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Fundamental theorem of asset pricing with acceptable risk in markets with frictions

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Abstract
We study the range of prices at which a rational agent should contemplate transacting a financial contract outside a given market. Trading is subject to nonproportional transaction costs and portfolio constraints, and full replication by way of market instruments is not always possible. Rationality is defined in terms of consistency with market prices and acceptable risk thresholds. We obtain a direct and a dual description of market-consistent prices with acceptable risk. The dual characterisation requires an appropriate extension of the classical fundamental theorem of asset pricing where the role of arbitrage opportunities is played by good deals, i.e., costless investment opportunities with acceptable risk–reward tradeoff. In particular, we highlight the importance of scalable good deals, i.e., investment opportunities that are good deals regardless of their volume.

Keywords Arbitrage pricing · Good deal pricing · Transaction costs · Portfolio constraints · Risk measures

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1 Introduction

One of the fundamental goals of financial economics is to investigate at which price(s) a rational agent should contemplate transacting a financial contract. The point of departure for the classical theory of arbitrage pricing is the assumption that agents are

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wealth maximisers and have access to a market where a number of basic financial assets are traded in an arbitrage-free way. In this setting, as is well known, the range of rational prices coincides with the interval of arbitrage-free prices. Since the pioneering contributions of Black and Scholes [12], Merton [42], Cox and Ross [23], Rubinstein [49], Ross [48], Harrison and Kreps [30], Kreps [39], this framework has successfully been extended in several directions. A prominent line of research has focused on the theory of good deal pricing, initiated by Cochrane and Saá-Requejo [22] and Bernardo and Ledoit [9] and based on the idea of restricting the interval of arbitrage-free prices by incorporating individual “preferences” into the pricing problem. This leads to tighter pricing bounds called good deal bounds. In this setting, arbitrage opportunities are replaced by good deals, i.e., investment opportunities that require no funding costs and deliver terminal payoffs that are sufficiently attractive based on the agent’s “preferences”. Differently from arbitrage opportunities, good deals may expose to downside risk, and the agent’s task is therefore that of determining acceptable risk thresholds. Several ways to define risk thresholds have been considered in the literature, e.g. through Sharpe ratios in Cochrane and Saá-Requejo [22], Björk and Slinko [11] and Bion-Nadal and Di Nunno [10], gain–loss ratios in Bernardo and Ledoit [9], test probabilities in Carr et al. [14], utility functions in Černý and Hodges [18], Černý [16], Klöppel and Schweizer [35] and Arai [2], expected shortfall in Černý [17], distance functions in Bondarenko and Longarela [13], and acceptability indices in Madan and Černý [41]. A theory for general acceptance sets has been developed by Jaschke and Küchler [32], Černý and Hodges [18], Staum [50], Černý [17] and Cheridito et al. [20]. We also refer to Arai and Fukasawa [4] and Arai [3] for a study of optimal good deal pricing bounds. One can broadly distinguish between two research directions in the field. A first strand of literature starts by imposing suitable constraints on price deflators or, equivalently, martingale measures with the aim of restricting the interval of arbitrage-free prices. The resulting good deal bounds can be therefore expressed in dual terms. The rationale for discarding some arbitrage-free prices is that transacting at those prices would create good deals with respect to a suitable acceptance set. The task is precisely to characterise the corresponding acceptance set. A second strand of literature starts by tightening the superreplication price through a suitable enlargement of the cone of positive random variables which is replaced by a larger acceptance set. The task is to establish a dual description of the resulting good deal bounds. This is achieved by extending the fundamental theorem of asset pricing to a good deal pricing setting. In this paper, we follow the latter approach.

Our goal is to contribute to the literature on good deal pricing in a static setting by establishing a version of the fundamental theorem of asset pricing in incomplete markets with frictions where agents use general acceptance sets to define good deals. This generality requires developing a new strategy as the standard change-of-numeraire and exhaustion techniques employed in the classical proof of the fundamental theorem can no longer be exploited. The highlights of our work are the following:

– The point of departure is a clear and economically motivated definition of rational prices that is missing in the good deal pricing literature with the exception of Černý [17]. Our approach is different and inspired by Koch-Medina and Munari [37, Definition 8.2.1]. We assume that an agent willing to purchase a financial contract outside of the market will never accept to buy at a price at which he or she could find a better replicable payoff in the market. In the spirit of good deal pricing, the agent is prepared to accept a suitable "replication error" which is formally captured by an acceptance set. The corresponding rational prices are called *market-consistent prices*. In a frictionless setting where agents accept no "replication error", our notion boils down to the classical notion of an arbitrage-free price.

– We work under general convex transaction costs and portfolio constraints, which allows us to model both proportional and nonproportional frictions. The bulk of the literature has focused on frictionless markets or markets with proportional transaction costs. Portfolio constraints have been rarely considered. Moreover, instead of focusing on the set of payoffs attainable at zero cost as a whole, we state our results by explicitly disentangling the specific role played by transaction costs and portfolio constraints.

– We introduce the notion of *scalable good deals*, i.e., payoffs that are good deals independently of their size, which extends to a good deal pricing setting the notion of a scalable arbitrage opportunity by Pennanen [43]. The absence of scalable good deals is key to deriving our characterisations of market-consistent prices. This condition is weaker than the absence of good deals commonly stipulated in the literature. In particular, there are situations where absence of arbitrage is sufficient to ensure absence of scalable good deals. We also argue that absence of scalable good deals is economically sounder than absence of good deals.

– We adapt the classical notion of a price deflator to our good deal setting with frictions and introduce the class of *strictly consistent price deflators*, which correspond to the Riesz densities of a pricing rule in a complete frictionless market where the basic traded assets are "priced" in accordance with their (suitably adjusted, in the presence of nonproportional frictions) bid–ask spreads and every nonzero acceptable payoff has a strictly positive "price". This is different from similar notions in the literature where no bid–ask spread adjustments are considered and acceptable payoffs, including positive payoffs, are often assumed to have a nonnegative "price" only.

– We establish direct and dual characterisations of market-consistent prices. The direct characterisation is based on the analysis of superreplication prices and extends to a good deal pricing setting the classical findings of Bensaid et al. [8] in markets with frictions. The dual characterisation is based on a general version of the fundamental theorem of asset pricing which establishes equivalence between absence of scalable good deals and existence of strictly consistent price deflators under suitable assumptions on the underlying model space. This extends to a good deal pricing setting the static version of the fundamental theorem obtained by Pennanen [43]. We provide a detailed comparison with the literature to highlight how our result extends and sharpens the various formulations of the fundamental theorem in the good deal pricing literature. The only work on good deal pricing featuring a strong result with strictly consistent price deflators is Černý and Hodges [18]. In that paper, the market is frictionless and the acceptance set is assumed to be boundedly generated, a condition that often forces the underlying probability space to be finite.

The paper is organised as follows. In Sect. 2, we describe the market model and the agent's acceptance set, and we introduce the notion of market-consistent prices with acceptable risk. In Sect. 3, we focus on good deals and show a number of sufficient conditions for the absence of scalable good deals. In Sect. 4, we establish a direct and a dual characterisation of market-consistent prices with acceptable risk (Propositions 4.4 and 4.19). The dual characterisation is based on our general version of the fundamental theorem of asset pricing (Theorem 4.15) and the corresponding superreplication theorem (Theorem 4.18), which are from a technical perspective the highlights of the paper. Throughout, we prove sharpness of our results by means of suitable examples which are always presented in the simplest possible setting, namely that of a two-states model, to demonstrate their general validity.

2 The pricing problem

In this section, we describe the underlying mathematical framework for our pricing problem. The bulk of the presentation is aligned with our reference literature on good deal pricing; see e.g. Carr et al. [14], Jaschke and Küchler [32], Černý and Hodges [18], Staum [50], Černý [17], Madan and Černý [41]. We highlight discrepancies where needed.

2.1 The market model

We consider a one-period financial market, and we model uncertainty about the terminal state of the economy by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote by L^1 the space of integrable random variables modulo almost sure equality under $\mathbb{P}$ and equip it with its canonical lattice and norm structure. Unless otherwise specified, every topological property of subsets of L^1 is with respect to the norm topology. The set of nonnegative integrable random variables is denoted by L^1_+ and is referred to as the standard positive cone. Similarly, for $\mathcal{L} \subseteq L^1$, we define $\mathcal{L}_+ := \mathcal{L} \cap L^1_+$. The L^1-norm is denoted by $\|\cdot\|_1$ and the expectation under $\mathbb{P}$ simply by $\mathbb{E}$. The elements of L^1 represent *payoffs* of financial contracts at the terminal date. We identify the elements of $\mathbb{R}$ with constant payoffs and refer to them as *risk-free payoffs*.

Throughout the paper, we consider an agent who has access to a financial market where a finite number of basic assets are traded. We denote by $\mathcal{S} \subseteq L^1$ the (nonzero) vector space spanned by the payoffs of the basic assets. The elements of $\mathcal{S}$ are called *replicable payoffs*. Contrary to most of the good deal pricing literature, we do not assume the existence of risk-free replicable payoffs. To each replicable payoff, we associate an ask price via a *pricing rule* $\pi : \mathcal{S} \to (-\infty, \infty]$. In line with the literature, we allow nonfinite prices to account for the existence of physical limitations in the availability of replicable payoffs. These limitations affect every agent. Moreover, we fix a nonempty set $\mathcal{M} \subseteq \mathcal{S}$ consisting of those replicable payoffs that can be bought by our agent. The elements of $\mathcal{M}$ are called *attainable payoffs* and account for the existence of e.g. regulatory limitations in the purchase of replicable payoffs. These limitations are specific to our agent. Even though the agent has access to a (possibly strict) subset of $\mathcal{S}$ only, it is mathematically convenient to define π on the entire set $\mathcal{S}$ to exploit its natural vector space structure. **Throughout the paper**, we work under the following assumptions on the market primitives.

Assumption 2.1 We assume that π is convex, lower semicontinuous and $\pi(0)=0$. Moreover, we assume that $\mathcal{M}$ is convex, closed and $0\in\mathcal{M}$.

This setting is compatible with a variety of market models encountered in the literature. In particular, we refer to Jouini and Kallal [33] for examples of markets with proportional frictions and to Çetin and Rogers [19] and Pennanen [43] for examples of markets with nonproportional frictions. Recall that frictions are proportional when both π and $\mathcal{M}$ are conic as defined in the Appendix.

Remark 2.2 It is worth highlighting the following basic facts about the space of replicable payoffs. In our framework, the space $\mathcal{S}$ is naturally equipped with the relative topology induced by the norm topology of L^1. Because $\mathcal{S}$ is finite-dimensional, this topology coincides with any other Hausdorff linear topology defined on $\mathcal{S}$; see e.g. Aliprantis and Border [1, Theorem 5.21]. This makes it possible to apply to $\mathcal{S}$ the entire range of notions and results from Euclidean spaces. In particular, we freely use that any closed and bounded subset of $\mathcal{S}$ is compact and that for every vector subspace $\mathcal{N}\subseteq\mathcal{S}$, there exists a vector subspace $\mathcal{N}^\perp\subseteq\mathcal{S}$, called the orthogonal complement of $\mathcal{N}$, such that $\mathcal{N}+\mathcal{N}^\perp=\mathcal{S}$ and $\mathcal{N}\cap\mathcal{N}^\perp=\{0\}$.

2.2 The acceptance set

To determine the range of rational prices for a financial contract to purchase outside of the market, the agent compares its payoff with those attainable payoffs that are "preferable" to it and uses the corresponding market prices to set an upper bound on rational prices. In line with good deal pricing theory, we define said "preference" relationship by means of an acceptance set $\mathcal{A}\subseteq L^1$. More precisely, we assume that $Z\in\mathcal{M}$ is "preferable" to $X\in L^1$ whenever $Z-X\in\mathcal{A}$. It should be noted that the relation induced by $\mathcal{A}$ is not a preference relation in a technical sense unless $\mathcal{A}$ is a convex cone. The bulk of the good deal pricing literature has focused on this special case. This is, however, unsatisfactory as there exist relevant acceptance sets that are convex but fail to be conic, e.g. acceptance sets defined through utility functions or stochastic dominance. To include these examples, we follow Černý and Hodges [18] and Staum [50] and dispense with conicity. In this case, we find it necessary to consequently dispense with the language of "preferences" and to provide a new, more general interpretation to the acceptance set. In this paper, we interpret $\mathcal{A}$ as the set of all replication errors that are deemed acceptable by the agent. In other words, the agent tries to replicate X by means of attainable payoffs Z available in the market and uses the acceptance set to determine whether the residual payoff $Z-X$ is acceptable or not. If $\mathcal{A}=L^1_+$, the agent accepts no downside risk in the replication procedure. This choice corresponds to the classical setting of arbitrage pricing. The elements of $\mathcal{A}$ are called *acceptable payoffs*. We assume that every payoff dominating an acceptable payoff is also acceptable and that convex combinations of acceptable payoffs remain acceptable. The first property corresponds to the usual monotonicity requirement from risk measure theory; see e.g. Artzner et al. [5].

Assumption 2.3 The set $\mathcal{A}$ is a strict, closed, convex subset of L^1 and satisfies $0\in\mathcal{A}$ as well as $\mathcal{A}+L^1_+\subseteq\mathcal{A}$.

These assumptions are satisfied by all standard convex acceptance sets encountered in risk measure theory, including acceptance sets based on expected shortfall, gain–loss ratios, test scenarios and test probabilities, utility functions and second-order stochastic dominance. We refer to Levy [40], Bernardo and Ledoit [9], Carr et al. [14], Černý and Hodges [18], Černý [16], Klöppel and Schweizer [35], Černý [17] and Arai [2] for applications of the aforementioned acceptance sets in a pricing context.

2.3 Market-consistent prices with acceptable risk

The agent's problem is to determine the range of prices at which he or she should contemplate purchasing a financial contract outside of the market. The candidate prices should satisfy the following rationality requirements. On the one hand, they should be *consistent with the market*, i.e., the agent should not be willing to transact if the market offers a better contract at a lower price. On the other hand, they should be *consistent with individual "preferences"*, i.e., the agent should determine when a marketed contract is better based on his or her pre-specified criterion of acceptability. This leads to the following definition.

Definition 2.4 A number $p \in \mathbb{R}$ is a *market-consistent (buyer) price (with acceptable risk)* for $X \in L^1$ if

(1) $p < \pi(Z)$ for every $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A} \setminus \{0\}$;

(2) $p \leq \pi(X)$ whenever $X \in \mathcal{M}$.

We denote by $\mathrm{MCP}(X)$ the set of market-consistent prices for X.

The set of market-consistent prices for a payoff $X \in L^1$ is an interval that is bounded to the right. The upper bound is called *superreplication price (with acceptable risk)* of X and is given by

$$\pi^+(X) := \inf\{\pi(Z) : Z \in \mathcal{M}, Z - X \in \mathcal{A}\}.$$

Remark 2.5 (i) The notion of a market-consistent price is formulated from a buyer's perspective, but can easily be adapted to a seller. One may thus wonder why, differently from arbitrage pricing, we do not focus on prices that are simultaneously market-consistent for both parties. From an economic perspective, this is because the choice of the acceptance set is based on individual "preferences", implying that the general financial situation is that of a buyer and seller equipped with different acceptance sets. From a mathematical perspective, the buyer's and seller's problems are related to each other and one can easily adapt our results to obtain the corresponding results for seller prices.

(ii) In the good deal pricing literature, the focus is typically on superreplication prices and the notion of a rational price is not explicitly discussed. The exception is Černý [17] where, in line with classical arbitrage pricing theory, rational prices are defined through extensions of the pricing rule preserving the absence of (suitably defined) good deals. Even though the pricing rule is not linear, the extension is assumed to be linear in the direction of the payoff that is "added" to the market. Our definition, inspired by the notion of market-consistency in Koch-Medina and Munari

[37, Definition 8.2.1] in a frictionless arbitrage pricing setting, is not based on market extensions and does not require the absence of good deals, which, differently from the absence of arbitrage opportunities, is a debatable requirement; see Sect. 3.

(iii) Note that in the definition of a market-consistent price, condition (1) need not imply condition (2), which is a natural requirement for a market-consistent price of an attainable payoff. The implication holds if for instance $\mathcal{A}$ and $\mathcal{M}$ have nonzero intersection and π and $\mathcal{M}$ are both conic.

3 Good deals

A good deal is any nonzero acceptable payoff that is attainable and can be acquired at zero cost. As such, a good deal constitutes a natural generalisation of an arbitrage opportunity, which corresponds to the situation where the acceptance set reduces to the standard positive cone. An important class of good deals is that of scalable good deals, i.e., payoffs that are good deals independently of their size. The notion of a good deal has appeared, sometimes with a slightly different meaning, under various names in the literature, including good deal in Cochrane and Saá-Requejo [22], Černý and Hodges [18], Björk and Slinko [11], Klöppel and Schweizer [35], Bion-Nadal and Di Nunno [10] and Baes et al. [6], good deal of first kind in Jaschke and Küchler [32], good opportunity in Bernardo and Ledoit [9], acceptable opportunity in Carr et al. [14]. The notion of a scalable good deal is a direct extension of that of a scalable arbitrage opportunity introduced by Pennanen [43] and appeared in a frictionless setting in Baes et al. [6]. The formal notions are recorded in the next definition. Here, we define the *recession cones* of $\mathcal{M}$ and $\mathcal{A}$ by

$$\mathcal{M}^\infty := \bigcap_{\lambda>0} \lambda\mathcal{M}, \qquad \mathcal{A}^\infty := \bigcap_{\lambda>0} \lambda\mathcal{A}.$$

Moreover, the *recession map* of π is the map $\pi^\infty : \mathcal{S} \to (-\infty, \infty]$ defined by

$$\pi^\infty(X) := \sup_{\lambda>0} \frac{\pi(\lambda)}{\lambda}.$$

Note that $\mathcal{M}^\infty$ and $\mathcal{A}^\infty$ are the largest convex cones contained in $\mathcal{M}$ and $\mathcal{A}$, respectively. Similarly, π^∞ is the smallest sublinear map defined on $\mathcal{S}$ such that $\pi \le \pi^\infty$ on $\mathcal{S}$. We collect more properties of recession cones and maps in the Appendix.

Definition 3.1 We say that a nonzero replicable payoff $X \in \mathcal{S}$ is

(1) a *good deal (with respect to $\mathcal{A}$)* if $X \in \mathcal{A} \cap \mathcal{M}$ and $\pi(X) \le 0$;

(2) a *scalable good deal (with respect to $\mathcal{A}$)* if $X \in \mathcal{A}^\infty \cap \mathcal{M}^\infty$ and $\pi^\infty(X) \le 0$;

(3) a *strong scalable good deal (with respect to $\mathcal{A}$)* if X is a scalable good deal while $-X$ is not.

We replace the term "good deal" with "arbitrage opportunity" whenever $\mathcal{A} = L^1_+$.

Remark 3.2 Note that if $X \in L^1$ is a strong scalable good deal, then by definition, there exists $\lambda > 0$ such that $-\lambda X$ is not a good deal. However, this "short" position can be completely offset at zero cost by acquiring the attainable payoff λX. This is what makes the scalable good deal "strong".

It is clear that every strong scalable good deal is a scalable good deal, which in turn is a good deal. It is also clear that every (scalable) arbitrage opportunity is a (scalable) good deal. The absence of scalable good deals will be key in what follows. In the classical setting of arbitrage pricing, an arbitrage opportunity constitutes an anomaly in the market because every rational agent will seek to exploit it, thereby raising its demand until prices also rise and the arbitrage opportunity eventually vanishes. The situation is quite different when we consider good deals as there might be no consensus across agents in the identification of a common criterion of acceptability, thereby casting doubts on the economic foundation of the absence of good deals. The key observation here is that to develop our theory, we only need the weaker condition of absence of (strong) scalable good deals. As shown by the next proposition, whose simple proof is omitted, this condition holds in a number of standard situations and is sometimes implied by the absence of (scalable) arbitrage opportunities. More precisely, the condition $\mathcal{M}^\infty \subseteq \mathcal{S}_+$ is typically implied by caps on short positions (it holds if the payoffs of the basic assets are positive and short selling is possible, but restricted for each asset). The condition $\pi^\infty = \infty$ on $\mathcal{M}^\infty \setminus \{0\}$ holds whenever the pricing rule π is not sublinear on the cone generated by any nonzero element of $\mathcal{M}^\infty$. Finally, the condition $\mathcal{M}^\infty = \{0\}$ is satisfied whenever there are caps on short and long positions alike.

Proposition 3.3 *Assume that one of the following conditions holds*:
(i) $\mathcal{A}^\infty = L^1_+$ *and there exists no scalable arbitrage opportunity.*
(ii) $\mathcal{M}^\infty \subseteq \mathcal{S}_+$ *and there exists no scalable arbitrage opportunity.*
(iii) $\pi^\infty(X) = \infty$ *for every nonzero* $X \in \mathcal{M}^\infty$.
(iv) $\mathcal{M}^\infty = \{0\}$.
Then there exists no scalable good deal.

The next proposition records for ease of reference an equivalent condition for the absence of strong scalable good deals, which is a one-period equivalent to the condition in Pennanen [44, Theorem 8]. In that paper, the condition is expressed in terms of portfolios instead of payoffs and the acceptance set is the standard positive cone. The straightforward verification is omitted.

Proposition 3.4 *There exists no strong scalable good deal if and only if the set* $\mathcal{N} := \mathcal{A}^\infty \cap \{X \in \mathcal{M}^\infty : \pi^\infty(X) \leq 0\}$ *is a vector space.*

4 Fundamental theorem of asset pricing

In this section, we establish a direct and a dual characterisation of market-consistent prices. Most of this section is new and both extends and sharpens the corresponding results in the good deal pricing literature. We refer to the dedicated remarks for a detailed comparison with the literature.

4.1 A key auxiliary set

It is immediate to see that for every payoff $X \in L^1$, we can rewrite $\pi^+(X)$ as

$$\pi^+(X) = \inf\{m \in \mathbb{R} : (X, m) \in \mathcal{C}\}, \tag{4.1}$$

where the set $\mathcal{C}$ consists of all the payoff–price couples featuring payoffs that can be superreplicated with acceptable risk by means of admissible payoffs available in the market for that price. Formally, the set is given by

$$\mathcal{C} := \{(X, m) \in L^1 \times \mathbb{R} : \exists Z \in \mathcal{M} \text{ with } Z - X \in \mathcal{A} \text{ and } \pi(Z) \leq m\}.$$

This set plays the same role that in classical arbitrage pricing theory is played by the set of payoffs that can be superreplicated at zero cost. To see the link, consider a frictionless market, i.e., a market where π is linear and $\mathcal{M} = \mathcal{S}$, and assume that $\mathcal{A} = L^1_+$. The set of payoffs that can be superreplicated at zero cost is given by

$$\mathcal{K} := \{X \in L^1 : \exists Z \in \mathcal{S} \text{ with } Z - X \in L^1_+ \text{ and } \pi(Z) \leq 0\}.$$

It is easily verified that by taking any $U \in \mathcal{S}$ satisfying $\pi(U) = 1$, we can rewrite $\mathcal{C}$ as

$$\mathcal{C} = \{(X, m) \in L^1 \times \mathbb{R} : X - mU \in \mathcal{K}\}.$$

In this classical setting, it is well known that the absence of arbitrage opportunities implies closedness of $\mathcal{K}$, and hence of $\mathcal{C}$. This is key to establish the classical fundamental theorem of asset pricing; see e.g. Föllmer and Schied [24, Chap. 1]. The closedness of $\mathcal{C}$ in our general framework will allow us to establish a general version of the fundamental theorem in the next subsections.

Lemma 4.1 *If there is no strong scalable good deal, then $\mathcal{C}$ is closed and $(0, -n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$.*

Proof Set $\mathcal{N} := \{X \in \mathcal{A}^\infty \cap \mathcal{M}^\infty : \pi^\infty(X) \leq 0\}$ and denote by $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in $\mathcal{S}$; see Remark 2.2. We claim that for every $(X, m) \in \mathcal{C}$, there exists $Z \in \mathcal{M} \cap \mathcal{N}^\perp$ such that $Z - X \in \mathcal{A}$ and $\pi(Z) \leq m$. To see this, note that we find $W \in \mathcal{M}$ such that $W - X \in \mathcal{A}$ and $\pi(W) \leq m$. We can write $W = W_\mathcal{N} + W_{\mathcal{N}^\perp}$ for unique elements $W_\mathcal{N} \in \mathcal{N}$ and $W_{\mathcal{N}^\perp} \in \mathcal{N}^\perp$. Note that $W_\mathcal{N}$ belongs to $-\mathcal{N}$ because the set $\mathcal{N}$ is a vector space by Proposition 3.4. Hence, setting $Z := W_{\mathcal{N}^\perp}$, we infer that $Z = W - W_\mathcal{N} \in \mathcal{M} + \mathcal{M}^\infty \subseteq \mathcal{M}$ as well as $Z - X = (W - X) - W_\mathcal{N} \in \mathcal{A} + \mathcal{A}^\infty \subseteq \mathcal{A}$ by (A.1). Moreover, combining (A.1) with (A.2) gives $\pi(Z) = \pi(W - W_\mathcal{N}) \leq m$. This shows the desired claim.

Next, we establish closedness. To this end, take a sequence $(X_n, m_n) \subseteq \mathcal{C}$ and a point $(X, m) \in L^1 \times \mathbb{R}$ and assume that $(X_n, m_n) \to (X, m)$. By assumption, we find a sequence $(Z_n) \subseteq \mathcal{M}$ such that $Z_n - X_n \in \mathcal{A}$ and $\pi(Z_n) \leq m_n$ for every $n \in \mathbb{N}$. Without loss of generality, we can assume that $(Z_n) \subseteq \mathcal{N}^\perp$. Now suppose that (Z_n) is unbounded. In this case, we find a subsequence of (Z_n) consisting of nonzero elements with diverging norms. For convenience, we still denote this subsequence by (Z_n). Since the unit sphere in $\mathcal{S}$ is compact, we can assume that $\frac{Z_n}{\|Z_n\|_1} \to Z$ for a suitable nonzero $Z \in \mathcal{M}^\infty$ by (A.1). As $X_n \to X$, we have $\frac{Z_n - X_n}{\|Z_n\|_1} \to Z$. This implies that $Z \in \mathcal{A}^\infty$ again by (A.1). We claim that $\pi^\infty(Z) \leq 0$. Otherwise, we can find $\lambda > 0$ such that $\pi(\lambda Z) > 0$. Without loss of generality, we may assume that $\|Z_n\|_1 > \lambda$ for every $n \in \mathbb{N}$. By lower semicontinuity and convexity of π, we derive the contradiction

$$0 < \pi(\lambda Z) \leq \liminf_{n\to\infty} \pi\left(\frac{\lambda Z_n}{\|Z_n\|_1}\right) \leq \liminf_{n\to\infty} \frac{\lambda \pi(Z_n)}{\|Z_n\|_1} \leq \lim_{n\to\infty} \frac{\lambda m_n}{\|Z_n\|_1} = 0.$$

This yields $\pi^\infty(Z) \leq 0$. As a result, it follows that Z belongs to $\mathcal{N}$. However, this is not possible because Z is a nonzero element in $\mathcal{N}^\perp$. To avoid this contradiction, (Z_n) must be bounded and hence must admit a convergent subsequence, which we still denote by (Z_n) for convenience. By closedness of $\mathcal{M}$, the limit Z also belongs to $\mathcal{M}$. As $Z_n - X_n \to Z - X$, it follows that $Z - X \in \mathcal{A}$ by closedness of $\mathcal{A}$. Moreover,

$$\pi(Z) \leq \liminf_{n\to\infty} \pi(Z_n) \leq \lim_{n\to\infty} m_n = m$$

by lower semicontinuity of π. This yields $(X, m) \in \mathcal{C}$ and establishes that $\mathcal{C}$ is closed.

Finally, we show that $(0, -n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$. To this end, assume to the contrary that for every $n \in \mathbb{N}$, there exists $Z_n \in \mathcal{A} \cap \mathcal{M}$ with $\pi(Z_n) \leq -n$. If the sequence (Z_n) is bounded, then we may assume without loss of generality that $Z_n \to Z$ for some $Z \in \mathcal{A} \cap \mathcal{M}$. The lower semicontinuity of π implies $\pi(Z) \leq \liminf_{n\to\infty} \pi(Z_n) = -\infty$, which cannot hold. Hence the sequence (Z_n) must be unbounded. As argued above, we can assume without loss of generality that (Z_n) is contained in $\mathcal{N}^\perp$ and has strictly positive divergent norms satisfying $\frac{Z_n}{\|Z_n\|_1} \to Z$ for some nonzero Z belonging to $\mathcal{A}^\infty \cap \mathcal{M}^\infty$. Moreover, as above, $\pi^\infty(Z) \leq 0$. As a consequence, it follows that Z belongs to $\mathcal{N}$ as well. However, this is not possible because Z is a nonzero element in $\mathcal{N}^\perp$. Hence we must have $(0, -n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$, concluding the proof. □

4.2 Direct characterisation of market-consistent prices

To obtain a direct characterisation of market-consistent prices, we investigate when the superreplication price, which is the upper bound of the set of market-consistent prices, is itself a market-consistent price. In Example 4.5, we show that in general, the superreplication price can be market-consistent or not, regardless of whether the underlying payoff is attainable or not. This is based on the following simple characterisation of market-consistency.

Proposition 4.2 *For every $X \in L^1$ with $\pi^+(X) \in \mathbb{R}$, we have $\pi^+(X) \in \mathrm{MCP}(X)$ if and only if $(\mathcal{A} + X) \cap \{Z \in \mathcal{M} : \pi(Z) = \pi^+(X)\} \subseteq \{X\}$.*

Proof First, assume that $(\mathcal{A} + X) \cap \{Z \in \mathcal{M} : \pi(Z) = \pi^+(X)\} \subseteq \{X\}$. Then for every $Z \in \mathcal{M}$ satisfying $Z - X \in \mathcal{A} \setminus \{0\}$, we must have $\pi^+(X) < \pi(Z)$. Since $\pi^+(X) \leq \pi(X)$ whenever $X \in \mathcal{M}$, it follows that $\pi^+(X) \in \mathrm{MCP}(X)$, proving the "if" implication. Conversely, assume that $\pi^+(X) \in \mathrm{MCP}(X)$ and take any payoff $Z \in (\mathcal{A} + X) \cap \mathcal{M}$. If we happen to have $\pi(Z) = \pi^+(X)$, then Z must be equal to X by market-consistency of $\pi^+(X)$. This proves the "only if" implication. □

Proposition 4.2 shows that market-consistency of the superreplication price is strongly linked with the attainability of the infimum in the definition of superreplication price. We therefore target sufficient conditions for this to hold.

Proposition 4.3 *If there exists no strong scalable good deal, then for every* $X \in L^1$ *with* $\pi^+(X) < \infty$, *there exists* $Z \in \mathcal{M}$ *such that* $Z - X \in \mathcal{A}$ *and* $\pi(Z) = \pi^+(X)$.

Proof First of all, we note that π^+ is lower semicontinuous by (4.1) because $\mathcal{C}$ is closed due to Lemma 4.1. Next, we claim that π^+ does not attain the value $-\infty$. To this end, note first that $\pi^+(0) > -\infty$ by Lemma 4.1. Since $\pi^+(0) \leq 0$, it follows that π^+ is finite at 0. It is readily seen that π^+ is convex. Hence, being lower semicontinuous, π^+ can never attain the value $-\infty$ on the space L^1. To show the assertion, take a payoff $X \in L^1$ such that $\pi^+(X) < \infty$. Since $\pi^+(X)$ is finite, it follows from the closedness of $\mathcal{C}$ established in Lemma 4.1 that the infimum in (4.1) is attained. By the definition of $\mathcal{C}$, this implies that $\pi^+(X) = \pi(Z)$ for a suitable $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A}$, concluding the proof. □

The next result provides a characterisation of market-consistent prices under the assumption that the market does not admit strong scalable good deals. In this case, we show that for a payoff outside $\mathcal{M}$, the superreplication price is never market-consistent and hence the set of market-consistent prices is an open interval. For a payoff in $\mathcal{M}$, the superreplication price may or may not be market-consistent, so that the corresponding set of market-consistent prices may or may not be a closed interval.

Proposition 4.4 *If there exists no strong scalable good deal, then for every* $X \in L^1$, *we have* $\mathrm{MCP}(X) \neq \emptyset$ *and the following statements hold*:

(i) *If* $X \in \mathcal{M}$, *then* $\pi^+(X) \leq \pi(X)$ *and both alternatives* $\pi^+(X) \notin \mathrm{MCP}(X)$ *as well as* $\pi^+(X) \in \mathrm{MCP}(X)$ *can hold.*

(ii) *If* $X \in \mathcal{M}$ *and* $\pi^+(X) \notin \mathrm{MCP}(X)$, *then both alternatives* $\pi^+(X) = \pi(X)$ *as well as* $\pi^+(X) < \pi(X)$ *can hold.*

(iii) *If* $X \in \mathcal{M}$ *and* $\pi^+(X) \in \mathrm{MCP}(X)$, *then* $\pi^+(X) = \pi(X)$.

(iv) *If* $X \notin \mathcal{M}$, *then* $\pi^+(X) \notin \mathrm{MCP}(X)$.

The alternatives in (i) *and* (ii) *can hold even if there exists no good deal.*

Proof It follows from Proposition 4.3 that $\pi^+(X) > -\infty$ for every $X \in L^1$, showing that $\mathrm{MCP}(X) \neq \emptyset$. Now take $X \in \mathcal{M}$ and observe that $\pi^+(X) \leq \pi(X)$. It is shown in Example 4.5 that all the situations in (i) and (ii) may hold (even if there exist no good deals). To establish (iii) and (iv), take an arbitrary $X \in L^1$ and assume that $\pi^+(X) \in \mathrm{MCP}(X)$. By Proposition 4.3, $(\mathcal{A} + X) \cap \{Z \in \mathcal{M} : \pi(Z) = \pi^+(X)\}$ is not empty. It then follows from Proposition 4.2 that X must belong to $\mathcal{M}$ and $\pi^+(X) = \pi(X)$, establishing the desired implications. □

Example 4.5 Let $\Omega = \{\omega_1, \omega_2\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}[\omega_1] = \mathbb{P}[\omega_2] = \frac{1}{2}$. In this simple setting, we identify every element of L^1 with a vector of $\mathbb{R}^2$. Set $\mathcal{S} = \mathbb{R}^2$ and consider the acceptance set defined by

$$\mathcal{A} = \left\{(x, y) \in \mathbb{R}^2 : y \geq \max\{-x, 0\}\right\}.$$

(i) Set $\pi(x,y)=\max\{2x+y,x+2y\}$ for $(x,y)\in\mathbb{R}^2$ and $\mathcal{M}=\mathbb{R}^2$. It is immediate to verify that no good deal exists. Set $X=(-2,1)\in\mathcal{M}$ and observe that $\pi^+(X)=0$ and

$$(\mathcal{A}+X)\cap\{Z\in\mathcal{M}:\pi(Z)=0\}=\{X\}.$$

It follows from Proposition 4.2 that $\pi^+(X)\in\mathrm{MCP}(X)$. Now take $Y=(1,-2)\in\mathcal{M}$. In this case, an explicit calculation shows that

$$\pi^+(Y)=\inf_{x\in\mathbb{R}}\max\left\{2x-2+\max\{1-x,0\},x-4+2\max\{1-x,0\}\right\}=-\frac{3}{2}.$$

Moreover, setting $W=(-\frac{1}{2},-\frac{1}{2})\in\mathcal{M}$, we have

$$(\mathcal{A}+Y)\cap\left\{Z\in\mathcal{M}:\pi(Z)=-\frac{3}{2}\right\}=\{W\}.$$

It follows from Proposition 4.2 that $\pi^+(Y)\notin\mathrm{MCP}(Y)$. Note also that $\pi(X)=\pi^+(X)$ and $\pi(Y)>\pi^+(Y)$.

(ii) Set $\pi(x,y)=\max\{x+y,x+2y\}$ on $\mathbb{R}^2$ and $\mathcal{M}=\{(x,y)\in\mathbb{R}^2:x\le 1\}$. Observe that no good deal exists. Set $X=(1,-1)\in\mathcal{M}$ and $Y=(2,-2)\notin\mathcal{M}$. Then $\pi^+(X)=\pi^+(Y)=0$ and

$$\begin{aligned}(\mathcal{A}+X)\cap\{Z\in\mathcal{M}:\pi(Z)=0\}&=(\mathcal{A}+Y)\cap\{Z\in\mathcal{M}:\pi(Z)=0\}\\&=\{\lambda X:\lambda\in[0,1]\}.\end{aligned}$$

It follows from Proposition 4.2 that $\pi^+(X)\notin\mathrm{MCP}(X)$ and $\pi^+(Y)\notin\mathrm{MCP}(Y)$. Note also that $\pi(X)=0$ so that $\pi(X)=\pi^+(X)$.

(iii) Set $\pi(x,y)=e^x-1$ for every $(x,y)\in\mathbb{R}^2$ and $\mathcal{M}=\mathbb{R}\times\mathbb{R}_+$. Any $X\in\mathbb{R}^2$ satisfies $\pi^+(X)=-1$ and

$$(\mathcal{A}+X)\cap\{Z\in\mathcal{M}:\pi(Z)=-1\}=\emptyset.$$

It follows from Proposition 4.2 that $\pi^+(X)\in\mathrm{MCP}(X)$ regardless of whether X belongs to $\mathcal{M}$ or not. Note that in this case, there exist strong scalable good deals.

Proposition 4.4 unveils a stark contrast between our general setting and the classical frictionless setting, where the superreplication price of every replicable payoff is market-consistent and coincides with the associated replication cost. In our case, for an attainable payoff, the superreplication price may be strictly lower than the associated replication cost. This is in line with the findings in Bensaid et al. [8], where the focus is on a multiperiod Cox–Ross–Rubinstein model with proportional transaction costs and no portfolio constraints, and the acceptance set is the standard positive cone. As explained in [8], the said discrepancy is a direct consequence of the fact that trading is costly and it may therefore “pay to weigh the benefits of replication against those of potential savings on transaction costs”. What also follows from the previous result and was only implicitly highlighted in [8] is that contrary to the frictionless case, the superreplication price of an attainable payoff, and a fortiori its replication cost, may fail to be market-consistent. This is another implication of transaction costs, which allow the infimum in the definition of the superreplication price to be attained by multiple replicable payoffs even if the market admits no good deals.

4.3 Consistent price deflators

In this section, we start our journey towards a dual characterisation of market-consistent prices with acceptable risk. Our results are expressed in terms of suitable dual elements called (strictly) consistent price deflators. These notions are encountered in the literature under special assumptions on the market model and/or on the acceptance set. In a frictionless setting, a consistent price deflator corresponds to a representative state pricing function in Carr et al. [14] and to a Riesz density of a no-good-deal pricing functional in Černý and Hodges [18]. In a market with proportional frictions, it corresponds to a Riesz density of an underlying frictionless pricing rule in Jouini and Kallal [33], to a consistent price system in Jaschke and Küchler [32], to a consistent pricing kernel in Staum [50], and is related to a risk-neutral measure in Černý [17]. In a market with nonproportional frictions, it corresponds to a marginal price deflator in Pennanen [43]. Strictly consistent price deflators have been considered in Jouini and Kallal [33], Černý and Hodges [18] and Pennanen [43]. Note that the acceptance set in [33] and [43] is the standard positive cone. The formal definition of a price deflator is as follows. We denote by L^∞ the space of bounded random variables modulo almost sure equality under $\mathbb{P}$ and by $\|\cdot\|_\infty$ its standard norm. It is also convenient to introduce the maps $\gamma_{\pi,\mathcal{M}}: L^\infty \to (-\infty,\infty]$ and $\gamma_{\mathcal{A}}: L^\infty \to [-\infty,\infty)$ defined by

$$\gamma_{\pi,\mathcal{M}}(Y) := \sup_{X\in\mathcal{M}} \big(\mathbb{E}[XY] - \pi(X)\big),$$

$$\gamma_{\mathcal{A}}(Y) := \inf_{X\in\mathcal{A}} \mathbb{E}[XY].$$

Note that $\gamma_{\pi,\mathcal{M}}$ coincides with the conjugate function of the restriction to $\mathcal{M}$ of the pricing rule π, whereas $\gamma_{\mathcal{A}}$ is up to a sign the support function of the set $-\mathcal{A}$.

Definition 4.6 We say that $D \in L^\infty$ is a *price deflator* if $\gamma_{\pi,\mathcal{M}}(D) < \infty$. In this case, D is called

(1) *weakly consistent (with $\mathcal{A}$)* if $\gamma_{\mathcal{A}}(D) > -\infty$;

(2) *consistent (with $\mathcal{A}$)* if $\gamma_{\mathcal{A}}(D) \geq 0$;

(3) *strictly consistent (with $\mathcal{A}$)* if $\mathbb{E}[DX] > 0$ for every nonzero $X \in \mathcal{A}$.

We define the sets of weakly and strictly consistent price deflators by

$$\mathcal{D} := \{D \in L^\infty : D \text{ is a weakly consistent price deflator}\},$$

$$\mathcal{D}_{\mathrm{str}} := \{D \in L^\infty : D \text{ is a strictly consistent price deflator}\}.$$

A price deflator is a natural extension of a classical price deflator to our market with frictions. To illustrate this, consider a price deflator $D \in L^\infty$ and define the functional $\psi(X) := \mathbb{E}[DX]$ for $X \in L^1$. By definition,

$$-\pi(-X) - \gamma_{\pi,\mathcal{M}}(D) \leq \psi(X) \leq \pi(X) + \gamma_{\pi,\mathcal{M}}(D), \qquad \forall X \in \mathcal{M}\cap(-\mathcal{M}).$$

The functional ψ can therefore be viewed as the pricing rule of an "artificial" frictionless market where every payoff in L^1 is "replicable" and the attainable payoffs are "priced", up to a suitable enlargement of size $2\gamma_{\pi,\mathcal{M}}(D)$, consistently with their market bid–ask spread. No enlargement is needed, i.e., $\gamma_{\pi,\mathcal{M}}(D)=0$, when ψ is already dominated from above by π. This happens for instance if both π and $\mathcal{M}$ are conic in the first place. In particular, this holds if π is linear and $\mathcal{M}$ coincides with the entire space $\mathcal{S}$, in which case ψ is a linear extension of the pricing rule beyond the space of replicable payoffs. Consistency with the acceptance set is of course specific to good deal pricing theory. If D is weakly consistent, then

$$\psi(X) \geq \gamma_{\mathcal{A}}(D), \qquad \forall X \in \mathcal{A}.$$

This means that the prices of acceptable payoffs in the "artificial" frictionless market with pricing rule ψ cannot be arbitrarily negative. A simple situation where such "artificial" prices are nonnegative is when $\mathcal{A}$ is a cone in the first place. In this case, weak consistency is equivalent to consistency. In particular, if $\mathcal{A}$ is taken to be the standard positive cone, then (strict) consistency boils down to the (strict) positivity of ψ or, equivalently, of D. This shows that a (strictly) consistent price deflator is a direct extension of a (strictly) positive price deflator in the classical theory.

Remark 4.7 In a market where some attainable payoff is frictionless, every price deflator can be represented in terms of a probability measure. In order to see this, let $D \in L^\infty$ be a (strictly positive) price deflator and consider a strictly positive payoff $U \in \mathcal{M}^\infty \cap (-\mathcal{M}^\infty)$ such that π is linear along U and satisfies $\pi(U) > 0$. It follows from the preceding discussion that $\mathbb{E}[DU]=\pi(U)$. Then we find a probability measure $\mathbb{Q}$ that is absolutely continuous with respect to (equivalent to) $\mathbb{P}$ and satisfies

$$\frac{\mathbb{E}[DX]}{\pi(U)} = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{U}\right], \qquad \forall X \in L^1.$$

The probability $\mathbb{Q}$ plays the role of an (equivalent) pricing measure from arbitrage pricing theory.

We now show that the existence of strictly consistent price deflators always implies the absence of scalable good deals. However, contrary to the classical frictionless setting, it does not generally imply the absence of good deals unless the price deflators satisfy suitable extra assumptions.

Proposition 4.8 *If there exists a strictly consistent price deflator $D \in L^\infty$, then there exists no scalable good deal. If additionally $\mathbb{E}[DX] \leq \pi(X)$ for every $X \in \mathcal{M}$, then there exists no good deal either.*

Proof Take a nonzero payoff $X \in \mathcal{A} \cap \mathcal{M}^\infty$. To prove that no scalable good deal exists, we show that $\pi^\infty(X) > 0$. To this end, note that by the definition of a price deflator,

$$\begin{aligned}\sup_{n\in\mathbb{N}}\Big(n\big(\mathbb{E}[DX]-\pi^\infty(X)\big)\Big) &= \sup_{n\in\mathbb{N}}\big(\mathbb{E}[D(nX)]-\pi^\infty(nX)\big)\\ &\leq \sup_{n\in\mathbb{N}}\big(\mathbb{E}[D(nX)]-\pi(nX)\big) < \infty,\end{aligned}$$

where we used that π^∞ dominates π. This is only possible if $\mathbb{E}[DX] - \pi^\infty(X) \le 0$. As a result, we obtain $\pi^\infty(X) \ge \mathbb{E}[DX] > 0$. Next, assume that $\mathbb{E}[DX] \le \pi(X)$ for every payoff $X \in \mathcal{M}$ and take a nonzero payoff $X \in \mathcal{A} \cap \mathcal{M}$. Then we obtain that $\pi(X) \ge \mathbb{E}[DX] > 0$, showing that no good deal exists. □

Example 4.9 We work in the setting of Example 4.5 and take the same $\mathcal{S}$ and $\mathcal{A}$. We show that the existence of strictly consistent price deflators is not sufficient to rule out good deals. In view of Proposition 4.8, this can occur only if either the pricing rule or the set of attainable payoffs fails to be conic and the supremum in Definition 4.6 is strictly positive. We provide an example in both cases.

(i) Set $\pi(x, y) = x + y^2$ on $\mathbb{R}^2$ and $\mathcal{M} = \mathbb{R}^2$. Note that $\mathcal{M}$ is conic while π is not. It is clear that $D = (2, 4)$ is a strictly consistent price deflator. In particular,

$$\sup_{X \in \mathcal{M}} \big(\mathbb{E}[DX] - \pi(X)\big) = \sup_{y \in \mathbb{R}} (2y - y^2) = 1.$$

However, $X = (-1, 1) \in \mathcal{A} \cap \mathcal{M}$ satisfies $\pi(X) = 0$ and is thus a good deal.

(ii) Set $\pi(x, y) = x + y$ on $\mathbb{R}^2$ and $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : x \ge -1, 0 \le y \le 1\}$. Note that π is conic while $\mathcal{M}$ is not. It is clear that $D = (2, 4)$ is a strictly consistent price deflator. In particular,

$$\sup_{X \in \mathcal{M}} \big(\mathbb{E}[DX] - \pi(X)\big) = \sup_{0 \le y \le 1} y = 1.$$

However, $X = (-1, 1) \in \mathcal{A} \cap \mathcal{M}$ satisfies $\pi(X) = 0$ and is thus a good deal.

4.4 Fundamental theorem of asset pricing

We now turn to the more challenging problem of investigating if and under which assumptions the converse of Proposition 4.8 holds, i.e., the absence of scalable good deals implies the existence of strictly consistent price deflators. A key role is again played by the set $\mathcal{C}$ introduced in Sect. 4.1. We start with two preparatory results. The first shows that price deflators appear naturally in the dual representation of $\mathcal{C}$. The second provides a useful equivalent condition for the absence of scalable good deals in terms of $\mathcal{C}$. When $\mathcal{A} = L^1_+$, this condition corresponds to the "no scalable arbitrage" condition in Pennanen [43]. We denote by $\mathrm{cl}(\mathcal{C})$ the closure of $\mathcal{C}$ with respect to the natural product topology on $L^1 \times \mathbb{R}$ and define the (upper) support function and barrier cone of $\mathcal{C}$ by

$$\sigma_{\mathcal{C}}(Y, r) := \sup_{(X,m) \in \mathcal{C}} (\mathbb{E}[XY] + mr), \qquad (Y, r) \in L^\infty \times \mathbb{R},$$

$$\mathrm{bar}(\mathcal{C}) := \{(Y, r) \in L^\infty \times \mathbb{R} : \sigma_{\mathcal{C}}(Y, r) < \infty\}.$$

In what follows, we freely use some basic properties of support functions and barrier cones and refer to the Appendix for the necessary details.

Lemma 4.10 *The sets $\mathcal{C}$ and $\mathcal{D}$ are convex and the following statements hold*:
(i) $(-\mathcal{A}\times\mathbb{R}_+)\subseteq\mathcal{C}$ *and* $\text{bar}(\mathcal{C})\subseteq L^\infty_+\times-\mathbb{R}_+$.
(ii) $\sigma_\mathcal{C}(Y,-1)=\gamma_{\pi,\mathcal{M}}(Y)-\gamma_\mathcal{A}(Y)$ *for every* $Y\in L^\infty$.
(iii) $\mathcal{D}=\{Y\in L^\infty_+:\sigma_\mathcal{C}(Y,-1)<\infty\}=\{Y\in L^\infty_+:(Y,-1)\in\text{bar}(\mathcal{C})\}$.
(iv) *If* $(0,-n)\notin\text{cl}(\mathcal{C})$ *for some* $n\in\mathbb{N}$, *then we can represent* $\text{cl}(\mathcal{C})$ *as*

$$\text{cl}(\mathcal{C})=\bigcap_{Y\in\mathcal{D}}\{(X,m)\in L^1\times\mathbb{R}:\mathbb{E}[XY]-m\leq\gamma_{\pi,\mathcal{M}}(Y)-\gamma_\mathcal{A}(Y)\}.$$

Proof The convexity of $\mathcal{C}$ and $\mathcal{D}$ is clear. Points (i)–(iii) follow easily once we observe that $\mathcal{C}=\{(Z,m)\in\mathcal{M}\times\mathbb{R}:\pi(Z)\leq m\}+((-\mathcal{A})\times\mathbb{R}_+)$. Note that no problems with nonfinite values arise as $0\in\mathcal{M}$, $\pi(0)=0$ and $\mathcal{A}$ contains the cone of positive random variables. To show (iv), assume that $\text{cl}(\mathcal{C})$ is strictly contained in $L^1\times\mathbb{R}$. The dual representation of closed convex sets in Aliprantis and Border [1, Theorem 7.51] yields

$$\text{cl}(\mathcal{C})=\bigcap_{(Y,r)\in L^\infty\times\mathbb{R}}\{(X,m)\in L^1\times\mathbb{R}:\mathbb{E}[XY]+mr\leq\sigma_\mathcal{C}(Y,r)\}. \tag{4.2}$$

Here we have used that $\sigma_{\text{cl}(\mathcal{C})}=\sigma_\mathcal{C}$. We claim that $\text{bar}(\mathcal{C})\cap(L^\infty\times(-\infty,0))\neq\emptyset$. To show this, take $n\in\mathbb{N}$ such that $(0,-n)\notin\text{cl}(\mathcal{C})$. Then it follows from (4.2) that there must exist $(Y,r)\in\text{bar}(\mathcal{C})$ satisfying $-nr=\mathbb{E}[0Y]-nr>\sigma_\mathcal{C}(Y,r)\geq 0$. This establishes the desired claim. Now recall from (i) that $\text{bar}(\mathcal{C})\subseteq L^\infty_+\times-\mathbb{R}_+$. Since $\sigma_\mathcal{C}$ is sublinear and $\text{bar}(\mathcal{C})$ is a convex cone, it follows that

$$\text{cl}(\mathcal{C})=\bigcap_{Y\in L^\infty_+}\{(X,m)\in L^1\times\mathbb{R}:\mathbb{E}[XY]-m\leq\sigma_\mathcal{C}(Y,-1)\}.$$

The desired representation is now a direct consequence of (ii). □

As recalled in the Appendix, the acceptance set $\mathcal{A}$ is pointed if $\mathcal{A}\cap(-\mathcal{A})=\{0\}$.

Lemma 4.11 *Let $\mathcal{A}$ be a pointed cone. Then there exists no scalable good deal if and only if for every nonzero $X\in\mathcal{A}$, there is $\lambda>0$ such that $(\lambda X,0)\notin\mathcal{C}$.*

Proof First, take a nonzero $X\in\mathcal{A}$ and let $\lambda>0$ satisfy $\lambda X\notin\{Z\in\mathcal{M}:\pi(Z)\leq 0\}$. This yields $X\notin\{Z\in\mathcal{M}^\infty:\pi^\infty(Z)\leq 0\}$. As a result, the "if" implication holds. To prove the converse, assume that no scalable good deal exists. First, we claim that $\{Z\in\mathcal{A}\cap\mathcal{M}:\pi(Z)\leq 0\}$ is bounded. If this is not the case, for every $n\in\mathbb{N}$, we find $Z_n\in\mathcal{A}\cap\mathcal{M}$ with $\pi(Z_n)\leq 0$ and $\|Z_n\|_1\geq n$. As the unit sphere in $\mathcal{S}$ is compact, there exists a nonzero $Z\in\mathcal{S}$ such that $\frac{Z_n}{\|Z_n\|_1}\to Z$. Note that $Z\in\mathcal{A}^\infty\cap\mathcal{M}^\infty$ by (A.1). Note also that the lower semicontinuity and convexity of π yield

$$\pi(Z)\leq\liminf_{n\to\infty}\pi\left(\frac{Z_n}{\|Z_n\|}\right)\leq\liminf_{n\to\infty}\frac{\pi(Z_n)}{\|Z_n\|}\leq 0.$$

This shows that Z is a scalable good deal, contradicting our assumption. Hence the set $\{Z\in\mathcal{A}\cap\mathcal{M}:\pi(Z)\leq 0\}$ is indeed bounded. Now suppose that we find a nonzero

$X \in \mathcal{A}$ such that for every $\lambda > 0$, there is $Z_\lambda \in \mathcal{M}$ with $\pi(Z_\lambda) \leq 0$ and $Z_\lambda - \lambda X \in \mathcal{A}$. In particular, $Z_\lambda \in \mathcal{A}$ and $\frac{Z_\lambda}{\lambda} \in \mathcal{A} + X$ for every $\lambda > 0$. As $(\mathcal{A} + X) \cap \mathcal{S}$ is closed and does not contain the zero payoff, $\| \cdot \|_1$ must be bounded from below by a suitable $\varepsilon > 0$ on the set $(\mathcal{A} + X) \cap \mathcal{S}$. In particular, $\frac{\|Z_\lambda\|_1}{\lambda} \geq \varepsilon$ for every $\lambda > 0$. This implies that $\{Z_\lambda : \lambda > 0\}$ is unbounded, in contrast to what was proved above. As a result, for every nonzero $X \in \mathcal{A}$, there must be $\lambda > 0$ such that $(\mathcal{A} + \lambda X) \cap \{Z \in \mathcal{M} : \pi(Z) \leq 0\}$ is empty. This yields the "only if" implication. □

The key tool to establish existence of strictly consistent price deflators is the general version of the classical results by Yan [52] and Kreps [39] recorded in Theorem A.1 in the Appendix. The "conification" appearing there leads us to work with the modified acceptance set

$$\mathcal{K}(\mathcal{A}) := \mathrm{cl}(\{\lambda X : \lambda \geq 0, X \in \mathcal{A}\}),$$

where cl now denotes the closure in L^1. A similar conification was considered in Černý and Hodges [18] and Staum [50] and is necessary to obtain a version of the fundamental theorem for nonconic acceptance sets. The next lemma records useful information about this enlarged acceptance set. We omit the simple proof.

Lemma 4.12 *The set $\mathcal{K}(\mathcal{A})$ is conic and satisfies the properties in Assumption 2.3 if it is a strict subset of L^1. In particular, if $\mathcal{A}$ is a cone, then $\mathcal{K}(\mathcal{A}) = \mathcal{A}$.*

We are finally in a position to state sufficient conditions for the existence of strictly consistent price deflators. As a first step, we provide two sets of sufficient conditions for the existence of consistent price deflators that are strictly positive. In order to move from strict positivity to strict consistency, we need an additional assumption on the model space L^1, namely separability. We refer to the accompanying remark for a detailed discussion about the proof strategy and the separability assumption.

Theorem 4.13 *Assume that one of the following conditions holds*:

(i) *$\mathcal{A}$ is a pointed cone and there exists no scalable good deal.*

(ii) *$\mathcal{K}(\mathcal{A})$ is pointed and there exists no scalable good deal with respect to $\mathcal{K}(\mathcal{A})$.*

Then there exists a strictly positive consistent price deflator $D \in L^\infty$. If in addition L^1 is separable with respect to its norm topology, then D can be taken to be strictly consistent.

Proof It follows from Proposition 4.12 that $\mathcal{K}(\mathcal{A})$ is a closed conic acceptance set. Note that every price deflator $D \in L^\infty$ that is (strictly) consistent with $\mathcal{K}(\mathcal{A})$ is also (strictly) consistent with $\mathcal{A}$. As a result, it suffices to prove the stated claims under condition (i). To this end, assume that $\mathcal{A}$ is a pointed cone and there exists no scalable good deal (with respect to $\mathcal{A}$). To establish existence of strictly positive respectively strictly consistent price deflators, we apply Theorem A.1 to $\mathcal{L} = L^1_+$ respectively $\mathcal{L} = \mathcal{A}$ and $\mathcal{L}' = \mathcal{D}$. Note that in the notation of that result, $\mathcal{L}' \cap (-\mathrm{bar}(\mathcal{K}_\mathcal{L})) = \mathcal{D}$ in either case by Lemma 4.10. **Throughout the proof**, convergence in L^∞ is always understood with respect to $\sigma(L^\infty, L^1)$. As a first step, we verify the completeness property in Theorem A.1. More precisely, we show that

$$\text{for every } (Y_n) \subseteq \mathcal{D}, \text{ we find } (\lambda_n) \subseteq (0,\infty) \text{ and } Y \in \mathcal{D} \text{ with } \sum_{k=1}^{n} \lambda_k Y_k \to Y. \tag{4.3}$$

To this end, recall that $\mathcal{D} \subseteq L^\infty_+$ by Lemma 4.10 and note that $\sigma_\mathcal{C}(Y,-1) \geq 0$ for every $Y \in \mathcal{D}$. Let $\alpha_n = (1+\|Y_n\|_\infty)^{-1}(1+\sigma_\mathcal{C}(Y_n,-1))^{-1}2^{-n} > 0$ and $S_n = \sum_{k=1}^n \alpha_k Y_k$ for every $n \in \mathbb{N}$. As a result, we find $Z \in L^\infty$ such that $S_n \to Z$ with respect to $\|\cdot\|_\infty$ and hence for $\sigma(L^\infty, L^1)$. To conclude the proof, note that $\sum_{k=1}^n \alpha_k \to r$ for some $r > 0$ and $\sigma_\mathcal{C}(Z,-r) \leq \liminf_{n\to\infty} \sum_{k=1}^n \alpha_k \sigma_\mathcal{C}(Y_k,-1) < \infty$ by $\sigma(L^\infty, L^1)$-lower semicontinuity and sublinearity of $\sigma_\mathcal{C}$. This yields $(Z,-r) \in \mathrm{bar}(\mathcal{C})$. The desired convergence in (4.3) holds by setting $\lambda_n = \frac{\alpha_n}{r} > 0$ for every $n \in \mathbb{N}$ and $Y = \frac{Z}{r} \in \mathcal{D}$.

As a next step, we show that we can always find a strictly positive consistent price deflator. In view of (4.3), it follows that it suffices to establish the countable separation property in Theorem A.1. More precisely, it suffices to find a sequence $(Y_n) \subseteq \mathcal{D}$ such that

$$\text{for every nonzero } X \in L^1_+, \text{ there exists } n \in \mathbb{N} \text{ such that } \mathbb{E}[XY_n] > 0. \tag{4.4}$$

By Lemma 4.11, for every nonzero $X \in L^1_+$, there exists $\lambda > 0$ with $(\lambda X, 0) \notin \mathcal{C}$. Since $\mathcal{C}$ is closed and $(0,-n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$ by Lemma 4.1, we can use the representation of (the closure of) $\mathcal{C}$ in Lemma 4.10 to find an element $Y_X \in \mathcal{D}$ such that $\mathbb{E}[\lambda X Y_X] > \sigma_\mathcal{C}(Y_X, 1) \geq 0$. Equivalently, we have that

$$\text{for every nonzero } X \in L^1_+, \text{ there exists } Y_X \in \mathcal{D} \text{ such that } \mathbb{E}[XY_X] > 0. \tag{4.5}$$

To establish (4.4), we start by showing that the family $\mathcal{G} = \{\{Y > 0\} : Y \in \mathcal{D}\}$ is nonempty and closed under countable unions. First, (4.5) yields $\mathcal{G} \neq \emptyset$. To show that $\mathcal{G}$ is closed under countable unions, take a sequence $(Y_n) \subseteq \mathcal{D} \setminus \{0\}$. By (4.3), we find a sequence $(\lambda_n) \subseteq (0,\infty)$ and an element $Y \in \mathcal{D}$ such that $S_n := \sum_{k=1}^n \lambda_k Y_k \to Y$. It is easy to see that

$$\{Y > 0\} = \bigcup_{n\in\mathbb{N}} \{Y_n > 0\} \qquad \mathbb{P}\text{-almost surely.} \tag{4.6}$$

Indeed, consider first the event $E = \{Y > 0\} \cap \bigcap_{n\in\mathbb{N}} \{Y_n = 0\}$. We must have $\mathbb{P}[E] = 0$, for otherwise $0 < \mathbb{E}[1_E Y] = \lim_{n\to\infty} \mathbb{E}[1_E S_n] = 0$. As a result, the inclusion "$\subseteq$" in (4.6) holds. Next, we claim that $\mathbb{P}[Y \geq S_n] = 1$ for every $n \in \mathbb{N}$. If not, we find $k \in \mathbb{N}$ and $\varepsilon > 0$ such that with $E = \{Y \leq S_k - \varepsilon\}$, we have

$$0 < \varepsilon\mathbb{P}[E] \leq \mathbb{E}[1_E(S_k - Y)] \leq \lim_{n\to\infty} \mathbb{E}[1_E(S_n - Y)] = 0.$$

This delivers the inclusion "$\supseteq$" in (4.6) and shows that $\mathcal{G}$ is closed under countable unions as desired. Now set $s = \sup\{\mathbb{P}[E] : E \in \mathcal{G}\}$. Take any sequence $(Y_n) \subseteq \mathcal{D}$ such that $\mathbb{P}[Y_n > 0] \uparrow s$. By closedness under countable unions, there must exist $Y^* \in \mathcal{D}$ such that $\{Y^* > 0\} = \bigcup_{n\in\mathbb{N}} \{Y_n > 0\}$ $\mathbb{P}$-almost surely. Take an arbitrary nonzero $X \in L^1_+$ and assume that $\mathbb{E}[XY_n] = 0$ for every $n \in \mathbb{N}$. This would imply that $\mathbb{E}[XY^*] = 0$, and thus the element $\frac{1}{2}Y^* + \frac{1}{2}Y_X \in \mathcal{D}$ would satisfy

$$\mathbb{P}\left[\frac{1}{2}Y^* + \frac{1}{2}Y_X > 0\right] \geq \mathbb{P}[Y^* > 0] + \mathbb{P}[\{Y^* = 0\} \cap \{Y_X > 0\}] > \mathbb{P}[Y^* > 0] = s,$$

which cannot hold. Thus we must have $\mathbb{E}[XY_n] > 0$ for some $n \in \mathbb{N}$, showing (4.4).

To conclude the proof, we show that there exists a strictly consistent price deflator if we additionally assume that L^1 is separable with respect to its norm topology. In view of (4.3), it follows from Theorem A.1 that we only have to exhibit a sequence $(Y_n) \subseteq \mathcal{D}$ such that

$$\text{for every nonzero } X \in \mathcal{A}, \text{ there exists } n \in \mathbb{N} \text{ such that } \mathbb{E}[XY_n] > 0. \tag{4.7}$$

Repeating the argument that led to (4.5), we obtain that

$$\text{for every nonzero } X \in \mathcal{A}, \text{ there exists } Y_X \in \mathcal{D} \text{ such that } \mathbb{E}[XY_X] > 0. \tag{4.8}$$

For every nonzero $X \in \mathcal{A}$, define $Z_X := \frac{Y_X}{\|Y_X\|_\infty}$. By separability, the unit ball in L^∞ is $\sigma(L^\infty, L^1)$-metrisable by Aliprantis and Border [1, Theorem 6.30]. Being $\sigma(L^\infty, L^1)$-compact by virtue of the Banach–Alaoglu theorem, see e.g. [1, Theorem 6.21], the unit ball together with any of its subsets is therefore $\sigma(L^\infty, L^1)$-separable. In particular, this is true for $\{Z_X : X \in \mathcal{A} \setminus \{0\}\}$. Let $\{Z_{X_n} : n \in \mathbb{N}\}$ be a countable $\sigma(L^\infty, L^1)$-dense subset. Then for every nonzero $X \in \mathcal{A}$, it follows immediately from (4.8) that we must have $\mathbb{E}[XY_{X_n}] > 0$ for some $n \in \mathbb{N}$ by $\sigma(L^\infty, L^1)$-density. This delivers (4.7). □

Remark 4.14 (i) The pointedness condition, which is clearly necessary for the existence of strictly consistent price deflators, is satisfied by many standard acceptance sets. For instance, by Bellini et al. [7, Proposition 5.9], pointedness holds whenever $\mathcal{A}$ is a law-invariant cone such that $\mathcal{A} \neq \{X \in L^1 : \mathbb{E}[X] \geq 0\}$. Incidentally, note that under pointedness, the absence of scalable good deals is equivalent to the generally weaker absence of strong scalable good deals.

(ii) A simple sufficient condition for separability of L^1 is that $\mathcal{F}$ is countably generated, e.g. if $\mathcal{F}$ is the σ-field generated by the basic payoffs spanning $\mathcal{S}$. We refer to Aliprantis and Border [1, Theorem 13.16] for a characterisation of separability in the nonatomic setting.

(iii) To see that the "conification" of the acceptance set is necessary to ensure existence of strictly consistent price deflators in the nonconic case, one can observe that every strictly consistent price deflator for $\mathcal{A}$ is automatically strictly consistent for $\mathcal{K}(\mathcal{A})$. This is also true for the more natural "conified" acceptance set $\{\lambda X : \lambda \geq 0, X \in \mathcal{A}\}$, but closedness is necessary for our arguments.

(iv) The proof of existence of *strictly positive* consistent price deflators builds on the exhaustion argument underpinning the classical result on equivalent probability measures in Halmos and Savage [29]. In fact, a direct application of that result provides an alternative proof of the countable separation property in (4.4). To see this, note that every element $Y_X \in \mathcal{D}$ in (4.5) is associated with a probability measure on $(\Omega, \mathcal{F})$ defined by $d\mathbb{P}_X := \frac{Y_X}{\mathbb{E}_{\mathbb{P}}[Y_X]} d\mathbb{P}$. Since the family of these probability measures is dominated by $\mathbb{P}$, it follows from [29, Lemma 7] that there exists a sequence $(X_n) \subseteq L^1_+ \setminus \{0\}$ such that for every $E \in \mathcal{F}$, we have $\mathbb{P}_{X_n}[E] = 0$ for every $n \in \mathbb{N}$ if and only if $\mathbb{P}_X[E] = 0$ for every nonzero $X \in L^1_+$. For every nonzero $X \in L^1_+$, we clearly have $\mathbb{P}_X[X > 0] > 0$, and hence there must exist $n \in \mathbb{N}$ such that $\mathbb{P}_{X_n}[X > 0] > 0$ or, equivalently, $\mathbb{E}[XY_{X_n}] > 0$. The countable separation property is thus fulfilled by the sequence (Y_{X_n}). It is worth noting that neither this argument nor the argument in the proof above can be used to ensure existence of *strictly consistent* price deflators when the acceptance sets is strictly larger than the positive cone and thus contains nonpositive payoffs. This is because controlling probabilities alone is not sufficient to control the sign of expectations. To deal with strict consistency in the general case, we therefore had to pursue a different strategy based on separability of L^1, which was inspired by the original work of Kreps [39] and by the related work of Clark [21] in the setting of frictionless markets.

We are finally in a position to establish the announced version of the fundamental theorem of asset pricing for markets with frictions and general acceptance sets, which we state in the usual form of an equivalence. The theorem follows at once by combining Proposition 4.8 and Theorem 4.13 and is split into three parts. In the first part, we focus on the situation where the acceptance set is the positive cone. In this case, we obtain a different proof of the one-period version of the fundamental theorem in markets with frictions established in Pennanen [43, Theorem 5.4]. As already said, the absence of scalable arbitrage opportunities corresponds to the "no scalable arbitrage" condition, and a price deflator corresponds to a marginal price deflator in [43]. In the second and third parts, we focus on conic and nonconic acceptance sets respectively. The corresponding versions of the fundamental theorem are new. We refer to Example 4.16 below for a proof of the necessity of our assumptions on the acceptance set.

Theorem 4.15 (i) *There exists no scalable arbitrage opportunity if and only if there exists a strictly positive price deflator in* L^∞.

(ii) *Let* L^1 *be separable with respect to its norm topology* (*e.g. if* $\mathcal{F}$ *is countably generated*).

(a) *Let* $\mathcal{A}$ *be a pointed cone. Then there exists no scalable good deal if and only if there exists a strictly consistent price deflator in* L^∞.

(b) *Let* $\mathcal{K}(\mathcal{A})$ *be pointed. If there exists no scalable good deal with respect to* $\mathcal{K}(\mathcal{A})$, *then there exists a strictly consistent price deflator in* L^∞. *If there exists a strictly consistent price deflator in* L^∞, *then there exists no scalable good deal* (*with respect to* $\mathcal{A}$).

Example 4.16 We prove necessity of our assumptions on $\mathcal{A}$ in the setting of Example 4.5.

(i) Set $\mathcal{M} = \mathbb{R}^2$ and $\pi(x, y) = \max\{x, y\}$ on $\mathbb{R}^2$ and define

$$\mathcal{A} = \mathbb{R}^2_+ \cup \{(x, y) \in \mathbb{R}^2 : x < 0, y \geq x^2\}.$$

Note that $\mathcal{A}$ is not a cone and no scalable good deal exists. However, there exists no strictly consistent price deflator $D=(d_1,d_2)$. Indeed, for every $\lambda>0$, we could otherwise take $X_\lambda=(-\lambda,\lambda^2)\in\mathcal{A}$ and note that $\mathbb{E}[DX_\lambda]>0$ implies $d_2\lambda>d_1$, which contradicts the strict positivity and hence the strict consistency of D. This shows that if we remove conicity, the "only if" implication in Theorem 4.15 (a) generally fails. It also shows that the converse of the second implication in (b) generally fails as well.

(ii) Set $\mathcal{M}=\mathbb{R}^2$ and $\pi(x,y)=x+y$ on $\mathbb{R}^2$ and define

$$\mathcal{A}=\mathbb{R}^2_+\cup\{(x,y)\in\mathbb{R}^2 : x<0,\, y\geq e^{-x}-1\}.$$

Note that $\mathcal{A}$ is not a cone and $\mathcal{K}(\mathcal{A})=\mathbb{R}^2_+\cup\{(x,y)\in\mathbb{R}^2 : x<0,\, y\geq -x\}$ is pointed. Note also that $D=(2,2)$ is a (in fact, the only) strictly consistent price deflator. However, $X=(-1,1)\in\mathcal{K}(\mathcal{A})\cap\mathcal{M}$ satisfies $\pi(X)=0$ and is therefore a scalable good deal with respect to $\mathcal{K}(\mathcal{A})$. This shows that the converse of the first implication in Theorem 4.15 (b) generally fails.

Remark 4.17 We now provide a detailed comparison of our version of the fundamental theorem of asset pricing with the various versions obtained in the good deal pricing literature.

(i) The focus of Carr et al. [14] is on one-period frictionless markets. The reference acceptance set is convex and defined in terms of finitely many test probabilities. The reference probability space is finite. In Theorem 1, the authors establish a fundamental theorem under the absence of a special type of good deals that is specific to the polyhedral structure of the acceptance set and stronger than the absence of scalable good deals. The statement is in terms of representative state pricing functions, which correspond to special (in general not strictly) consistent price deflators.

(ii) The focus of Jaschke and Küchler [32] is on multi-period markets with proportional frictions and admitting a frictionless asset. The reference acceptance set is assumed to be a convex cone. The reference probability space is general. In fact, the payoff space is an abstract topological vector space. In Corollary 8, the authors establish a fundamental theorem under the assumption of absence of good deals of the second kind. In our setting, this is equivalent to the absence of payoffs $X\in\mathcal{A}\cap\mathcal{M}$ such that $\pi(X)<0$. The statement is in terms of consistent (not strictly consistent) price deflators. To deal with the infinite-dimensionality of $\mathcal{M}$, which follows from the multi-period nature of the market model, the fundamental theorem is stated under an additional assumption that corresponds to the closedness of $\mathcal{C}$. No sufficient conditions for this are provided. It should be noted that the absence of good deals of the second kind is not sufficient to ensure closedness of $\mathcal{C}$ even when $\mathcal{M}$ is finite-dimensional. To show this, let $\Omega=\{\omega_1,\omega_2,\omega_3\}$ and assume that $\mathcal{F}$ is the power set of Ω and $\mathbb{P}[\omega_1]=\mathbb{P}[\omega_2]=\mathbb{P}[\omega_3]=\frac{1}{3}$. We identify every element of L^1 with a vector of $\mathbb{R}^3$. Let $\mathcal{M}$ coincide with $\mathcal{S}=\{(x,y,z)\in\mathbb{R}^3 : x=0\}$ and let $\pi:\mathcal{S}\to\mathbb{R}$ be defined by $\pi(x,y,z)=y$ for $(x,y,z)\in\mathbb{R}^3$. Consider the closed convex conic acceptance set

$$\mathcal{A}=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+6xy+2\sqrt{6}xz+2\sqrt{6}yz\geq 0,\, \sqrt{3}x+\sqrt{3}y+\sqrt{2}z\geq 0\},$$

obtained by rotating the cone $\mathcal{A}'=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2\leq 3z^2, z\geq 0\}$ by $\pi/3$ around the direction $(-1,1,0)$. It is easy to verify that if $X\in\mathcal{A}\cap\mathcal{M}$, then

$\pi(X) \geq 0$, and hence there are no good deals of the second kind. We show that $\mathcal{C}$ is not closed. For every $n \in \mathbb{N}$, set $X_n = (1 - \frac{1}{n}, -1, 0)$ and note that $(X_n, 0) \in \mathcal{C}$ as $Z_n = (0, 0, n^2) \in \mathcal{M}$ satisfies $\pi(Z_n) = 0$ and $Z_n - X_n \in \mathcal{A}$. Clearly, we have $(X_n, 0) \to (X, 0)$ with $X = (1, -1, 0)$. We conclude that $\mathcal{C}$ is not closed as $(X, 0) \notin \mathcal{C}$.

(iii) The focus of Černý and Hodges [18] is on one-period frictionless markets and convex acceptance sets. The reference probability space is general. In fact, the payoff space is an abstract locally convex topological vector space. In Theorem 2.5, the authors establish a fundamental theorem under the absence of good deals with respect to the “conified” acceptance set. The statement is expressed in terms of strictly consistent price deflators and is proved under the additional assumption that the payoff space is an L^p-space for some $1 < p < \infty$ and that $\mathcal{A}$ is boundedly generated, i.e., included in the cone generated by a bounded set. This condition typically fails when the underlying probability space is not finite.

(iv) The focus of Staum [50] is on multi-period markets with convex frictions. The reference acceptance set is convex. The reference probability space is general. In fact, the payoff space is an abstract locally convex topological vector space. In Theorem 6.2, the author establishes a fundamental theorem under the assumption that for all payoffs $X \in L^1$ and nonzero $Z \in L^1_+$,

$$\inf\{\pi(Z) : Z \in \mathcal{M}, Z - X \in \mathcal{A}\} + \inf\{\pi(Z) : Z \in \mathcal{M}, Z - X \in L^1_+\} > 0.$$

The link with the absence of good deals is not discussed. The statement is in terms of strictly positive (not strictly consistent) price deflators. To deal with the infinite-dimensionality of $\mathcal{M}$, which follows from the multi-period nature of the market model, the fundamental theorem is stated under the additional assumption that π^+ is lower semicontinuous. Sufficient conditions for this are provided when the payoff space is L^∞ (with respect to the standard norm topology). Unfortunately, the proof of Lemma 6.1, which is key to deriving the fundamental theorem, is flawed. On the one side, Zorn’s lemma is evoked to infer that a family of sets that is closed under countable unions admits a maximal element. However, this is not true as illustrated for instance by the family of all countable subsets of $\mathbb{R}$. On the other side, it is tacitly assumed that for a generic dual pair $(\mathcal{X}, \mathcal{X}')$, the series $\sum_{n\in\mathbb{N}} 2^{-n} Y_n$ converges in the topology $\sigma(\mathcal{X}', \mathcal{X})$ for every choice of $(Y_n) \subseteq \mathcal{X}'$, which cannot hold unless special assumptions are required of the pair $(\mathcal{X}, \mathcal{X}')$. The underlying strategy of reproducing the exhaustion argument used in the classical proof of the fundamental theorem seems unlikely to work because it heavily relies on the existence of a (dominating) probability measure and, as highlighted in Remark 4.14, breaks down in the presence of nonpositive acceptable payoffs.

(v) The focus of Černý [17] is on one-period markets with convex frictions. The reference acceptance set is a convex cone. The reference payoff space is tailored to the chosen acceptance set by way of a duality construction which often delivers standard L^p-spaces, for example when the acceptance set is based on expected shortfall. In Theorem 3.1, the author establishes a version of the fundamental theorem under the absence of special good deals. In our setting, they correspond to payoffs $X \in \mathcal{M}$ with $\pi(X) \leq 0$ and

$$\inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\} < 0.$$

The statement is in terms of a special class of (not necessarily strictly positive) price deflators. The proof uses the additional assumption that the barrier cone of the acceptance set is compactly generated.

(vi) The focus of Madan and Černý [41] is on one-period frictionless markets. The reference acceptance set is induced by an acceptability index. The reference payoff space consists of suitably integrable random variables. In Theorem 1, the authors provide a version of the fundamental theorem under the absence of good deals. The statement is in terms of (not necessarily strictly positive) price deflators.

(vii) The focus of Cheridito et al. [20] is on multi-period markets with general frictions and admitting a frictionless asset. The reference acceptance set is also general, but is required to ensure convexity of a set that in our notation corresponds to

$$\{X \in L^1 : \exists Z \in \mathcal{M} \text{ with } \pi(Z) \leq 0 \text{ and } Z - X \in \mathcal{A}\} = \{X \in L^1 : (X, 0) \in \mathcal{C}\}.$$

The payoff space consists of suitably regular stochastic processes. Notably, no dominating probability measure is assumed to exist. In Theorem 2.1, the authors establish a fundamental theorem under the absence of a suitable class of strong good deals. To deal with the infinite-dimensionality of $\mathcal{M}$, which follows from the multi-period nature of the market model, the fundamental theorem is stated under additional regularity assumptions on the market model and the acceptance set ensuring finiteness of superreplication prices of special call options. The statement is in terms of (not necessarily strictly) consistent price deflators.

(viii) We also highlight a link with the recent work by Herdegen and Khan [31]. The focus of that paper is on mean–risk portfolio problems in the context of a one-period frictionless market. The authors establish a variety of conditions for the existence of optimal portfolios under the assumption of absence of "ρ-arbitrage", where ρ is a coherent risk measure. Under mild assumptions on the risk measure, this notion is equivalent to that of a good deal provided the acceptance set consists of all payoffs with nonpositive risk. In this sense, [31, Theorem 4.20] can be viewed as a version of the fundamental theorem in a good deal pricing setting. The theorem is formulated in our language in terms of consistent price deflators satisfying some special dual conditions. As these conditions are only indirectly related to the risk measure and hence to its acceptance set, it is not clear to which extent they are linked with strict consistency of price deflators. We leave a full clarification of this to future research.

4.5 Superreplication duality

In this section, we derive a dual representation of superreplication prices based on consistent price deflators. We refer to Jaschke and Küchler [32, Corollary 8], Staum [50, Theorem 4.1] and Cheridito et al. [20, Theorem 2.1] for similar representations under the assumption of absence of good deals. We also refer to Frittelli and Scandolo [25, Proposition 3.9] for a similar representation in a risk measure setting. These representations were obtained under the assumption of lower semicontinuity of π^+. As mentioned in the proof of Proposition 4.3, a sufficient condition for this to hold is the absence of strong scalable good deals. In a second step, we improve the dual representation by replacing consistency with strict consistency. In a frictionless setting where the acceptance set is the standard positive cone, this is equivalent to moving from price deflators to strictly positive price deflators. This sharper result thus extends the classical result on superreplication duality to markets with frictions and general acceptance sets.

Theorem 4.18 *The following statements hold*:

(i) *If there exists no strong scalable good deal, then for every $X \in L^1$,*

$$\pi^+(X) = \sup_{D \in \mathcal{D}} \big(\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D)\big).$$

(ii) *If there exists no scalable good deal and if either $\mathcal{A} = L^1_+$ or $\mathcal{A}$ is a pointed cone and L^1 is separable with respect to its norm topology, then for every $X \in L^1$,*

$$\pi^+(X) = \sup_{D \in \mathcal{D}_{\text{str}}} \big(\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)\big). \tag{4.9}$$

Proof Assume the market is free of strong scalable good deals. It follows from Lemma 4.1 that $\mathcal{C}$ is closed and $(0, -n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$. Now take an arbitrary $X \in L^1$. From (4.1) and from the representation of (the closure of) $\mathcal{C}$ obtained in Lemma 4.10, we infer that

$$\begin{aligned}\pi^+(X) &= \inf\{m \in \mathbb{R} : \mathbb{E}[DX] - m - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D) \le 0, \forall D \in \mathcal{D}\}\\ &= \inf\{m \in \mathbb{R} : m \ge \mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D), \forall D \in \mathcal{D}\}\\ &= \sup\{\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D) : D \in \mathcal{D}\}.\end{aligned}$$

This proves (i). Now let the assumptions in (ii) hold. It follows from Theorem 4.13 that $\mathcal{D}_{\text{str}}$ is nonempty. Moreover, by Lemma 4.1, $\mathcal{C}$ is closed and $(0, -n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$. We claim that the representation in Lemma 4.10 for (the closure of) $\mathcal{C}$ can be rewritten as

$$\mathcal{C} = \bigcap_{Y \in \mathcal{D}_{\text{str}}} \{(X, m) \in L^1 \times \mathbb{R} : \mathbb{E}[XY] - m \le \gamma_{\pi,\mathcal{M}}(Y)\}. \tag{4.10}$$

Note that $\gamma_{\mathcal{A}}(Y) = 0$ for every $Y \in \mathcal{D}$ by conicity of $\mathcal{A}$. Clearly, we only need to establish the inclusion "$\supseteq$" in (4.10). To this end, take any $(X, m) \in L^1 \times \mathbb{R}$ such that $\mathbb{E}[XY] - m \le \gamma_{\pi,\mathcal{M}}(Y)$ for every $Y \in \mathcal{D}_{\text{str}}$. Fix $Y^* \in \mathcal{D}_{\text{str}}$ and take any $Y \in \mathcal{D}$. For every $\lambda \in (0, 1)$, we have $\lambda Y^* + (1 - \lambda)Y \in \mathcal{D}_{\text{str}}$ so that

$$\begin{aligned}\lambda(\mathbb{E}[XY^*] - m) + (1 - \lambda)(\mathbb{E}[XY] - m) &= \mathbb{E}\big[X\big(\lambda Y^* + (1 - \lambda)Y\big)\big] - m\\ &\le \gamma_{\pi,\mathcal{M}}\big(\lambda Y^* + (1 - \lambda)Y\big)\\ &\le \lambda\gamma_{\pi,\mathcal{M}}(Y^*) + (1 - \lambda)\gamma_{\pi,\mathcal{M}}(Y).\end{aligned}$$

Letting $\lambda \downarrow 0$ delivers $\mathbb{E}[XY] - m \le \gamma_{\pi,\mathcal{M}}(Y)$ and shows the desired inclusion. Now take any payoff $X \in L^1$. It follows from (4.1) and (4.10) that

$$\begin{aligned}\pi^+(X) &= \inf\{m \in \mathbb{R} : \mathbb{E}[DX] - m \le \gamma_{\pi,\mathcal{M}}(D), \forall D \in \mathcal{D}_{\text{str}}\}\\ &= \inf\{m \in \mathbb{R} : m \ge \mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D), \forall D \in \mathcal{D}_{\text{str}}\}\\ &= \sup\{\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D) : D \in \mathcal{D}_{\text{str}}\}.\end{aligned}$$

This establishes (ii) and concludes the proof. □

4.6 Dual characterisation of market-consistent prices

The fundamental theorem also allows us to derive a dual characterisation of market-consistent prices with acceptable risk, which extends the classical characterisation of arbitrage-free prices in terms of strictly positive price deflators. We complement this by showing that contrary to the standard frictionless setting, for an attainable payoff with market-consistent superreplication price, the supremum in the dual representation of the corresponding superreplication price need not be attained. Interestingly, this implies that a dual characterisation of market-consistent prices for replicable payoffs in terms of strictly consistent price deflators is not always possible.

Proposition 4.19 *If there exists no scalable good deal and if either $\mathcal{A} = L^1_+$ or $\mathcal{A}$ is a pointed cone and L^1 is separable with respect to its norm topology, then the following statements hold for every $X \in L^1$:*

(i) *If $\pi^+(X) \in \mathrm{MCP}(X)$ and the supremum in (4.9) is attained or if we have $\pi^+(X) \notin \mathrm{MCP}(X)$, then*

$$\mathrm{MCP}(X) = \{p \in \mathbb{R} : \exists D \in \mathcal{D}_{\mathrm{str}} \text{ with } p \leq \mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)\}. \tag{4.11}$$

(ii) *If $\pi^+(X) \in \mathrm{MCP}(X)$ and the supremum in (4.9) is not attained, then only the inclusion "$\supseteq$" in (4.11) always holds. The inclusion "$\subseteq$" in (4.11) can fail to hold even if both π and $\mathcal{M}$ are conic and there exists no good deal.*

Proof It follows from Theorem 4.13 that $\mathcal{D}_{\mathrm{str}}$ is nonempty. First, we show the inclusion "$\supseteq$" in (4.11). Let $D \in \mathcal{D}_{\mathrm{str}}$. Note that for every attainable payoff $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A} \setminus \{0\}$, we have

$$\begin{aligned}\pi(Z) \geq \mathbb{E}[DZ] - \gamma_{\pi,\mathcal{M}}(D) &= \mathbb{E}[D(Z - X)] + \mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)\\ &> \mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)\end{aligned}$$

by strict consistency. Note also that $\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D) \leq \pi(X)$ if $X \in \mathcal{M}$. This shows that $\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)$ is a market-consistent price for X and yields the desired inclusion. Now recall that $\pi^+(X)$ is the supremum of the set $\mathrm{MCP}(X)$. If $\pi^+(X)$ belongs to $\mathrm{MCP}(X)$, then the inclusion "$\supseteq$" in (4.11) is an equality if and only if the supremum in (4.9) is attained. We refer to Example 4.20 below for a concrete situation where the latter condition fails even if both π and $\mathcal{M}$ are conic and the market admits no good deals. Finally, assume that $\pi^+(X)$ does not belong to $\mathrm{MCP}(X)$. To complete the proof we only have to show the inclusion "$\subseteq$" in (4.11). To this end, take an arbitrary market-consistent price $p \in \mathrm{MCP}(X)$ and note that we must have $p < \pi^+(X)$. Hence it follows from the representation (4.9) that $p < \mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)$ for a suitable $D \in \mathcal{D}_{\mathrm{str}}$. This concludes the proof. □

Example 4.20 We work in the setting of Example 4.5. Take $\mathcal{A} = \mathbb{R}^2_+$ and $\mathcal{S} = \mathbb{R}^2$, and define $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq -x\}$ and

$$\pi(x, y) = \begin{cases} -\sqrt{x^2 + xy} & \text{if } (x, y) \in \mathcal{M}, \\ \infty & \text{otherwise.} \end{cases}$$

Note that π is convex because it is continuous on $\mathcal{M}$ and its Hessian matrix in the interior of $\mathcal{M}$ has nonnegative eigenvalues, namely 0 and $\frac{1}{4}(x^2+y^2)(x^2+xy)^{-3/2}$. Both $\mathcal{A}$ and $\mathcal{M}$ are cones and π is conic. Moreover, there exists no good deal. A direct inspection shows that strictly consistent price deflators $D \in \mathbb{R}^2$ exist, e.g. $D = (2, 1)$, and satisfy $\gamma_{\pi,\mathcal{M}}(D) = 0$ by conicity. Now set $X = (-1, 1) \in \mathcal{M}$. We have $\pi^+(X) = \pi(X) = 0$ since $(\mathcal{A}+X) \cap \mathcal{M} = \{X\}$. This also yields $0 \in \mathrm{MCP}(X)$ by Proposition 4.2. We show that there is no $D = (d_1, d_2) \in \mathcal{D}_{\mathrm{str}}$ such that $\mathbb{E}[DX] = 0$. Indeed, we should otherwise have $d_1 = d_2$ and taking $Z_\lambda = (-1, \lambda) \in \mathcal{M}$ for $\lambda \in (0, 1)$ would deliver

$$\sup_{0<\lambda<1} \big(\mathbb{E}[DZ_\lambda] - \pi(Z_\lambda)\big) \leq 0 \qquad \Longrightarrow \qquad d_1 \geq \sup_{0<\lambda<1} \frac{2}{\sqrt{1-\lambda}} = \infty.$$

As a result, the supremum in (4.9) is not attained.

The next example shows that conicity is necessary for both Theorem 4.18 and Proposition 4.19 to hold.

Example 4.21 We work in the setting of Example 4.5. Define $\pi(x, y) = \max\{x, x+y\}$ on $\mathbb{R}^2$ and set $\mathcal{M} = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$ and

$$\mathcal{A} = \big\{(x, y) \in \mathbb{R}^2 : y \geq \max\{-2x, 0\}, x \geq -1\big\}.$$

Note that π and $\mathcal{M}$ are both conic while $\mathcal{A}$ is not. Note also that there exists no good deal. It is not difficult to verify that strictly consistent price deflators exist. Indeed, for a strictly positive $D = (d_1, d_2)$,

$$\begin{cases} \sup\{\mathbb{E}[DX] - \pi(X) : X \in \mathcal{M}\} < \infty, \\ \mathbb{E}[DX] > 0 \text{ for every nonzero } X \in \mathcal{A} \end{cases} \qquad \Longleftrightarrow \qquad \begin{cases} d_1 = 2, \\ 1 < d_2 \leq 2. \end{cases}$$

Set $X = (2, -4)$. Since $(\mathcal{A} + X) \cap \mathcal{M} = \{(x, y) \in \mathbb{R}^2 : x \geq 1, y \geq 0\}$, we see that $\pi^+(X) = \pi(1, 0) = 1$. As X does not belong to $\mathcal{M}$, we have $\mathrm{MCP}(X) = (-\infty, 1)$ by Proposition 4.4. Both (4.9) and (4.11) fail, since for every strictly consistent price deflator $D = (d_1, d_2)$, we have $\gamma_{\pi,\mathcal{M}}(D) = 0$ by conicity and

$$\sup_{D\in\mathcal{D}_{\mathrm{str}}} \big(\mathbb{E}[DX] - \gamma_{\pi,\mathcal{M}}(D)\big) = \sup_{1<d_2\leq 2} (2 - 2d_2) = 0.$$

5 Conclusions

In this paper, we have established a version of the fundamental theorem of asset pricing in incomplete markets with frictions where agents use general acceptance sets to define good deals based on their individual "preferences". The basic result states that absence of scalable good deals is equivalent to the existence of strictly consistent price deflators. This extends and sharpens the existing versions of the fundamental theorem in the good deal pricing literature and allows deriving an appropriate version of the classical superreplication duality. Even though our focus is on one-period models, we have to cope with technical challenges as the standard techniques used in arbitrage pricing, e.g. changes of numeraire and exhaustion arguments, break down in the presence of general acceptance sets. We conclude by collecting some general remarks about possible extensions of our results beyond the setting of integrable payoffs and beyond one-period market models. Here, we denote by L^0 the space of random variables modulo almost sure equality under $\mathbb{P}$ and equip it with its canonical vector and lattice structure.

– In the paper, the reference payoff space is L^1. One may wonder whether a simple change of probability would not allow working in the more natural space L^0. Indeed, assume that $\mathcal{S} \subseteq L^0$ and define

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \frac{S}{\mathbb{E}[S]}, \qquad S := \left(1 + \sum_{i=1}^{N} |S_i|\right)^{-1},$$

where $S_1, \ldots, S_N \in L^0$ are the payoffs of the basic assets. It is immediate to see that the probability $\mathbb{Q}$ is equivalent to $\mathbb{P}$ and every payoff in $\mathcal{S}$ is integrable with respect to $\mathbb{Q}$. As a consequence, it is possible to apply our results to the model space $L^1(\mathbb{Q})$ of $\mathbb{Q}$-integrable random variables. This is reminiscent of what is done in arbitrage pricing theory; see e.g. Föllmer and Schied [24, proof of Theorem 1.7]. The problem with this approach is that the acceptance set $\mathcal{A}$ often depends explicitly on the natural probability $\mathbb{P}$ and its (topological) properties are typically lost after we pass to $\mathbb{Q}$. Most importantly, the set $\mathcal{A} \cap L^1(\mathbb{Q})$ is seldom closed with respect to the norm topology of $L^1(\mathbb{Q})$. Interestingly, this issue does not arise in arbitrage pricing theory because the acceptance set used there, namely the set of positive random variables, is invariant with respect to changes of equivalent probability. More generally, the change of probability would not be problematic if the acceptance set were invariant with respect to changes of the numeraire. Unfortunately, as shown in Koch-Medina et al. [38], numeraire-invariance is only satisfied by acceptance sets based on test scenarios, which are not pointed (unless they coincide with the canonical positive cone) and hence do not admit strictly consistent price deflators; see Remark 4.14.

– The results in Sects. 2, 3, 4.1 and 4.2 continue to hold if L^1 is replaced by any real vector space $\mathcal{X} \subseteq L^0$ equipped with a linear Hausdorff topology. In particular, we may take $\mathcal{X} = L^0$ equipped with the usual topology of convergence in probability. (Note that having finite dimension, the space $\mathcal{S}$ of replicable payoffs remains normable regardless of the choice of $\mathcal{X}$.)

– The results in Sects. 4.3–4.6 continue to hold if the pair (L^1, L^∞) is replaced by any pair $(\mathcal{X}, \mathcal{X}')$ of real vector spaces satisfying $L^\infty \subseteq \mathcal{X}, \mathcal{X}' \subseteq L^1$ that are in separating duality through the bilinear form $(X, Y) \mapsto \mathbb{E}[XY]$ and that are equipped with the weak topologies $\sigma(\mathcal{X}, \mathcal{X}')$ and $\sigma(\mathcal{X}', \mathcal{X})$. Moreover, the space $\mathcal{X}'$ must be the norm dual of a normed space $\mathcal{Y} \subseteq L^1$ (which need not coincide with $\mathcal{X}$), and the topology $\sigma(\mathcal{X}', \mathcal{X})$ must be weaker than the weak-star topology $\sigma(\mathcal{X}', \mathcal{Y})$. For concreteness, the payoff space $\mathcal{X}$ could be any Lebesgue space or more generally any Orlicz space and we could take $\mathcal{X}' = L^\infty$ and $\mathcal{Y} = L^1$. The separability assumption for L^1 in Sects. 4.4–4.6 is replaced by separability of $\mathcal{Y}$ with respect to its norm topology. This is important because in the example above, $\mathcal{Y} = L^1$ may be separable while the Orlicz space $\mathcal{X}$ may fail to be separable, both with respect to the respective norm topologies; see e.g. Rao and Ren [45, Theorem 1 in Sect. 3.5]. In the abstract setting, the acceptance set must be $\sigma(\mathcal{X},\mathcal{X}')$-closed. For the common payoff spaces and acceptance sets, this requirement is fulfilled even in the (generally restrictive) situation where $\mathcal{X}'$ is a small space. For concreteness, let $(\Omega,\mathcal{F},\mathbb{P})$ be nonatomic and let $\mathcal{X}$ be an Orlicz space. Moreover, let $\mathcal{X}'=L^\infty$. The set $\mathcal{A}$ is closed with respect to $\sigma(\mathcal{X},\mathcal{X}')$ in any of the following cases:

(a) $\mathcal{A}$ is closed with respect to the norm topology of L^1.

(b) $\mathcal{A}$ is law-invariant (under $\mathbb{P}$) or surplus-invariant (see below) and for all $(X_n)\subseteq\mathcal{A}\cap\mathcal{X}$ and $X\in\mathcal{X}$ with $X_n\to X$ $\mathbb{P}$-almost surely and $\sup_{n\in\mathbb{N}}|X_n|\in\mathcal{X}$, we have $X\in\mathcal{A}$.

The condition in (a) clearly implies $\sigma(\mathcal{X},\mathcal{X}')$-closedness of $\mathcal{A}$. In (b), law-invariance is standard and stipulates that acceptability is only driven by the probability distribution of a payoff, while surplus-invariance, introduced in Koch-Medina et al. [36] and studied more thoroughly in Koch-Medina et al. [38], stipulates that acceptability is only driven by the downside profile of a payoff. The closedness under dominated $\mathbb{P}$-almost sure convergence is sometimes referred to as Fatou closedness. In these cases, the desired $\sigma(\mathcal{X},\mathcal{X}')$-closedness of $\mathcal{A}$ follows from the results in Svindland [51] and Gao et al. [26] under law-invariance and from those in Gao and Munari [27] under surplus-invariance.

– Our mathematical formulation of the pricing problem is also compatible with multi-period (both discrete and continuous) market models. The only difference is that being the space of terminal values of self-financing trading strategies, $\mathcal{S}$ is never finite-dimensional in the multi-period case (unless we restrict attention to buy-and-hold strategies only). A direct inspection shows that our key results can be easily extended to a multi-period setting (in fact, the proofs remain identical) provided that Lemma 4.1 on closedness of $\mathcal{C}$ and Lemma 4.11 featuring an equivalent formulation of the absence of scalable good deals can be also extended. The current proofs of those results explicitly rely on the finite-dimensionality of $\mathcal{S}$. Recalling that $\mathcal{C}$ plays the role of the set of payoffs that can be superreplicated at zero cost in arbitrage pricing theory, one may wonder whether classical arguments of Komlós or randomised Bolzano–Weierstrass type could not be used to achieve the desired extension. The problem with these arguments is that they rely on closedness with respect to pointwise convergence, which is fulfilled by the acceptance set underlying arbitrage pricing theory, namely the set of positive random variables, but fails to hold for most acceptance sets encountered in good deal pricing theory. Closedness of the positive cone with respect to pointwise convergence is also crucial in the proof of the fundamental theorem in Pennanen [43]. A possible way to overcome this issue is to replace the set $\mathcal{C}$ with its closure. In view of Lemma 4.11, this would mean replacing absence of scalable good deals with a stronger condition that is reminiscent of the “no-free lunch” condition used in arbitrage pricing in continuous time. We leave this investigation for future research.

Appendix

In this appendix, we recall some standard notions from functional analysis and refer to e.g. Aliprantis and Border [1, Chaps. 5–7] and Zălinescu [53, Chap. 1] for the necessary mathematical background.

We use the convention $0 \cdot \infty = 0$. Let $\mathcal{X}$ be a real vector space equipped with a linear Hausdorff topology. A set $\mathcal{C} \subseteq \mathcal{X}$ is *pointed* if $\mathcal{C} \cap (-\mathcal{C}) = \{0\}$, *convex* if $\lambda\mathcal{C} + (1-\lambda)\mathcal{C} \subseteq \mathcal{C}$ for every $\lambda \in (0,1)$, and *conic* (or a *cone*) if $\lambda\mathcal{C} \subseteq \mathcal{C}$ for every $\lambda \in [0,\infty)$. If $\mathcal{C}$ is convex and $0 \in \mathcal{C}$, its *recession cone* is

$$\mathcal{C}^\infty := \bigcap_{\lambda>0} \lambda\mathcal{C}.$$

The set $\mathcal{C}^\infty$ is the largest convex cone contained in $\mathcal{C}$. If $\mathcal{C}$ is additionally closed, then $\mathcal{C}^\infty$ is also closed. In this case, we can equivalently express $\mathcal{C}^\infty$ as

$$\begin{aligned}\mathcal{C}^\infty = \{X \in \mathcal{X} : \exists \text{ nets } (X_\alpha) \subseteq \mathcal{C} \text{ with } (\lambda_\alpha) \subseteq [0,\infty) \\ \text{and } \lambda_\alpha \to 0, \lambda_\alpha X_\alpha \to X\}.\end{aligned} \tag{A.1}$$

A map $\varphi : \mathcal{X} \to (-\infty,\infty]$ is *convex* if $\varphi(\lambda X + (1-\lambda)Y) \le \lambda\varphi(X) + (1-\lambda)\varphi(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$, *conic* if $\varphi(\lambda X) = \lambda\varphi(X)$ for all $X \in \mathcal{X}$ and $\lambda \in [0,\infty)$, *sublinear* if φ is both convex and conic, and *lower semicontinuous* if for every net $(X_\alpha) \subseteq \mathcal{X}$ and every $X \in \mathcal{X}$ with $X_\alpha \to X$, we have $\varphi(X) \le \liminf_\alpha \varphi(X_\alpha)$. This is equivalent to $\{X \in \mathcal{X} : \varphi(X) \le m\}$ being closed for every $m \in \mathbb{R}$. If φ is convex and $\varphi(0) = 0$, its *recession map* $\varphi^\infty : \mathcal{X} \to (-\infty,\infty]$ is

$$\varphi^\infty(X) := \sup_{\lambda>0} \frac{\varphi(\lambda X)}{\lambda}.$$

The map φ^∞ is the smallest sublinear map dominating φ from above in a pointwise sense. If φ is additionally lower semicontinuous, then φ^∞ is also lower semicontinuous and for every $m \in \mathbb{R}$, we have

$$\{X \in \mathcal{X} : \varphi(X) \le m\}^\infty = \{X \in \mathcal{X} : \varphi^\infty(X) \le 0\}. \tag{A.2}$$

Let $\mathcal{X}'$ be another real vector space. Given a bilinear mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X}' \to \mathbb{R}$, we say that $\mathcal{X}$ and $\mathcal{X}'$ are in separating duality if we have $\langle X, Y\rangle = 0$ for every $Y \in \mathcal{X}'$ if and only if $X = 0$ and, similarly, $\langle X, Y\rangle = 0$ for every $X \in \mathcal{X}$ if and only if $Y = 0$. We denote by $\sigma(\mathcal{X}, \mathcal{X}')$ the weakest linear topology on $\mathcal{X}$ such that $\langle \cdot, Y\rangle$ is continuous for every $Y \in \mathcal{X}'$. Similarly, $\sigma(\mathcal{X}', \mathcal{X})$ is the weakest linear topology on $\mathcal{X}'$ such that $\langle X, \cdot\rangle$ is continuous for every $X \in \mathcal{X}$. The *(upper) support functional* and the *barrier cone* of a (nonempty) set $\mathcal{C} \subseteq \mathcal{X}$ are the map $\sigma_{\mathcal{C}} : \mathcal{X}' \to (-\infty,\infty]$ and the set given by

$$\sigma_{\mathcal{C}}(Y) := \sup_{X\in\mathcal{C}} \langle X, Y\rangle, \qquad \mathrm{bar}(\mathcal{C}) := \{Y \in \mathcal{X}' : \sigma_{\mathcal{C}}(Y) < \infty\}.$$

The map $\sigma_{\mathcal{C}}$ is sublinear and $\sigma(\mathcal{X}', \mathcal{X})$-lower semicontinuous. The set $\mathrm{bar}(\mathcal{C})$ is a convex cone that, unless $\mathcal{C}$ is a cone, may fail to be $\sigma(\mathcal{X}', \mathcal{X})$-closed. The following result records a general version of the classical results by Yan [52] and Kreps [39]. We refer to Clark [21], Jouini et al. [34], Rokhlin [46], Cassese [15], Rokhlin [47] and Gao and Xanthos [28] for a variety of versions of the same principle.

Theorem A.1 *Let $\mathcal{X}$ and $\mathcal{X}'$ be real topological vector spaces which are in separating duality through $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X}' \to \mathbb{R}$. Let $\mathcal{L}\subseteq\mathcal{X}$ and $\mathcal{L}'\subseteq\mathcal{X}'$ and set $\mathcal{K}_{\mathcal{L}} := \{\lambda X : \lambda \geq 0, X \in \mathcal{L}\}$. Assume that the following properties hold*:

(i) (Completeness) *For every sequence $(Y_n)\subseteq\mathcal{L}'$, there exist a sequence (λ_n) in $(0,\infty)$ and $Y\in\mathcal{L}'$ such that $\sum_{k=1}^{n}\lambda_k Y_k \to Y$ with respect to $\sigma(\mathcal{X}',\mathcal{X})$.*

(ii) (Countable separation) *There exists a sequence $(Y_n)\subseteq\mathcal{L}'\cap(-\operatorname{bar}(\mathcal{K}_{\mathcal{L}}))$ such that for every nonzero $X\in\mathcal{L}$, we have $\langle X, Y_n\rangle > 0$ for some $n\in\mathbb{N}$.*

Then there exists $Y\in\mathcal{L}'$ such that $\langle X, Y\rangle > 0$ for every nonzero $X\in\mathcal{L}$.

Proof By (ii), there exists a sequence $(Y_n)\subseteq\mathcal{L}'\cap(-\operatorname{bar}(\mathcal{K}_{\mathcal{L}}))$ such that for every nonzero $X\in\mathcal{L}$, we have $\langle X, Y_n\rangle > 0$ for some $n\in\mathbb{N}$. In particular, note that $\langle X, Y_n\rangle \geq 0$ for all $X\in\mathcal{L}$ and $n\in\mathbb{N}$ because $(Y_n)\subseteq -\operatorname{bar}(\mathcal{K}_{\mathcal{L}})$. Moreover, by (i), there exist a sequence $(\lambda_n)\subseteq(0,\infty)$ and $Y\in\mathcal{L}'$ such that $\sum_{k=1}^{n}\lambda_k Y_k \to Y$. It is immediate to see that $\langle X, Y\rangle > 0$ for every nonzero $X\in\mathcal{L}$. □

Remark A.2 The preceding theorem extends the abstract version of the Kreps–Yan theorem from Jouini et al. [34]. In that paper, $\mathcal{L}$ was assumed to be a pointed convex cone satisfying $\mathcal{L}-\mathcal{L}=\mathcal{X}$ and $\mathcal{L}'$ was taken to coincide with $-\operatorname{bar}(\mathcal{L})$. Incidentally, note that pointedness is automatically implied by the countable separation property (regardless of the special choice of $\mathcal{L}$).

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Declarations

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