and $[x] \notin T_{Y,[y]}$;

(b) $T_{\tilde{F}_y,x} \subseteq K^1_{x^\alpha y^\beta}$ whenever $\alpha + \beta = k$ and $\beta \geq 1$, where $\tilde{F}_y$ denotes the affine cone over F_y.

Proof. For (a), let $p = ([x],[y]) \in \Theta$ be a general point. By Proposition 1.2.7 we have that F_y is a cone and $[y]$ is a vertex for it, so that the line $\langle [x],[y] \rangle$ is contained in F_y. This means that $[y]$ is a tangent vector in $[x]$, i.e. $[y] \in T_{F_y,[x]}$.

For the second claim of (a), let us suppose by contradiction that $[x] \in T_{Y,[y]}$, where we can assume that $x^{k+1} \neq 0$, by the generality of p. Let $\tilde{\Theta}$ be the lifting to $R^1 \times R^1$ of $\Theta \subset \mathbb{P}(R^1) \times \mathbb{P}(R^1)$ and let $\tilde{\pi}_2$

be the projection on the second factor from $\tilde{\Theta}$. By construction, we have that $\tilde{\Theta} \subseteq \{(x, y) \,|\, x^k y = 0\}$. Let $\tilde{Y}$ be the affine cone of Y. Since $\pi_2 : \Theta \to Y$ is surjective, we have $\tilde{\pi}_2(\tilde{\Theta}) = \tilde{Y}$ and that, for $p = (x, y) \in \tilde{\Theta}$ general,

$$d_p\tilde{\pi}_2 : T_{\tilde{\Theta},p} \to T_{\tilde{Y},y}$$

is surjective.

By assumption we have also that $x \in T_{\tilde{Y},y}$ so there exists a tangent vector to $\tilde{\Theta}$ at p of the form (v, x). Since points of $\tilde{\Theta}$ satisfy $x^k y = 0$ we have

$$0 \equiv (x + tv + t^2(\cdots))^k(y + tx + t^2(\cdots)) \pmod{t^2}$$

which yields $x^{k+1} = 0$. But since this is impossible by the generality of x, we obtain that $[x] \notin T_{Y,[y]}$.

For (b), first of all, notice that by Proposition 1.2.3 we have $\tilde{F}_y \subseteq \{x \in R^1 \,|\, x^i y^j = 0\}$ for all $i + j = k + 1$ and $i, j \geq 1$. Hence, for p general, if an element $v \in R^1$ belongs to $T_{\tilde{F}_y,x}$ then the following relation must be satisfied

$$0 \equiv (x + tv + t^2(\cdots))^i y^j = itvx^{i-1}y^j + t^2(\cdots) \pmod{t^2}.$$

Hence, we have that $v \in K^1_{x^\alpha y^\beta}$, with $\alpha + \beta = k$ and $\beta \geq 1$. □

Let us now set $R = \oplus_{i=0}^N R^i$ as a complete intersection SAGA presented by forms of degree $d - 1$, with codimension $n + 1$ and socle in degree $N = (n + 1)(d - 2)$. Now we will prove some results giving restrictions on the dimensions of Y and of the general fiber F_y with $[y] \in Y$. We are ultimately interested into the case where $d = 3$; nevertheless, let us stress that the following proposition (2.2.3) holds for every $d \geq 3$.

Proposition 2.2.3. *If we assume $n > \frac{k}{d-2}$, then*

$$\dim(F_y) \leq n - 2.$$

In particular, if $d = 3$, then F_y cannot be an hypersurface.

Proof. Recall that $\dim(F_y) \leq n - 1$ by Proposition 1.2.7(d) so we have to rule out only the case $\dim(F_y) = n - 1$.

Let us assume, by contradiction, that F_y is an hypersurface. Hence, by denoting with $\tilde{F}_y$ the affine cone over F_y, we have $\dim(\tilde{F}_y) = n$.

Recall that $\Theta =: \Theta_k \subseteq \Gamma_k = \{([x], [y]) \,|\, x^k y = 0\}$ by assumption. We will show that the multiplication map $x^{k-1} \cdot : R^1 \to R^k$ is never injective so we can define, as we have done for Θ_k, an incidence correspondence Γ_{k-1} with a unique irreducible component Θ_{k-1} which dominates $\mathbb{P}(R^1)$ via its first projection. Moreover, we will have $\Theta_k = \Theta_{k-1}$ so F_y is also the fiber of the second projection from Θ_{k-1} and we can iterate this process.

We claim now that $\Theta \subseteq \Gamma_{k-1}$. If $p = ([x], [y]) \in \Theta$ is general, by using Lemma 2.2.2 and Proposition 2.1.1 we can conclude

$$n = \dim(T_{\tilde{F}_y,x}) \leq \dim(K^1_{x^{k-1}y}) \leq \frac{k}{d-2}$$

unless $x^{k-1}y = 0$. Since, by hypothesis, we have that $n > k/(d-2)$, the only possibility is that $x^{k-1}y = 0$. In particular we have shown that $\Theta \subseteq \{([x],[y]) \mid x^{k-1}y = 0\} = \Gamma_{k-1}$ as claimed.

Then, Θ is contained in Θ_{k-1} since it dominates $\mathbb{P}(R^1)$. On the other hand, since $\Gamma_{k-1} \subseteq \Gamma_k$, we have also the other inclusion: $\Theta = \Theta_{k-1}$. In particular, the varieties Y and F_y defined for Θ are the same as the ones defined for Θ_{k-1}. Then, by reasoning as before, we obtain $n = \dim(T_{\tilde{F}_y,x}) \leq \dim(K^1_{x^{k-2}y})$. If we assume that $x^{k-2}y \neq 0$ for p general in Θ, by Proposition 2.1.1 we would obtain $n \leq (k-1)/(d-2) \leq k/(d-2)$ which is, as before, incompatible with the hypothesis on n. Then $\Theta \subseteq \Gamma_{k-2}$ and we can iterate this process.

By recursion, we reduce ourselves to the case with $k = 1$. We can then see Θ as a subvariety of Γ_1 which dominates $\mathbb{P}(R^1)$ via its first projection. This implies the failure of the weak Lefschetz Property in degree 1. Then, by Remark 2.2.1, we get a contradiction: F_y has dimension at most $n-2$, as claimed.

For the second statement, let us notice that if $d = 3$, the required condition comes to be $n > k$, which is always satisfied since $n = N-1$ and k is at most $N-2$ by construction. Hence, we always have that the dimension of the general fiber F_y is at most $n-2$. □

Let us observe that, from the above Proposition 2.2.3 and from Proposition 1.2.7, we automatically obtain also a new lower bound for the dimension of Y, when the condition in the hypotheses is satisfied. In particular, in the case where $d = 3$, we get that $2 \leq \dim(Y)$.
Let us now show another result, which gives a new upper bound for the dimension of the variety Y, when $d = 3$.

Proposition 2.2.4. *Assume that $d = 3$. Then, the dimension of Y is at most $n-3$.*

Proof. First of all, let us notice that if $k \leq N-3 = n-2$, then by Proposition 1.2.7(a), Y is contained in $\mathcal{N}_{k+1} \subseteq \mathcal{N}_{n-1}$, whose dimension is at most $n-3$ (see Corollary 2.1.2 and, in particular, Properties (2.1)). Hence, we easily get that in this case $\dim(Y) \leq n-3$ as claimed.

Let us now consider the remaining case: $k = N-2 = n-1$. Recall that $\dim(Y) \leq n-2$ by Corollary 1.3.7 so, to conclude the proof, we only have to rule out the case where $\dim(Y) = n-2$. Assume by contradiction that $\dim(Y) = n-2$. Since $k = n-1$, proceeding as above, we have that Y is contained in $\mathcal{N}_n$, which has dimension at most $n-2$. Hence, we get that Y is a component of $\mathcal{N}_n$. Then, if $[y] \in Y$ is a general point ($y^{n-1} \neq 0$, since $Y \not\subseteq \mathcal{N}_{n-1}$, for dimension reasons), we can write $T_y(Y) = T_y(\mathcal{N}_n) = \mathbb{P}(K^1_{y^{n-1}})$ by Lemma 2.1.3. However, by Proposition 1.2.3 we know that $[x] \in F_y$ belongs to $\mathbb{P}(K^1_{y^{n-1}})$ and then to $T_y(Y)$, contradicting Lemma 2.2.2(a). □

Let us now conclude this section with another technical results for complete intersection SAGAs presented by quadrics, which links the dimension of the general fiber F_y to the "nihilpotent order" of the variety Y, with which we mean the minimum integer i such that $Y \subseteq \mathcal{N}_i$.

Proposition 2.2.5. *Assume that $d = 3$. Then, the following conditions are not compatible:*

(a) for $y \in Y$ general, $\dim(F_y) = k-1$;

(b) $Y \not\subseteq \mathcal{N}_{k-1}$.

Proof. Let us recall that by construction $2 \leq k \leq N-2$ and that by Proposition 1.2.3 we have that $x^\alpha y^\beta = 0$ for every α, β such that $\alpha + \beta = k+1$ and $\beta \geq 1$; in particular (Proposition 1.2.7(a)) we have $y^{k+1} = 0$ and $Y \subseteq \mathcal{N}_{k+1}$.

Let us assume by contradiction that both conditions (a) and (b) hold. By (b) we have $y^{k-1} \neq 0$ for $y \in Y$ general so for $([x],[y])$ general in Θ we have $xy^{k-1} \neq 0$. Indeed, otherwise, F_y would be contained in $\mathbb{P}(K^1_{y^{k-1}})$, whose dimension is at most $k-2$ (see Properties (2.1)), which is impossible by assumption. As a consequence, we have that

$$\text{for} \quad ([x],[y]) \in \Theta \quad \text{general}, \quad x^\alpha y^\beta \neq 0 \quad \text{for} \quad \alpha + \beta = k \quad \text{with} \quad \alpha, \beta \geq 1 \tag{2.3}$$

since, otherwise, by using the same argument as the one in the proof of Proposition 1.2.3, we would also obtain that $xy^{k-1} = 0$.

By property (2.3) and since for y general $\dim(F_y) = k-1$ by assumption, we also have

$$T_x(F_y) = \mathbb{P}(K^1_{x^{k-1}y}) = \mathbb{P}(K^1_{x^{k-2}y^2}) \tag{2.4}$$

by Lemma 2.2.2.

Let us now claim that

$$\text{for} \quad ([x],[y]) \in \Theta \quad \text{general}, \quad T_y(Y) \subseteq T_x(F_y). \tag{2.5}$$

To show this, first of all, recall that $Y \not\subseteq \mathcal{N}_{k-1}$ and $Y \subseteq \mathcal{N}_{k+1}$. Let us now consider two cases:

$$1) \quad Y \not\subseteq \mathcal{N}_k \qquad \text{and} \qquad 2) \quad Y \subseteq \mathcal{N}_k.$$

Assume that $p = ([x],[y]) \in \Theta$ is general (so that $[y]$ is general in Y and $[x]$ is general in F_y). In the first case, since Y is contained in $\mathcal{N}_{k+1}$, we have $T_y(Y) \subseteq \mathbb{P}(K^1_{y^k})$ by Lemma 2.1.3. Moreover, we have that $\mathbb{P}(K^1_{y^k}) = T_x(F_y)$ since $y^k \neq 0$ and $\dim(F_y) = k-1$. Analogously, for the second case we have $T_y(Y) \subseteq \mathbb{P}(K^1_{y^{k-1}}) \subseteq \mathbb{P}(K^1_{xy^{k-1}}) = T_x(F_y)$. Here, we have used that $xy^{k-1} \neq 0$ since p is general (by property (2.3)).

Consider, as in Lemma 2.2.2, the affine cone $\tilde{Y}$ of Y, the lifting $\tilde{\Theta}$ of $\Theta \subset \mathbb{P}(R^1) \times \mathbb{P}(R^1)$ to $R^1 \times R^1$ and its projection $\tilde{\pi}_2$ on the second factor. By construction, we have that $\tilde{\Theta} \subseteq \{(x,y) \,|\, x^\alpha y^\beta = 0\} = \tilde{\Gamma}_{\alpha,\beta}$ whenever $\alpha + \beta = k+1$ and $\beta \geq 1$. As in Lemma 2.2.2, for $p = (x,y) \in \tilde{\Theta}$ general, the differential map

$$d_p\tilde{\pi}_2 : T_{\tilde{\Theta},p} \to T_{\tilde{Y},y}$$

is surjective.

Let us take any w in $T_{\tilde{Y},y}$. By Property (2.5), we have that its class $[w]$ belongs to $T_x(F_y)$. Moreover, by the surjectivity of $d_p\tilde{\pi}_2$, we can take in $T_{\tilde{\Theta},p}$ an element of the form (v,w).

The tangent space to $\tilde{\Theta}$ in p is a subspace of the tangent space $T_{\alpha,\beta} = T_{\tilde{\Gamma}_{\alpha,\beta},p}$ to the locus $\tilde{\Gamma}_{\alpha,\beta}$ in p, so we have $(v,w) \in T_{\alpha,\beta}$. Then, we have

$$0 \equiv (x + tv + t^2(\cdots))^\alpha (y + tw + t^2(\cdots))^\beta \pmod{t^2}$$

for α, β as above. In particular, by taking (α, β) equal to $(k-1, 2)$ and $(k, 1)$ we obtain the following relations satisfied by (v, w):

$$(k-1)vx^{k-2}y^2 + 2x^{k-1}yw = 0 \qquad kvx^{k-1}y + x^k w = 0.$$

Since $[w] \in T_x(F_y)$, by property (2.4), we have $x^{k-1}yw = 0$. Then, from the first equation we get $v \in K^1_{x^{k-2}y^2} = K^1_{x^{k-1}y}$ (again by property (2.4)). Hence the second equation yields $x^k w = 0$. In conclusion, we have proved that $T_y(Y) \subset \mathbb{P}(K^1_{x^k}) = \pi_1^{-1}([x])$. We stress that the last equality holds since $[x]$ is general and then, the whole fiber over $[x]$ with respect to p_1 is contained in Θ so $\mathbb{P}(K^1_{x^k}) = p_1^{-1}([x]) = \pi_1^{-1}([x])$.

This easily brings to a contradiction. Indeed, the above property implies that $\dim(Y) \leq \dim(\pi_1^{-1}([x]))$ for $[x] \in \mathbb{P}(R^1)$ general, and since

$$\dim(\Theta) = \dim(Y) + \dim(F_y) = \dim(\pi_1^{-1}([x])) + n$$

we also get $\dim(Y) \leq \dim(Y) + \dim(F_y) - n$, which is impossible by Proposition 1.2.7. □

2.3 Proof of Theorem C

In this section we will prove Theorem C from the Introduction (see Theorems 2.3.2 and 2.3.4). In particular, we will show that a complete intersection SAGA presented by quadrics satisfies the whole strong Lefschetz property (both in degree 1 and in degree 2) if it has codimension 5 and, in the case of codimension 6, it satisfies the strong Lefschetz property in degree 1. (Observe that here we are using the definition of strong Lefschetz property in narrow sense (see definition 1.1.6).)

Let us firstly consider the former case, the one with codimension $n + 1 = 5$. We have thus $n = 4$ and $d = 3$: in particular, we are dealing with standard Artinian Gorenstein algebras which are quotients of $S = \mathbb{K}[x_0, \cdots, x_4]$ by ideals generated by a regular sequence of length 5 whose elements have degree 2. Under these assumptions we have $I = (I^2)$, $N = 5$ and

$$R = S/I = R^0 \oplus R^1 \oplus R^2 \oplus R^3 \oplus R^4 \oplus R^5$$

with $(\dim(R^i))_{i=0}^5 = (1, 5, 10, 10, 5, 1)$. For simplicity, if $\alpha \in R^c$, we will define by $\mu_i(\alpha)$ to be the multiplication map by α from R^i to R^{i+c}. In particular we have $K^i_\alpha = \ker(\mu_i(\alpha))$.

Let us start with the following technical result we will need in what follows.

Proposition 2.3.1. *Let $[x] \in \mathbb{P}(R^1)$ and $[q] \in \mathbb{P}(R^2)$ such that $qx = 0$. Let $W \subset K^2_q$ be a subspace with $\dim(W) \geq 4$. Then $W \cap (x \cdot R^1) \neq \{0\}$.*

Proof. Consider the quotient $R_q = R/(0 : q)$, i.e. the quotient of R by the ideal J such that $J^i = K^i_q$. This is a SAGA with socle in degree $N_q = N - \deg(q) = 3$ by Lemma 1.1.9. Since $xq = 0$ by hypothesis, we have $K^1_q \neq 0$. By Proposition 2.1.1 we have $\dim(K^1_q) \leq 2$ and so $\dim(R^1_q) \in \{3, 4\}$. Since $\dim(K^2_q) = \dim(R^2) - \dim(R^2_q) = 10 - \dim(R^1_q)$ we have that $\dim(K^2_q) \in \{6, 7\}$. In particular, $\dim(K^2_q) \leq 7$. Consider $W \subseteq K^2_q$ of dimension 4 and the subspace $V = x \cdot R^1$. By Proposition 2.1.1 we have that V has dimension at least $5 - 1 = 4$ and, by construction, is a subspace of K^2_q. Then $W \cap V$ has dimension at least 1 as claimed. □

Let us now prove the first part of Theorem.

Theorem 2.3.2 (Theorem C)**.** *Let R be as above. Then R satisfies the strong Lefschetz property, i.e. the general element $x \in R^1$ is such that*

(SLP_1) $\mu_1(x^3) = x^3 \cdot : R^1 \to R^4$

(SLP_2) $\mu_2(x) = x \cdot : R^2 \to R^3$

are both isomorphisms.

Proof. The proof is organized in two steps: first of all we will prove that SLP_1 implies SLP_2 and then that SLP_1 holds.

Step 1: $SLP_1 \implies SLP_2$. We will proceed by contradiction by assuming that SLP_2 is false, i.e. that for all $x \in R^1$ we have $K_x^2 \neq \{0\}$. We can consider the incidence variety

$$\Gamma^{(1,2)} := \Gamma^{(1,2)}_{1,1} = \{([x],[q]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^2) \,|\, xq = 0\}$$

and its projections p_1 and p_2 (see the general construction in Section 1.2). Since SLP_2 does not hold we have that p_1 is dominant. Hence, as done in Section 1.2 one has that there exists a unique irreducible component Θ of $\Gamma^{1,2}$ that dominates $\mathbb{P}(R^1)$ via the first projection. Again, let us call π_i the restrictions to Θ of such projections p_i and let Y be the image of Θ via π_2. Since π_1 is dominant and $\mathbb{P}(R^1) \simeq \mathbb{P}^4$, we have that Θ has dimension at least 4. Moreover, if $[q] \in \mathbb{P}(R^2)$, we have $\pi_2^{-1}([q]) \subseteq \mathbb{P}(K_q^1) \times [q]$ so its dimension is at most 1 by Proposition 2.1.1. Then, for $[q] \in Y$ general,

$$\dim(Y) = \dim(\Theta) - \dim(\Theta \cap \pi_2^{-1}([q])) \geq 4 - 1 = 3.$$

We claim now that $Y \subseteq \mathcal{N}_2^{(2)} := \{[q] \in \mathbb{P}(R^2) \,|\, q^2 = 0\}$. Let $([x],[q])$ be a generic point in Θ. Proceeding as in Proposition 1.2.3, since $\pi_1 : \Theta \to \mathbb{P}(R^1)$ is dominant, for any $v \in R^1$, we can find $\beta(t) = q + tw + t^2(\cdots) \in Y \subset \mathbb{P}(R^2)$ such that $(x+tv)\beta(t) = 0$. Then, by considering the expansion of this relation modulo t^2 we obtain

$$xw + qv = 0 \text{ for all } v \in R^1. \tag{2.6}$$

Then, by multiplying by q, one gets $q^2 v = 0$ for all $v \in R^1$. By Gorenstein duality we have $q^2 = 0$ so $Y \subseteq \mathcal{N}_2^{(2)}$ as claimed.

Since $p = ([x],[q])$ was general, we can also assume that $[q]$ is smooth for Y and $\mathcal{N}_2^{(2)}$ and that the differential $d_p\pi_2 : T_{\Theta,p} \to T_{Y,q}$ is surjective. Then, as in Corollary 2.1.2, one can show that the Zarisky tangent space to the affine cone $\tilde{\mathcal{N}}_2^{(2)}$ of $\mathcal{N}_2^{(2)}$ at q is $T_{\tilde{\mathcal{N}}_2^{(2)},q} \simeq K_q^2$. Since $\dim(Y) \geq 3$ and $Y \subseteq \mathcal{N}_2^{(2)}$ we can find three tangent vectors w_1, w_2, w_3 such that $W = \langle w_1, w_2, w_3, q\rangle$ is a 4-dimensional subspace of $T_{\tilde{Y},q} \subseteq K_q^2$, where $\tilde{Y}$ is the affine cone of Y. Then, by Proposition 2.3.1, we have $W \cap (x \cdot R^1) \neq \{0\}$ so we can find $\eta \in R^1 \setminus \{0\}$ such that $x\eta \in W$. Notice that $x\eta$ cannot be equal to q since, otherwise, we would have that for $[x] \in \mathbb{P}(R^1)$ general $0 = xq = x^2\eta$ and then $x^3\eta = 0$: this is impossible since we are assuming SLP_1. Then $x\eta$ is not 0 as tangent vector in $T_{Y,q}$. By the surjectivity of the differential map $d_p\pi_2$, there exists $v \in R^1$ such that

$$(v, x\eta) \in T_{\Theta,p} \subseteq T_{\mathbb{P}(R^1),[x]} \times T_{\mathbb{P}(R^2),[q]}$$

so there is a curve $(\alpha(t), \beta(t)) \subseteq \Theta$ passing through p with tangent vector $(v, x\eta)$. By expanding at the first order one gets a relation as the one in Equation (2.6): $x^2\eta + qv = 0$. If we multiply by x we have $x^3\eta = 0$. However, this is only possible for x special since we are assuming SLP_1 and so it leads to a contradiction.

Step 2: SLP_1 holds. Assume, by contradiction, that SLP_1 does not hold. Then, as in the construction described in Section 1.2, we can consider $\Gamma_{N-2} = \Gamma_3 = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \,|\, x^3y = 0\}$ which dominates $\mathbb{P}(R^1)$ via its projection p_1. Let us consider $\Theta \subseteq \Gamma_3$, the unique irreducible component that dominates $\mathbb{P}(R^1)$ via the first projection, let $\pi_i = p_{i|\Theta}$, $Y = \pi_2(\Theta)$ and F_y the first projection of the fiber through π_2 over the point $y \in Y$. By Proposition 1.2.7, we have some immediate restrictions to the values of the dimensions of Y and of the general fiber F_y:

$$1 \leq \dim(Y) \leq 3 \qquad 1 \leq \dim(F_y) \leq 3 \qquad \dim(Y) + \dim(F_y) = \dim(\Theta) \geq 4.$$

Moreover, by Corollary 1.3.7 of Section 1.3 (observe that here we have $k = N-2$), we can improve the above bounds, obtaining

$$1 \leq \dim(Y) \leq 2 \qquad 2 \leq \dim(F_y) \leq 3 \qquad \dim(Y) + \dim(F_y) = \dim(\Theta) \geq 4. \tag{2.7}$$

Let us now list in the following table the possible values of the pairs $(\dim(Y), \dim(F_y))$, for $y \in Y$ general point:

$\dim(Y)$ VS $\dim(F_y)$	2	3
1	(2.7)	??
2	??	??

While the case $(\dim(Y), \dim(F_y)) = (1,2)$ has been already ruled out by inequalities 2.7, we still have three possibilities that could occur. However, let us observe that by Proposition 2.2.3 we know that the general fiber F_y can not be a hypersurface in $\mathbb{P}(R^1) \simeq \mathbb{P}^4$, hence it can not have dimension 3. Let us analyse the remaining case, namely $\dim(Y) = \dim(F_y) = 2$. But by Proposition 2.2.4, we have that $\dim(Y) \neq n - 2 = 2$.

In other words, we can get rid of all the possibilities in the above table, by using the results proved in the previous sections:

$\dim(Y)$ VS $\dim(F_y)$	2	3
1	(2.7)	Prop. 2.2.3
2	Prop. 2.2.4	Prop. 2.2.3

Hence, there are no possibilities for Y and F_y to exists: this means that the first projection $\pi_1 : \Gamma_3 \dashrightarrow \mathbb{P}(R^1)$ cannot be surjective, and so, equivalently (see Remark 1.2.1), that the strong Lefschetz property in degree 1 holds.

$\square$

Since Jacobian rings of smooth cubic threefolds represent a special case of complete intersection SAGAs presented by quadrics with codimension 5, we have the following obvious but important consequence:

Corollary 2.3.3. *The Jacobian ring of a smooth cubic threefold satisfies the strong Lefschetz property.*

With the same idea of strategy we have followed to prove the previous theorem, we can deal with the case of complete intersection SAGAs presented by quadrics of codimension 6. In this case we have $n = 5$ and $d = 3$: we are dealing with standard Artinian Gorenstein algebras, that are quotients of $S = \mathbb{K}[x_0, \cdots, x_5]$ by ideals generated by a regular sequence of length 6 whose elements have degree 2. In this situation, we have $I = (I^2)$, $N = 6$ and

$$R = S/I = R^0 \oplus R^1 \oplus R^2 \oplus R^3 \oplus R^4 \oplus R^5 \oplus R^6$$

with $(\dim(R^i))_{i=0}^6 = (1, 6, 15, 20, 15, 6, 1)$.

Theorem 2.3.4 (Theorem C)**.** *Let R be a complete intersection SAGA presented by quadrics of codimension* 6*. Then R satisfies the strong Lefschetz property in degree 1 (SLP_1), i.e. the general element $x \in R^1$ is such that the map $\mu_1(x^4) : R^1 \to R^5$ is an isomorphism.*

Proof. Let us assume by contradiction that the statement does not hold: the map $\mu_1(x^4) : R^1 \to R^5$ is never injective for $x \in R^1$. Then we are again in the situation described in the construction of Section 1.2 with $k = N - 2 = 4$ and $\Gamma_4 = \Gamma_{4,1}^{(1,1)} = \{([x], [y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \mid x^4 y = 0\}$: since the first projection from Γ_4 is surjective by assumption, we can define Θ, Y and F_y as usual. Let us now focus on the dimensions of Y and of F_y for general $[y] \in Y$.

First of all, let us recall that, by Proposition 1.2.7 and Corollary 1.3.7, we have

$$1 \leq \dim(Y) \leq 3 \qquad 2 \leq \dim(F_y) \leq 4 \qquad \dim(Y) + \dim(F_y) = \dim(\Theta) \geq 5. \tag{2.8}$$

As we have done in the proof of the previous theorem 2.3.2, let us now list in the following table the possible values of the pairs $(\dim(Y), \dim(F_y))$. By using the constraints (2.8) and the various results proved in the previous sections, we can rule out all the possibilities: in the table below we specify which result excludes each pair.

$\dim(Y)$ VS $\dim(F_y)$	2	3	4
1	(2.8)	(2.8)	Prop. 2.2.3
2	(2.8)	Prop. 2.2.5 + bounds (2.1)	Prop. 2.2.3
3	Prop. 2.2.4	Prop. 2.2.4	Prop. 2.2.3

In particular, the case $(\dim(Y), \dim(F_y)) = (2, 3)$ can't be attended because of Proposition 2.2.5. Indeed, if we assume that the dimension of the general fiber F_y is $n - 2 = 3$, we then have that Y has to be contained in the nihilpotent locus $\mathcal{N}_3$, whose dimension by bounds (2.1) is at most 1: Y can't have dimension equal to 2.

Finally, since no pair as above is possible for our framework, we get a contradiction and this concludes the proof. □

As in the previous case, we have the following easy but important consequence.

Corollary 2.3.5. *The jacobian ring of a smooth cubic fourfold satisfies the strong Lefschetz property in degree* 1*.*

Remark 2.3.6. *Our techniques do not apparently allow us to show also the validity of the strong Lefschetz property in degree* 2 *for complete intersection SAGA presented by quadrics with codimension* 6*. Indeed, if we for example proceed as we have done for the case of codimension* 5*, by supposing the failure of such a property and consequently constructing the usual framework, we can not say almost anything about the variety* Y*. For example, we can not obtain the inclusion of* Y *in any nihilpotent locus. Indeed, the failure of the strong Lefschetz property in degree* 2 *means that the multiplication map* $\mu_2(x^2) : R^2 \to R^4$ *is never injective for all* $x \in R^1$*: we can then consider*

$$\Gamma = \Gamma_{2,1}^{(1,2)} = \{([x], [q]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^2) \mid x^2 q = 0\},$$

whose first projection over $\mathbb{P}(R^1)$ *is surjective and where* Y *is defined as we have usually done. In analogy with the case of codimension* 5*, we would like to show (and we also expect) that* $Y \subset \mathcal{N}_3^{(2)}$*. But if we try to apply the method used in the proof of Proposition 1.2.3 to this specific situation, we can find that every point* $([x], [q]) \in \Gamma$ *satisfies also the equation* $xq^2 = 0$*. At this point, we can say that for all* $v \in R^1$ *there exists an element* $w \in R^2$ *such that*

$$(x + tv)(q + tw)^2 = 0 \ mod\ t^2$$

and from this we can get

$$vq^2 + 2xqw = 0.$$

But now multiplication by q *does not make sense anymore: since* vq^3 *would be an element of* R^7 *and hence naturally zero, we can not divide by* v *and obtain the desired equation. This fact seems to stop any reasoning at the beginning, since we do not have apparently instruments to work with this variety* Y *in an appropriate way, as we have done in the other case.*

2.4 Some results in higher codimension

In this section, our aim is to obtain some results concerning Lefschetz properties for complete intersection SAGAs presented by quadrics with codimension equal to $n + 1 \geq 4$ by using techniques and results developed in the previous sections. In particular, we will show Theorem D:

Theorem 2.4.1 ((Theorem D))**.** *Let* R *be a complete intersection SAGA of codimension* $n + 1$ *presented by quadrics. Let* $k \in \{2, 3, 4\}$*. If* $n \geq k + 1$ *we have that* R *satisfies* $SLP_1(k)$*.*

We will show the above theorem by splitting up the proof in 3 cases, which will be treated in the Propositions 2.4.2, 2.4.3 and 2.4.5 respectively, according to the value of k.

Proposition 2.4.2. *Property* $SLP_1(2)$ *holds for every* $n \geq 3$*.*

Proof. Let us assume by contradiction that the multiplication map $x^2\cdot : R^1 \to R^3$ is not injective for $[x] \in \mathbb{P}(R^1)$. Then we can consider the locus $\Gamma_2 = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \mid x^2y = 0\}$, with the corresponding varieties Θ, Y and F_y defined as we have usually done. By Proposition 1.2.7(a), we obtain that $Y \subseteq \mathcal{N}_3$. Then, since by Corollary 2.1.2 we get that $\dim(\mathcal{N}_3) \leq 1$ and by Proposition 1.2.7(d) we know that $\dim(Y) \neq 0$, we have $\dim(Y) = 1$: hence for $[y] \in Y$ general F_y must be a hypersurface, which is not possible by Proposition 2.2.3. □

Proposition 2.4.3. *Property $SLP_1(3)$ holds for every $n \geq 4$.*

Proof. Let us assume by contradiction that the multiplication map $x^3\cdot : R^1 \to R^4$ is not injective for $[x] \in \mathbb{P}(R^1)$. As in the proof of Proposition 2.4.2, let us construct Γ_3, Θ, Y and F_y. In this case, by Proposition 1.2.7 we have that $Y \subseteq \mathcal{N}_4$ so $\dim(Y) \leq 2$, by Corollary 2.1.2. If $[y] \in Y$ is general, then the only possible value for $(\dim(Y), \dim(F_y))$ is $(2, n-2)$ since F_y can not be a hypersurface by Proposition 2.2.3. By dimension reasons, $Y \not\subseteq \mathcal{N}_3$, thus we have that the general element $[y]$ of Y is such that $y^3 \neq 0$. Then, since by Proposition 1.2.3 we know that for every $([x],[y]) \in \Theta$ also the equation $xy^3 = 0$ is satisfied, we get F_y must be contained in $\mathbb{P}(K^1_{y^3})$, whose dimension is at most 2, by Proposition 2.1.1. Then we have proved that $n - 2 = \dim(F_y) \leq 2$ which is impossible for $n \geq 5$. The case where $n = 4$ corresponds to the strong Lefschetz property (in narrow sense) for complete intersection SAGAs presented by quadrics with codimension 5, which has already been proved in Theorem 2.3.2 (Theorem C). □

Before showing the analogous result for the $SLP_1(4)$, let us prove the following:

Lemma 2.4.4. *Let R be a complete intersection SAGA presented by quadrics of codimension $n+1$ and consider $4 \leq k \leq n-1$. Assume that R does not satisfy $SLP_1(k)$ so one can consider the varieties Γ_k, Θ, Y and F_y constructed as in Section 1.2. For $[y] \in Y$ general we have the following properties:*

(a) If R satisfies $SLP_1(k-1)$, then $\dim(F_y) \leq k-1$;

(b) $(\dim(Y), \dim(F_y)) \neq (k-1, k-1)$.

Proof. For (a), let us assume by contradiction that for $[y] \in Y$ general, $\dim(F_y) = h \geq k$. Then, by Lemma 2.2.2, we have that for $[x] \in F_y$ general

$$T_{[x]}(F_y) = \mathbb{P}(T_{\tilde{F}_y,x}) \subseteq \mathbb{P}(K^1_{x^\alpha y^\beta}),$$

where $\alpha + \beta = k$, with $\beta \geq 1$ and $\tilde{F}_y$ is the affine cone over F_y. But since $SLP_1(k-1)$ holds for R by hypothesis, for $([x],[y]) \in \Theta$ general, we have that $x^{k-1}y \neq 0$, and so

$$h = \dim(F_y) \leq \dim(\mathbb{P}(K^1_{x^{k-1}y})) \leq k-1,$$

where the last inequality comes from Proposition 2.1.1. This is clearly impossible by the assumptions over h.

For (b), let us consider $[y] \in Y$ general and assume by contradiction that $\dim(Y) = \dim(F_y) = k-1$. By Proposition 1.2.7(a) we get that $Y \subseteq \mathcal{N}_{k+1}$, and by Corollary 2.1.2 we deduce that Y is an irreducible component of $\mathcal{N}_{k+1}$ and for dimension reasons we have that $Y \not\subset \mathcal{N}_k$, hence $y^k \neq 0$ for $[y] \in Y$ general. By reasoning as in the proof of Proposition 2.4.3 and by Proposition 2.1.1, we get $F_y = \mathbb{P}(K^1_{y^k})$. Moreover, since $[y]$ is general in Y and Y is an irreducible component of $\mathcal{N}_{k+1}$, by Corollary 2.1.3 we have $T_{[y]}(Y) = \mathbb{P}(K^1_{y^k}) = F_y$. With these conditions, we can proceed as in the proof of Proposition 2.2.5 and consider for example the equations

$$0 \equiv (x+tv)^{k-1}(y+tw)^2 \pmod{t^2} \qquad 0 \equiv (x+tv)^k(y+tw) \pmod{t^2},$$

where $w \in T_{\tilde{Y},y}$ and $(v,w) \in T_{\tilde{\Theta},(x,y)}$. In this way, we get that $Y \subseteq \pi_1^{-1}([x])$, where, as usual, $\pi_1 : \Theta \subseteq \mathbb{P}(R^1) \times \mathbb{P}(R^1) \to \mathbb{P}(R^1)$ is the first projection. This leads to a contradiction as shown in Proposition 2.2.5. □

We can now show the last case we need to prove Theorem 2.4.1.

Proposition 2.4.5. *Property $SLP_1(4)$ holds for every $n \geq 5$.*

Proof. First of all, let us notice that the statement for $n = 5$ corresponds to the strong Lefschetz property (in narrow sense) for complete intersection SAGAs presented by quadrics with codimension 6, which has already been proved in Theorem 2.3.4 (Theorem C). We have to prove $SLP_1(4)$ for $n \geq 6$.

Let us assume that for $x \in R^1$, the multiplication map $x^4\cdot : R^1 \to R^5$ is not injective. As usual, we can then consider the incidence correspondence $\Gamma_4 = \{([x],[y]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \mid x^4 y = 0\}$ and the corresponding varieties Θ, Y and F_y, for $[y] \in Y$ general.
By Proposition 1.2.7(a), we get that $Y \subseteq \mathcal{N}_5$, so $\dim(Y) \leq 3$ by Corollary 2.1.2. By using the bounds of Proposition 1.2.7 and Proposition 2.2.3, the only possible cases for the values of $(\dim(Y), \dim(F_y))$ are

$$(2, n-2) \qquad (3, n-2) \qquad (3, n-3).$$

By Proposition 2.4.3, we know that $SLP_1(3)$ holds for $n \geq 6$. Then, by Lemma 2.4.4(a), we get that $\dim(F_y)$ is at most 3: the cases $(\dim(Y), \dim(F_y)) = (2, n-2)$ and $(\dim(Y), \dim(F_y)) = (3, n-2)$ can not occur for every $n \geq 6$. We also have that $(\dim(Y), \dim(F_y)) \neq (3, n-3)$ for every $n \geq 7$.

The only case we have still to analyze is the one with $n = 6$ and $\dim(Y) = \dim(F_y) = 3$. By considering Lemma 2.4.4(b), we can rule out this last possibility too: $SLP_1(4)$ holds for R, for every $n \geq 5$. □

We conclude this section by observing how much can be easily said, by using these methods, for the SLP_1 (in narrow sense) for a complete intersection SAGA of codimension 7 (i.e. $n = 6$) presented by quadrics (e.g. for the Jacobian ring of a smooth cubic fivefold).

Corollary 2.4.6. *Let R be a complete intersection SAGA of codimension 7 presented by quadrics. We have that $SLP_1(4)$ holds. Moreover, if $SLP_1(5)$ does not hold (i.e. if SLP_1 does not hold), then one can construct the varieties Γ_5, Θ, Y and F_y as usual and we have, for $[y] \in Y$ general, $(\dim(Y), \dim(F_y)) \in \{(2,4),(3,3)\}$.*

Proof. Property $SLP_1(4)$ holds by Theorem 2.4.1. Assume that $SLP_1(5)$ does not hold for R. Then. by Proposition 1.2.7 and Corollary 1.3.7, we have

$$1 \leq \dim(Y) \leq 4 \qquad 2 \leq \dim(F_y) \leq 5 \qquad \dim(Y) + \dim(F_y) = \dim(\Theta) \geq 6. \tag{2.9}$$

As in Theorem 2.3.4, we put in a table the possible values of the pairs $(\dim(Y), \dim(F_y))$ and we specify which result rules out the corresponding case.

$\dim(Y)$ VS $\dim(F_y)$	2	3	4	5
1	(2.9)	(2.9)	(2.9)	Prop. 2.2.3
2	(2.9)	(2.9)		Prop. 2.2.3
3	(2.9)		Prop. 2.2.5 + Bounds (2.1)	Prop. 2.2.3
4	Prop. 2.2.4	Prop. 2.2.4	Prop. 2.2.4	Prop. 2.2.3

This concludes the proof. □

2.5 A lifting criterion for weak Lefschetz property

It is known (see, for example, [HMM+13, Proposition 3.11]) that, the SLP for a graded algebra is inherited to its quotients by suitable conductor ideals. Let us recall that given an element $\alpha \in R^e \setminus \{0\}$ we call conductor ideal of α the ideal $(0 : \alpha) = \oplus_{i=0}^{N} \ker(\alpha\cdot : R^i \to R^{i+e})$, where R is any SAGA with socle in degree N (see the definition in Lemma 1.1.9). In this section we prove a sort of converse for the weak Lefschetz property in degree 2 (WLP_2) for complete intersection SAGAs presented by quadrics. More precisely, we will give a criterion to reduce the proof of WLP_2 for a SAGA R as above to a suitable quotient of R, modulo the existence of a non-Lefschetz element. We stress, moreover, that this criterion works for any codimension.

Elements of $\mathbb{P}(R^1)$ for which the corresponding multiplication map is not of maximal rank play here an important role, so it is convenient to introduce the following subschemes of $\mathbb{P}(R^1)$.

Definition 2.5.1. *Let R be any SAGA with socle in degree N. For $1 \leq a \leq N-1$ we define the* **Lefschetz locus in degree** a *to be*

$$\mathcal{L}_a := \{[x] \in \mathbb{P}(R^1) \,|\, x\cdot : R^a \to R^{a+1} \text{ has maximal rank}\} \subset \mathbb{P}(R^1).$$

An element $[x] \in \mathbb{P}(R^1)$ is called **Lefschetz element in degree** a *if $[x] \in \mathcal{L}_a$. On the contrary, elements not in $\mathcal{L}_a$ are called* **non-Lefschetz elements (in degree** a**)**.

Geometric results on these loci can be found, for example, in [AR19] and [BMMRN18].

Remark 2.5.2. *Let us stress that if we consider a non-Lefschetz element in degree 1 $[z]$, i.e. $[z] \in \mathbb{P}(R^1) \setminus \mathcal{L}_1$, we have that the multiplication map $z\cdot : R^1 \to R^2$ is not injective. Hence, we have a non trivial kernel of such a map, i.e. $K_z^1 \neq \{0\}$. If R is a complete intersection SAGA presented by forms of degree $d-1$, by Proposition 2.1.1, we know that $\dim(K_z^1) \leq \frac{1}{d-2}$. If $d \geq 4$ the only possibility is that $\dim(K_z^1) = 0$ and so $K_z^1 = \{0\}$ (actually in this case the multiplication by z corresponds to the multiplication map $S^1 \to S^2$, that is clearly always injective). If $d = 3$, we have $\dim(K_z^1) \leq 1$ and if*

$[z]$ is a non-Lefschetz element we get the equality and in particular there exists an element $[w] \in \mathbb{P}(R^1)$ such that $\mathbb{P}(K_z^1) = [w]$, i.e. $zw = 0$ and $K_z^1 = \langle w \rangle$. Clearly, $[w]$ itself is a non-Lefschetz element in degree 1 with $\mathbb{P}(K_w^1) = [z]$.

Let us start by setting as usual $S = \mathbb{K}[x_0, \dots, x_n]$ and by proving the following result.

Lemma 2.5.3. *Assume that $R = S/I$ is a complete intersection SAGA of codimension $n+1$ presented by quadrics (so the socle is in degree $N = n+1$). Assume that there exists a non-Lefschetz element in degree 1 $[z]$, such that $zw = 0$ for $w \neq 0$. Then $(z) = (0 : w)$ and $\bar{R} = R/(z)$ is a complete intersection SAGA of codimension n presented by quadrics. In particular, $\dim(K_w^s) = \dim(R^{s-1}) - \dim(K_z^{s-1})$.*

Proof. By definition, we have $(z) \subseteq (0 : w)$, so we can define an epimorphism of graded $\mathbb{K}$-algebras

$$\varphi : \bar{R} := R/(z) \to \tilde{R} := R/(0 : w).$$

By Lemma 1.1.9, the latter is a SAGA of codimension n and socle in degree $\tilde{N} = n$. By considering $\bar{R}$, it is clearly an Artinian standard algebra of codimension n. We want to show that $\bar{R}$ is also a complete intersection SAGA presented by quadrics. By hypothesis, we know that $zw \in I$, so we can complete $\{zw\}$ to a regular sequence of the form $\{g_0, \cdots, g_{n-1}, zw\}$ spanning I. Notice that, by construction, $g_0, \cdots, g_{n-1}$ do not belong to the ideal (z) and the reductions $\bar{g}_i$ of g_i modulo (z) are a regular sequence of quadrics in the polynomial ring $\bar{S} = S/(z)$.

Hence we have

$$\bar{R} = R/(z) = \frac{S/I}{(z)} \simeq \frac{\bar{S}}{(\bar{g}_0, \cdots, \bar{g}_{n-1})}$$

so $\bar{R}$ is a complete intersection SAGA presented by quadrics. In particular, it has socle in degree $\bar{N} = n = \tilde{N}$.

Since φ is an epimorphism and preserve the degrees, the image of a generator $\bar{\sigma}$ of $\bar{R}^n$ is a non-zero multiple of the generator $\tilde{\sigma}$ of $\tilde{R}^n$. This also implies the injectivity of φ. Indeed, let us take a non-zero element $x \in \bar{R}^i$. There exists $y \in \bar{R}^{n-i}$ such that $xy = \bar{\sigma}$. Hence, we have

$$\lambda\tilde{\sigma} = \varphi(\bar{\sigma}) = \varphi(xy) = \varphi(x)\varphi(y),$$

and so we get that $\varphi(x)$ can not be zero and φ is an isomorphism. In particular, for all s,

$$R^{s-1} \cdot z = (z)_s = (0 : w)_s = K_w^s$$

then we clearly have

$$\dim(K_w^s) = \dim(R^{s-1}z) = \dim(R^{s-1}) - \dim(K_z^{s-1})$$

as claimed. □

We can now prove the following "lifting criterion":

Theorem 2.5.4. *Let $R = S/I$ be a complete intersection SAGA of codimension $n+1 \geq 6$ presented by quadrics and assume that z is a non-Lefschetz element (in degree 1) for R. If $\bar{R} = R/(z)$ satisfies the WLP_2, then the same holds for R.*

Proof. First of all, by Lemma 2.5.3, we have that $\bar{R} = R/(z) = R/(0 : w)$ is a complete intersection SAGA presented by quadrics and $K_z^1 = \langle w \rangle$. Let pr_z be the projection $R \to \bar{R}$.

Assume, by contradiction, that WLP_2 holds for $\bar{R}$ but not for R. In particular, for all $x \in R^1$, the multiplication map $x\cdot : R^2 \to R^3$ has non trivial kernel, i.e. $K_x^2 \neq \{0\}$. Consider the incidence correspondence

$$\Gamma = \{([x],[v]) \in \mathbb{P}(R^1) \times \mathbb{P}(R^1) \,|\, xvz = 0\}$$

with its projections p_1 and p_2 on the factors.

We claim that p_1 is surjective. Since Γ is a closed subset, it is enough to show that for $[x] \in \mathbb{P}(R^1)$ general there exists $[v] \in \mathbb{P}(R^1)$ such that $xvz = 0$. Let x be a general element of R^1. As $K_x^2 \neq 0$ we have that there exists $[q] \in \mathbb{P}(R^2)$ such that $xq = 0$ in R. Then we have also $\mathrm{pr}_z(xq) = \overline{xq} = 0$ in $\bar{R}$. Since $[x]$ is general in $\mathbb{P}(R^1)$, then the same holds for $[\bar{x}] \in \mathbb{P}(\bar{R}^1)$, so we get $\bar{q} = 0$ in $\bar{R}^2$, as WLP_2 holds for $\bar{R}$ by assumption. Then, by Lemma 2.5.3, we have $q \in (0 : w)_2 = (z)_2 = z \cdot R^1$ so there exists $[v] \in \mathbb{P}(R^1)$ such that $0 = xq = xvz$ as claimed.

In analogy with what happens for the construction described in Section 1.2, we have that there exists a unique irreducible component Θ' of Γ which dominates $\mathbb{P}(R^1)$ via π_1, where we set π_i to be the restriction of p_i to Θ' for $i = 1, 2$. We have that for $[x] \in \mathbb{P}(R^1)$ general

$$\pi_1^{-1}([x]) = p_1^{-1}([x]) = [x] \times \mathbb{P}(K_{xz}^1)$$

so the general fiber of π_1 has dimension at most 1 by Proposition 2.1.1.

Let us now show that the general fiber of π_1 has dimension 1. Consider $[x] \in \mathbb{P}(R^1)$ general. Firstly, let us observe that $([x],[w])$ belongs to $p_1^{-1}([x]) = \pi_1^{-1}([x])$ since $zw = 0$. As shown above, there exists $[q] \in \mathbb{P}(R^2)$ such that $xq = 0$ and $q = zv$ for suitable $[v] \in \mathbb{P}(R^1)$. Moreover $[v] \neq [w]$ since, otherwise, $[q]$ would be zero, hence $\pi_1^{-1}([x]) = \langle [w], [v] \rangle$ as claimed.

By considering the second projection π_2, we have that for $[v]$ general in $Y' = \pi_2(\Theta')$, the fiber $\pi_2^{-1}([v])$ is such that

$$\pi_2^{-1}([v]) \subseteq p_2^{-1}([v]) = \mathbb{P}(K_{vz}^1) \times [v],$$

which has dimension at most 1 by Proposition 2.1.1. Since π_1 is dominant, for $([x],[v]) \in \Theta'$ general we have

$$n + 1 = \dim(\mathbb{P}(R^1)) + \dim(\pi_1^{-1}([x])) = \dim(\Theta') = \dim(Y') + \dim(\pi_2^{-1}([v])).$$

Since $\dim(Y') \leq n$ and $\dim(\pi_2^{-1}([v])) \leq 1$, for v general, the only possibility is to have $\dim(\pi_2^{-1}([v])) = 1$ and $Y' = \mathbb{P}(R^1)$.

We will show now that having $Y' = \mathbb{P}(R^1)$ gives a contradiction. First of all, by reasoning as in the proof of Proposition 1.2.3, one can prove that

$$Y' \subseteq \{[v] \in \mathbb{P}(R^1) \mid v^2 z = 0\}.$$

Since $Y' = \mathbb{P}(R^1)$ and squares of elements of R^1 generates R^2 (as R is standard), we have that $z \cdot R^2 = 0$. This is impossible by Gorenstein duality, since $z \neq 0$. $\square$

In the statement of Theorem 2.5.4 we require $n+1 \geq 6$ since for codimension 5 the WLP_2 has already been proved (in [AR19] or as consequence of SLP proved in Theorem 2.3.2) and in even smaller codimension, it easily follows from WLP_1 that is known to hold. From this, we get the following consequence:

Corollary 2.5.5. *Let R be a complete intersection SAGA of codimension 6 presented by quadrics (e.g. R is the jacobian ring of a cubic fourfold). If $\mathcal{L}_1$ is not the whole $\mathbb{P}(R^1)$, R satisfies WLP.*

2.6 Nihilpotent loci and geometrical properties

In this section we will study geometric properties of the nihilpotent loci $\mathcal{N}_k \subseteq \mathbb{P}(R^1)$ where R is a complete intersection SAGA of codimension $n+1$ presented by quadrics. We stress that we don't make any assumptions about the validity of any weak or strong Lefschetz property for R in this section.

We recall that the nihilpotent loci (in $\mathbb{P}(R^1)$) are defined as $\mathcal{N}_k = \{[x] \in \mathbb{P}(R^1) \mid x^k = 0\}$. Moreover, if $X \subset \mathbb{P}^r$ is non-empty, we denote by $\mathrm{Sec}^k(X) \subseteq \mathbb{P}^r$ the k-secant variety associated to X, i.e.

$$\mathrm{Sec}^k(X) := \overline{\bigcup_{p_1,\ldots,p_k \in X} \langle p_1, \ldots, p_k \rangle}$$

where $\langle p_1, \ldots, p_k \rangle$ is the linear span of the points $p_1, \ldots, p_k$. For brevity, we set $\mathrm{Sec}^2(X) := \mathrm{Sec}(X)$. The interested reader can refer to [Rus16, Chapter 1] for various properties of these classical loci (although the definition considered is slightly different from the one adopted by us).

We stress that, like the non-Lesfschetz loci $\mathbb{P}(R^1) \setminus \mathcal{L}_k$, the nihilpotent loci $\mathcal{N}_k$ are expected to be empty when k is small for R general. Hence it is interesting to study these loci when R is "special". For example, these loci give a lot of information for SAGAs for which some Lefschetz properties do not hold.

Let us start by analyzing the locus $\mathcal{N}_2 \subseteq \mathbb{P}(R^1) \simeq \mathbb{P}^n$. We recall that by Corollary 2.1.2 and, in particular, by bounds (2.1) we have that $\dim(\mathcal{N}_2) \leq 0$ so it is either empty or it is the union of a finite number of points. These points have to satisfy the following:

Proposition 2.6.1. *Assume that $[t_1], \ldots, [t_k] \in \mathcal{N}_2$ are distinct points. Then $\Pi_{i=1}^k t_i \neq 0$ in R and $[t_1], \ldots, [t_k]$ are in general position in $\mathbb{P}(R^1)$. In particular, $\#\mathcal{N}_2 \leq n+1 = N$.*

Proof. The statement is trivially true for $k=1$. If $k=2$ the only statement one has to check is that $t_1 t_2 \neq 0$. This is true since $\dim(K^1_{t_1}) \leq 1$ by Proposition 2.1.1 and $t_1 \in K^1_{t_1}$, hence $K^1_{t_1} = \langle t_1 \rangle$, but $[t_1] \neq [t_2]$. We will then proceed by induction assuming that the claim is true till $k-1$.

Let $T = \{[t_1], \ldots, [t_k]\}$ be a set of k distinct points of $\mathcal{N}_2$. By contradiction, let us assume that either (A_1) or (A_2) holds, where

(A_1) $\{t_1, \ldots, t_k\}$ are linearly dependent

(A_2) $\Pi_{i=1}^k t_i = 0$.

First of all, we claim that (A_2) is equivalent to (A_1). By induction hypothesis, for $\{z_1, \dots, z_{k-1}\} \subset \{t_1, \dots, t_k\}$ with $[z_i] \neq [z_j]$ for all $i \neq j$, we have $\Pi_{i=1}^{k-1} z_i \neq 0$, so $K^1_{z_1 \cdots z_{k-1}}$ has dimension at most $k-1$ by Proposition 2.1.1. Since $z_i^2 = 0$ by assumption, we have $K^1_{z_1 \cdots z_{k-1}} = \langle z_1, \dots, z_{k-1} \rangle$. Then, (A_1) holds if and only if we have, up to a permutation of the elements, $t_k \in \langle t_1, \dots, t_{k-1} \rangle = K^1_{t_1 \cdots t_{k-1}}$ and this is equivalent to $\Pi_{i=1}^{k} t_i = 0$, i.e. (A_2).

Hence, let us suppose that $t_k \in \langle t_1, \dots, t_{k-1} \rangle$, so we can write $t_k = \sum_{i=1}^{k-1} a_i t_i$. Then we have

$$0 = t_k^2 = 2 \sum_{1 \leq i < j \leq k-1} a_i a_j t_i t_j.$$

If $k = 3$ we have $0 = t_3^2 = 2a_1 a_2 t_1 t_2$ so, since $t_1 t_2 \neq 0$ by induction hypothesis, we have either $a_1 = 0$ or $a_2 = 0$. This implies either $\{t_1, t_3\}$ or $\{t_2, t_3\}$ linearly dependent, and we get a contradiction since this is against the induction hypothesis. If $k \geq 4$, by multiplying by $\Pi_{i=1}^{k-3} t_i$, we get

$$0 = 2a_{k-2} a_{k-1} \Pi_{i=1}^{k-1} t_i.$$

Since $\Pi_{i=1}^{k-1} t_i \neq 0$ by induction hypothesis, we have either $a_{k-2} = 0$ or $a_{k-1} = 0$ and we have a contradiction as in the case $k = 3$. $\square$

By considering the Fermat hypersurface $X = V(F)$ in $\mathbb{P}^n$, one can easily see that, for the Jacobian ring $R = S/J(F)$, the set $\mathcal{N}_2$ consists of exactly $n+1$ independent points. However, also the converse is true, as shown by the following:

Corollary 2.6.2. *Assume that $\#\mathcal{N}_2 = n+1$. Then R is the Jacobian ring of a cubic hypersurface X projectively equivalent to the Fermat cubic hypersurface in $\mathbb{P}^n$.*

Proof. By assumption we have that $\mathcal{N}_2 = \{[t_0], \dots, [t_n]\}$. By Proposition 2.6.1, $\{t_0, \dots, t_n\}$ are $n+1$ linearly independent forms so $R = S/I$ with $S = \mathbb{K}[t_0, \dots, t_n]$. On the other hand, in S we have $t_i^2 \in I$ and $\{t_0^2, \dots, t_n^2\}$ is a regular sequence which generates I as ideal of S. Then, if we set $F = \sum_{i=0}^{n} t_i^3$, we have that I is the Jacobian ideal of the Fermat cubic hypersurface $X = V(F)$ as claimed. $\square$

Remark 2.6.3. *We have $\mathrm{Sec}^k(\mathcal{N}_2) \subseteq \mathcal{N}_{k+1}$. Indeed, if $[t_1], \dots, [t_k] \in \mathcal{N}_2$ we have $t_i^2 = 0$. In particular, every monomial of degree $k+1$ in the variables t_i is identically 0. Then $(\sum_{i=1}^{k} a_i t_i)^{k+1} \equiv 0$ for all $a_1, \dots, a_k \in \mathbb{K}$ so $\mathrm{Sec}^k(\mathcal{N}_2) \subseteq \mathcal{N}_{k+1}$. More generally,*

$$\text{whenever } r > k(a-1) \text{ one has } \mathrm{Sec}^k(\mathcal{N}_a) \subseteq \mathcal{N}_r.$$

Indeed, consider $[t_1], \dots, [t_k] \in \mathcal{N}_a$ and let $m = \prod_{i=1}^{k} t_i^{\alpha_i}$ with $\sum_{i=1}^{k} \alpha_i = r$. We have $m = 0$ if there exists i such that $\alpha_i \geq a$. On the other hand, this always happens if $r > k(a-1)$: if $\alpha_i \leq (a-1)$ for all i, we would have

$$r = \sum_{i=1}^{k} \alpha_i \leq \sum_{i=1}^{k} (a-1) = k(a-1) < r$$

which gives a contradiction.

With the following Lemma, let us study the geometry of the nihilpotent locus $\mathcal{N}_3$:

Lemma 2.6.4. *If L is a line contained in $\mathcal{N}_3$, then $L \subseteq \mathrm{Sec}(\mathcal{N}_2)$, i.e. a line in $\mathcal{N}_3$ is a line joining two different points of $\mathcal{N}_2$.*

Proof. Assume that L is a line in $\mathcal{N}_3$. Since the dimension of $\mathcal{N}_3$ is at most 1 by bounds (2.1), we have that L is a component of $\mathcal{N}_3$. As $\dim(\mathcal{N}_2) \leq 0$ and $\dim(K^1_x) \leq 1$ for any $x \in R^1$ by Proposition 2.1.1(a), we can find $[v], [w] \in L$ such that $[v] \neq [w]$, $[v], [w] \notin \mathcal{N}_2$ and $vw \neq 0$. By hypothesis, we have that $(v + tw)^3 = 0$ for all $t \in \mathbb{K}$ so $v^3 = v^2 w = vw^2 = w^3 = 0$. Then, $K^1_{v^2}, K^1_{w^2}$ and K^1_{vw} contain $\langle v, w\rangle$. On the other hand, these subspaces have dimension at most 2 by Proposition 2.1.1(a) so they coincide with $\langle v, w\rangle$. By Proposition 2.1.1(b), there exist $\lambda, \mu \in \mathbb{K}$ such that

$$v^2 = \lambda w^2 \qquad \text{and} \qquad vw = \mu w^2 \tag{2.10}$$

so we have $(v + tw)^2 = v^2 + 2tvw + t^2w^2 = w^2(t^2 + 2\mu t + \lambda)$.

We claim that $t^2 + 2\mu t + \lambda$ has two distinct roots so L is indeed a line contained in $\mathrm{Sec}(\mathcal{N}_2)$. Assume, on the contrary, that $t^2 + 2\mu t + \lambda$ is a square. This implies that $\mu^2 = \lambda$. Then, from the Equations (2.10), we obtain

$$v(v - \mu w) = 0 \qquad w(v - \mu w) = 0$$

so $v - \mu w \in K^1_v \cap K^1_w$. By Proposition 2.1.1 we can conclude that $[v] = [w]$ which is against our assumptions. □

We will generalize this result in Theorem 2.6.6 by considering suitable linear subspaces contained in $\mathcal{N}_k$. We need first the following technical lemma.

Lemma 2.6.5. *Let $k \geq 2$ and let T be an hypersurface in $\mathbb{P}^k \subset \mathbb{P}(R^1)$. Assume either that*

1. *$0 \leq s \leq k - 1$ or*
2. *$s = k$ and the support of T is not contained in the union of 2 different hyperplanes.*

Then there exist $[x_0], \ldots, [x_s] \in T$ which are linearly independent and such that $\Pi^s_{i=0} x_i \neq 0$.

Proof. The statement of the lemma is clearly true for $s = 0$. We will proceed by induction on $s \leq k$. Then let us assume that there are $[x_0], \ldots, [x_{s-1}] \in T$ which are linearly independent and with $y = x_0 \cdots x_{s-1} \neq 0$. Consider the linear spaces $\tau_1 = \langle x_0, \ldots, x_{s-1}\rangle$ and $\tau_2 = \mathbb{P}(K^1_y)$. We are done if we prove that $U = T \setminus (\tau_1 \cup \tau_2)$ is not empty. By construction we have $\dim(\tau_1) = s - 1$ and $\dim(\tau_2) \leq s - 1$ by Proposition 2.1.1. Hence, if $s < k$, U is an open dense subset of T. If $s = k$ and the support of T is not contained in the union of 2 different hyperplanes, there exists an irreducible component C of T which is different from τ_1 and τ_2. Then $C \setminus (\tau_1 \cup \tau_2)$ is not empty so U is again not empty as claimed. □

Theorem 2.6.6. *Assume that π is a $(k - 1)$-plane contained in $\mathcal{N}_{k+1}$. Then*

(A) $T_k = \pi \cap \mathcal{N}_k$ is an hypersurface of degree k (with possible multiple components) in π;

(B) there exist $[x_0], \ldots, [x_{k-1}]$ in T_k which are linearly independent, $\Pi^{k-1}_{i=0} x_k \neq 0$.

In particular, T_k is non degenerate in π and $\pi \subseteq \mathrm{Sec}^k(\mathcal{N}_k)$.

Proof. Notice, first of all, that $\dim(\mathcal{N}_k) \leq k-2$ by bounds (2.1), so $\pi \setminus \mathcal{N}_k \neq \emptyset$. Then, we can find $\{x_0, \ldots, x_{k-1}\}$ linearly independent which span π and such that $x_{k-1} \notin \mathcal{N}_k$. Since $\pi \subseteq \mathcal{N}_{k+1}$, we have that

$$(\alpha_0 x_0 + \cdots + \alpha_{k-1}x_{k-1})^{k+1} \equiv 0 \qquad \forall \alpha_0, \ldots, \alpha_{k-1} \in \mathbb{K}$$

and this is equivalent to say that all monomials of degree $k+1$ in the variables $x_0, \ldots, x_{k-1}$ are 0. Then, if m is a monomial of degree k in these variables, either $m = 0$ or $K^1_m = \langle x_0, \ldots, x_{k-1}\rangle$, by Proposition 2.1.1. In particular, we have that for each monomial of degree k there exists $\lambda_m \in \mathbb{K}$ with $m = \lambda_m x_{k-1}^k$ (recall that we assumed $x_{k-1} \notin \mathcal{N}_k$). Then

$$(\alpha_0 x_0 + \cdots + \alpha_{k-1}x_{k-1})^{k} = p_k(\underline{\alpha}) x_{k-1}^k \tag{2.11}$$

where $p_k(\underline{\alpha})$ is a homogeneous polynomial of degree k in the variables $\alpha_0, \ldots, \alpha_{k-1}$. It is not 0 since the coefficient of α_{k-1}^k is 1 by construction. By Equation (2.11), $T_k = \pi \cap \mathcal{N}_k$ is described by the vanishing of $p_k(\underline{\alpha})$. In particular, $\mathcal{N}_k$ is not empty and we have also proved (A).

For (B), if T_k has support which is not contained in 2 different hyperplanes, the thesis follows directly from Lemma 2.6.5 so we have to discuss only the cases

$$(B_1): \ \mathrm{Supp}(T_k) = H_1 \cup H_2 \qquad \text{and} \qquad (B_2): \ \mathrm{Supp}(T_k) = H_1$$

where H_1 and H_2 are distinct hyperplanes.

In both cases (B_1) and (B_2), there is an hyperplane H_1 of π contained in T_k. We recall that T_k is contained in $\mathcal{N}_k$ by construction. By Lemma 2.6.5 applied to $H_1 \subset \pi$ we can find $[x_0], \ldots, [x_{k-2}] \in H_1$ which are linearly independent and such that $y = \prod_{i=0}^{k-2} x_i \neq 0$. Since $H_1 = \langle [x_0], \ldots, [x_{k-2}]\rangle$ and $H_1 \subset \mathcal{N}_k$ we have that all monomials of degree k in the variables $x_0, \ldots, x_{k-2}$ are 0. Then, by Proposition 2.1.1, $K^1_y = \langle x_0, \ldots, x_{k-2}\rangle$ so $H_1 = \mathbb{P}(K^1_y)$.

If we are in case (B_1) we can then choose x_{k-1} in $H_2 \setminus H_1$ and $\{x_0, \ldots, x_{k-2}, x_{k-1}\}$ is a set of points with the desired properties. We claim now that case (B_2) can not occur. Assume, by contradiction, that $\mathrm{Supp}(T_k)$ is the hyperplane $H_1 = \mathbb{P}(K^1_y)$. Then for any x_{k-1} in $\pi \setminus H_1$ we have that $\pi = \langle x_0, \ldots, x_{k-2}, x_{k-1}\rangle$, $x_{k-1}^k \neq 0$ and $y x_{k-1} \neq 0$. With this choice of the x_i's, the polynomial $p_k(\underline{\alpha})$ of Equation (2.11) is proportional to α_{k-1}^k since $T_k = \pi \cap \mathcal{N}_k$ has support on H_1. On the other hand the coefficient of $\prod_{i=0}^{k-1} \alpha_i$ can not be zero since $\prod_{i=0}^{k-1} x_i = y x_{k-1} \neq 0$. □

We conclude this section by presenting some examples in order to make the phenomenology of the nihilpotent loci clearer (some computations have been made by using the computer algebra software Magma). We set $S = \mathbb{K}[x_0, \ldots, x_n] = \bigoplus_{k \geq 0} S^k$ and we define $\{w_0, \ldots, w_n\}$ to be the projective coordinates on $\mathbb{P}(R^1)$ induced by the basis $\{x_0, \ldots, x_n\}$ of $R^1 = S^1$.

Example 2.6.7. *Let X be the Fermat cubic in $\mathbb{P}^n$ and consider the Jacobian ring R of X. For any $2 \leq k \leq n$ we have $(\sum_i w_i x_i)^k \in J^k$ if and only if all monomials in the $\{w_i\}$ of degree k without multiple factors vanish. This is true whenever any set of $n-k+1$ variables is zero. With these arguments one can prove that $\mathcal{N}_k$ is the union of the coordinated planes of dimension $k-2$. In particular, $\mathcal{N}_k = \mathrm{Sec}^k(\mathcal{N}_2)$ and $\mathrm{Sing}(\mathcal{N}_k) = \mathcal{N}_{k-1}$ for $k \geq 2$.*

Example 2.6.8. *Consider the smooth cubic surface $X = V(f)$ with $f = x_0^3 + x_1^3 + x_2^3 + x_3^3 + 6x_0x_1x_2$ and consider the Jacobian ring R of X. One has that there are 4 points in $\mathcal{N}_2$ so, by Corollary 2.6.2, X is the Fermat cubic up to a projective transformation. Indeed, if λ is a non trivial third root of 1, we have*

$$(x_0 + x_1 + x_2)^3 + (x_0 - (\lambda + 1)x_1 + \lambda x_2)^3 + (x_0 + \lambda x_1 - (\lambda + 1)x_2)^3 + 3x_3^3 = 3f.$$

Example 2.6.9. *Consider the smooth cubic surface $X = V(f)$ with $f = x_0^3 + x_1^3 + x_2^3 + x_3^3 + 3x_0x_1x_2$ and consider the Jacobian ring R of X. If $P = [0 : 0 : 0 : 1]$ and $C = V(w_3, g)$ is the smooth plane cubic with $g := w_0^3 + w_1^3 + w_2^3 - 6w_0w_1w_2$, we have (considering the reduced structure)*

$$\mathcal{N}_2 = \{P\} \qquad \mathcal{N}_3 = \{P\} \cup C \qquad \mathcal{N}_4 = V(w_3) \cup V(g).$$

In particular, $\mathcal{N}_4$ is the union of the plane containing the cubic curve C and the cone with vertex P generated by C. Notice that $\mathcal{N}_3$ does not have pure dimension.

Example 2.6.10. *Consider the smooth cubic surface $X = V(f)$ with $f = x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_0(x_1^2 + x_2^2 + x_3^2)$ and let R be its Jacobian ring. One can show that, in this case, $\mathcal{N}_2$ and $\mathcal{N}_3$ are both empty whereas $\mathcal{N}_4$ is a smooth quartic hypersurface.*

Example 2.6.11. *Consider the regular sequence $\{x_0^2, x_1^2, x_2^2, x_3^2 + 2x_0x_1\}$ in $S = \mathbb{K}[x_0, \dots, x_3]$, the ideal J spanned by it and set $R = S/J$. Notice that J is not the Jacobian ideal of a cubic surface. Let P_i be the coordinated points and consider the conic $C = V(w_2, g)$ with $g := w_3^2 - 3w_0w_1$. Then we have*

$$\mathcal{N}_2 = \{P_0, P_1, P_2\} \qquad \mathcal{N}_3 = \langle < P_0, P_1 > \rangle \cup \langle < P_0, P_2 > \rangle \cup \langle < P_1, P_2 > \rangle \cup C$$

$$\mathcal{N}_4 = V(w_3) \cup V(w_2) \cup V(g).$$

In particular, $\mathcal{N}_4$ is the union of two planes (the first one - $V(w_3)$ - contains P_0, P_1 and P_2 and the lines joining these points whereas the second - $V(w_2)$ - is the plane containing the conic C and the line $\langle P_0, P_1 \rangle$) and $V(g)$ (which is a quadric cone with vertex P_2). Notice that, as varieties, we have $\mathrm{Sing}(\mathcal{N}_k) = \mathcal{N}_{k-1}$ *for* $k = 2, 3, 4$.

Chapter 3

From Gorenstein algebras to Hessian hypersurfaces

In this third chapter we keep on analyzing SAGAs and also smooth cubic hypersurfaces. However, we do this from a different perspective than the one characterizing the previous chapters. In particular, in what follows we will change the kind of SAGA under analysis: we will not consider any more complete intersection SAGAs presented by quadrics (and so, in particular, jacobian rings of smooth cubic hypersurfaces). Instead, we will consider SAGAs defined as quotients of the differential operators ring over the annihilator of a cubic polynomial (see Example 1.1.4). In this setting, by studying the Lefschetz properties and, in particular, the non-Lefschetz loci (see Definition 2.5.1), it turns out that the non-Lefschetz locus in degree 1 of such a SAGA coincides (up to isomorphism) to the Hessian hypersurface associated to the cubic polynomial defining the SAGA, i.e. to the hypersurface defined as the zero locus of the determinant of the Hessian matrix of such a cubic polynomial. After this comparison and analysis, we will start a deep study of these Hessian hypersurfaces associated to smooth cubic hypersurfaces. We will analyze their singularities and desingularizations and we will describe the singular locus of the Hessian hypersurfaces associated to a general smooth cubic fourfold.

In particular, in Section 3.1 we will present and describe the connection between the non-Lefschetz locus of these particular Gorenstein algebras and the Hessian locus of a cubic hypersurface. In Section 3.2, we will put into the picture another natural connection between Hessians of cubic polynomial and quadratic forms, which will be very useful in the following analysis. Moreover, we will present the first results concerning the geometry of such Hessian hypersurfaces, like the expected dimension of their singular loci. In Section 3.3, we will describe in a more detailed way these singular loci and we will present a desingularization for the general Hessian hypersurface (we will prove Theorem E and Theorem F from the Introduction). This analysis will move on in Section 3.4, while in Section 3.5 we will look at these singularities as degeneracy loci of specific symmetric maps between vector bundles. From this perspective and from the construction of suitable non-trivial covers over these loci, we will get new geometric information, which, in particular, will be applied, in Section 3.6, for the description of the singular locus of the Hessian hypersurface associated to a general smooth cubic fourfold: in this last section we will prove Theorem G.

In this chapter, as we have done in the previous ones, we will work over an algebraically closed field

$\mathbb{K}$ of characteristic 0 up to Section 3.4, while in the last two sections, we will work over the field $\mathbb{C}$ of complex numbers.

3.1 Hessians and Gorenstein algebras

In this first section, we will show the natural connection between the non-Lefschetz locus of specific SAGAs and the zero locus of the determinant of the Hessian matrix of cubic polynomials.

As done in Section 1.1 (see Example 1.1.4), we set

$$y_i := \partial_{x_i} = \frac{\partial}{\partial x_i} \qquad \text{and} \qquad \partial_v = \sum_{i=0}^{n} v_i \partial_{x_i} = \sum_{i=0}^{n} v_i y_i$$

where $v = \sum_{k=0}^{n} v_k e_k \in \mathbb{K}^{n+1}$ with $\{e_i\}_{i=0,\dots,n}$ as the standard basis of $\mathbb{K}^{n+1}$. If one considers the graded algebra $D = \mathbb{K}[y_0, \dots, y_n] = \bigoplus_{k\geq 0} D^k$, one has a natural pairing $S \times D \to S$ where elements of D act as differential operators on $S = \mathbb{K}[x_0, \dots, x_n]$. Observe also that there is a natural identification

$$v = \sum_{k=0}^{n} v_k e_k \quad \longleftrightarrow \quad \partial_v = \sum_{i=0}^{n} v_i y_i \tag{3.1}$$

For each $f \in S^d$ one can define the gradient $\nabla(f) = (y_i f)_{i=0,\dots,n} \in (S^{d-1})^{\oplus n+1}$ and the Hessian matrix of f and the hessian of f, i.e

$$\mathrm{Hess}(f) = (y_i y_j f)_{i,j=0,\dots,n} \in \mathrm{M}_{n+1}^{sym}(S^{d-2}) \qquad \text{and} \qquad \mathrm{hess}(f) = \det(\mathrm{Hess}(f)) \in S^{(n+1)(d-2)},$$

where $\mathrm{M}_{n+1}^{sym}(S^{d-2})$ denotes the set of square symmetric matrices of order $n+1$ whose entries are homogeneous polynomials of degree $d-2$ (eventually zero).

Definition 3.1.1. *The zero locus of the determinant of the Hessian matrix of a polynomial f is said to be the Hessian hypersurface $\mathcal{H}_f$ associated to f, i.e. $\mathcal{H}_f = V(\mathrm{hess}(f))$.*

(We will often write $\mathcal{H}$, instead of $\mathcal{H}_f$, when it is clear from the context which $f \in S$ we are referring to.)

For any $d \geq 2$, one can consider the subloci of $\mathbb{P}(S^d)$ given by

$$\mathcal{C}_{sing} = \{[f] \,|\, V(f) \text{ is singular}\} \qquad \mathcal{C}_{cone} = \{[f] \,|\, V(f) \text{ is a cone}\} \qquad \text{and} \qquad \mathcal{C}_{GN} = \{[f] \,|\, \mathrm{hess}(f) = 0\}.$$

The latter is called the Gordan-Noether locus and it is well known that

$$\mathcal{C}_{cone} \subseteq \mathcal{C}_{GN} \subseteq \mathcal{C}_{sing}$$

and that $\mathcal{C}_{sing}$ is a divisor in $\mathbb{P}(S^d)$. Moreover, the first inclusion is strict unless $d = 2$ or, by the Gordan-Nother's Theorem 1.1.22, $d \geq 3$ and $n \leq 3$. The second inclusion is an equality for $d = 2$, but is again strict for $d \geq 3$.
From now on, we will deal with the case of cubic polynomials, i.e. we will set $d = 3$.

Let us recall the differential Euler identity (see Lemma 1.4.2), that we can now state in the following way. Let $v = \sum_k v_k e_k \in \mathbb{K}^{n+1}$ and consider $\partial_v \in D^1$, then

$$\forall G \in S^m \qquad (\partial_v)^m(G) = m! \cdot G(v). \tag{3.2}$$

We have the following easy but useful result:

Lemma 3.1.2. *Let f be an element in S^3. Then the following hold:*

(a) For all $v, w \in \mathbb{K}^{n+1}$ we have

$$\mathrm{Hess}(f)_{ij}|_{e_k} = \frac{\partial^2 f}{\partial x_i \partial x_j}|_{e_k} = \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}(= y_i y_j y_k(f)) \qquad \textit{and} \qquad \mathrm{Hess}(f)|_v \cdot w = \nabla(\partial_v \partial_w(f)).$$

In particular, $\mathrm{Hess}(f)|_v \cdot w = \mathrm{Hess}(f)|_w \cdot v$.

(b) For all $v \in \mathbb{K}^{n+1}$ one has $2\nabla(f)|_v = \mathrm{Hess}(f)|_v \cdot v$. In particular, assuming $f \neq 0$, $[v] \in \mathbb{P}^n$ is singular for $V(f)$ if and only if $\mathrm{Hess}(f)|_v \cdot v = 0$.

(c) For all $v, w \in \mathbb{K}^{n+1}$ we have $w^T \cdot \mathrm{Hess}(f)|_v \cdot w = 2(\partial_v f)|_w$.

Proof. **(a)** Since $f \in S^3$, we have $\partial_v \partial_w(f) \in S^1$ and an element $g \in S^1$ is identified by its gradient $\nabla(g)$ by the differential Euler relation (3.2). More precisely, one can easily see that if $g = \sum_k a_k x_k$ then $g|_{e_k} = a_k = y_k g$. This proves the first equality. By $\mathbb{K}$-bilinearity, in order to prove the second equality, it is enough to consider the case $v = e_i$ and $w = e_j$. We have

$$\mathrm{Hess}(f)|_{e_i} \cdot e_j = (\mathrm{Hess}(f)|_{e_i})^j = (\mathrm{Hess}(f)_{kj}|_{e_i})_{k=0}^n = (y_i y_j y_k(f))_{k=0}^n = \nabla(y_i y_j(f))$$

where, if M is a matrix, we set M^j to be its j-th column and M_{ij} to be the i-th entry of M^j.

(b) From (a) we have

$$\mathrm{Hess}(f)|_v \cdot v = \nabla({\partial_v}^2(f)) = (y_k {\partial_v}^2 f)_{k=0}^n = ({\partial_v}^2(y_k f))_{k=0}^n.$$

Now, since $y_k f \in S^2$, by the Euler differential identity (3.2) we have ${\partial_v}^2(y_k f) = 2y_k(f)|_v$, which proves the claim.

(c) Using (a), (b) and the symmetry of $\mathrm{Hess}(f)$, we obtain

$$w^T \cdot \mathrm{Hess}(f)|_v \cdot w = w^T \cdot \mathrm{Hess}(f)|_w \cdot v = v^T \cdot \mathrm{Hess}(f)|_w \cdot w = 2v^T \cdot \nabla(f)|_w.$$

On the other hand, $v^T \cdot \nabla(f) = \partial_v(f)$, so we get the claim. □

In particular, a cubic $V(f) \subseteq \mathbb{P}^n$ is a cone if and only if there exists $[v] \in \mathbb{P}^n$ such that $\mathrm{Hess}(f)|_v \equiv 0$.

Let us now focus on the setting of SAGAs. Let us consider a homogeneous polynomial f of degree $\deg(f) = 3$, i.e. $f \in S^3$, and let us take the SAGA A defined as

$$A = \frac{D}{\mathrm{Ann}_D(f)} = A^0 \oplus A^1 \oplus A^2 \oplus A^3,$$

as described in Example 1.1.4. Let us recall that such a SAGA has socle in degree $N = \deg(f) = 3$ and, moreover, that by Macaulay's Inverse Systems for every SAGA R with socle in degree 3 there exists a cubic polynomial g such that R can be written as the quotient $D/\mathrm{Ann}_D(g)$. Suppose now that the polynomial f is such that the associated hypersurface $V(f) \subset \mathbb{P}^n$ is smooth: in particular it is not a cone and so $(\mathrm{Ann}_D(f))_1 = (0)$. In this case, by definition of SAGA (see Definition 1.1.3), we have $(\dim(A^i))_{i=0,\cdots,3} = (1, n+1, n+1, 1)$. Here, for the SAGA A the only Lefschetz property which makes sense to define is the strong Lefschetz property (in degree 1). Then A satisfies the strong (and weak) Lefschetz property if for $v \in A^1$ general the multiplication map $\mu_1(v) = v\cdot : A^1 \to A^2$ is a bijection. (Here, for simplicity, we identify v with ∂_v as in (3.1).)
Let us then assume that $[v] \in \mathbb{P}(A^1)$ is such that the corresponding multiplication $\mu_1(v)$ is not injective, i.e. $[v] \in \mathbb{P}(A^1)\setminus\mathcal{L}_1$ is a non-Lefschetz element (in degree 1) (see Definition 2.5.1). Then there exists an element (another non-Lefschetz element) $[w]$ $(= \partial_w)$ in $\mathbb{P}(A^1)$ such that $v \cdot w = 0$ in A^2, i.e. $vw(f) = 0$. Now, by Lemma 3.1.2(a) we have that $\mathrm{Hess}(f)|_v \cdot w = \nabla(vw(f))$, but since $vw(f)$ is a homogeneous polynomial of degree 1 we get that $\nabla(vw(f)) = 0$ as a vector if and only if $vw(f) = 0$ as a polynomial. In particular,

$$\mathrm{Hess}(f)|_v \cdot w = 0 \qquad \Longleftrightarrow \qquad vw(f) = 0.$$

In other words, we have proved that if an element $[v] \in \mathbb{P}(A^1)$ (seen as differential operator of degree 1) is a non-Lefschetz element for the SAGA A, then the corresponding $v \in \mathbb{K}^{n+1}$ (under identification (3.1)) is such that the matrix $\mathrm{Hess}(f)|_v$ has non-maximal rank, i.e. $\mathrm{Rank}(\mathrm{Hess}(f)|_v) \leq n$. Finally, by linear algebra, this happens if and only if $[v] \in \mathcal{H}_f = V(\mathrm{hess}(f))$.
Moreover, if the polynomial f is such that $V(f)$ is smooth, than $[f] \notin \mathcal{C}_{GN}$, hence the determinant $\mathrm{hess}(f)$ is not identically zero. In this case, the general element in $\mathbb{P}^n$ doesn't belong to the zero locus $V(\mathrm{hess}(f))$, then, by the above argument, we have also that the general element in $\mathbb{P}(A^1)$ is a Lefschetz element, i.e. $A = D/\mathrm{Ann}_D(f)$ satisfies the strong (and weak) Lefschetz property.
We have then proved the following:

Proposition 3.1.3. *If $f \in S^3$ is a homogeneous polynomial whose associated hypersurface $V(f)$ is smooth, then the SAGA $A = D/\mathrm{Ann}_D(f)$ satisfies the strong Lefschetz property. Moreover, the non-Lefschetz locus (in degree 1) of A coincides (up to isomorphism) to the Hessian hypersurface $\mathcal{H}_f$.*

3.2 Hessians and quadrics

In this section, we describe another natural comparison: the one between quadratic forms and Hessian matrices of cubic forms. Moreover, we derive the first results about, for example, the expected dimension of the singular locus of the Hessian hypersurface associated to a cubic form.
First of all, let us observe that such a Hessian matrix is a symmetric square matrix of order $n+1$ whose entries are homogeneous polynomials of degree 1: after evaluation in a point $v \in \mathbb{K}^{n+1}$, it can be seen as the matrix associated to a quadratic form over $\mathbb{K}^{n+1}$. In particular, as a consequence of the previous Section 3.1 and Lemma 3.1.2, we have that for any $[f] \notin \mathcal{C}_{sing}$ we have a commutative

diagram

$$
\begin{tikzcd}
 & \mathbb{P}^n \arrow[dl, hook', "\tau_1"'] \arrow[dr, hook, "\tau_2"] & \\
\mathbb{P}(S^2) & & \mathbb{P}(\mathrm{M}_{n+1}^{sym}(\mathbb{K})) \arrow[ll, "\simeq"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & {[v]} \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
{[\partial_v(f)]} & & {[\mathrm{Hess}(f)|_v]} \arrow[ll, mapsto, "\simeq"']
\end{tikzcd}
\tag{3.3}
$$

where the diagonal arrows are linear embeddings of $\mathbb{P}^n$, while the horizontal map is the canonical isomorphism $[M] \mapsto [x^T M x]$ which identifies a symmetric matrix M with the quadratic form represented by M. We are interested in studying the geometry of the loci

$$
\mathcal{D}_k(f) = \{[x] \in \mathbb{P}^n \mid \mathrm{Rank}(\mathrm{Hess}(f)|_x) \leq k\} \tag{3.4}
$$

for a general $[f] \in \mathbb{P}(S^3)$, which give a natural stratification not only of the whole projective space $\mathbb{P}^n$, but also of the Hessian locus $\mathcal{H} = V(\mathrm{hess}(f))$, which coincides, as observed in the previous Section 3.1, with $\mathcal{D}_n(f)$. Since the rank of a matrix $M \in \mathrm{M}_{n+1}^{sym}(\mathbb{K})$ and that of the quadratic form $x^T M x$ are the same, it can be useful to study the image of $\mathcal{D}_k$ via the linear embedding τ_1 (or τ_2). Passing from one map to the other will be useful to catch different features of the objects we want to study. If we consider the Jacobian ideal J_f given by f, we have that the image of τ_1 is exactly $\mathbb{P}(J_f^2)$. Then, if we define

$$
\mathcal{Q}_k = \{[q] \in \mathbb{P}(S^2) \mid \mathrm{Rank}(q) \leq k\}
$$

it is clear that $\tau_1(\mathcal{D}_k(f)) = \mathbb{P}(J_f^2) \cap \mathcal{Q}_k$. In what follows, for brevity, we will not specify the linear embedding τ_i (for $i = 1, 2$) in the identification of the loci $\mathcal{D}_k$ with their images. Moreover, we will write simply $\mathcal{D}_k$ instead of $\mathcal{D}_k(f)$ and J^k instead of J_f^k when it is clear from the context which $[f] \in \mathbb{P}(S^3)$ we are considering. In light of this, we recall some important facts about the geometry of the loci $\mathcal{Q}_k$.

Lemma 3.2.1. *For any $1 \leq k \leq n+1$, $\mathcal{Q}_k$ is a closed subvariety of $\mathbb{P}(S^2)$. Moreover*

- *We have $\mathrm{codim}_{\mathbb{P}(S^2)} \mathcal{Q}_k = \binom{n+2-k}{2}$ and $\dim \mathcal{Q}_k = kn - \frac{(k-1)(k-2)}{2}$;*
- *The degree of $\mathcal{Q}_k$ as variety inside $\mathbb{P}(S^2)$ is given by the formula*

$$
\deg(\mathcal{Q}_k) = \prod_{t=0}^{n-k} \frac{\binom{n+t+1}{n-k-t+1}}{\binom{2t+1}{t}}.
$$

- *For $1 \leq k \leq n$, the singular locus of $\mathcal{Q}_k$ coincides with $\mathcal{Q}_{k-1}$.*

Proof. See [Har95, Chapter 22] and [HT84b] for the formula of the degree of $\mathcal{Q}_k$. □

Notice that from the above description, it is clear that $\mathcal{D}_{k-1} \subseteq \mathrm{Sing}(\mathcal{D}_k)$ and one might expect that the equality holds. Actually, we will prove that this is true when f is general (see Theorem 3.4.1 and Corollary 3.4.2) although it does not hold for all $[f] \notin \mathcal{C}_{sing}$ (see Remark 3.4.3).

Remark 3.2.2. *Let us now observe that if we consider an element $[f] \in \mathcal{C}_{GN} \setminus \mathcal{C}_{cone}$, since in this case $\mathrm{hess}(f) \equiv 0$, we have that $\mathcal{D}_n(f)$ is the whole projective space $\mathbb{P}^n$, i.e. $\mathbb{P}(J^2) \subset \mathcal{Q}_n$. In other words, from the point of view of Section 3.1, in this case we have that every element $[v] \in \mathbb{P}(A^1)$*

is a non-Lefschetz element (in degree 1*) for the SAGA* $A = D/\mathrm{Ann}_D(f)$*, i.e.* A *does not satisfy the strong Lefschetz property (in degree* 1*). Notice that this fact has already been stated (and proved) in Lemma 1.4.3.*

Let us then take $[f] \in \mathbb{P}(S^3)$ such that $V(f)$ is smooth and consider the first level of the stratification of $\mathcal{H}_f$ presented above, i.e. the variety $\mathcal{D}_{n-1}(f)$. We can immediately observe that in $\mathbb{P}(S^2)$, which has dimension $\binom{n+2}{2} - 1$, we have the subspaces $\mathbb{P}(J_f^2)$ and $\mathcal{Q}_{n-1}$, whose dimension, are n and $\binom{n+2}{2} - 4$ respectively (from Lemma 3.2.1). Hence, we easily get that the expected dimension of $\mathcal{D}_{n-1}(f) = \mathbb{P}(J^2) \cap \mathcal{Q}_{n-1}$ is

$$\mathrm{Edim}(\mathcal{D}_{n-1}) = \binom{n+2}{2} - 4 + n - \left(\binom{n+2}{2} - 1\right) = n - 3.$$

Since $\mathcal{D}_{n-1} \subseteq \mathrm{Sing}(\mathcal{D}_n) = \mathrm{Sing}(\mathcal{H})$ and $\dim(\mathcal{D}_{n-1}) \geq n-3$, we have the following:

Proposition 3.2.3. *For all* $[f] \notin \mathcal{C}_{sing}$ *we have that the Hessian hypersurface* $\mathcal{H}_f$ *has singular locus of dimension at least* $n-3$ *(i.e.* $\mathrm{Sing}(\mathcal{H}_f)$ *has codimension at most* 2 *in* $\mathcal{H}_f$*). In particular, if* $n \geq 3$*, the Hessian variety is singular.*

The inclusion $\mathcal{D}_{n-1} \subseteq \mathrm{Sing}(\mathcal{H}_f)$ can also be obtained by using Jacobi's formula, which controls the derivatives of the determinant of the Hessian matrix. We will show, generalizing a result in [AR96], that for all $[f] \notin \mathcal{C}_{sing}$ we actually have $\mathcal{D}_{n-1} = \mathrm{Sing}(\mathcal{H}_f)$ (see Theorem 3.3.5) and that $\mathcal{D}_{n-1}$ has the expected dimension, when $[f]$ is general (see Section 3.4).

Remark 3.2.4. *When* $n \leq 4$*, the above mentioned results are known. More precisely:*

- *For* $n = 2$ *it is well known that the Hessian curve associated to the general cubic plane curve is smooth.*
- *For* $n = 3$*, the Hessian surface* $\mathcal{H}$ *associated to the general cubic surface is singular in* 10 *points, which are nodes for* $\mathcal{H}$ *(see* [DvG07]*);*
- *For* $n = 4$*, in* [AR96, Appendix IV] *is proved that the Hessian hypersurface associated to the general cubic threefold is singular along a curve.*

We conclude this section with a remark about the expected dimension of the loci $D_k(f)$.

Remark 3.2.5. *As we have done above, one can argue that for* $[f] \in \mathbb{P}(S^3)$*,*

$$\mathrm{Edim}(\mathcal{D}_k(f)) = n - \binom{n-k+2}{2}$$

is the expected dimension of $\mathcal{D}_k(f) = \mathbb{P}(J_f^2) \cap \mathcal{Q}_k$*. In particular the expected codimension of* $\mathcal{D}_k(f)$ *is exactly the codimension of* $\mathcal{Q}_k$ *in* $\mathbb{P}(S^2)$*:* $\mathrm{Ecodim}_{\mathbb{P}^n}(\mathcal{D}_k(f)) = \mathrm{codim}_{\mathbb{P}(S^2)}(\mathcal{Q}_k)$*.*

3.3 Singular loci and desingularizations

Set $U = \mathbb{P}(S^3) \setminus \mathcal{C}_{sing}$, i.e. the open set parametrizing smooth cubics in $\mathbb{P}^n$ and consider $[f] \in U$. As we have seen, for $n \geq 3$, the Hessian variety $\mathcal{H}_f$ of $[f]$ is singular with $\dim(\mathrm{Sing}(\mathcal{H}_f)) \geq n-3$. The aim of this section is twofold. Firstly, we want to show that for all $[f] \in U$ the singular locus of $\mathcal{H}$ coincides with $\mathcal{D}_{n-1}$. Secondly, we want to describe a way to desingularize $\mathcal{H}_f$ for the general $[f] \in U$. For both results, it will be central the following variety.

For any $[f] \in \mathbb{P}(S^3)$ define

$$\Gamma_f = \{([x],[y]) \in \mathbb{P}^n \times \mathbb{P}^n \mid \mathrm{Hess}(f)|_x \cdot y = 0\} \tag{3.5}$$

and denote by π_1 and π_2 the natural projections from Γ_f on the factors. For brevity, we will simply write Γ when it is clear from the context which $[f] \in \mathbb{P}(S^3)$ we are considering.

Lemma 3.3.1. *The morphism $\tau([x],[y]) = ([y],[x])$ induces a natural involution on Γ which acts freely on Γ if and only if $[f] \notin \mathcal{C}_{sing}$. Moreover, the fiber over $[v] \in \mathbb{P}^n$ is $\mathbb{P}(\mathrm{Ker}(\mathrm{Hess}(f)|_v))$. In particular, the image of π_i is the Hessian locus $\mathcal{H} = \mathcal{D}_n(f)$ and π_i is an isomorphism over the open $\mathcal{H} \setminus \mathcal{D}_{n-1}(f)$.*

Proof. The involution τ on $\mathbb{P}^n \times \mathbb{P}^n$ descends to an involution on Γ since $\mathrm{Hess}(f)|_v \cdot w = \mathrm{Hess}(f)|_w \cdot v$ as proved in Lemma 3.1.2(a). A point $([v],[w]) \in \Gamma$ is a fixed point if and only if $[v] = [w]$ so τ has a fixed point if and only if there exists $[v] \in \mathbb{P}^n$ such that $\mathrm{Hess}(f)|_v \cdot v = 0$, but by Lemma 3.1.2(b), this happens if and only if $V(f)$ is singular. The fiber of π_1 over $[v] \in \mathbb{P}^n$ is $[v] \times \mathbb{P}(\mathrm{Ker}(\mathrm{Hess}(f)|_v))$ by definition of Γ. Then, clearly, $\pi_1^{-1}([v])$ is not empty if and only if we can find a non trivial element in $\mathrm{Ker}(\mathrm{Hess}(f)|_v)$, i.e. if and only if the rank of $\mathrm{Hess}(f)|_v$ is not maximal. This happens exactly when $[v] \in \mathcal{H}$ by definition of $\mathcal{H}$. On $\mathcal{H}^s = \mathcal{H} \setminus \mathcal{D}_{n-1}$ we have only points such that $\mathrm{Rank}(\mathrm{Hess}(f)|_v) = n$ so $\mathrm{Ker}(\mathrm{Hess}(f)|_v)$ has dimension 1. Hence, $\pi_1|_{\pi_1^{-1}(\mathcal{H}^s)} : \pi_1^{-1}(\mathcal{H}^s) \to \mathcal{H}^s$ is an isomorphism. The claim for the second projection follows since $\pi_i \circ \tau = \pi_{3-i}$, for $i = 1, 2$. □

The variety Γ has already been used by Adler in [AR96] in order to desingularize the Hessian locus for $n = 4$. The approach used by Adler involved the study of a specific case, namely the case of the Klein cubic $f_0 = x_0x_4^2 + x_1x_0^2 + x_2x_1^2 + x_3x_2^2 + x_4x_3^2$, and to prove that Γ_{f_0} is smooth. Then, the result holds also for $[f] \in U$ general. Unfortunately, this approach cannot be carried out completely for any n. Nevertheless, the methods used in [AR96] can be used and generalised in order to prove that $\mathrm{Sing}(\mathcal{H}) = \mathcal{D}_{n-1}$ as we will do in the next subsection 3.3.1. Instead, in subsection 3.3.2 we propose a different approach in order to describe the desingularization of $\mathcal{H}$ for any n.

Let us now stress that some objects introduced above, as the Hessian hypersurface $\mathcal{H}_f$ or the variety Γ_f, have another equivalent description (up to isomorphism). Indeed, from Sections 3.1, the pairing $S \times D \to S$ gives a duality $(S^1)^* = D^1$ (where $\{y_i\}_{i=0,\dots,n}$ is the dual basis of $\{x_i\}_{i=0,\dots,n}$) which induces a canonical isomorphism (see identification (3.1))

$$\alpha : \mathbb{P}^n \to \mathbb{P}(D^1) \qquad [v] \mapsto \alpha([v]) = [\partial_v].$$

Lemma 3.3.2. *We have*

$$\mathcal{H}_f = \{[v] \in \mathbb{P}^n \mid \partial_v(f) \text{ has rank at most } n\}, \qquad \Gamma_f = \{([v],[w]) \in \mathbb{P}^n \times \mathbb{P}^n \mid \partial_w\partial_v(f) = 0\}$$

and

$$\mathbb{P}(\mathrm{Ker}(\mathrm{Hess}(f)|_v)) = \{[w] \in \mathbb{P}^n \mid \partial_w \partial_v(f) = 0\}.$$

Proof. Consider $[v] \in \mathbb{P}^n$. One has $[v] \in \mathcal{H} \iff \det(\mathrm{Hess}(f)|_v) = \mathrm{hess}(f)|_v = 0$ by definition. On the other hand this is equivalent to say that $\mathrm{Hess}(f)|_v$ has rank at most n. Using the identification described in diagram (3.3), this is equivalent to ask that $\mathrm{Rank}(\partial_v(f)) \le n$. The second and third claim follow directly from $\mathrm{Hess}(f)|_v \cdot w = \nabla(\partial_v \partial_w(f))$ (see Lemma 3.1.2) since the vanishing of $\nabla(g)$ for $g \in S^1$ is equivalent to the vanishing of g. □

Remark 3.3.3. *Let us observe that this description of the Hessian hypersurface is known, in literature, as the Steinerian hypersurface (see* [Dol12, Section 1.1.6]*). Moreover, in the case of a smooth cubic hypersurface $V(f)$ the associated Hessian $\mathcal{H}_f$ and the Steinerian hypersurface coincide (see* [Dol12, Theorem 3.2.1]*).*

One can then also observe that the variety Γ_f coincides exactly with the incidence correspondence Γ_k (for $k = 1$) introduced in Section 1.2, with respect to the SAGA $A = D/\mathrm{Ann}_D(f)$ and that, moreover, the involution τ described in Lemma 3.3.1 has already been defined in Lemma 1.2.8.
In the following, we will use both the descriptions of the variety Γ_f given so far.

3.3.1 Description of the singular locus

In this subsection we generalize the method used in [AR96] for $n = 4$, and we describe the singular locus of the Hessian hypersurface associated to a smooth cubic polynomial $[f]$ for any n. Recall that

$$\mathcal{D}_{n-1}(f) = \{[x] \in \mathbb{P}^n \mid \mathrm{Rank}(\mathrm{Hess}(f)_{|x}) \le n-1\} \subseteq \mathrm{Sing}(\mathcal{H}).$$

We want to show that the other inclusion holds too, for any $[f] \in U$.
We will use the following:

Lemma 3.3.4. *Let A and B two symmetric matrices of order m with coefficients in a field $\mathbb{K}$ and and consider the block matrix $M = (A|B)$. We have*

$$\mathrm{Rank}(M) < m \quad \iff \quad \mathrm{Ker}(A) \cap \mathrm{Ker}(B) \neq \{0\}.$$

Proof. Firstly, let us suppose that $\mathrm{Ker}(A) \cap \mathrm{Ker}(B) \neq \{0\}$, so that we can take a non trivial element $v \in \mathbb{K}^m$ in this intersection. Up to a change of coordinates, we can assume that $v = e_1$, the first element of the canonical basis of $\mathbb{K}^m$. Then we have $A \cdot e_1 = A^1 = B \cdot e_1 = B^1 = 0$ and, by symmetry, also $A_1 = B_1 = 0$, where, given a matrix C, C^i and C_j denote the i-th column and the j-th row of C respectively. From this we get that $M_1 = 0$ and so the rank of M can not be maximal.
For the converse, let us assume that $\mathrm{Rank}(M) < m$: this means that the subspace generated by the rows of M has dimension strictly smaller than m. Then, there exists $N \in \mathrm{GL}_m(\mathbb{K})$ such that $N \cdot M$ has the first row $(N \cdot M)_1$ identically 0. Since $N \cdot M = (N \cdot A|N \cdot B)$ we have $e_1^T \cdot N \cdot A = e_1^T \cdot N \cdot B = 0$ thus, being A and B symmetric, we clearly have $v = N^T \cdot e_1 \in \mathrm{Ker}(A) \cap \mathrm{Ker}(B)$ with $v \neq 0$. □

We can then prove Theorem E:

Theorem 3.3.5 (Theorem E)**.** *For any $[f] \in U$ we have*

$$\mathcal{D}_{n-1}(f) = \mathrm{Sing}(\mathcal{D}_n(f)),$$

i.e. the singular locus of the Hessian variety $\mathcal{H}_f$ is $\mathcal{D}_{n-1}(f)$.

Proof. As recalled before, it is enough to show that for every $[f] \in U$ the inclusion $\mathrm{Sing}(\mathcal{H}) \subseteq \mathcal{D}_{n-1}$ holds ($\mathcal{H} = \mathcal{H}_f$).
First of all, let us define $\mathcal{H}^s = \mathcal{H} \setminus \mathcal{D}_{n-1}$, an open set of $\mathcal{H}$ that can be described as

$$\mathcal{H}^s = \{[x] \in \mathbb{P}^n \mid \mathrm{Rank}(\mathrm{Hess}(f)|_x) = n\}.$$

Since $[f] \in U$, one can easily show that $\mathcal{H}^s$ is not empty. If we prove that each point in $\mathcal{H}^s$ is a smooth point for $\mathcal{H}$, then we are done.

Let us now consider the variety Γ defined in (3.5) with the two projections π_i, $i = 1, 2$. By defining $\Gamma^s = \pi_1^{-1}(\mathcal{H}^s)$, we know by Lemma 3.3.1 that $\pi_1|_{\Gamma^s} : \Gamma^s \to \mathcal{H}^s$ is an isomorphism with inverse map given by $[x] \mapsto ([x], \mathbb{P}(\mathrm{Ker}(\mathrm{Hess}(f)|_x)))$.

Then we have that a point $[x] \in \mathcal{H}^s$ is smooth in $\mathcal{H}^s$ (so in $\mathcal{H}$) if and only if the point $\pi_1^{-1}([x]) = ([x], [y]) = ([x], \mathbb{P}(\mathrm{Ker}(\mathrm{Hess}(f)|_x)))$ is smooth in Γ. We can now consider the bihomogeneous lifting of Γ to $\mathbb{K}^{n+1} \times \mathbb{K}^{n+1}$, i.e.

$$\tilde{\Gamma} = \{(x, y) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \mid \mathrm{Hess}(f)|_x \cdot y = 0\}.$$

For our aim, we can also check the smoothness of $\tilde{\Gamma}$ in the point $\tilde{p} = (x, y) \in \tilde{\Gamma}$. We can easily describe $\tilde{\Gamma}$ as the zero locus of a suitable function: indeed, we can write $\tilde{\Gamma} = V(\tilde{F})$, where

$$\tilde{F} : \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \to \mathbb{K}^{n+1} \qquad \tilde{F}((x, y)) = \mathrm{Hess}(f)|_x \cdot y.$$

Let us now observe that, by Lemma 3.1.2(a), we have the equality $\mathrm{Hess}(f)|_x \cdot y = \mathrm{Hess}(f)|_y \cdot x$; from this one easily gets that the Jacobian matrix of the map $\tilde{F}$ in $\tilde{p} = (x, y) \in \tilde{\Gamma}$ can be described as

$$J(\tilde{F})|_{\tilde{p}} = (\mathrm{Hess}(f)|_y \mid \mathrm{Hess}(f)|_x),$$

a matrix in $\mathrm{M}_{(n+1)\times 2(n+1)}(\mathbb{K})$. Let us now assume, by contradiction, that $\tilde{p} \in \tilde{\Gamma}$ is a singular point: this implies that the matrix $J(\tilde{F})|_{\tilde{p}}$ is not of maximal rank (i.e. it has rank smaller or equal than n). Observe that, since $\tilde{p} = (x, y)$ and $[x] \in \mathcal{H}^s$, we have that $\mathrm{Rank}(\mathrm{Hess}(f)|_x) = n$, and so $\mathrm{Rank}(J(\tilde{F})|_{\tilde{p}}) = n$. By Lemma 3.3.4, we get that the intersection $\mathrm{Ker}(\mathrm{Hess}(f)|_y) \cap \mathrm{Ker}(\mathrm{Hess}(f)|_x)$ is not trivial and in particular equal to $\langle y \rangle$, since $\mathrm{Ker}(\mathrm{Hess}(f)|_x) = \langle y \rangle$. Then we get that $\mathrm{Hess}(f)|_y \cdot y = 0$, that is a contradiction by Lemma 3.1.2(b), since we are considering $[f] \in U$. Hence, $\tilde{p}$ is a smooth point of $\tilde{\Gamma}$, as claimed. $\square$

The techniques used in [AR96, Appendix IV] can be further generalized in order to characterize the cubics $[f] \in U$ such that Γ_f is singular. We report the method here for completion, generalising it for any value of n.

Let us define, for any $[x] \in \mathbb{P}^n$,

$$\iota([x]) = \mathbb{P}(\mathrm{Ker}(\mathrm{Hess}(f)|_x)).$$

Notice that $\iota([x]) \neq \emptyset$ if and only if $[x] \in \mathcal{H}$ and that $\iota([x]) \subset \mathcal{H}$ since, for any $[y] \in \iota([x])$, we have $[x] \in \iota[y]$ by Lemma 3.1.2(a).

Definition 3.3.6. *Let $[f] \in \mathbb{P}(S^3)$. We say that $T = ([x],[y],[z]) \in \mathcal{H}_f^3$ is a triangle (in $\mathcal{H}_f$) if $[x] \in \iota([y]), [y] \in \iota([z])$ and $[z] \in \iota([x])$.*

Notice that if T is a triangle in $\mathcal{H}$, then any permutation of the triple T is so, by Lemma 3.1.2(a). Moreover, two elements of T cannot be equal since $V(f)$ is smooth, by 3.1.2(b), and also all "vertices" of T actually belong to $\mathrm{Sing}(\mathcal{H})$.

Lemma 3.3.7. *For any $[f] \in \mathbb{P}(S^3)$, we have that the point $([x],[y])$ is singular for Γ_f if and only if there exists $[z]$ such that $T = ([x],[y],[z])$ is a triangle in $\mathcal{H}_f$.*

Proof. Let us consider again the bihomogeneous lifting $\tilde{\Gamma}$ and let us assume that $(x,y) \in \tilde{\Gamma}$ is singular for $\tilde{\Gamma}$, with $x, y \neq 0$. As observed in the proof of the previous Theorem 3.3.5, this is equivalent to asking that $\mathrm{Rank}(\mathrm{Hess}(f)|_y, \mathrm{Hess}(f)|_x) < n$ and this happens exactly if there exists $[z] \in \iota([x]) \cap \iota([y])$. By Lemma 3.1.2(b), we also have that $[x] \in \iota([z])$ and $[y] \in \iota([z])$. But since (x,y) is a point of $\tilde{\Gamma}$ we also have that $[x] \in \iota([y])$. Then, $T = ([x],[y],[z])$ is a triangle in $\mathcal{H}$. □

Adler in [AR96] uses this characterization of the singularities in Γ through the existence of some triangle T in $\mathcal{H}$ to show the smoothness for the variety Γ associated to the Klein cubic. This has been done by studying the rich and particular geometry of this specific cubic and by using also some refined geometrical and algebraic techniques. However, we think that this approach can not be easily exploited and generalized to any dimension. For this reason, in the next subsection we will present a different strategy.

3.3.2 Desingularizing the general Hessian hypersurface

In this subsection, we will prove Theorem F from the Introduction. Here the description of Γ_f presented in Lemma 3.3.2 will be used.

Theorem 3.3.8 (Theorem F)**.** *Let $[f] \in U$ be general. Then Γ_f is smooth and $\pi_i : \Gamma_f \to \mathcal{H}$ is a desingularization of $\mathcal{H}$.*

Proof. Let us consider the incidence variety

$$W = \{([v],[w],[f]) \in \mathbb{P}^n \times \mathbb{P}^n \times U \quad | \quad \partial_v \partial_w(f) = 0\}.$$

First of all, we want to show that W is smooth. In order to do this, it is enough to consider the lifting

$$\tilde{W} = \{(v,w,f) \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \times \tilde{U} \quad | \quad v, w \neq 0, \ \partial_v \partial_w(f) = 0\}$$

where $\tilde{U} = \{f \in S^3 \setminus \{0\} \,|\, V(f) \text{ is smooth}\}$, and to prove that $\tilde{W}$ is smooth. To show that $\tilde{W}$ is smooth we will prove that the Zariski tangent space of $\tilde{W}$ has constant dimension in any point of $\tilde{W}$.

Fix $p = (v,w,f) \in \tilde{W}$. The Zariski tangent space of $\tilde{W}$ in p is contained in $T_p(\mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \times \tilde{U}) = \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \times S^3$ and it is described as the set of triples (v', w', f') such that

$$\partial_{v+tv'} \partial_{w+tw'}(f + tf') \equiv 0 \mod t^2.$$

Expanding this relation we have

$$\partial_{v+tv'}\partial_{w+tw'}(f+tf') = \partial_v\partial_w(f) + t\left(\partial_v\partial_{w'}(f) + \partial_{v'}\partial_w(f) + \partial_v\partial_w(f')\right) + t^2(\cdots)$$

so, for any $(v', w', f') \in \mathbb{K}^{n+1} \times \mathbb{K}^{n+1} \times S^3$ we have

$$(v', w', f') \in T_p\tilde{W} \qquad \Longleftrightarrow \qquad (\star): \quad \partial_v\partial_{w'}(f) + \partial_{v'}\partial_w(f) + \partial_v\partial_w(f') = 0. \tag{3.6}$$

Let $\sigma : \tilde{W} \to \mathbb{K}^{n+1} \times \mathbb{K}^{n+1}$ be the projection on the first two factors.

Claim (I): the image of σ is the open subset

$$V = \{(v, w) \quad | \quad v \text{ and } w \text{ are neither proportional nor } 0\}.$$

Indeed, notice that the fiber of σ over (v, w) with $v, w \neq 0$ is

$$\sigma^{-1}(v, w) = (v, w) \times \{f \in \tilde{U} \,|\, \partial_v\partial_w(f) = 0\}.$$

If v and w are proportional and $f \in \sigma^{-1}(v, w)$, we would have $[f] \in S^3$ such that $V(f)$ is smooth and $\partial_v\partial_v(f) = 0$. This is impossible by Lemma 3.1.2(b), so the image of σ is in V. If v and w are not proportional, one can change coordinates and assume $(v, w) = (e_0, e_1)$. Then, the Fermat cubic $\sum_i x_i^3$ is in the fiber which is, therefore, not empty.

Claim (II): for all $p \in \tilde{W}$ the differential $d_p\sigma$ is surjective and $\mathrm{Ker}(d_p\sigma)$ has constant dimension. Indeed, for any $v', w' \in \mathbb{K}^{n+1}$, we have

$$(d_p\sigma)^{-1}((v', w')) = (v', w') \times \{f' \in S^3 \quad | \quad (v', w', f') \text{ satisfies } (\star)\}.$$

Notice that, for any $v, w \neq 0$, the sequence

$$0 \to \mathrm{Ann}_{S^3}(\partial_v\partial_w) \to S^3 \xrightarrow{\partial_v\partial_w} S^1 \to 0$$

is exact. Then, since $\partial_v\partial_{w'}(f) + \partial_{v'}\partial_w(f)$ is determined by the data $p = (v, w, f)$ and (v', w'), the surjectivity of $\partial_v\partial_w$ in the above sequence gives the surjectivity of $d_p\sigma$. The exactness of the above sequence also implies that $\dim(\mathrm{Ann}_{S^3}(\partial_v\partial_w))$ is constant. By Equation (3.6) we have

$$\mathrm{Ker}(d_p\sigma) = (0, 0) \times \{f' \in S^3 \quad | \quad \partial_v\partial_w(f') = 0\} = (0, 0) \times \mathrm{Ann}_{S^3}(\partial_v\partial_w)$$

so we have proved the claim.

Summing up, $\sigma : \tilde{W} \to V$ is a surjective morphism which is submersive and with $\mathrm{Ker}(d_p(\sigma))$ which has constant dimension. Since the target V is smooth, we have that the dimension of $T_p\tilde{W}$ is then constant, so $\tilde{W}$ (and W) is smooth as claimed.

Consider now the projection $\pi_3 : W \to U \subset \mathbb{P}(S^3)$. Observe that the fiber of π_3 over $[f] \in U$ is exactly the variety Γ_f associated to $[f] \in U$ so π_3 is surjective. Hence, by generic smoothness (see, for example, [Har77, Corollary 10.7]) we have that the general fiber of π_3 is smooth, i.e. Γ_f is smooth, for $[f] \in U$ general. $\square$

3.4 Smoothness of $\mathcal{D}_k(f) \setminus \mathcal{D}_{k-1}(f)$ for $[f]$ general

In this section we want to prove, under suitable assumptions, that for $[f] \in U = \mathbb{P}(S^3) \setminus \mathcal{C}_{sing}$ general, $\mathcal{D}_k(f) \setminus \mathcal{D}_{k-1}(f)$ is smooth and has the expected dimension. The approach will be similar to the one used in Section 3.3 in order to prove Theorem 3.3.8.

As recalled in Lemma 3.2.1, for any $1 \leq k \leq n$, the variety $\mathcal{Q}_k$ parametrizing quadrics in $\mathbb{P}^n$ of rank at most k is singular exactly along $\mathcal{Q}_{k-1}$. Hence, $\mathcal{Q}_k^s = \mathcal{Q}_k \setminus \mathcal{Q}_{k-1}$ is the smooth locus of $\mathcal{Q}_k$. Then,

$$\tilde{\mathcal{Q}}_k = \{q \in S^2 \setminus \{0\} \,|\, \mathrm{Rank}(q) \leq k\} \qquad \text{and} \qquad \tilde{\mathcal{Q}}_k^s = \{q \in \tilde{\mathcal{Q}}_k \,|\, \mathrm{Rank}(q) = k\}$$

are the affine cones (with the origin removed) of $\mathcal{Q}_k$ and $\mathcal{Q}_k^s$ respectively and $\tilde{\mathcal{Q}}_k^s$ is smooth. Set

$$\mathcal{J}_k = \{(q, v, f) \in \tilde{\mathcal{Q}}_k \times \mathbb{K}^{n+1} \times S^3 \quad | \quad v, f \neq 0 \quad \text{and} \quad \partial_v(f) = q\}$$

and $\mathcal{J}_k^s = \{(q, v, f) \in \mathcal{J}_k \,|\, q \in \tilde{\mathcal{Q}}_k^s\}$.

Theorem 3.4.1. *Set $c_{n,k} = \mathrm{codim}_{\mathbb{P}(S^2)}(\mathcal{Q}_k) = \binom{n-k+2}{2}$. For all $1 \leq k \leq n$, the following hold:*

(a) *$\mathcal{J}_k$ and $\mathcal{J}_k^s$ are irreducible of dimension $\dim(S^3) + n - c_{n,k} + 1$;*

(b) *$\mathcal{J}_k^s$ is smooth.*

Set $\mathcal{D}_k^s(f) = \mathcal{D}_k(f) \setminus \mathcal{D}_{k-1}(f)$ for $[f] \in U$. When $[f]$ is general then either $\mathcal{D}_k^s(f) = \emptyset$ or the following hold:

(c) *$\mathcal{D}_k^s(f)$ is smooth, i.e. $\mathrm{Sing}(\mathcal{D}_k(f)) = \mathcal{D}_{k-1}(f)$;*

(d) *We have that $\mathrm{codim}_{\mathbb{P}(S^2)}(\mathcal{Q}_k) = \mathrm{codim}(\mathcal{D}_k(f)) = c_{n,k}$. In particular, $\mathcal{D}_k(f)$ has the expected dimension;*

(e) $\deg(\mathcal{D}_k(f)) = \deg(\mathcal{Q}_k) = \prod_{t=0}^{n-k} \binom{n+t+1}{n-k-t+1} / \binom{2t+1}{t}$.

Proof. First of all, consider the projection

$$\sigma : \mathcal{J}_k \to \tilde{\mathcal{Q}}_k \times (\mathbb{K}^{n+1} \setminus \{0\})$$

and observe that for any $v \neq 0$ we have an exact sequence

$$0 \to \mathrm{Ann}_{S^3}(\partial_v) \to S^3 \xrightarrow{\partial_v} S^2 \to 0 \tag{3.7}$$

since, given (q, v) with $q \in S^2$, $v \neq 0$, one can simply "integrate" q in the direction of v in order to get an f such that $\partial_v(f) = q$. In particular, $a_{n,k} := \dim(\mathrm{Ann}_{S^3}(\partial_v)) = \dim(S^3) - \dim(S^2)$ does not depend on v.

Claim (a): Being ∂_v linear, the fiber of σ over (q, v) is

$$\sigma^{-1}(q, v) = (q, v) \times \{f \,|\, \partial_v(f) = q\} = (q, v) \times (f_0 + \mathrm{Ann}_{S^3}(\partial_v)) \setminus \{0\}$$

where f_0 is any fixed primitive of q in the direction of v. In particular $\sigma^{-1}(q, v)$ is an affine space of dimension $a_{n,k}$, possibly with its origin removed. Thus, since the fibers of σ are irreducible and equidimensional and the target is irreducible (of dimension $\dim(S^2) - c_{n,k} + n + 1$), we can conclude that $\mathcal{J}_k$ is irreducible too and it has the desired dimension. Moreover, the same holds if we restrict our attention to $\mathcal{J}_k^s$ and use the same argument.

Claim (b): It is clear from the above discussion that σ is surjective. Let $p = (q, v, f)$ be any point in $\mathcal{J}_k^s$. The Zariski tangent space to $\mathcal{J}_k^s$ in p is a subspace of

$$T_p(\tilde{\mathcal{Q}}_k^s \times \mathbb{K}^{n+1} \times S^3) = T_q(\tilde{\mathcal{Q}}_k^s) \times \mathbb{K}^{n+1} \times S^3$$

and $(q', v', f') \in T_p\mathcal{J}_k$ if and only if $\partial_{v+tv'}(f + tf') = q + tq' \mod t^2$, i.e. if and only if

$$(\star): \qquad \partial_{v'}(f) + \partial_v(f') = q'$$

holds. The differential $d_p\sigma$ is the map sending (q', v', f') to (q', v'). By the description of the Zariski tangent space it is easy to see that

$$(d_p\sigma)^{-1}(q', v') = (q', v') \times \{f' \in S^3 \mid \partial_v(f') = q' - \partial_{v'}(f)\} \qquad \text{and} \qquad \mathrm{Ker}(d_p\sigma) \simeq \mathrm{Ann}_{S^3}(\partial_v)$$

so from exact sequence (3.7) we have that $d_p\sigma$ is surjective and $\dim(\mathrm{Ker}(d_p\sigma)) = a_{n,k}$ is constant.

Hence, $\sigma|_{\mathcal{J}_k^s} : \mathcal{J}_k^s \to \tilde{\mathcal{Q}}_k^s \times (\mathbb{K}^{n+1} \setminus \{0\})$ is a surjective and submersive map on a smooth target and its differential has kernel of constant dimension $a_{n,k}$. Hence, the Zariski tangent space of $\mathcal{J}_k^s$ has constant dimension equal to $a_{n,k} + \dim(\tilde{\mathcal{Q}}_k^s) + \dim(\mathbb{K}^{n+1}) = \dim(S^3) + n - c_{n,k} + 1 = \dim(\mathcal{J}_k^s)$. In particular, $\mathcal{J}_k^s$ is smooth as claimed.

From now on, let us assume that $\mathcal{D}_k^s(f) = \mathcal{D}_k(f) \setminus \mathcal{D}_{k-1}(f) \neq \emptyset$ for $[f] \in U$ general.

Claim (c)+(d): Consider the map $\pi_3 : \mathcal{J}_k^s \to S^3 \setminus \{0\}$. The fiber of π_3 over $f \in S^3 \setminus \{0\}$, is

$$V_f = \pi_3^{-1}(f) = \{(\partial_v(f), v) \quad | \quad v \neq 0, \quad \partial_v(f) \in \tilde{\mathcal{Q}}_k^s\} \times \{f\}$$

so, by the above assumption, π_3 is dominant. By (b) $\mathcal{J}_k^s$ is smooth so, by generic smoothness, the general fiber of π_3 is smooth, too. In particular, for $[f] \in U$ general we have that $V_f = \pi_3^{-1}(f)$ is smooth of dimension $n - c_{n,k} + 1$ (by (a)). Fix a general $[f] \in U$ and consider the restriction $\beta : V_f \to \mathbb{K}^{n+1} \setminus \{0\}$ of the second projection π_2 to V_f. We claim that β is an embedding and that $\beta(V_f)$ is the affine cone of $\mathcal{D}_k^s(f)$ with the origin removed.

Assume that $\beta(\partial_v(f), v) = \beta(\partial_w(f), w)$. Then $v = w$, so β is injective. The tangent in $p = (q, v)$ to V_f is given by the vectors (q', v') such that $q' \in T_q\tilde{\mathcal{Q}}_k^s$, $v' \in \mathbb{K}^{n+1}$ and $\partial_{v'}(f) = q'$ (see $(\star)$). If $d_p\beta(q', v') = 0$ then (q', v') is such that $v' = 0$ and then $q' = 0$. In particular, β is an embedding. Hence $\beta(V_f)$ is smooth of dimension $n - c_{n,k} + 1$. On the other hand,

$$\beta(V_f) = \{v \in \mathbb{K}^{n+1} \setminus \{0\} \quad | \quad \partial_v(f) \in \mathcal{Q}_k^s\}$$

is the affine cone of $\mathcal{D}_k^s(f)$ with the origin removed so $\mathcal{D}_k^s(f)$ is smooth and has dimension $n - c_{n,k}$. In particular, $\mathrm{Sing}(\mathcal{D}_k(f)) = \mathcal{D}_{k-1}(f)$.

Claim (e): This follows by Lemma 3.2.1 since $\mathcal{D}_k(f) = \mathbb{P}(J_f^2) \cap \mathcal{Q}_k$. □

Corollary 3.4.2. *Assume that $\mathcal{D}_k(f)$ is non-empty for $[f] \in U$ general. Then $\mathcal{D}_k(f) \setminus \mathcal{D}_{k-1}(f)$ is non-empty too for $[f] \in U$ general. In particular, for $[f] \in U$ general, $\mathcal{D}_k(f)$ is reduced, has the expected dimension and* $\mathrm{Sing}(\mathcal{D}_k(f)) = \mathcal{D}_{k-1}(f)$.

Proof. Let $h = \min\{j \mid \mathcal{D}_j(f) \neq \emptyset \quad \text{for general } [f] \in U\}$ so $h \leq k$. The claim is clearly true for $k = h$ so for general $[f] \in U$, $\mathcal{D}_h(f)$ is smooth and has the expected dimension. Let us then take $h+1$. Then

$$\dim(\mathcal{D}_h(f)) = \operatorname{Edim}(\mathcal{D}_h(f)) < \operatorname{Edim}(\mathcal{D}_{h+1}(f)) \leq \dim(\mathcal{D}_{h+1}(f))$$

where the strict inequality in the middle follows from Remark 3.2.5. Hence $\mathcal{D}_{h+1}(f) \setminus \mathcal{D}_h(f) \neq \emptyset$ for $[f]$ general. By recursion one has the thesis. □

Remark 3.4.3. *Notice that, unless $k = n$ (see Theorem 3.3.5), we don't have $\operatorname{Sing}(\mathcal{D}_k(f)) = \mathcal{D}_{k-1}(f)$ for all $[f] \in U$. Indeed, for example, if we consider the Klein cubic fourfold (i.e. $n = 5$ and $f = x_5^2 x_0 + \sum_{i=0}^4 x_i^2 x_{i+1}$) we have that $V(f)$ is smooth, $\mathcal{D}_4(f) \setminus \mathcal{D}_3(f)$ is not empty but it is not smooth. The same holds for the expected dimension: for example, for all $n \geq 2$, the dimension of the singular locus of the Hessian hypersurface associated to the Fermat cubic is 1 more than the expected dimension.*

3.5 Degeneracy Loci

From this section, till the end of this thesis, we will work on the field $\mathbb{C}$ of complex numbers. In this section, we will study these loci $\mathcal{D}_k$ from a different, although natural, perspective. In particular, one can think of these varieties as of degeneracy loci of a specific vector bundle map. For a general treatment of this subject, the reader can refer to [Tu86] and [Laz04].

Let X be a smooth variety of dimension n. We are interested in degeneracy loci of a symmetric morphism between vector bundles (symmetric over each fiber) on X, i.e. a morphism

$$\varphi : E \to E^* \otimes L \quad \text{such that} \quad \varphi = \varphi^* \otimes \operatorname{Id}_L,$$

where E and L are a vector bundle of rank e and a line bundle over X respectively. Then one can define the degeneracy loci at order k for such a map as

$$\mathcal{D}'_k(\varphi) = \{x \in X \mid \operatorname{Rank}(\varphi_x) \leq k\}.$$

Let us summarize in the following theorem, some important results about the non-emptiness and the connectedness of these degeneracy loci (see [FL83], [HT90] and [Tu89]).

Theorem 3.5.1. *Let $\varphi : E \to E^* \otimes L$ be a symmetric morphism of vector bundles of rank e and consider $k \leq e$. If $n \geq \binom{e-k+1}{2}$, then $\mathcal{D}'_k(\varphi)$ is non-empty.*
Moreover, if $(\operatorname{Sym}^2 E^*) \otimes L$ is ample, the following hold:

(a) if k is even and $n - \binom{e-k+1}{2} \geq 1$, then $\mathcal{D}'_k(\varphi)$ is connected;

(b) if k is odd and $n - \binom{e-k+1}{2} \geq e - k$, then $\mathcal{D}'_k(\varphi)$ is connected.

In particular, we can consider a symmetric matrix $M \in \mathrm{M}^{sym}_{n+1}(S^1)$ and the symmetric homomorphism of vector bundles induced by M, namely

$$\varphi_M : \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{M\cdot} \mathcal{O}_{\mathbb{P}^n}^{n+1}(1).$$

By taking $M = \operatorname{Hess}(f)$ with $[f] \in \mathbb{P}(S^3)$, we have that the locus $\mathcal{D}_k(f)$ introduced in Section 3.2 is exactly the degeneracy locus $\mathcal{D}'_k(\varphi_{\operatorname{Hess}(f)})$ introduced above. By Theorem 3.5.1, recalling that $\operatorname{Sing}(\mathcal{H}) = \mathcal{D}_{n-1}$ by 3.3.5, we then have

Proposition 3.5.2. *The following hold:*

(a) if $n \geq 3$, $\mathrm{Sing}(\mathcal{H})$ is non-empty for all $[f] \in \mathbb{P}(S^3)$. Moreover, if $n \geq 4$, it is also connected.

(b) if $3 \leq n \leq 5$, for $[f] \in U$ general, $\mathrm{Sing}(\mathcal{H})$ is smooth of dimension $n-3$.

Proof. Claim (a): the non-emptyness of $\mathcal{D}_{n-1}(f)$ follows directly from Theorem 3.5.1 or from the argument about the expected dimension of Remark 3.2.5. The connectedness for $n \geq 5$ follows again from Theorem 3.5.1 while the case $n=4$ has been treated in [AR96]. For claim (b), if $n = 3, 4$ the result is already known (see Remark 3.2.4) so we can assume $n = 5$. In this case, by (a) we have that $\mathcal{D}_4 \neq \emptyset$. On the other hand, by [RV17, Lemma 2.1] we have that $\mathcal{D}_3 = \emptyset$ when $[f]$ is general. Then, the claim follows from Theorem 3.4.1. □

Let us now consider the case $k = n-2$: we know that $\mathcal{D}_{n-2}$ is contained in the singular locus of $\mathrm{Sing}(\mathcal{H})$. By Theorem 3.5.1 we can easily see that in this case the condition $n \geq \binom{n+1-(n-2)+1}{2} = 6$ tells us that for every $n \geq 6$ the singular locus of the Hessian hypersurface associated to any $[f] \in \mathbb{P}(S^3)$ is itself singular. Then, by Proposition 3.5.2, the case $n = 5$ is the last one where the singular locus of $\mathcal{H}$ is smooth generically. On the other hand, it is also the first one which has still to be analysed (by Remark 3.2.4). We will focus on this specific case in Section 3.6.

Let us now consider again the map $\varphi_M = M\cdot$ where $M \in \mathrm{M}^{sym}_{n+1}(S^1)$ as above. Let us observe that by following the strategy presented with a more general flavour in [HT84a] one can compute the odd Chern classes of $Z := \mathcal{D}'_k(\varphi_M)$ assuming that Z is smooth. In this case, indeed one has that $\mathcal{D}'_{k-1}(\varphi_M)$ is empty. We can then consider the following exact sequence on Z

$$0 \to B \to \mathcal{O}_Z^{n+1} \xrightarrow{\alpha} \mathcal{O}_Z^{n+1}(1) \to C \to 0 \tag{3.8}$$

where α is the restriction to Z of φ_M, and $B = \ker(\alpha)$ and $C = \mathrm{coker}(\alpha)$ are locally free (on Z) of rank $n+1-k$. Since α is symmetric, by dualizing and tensoring with $\mathcal{O}_Z(1)$ we get

$$C = B^* \otimes \mathcal{O}_Z(1).$$

Starting from this we derive an explicit relation satisfied by the canonical divisor K_Z which will be used in Section 3.6.

Proposition 3.5.3. *Assume that $Z = \mathcal{D}'_k(\varphi_M)$ is smooth. Then we have*

$$2K_Z = (n+1)(n-k)H|_Z$$

where H denotes the hyperplane class of $\mathbb{P}^n$.

Proof. As $\mathcal{N}_{Z/\mathbb{P}^n} = (\mathrm{Sym}^2 B^*) \otimes \mathcal{O}_Z(1)$ (see [HT84a]), we get

$$c_1(\mathcal{N}_{Z/\mathbb{P}^n}) = \mathrm{Rank}(\mathrm{Sym}^2 B^*) c_1(\mathcal{O}_Z(1)) + c_1(\mathrm{Sym}^2 B^*) = \binom{n+2-k}{2} H|_Z - (n+2-k)c_1(B).$$

Since $c_1(Z) = c_1(\mathbb{P}^n)|_Z - c_1(\mathcal{N}_{Z/\mathbb{P}^n})$ by the normal exact sequence we have

$$K_Z = -c_1(Z) = -(n+1)H|_Z + \binom{n+2-k}{2} H|_Z - (n+2-k)c_1(B).$$

Since $C = B^* \otimes \mathcal{O}_Z(1)$, we obtain $c_1(C) = (n+1-k)H|_Z - c_1(B)$. From this relation and from the exact sequence (3.8), one easily gets that $2c_1(B) = -kH|_Z$ which allows us to conclude. □

Remark 3.5.4. *Let us notice that Propositions 3.5.2,3.5.3 and Theorem 3.4.1 give a description of* $\mathrm{Sing}(\mathcal{H})$*, where* $\mathcal{H}$ *is the Hessian hypersurface in* $\mathbb{P}^4$ *associated to a general cubic threefold. Indeed, one gets that it is a smooth, irreducible curve of degree* 20 *and genus* 26*, as shown in* [AR96] *with different techniques.*

In the next section 3.6, we will deal with the case $n = 5$ and try to classify the surface $Z = \mathcal{D}_4(f)$ arising as singular locus of the Hessian variety of a general smooth cubic fourfold $V(f)$. In order to do this we will need a construction related to degeneracy loci and covers, that we will develop in the next subsections and that is inspired by a natural question coming from Proposition 3.5.3: one might wonder whether in this case $K_Z = 3H|_Z$ or not, since we know that $2K_Z = 6H|_Z$.

3.5.1 Covers and connectedness

In this subsection we present a general construction that allows us to describe the existence of $2:1$ covers for suitable degeneracy loci of symmetric maps between vector bundles.

Let us start by considering a vector bundle E of rank $e+1$ on an irreducible projective variety X of dimension n and a symmetric map $\varphi : E \to E^* \otimes L$ where L is a line bundle. For any m with $1 \leq m \leq e+1$ one can consider the relative Grassmannian $\pi : \mathbb{G} = G(m, E) \to X$ associated to E, a fiber bundle whose fiber over x is the Grassmannian $G(m, E_x)$ of m-dimensional subspaces of E_x. We will denote by S_E and Q_E the tautological bundle and the universal bundle of $\mathbb{G}$ respectively, which fit into the exact sequence

$$0 \longrightarrow S_E \xrightarrow{\alpha} \pi^*E \longrightarrow Q_E \longrightarrow 0 \tag{3.9}$$

where $\alpha_W : (S_E)_W \simeq W \to (\pi^*E)_W \simeq E_{\pi(W)}$ is the natural inclusion of the subspace $W \subseteq E_{\pi(W)}$ for any $W \in \mathbb{G}$.

Denote by $\alpha^* \otimes \mathrm{id}_{\pi^*L} : \pi^*(E^* \otimes L) \to S_E^* \otimes \pi^*(L)$ the map obtained by dualizing α and then tensoring by $\pi^*(L)$. Then from the diagram

$$\begin{tikzcd}
0 \arrow[r] & Q_E^* \otimes \pi^*L \arrow[r] & \pi^*(E^* \otimes L) \arrow[r, "\alpha^*\otimes\mathrm{id}_{\pi^*(L)}"] & S_E^* \otimes \pi^*(L) \arrow[r] & 0 \\
0 \arrow[r] & S_E \arrow[r, "\alpha"'] \arrow[urr, bend left, "\psi"] & \pi^*E \arrow[u, "\pi^*(\varphi)"] \arrow[r] & Q_E \arrow[r] & 0
\end{tikzcd}$$

one can define the morphism $\psi : S_E \to S_E^* \otimes \pi^*(L)$, as $\psi = (\alpha^* \otimes \mathrm{id}_{\pi^*(L)}) \circ \pi^*(\varphi) \circ \alpha$.

Remark 3.5.5. *As* φ *is symmetric, we have* $\varphi^* \otimes \mathrm{id}_{\pi^*(L)} = \varphi$*, so* $\pi^*(\varphi^* \otimes \mathrm{id}_{\pi^*(L)}) = \pi^*\varphi$*. Hence*

$$\psi^* \otimes \mathrm{id}_{\pi^*L} = (\alpha^* \otimes \mathrm{id}_{\pi^*(L)}) \circ ((\pi^*(\varphi))^* \otimes \mathrm{id}_{\pi^*(L)}) \circ \alpha = (\alpha^* \otimes \mathrm{id}_{\pi^*(L)}) \circ \pi^*(\varphi) \circ \alpha = \psi$$

so ψ *is a symmetric morphism of vector bundles on* $\mathbb{G}$*. Notice, moreover, that we can interpret* ψ *as a section of the bundle* $\mathrm{Hom}^{sym}(S_E, S_E^* \otimes \pi^*(L)) = \mathrm{Sym}^2(S_E^*) \otimes \pi^*(L)$*, i.e.* $\psi \in H^0(\mathrm{Sym}^2(S_E^*) \otimes \pi^*(L))$*.*

We fix now an integer k such that the degeneracy locus $\mathcal{D}'_k(\varphi) = \mathcal{D}'_k$ is non-empty. If $x \in \mathcal{D}'_k$ then $\varphi_x : E_x \to E_x^* \otimes L_x \simeq E_x^*$ is a symmetric morphism. Then, φ_x can be thought either as a symmetric bilinear form or as a polynomial of degree 2 which identifies a quadric $Q_x \subseteq \mathbb{P}^e$. By construction, Q_x is a quadric of rank at most k. Recall that a linear subspace in $\mathbb{P}^e$ is called isotropic with respect to Q_x if it is contained in Q_x.

Remark 3.5.6. *Assume that Q_x has rank exactly k. Then Q_x contains either one or two families of maximal isotropic subspaces. More precisely, Q_x always contains:*

- *two irreducible $\binom{h}{2}$-dimensional families of maximal isotropic $(e-h)$-planes if $k = 2h$;*
- *one irreducible $\binom{h+1}{2}$-dimensional family of maximal isotropic $(e-h-1)$-planes if $k = 2h+1$.*

In order to see this, notice that the vertex of Q_x is a $(e-k)$-plane so, by cutting Q_x with a general $(k-1)$-plane Λ, one has a smooth quadric of dimension equal to $k-2$. Then $Q_x \cap \Lambda$ contains either one family of dimension $\binom{h+1}{2}$ or two families of dimension $\binom{h}{2}$ of $(h-1)$-planes depending on the parity of k (see [GH94, Chapter 6, page 735]*). One then concludes by observing that families of linear spaces in $Q_x \cap \Lambda$ and in Q_x are in bijection via joint union with the vertex of Q_x. If the rank of Q_x is strictly less than k, with the same method, one can show that Q_x always contains $(e-h)$-planes when $k = 2h$ or $(e-h-1)$-planes when $k = 2h+1$.*

From now on set

$$m = \begin{cases} e-h & \text{if } k = 2h+1 \\ e-h+1 & \text{if } k = 2h \end{cases}$$

so that, from Remark 3.5.6, the quadric Q_x contains (possibly non-maximal) isotropic subspaces of dimension $m-1$ for all $x \in \mathcal{D}'_k$.

Lemma 3.5.7. *Let $W \in \mathbb{G} = G(m, E)$ and set $x = \pi(W)$, i.e. $W \subseteq E_x$ is an m-dimensional linear subspace of E_x. Then $\mathbb{P}(W)$ is an isotropic subspace for Q_x if and only if $\psi_W \equiv 0$.*

Proof. Recall that $\alpha_W : W \to E_x$ is the inclusion of W in E_x. Hence, $(\alpha^*)_W = (\alpha_W)^* : E_x^* \to W^*$ takes a linear form on E_x and restrict it to W. Since $\pi^*(\varphi)_W : E_x \to E_x^* \otimes L_x \simeq E_x^*$ is the map sending $v \in E_x$ to $\varphi_x(v)$ one has that

$$\forall\, W \in \mathbb{G}(m, E) \text{ and } \forall\, v \in (S_E)_W = W \qquad \psi_W(v) = \varphi_x(v)|_W,$$

i.e. the linear form $\varphi_x(v)$ on E_x restricted to W. Since $\mathbb{P}(W)$ is an isotropic subspace for Q_x if and only if $\forall v, v' \in W$ one has $\varphi_x(v)(v') = 0$, we have that $\mathbb{P}(W)$ is isotropic for Q_x if and only if $\psi_W \equiv 0$. □

Let us now define the zero locus of the symmetric morphism ψ as $T = Z(\psi) = \{W \in \mathbb{G} \,|\, \psi_W \equiv 0\}$.

Remark 3.5.8. *One can compute the expected dimension of T as*

$$\mathrm{Edim}(T) = \dim(\mathbb{G}) - \mathrm{Rank}(\mathrm{Sym}^2(S_E^*) \otimes \pi^*(L)) = \dim(X) + m(e+1-m) - \binom{m+1}{2}$$

since $T = Z(\psi)$, where ψ is section of the bundle $\mathrm{Sym}^2(S_E^*) \otimes \pi^*(L)$ and $\mathrm{Rank}(S_E^*) = m$.

By construction and, in particular, by Lemma 3.5.7, it is then clear that $\mathcal{D}'_k = \mathcal{D}'_k(\varphi) \subseteq \pi(T)$. Indeed, to any point $x \in \mathcal{D}'_k$ corresponds a quadric Q_x which contains at least one family of isotropic $(m-1)$-planes: for every such plane W, which belongs to T by Lemma 3.5.7 we have that $\psi_W \equiv 0$ and $\pi(W) = x$.

Let us now consider the following easy result of linear algebra, which will allow us to have, in a specific case, also the other inclusion:

Lemma 3.5.9. *Let $W \subseteq V$ be two vector spaces of dimension l and $e+1$ respectively, $\iota : W \to V$ the natural inclusion and let $\eta : V \to V^*$ be a linear map. If $\zeta = \iota^* \circ \eta \circ \iota : W \to W^*$ is the zero map, then the rank of η is at most $2(e+1-l)$.*

Proof. If we define $\tilde{\eta} : W \to V^*$ to be the composition $\eta \circ \iota$, since $\zeta \equiv 0$, we have that $\mathrm{Im}(\tilde{\eta}) \subseteq \mathrm{Ker}(\iota^*)$, which has dimension $e+1-l$, and so $\dim(\mathrm{Im}(\tilde{\eta})) \leq e+1-l$ and also $\dim(\mathrm{Ker}(\tilde{\eta})) \geq l-(e+1-l) = 2l-e-1$. Since ι is injective, we also have that $\dim(\iota(\mathrm{Ker}(\tilde{\eta}))) \geq 2l-e-1$, but $\iota(\mathrm{Ker}(\tilde{\eta})) = W \cap \mathrm{Ker}(\eta) \subseteq \mathrm{Ker}(\eta)$. From this we get that $\mathrm{Ker}(\eta)$ has dimension at least $2l-e-1$ and so $\mathrm{Rank}(\eta) \leq (e+1)-(2l-e-1)$, so we get the claim. □

In particular, for $k = 2h$ even, by setting $l = e-h+1$ in the above Lemma 3.5.9, one also has that $\pi(Z(\psi)) \subseteq \mathcal{D}'_k(\varphi)$:

Corollary 3.5.10. *Assume that $k = 2h$ is even. Then $\pi(Z(\psi)) = \mathcal{D}'_k(\varphi)$.*

Assume, as above, that $k = 2h$ is such that the degeneracy locus $\mathcal{D}'_k(\varphi) = \mathcal{D}'_k$ is non-empty and denote by $\pi : Z(\psi) \to \mathcal{D}'_k(\varphi)$ also the restriction of $\pi : \mathbb{G} \to \mathbb{P}^e$ and consider $m = e-h+1$. The fiber of π over $[x] \in \mathcal{D}'_k$ is, by construction, a variety parametrizing the isotropic $(m-1)$-planes in Q_x. If $[x] \in \mathcal{D}'_k \setminus \mathcal{D}'_{k-1}$, $\pi^{-1}([x])$ parametrizes maximal isotropic $(m-1)$-planes in Q_x so it has two irreducible disjoint components of dimension $\binom{h}{2}$ by Remark 3.5.6. One can consider the Stein factorization of π, i.e.

$$\begin{tikzcd} & T = Z(\psi) \arrow[ld, "\alpha"'] \arrow[r, hook] \arrow[d, "\pi"] & \mathbb{G} \arrow[d, "\pi"] \\ \tilde{Z} \arrow[rd, "\beta"'] & & \\ & Z = \mathcal{D}'_k(\varphi) \arrow[r, hook] & \mathbb{P}^e \end{tikzcd} \tag{3.10}$$

where α has connected fibers and β is finite. From the above discussion, one can see that the map β is a $2:1$ morphism, whose possible ramification lies in $\beta^{-1}(\mathcal{D}'_{k-1})$.

It is then interesting to focus on the case where $\mathcal{D}'_{k-1}$ is empty (this happens, for example, when $\mathcal{D}'_k$ is smooth) and $Z = \mathcal{D}'_k$ is connected: in this situation one can wonder whether the finite map β, which is then an unramified $2:1$ cover, is non-trivial. This covering is trivial if and only if $\tilde{Z}$ is not connected, i.e. if and only if T is not connected since α has connected fibers. Notice that, even if we assume that E and k satisfy the hypotheses of Theorem 3.5.1(a) we cannot guarantee the connectedness of $T = Z(\psi) = \mathcal{D}'_0(\psi)$ since $(\mathrm{Sym}^2 S_E^*) \otimes \pi^* L$ is not ample in general.

Let us now propose a sufficient condition which allows us to obtain the connectedness of T, under suitable assumptions that will be satisfied in the case we will consider.

Lemma 3.5.11. *Assume that*

$$0 \to \mathcal{F}_p \xrightarrow{\lambda_{p-1}} \cdots \xrightarrow{\lambda_2} \mathcal{F}_2 \xrightarrow{\lambda_1} \mathcal{F}_1 \xrightarrow{\lambda_0} \mathcal{F}_0 \to 0$$

is an exact sequence of sheaves. Let $k \geq 0$ and assume that $H^{j+k-1}(\mathcal{F}_j) = 0$ for all j such that $1 \leq j \leq p$. Then one has $H^k(\mathcal{F}_0) = 0$.

Proof. Split the starting exact sequence into short exact sequences of the form

$$(\star_j): \qquad 0 \to K_j \to \mathcal{F}_j \to K_{j-1} \to 0$$

for $1 \leq j \leq p-1$ with $K_0 = \mathcal{F}_0$ and $K_{p-1} = \mathcal{F}_p$:

$$\begin{tikzcd}
 & & K_{p-1} \arrow[dr, hook] & & & & K_2 \arrow[dr, hook] & & & & K_0 \arrow[dr, "="] & & \\
0 \arrow[r] & \mathcal{F}_p \arrow[ur, "="] \arrow[rr, "\lambda_{p-1}"] & & \mathcal{F}_{p-1} \arrow[rr, "\lambda_{p-2}"] \arrow[dr, hook] & & \cdots \arrow[rr, "\lambda_2"] & & \mathcal{F}_2 \arrow[rr, "\lambda_1"] \arrow[dr] & & \mathcal{F}_1 \arrow[rr, two heads, "\lambda_0"] \arrow[ur] & & \mathcal{F}_0 \arrow[r] & 0. \\
 & & & & K_{p-2} \arrow[uurr, dotted, no head] & & & & K_1 \arrow[ur, hook] & & & &
\end{tikzcd}$$

From exact sequence $(\star_{p-1})$ and by assumption we have

$$\cdots \to \underbrace{H^{p-2+k}(\mathcal{F}_{p-1})}_{=0} \to H^{p-2+k}(K_{p-2}) \to \underbrace{H^{p-1+k}(\mathcal{F}_p)}_{=0} \to \cdots$$

so $H^{p-2+k}(K_{p-2}) = 0$. By a recursive argument we can show that if we have $H^{j+k}(K_j) = 0$ and $H^{j-1+k}(\mathcal{F}_j) = 0$ (the latter is true by assumption), then also $H^{j-1+k}(K_{j-1}) = 0$ holds. This claim follows easily from the long exact sequence in cohomology induced by $(\star_j)$:

$$\cdots \to \underbrace{H^{j-1+k}(\mathcal{F}_j)}_{=0} \to H^{j-1+k}(K_{j-1}) \to \underbrace{H^{j+k}(K_j)}_{=0} \to \cdots .$$

At the end of this process we get $0 = H^k(K_0) = H^k(\mathcal{F}_0)$ as desired. □

Corollary 3.5.12. *Let X be a smooth connected variety and T a smooth subvariety of X which is the zero locus of a non-trivial section θ of a vector bundle $\mathcal{P}$ on X of rank p.*

(a) For any $M \in \mathrm{Pic}(X)$ and $k \geq 0$, if $H^{j+k-1}(M \otimes \bigwedge^j \mathcal{P}^*) = 0$ for $1 \leq j \leq p$, then one has $H^k(\mathcal{I}_{T/X} \otimes M) = 0$;

(b) If $H^j(\bigwedge^j \mathcal{P}^*) = 0$ for all j, then $h^0(\mathcal{O}_T) = 1$;

(c) For any $k \geq 1$, if $h^k(\mathcal{O}_X) = 0$ and $H^{j+k}(\bigwedge^j \mathcal{P}^*) = 0$ for all j, then $h^k(\mathcal{O}_T) = 0$.

Proof. Since Z is smooth by assumption, the Koszul sequence induced by θ

$$0 \longrightarrow \bigwedge^p \mathcal{P}^* \xrightarrow{\lambda_{p-1}} \cdots \xrightarrow{\lambda_2} \bigwedge^2 \mathcal{P}^* \xrightarrow{\lambda_1} \mathcal{P}^* \xrightarrow{\lambda_0} \mathcal{I}_{T/X} \longrightarrow 0$$

is exact (see for example [GH94, Chapter 5, pag. 687]). By tensoring with $M \in \mathrm{Pic}(X)$, we preserve the exactness of such sequence, so (a) follows directly from Lemma 3.5.11.
For (b), let us consider the exact sequence

$$0 \to \mathcal{I}_{T/X} \to \mathcal{O}_X \to \mathcal{O}_T \to 0.$$

Since X is connected we have an exact sequence

$$0 \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_T) \to H^1(\mathcal{I}_{T/X})$$

hence the vanishing of $H^1(\mathcal{I}_{T/X})$ implies the connectedness of T. But since $H^j(\bigwedge^j \mathcal{P}^*) = 0$ by assumption, we can conclude, by using (a) (for $k = 1$).
For (c), from the same sequence, we have an injection

$$0 = H^k(\mathcal{O}_X) \to H^k(\mathcal{O}_T) \hookrightarrow H^{k+1}(\mathcal{I}_{T/X}).$$

We can then conclude since, under our assumption, we get that $H^{k+1}(I_{T/X})$ is zero, again by using (a). □

Hence, if we assume that the degeneracy locus $Z = \mathcal{D}'_k(\varphi)$ is connected and smooth, one can show that the map $\beta : \tilde{Z} \to Z$ introduced above is a non-trivial unramified covering, by proving that $H^j(\bigwedge^j \mathcal{P}^*) = 0$ with $\mathcal{P} = \mathrm{Sym}^2(S_E^*) \otimes \pi^* L$ whenever $1 \leq j \leq \binom{m+1}{2}$.

3.5.2 An application: the case $n = 5$

As an application of the above discussion, let us set $X = \mathbb{P}^5$ and $S = \mathbb{C}[x_0, \cdots, x_5]$ and let us consider $\varphi = \varphi_M : \mathcal{O}_{\mathbb{P}^5}^6 \to \mathcal{O}_{\mathbb{P}^5}(1)^6$ where $M \in \mathrm{M}_6^{sym}(S^1)$. For example, one can take $M = \mathrm{Hess}(f)$ for $[f] \in \mathbb{P}(S^3)$ general.

As we have done in the previous sections, the degeneracy locus $Z = \mathcal{D}'_4(\varphi)$ is a surface. In particular, let us assume that such a surface Z is smooth (this happens for M general by Proposition 3.5.2), so that $\mathcal{D}'_3(\varphi) = \emptyset$. Here with respect to the notations of Subsection 3.5.1 (and the objects in Diagram (3.10)), we have $k = 4$ and $h = 2$, so $\tilde{Z}$ is a smooth surface which is an unramified double cover of Z. The fiber of $\alpha : T \to \tilde{Z}$ over $p \in \beta^{-1}([x])$ is a $\mathbb{P}^1$ which parametrizes one of the two irreducible families of maximal isotropic 3-planes contained in the quadric Q_x. Hence, α has irreducible and equidimensional fibers so Z is a smooth threefold and we can apply Corollary 3.5.12 in order to study the connectedness of the covering β. Indeed, we have that $\tilde{Z}$ is connected if T is so and this is equivalent to ask $h^0(\mathcal{O}_T) = 1$.

Notice that since $E = \mathcal{O}_{\mathbb{P}^5}^6$, we have $\mathbb{G} = G(4, \mathcal{O}_{\mathbb{P}^5}^6) = G(3, \mathbb{P}^5) \times \mathbb{P}^5$ so $h^1(\mathcal{O}_{\mathbb{G}}) = 0$. We denote by π_1 and π_2 the two natural projections. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on $G(3, \mathbb{P}^5)$ and $\mathbb{P}^5$ respectively, we set $\pi_1^*(\mathcal{F}) \otimes \pi_2^*(\mathcal{G}) := \mathcal{F} \boxtimes \mathcal{G}$ for brevity. Recall that T is the zero locus of a section of the vector bundle $\mathcal{P} = \mathrm{Sym}^2(S^*_{\mathcal{O}_{\mathbb{P}^5}^6}) \otimes \pi^* \mathcal{O}_{\mathbb{P}^5}(1)$ which, in this case, can be written as $\mathcal{P} = \mathrm{Sym}^2(S^*) \boxtimes \mathcal{O}_{\mathbb{P}^5}(1)$ where S is the tautological bundle of $G(3, \mathbb{P}^5)$. Thus, we have

$$\bigwedge^j \mathcal{P}^* = \bigwedge^j \left(\mathrm{Sym}^2(S) \boxtimes \mathcal{O}_{\mathbb{P}^5}(-1)\right) = \left(\bigwedge^j \mathrm{Sym}^2(S)\right) \boxtimes \mathcal{O}_{\mathbb{P}^5}(-j) \tag{3.11}$$

so it is clear that the vanishing of the cohomology group $H^i(\bigwedge^j \mathcal{P})$ strongly depends on the vanishing of some cohomology groups of $\bigwedge^j \mathrm{Sym}^2(S)$ on the Grassmannian $G(3,\mathbb{P}^5)$. The latter can be tested using Bott's Theorem on homogeneous vector bundles on Grassmannians. The interested reader can refer to [FH91] and [Ott89]. An explicit and standard, but long computation (see Appendix A for a detailed proof) shows the following:

Proposition 3.5.13. *One has $H^i(\bigwedge^j \mathrm{Sym}^2 S) = 0$ for all pairs (i,j) with $i \geq 0$, $0 \leq j \leq 10$ except for the cases where $(i,j) \in \{(2,2),(2,3),(2,4),(4,5),(4,6),(4,7),(6,9)\}$. For these cases, $H^i(\bigwedge^j \mathrm{Sym}^2 S) \neq 0$.*

For all $d \in \mathbb{Z}$, from Equation (3.11) and by Künnet's Theorem we have

$$H^i\left(\pi_2^*\mathcal{O}_{\mathbb{P}^5}(d) \otimes \bigwedge^j \mathcal{P}^*\right) \simeq \bigoplus_{a+b=i}\left(H^a\left(\bigwedge^j \mathrm{Sym}^2 S\right) \otimes H^b(\mathcal{O}_{\mathbb{P}^5}(d-j))\right). \tag{3.12}$$

In particular, since for $j \geq 1$ the group $H^b(\mathcal{O}_{\mathbb{P}^5}(-j))$ is trivial whenever $b \neq 5$, we have

$$H^j\left(\bigwedge^j \mathcal{P}^*\right) \simeq H^{j-5}\left(\bigwedge^j \mathrm{Sym}^2 S\right) \otimes H^5(\mathcal{O}_{\mathbb{P}^5}(-j)).$$

Hence, by Proposition 3.5.13 and since $H^5(\mathcal{O}_{\mathbb{P}^5}(-j)) \simeq H^0(\mathcal{O}_{\mathbb{P}^5}(j-6))^* = 0$ for $j < 6$, we obtain $H^j\left(\bigwedge^j \mathcal{P}^*\right) = 0$. Then, using Corollary 3.5.12(b) we have that T is connected and the same holds for $\tilde{Z}$. With the same reasoning just used, we also get that $H^{j+1}(\bigwedge^j \mathcal{P}^*) = 0$ for $j \geq 1$ so we can conclude, again by Corollary 3.5.12(c), that $h^1(\mathcal{O}_T) = 0$.

Similarly, for $d \geq 0$ one has

$$H^{j-1}\left(\pi_2^*(\mathcal{O}_{\mathbb{P}^5}(d)) \otimes \bigwedge^j \mathcal{P}^*\right) \simeq \begin{cases} \text{if } j \leq d & H^{j-1}\left(\bigwedge^j \mathrm{Sym}^2 S\right) \otimes H^0(\mathcal{O}_{\mathbb{P}^5}(d-j)) \\ \text{if } j > d & H^{j-6}\left(\bigwedge^j \mathrm{Sym}^2 S\right) \otimes H^5(\mathcal{O}_{\mathbb{P}^5}(d-j)) \end{cases}$$

which is always trivial if $d \leq 2$ by Proposition 3.5.13. Hence, by Corollary 3.5.12(a), one has $H^0(\mathcal{I}_{T/\mathbb{G}} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(d)) = 0$ for $d = 1,2$.

Summing up, one has the following

Proposition 3.5.14. *Assume that $M \in \mathrm{M}_6^{sym}(S^1)$ is general. Then the variety T constructed above is a smooth connected threefold with $h^1(\mathcal{O}_T) = 0$, $h^0(\mathcal{I}_{T/\mathbb{G}} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(1)) = 0$ and $h^0(\mathcal{I}_{T/\mathbb{G}} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(2)) = 0$. Moreover, $\tilde{Z}$ is a connected surface so $\beta : \tilde{Z} \to Z$ is a non-trivial unramified double cover.*

3.6 The case of a general smooth cubic fourfold

In this last section we will focus on the case $n = 5$ and we will study the properties of the singular locus $Z(=\mathcal{D}_{n-1=4}(f))$ of the Hessian variety $\mathcal{H}_f$ associated to the general smooth cubic fourfold $X = V(f) \subset \mathbb{P}^5$, for $[f] \in U = \mathbb{P}(S^3) \setminus \mathcal{C}_{sing}$, where $S = \mathbb{C}[x_0, \cdots, x_5]$: in particular, we will prove Theorem G from the Introduction. For the topics of this section, the reader can refer for example to [Bea96, GH94, Har77].

The starting point of this analysis is based on some results obtained more generally in the previous sections which we sum up in this lemma:

Lemma 3.6.1. *Assume that $[f] \in U$ is general and denote by H the hyperplane class of $\mathbb{P}^5$. We have the following:*

(a) *$\mathcal{H}_f = \mathcal{D}_5(f)$ has singular locus $Z = \mathrm{Sing}(\mathcal{H}_f) = \mathcal{D}_4(f)$ which has dimension 2;*

(b) *Z is connected, smooth and it is cut by 21 quintic hypersurfaces;*

(c) *As subvariety of $\mathbb{P}^5$, Z has degree 35;*

(d) *There exists $\eta \in \mathrm{Pic}^0(Z)[2]$ such that $3H_{|Z} + \eta = K_Z$.*

In particular, Z is a minimal surface of general type.

Proof. Most of the information for (a), (b) and (c) follows from Theorem 3.4.1 and Propositions 3.5.2. The surface Z is the degeneracy locus at rank 4 of the Hessian matrix $\mathrm{Hess}(f)$, which is a symmetric matrix of order 6, so it is defined by the vanishing of the 36 minors of $\mathrm{Hess}(f)$ of order 5. Each minor gives a quintic equation and, by symmetry, it is enough to consider only 21 of them. For (d), by Proposition 3.5.3 we have $2K_Z = 6H_{|Z}$ so there exists a possibly trivial torsion line bundle η of order 2 such that $K_Z = 3H_{|Z} + \eta$. In particular, K_Z is numerically equivalent to $3H_{|Z}$ which is ample since $3H$ is ample on $\mathbb{P}^5$ so K_Z is ample too by Moishezon-Nakai criterion (see [Har77, Theorem 1.10, pag.365]). Hence, Z is a minimal surface, since we don't have (-1)-curves. Moreover, it is of general type (see for example [Bea96, Prop. X.1]). □

We stress that K_Z is not linearly equivalent to $3H_{|Z}$ as we will show later. Now, let us compute the main invariants of the surface Z:

Proposition 3.6.2. *Let Z be as above. Then*

(a) *The (topological) Euler characteristic of Z is $e(Z) = 357$;*

(b) *The Hilbert polynomial of Z is $\chi(\mathcal{O}_Z(n)) = \frac{35}{2}n^2 - \frac{105}{2}n + 56$;*

(c) *There exists a non-trivial unramified double covering $\tilde{Z} \to Z$;*

(d) *Z is regular (i.e. it has irregularity $q(Z) = 0$), its geometric genus is $p_g(Z) = 55$ and $h^{1,1}(Z)(= \dim(H^1(Z, \Omega^1_Z)) = 245$;*

(e) *$h^0(I_{Z/\mathbb{P}^5}(1)) = h^0(I_{Z/\mathbb{P}^5}(2)) = 0$.*

Proof. Let us start with the computation of $e(Z)$. One can compute $e(Z)$ by using a computer algebra software (we will use this approach later in order to compute some cohomology groups), but actually here we adapt a formula presented and proved in [Pra88, Proposition 7.13], which in our specific case is

$$c_2(Z) = \sum_{(i_1,i_2)} (-1)^{i_1+i_2}((i_1+1, i_2))Q_{(i_1+2,i_2+1)}\left(\mathcal{O}^6_{\mathbb{P}^5}(1/2)\right) c_{2-i_1-i_2}(\mathbb{P}^5)$$

where (i_1, i_2) ranges in $\{(2,0),(1,1),(1,0),(0,0)\}$ and where $\mathcal{O}_{\mathbb{P}^5}(1/2)$ is a formal line bundle whose square is $\mathcal{O}_{\mathbb{P}^5}(1)$ (one can be more precise by invoking squaring principle). For brevity, we don't report the definition of the coefficients $((a,b))$ and of the Q-Schur polynomial $Q_{(a,b)}$. Nevertheless, we give the values of the non-vanishing Q-Schur polynomial evaluated in $\mathcal{O}^6_{\mathbb{P}^5}(1/2)$ and of the coefficients $((a,b))$ appearing in the above formula:

$$\begin{array}{cccc} Q_{2,1} = 35H^3 & Q_{3,1} = 105H^4 & Q_{4,1} = \frac{777}{4}H^5 & Q_{3,2} = \frac{483}{4}H^5 \\ ((1,0)) = 1 & ((2,0)) = 3 & ((3,0)) = 7 & ((2,1)) = 3 \end{array}$$

Summing up and developing the computation, one gets $e(Z) = \deg(c_2(Z)) = 357$.

By Lemma 3.6.1(c) we have that Z, as subvariety of $\mathbb{P}^5$, has degree 35. Hence we have $H^2_{|Z} = 35$. As $K_Z \equiv_{num} 3H_{|Z}$ by Lemma 3.6.1(d), we have that $K_Z^2 = 315$ and $K_Z \cdot H_{|Z} = 105$. Then we have

$$\chi(\mathcal{O}_Z) = \frac{e(Z) + K_Z^2}{12} = \frac{357 + 315}{12} = 56$$

by Noether's formula (see for example [Bea96, I.14]) and

$$\chi(\mathcal{O}_Z(n)) = 56 + \frac{1}{2}(nH_{|Z})(nH_{|Z} - K_Z) = 56 + \frac{1}{2}((H^2_{|Z})n^2 - (H_{|Z}\cdot K_Z)n) = \frac{35}{2}n^2 - \frac{105}{2}n + 56.$$

We can apply the double cover construction of Subsections 3.5.1 and 3.5.2 to $Z = \mathcal{D}_4(f) = \mathcal{D}'(\varphi_{\mathrm{Hess}(f)})$ in order to construct the threefold T, the unramified double covering $\beta : \tilde{Z} \to Z$ and the morphisms π and α (see Diagram (3.10)). We can apply Proposition 3.5.14 in order to see that $h^1(\mathcal{O}_T) = 0$ and that $\beta : \tilde{Z} \to Z$ is indeed a non-trivial unramified double covering. Since $\pi : T \to Z$ is surjective, we have that $\pi^* : H^0(\Omega^1_Z) \to H^0(\Omega^1_T)$ is injective. As $h^1(\mathcal{O}_T) = h^0(\Omega^1_T) = 0$ we have then $q(Z) = h^1(\mathcal{O}_Z) = 0$. The last two invariants, namely $h^{1,1}(Z)$ and $p_g(Z)$, are easily computed knowing that $\chi(\mathcal{O}_Z) = 56$, $q(Z) = 0$ and $e(Z) = 357$. Indeed, we have that $56 = \chi(\mathcal{O}_Z) = 1 - q(Z) + p_g(Z)$, from which we get that $p_g(Z) = 55$; moreover, $357 = e(Z) = 2 - 4q(Z) + 2p_g(Z) + h^{1,1}(Z)$ and one obtains that $h^{1,1}(Z) = 245$.

Claim (e) follows since a non-trivial section of $\mathcal{I}_{Z/\mathbb{P}^5}(d)$ induces, via pullback, a non-trivial section of $\mathcal{I}_{T/\mathbb{G}} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(d)$ and we know by Proposition 3.5.14 that $h^0(\mathcal{I}_{T/\mathbb{G}} \otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(d)) = 0$ for $d = 0, 1$. □

The approach described in Subsection 3.5.2 is not powerful enough to prove the vanishing $h^0(\mathcal{I}_{T/\mathbb{G}}\otimes \pi_2^*\mathcal{O}_{\mathbb{P}^5}(d)) = 0$ for $d = 3, 4$ and so we cannot use this method in order to conclude that $h^0(\mathcal{I}_{Z/\mathbb{P}^5}(d)) = 0$ for $d = 3, 4$. Nevertheless, by semicontinuity it is enough to establish the vanishing for a single example in order to have it for the general one. With this approach, one can use a computer algebra software (like *Magma*) in order to compute the Hilbert series $h_{Z_0}(t)$ (see for example [Har95, Lecture 13]) associated to the surface Z_0 for a specific case, namely the Klein cubic fourfold

$$X_0 = \{f_0 = 0\} \quad \text{where } f_0 = x_0^2x_1 + x_1^2x_2 + x_2^2x_3 + x_3^2x_4 + x_4^2x_5 + x_5^2x_0.$$

Defining, for brevity, $x_k = x_i$ for any $k \in \mathbb{Z}$, $i \in \{0, \dots, 5\}$ if and only if $k \equiv i \mod 6$, one has

$$\mathrm{hess}(f_0) = \sum_{i=0}^{2} x_i^3x_{i+3}^3 - x_0x_1x_2x_3x_4x_5 + \sum_{i=0}^{5} x_ix_{i+1}^3x_{i+3}^2 - \sum_{i=0}^{5} x_ix_{i+1}x_{i+2}x_{i+3}^3 - \sum_{i=0}^{1} x_i^2x_{i+2}^2x_{i+4}^2.$$

Let us observe that $Z_0 = \mathcal{D}_4(f_0)$ has the expected dimension: this fact is proved directly in Appendix B, by using the same reasoning presented in [AR96] for the case of the Klein threefold. By using *Magma*, one obtains

$$h_{Z_0}(t) = \frac{15t^4 + 10t^3 + 6t^2 + 3t + 1}{(1-t)^3} = \sum_{i=0}^{\infty} h^{(i)}_{Z_0} t^i = 1 + 6t + 21t^2 + 56t^3 + 126t^4 + 231t^5 \mod t^6. \quad (3.13)$$

Remark 3.6.3. *Let us recall that, by definition, the coefficients $h^{(i)}_{Z_0}$ of the above Hilbert series represent the dimensions of $S^{(i)}_{Z_0}$, the degree i part of the homogeneous coordinate ring S_{Z_0} of Z_0, i.e. $S/I_{Z_0/\mathbb{P}^5}$. Moreover, let us observe that $S^{(i)}_{Z_0}$ coincides also with the image of $S^i = H^0(\mathcal{O}_{\mathbb{P}^5}(i))$ in $H^0(O_{Z_0}(i))$ via the map induced by the exact sequence*

$$0 \to \mathcal{I}_{Z_0/\mathbb{P}^5}(i) \to \mathcal{O}_{\mathbb{P}^5}(i) \to \mathcal{O}_{Z_0}(i) \to 0.$$

Actually, this is also the Hilbert series for Z, singular locus of $\mathcal{H}_f$ for $[f] \in U$ general, since $h_Z(t)$ is constant for flat families (and we are considering degeneracy loci associated to a morphism of vector bundles). This has several consequences.

Proposition 3.6.4. *Let $[f] \in U$ be general and let Z be as above. Then*

(a) The 2-torsion element η such that $K_Z = 3H_{|Z} + \eta$ is non-trivial;

(b) $h^0(\mathcal{I}_{Z/\mathbb{P}^5}(d)) = 0$ for $d \leq 4$ and $h^0(\mathcal{I}_{Z/\mathbb{P}^5}(5)) = 21$.

Proof. In order to prove (a), notice that, by Remark 3.6.3 the coefficient $h^{(3)}_Z$ of the Hilbert series of Z is 56, which equals, by definition, the dimension of $S^{(3)}_Z$, that is the image of $S^3 = H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ in $H^0(\mathcal{O}_Z(3))$ via the map induced by the exact sequence

$$0 \to \mathcal{I}_{Z/\mathbb{P}^5}(3) \to \mathcal{O}_{\mathbb{P}^5}(3) \to \mathcal{O}_Z(3) \to 0.$$

Then, if $K_Z = 3H_{|Z}$ we would have a contradiction since we would obtain, by Proposition 3.6.2, $55 = p_g(Z) = h^0(\mathcal{O}_Z(3)) \geq 56$. Hence η is a non-trivial 2-torsion element of $\mathrm{Pic}(Z)$.

For (b), again by Remark 3.6.3, notice that $h^{(d)}_Z = \dim(S^d) = h^0(\mathcal{O}_{\mathbb{P}^5}(d))$ for $d \leq 4$. Hence $H^0(\mathcal{I}_{Z/\mathbb{P}^5}(d))$, which equals the kernel of the map $S^d \to H^0(\mathcal{O}_Z(d))$, is trivial for $d \leq 4$. For $d = 5$ one has $h^{(5)}_Z = 231 = 252 - 21 = \dim(S^5) - 21$ so $h^0(\mathcal{I}_{Z/\mathbb{P}^5}(5)) = 21$ with the same argument as before. $\square$

Recall that we proved in Subsection 3.5.2 that Z has a natural non-trivial unramified double cover. This corresponds to a 2-torsion line bundles η' on Z. An intriguing question is whether η and η' coincide. We conjecture the following:

Conjecture 3.6.5. *We have $\eta = \eta'$ for $[f] \in U$ general.*

We conclude this section by exploiting again *Magma* in order to obtain the following data which hold for Z associated to $[f] \in U$ general:

d	0	1	2	3
$h^0(\mathcal{O}_Z(d))$	1	6	21	56
$h^1(\mathcal{O}_Z(d))$	0	0	0	0
$h^2(\mathcal{O}_Z(d))$	55	15	0	0

(3.14)

By using this, we can show the following result, concerning the projective normality of Z (in the general case). Let us recall that to prove this property we have to show that for every $d \geq 0$ the map $H^0(\mathbb{P}^5, \mathcal{O}_{\mathbb{P}^5}(d)) \to H^0(Z, \mathcal{O}_Z(d))$ is surjective, or equivalently, that the group $H^1(\mathcal{I}_{Z/\mathbb{P}^5}(d))$ is trivial for every $d \geq 0$ (see for example [Har77, Ex. 5.14]).

Proposition 3.6.6. *Let $[f] \in U$ be general and let Z be as above. Then Z is projectively normal.*

Proof. Recall that if $S_Z = S/I_{Z/\mathbb{P}^5}$ is the homogeneous coordinate ring of Z, we have, for each $d \geq 0$, an exact sequence of vector spaces

$$0 \to S_Z^{(d)} \to H^0(\mathcal{O}_Z(d)) \to H^1(\mathcal{I}_{Z/\mathbb{P}^5}(d)) \to 0.$$

From these sequences one has

$$\sum_{d=0}^{+\infty} h^0(\mathcal{O}_Z(d))t^d = h_Z(t) + \sum_{d=0}^{+\infty} h^1(\mathcal{I}_{Z/\mathbb{P}^5}(d))t^d.$$

Since $dH_{|Z} \equiv_{num} K_Z + (d-3)H|_Z$ and $(d-3)H_{|Z}$ is ample for $d \geq 4$, by Kodaira vanishing, one has $H^p(\mathcal{O}_Z(d)) = 0$ for $d \geq 4$ and $p = 1, 2$. In particular, using also Table (3.14), one has

$$\sum_{d=0}^{+\infty} h^0(\mathcal{O}_Z(d))t^d = \sum_{d=0}^{+\infty} \chi(\mathcal{O}_Z(d))t^d - (55 + 15t) = \frac{7(18t^2 - 21t + 8)}{(1-t)^3} - 55 - 15t.$$

One can easily check that the latter series coincides with $h_Z(t)$ (see Equation (3.13)) so one can conclude that $h^1(\mathcal{I}_{Z/\mathbb{P}^5}(d)) = 0$ for all $d \geq 0$. This is equivalent to the projective normality of Z. $\square$

Appendix A

Cohomology on Grassmannians

The aim of this first appendix is to prove Proposition 3.5.13 from Section 3.5: in order to do this, we will essentially use tools from representation theory. For this part we refer to notation and approach used by G.Ottaviani in [Ott89], while for some basic definitions and a more detailed explanation of these topics, one can refer to [FH91]. In this appendix, we work on the field $\mathbb{C}$ of complex numbers.

Even if we are interested on some cohomology groups over the Grassmannian of projective $3-$planes in $\mathbb{P}^5$, before considering this specific case, let us introduce, in a more general setting, a possible strategy to study cohomology groups over a Grassmannian $G(k,n)$ of projective $k-$planes in $\mathbb{P}^n$. As done in [Ott89], we can think of $G(k,n)$ as of the complex homogeneous manifold $SL_{n+1}(\mathbb{C})/P$ where

$$P = \left\{ \begin{bmatrix} h_1 & 0 \\ h_3 & h_4 \end{bmatrix} \in SL_{n+1}(\mathbb{C}) \mid h_4 \in GL_{k+1}(\mathbb{C}) \right\}.$$

We can then consider the simple Lie algebra $\mathfrak{sl}_{n+1}(\mathbb{C}) = \{A \in M_{n+1}(\mathbb{C}) \mid \mathrm{Tr}(A) = 0\}$ associated to $SL_{n+1}(\mathbb{C})$ and $\mathfrak{p}$ the one associated to P. We can also take

$$\mathfrak{h} = \{A \in \mathfrak{sl}_{n+1}(\mathbb{C}) \mid A \text{ is diagonal}\} \subset \mathfrak{sl}_{n+1}(\mathbb{C}),$$

as a Cartan subalgebra of $\mathfrak{sl}_{n+1}(\mathbb{C})$.

In $\mathfrak{gl}_{n+1}(\mathbb{C}) = M_{n+1}(\mathbb{C})$ (the Lie algebra associated to $GL_{n+1}(\mathbb{C})$), let us define $E_{i,j}$ to be the matrix with all trivial coefficients but the one at the entry (i,j) which is equal to 1 and let $\{\epsilon_{i,j}\}$ be the dual basis of $\{E_{i,j}\}$. Then, as a basis of $\mathfrak{h}$ we can take $\{x_i\}_{i=0,\dots,n}$, where $x_i := E_{i,i} - E_{i+1,i+1}$. Let us call $\{\lambda_i\}_{i=0,\dots,n} \subset \mathfrak{h}^*$ the dual basis of $\{x_i\}_i$. By setting $\alpha_i := \epsilon_{i,i} - \epsilon_{i+1,i+1}$ and by considering $\frac{1}{2(n+1)}(\ ,\ )$ to be the Killing form in $\mathfrak{h}^*$, it is well known that

$$(\lambda_i, \alpha_j) = \delta_{i,j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}.$$

It is also well known that the set $\{\alpha_1, \dots, \alpha_n\}$ gives a basis of the root system Φ of $\mathfrak{sl}(n+1)$ with respect to $\mathfrak{h}$, where $\Phi = \Phi^+ \cup \Phi^-$ and

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j \mid 1 \leq i \leq j \leq n\}$$

is the set of positive roots and $\Phi^- = -\Phi^+$. Let us now recollect some definitions that will be very useful in what follows:

Definition A.0.1. *(a) Given a finite dimensional $\mathbb{K}-$vector space V and a representation $\rho : P \to GL(V)$ we say that $\lambda \in \mathfrak{h}^*$ is a* **weight** *for $D(\rho)$ if the space*

$$V_\lambda := \{\underline{v} \in V \mid D(\rho(H)) \cdot \underline{v} = \lambda(H) \cdot \underline{v} \;\forall\; H \in \mathfrak{h}\}$$

is not trivial, where D denotes the operator of differentiation.

(b) A weight $\lambda = \sum_{i=1}^n n_i\lambda_i$ with $n_i \in \mathbb{Z}$ is said to be **singular** *if $(\lambda, \alpha) = 0$ for at least one $\alpha \in \Phi$.*

(c) A weight λ is said to be **regular with index p** *if it is not singular and there are exactly p roots $\alpha \in \Phi^+$ such that $(\lambda, \alpha) < 0$.*

(d) If μ and ν are two distinct weights of the same representation, we say that μ is **higher than** *ν ($\mu \geq \nu$) if $\mu - \nu$ can be written as a linear combination $\sum_j m_j\alpha_j$, where all the coefficients m_j are non-negative.*

(e) A weight μ is a **highest weight** *if there do not exist other weights higher than μ.*

(f) A **homogeneous vector bundle** *E_ρ of rank r on $G(k, n) \simeq SL(n + 1)/P$ is a bundle arising from a representation $\rho : P \to GL(r)$.*

Since $\{\alpha_j\}_{j=1,\cdots,n}$ is a collection of elements of $\mathfrak{h}^*$ and $\{\lambda_i\}_{i=1,\cdots n}$ is a basis for $\mathfrak{h}^*$, we can write α_j as a linear combination of the λ_i's:

$$\alpha_j = \sum_{i,j} a_{i,j}\lambda_i, \quad \text{for } a_{i,j} \in \mathbb{K}.$$

Since $\{\lambda_i\}_i$ and $\{x_k\}_k$ are dual to each other, if we apply α_j to x_k, we simply get a_k. On the other hand, we know by construction that $\alpha_j = \epsilon_{i,i} - \epsilon_{i+1,i+1}$ and that $x_k = E_{k,k} - E_{k+1,k+1}$. Hence, by an easy calculation, one can write, in a compact way, $\alpha_j = -\lambda_{j-1} + 2\lambda_j - \lambda_{j+1}$, where we set λ_0 and λ_{n+1} to be zero.

Finally, by setting $\delta := \sum_{i=1}^n \lambda_i$, we can now state the following theorem due to Bott, which will be central in what follows (see [Ott89] or [Bot57, Theorem IV'] for the original statement):

Theorem A.0.2. *Let E_ρ be a homogeneous vector bundle on $G(k, n) \simeq SL(n + 1)/P$, defined by an irreducible representation ρ and let λ be the highest weight of $D(\rho) : \mathfrak{p} \to \mathfrak{gl}(r)$.*

1. *If $\lambda + \delta$ is singular then $H^i(G(k, n), E_\rho) = 0$ for all i.*

2. *If $\lambda + \delta$ is regular with index p then $H^i(G(k, n), E_\rho) = 0$ for all $i \neq p$.*

Actually, this theorem gives also the dimension of $H^p(G(k, n); E_\rho)$ in the case where $\lambda+\delta$ is regular with index p, but now we are not interested in such a result, so we are skipping this part.

After all this general introduction, let us now focus on our specific case. Since we are working with bundles over the Grassmannian $G(3,5)$, we are interested in the case where $k=3, n=5$, hence we have

$$P = \left\{ \begin{bmatrix} h_1 & 0 \\ h_3 & h_4 \end{bmatrix} \in SL_6(\mathbb{C}) \mid h_4 \in GL_4(\mathbb{C}) \right\}.$$

Let us now consider the tautological bundle S of the Grassmannian $G(3,5)$, i.e. the bundle fitting into the exact sequence

$$0 \to S \to G(3,5) \to Q \to 0$$

and such that for a $3-$plane W, element of $G(3,5)$, we have that $S_W \simeq W$. This tautological bundle S is associated to the standard representation, i.e. the representation $\rho : P \to GL(V)$, such that for $A \in P$ we have $\rho(A) = h_4$.

We have actually to consider the representation $D_e(\rho) = D(\rho)$ (where e denotes the identity element of P) over the associated Lie algebras: $D(\rho) : \mathfrak{p} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V) = M_4(\mathbb{C})$ and $\mathfrak{p} = \left\{ M = \begin{bmatrix} h_1 & 0 \\ h_3 & h_4 \end{bmatrix} \mid \mathrm{Tr}(M) = 0 \right\}$, sending $M \in \mathfrak{p}$ to

$$\lim_{t \to 0} \frac{\rho(I + tM) - \rho(I)}{t} = \frac{I_4 + th_4 - I_4}{t} = h_4,$$

so in this case actually $D(\rho) = \rho$.

Let us now look at the possible weights of $D(\rho) = \rho$: let us look at the spaces

$$V_\lambda = \{ \underline{v} \in V \mid \underbrace{D(\rho)(H)}_{=h_4} \cdot \underline{v} = \lambda(H) \cdot \underline{v} \quad \forall\, H \in \mathfrak{h} \}.$$

Since $\{x_i\}_{i=1,\dots,5}$ is a basis for $\mathfrak{h}$, we can write $H = \sum_{i=1}^5 b_i x_i$ for some coefficients $b_i \in \mathbb{K}$ and so, by definition of the x_i's one obtains that $D(\rho)(H) = h_4$ is of the form

$$h_4 = \mathrm{diag}(b_3 - b_2,\ b_4 - b_3,\ b_5 - b_4,\ -b_5)$$

and writing $\lambda = \sum_{i=1}^5 n_i \lambda_i$, since $\{\lambda_i\}_i$ is dual with respect to $\{x_i\}_i$, we get also that $\lambda(H) = \sum_{i=1}^5 n_i b_i$ and we can describe the space V_λ as the kernel, for every $\{b_i\}_i$

$$\mathrm{Ker} \begin{bmatrix} (b_3 - b_2) - \sum_{i=1}^5 n_i b_i & 0 & 0 & 0 \\ 0 & (b_4 - b_3) - \sum_{i=1}^5 n_i b_i & 0 & 0 \\ 0 & 0 & (b_5 - b_4) - \sum_{i=1}^5 n_i b_i & 0 \\ 0 & 0 & 0 & (-b_5) - \sum_{i=1}^5 n_i b_i \end{bmatrix}.$$

To make this kernel not trivial for every b_i, one of the elements on the diagonal of the above matrix has to be zero. One can then easily see that the weights associated to such a representation are

$$\mu_1 := \lambda_3 - \lambda_2, \qquad \mu_2 = \lambda_4 - \lambda_3, \qquad \mu_3 = \lambda_5 - \lambda_4, \qquad \mu_4 = -\lambda_5.$$

Let us now look for the highest weight. It is known (see for example [FH91, Proposition 14.13]) that for each irreducible component of a representation there exists a unique highest weight. Moreover the standard representation is irreducible. Recalling that for all $i = 1, \dots, 5$ we can write $\alpha_i = -\lambda_{i-1} + 2\lambda_i - \lambda_{i+1}$, where $\lambda_0 = \lambda_6 = 0$, let us observe that:

- $\mu_3 - \mu_4 = -\lambda_4 + 2\lambda_5 = \alpha_5$, so $\mu_3 \geq \mu_4$
- $\mu_2 - \mu_3 = -\lambda_3 + 2\lambda_4 - \lambda_5 = \alpha_4$, so $\mu_2 \geq \mu_3$
- $\mu_1 - \mu_2 = -\lambda_2 + 2\lambda_3 - \lambda_4 = \alpha_3$, so $\mu_1 \geq \mu_2$.

From these relations, one can easily see that μ_1 is higher than any other μ_j, with $j = 2, 3, 4$: μ_1 is the highest weight. Then, we can now apply Bott's Theorem A.0.2: let us test the singularity of $\mu_1 + \delta = \lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5$. Since, as we said above, $(\lambda_i, \alpha_j) = \delta_{ij}$, we can then easily see that

$$(\mu_1 + \delta, \alpha_2) = (\lambda_1 + 2\lambda_3 + \lambda_4 + \lambda_5, \alpha_2) = 0 \quad \rightarrow \mu_1 + \delta \text{ is singular},$$

so by Bott's theorem we get that

$$H^i(G(3,5), S) = 0 \;\; \forall\, i,$$

(recall that the tautological bundle S is the associated one to the standard representation we are considering).

The next step is to calculate the cohomology groups of the bundle $\mathrm{Sym}^2 S$, which is associated to the representation $\mathrm{Sym}^2\rho$, that we will denote by $\rho^{(2)}$. For the computation of the corresponding weights, we can use the following result (see [FH91, Chapter 13]):

Proposition A.0.3. *If we consider the symmetric product $\mathrm{Sym}^2\rho$ of a representation ρ, the corresponding weights are given by the sums of the weights of $D(\rho)$.*
Moreover, if we consider the k-th exterior product $\bigwedge^k \rho$ the corresponding weights are given by the sums of k distinct weights of $D(\rho)$.

Let us then indicate with $\mu_{ij} := \mu_i + \mu_j$ with $i, j = 1, \cdots, 4$ the weights of $\rho^{(2)}$. By studying, as done before, the differences between every couple of weights we can create a diagram with the rows that describe the order relation we introduced in Definition A.0.1, i.e. $\mu_{ij} \rightarrow \mu_{hk}$ means that $\mu_{ij} \geq \mu_{hk}$, (i.e. $\mu_i + \mu_j - \mu_h - \mu_k$ is a linear combination of positive roots with non-negative coefficients). Clearly, in such a diagram, the transitivity holds.

In particular, we have:

$$\begin{tikzcd}
 & & & \mu_{11} \arrow[dl] \\
 & & \mu_{12} \arrow[dl] \arrow[dr] & \\
 & \mu_{13} \arrow[dl] \arrow[dr] & & \mu_{22} \arrow[dl] \\
\mu_{14} \arrow[dr] & & \mu_{23} \arrow[dl] \arrow[dr] & \\
 & \mu_{24} \arrow[dr] & & \mu_{33} \arrow[dl] \\
 & & \mu_{34} \arrow[dr] & \\
 & & & \mu_{44}
\end{tikzcd} \tag{A.1}$$

It is then clear that every weight is higher than μ_{44} and that μ_{11} is higher than any other weight: we then have one maximal weight for $\rho^{(2)}$, namely μ_{11}, so such a representation is irreducible. We then have that $\mu_{11} + \delta = \lambda_1 - \lambda_2 + 3\lambda_3 + \lambda_4 + \lambda_5$ and we can easily see that

$$(\mu_{11} + \delta, \alpha_1 + \alpha_2) = (\lambda_1 - \lambda_2 + 3\lambda_3 + \lambda_4 + \lambda_5, \alpha_1 + \alpha_2) = 1 - 1 = 0,$$

so the element $\mu_{11}+\delta$ is singular (see Definition A.0.1(b)), then by using again Bott's theorem A.0.2 we get that

$$H^i(G(3,5),\mathrm{Sym}^2 S)=0\ \forall\ i.$$

Let us the consider the homogeneous bundles obtained as the exterior products of the symmetric product $\mathrm{Sym}^2 S$: they obviously correspond to the representations obtained as the exterior products of the symmetric product representation $\rho^{(2)}$ we have just considered. Let us refer to these representations as

$$\sigma_p := \bigwedge^p \rho^{(2)}, \qquad \text{associated to the bundle } \bigwedge^p \mathrm{Sym}^2 S,$$

where $D(\sigma_p):\mathfrak{p}\to\mathfrak{gl}(m)$, where $m:=\dim(\bigwedge^p \mathrm{Sym}^2 V)$.
Let us then start with the case $p=2$: by Proposition A.0.3 we know that the weights associated to σ_2 are given by sums of two different weights corresponding to $\rho^{(2)}$, then they are of the form $\mu_{ij}+\mu_{hk}$, with $i,j,h,k\in\{1,\cdots,4\}$ and $(i,j)\neq(h,k)$. From diagram (A.1), it is then clear that in this case the highest weight is given by $\nu:=\mu_{11}+\mu_{12}=3\mu_1+\mu_2$. Then we have that

$$\nu+\delta=\lambda_1-2\lambda_2+3\lambda_3+2\lambda_4+\lambda_5.$$

One can observe that in this case, there exist no elements α in Φ such that $(\nu+\delta,\alpha)=0$. Hence, $\nu+\delta$ is not singular: in order to compute its index, we are now supposed to find the cardinality of the set of elements α in Φ^+ such that $(\nu+\delta,\alpha)$ is strictly negative. One can easily see that there only two such α's, namely α_2 and $\alpha_1+\alpha_2$:

$$(\nu+\delta,\alpha_2)=(\lambda_1-2\lambda_2+3\lambda_3+2\lambda_4+\lambda_5,\alpha_2)=-2,$$

$$(\nu+\delta,\alpha_1+\alpha_2)=(\lambda_1-2\lambda_2+3\lambda_3+2\lambda_4+\lambda_5,\alpha_1+\alpha_2)=+1-2=-1.$$

By Definition A.0.1(c), we then have that $\nu+\delta$ is regular with index 2. Then by Bott's theorem A.0.2, we get

$$H^i\left(G(3,5),\bigwedge^2 \mathrm{Sym}^2 S\right)=0 \qquad \forall\ i\neq 2.$$

Let us now consider the case $p=3$. From Proposition A.0.3, we know that the weights are sums of three distinct weights of $\rho^{(2)}$ and, from the diagram (A.1), we can see that in this case we do not have a unique choice for the highest weight: we have two possibilities, namely

$$\nu_1=\mu_{11}+\mu_{12}+\mu_{13}=4\mu_1+\mu_2+\mu_3 \qquad \text{and} \qquad \nu_2=\mu_{11}+\mu_{12}+\mu_{22}=3\mu_1+3\mu_2.$$

Indeed, ν_1 and ν_2 are higher than any other possible weight, but ν_1 and ν_2 are not comparable: if we consider $\nu_1-\nu_2$ we get $\mu_{13}-\mu_{22}=\mu_1-\mu_2+\mu_3-\mu_2=\alpha_3-\alpha_4$ and obviously $\mu_{22}-\mu_{13}=\alpha_4-\alpha_3$, so neither $\mu_{13}\geq\mu_{22}$ nor the converse holds. Then, in this case, the representation σ_3 we are considering splits up into two irreducible subrepresentations, with these as highest weights. We have then to analyze both the components. For the first one, with ν_1 as highest weight, we have that

$$\nu_1+\delta=\lambda_1-3\lambda_2+4\lambda_3+\lambda_4+2\lambda_5.$$

We can then easily see as above that $\nu_1 + \delta$ is not singular and that the elements $\alpha \in \Phi^+$ such that $(\nu_1 + \delta, \alpha) < 0$ are exactly $\alpha_1, \alpha_1 + \alpha_2$: $\nu_1 + \delta$ is regular with index 2.
For the second component, with ν_2 as highest weight, we have

$$\nu_2 + \delta = \lambda_1 - 2\lambda_2 + \lambda_3 + 4\lambda_4 + \lambda_5,$$

which is singular, since $(\nu_2 + \delta, \alpha_1 + \alpha_2 + \alpha_3) = 0$. Hence, by Bott's Theorem A.0.2, we get that

$$H^i\left(G(3,5), \bigwedge^3 \mathrm{Sym}^2 S\right) = 0 \quad \forall i \neq 2.$$

Analogously, the same behavior arises in the cases where $p = 4, 5, 6, 7$. Indeed, for these values of p we have two different highest weights corresponding to two irreducible subresentations and one can show:

(p=4) $\nu_1 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14}$ and $\nu_2 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{22}$; moreover, $\nu_2 + \delta$ is singular since $(\nu_2 + \delta, \alpha_1 + \alpha_2 + \alpha_3) = 0$, while $\nu_1 + \delta$ is not singular with index 2 (one has $(\nu_1 + \delta, \alpha_2) < 0$ and $(\nu_1 + \delta, \alpha_1 + \alpha_2) < 0$)

(p=5) $\nu_1 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{22}$ and $\nu_2 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{22} + \mu_{23}$. One can easily show, as done above, that $\nu_1 + \delta$ is singular, while $\nu_2 + \delta$ is regular with index 4

(p=6) $\nu_1 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{22} + \mu_{23}$ and $\nu_2 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{22} + \mu_{23} + \mu_{33}$; here, we have that $\nu_1 + \delta$ is regular with index 4, while $\nu_2 + \delta$ is singular

(p=7) $\nu_1 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{22} + \mu_{23} + \mu_{24}$ and $\nu_2 = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{22} + \mu_{23} + \mu_{33}$. In this case, $\nu_1 + \delta$ is regular with index 4, while $\nu_2 + \delta$ is singular.

By applying Bott's Theorem A.0.2, we get:

$$H^i\left(G(3,5), \bigwedge^4 \mathrm{Sym}^2 S\right) = 0 \quad \forall\ i \neq 2 \qquad H^i\left(G(3,5), \bigwedge^5 \mathrm{Sym}^2 S\right) = 0 \quad \forall\ i \neq 4.$$

$$H^i\left(G(3,5), \bigwedge^6 \mathrm{Sym}^2\right) = 0 \quad \forall\ i \neq 4 \qquad H^i\left(G(3,5), \bigwedge^7 \mathrm{Sym}^2 S\right) = 0 \quad \forall\ i \neq 4.$$

Let us now consider $p = 8$. In this case, we have an irreducible representation, with only one highest weight, namely

$$\nu = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{22} + \mu_{23} + \mu_{24} + \mu_{33}.$$

And by taking

$$\nu + \delta = \lambda_1 - 4\lambda_2 + \lambda_3 + 2\lambda_4 + 3\lambda_5,$$

we have that it is singular, since $(\nu + \delta, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) = 0$. Hence, by Bott's Theorem A.0.2, we get

$$H^i\left(G(3,5), \bigwedge^8 \mathrm{Sym}^2 S\right) = 0 \quad \forall\ i.$$

Similarly, for $p = 9, 10$ we have a unique highest weight ν and, in particular, one gets (p=9) $\nu = \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} + \mu_{22} + \mu_{23} + \mu_{24} + \mu_{33} + \mu_{34}$ and $\nu + \delta$ is not singular and the elements $\alpha \in \Phi^+$ such that $(\nu + \delta, \alpha) < 0$ are $\alpha_2,\ \alpha_1 + \alpha_2,\ \alpha_2 + \alpha_3,\ \alpha_1 + \alpha_2 + \alpha_3,\ \alpha_2 + \alpha_3 + \alpha_4$ and $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$: $\nu + \delta$ is regular with index 6

(p=10) $\nu = 5\mu_1 + 5\mu_2 + 5\mu_3 + 5\mu_4$ and $\nu + \delta$ is singular

By using Bott's Theorem A.0.2, we get

$$H^i\left(G(3,5), \bigwedge^9 \mathrm{Sym}^2 S\right) = 0 \quad \forall\, i \neq 6 \qquad H^i\left(G(3,5), \bigwedge^{10} \mathrm{Sym}^2 S\right) = 0 \quad \forall\, i.$$

By summing up all these results, we have thus proved Proposition 3.5.13, which describes the cohomology of S and of its symmetric and exterior powers, where S is the tautological bundle to the Grassmannian $G(3,5)$:

Proposition. *One has* $H^i(\bigwedge^j \mathrm{Sym}^2 S) = 0$ *for all pairs* (i,j) *with* $i \geq 0$, $0 \leq j \leq 10$ *except for the cases where* $(i,j) \in \{(2,2),(2,3),(2,4),(4,5),(4,6),(4,7),(6,9)\}$. *For these cases,* $H^i(\bigwedge^j \mathrm{Sym}^2 S) \neq 0$.

Appendix B

The Klein cubic fourfold

In this appendix, we will consider the Klein cubic fourfold $X = V(f) \subset \mathbb{P}^5_{\mathbb{C}}$, where

$$f = x_0^2 x_1 + x_1^2 x_2 + x_2^2 x_3 + x_3^2 x_4 + x_4^2 x_5 + x_5^2 x_0,$$

and its associated hessian hypersurface $\mathcal{H} = V(\det(\mathrm{Hess}(f)))$. As stated in Section 3.6, we will prove that $\mathcal{H}$ is singular along a surface, i.e. $\dim(\mathrm{Sing}(\mathcal{H})) = \mathrm{Edim}(\mathrm{Sing}(\mathcal{H})) = 2$. However, let us stress than in this case such a surface is not smooth (as we showed for the general cubic form in 6 variables). To this aim, here we use the same approach proposed in [AR96, Appendix IV], where Adler proves that the Klein cubic threefold has associated Hessian hypersurface which is singular along a curve. To avoid confusion, let us write $(x_0, x_1, x_2, x_3, x_4, x_5) = (t, v, w, x, y, z)$. Let us consider the matrix

$$H = \frac{1}{2}\mathrm{Hess}(f) = \begin{bmatrix} v & t & 0 & 0 & 0 & z \\ t & w & v & 0 & 0 & 0 \\ 0 & v & x & w & 0 & 0 \\ 0 & 0 & w & y & x & 0 \\ 0 & 0 & 0 & x & z & y \\ z & 0 & 0 & 0 & y & t \end{bmatrix}.$$

We will prove our claim by showing that there exist two hyperplanes $L_1,\ L_2$ such that $H_{|L_1 \cap L_2}$ has rank $n - 1 = 4$ only in finitely many points or, in other words, that there exist only finitely many matrices of this kind with rank at most 4. Indeed, this would mean that the locus $\mathcal{D}_4(f) = \mathrm{Sing}(\mathcal{H})$ has dimension 2: let us recall that such a dimension can't be strictly smaller than 2, since we know that $\dim(\mathcal{D}_4(f)) \geq 2$, by the computation of the expected dimension of such locus (see Proposition 3.2.3).

Let us then consider $L_1 = \{x = 0\}$ and $L_2 = \{z = 0\}$: on the intersection of these two hyperplanes the Hessian matrix has the form

$$H_{|x=z=0} = \begin{bmatrix} v & t & 0 & 0 & 0 & 0 \\ t & w & v & 0 & 0 & 0 \\ 0 & v & 0 & w & 0 & 0 \\ 0 & 0 & w & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & y & t \end{bmatrix}.$$

Since we want to analyze the case where the rank of such a matrix is 4, we have to study the vanishing of minors of order 5. If, for example, we take $H^{4,2}$, i.e. the order 5 submatrix obtained by ruling out the 4th row and the 2nd column of H, we see that its determinant is $-y^2wv^2 = 0$. We thus get that either $y = 0$ or $w = 0$ or $v = 0$: if two coordinates (x and z) are 0, then a third one must be 0 too. If we consider the symmetric action which f is invariant for, we can assume that either $v = 0$ or $w = 0$. In the first case, where $v = x = z = 0$, we get the matrix

$$H_{|v=x=z=0} = \begin{bmatrix} 0 & t & 0 & 0 & 0 & 0 \\ t & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 \\ 0 & 0 & w & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & y & t \end{bmatrix}.$$

By considering, for example, the submatrix $H^{3,3}$ and its determinant, we get that $-t^2y^3 = 0$. In the second case, where $z = x = w = 0$, we get

$$H_{|z=x=w=0} = \begin{bmatrix} v & t & 0 & 0 & 0 & 0 \\ t & 0 & v & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & y & t \end{bmatrix}$$

and from the determinant of $H^{4,4}$ we get $v^3y^2 = 0$. Thus, by supposing that two coordinates vanish, actually at least four are zero and as above we can assume that we have $z = x = v = t = 0$ or $z = x = w = v = 0$. In the first case, we get

$$H_{|z=x=v=t=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w & 0 & 0 \\ 0 & 0 & w & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & y & 0 \end{bmatrix}$$

and from $H^{1,1}$ we get that $w^3y^2 = 0$. In the second case,

$$H_{|z=x=w=v=0} = \begin{bmatrix} 0 & t & 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 & y & t \end{bmatrix}$$

and from $H^{3,3}$ we get that $t^2y^3 = 0$. Thus, we get that at least five coordinates are actually zero. But in $\mathbb{P}^5$, there are only 6 points with 5 coordinates equal to 0. Thus we have proved our claim.

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