Rend. Sem. Mat. Univ. Pol. Torino
Vol. xx, x (xxxx), 1 – 26

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LINEAR ALGEBRA AND TORIC DATA OF WEIGHTED PROJECTIVE SPACES

Abstract. This paper is devoted to give characterizations of suitable matrices associated with fans and polytopes defining a weighted projective space and switching rules between them.

Introduction

The aim of the present paper is to give characterizations of fans and polytopes defining a weighted projective space (*wps* for short) and switching rules between them, from a linear algebraic point of view.

After recalling some notation and preliminaries on toric varieties, the starting point here is the usual definition of a (complex) wps as a geometric quotient (see Definition 2), directly checking its natural toric structure. The bridge with the classical presentation of toric varieties via fans is then given by the Cox Theorem [8] Thm. 2.1.

Section 2 is devoted to the characterization of a wps's fan: up to permutations on generators it is possible to associate a $n \times (n+1)$ integer matrix with the fan of a n-dimensional wps (so called *fan matrix*) whose entries turn out to verify an amount of relations (see equivalent conditions in the Theorem 3, giving a linear algebraic characterization of a wps's fan). Until here almost nothing is new, since the equivalence of conditions (1) and (2) in Theorem 3 can be recovered from [4] and [2], while condition (3) can be deduced from [7] Thm. 3.6. Anyway, we were not able to find in the literature the relations between the fan generators $\mathbf{v}_j$'s of a wps $\mathbb{P}(Q)$, the associated primitive vectors $\mathbf{n}_j$'s and the reduction Q' of the weight vector Q, as explained in Lemma 1, although probably well-known to the experts. Notice that Lemma 1 can be used as a key step to get a completely combinatoric proof of the well-known Reduction Theorem 2 (see [18] Thm. 1.26 and its proof). Probably the most original result in Section 2 is the Proposition 5 where it is shown that the fan of a given wps $\mathbb{P}(Q)$ is *encoded* in the switching matrix giving the *Hermite normal form* (HNF for short) of the transposed weight vector Q^T. This section ends up with the Proposition 6 in which on the one hand (parts from (1) to (3)) we rewrite the Conrads's presentation of a wps's fan matrix, but proved by directly starting from relations given in the Theorem 3, and on the other hand (part (4)) we present a *Q-canonical* form for the fan of $\mathbb{P}(Q)$, only depending on the weights order in Q (see Remark 4), which can be simply obtained by the HNF of *any* fan matrix of $\mathbb{P}(Q)$. In our opinion this Proposition describes a clean and easy method to get a fan of $\mathbb{P}(Q)$ even by hand (see Example 1).

Section 3 is dedicated to characterize polytopes associated with a polarized wps.

*The authors were partially supported by the Local Project "Computational Algebra and Applications"(2007) and the MIUR-PRIN "Geometria delle varietà algebriche"2010 Research Funds

As far as we know, results of this section were not known before. Let $\mathcal{O}(1)$ be the *minimal* polarization given by a generator of the Picard group $\mathrm{Pic}(\mathbb{P}(Q))$. Then we draw a *fan-polytope correspondence* between fans of $\mathbb{P}(Q)$ and polytopes of $(\mathbb{P}(Q),\mathcal{O}(1))$: this is given, up to suitable weightings, in the one direction by taking the transposed inverse (so called *transverse*) of a maximal submatrix of a fan matrix, in the other direction by an obvious completion of the transposed adjoint matrix of the polytope matrix (see Definitions 5 and 7 and Remark 10). In our opinion this correspondence combined with the previous Proposition 6 provides a clean and easy method to get a polytope of $(\mathbb{P}(Q),\mathcal{O}(m))$, even by hand, up to the elementary computation of the inverse of a (possibly big) matrix (see Example 2). Main results of this section are given by the Theorem 4, which is a direct consequence of Lemma 1, and the Proposition 9. This section ends up with the Theorem 5 giving a linear algebraic characterization of a polarized wps's polytope. This result has to be thought of as the polytopal counterpart of Theorem 3.

1. Preliminaries and notation

1.1. Toric varieties

A *n–dimensional toric variety* is an algebraic normal variety X containing the *torus* $T := (\mathbb{C}^*)^n$ as a Zariski open subset such that the natural multiplicative self–action of the torus can be extended to an action $T \times X \to X$.

Let us quickly recall the classical approach to toric varieties by means of *cones* and *fans*. For proofs and details the interested reader is referred to the extensive treatments [11], [15], [17] and the recent and quite comprehensive [10].

As usual M denotes the *group of characters* $\chi : T \to \mathbb{C}^*$ of T and N the *group of 1–parameter subgroups* $\lambda : \mathbb{C}^* \to T$. It follows that M and N are n–dimensional dual lattices via the pairing

$$\begin{array}{ccc} M \times N & \longrightarrow & \mathrm{Hom}(\mathbb{C}^*,\mathbb{C}^*) \cong \mathbb{C}^* \\ (\chi,\lambda) & \longmapsto & \chi \circ \lambda \end{array}$$

which translates into the standard paring $\langle u,v\rangle = \sum u_i v_i$ under the identifications $M \cong \mathbb{Z}^n \cong N$ obtained by setting $\chi(\mathbf{t}) = \mathbf{t}^{\mathbf{u}} := \prod t_i^{u_i}$ and $\lambda(t) = t^{\mathbf{v}} := (t^{v_1},\ldots,t^{v_n})$.

Cones and affine toric varieties

Define $N_{\mathbb{R}} := N \otimes \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes \mathbb{R} \cong \mathrm{Hom}(N,\mathbb{Z}) \otimes \mathbb{R} \cong \mathrm{Hom}(N_{\mathbb{R}},\mathbb{R})$.

A *convex polyhedral cone* (or simply a *cone*) σ is the subset of $N_{\mathbb{R}}$ defined by

$$\sigma = \langle \mathbf{v}_1,\ldots,\mathbf{v}_s\rangle := \{r_1\mathbf{v}_1 + \cdots + r_s\mathbf{v}_s \in N_{\mathbb{R}} \mid r_i \in \mathbb{R}_{\geq 0}\}$$

The s vectors $\mathbf{v}_1,\ldots,\mathbf{v}_s \in N_{\mathbb{R}}$ are said *to generate* σ. A cone $\sigma = \langle \mathbf{v}_1,\ldots,\mathbf{v}_s\rangle$ is called *rational* if $\mathbf{v}_1,\ldots,\mathbf{v}_s \in N$, *simplicial* if $\mathbf{v}_1,\ldots,\mathbf{v}_s$ are $\mathbb{R}$–linear independent and *non-singular* if $\mathbf{v}_1,\ldots,\mathbf{v}_s$ can be extended by $n-s$ further elements of N to give a basis of the lattice N.

A cone σ is called *strictly convex* if it does not contain a linear subspace of positive dimension of $N_{\mathbb{R}}$.
The *dual cone* $\sigma^{\vee}$ *of* σ is the subset of $M_{\mathbb{R}}$ defined by

$$\sigma^{\vee} = \{\mathbf{u} \in M_{\mathbb{R}} \mid \forall\, \mathbf{v} \in \sigma \quad \langle \mathbf{u}, \mathbf{v} \rangle \geq 0\}$$

A *face* τ *of* σ (denoted by $\tau < \sigma$) is the subset defined by

$$\tau = \sigma \cap \mathbf{u}^{\perp} = \{\mathbf{v} \in \sigma \mid \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$$

for some $\mathbf{u} \in \sigma^{\vee}$. Observe that also τ is a cone.
Gordon's Lemma (see [15] §1.2, Proposition 1) ensures that the semigroup $S_{\sigma} := \sigma^{\vee} \cap M$ is *finitely generated*. Then also the associated $\mathbb{C}$–algebra $A_{\sigma} := \mathbb{C}[S_{\sigma}]$ is finitely generated. A choice of r generators gives a presentation of A_{σ}

$$A_{\sigma} \cong \mathbb{C}[X_1, \ldots, X_r]/I_{\sigma}$$

where I_{σ} is the ideal generated by the relations between generators. Then

$$U_{\sigma} := \mathcal{V}(I_{\sigma}) \subset \mathbb{C}^r$$

turns out to be an *affine toric variety*. In other terms an affine toric variety is given by $U_{\sigma} := \mathrm{Spec}(A_{\sigma})$. Since a closed point $x \in U_{\sigma}$ is an evaluation of elements in $\mathbb{C}[S_{\sigma}]$ satisfying the relations generating I_{σ}, then it can be identified with a semigroup morphism $x : S_{\sigma} \to \mathbb{C}$ assigned by thinking of $\mathbb{C}$ as a multiplicative semigroup. In particular the *characteristic morphism*

$$\begin{array}{rclc} x_{\sigma} & : \sigma^{\vee} \cap M & \longrightarrow & \mathbb{C} \\ & \mathbf{u} & \longmapsto & \begin{cases} 1 & \text{if } \mathbf{u} \in \sigma^{\perp} \\ 0 & \text{otherwise} \end{cases} \end{array} \tag{1}$$

which is well defined since $\sigma^{\perp} < \sigma^{\vee}$, defines a *characteristic point* $x_{\sigma} \in U_{\sigma}$ whose torus orbit O_{σ} turns out to be a $(n - \dim(\sigma))$–dimensional torus embedded in U_{σ} (see e.g. [15] §3).

Fans and toric varieties

A *fan* Σ is a finite set of cones $\sigma \subset N_{\mathbb{R}}$ such that

1. for any cone $\sigma \in \Sigma$ and for any face $\tau < \sigma$ then $\tau \in \Sigma$,
2. for any $\sigma, \tau \in \Sigma$ then $\sigma \cap \tau < \sigma$ and $\sigma \cap \tau < \tau$.

For every i with $0 \leq i \leq n$ denote by $\Sigma(i) \subset \Sigma$ the subset of i–dimensional cones, called the *i–skeleton of* Σ. A fan Σ is called *simplicial* if every cone $\sigma \in \Sigma$ is simplicial and *non-singular* if every such cone is non-singular. The *support* of a fan Σ is the subset $|\Sigma| \subset N_{\mathbb{R}}$ obtained as the union of all of its cones i.e.

$$|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma \subset N_{\mathbb{R}} \ .$$

If $|\Sigma| = N_{\mathbb{R}}$ then Σ will be called *complete* or *compact*.

Since for any face $\tau < \sigma$ the semigroup S_σ turns out to be a sub-semigroup of S_τ, there is an induced immersion $U_\tau \hookrightarrow U_\sigma$ between the associated affine toric varieties which embeds U_τ as a principal open subset of U_σ. Given a fan Σ one can construct *an associated toric variety* $X(\Sigma)$ by patching all the affine toric varieties $\{U_\sigma \mid \sigma \in \Sigma\}$ along the principal open subsets associated with any common face. Moreover *for every toric variety X there exists a fan Σ such that $X \cong X(\Sigma)$* (see [17] Theorem 1.5). It turns out that ([17] Theorems 1.10 and 1.11; [15] §2):

- $X(\Sigma)$ *is non-singular if and only if the fan Σ is non-singular,*
- $X(\Sigma)$ *is complete if and only if the fan Σ is complete.*

In the following a 1*–generated fan* Σ is a *fan generated by a set of* $n+1$ *integral vectors* i.e. a fan whose cones $\sigma \subset N \otimes \mathbb{R}$ are generated by any proper subset of a given finite subset $\{\mathbf{v}_0, \dots, \mathbf{v}_n\} \subset N$: we will write

$$\Sigma = \mathrm{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) \ . \tag{2}$$

Given a 1–generated fan $\Sigma = \mathrm{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$, the matrix $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ will be called *a fan matrix of* Σ. Notice that Σ determines V up to a permutations of columns, meaning that Σ admits $(n+1)!$ associated fan matrices.

If $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$ is a fan matrix of $\Sigma = \mathrm{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n)$ then we will denote the maximal square sub-matrices of V and the associated n–minors as follows

$$\forall\, 0 \le j \le n \quad V^j := (\mathbf{v}_0, \dots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_n) \ , \ V_j = \det(V^j) \ . \tag{3}$$

Polytopes and projective toric varieties

A *polytope* $\Delta \subset M_{\mathbb{R}}$ is the convex hull of a finite set of points. If this set is a subset of M then the polytope is called *integral*. Starting from an integral polytope one can construct a projective toric variety as follows. Here we will follow the approach of [1], which the interested reader is referred to for proofs and details (see also [9] §3.2.2).

For any $k \in \mathbb{N}$ one can define the dilated polytope $k\Delta := \{k\mathbf{u} \mid \mathbf{u} \in \Delta\}$. It is then possible to define a graded $\mathbb{C}$–algebra S_Δ, associated with the integral polytope Δ, as follows. For any $\mathbf{u} \in k\Delta \cap M$ consider the associated character $\chi^{\mathbf{u}} : \mathbf{t} \mapsto \mathbf{t}^{\mathbf{u}}$. Given $t \in \mathbb{C}^*$ consider the *monomial* $t^k\chi^{\mathbf{u}} : \mathbf{t} \mapsto t^k\mathbf{t}^{\mathbf{u}}$. It well defines a *monomial product* $t^{k_1}\chi^{\mathbf{u}_1} \cdot t^{k_2}\chi^{\mathbf{u}_2} := t^{k_1+k_2}\chi^{\mathbf{u}_1+\mathbf{u}_2}$ where $\mathbf{u}_1 + \mathbf{u}_2 \in (k_1+k_2)\Delta$. Let S_Δ be the $\mathbb{C}$–algebra generated by all monomials $\{t^k\chi^{\mathbf{u}} \mid k \in \mathbb{N} \ , \ \mathbf{u} \in k\Delta\}$ which is a graded object by setting $\deg(t^k\mathbf{u}) = k$.

The *projective* variety $\mathbb{P}_\Delta := \mathrm{Proj}(S_\Delta)$ turns out to be naturally a *toric variety* whose fan Σ_Δ can be recovered as follows. For any nonempty face $F \prec \Delta$ consider the cone

$$\check{\sigma}_F := \{r(\mathbf{u} - \mathbf{u}') \mid \mathbf{u} \in \Delta \ , \ \mathbf{u}' \in F \ , \ r \in \mathbb{R}_{\ge 0}\} \subset M_{\mathbb{R}}$$

and define $\sigma_F := \check{\sigma}_F^\vee \subset N_{\mathbb{R}}$. Then $\Sigma_\Delta := \{\sigma_F \mid F \prec \Delta\}$ turns out to be a fan, called the *normal fan* of the polytope Δ, such that there exists a very ample divisor H of $X(\Sigma_\Delta)$ for which $(X(\Sigma_\Delta),H) \cong (\mathbb{P}_\Delta, O(1)$, where $O(1)$ is the natural polarization of $\mathbb{P}_\Delta = \mathrm{Proj}(S_\Delta)$ (see [1] Proposition 1.1.2).

Viceversa a *projective toric variety* is the couple $(X(\Sigma),H)$ of a toric variety $X(\Sigma)$ and a polarization given by (the linear equivalence class of) a hyperplane section H. For any 1-cone $\rho \in \Sigma(1)$, consider the torus stable divisor $D_\rho := \overline{O}_\rho$ defined as the closure of the torus orbit of the characteristic point x_ρ, defined in (1). Since those divisors generate the Chow group of Weil divisors $A_{n-1}(X(\Sigma))$ (see [15] §3.4), there exist integer coefficients $a_\rho \in \mathbb{Z}$ such that $H = \sum_{\rho\in\Sigma(1)} a_\rho D_\rho$. It is then well defined the integral polytope

$$\Delta_H := \{\mathbf{u} \in M_\mathbb{R} \mid \forall \rho \in \Sigma(1)\ \langle \mathbf{u}, \mathbf{n}_\rho \rangle \geq -a_\rho\} \tag{4}$$

where $\mathbf{n}_\rho$ is the unique generator of the semigroup $\rho \cap N$. Then

$$(\mathbb{P}_{\Delta_H}, O(1)) \cong (X(\Sigma),H)\ .$$

1.2. Hermite normal form

It is well known that Hermite algorithm provides an effective way to determine a basis of a subgroup of $\mathbb{Z}^n$. We briefly recall the definition and the main properties. For details, see for example [6].

DEFINITION 1. *An $m \times n$ matrix $M = (m_{ij})$ with integral coefficients is in* Hermite normal form *(abbreviated* HNF*) if there exists $r \leq m$ and a strictly increasing map $f : \{1,\dots,r\} \to \{1,\dots,n\}$ satisfying the following properties:*

1. *For $1 \leq i \leq r$, $m_{i,f(i)} \geq 1$, $m_{ij} = 0$ if $j < f(i)$ and $0 \leq m_{i,f(k)} < m_{k,f(k)}$ if $i < k$.*
2. *The last $m-r$ rows of M are equal to* 0*.*

THEOREM 1 ([6] Theorem 2.4.3). *Let A be an $m \times n$ matrix with coefficients in $\mathbb{Z}$. Then there exists a unique $m \times n$ matrix $B = (b_{ij})$ in HNF of the form $B = U \cdot A$ where $U \in \mathrm{GL}(m,\mathbb{Z})$.*

We will refer to matrix B as the HNF of matrix A. The construction of B and U is effective, see [6, Algorithm 2.4.4], based on Eulid's algorithm for greatest common divisor. In the following two applications of this algorithm will be considered: for computing a fan of a given wps (see Prop. 5) and the so–called Q–canonical fan of $\mathbb{P}(Q)$ (see Prop. 6). At this purpose, a key theoretical tool is the following (for the proof see [6, §2.4.3])

PROPOSITION 1.

1. *Let L be a subgroup of $\mathbb{Z}^n$, $V = \{\mathbf{v}_1,\dots,\mathbf{v}_m\}$ a set of generators, and let A be the $m \times n$ matrix having $\mathbf{v}_1,\dots,\mathbf{v}_m$ as rows. Let B be the HNF of A. Then the nonzero rows of B are a basis of L.*

2. *Let A be a $m \times n$ matrix, and let $B = U \cdot A^T$ be the HNF of the transposed of A, and let r be the number of nonzero rows of B. Then a $\mathbb{Z}$-basis for the kernel of A is given by the last $m - r$ rows of U.*

1.3. Transversion of a matrix

In the following, given a matrix $A \in \mathrm{GL}(n,\mathbb{Q})$, the matrix obtained by taking the <u>trans</u>posed matrix of the in<u>verse</u> matrix

$$A^* := ((A)^{-1})^T$$

is called the *transverse matrix* of A. We will see in the following (see subsection 3.1, in particular Thm. 4) that *transversion* of a matrix describes, up to the multiplication by a diagonal matrix of weights, the passage from a fan to a polytope (and back) associated with the same weighted projective space $\mathbb{P}(Q)$.

Here are some elementary properties of transversion:

PROPOSITION 2. *Let A and B be matrices of* $\mathrm{GL}(n,\mathbb{Q})$. *Then:*

1. $(A^*)^* = A$ *i.e. transversion is an involution in* $\mathrm{GL}(n,\mathbb{Q})$,
2. $(A \cdot B)^* = A^* \cdot B^*$,
3. $\det(A^*) = 1/\det(A)$,
4. *if A is a upper (lower) triangular matrix then A^* is an lower (upper) triangular matrix,*
5. *if* $A \in \mathrm{GL}(n,\mathbb{Z})$ *then* $A^* \in \mathrm{GL}(n,\mathbb{Z})$ *too.*

1.4. Weighted projective spaces

In the present subsection we will briefly recall the definition and some well known fact about *weighted projective spaces* (*wps* in the following). Proofs and details can be recovered in the extensive treatments [12] [16], [13] and [3].

DEFINITION 2. *Set $Q := (q_0, \dots, q_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$ and consider the multiplicative group $\mu_Q := \mu_{q_0} \oplus \cdots \oplus \mu_{q_n}$ where μ_{q_i} is the group of q_i-th roots of unity. Consider the following action of μ_Q over the n–dimensional complex projective space $\mathbb{P}^n$*

$$\begin{array}{rccc} \mu_Q : & \mu_Q \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ & ((\zeta_j), [z_j]) & \longmapsto & [\zeta_j z_j] \end{array} .$$

Let $\Delta_Q \subset \mu_Q$ be the diagonal subgroup and consider the quotient group $\mathbb{W}_Q := \mu_Q / \Delta_Q$. Then the induced quotient space

$$\mathbb{P}(Q) := \mathbb{P}^n / \mathbb{W}_Q$$

*is called the Q–*weighted projective space *(Q-wps).*

REMARK 1. If q is the greatest common divisor of $(q_0,\dots,q_n)$ then

$$\Delta_Q \cong \mu_q$$

Therefore we get the canonical isomorphism

$$\mathbb{P}(Q) \cong \mathbb{P}\left(\frac{q_0}{q},\dots,\frac{q_n}{q}\right)$$

For this reason in the following *we will always assume that*

$$q = \gcd(q_0,\dots,q_n) = 1 \ .$$

Let us recall the following standard notation

$$\begin{array}{rcl} d_j & := & \gcd(q_0,\dots,q_{j-1},q_{j+1},\dots,q_n) \ , \\ a_j & := & \operatorname{lcm}(d_0,\dots,d_{j-1},d_{j+1},\dots,d_n) \ , \\ a & := & \operatorname{lcm}(a_0,\dots,a_n) \ . \end{array} \tag{5}$$

DEFINITION 3 (Weight vector). *In the following a* weight vector $Q=(q_0,\dots,q_n)$ *will denote a* $n+1$*–tuple of* coprime positive integer *numbers. Referring to notation defined in (5), a weight vector* Q *will be called* reduced *if* $d_j = 1$*, or equivalently* $a_j = 1$*, for any* $j = 0,\dots,n$*.*

REMARK 2. *Every weighted projective space is a toric variety*. In fact the natural torus action over $\mathbb{P}^n$ passes through the quotient as follows

$$\begin{array}{ccc} (\mathbb{C}^*)^n \times \mathbb{P}^n & \longrightarrow & \mathbb{P}^n \\ \downarrow{\scriptstyle \tau_Q\times\pi_Q} & & \downarrow{\scriptstyle \pi_Q} \\ (\mathbb{C}^*)^n \times \mathbb{P}^n(Q) & \longrightarrow & \mathbb{P}^n(Q) \end{array}$$

where π_Q is the natural quotient map and τ_Q is the quotient map associated with the action

$$\begin{array}{ccc} \mu_Q \times (\mathbb{C}^*)^n & \longrightarrow & (\mathbb{C}^*)^n \\ ((\zeta_j),(t_i)) & \longmapsto & \left(\zeta_0^{-1}\zeta_i t_i\right) \end{array}$$

Then the torus $(\mathbb{C}^*)^n$ can be embedded in $\mathbb{P}(Q)$ via the following map

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \hookrightarrow & \mathbb{P}(Q) \\ (t_1,\dots,t_n) & \longmapsto & [1:t_1:\dots:t_n] \end{array}$$

whose image is the open subset $\mathbb{P}(Q)\setminus\mathcal{V}\left(\prod_j z_j\right)$.

PROPOSITION 3. *Since* $\gcd(q_0,\dots,q_n)=1$*, the following facts are true:*

1. $\gcd(q_j,d_j) = 1$,

2. *if* $i \neq j$ *then* $\gcd(d_i, d_j) = 1$,
3. $a_j \mid q_j$,
4. $\gcd(a_j, d_j) = 1$,
5. $a_j d_j = a$,
6. *setting* $q'_j := q_j / a_j$*, then* $Q' = (q'_0, \ldots, q'_n)$ *is reduced;* Q' *is then called the* reduction of Q.

The proofs of these well known properties (see [13] 1.3.1) are elementary.

Instead we will prove the following property, which does not appear in the main treatments of the subject.

PROPOSITION 4. *Let* $Q = (q_0, \ldots, q_n)$ *be a weight vector and* $Q' = (q'_0, \ldots, q'_n)$ *be its reduction. Define*

$$\delta := \operatorname{lcm}(q_0, \ldots, q_n) \quad \textit{and} \quad \delta' := \operatorname{lcm}(q'_0, \ldots, q'_n) \ . \tag{6}$$

Then $\delta = a\delta'$*, where* a *is defined in* (5).

REMARK 3. The Proposition 4 still holds when $q := \gcd(q_0, \ldots, q_n) > 1$.

Proof of Proposition 4. We will prove that δ divides $a\delta'$ and, viceversa, $a\delta'$ divides δ. On the one hand, to show that δ divides $a\delta'$ it suffices to show that q_j divides $a\delta'$ for every j and this fact follows immediately by definitions since $q_j = a_j q'_j \mid a\delta'$.
On the other hand, by definitions of a and δ', to prove that $a\delta'$ divides δ it suffices to prove that $a_i q'_k \mid \delta$, which is $a_i \mid \frac{\delta}{q_k} a_k$, for every i, k. By the definition of a_i given in (5), the latter is obtained by showing that d_j divides $\frac{\delta}{q_k} a_k$ for every j, k.
If $j \neq k$ then d_j divides a_k and we are done.
Suppose now $j = k$. Let p be a prime dividing d_k and let p^t, p^r be the highest powers of p dividing d_k and q_k respectively. Then p^t divides q_i for every $i \neq k$, by the definition of d_k: in particular $p^t \mid \delta$. If $r \geq t$ then

$$\forall\, i \quad p^t \mid q_i \ \Rightarrow\ \forall\, j \quad p^t \mid d_j \ \Rightarrow\ \forall\, k \quad p^t \mid a_k \ .$$

If $r < t$ then p^{t-r} divides $\frac{\delta}{q_k}$; moreover p^r divides d_i for every $i \neq k$, since $p^r \mid q_k$ and $p^t \mid q_i$ for every $i \neq k$: then p^r divides a_k. Therefore p^t divides $\frac{\delta}{q_k} a_k$. Thus we proved that d_k divides $\frac{\delta}{q_k} a_k$. □

Let us now recall the following well-known result to which we will refer below as to the *Reduction Theorem*.

THEOREM 2 (Reduction Theorem ([12] §1,[13] 1.3.1)). *Let* $Q' = (q'_0, \ldots, q'_n)$ *be the reduced weight vector of* $Q = (q_0, \ldots, q_n)$*. Then*

$$\mathbb{P}(Q) \cong \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^* = \mathbb{P}(Q') \tag{7}$$

where the quotient is realized by means of the (reduced) action

$$\begin{array}{rccc} \nu_{Q'}: & \mathbb{C}^* \times \mathbb{C}^{n+1} & \longrightarrow & \mathbb{C}^{n+1} \\ & (t,(z_j)) & \longmapsto & \left(t^{q'_0} z_0, \ldots, t^{q'_n} z_n\right) \end{array} .$$

Let us end up this preliminary section with the following technical statement which will be useful below. Partial proofs of this result may recovered from [4] §2 and [14] Prop. 2.3. Moreover it is certainly well-know to experts. However for purposes of definiteness we include here a detailed proof.

LEMMA 1. *Let $Q = (q_0, \ldots, q_n)$ be a weight vector; let $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ be a set of vectors in $\mathbb{Q}^n$, generating $\mathbb{Q}^n$ and such that $\sum_{j=0}^n q_j \mathbf{v}_j = 0$. Let L be the lattice generated in $\mathbb{Q}^n$ by $\{\mathbf{v}_0, \ldots, \mathbf{v}_n\}$ and L' be the sublattice generated by $\{q_0\mathbf{v}_0, \ldots, q_n\mathbf{v}_n\}$. Then the following properties hold:*

(a) $[L : L'] = \prod_{j=0}^n q_j$;

(b) let $V := (v_{ij})$ be the $n \times (n+1)$ matrix whose columns are given by components of $\mathbf{v}_0, \ldots, \mathbf{v}_n$ over a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of L i.e. $\mathbf{v}_j = \sum_{i=1}^n = v_{ij}\mathbf{e}_i$, for every $j = 0, \ldots, n$, and denote by V_j the n-minor of V obtained by deleting the j-th column as in (3). Then

$$\forall\ j = 0, \ldots, n \quad V_j = (-1)^{\varepsilon + j} q_j \text{ , for a fixed } \varepsilon \in \{0, 1\} \text{ ,}$$

(c) $\forall\ j = 0, \ldots, n \quad \mathbf{v}_j = d_j \mathbf{n}_j$, where $\mathbf{n}_j$ is the generator of the semigroup $\langle \mathbf{v}_j \rangle \cap L$ and d_j is defined in (5); in particular L is the lattice generated by $\{\mathbf{n}_0, \ldots, \mathbf{n}_n\}$; moreover $\{\mathbf{n}_0, \ldots, \mathbf{n}_n\}$ satisfy the hypotheses of this Lemma with respect to the reduced weight vector Q' i.e. they generate $\mathbb{Q}^n$ and $\sum_{j=0}^n q'_j \mathbf{n}_j = 0$.

Proof. For (a), observe that L' has $q_1\mathbf{v}_1, \ldots, q_n\mathbf{v}_n$ as a basis. Then L' has index $\prod_{j=1}^n q_j$ in the lattice L_0 generated by $\mathbf{v}_1, \ldots, \mathbf{v}_n$. The quotient L/L_0 is cyclic generated by the image of $\mathbf{v}_0$, so that $[L : L_0]$ divides q_0. If $r\mathbf{v}_0 \in L_0$, with $r \in \mathbb{Z}$ then $r\mathbf{v}_0 = \sum_{j=1}^n s_j \mathbf{v}_j$ with $s_1, \ldots, s_n \in \mathbb{Z}$. Since $\gcd(q_0, \ldots, q_n) = 1$ then there exists $\lambda \in \mathbb{Z}$ such that $r = -\lambda q_0$, $s_i = \lambda q_i$ for $i = 1, \ldots, n$; in particular q_0 divides r, so that $[L : L_0] = q_0$ and $[L : L'] = [L : L_0][L_0 : L'] = \prod_{i=0}^n q_i$.

(b): for $j = 0, \ldots, n$, let L_j be the lattice generated by $\mathbf{v}_0, \ldots, \mathbf{v}_{j-1}, \mathbf{v}_{j+1}, \ldots, \mathbf{v}_n$. Then $|V_j| = [L : L_j] = q_j$, as we have shown in (a) for the case $j = 0$. Let $\varepsilon \in \{0, 1\}$ be such that $V_0 = (-1)^\varepsilon q_0$. Then

$$\forall\ j = 0, \ldots, n \quad V_j = (-1)^j \frac{q_j}{q_0} V_0 = (-1)^{\varepsilon + j} q_j$$

since $\sum_{j=0}^n q_j \mathbf{v}_j = 0$.

(c): we have

$$\forall\ j = 0, \ldots, n \quad q_j \mathbf{v}_j = -\sum_{k \neq j} q_k \mathbf{v}_k = -d_j \sum_{k \neq j} \tilde{q}_k \mathbf{v}_k$$

where $\tilde{q}_k := q_k / d_j \in \mathbb{N}$. By (1) in Proposition 3, $\gcd(q_j, d_j) = 1$ meaning that

$$\forall\ j = 0, \ldots, n \quad \exists\ \mathbf{v}'_j \in L\ :\ \mathbf{v}_j = d_j \mathbf{v}'_j \ .$$

Then (5) in Proposition 3 allows to write

$$0=\sum_{j=0}^{n} q_j\mathbf{v}_j=\sum_{j=0}^{n}\left(q_j'a_j\right)\left(d_j\mathbf{v}_j'\right)=a\sum_{j=0}^{n} q_j'\mathbf{v}_j' \Longrightarrow \sum_{j=0}^{n} q_j'\mathbf{v}_j'=0\ . \tag{8}$$

Moreover $\mathbf{v}_0',\ldots\mathbf{v}_n'$ generate L and (a) ensures that the following index

$$\left[L:\langle q_j'\mathbf{v}_j' \mid j=0,\ldots,n\rangle\right]=\prod_{j=0}^{n} q_j'. \tag{9}$$

Then the proof ends up by showing that, for all j, $\mathbf{v}_j'=\mathbf{n}_j$. With this goal in mind, consider $h_j\in\mathbb{N}$ such that $\mathbf{v}_j'=h_j\mathbf{n}_j$. If $V'=(v_{ij}')$ is the matrix of components of $\mathbf{v}_0',\ldots,\mathbf{v}_n'$, over the basis $\mathbf{e}_1,\ldots,\mathbf{e}_n$ of L, then

$$\forall\ j=0,\ldots,n \quad \left|V_j'\right|=q_j'$$

On the other hand

$$\mathbf{v}_0'\in h_0L \Longrightarrow \forall\ i=1,\ldots,n \quad h_0\mid v_{i0}' \Longrightarrow \forall\ k=1,\ldots,n \quad h_0\mid \left|V_k'\right|=q_k'\ .$$

Therefore (6) in Proposition 3 implies that

$$h_0 \mid \gcd\left(q_1',\ldots,q_n'\right)=1 \Longrightarrow h_0=1$$

Analogously $h_j=1$, for all $1\leq j\leq n$. Hence $\mathbf{v}_j'=\mathbf{n}_j$. □

2. Characterization of fans giving $\mathbb{P}(Q)$

2.1. Characterizing the fan

Let us fix an n-dimensional lattice N and a subset of $n+1$ vectors $\{\mathbf{v}_0,...,\mathbf{v}_n\}\subset N$. The following theorem is an application of the previous Lemma 1 to known results such as e.g. Lemma 2.11 in [2], Prop. 5.4 in [5] and Thm. 3.6 in [7].

THEOREM 3. *Let $Q=(q_0,\ldots,q_n)$ be a weight vector. Consider the fan* $\Sigma=\mathrm{fan}(\mathbf{v}_0,\ldots,\mathbf{v}_n)$ *and the associated matrix* $V=(\mathbf{v}_0,\ldots,\mathbf{v}_n)$ *with respect to a fixed basis of N. Then the following facts are equivalent:*

1. *Σ is a fan of $\mathbb{P}(Q)$,*
2. *$\sum_{j=0}^{n} q_j\mathbf{v}_j=0$ and the sub-lattice $N':=\langle q_0\mathbf{v}_0,\ldots,q_n\mathbf{v}_n\rangle\subset N$ has finite index*

$$[N:N']=\prod_{j=0}^{n} q_j\ ,$$

3. *$\forall\ j=0,\ldots,n \quad V_j=(-1)^{\varepsilon+j}q_j$, for a fixed $\varepsilon\in\{0,1\}$,*

4. $q_0\mathbf{v}_0 = -\sum_{i=1}^n q_i\mathbf{v}_i$ *and* $|V_0| := |\det(\mathbf{v}_1,\dots,\mathbf{v}_n)| = q_0$.

Proof of Theorem 3. $(1) \Rightarrow (2)$. A fan of the wps $\mathbb{P}(Q)$ with $Q = (q_0,\dots,q_n)$ is presented in [15] at the end of §2.3. Then, by Lemma 1(a) one may check that fan to satisfy conditions stated in (2).

$(2) \Rightarrow (3)$. This is Lemma 1(b).

$(3) \Rightarrow (4)$. For any $k = 1,\dots,n$ consider the $(n+1)\times(n+1)$ matrix

$$A_k := \left(\begin{array}{ccc} v_{k0} & \cdots & v_{kn} \\ \hline & & \\ & V & \end{array}\right) .$$

Since the first and the $(k+1)$-st rows of A_k are equal we get

$$\forall\, k = 1,\dots,n \quad 0 = \det(A_k) = \sum_{j=0}^n (-1)^j v_{kj} V_j \overset{(3)}{=} (-1)^\varepsilon \sum_{j=0}^n q_j v_{kj} \;\Rightarrow\; \sum_{j=0}^n q_j \mathbf{v}_j = 0 .$$

$(4) \Rightarrow (2)$. Since $|V_0| = q_0$ then $\{q_1\mathbf{v}_1,\dots,q_n\mathbf{v}_n\}$ is a basis of the sub-lattice N'. Hence

$$[N : N'] = |\det(q_1\mathbf{v}_1,\dots,q_n\mathbf{v}_n)| = \left(\prod_{i=1}^n q_i\right)|V_0| \overset{(4)}{=} \prod_{j=0}^n q_j .$$

$(2) \Rightarrow (1)$. First note that (2) and Lemma 1(a) imply that $\mathbf{v}_0,\dots,\mathbf{v}_n$ generate N. It follows by Lemma 1(c) that $\mathbf{n}_0,\dots,\mathbf{n}_n$ generate N and $\sum_{j=0}^n q'_j\mathbf{n}_j = 0$.

Let D_j be the torus invariant divisor associated with the 1-dimensional cone $\langle\mathbf{n}_j\rangle \in \Sigma(1)$ for the toric variety with fan Σ. Consider the sequence

$$0 \longrightarrow M \xrightarrow{div} \bigoplus_{j=0}^n \mathbb{Z}\cdot D_j \xrightarrow{d} \mathbb{Z} \longrightarrow 0 \tag{10}$$

where $div(\mathbf{m}) = \sum_{j=0}^n \langle\mathbf{m},\mathbf{n}_j\rangle D_j$ and $d(\sum_{j=0}^n b_jD_j) = \sum_{j=0}^n b_jq'_j$. Then div is injective since the $\mathbf{n}_j$ span $N = M^\vee$ and d is surjective since $\gcd(q'_0,\dots,q'_n) = 1$. Furthermore, $d\circ div = 0$ follows easily from $\sum_{j=0}^n q'_j\mathbf{n}_j = 0$.

Hence, to prove that (10) is exact, it suffices to show that $\ker(d) \subset \mathrm{Im}(div)$. Take $\sum_{j=0}^n b_jD_j \in \ker(d)$. Since $\mathbf{n}_1,\dots,\mathbf{n}_n$ are linearly independent over $\mathbb{Q}$, one can find $\mathbf{m} \in M\otimes\mathbb{Q}$ such that $\langle\mathbf{m},\mathbf{n}_i\rangle = b_i$ for $1 \le i \le n$. Since $\sum_{j=0}^n q'_j\mathbf{n}_j = 0$ and $\sum_{j=0}^n b_jq'_j = 0$, we obtain

$$q'_0\langle\mathbf{m},\mathbf{n}_0\rangle = -\sum_{i=1}^n \langle\mathbf{m},q'_i\mathbf{n}_i\rangle = -\sum_{i=1}^n q'_i\langle\mathbf{m},\mathbf{n}_i\rangle = -\sum_{i=1}^n q'_ib_i = q'_0b_0 .$$

It follows that $\langle\mathbf{m},\mathbf{n}_0\rangle = b_0$. Thus $\langle\mathbf{m},\mathbf{n}_j\rangle = b_j \in \mathbb{Z}$ for all j. This implies $\mathbf{m} \in M$ since $\mathbf{n}_0,\dots,\mathbf{n}_n$ span $N = M^\vee$, and the desired exactness follows immediately.

We are now in a position to apply the Cox Theorem, [8] Thm. 2.1, to give a geometric quotient description of $X(\Sigma)$. In particular the exact sequence (10) suffices to show that the *Chow group* of Weil divisors modulo rational equivalence for the toric variety $X(\Sigma)$ is given by $A_{n-1}(X) \cong \mathbb{Z}$ (see e.g. [15] §3.4). Let $S = \mathbb{C}[x_0,\ldots,x_n]$ be the polynomial ring obtained by associating the variable x_j with the 1-dimensional cone $\langle \mathbf{n}_j \rangle \in \Sigma(1)$. The grading on S is defined by setting

$$\deg(x_j) := \deg\left(d(D_j)\right) = q_j \ .$$

Since $\mathrm{Hom}(A_{n-1}(X),\mathbb{C}^*) \cong \mathbb{C}^*$, it is then possible to exhibit $X(\Sigma)$ as a geometric quotient

$$X(\Sigma) \cong \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \, \mathbb{C}^*$$

where the quotient is realized by means of the action $\nu_{Q'}$ in the statement of the Reduction Theorem 2. Then $X(\Sigma) \cong \mathbb{P}(Q') \cong \mathbb{P}(Q)$. □

The following definition will be useful in section 3, when speaking about the *fan-polytope correspondence*:

DEFINITION 4 (F–admissible matrices). *A matrix* $V \in \mathrm{Mat}(n,n+1,\mathbb{Z})$ *will be called* F–admissible *if it satisfies the following conditions*

1. *the matrix* $V = (\mathbf{v}_0,\ldots,\mathbf{v}_n)$ *admits only nonvanishing coprime maximal minors i.e.* $\forall\ j = 0,1,\ldots,n \quad V_j \neq 0$ *and* $\gcd(V_j \mid 0 \leq j \leq n) = 1$*;*
2. *the columns* $\mathbf{v}_j$ *of* V *satisfy one of the equivalent conditions (2), (3), (4) of Theorem 3 with respect to the weights* $q_j := |V_j|$.

The subset of F*–admissible matrices will be denoted by* $\mathfrak{V}_n \subset \mathrm{Mat}(n,n+1,\mathbb{Z})$.

2.2. Hermite normal form of weights and fans of $\mathbb{P}(Q)$

The following result, which is a direct consequence of Theorem 3, exhibit a rather surprising method to get a fan of a given wps $\mathbb{P}(Q)$: in fact this fan turns out to be *encoded* in the switching matrix giving the HNF of the transposed weight vector Q^T. Since such a matrix is obtained by a well known algorithm, based on Euclid's algorithm for greatest common divisor, [6] Algorithm 2.4.4, this gives a constructive method to produce a fan of $\mathbb{P}(Q)$ which can be performed by any procedures computing elementary linear algebra operations.

PROPOSITION 5. *Let* $Q = (q_0,\ldots,q_n)$ *be a weight vector,* B *the HNF of the transposed vector* Q^T *and* $U \in \mathrm{GL}(n+1,\mathbb{Z})$ *be such that* $U \cdot Q^T = B$. *Let* C *be the matrix consisting of the last* n *rows of* U *and let* $\mathbf{v}_j$ *be the* j^{th} *column vector of* C*, for* $0 \leq j \leq n$. *Let* L, L' *be the lattices generated in* $\mathbb{Z}^n$ *by* $\mathbf{v}_0,\ldots,\mathbf{v}_n$ *and* $q_0\mathbf{v}_0,\ldots,q_n\mathbf{v}_n$ *respectively. Then*

1. $B = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$*;*

2. *$L = \mathbb{Z}^n$;*

3. *$\sum_{j=0}^{n} q_j \mathbf{v}_j = 0$;*

4. *there exists $\varepsilon \in \{0,1\}$ such that $C_j = (-1)^{\varepsilon+j} q_j$ for all $0 \le j \le n$.*

5. *$[L : L'] = \prod_{j=0}^{n} q_j$.*

As a consequence of Theorem 3, $\mathrm{fan}(\mathbf{v}_0,\dots,\mathbf{v}_n)$ *is a fan of* $\mathbb{P}(Q)$.

Proof. The rank of Q is 1, so by definition of HNF, $B = \begin{pmatrix} a \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with $a \ge 1$. By the equality $Q^T = U^{-1} \cdot B$ we see that a must divide $q_0,\dots,q_n$, so that $a = 1$; this proves (1). Then (3) follows immediately by $U \cdot Q^T = B$.
To prove part (2), let $(c_0,\dots,c_n)$ be the 2nd column of U^{-1}. Then $U \cdot U^{-1} = I$ easily implies that $\sum_{j=0}^{n} c_j \mathbf{v}_j = \mathbf{e}_1$, the first standard basis vector of $\mathbb{Z}^n$. Columns $3,\dots,n+1$ of U^{-1} similarly show that $\mathbf{e}_2,\dots,\mathbf{e}_n$ are in the sublattice generated by $\mathbf{v}_0,\dots,\mathbf{v}_n$. Hence this sublattice must be $\mathbb{Z}^n$.
Finally parts (4) and (5) follow immediately from parts (2) and (3) of Lemma 1. □

2.3. A Q–canonical fan of $\mathbb{P}(Q)$

In the present subsection we want to use the characterization (4) in Theorem 3 to get a *Q–canonical fan* of $\mathbb{P}(Q)$, in the sense that the associated fan matrix is in HNF, up to a permutation of columns (see the following Remark 4). This fact presents the fan in a triangular shape and generated by as many as possible of the vectors $\mathbf{e}_1,\dots,\mathbf{e}_n$ in a given basis of the lattice N. Moreover it turns out to be a convenient procedure to get a fan of $\mathbb{P}(Q)$ by hands (see Example 1 below).

PROPOSITION 6. *Let $Q = (q_0,\dots,q_n)$ be a weight vector. For any j with $1 \le j \le n$, define $k_j := \gcd(q_0, q_j, q_{j+1},\dots,q_n)$. Then:*

1. *$k_j \mid k_{j+1}$,*

2. *either $k_n = (q_0,q_n) = 1$ or there exists a positive integer i, with $1 \le i \le n-1$, such that $k_i = 1$ and $k_{i+1} > 1$,*

3. *consider a upper triangular matrix $V^0 = (\mathbf{v}_1,\dots,\mathbf{v}_n) \in \mathrm{Mat}(n,n,\mathbb{Z})$ whose columns $\mathbf{v}_j$ are such that:*

$$\begin{array}{rrcl} \forall\, 1 \le j \le i-1 & \mathbf{v}_j & = & \mathbf{e}_j \\ \forall\, i \le j \le n-1 & v_{jj} & = & k_{j+1}/k_j \\ & v_{nn} & = & q_0/k_n \end{array}$$

where v_{kj} is the k-th entry of the column $\mathbf{v}_j$; then there exists a choice for v_{kj} with $i \leq j$ and $k < j$ such that V^0 can be completed to a matrix $V = (\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_n) \in \mathrm{Mat}(n, n+1, \mathbb{Z})$ whose columns satisfy the following condition

$$\sum_{j=0}^{n} q_j \mathbf{v}_j = 0 \ ;$$

in particular the columns of V satisfy condition (4) of Theorem 3, hence generate a fan of $\mathbb{P}(Q)$.

4. *there exists a unique choice of the previous matrices V and V^0 such that V^0 is in HNF with only nonnegative entries; then the column $\mathbf{v}_0$ in V admits only negative entries. Moreover the matrix $V' = (\mathbf{v}_1, \dots, \mathbf{v}_n, \mathbf{v}_0)$ is in HNF.*

Proof. (1) is obvious. (2) follows from (1) by recalling the hypothesis

$$k_1 = \gcd(q_0, q_1, \dots, q_n) = 1 \ .$$

To prove (3) we have to show the existence of an integral vector $\mathbf{v}_0 = \begin{pmatrix} v_{10} \\ \vdots \\ v_{n0} \end{pmatrix}$ and integers v_{jk} satisfying the following equations

$$\text{(11)} \qquad \forall\, 1 \leq j \leq i-1 \qquad q_0 v_{j0} + q_j + \sum_{k=i}^{n} q_k v_{jk} = 0$$

$$\text{(12)} \qquad \forall\, i \leq j \leq n-1 \qquad q_0 v_{j0} + q_j \frac{k_{j+1}}{k_j} + \sum_{k=j+1}^{n} q_k v_{jk} = 0$$

$$\text{(13)} \qquad q_0 v_{n0} + q_n \frac{q_0}{k_n} = 0 \ .$$

The last equation (13) is clearly satisfied by putting $v_{n0} = -q_n/k_n = -q_n/(q_0, q_n)$. The j-th equation in (12) admits integer solutions for $(v_{j0}, v_{j,j+1}, \dots, v_{jn})$ if and only if

$$\gcd(q_0, q_{j+1}, \dots, q_n) = k_{j+1} \mid q_j \frac{k_{j+1}}{k_j}$$

which is clearly true since $k_j \mid q_j$, by definition. Finally the j-th equation in (11) admits integer solutions for $(v_{j0}, v_{ji}, v_{j,i+1} \dots, v_{jn})$ if and only if

$$\forall\, 1 \leq j \leq i-1 \quad \gcd(q_0, q_i, \dots, q_n) = k_i \mid q_j$$

which is clearly true since $k_i = 1$, by the previous part (2). Recall now that V^0 is a triangular matrix, giving

$$\det(V^0) = \prod_{j=1}^{n} v_{jj} = k_{i+1} \cdot \frac{k_{i+2}}{k_{i+1}} \cdots \frac{k_n}{k_{n-1}} \cdot \frac{q_0}{k_n} = q_0$$

which is enough to get condition (4) of Theorem 3 for the columns of V.

To prove (4) let us first of all observe that, for any $1 \leq j \leq i-1$, $\mathbf{v}_j = \mathbf{e}_j$, meaning that the first $i-1$ columns of V^0 are composed of nonnegative entries satisfying the HNF conditions. Moreover V^0 is upper triangular. Then it remains to prove that there exists a unique choice for v_{jk} such that

$$\forall k : i \leq k \leq n \; , \; \forall j : j < k \quad 0 \leq v_{jk} < v_{kk} \; .$$

The j–th equation in (12) can be rewritten as follows

$$q_0 v_{j0} + q_n v_{jn} = -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-1} q_k v_{jk} \; .$$

Fixing variables v_{jk}, for $j+1 \leq k \leq n-1$, the previous diophantine equation admits solutions for v_{j0}, v_{jn} if and only if

$$k_n = \gcd(q_0, q_n) \mid -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-1} q_k v_{jk} \; . \tag{14}$$

Moreover, given a particular solution $v_{jn}^{(0)}$, all the possible integer solutions for v_{jn} are given by

$$v_{jn} = v_{jn}^{(0)} - \frac{q_0}{k_n} \cdot h_{jn} = v_{jn}^{(0)} - v_{nn} \cdot h_{jn} \; , \quad \forall \; h_{jn} \in \mathbb{Z} \; .$$

Divide $v_{jn}^{(0)}$ by v_{nn}. Then the remainder of such a division gives a unique choice for v_{jn} such that

$$\forall \; i \leq j \leq n-1 \quad 0 \leq v_{jn} < v_{nn} \; .$$

Analogously the j-th equation in (11) can be rewritten as follows

$$q_0 v_{j0} + q_n v_{jn} = -q_j - \sum_{k=i}^{n-1} q_k v_{jk}$$

and the same argument ensures the existence of a unique choice for v_{jn} such that

$$\forall \; 1 \leq j \leq i-1 \quad 0 \leq v_{jn} < v_{nn} \; .$$

Then the last column in V^0 can be uniquely chosen with non-negative entries satisfying the HNF condition. Iteratively, condition (14) is satisfied if and only if there exist integer solutions for x, v_{jk} in the diophantine equation

$$k_n x + q_{n-1} v_{j,n-1} = -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-2} q_k v_{jk}$$

which is if and only if

$$\gcd(k_n, q_{n-1}) = \gcd(q_0, q_{n-1}, q_n) =: k_{n-1} \mid -q_j \frac{k_{j+1}}{k_j} - \sum_{k=j+1}^{n-2} q_k v_{jk} \; .$$

In particular, given a solution $v^{(0)}_{j,n-1}$, all the possible integer solutions for $v_{j,n-1}$ are given by

$$v_{j,n-1} = v^{(0)}_{j,n-1} - \frac{k_n}{k_{n-1}} \cdot h_{j,n-1} = v^{(0)}_{j,n-1} - v_{n-1,n-1} \cdot h_{j,n-1}\ , \quad \forall\, h_{j,n-1} \in \mathbb{Z}\ .$$

Therefore, the division algorithm ensures the existence of a unique choice for $v_{j,n-1}$ such that

$$\forall\, i \le j \le n-1 \quad 0 \le v_{j,n-1} < v_{n-1,n-1}\ .$$

The same argument ensures the existence of a unique choice for $v_{j,n-1}$ such that

$$\forall\, 1 \le j \le i-1 \quad 0 \le v_{j,n-1} < v_{n-1,n-1}\ .$$

Then the $(n-1)$–st column in V^0 can be uniquely chosen with non-negative entries satisfying the HNF condition. By completing the iteration, V_0 can then be uniquely chosen in HNF. Consequently $\mathbf{v}_0$ has to necessarily admit only negative entries. To prove that V' is in HNF it suffices to observe that, for V', the function $f : \{1,\dots,n\} \to \{1,\dots,n+1\}$, in Definition 1, is given by setting $f(i) = i$, for any $1 \le\ i \le n$. Then V' is in HNF if and only V^0 is in HNF, since there are no condition for the entries of $\mathbf{v}_0$ which is the $(n+1)$–st column of V'. □

REMARK 4. When the weight vector Q is fixed, a significant consequence of Proposition 6 is that the fan of $\mathbb{P}(Q)$ presented in (4) is unique and is given by the HNF of a matrix V associated with any fan of $\mathbb{P}(Q)$. Clearly the uniqueness of the Q–canonical fan of $\mathbb{P}(Q)$ depends on the weights order in Q. Then we can't define a *canonical* fan of $\mathbb{P}(Q)$ but just a Q–canonical one.

EXAMPLE 1. Let us apply the Proposition 6 to produce by hand the Q-canonical fan (hence a fan) of $\mathbb{P}(Q)$ for $Q = (2,3,4,15,25)$. First of all observe that in this case

$$k_1 = \gcd(Q) = 1\ ,\ k_2 = d_1 = 1\ ,\ k_3 = \gcd(2,15,25) = 1\ ,\ k_4 = \gcd(2,25) = 1\ .$$

The matrix V' in Proposition 6(4) is in HNF, then it looks as follows

$$V' = \begin{pmatrix} 1 & 0 & 0 & v_{1,3} & v_{1,0} \\ 0 & 1 & 0 & v_{2,3} & v_{2,0} \\ 0 & 0 & 1 & v_{3,3} & v_{3,0} \\ 0 & 0 & 0 & 2 & v_{4,0} \end{pmatrix}$$

with $0 \le v_{k,3} \le 1$, for $1 \le k \le 3$. Moreover we get the following conditions

$$\begin{aligned}
0 &= q_0 v_{4,0} + \frac{q_0 q_4}{k_4} = 2v_{4,0} + 50 \ \Rightarrow\ v_{4,0} = -25 \\
0 &= q_0 v_{3,0} + \frac{k_3 q_3}{k_4} + q_4 v_{3,3} = 2v_{3,0} + 15 + 25 v_{3,3} \ \Rightarrow\ v_{3,3} = 1 \text{ and } v_{3,0} = -20 \\
0 &= q_0 v_{2,0} + \frac{k_2 q_2}{k_3} + q_4 v_{2,3} = 2v_{2,0} + 4 + 25 v_{2,3} \ \Rightarrow\ v_{2,3} = 0 \text{ and } v_{2,0} = -2 \\
0 &= q_0 v_{1,0} + \frac{k_1 q_1}{k_2} + q_4 v_{1,3} = 2v_{1,0} + 3 + 25 v_{1,3} \ \Rightarrow\ v_{1,3} = 1 \text{ and } v_{1,0} = -14
\end{aligned}$$

giving the following Q–canonical fan for $\mathbb{P}(2,3,4,15,25)$:

$$\Sigma = \text{fan}\left(\left(\begin{array}{c} -14 \\ -2 \\ -20 \\ -25 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 1 \\ 0 \\ 0 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \\ 1 \\ 0 \end{array}\right), \left(\begin{array}{c} 1 \\ 0 \\ 1 \\ 2 \end{array}\right)\right).$$

REMARK 5. Proposition 6 above has to be compared with results in §3 and §4 of [7]. In particular parts from (1) to (3) give a rewrite of Proposition 3.2, Remark 3.3 and Theorem 3.6 in [7]. For what concerns part (4), although it apparently looks to be related with Remark 4.3 and Theorem 4.5 in [7], it seems to us to be a rather new result in the literature. In fact Conrads, in §4 of [7], discusses HNFs of square matrices in $\mathrm{GL}(n,\mathbb{Q})\cap\mathrm{Mat}(n,n,\mathbb{Z})$ since he's interested in classifying isomorphism classes of simplices of a given type. Here the aim is quite different since we study HNF of the simplex itself, which turns out to be unique and then identifying the Q–canonical fan of $\mathbb{P}(Q)$.

3. Characterization of polytopes giving $\mathbb{P}(Q)$

3.1. From fans to polytopes and back

We shall use the following notation: given a $n\times(n+1)$ matrix $V=(\mathbf{v}_0,\dots,\mathbf{v}_n)=(v_{ij})$ with $1\leq i\leq n, 0\leq j\leq n$, the $n\times n$ sub-matrix of V obtained by removing the first column is denoted by $V^0=(\mathbf{v}_1,\dots,\mathbf{v}_n)=(v_{ik})$ with $1\leq i\leq n$ and $1\leq k\leq n$.

DEFINITION 5. *Let $V\in\mathrm{Mat}(n,n+1,\mathbb{Z})$ be a matrix whose maximal minors do not vanish i.e., in the same notation given above, $V_l\neq 0$ for every $0\leq l\leq n$. Consider the vector of absolute values of maximal minors $Q=(|V_0|,\dots,|V_n|)$. Recalling 1.3, the* $(0,Q)$-weighted transverse matrix *of V (or simply* weighted transverse*) is defined to be the following $n\times n$ rational matrix*

$$(V^0)^*_Q := (V^0)^*\cdot(\delta\, I^0_Q)$$

where $I^0_Q:=\mathrm{diag}(1/|V_1|,\dots,1/|V_n|)$ and $\delta:=\mathrm{lcm}(|V_0|,\dots,|V_n|)$.

REMARK 6. If $V\in\mathfrak{V}_n$, as defined in the Definition 4, then Theorem 4 below implicitly shows that the weighted transverse matrix $(V^0)^*_Q$ is a $n\times n$ *integral* matrix. In particular this fact is also proved explicitly in the following Proposition 7.

PROPOSITION 7. *If $V=(\mathbf{v}_0,\mathbf{v}_1,\dots,\mathbf{v}_n)$ is a fan matrix of $\mathbb{P}(Q)$, with $Q=(q_0,...,q_n)$, then the weighted transverse $(V^0)^*_Q$ has integral entries.*

Proof. Recall that the *adjoint matrix of* an invertible square matrix A is defined by setting $\mathrm{Adj}(A):=\det(A)\ A^{-1}$. Set $W=\mathrm{Adj}(V^0)$ and let $\mathbf{w}_i$ be the i-th row of W.

Observe that parts (3) and (4) in Theorem 3 give, for $i = 1, \dots, n$,

$$\begin{aligned} |\mathbf{w}_i \cdot \mathbf{v}_i| &= |\det\left(V^0\right)| = q_0 \\ \mathbf{w}_i \cdot \mathbf{v}_k &= 0 \quad \text{for } 1 \le k \le n \text{ and } k \ne i \\ |\mathbf{w}_i \cdot \mathbf{v}_0| &= |\det\left(V^0\right)| \frac{q_i}{q_0} = q_i \ , \end{aligned}$$

where the dot product is the usual matrix product. Therefore $\frac{\delta}{q_0 q_i} \mathbf{w}_i \cdot \mathbf{v}_j \in \mathbb{Z}$ for any $0 \le j \le n$. This means that

$$\forall\ 1 \le i \le n \quad \frac{\delta}{q_0 q_i} \mathbf{w}_i \in \mathbb{Z}^n$$

since $\mathcal{L}(\mathbf{v}_0, ..., \mathbf{v}_n) = \mathbb{Z}^n$. The proof concludes by transposing W. □

Let us denote by $\mathcal{O}_{\mathbb{P}(Q)}(1)$, or $\mathcal{O}(1)$ for short, the generator of the Picard group $\mathrm{Pic}(\mathbb{P}(Q)) \cong \mathbb{Z} \cdot \mathcal{O}_{\mathbb{P}(Q)}(1)$.

PROPOSITION 8. *If D_j is the torus invariant divisor associated with $\langle \mathbf{v}_j \rangle \in \Sigma(1)$ then $(\delta'/q'_j) D_j$ is an ample divisor in the linear system $|\mathcal{O}_{\mathbb{P}(Q)}(1)|$, where as usual $Q' = (q'_0, \dots, q'_n)$ is the reduced weight vector of Q and $\delta' = \mathrm{lcm}(Q')$.*

Proof. Recall the exact sequence (10) showing that the Chow group of $\mathbb{P}(Q)$ is given by $A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$. By construction, the morphism $d : \bigoplus_{i=0}^n \mathbb{Z} \cdot D_j \to \mathbb{Z}$ sends a Weil divisor $\sum_{j=0}^n b_j D_j$ to the generator $1 \in \mathbb{Z}$ if and only if $(b_0, \dots, b_n)$ is a solution of the diophantine equation $\sum_{j=0}^n q'_j x_j = 1$. It is a well known fact that the Picard group of a normal toric variety can be identified with the subgroup of $A_{n-1}(X)$ generated by the classes of torus invariant Cartier divisors (see e.g. [15] § 3.4, [10] § 4.2). In particular $\mathrm{Pic}(\mathbb{P}(Q)) \subset A_{n-1}(\mathbb{P}(Q)) \cong \mathbb{Z}$ is a free cyclic subgroup. Then a generator of $\mathrm{Pic}(\mathbb{P}(Q))$ is given by a suitable multiple kD of a generator D of $A_{n-1}(\mathbb{P}(Q))$, where k is the least positive integer number such that kD is Cartier. There is a Criterion to determine when a Weil divisor of a toric variety is a Cartier divisor ([17] Prop. 2.4) which applied to the case of $\mathbb{P}(Q)$ can be rewritten as follows:

$$(15) \quad \sum_{j=0}^{n} b_j D_j \text{ is a Cartier divisor} \iff \forall\ 0 \le l \le n\ \exists\ \mathbf{u}_l \in M\ :\ \forall\ j \ne l \quad \langle \mathbf{u}_l, \mathbf{n}_j \rangle = b_j$$

where $\mathbf{n}_j$ is a generator of the monoid $\langle \mathbf{v}_j \rangle \cap N$. Recall the exact sequence (10) and let $D = \sum_{j=0}^n b_j D_j$ be a generator of $A_{n-1}(\mathbb{P}(Q))$, i.e. $d(D) = 1$, and consider the positive integer multiple kD. Then (15) gives that kD is a Cartier divisor if and only if, for every $l = 0, \dots, n$,

$$(16) \qquad \exists\ \mathbf{u}_l \in M\ :\ \forall\ j \ne l \quad \langle \mathbf{u}_l, \mathbf{n}_j \rangle = k b_j\ .$$

Since $div(\mathbf{u}_l) = \sum_{j=0}^n \langle \mathbf{u}_l, \mathbf{n}_j \rangle D_j$, then (16) is equivalent to requiring that

$$(17) \qquad \exists\ \mathbf{u}_l \in M\ :\ div(\mathbf{u}_l) = \sum_{j \ne l} k b_j D_j + \langle \mathbf{u}_l, \mathbf{n}_l \rangle D_l = kD + (\langle \mathbf{u}_l, \mathbf{n}_l \rangle - k b_l) D_l\ .$$

The exactness of (10) ensures that (17) is equivalent to asking that

$$\begin{aligned} \exists\, \mathbf{u}_l \in M \quad &: \quad d\left(kD + \left(\langle \mathbf{u}_l, \mathbf{n}_l\rangle - kb_l\right) D_l\right) = 0 \\ &\Leftrightarrow \quad k = q'_l \left(kb_l - \langle \mathbf{u}_l, \mathbf{n}_l\rangle\right) \ . \end{aligned} \tag{18}$$

Then (18) gives that kD is Cartier if and only if $q'_l \mid k$ for every $0 \leq l \leq n$. Then the inclusion $\mathrm{Pic}(\mathbb{P}(Q)) \hookrightarrow A_{n-1}(\mathbb{P}(Q))$ turns out to be the multiplication by δ'. To complete the proof, notice that D_j and $q'_j D$ give the same class in $A_{n-1}(\mathbb{P}(Q))$: in fact $d(D_j - q'_j D) = 0$. Then $(\delta'/q'_j)D_j$ and $\delta' D$ give the generator of $\mathrm{Pic}(\mathbb{P}(Q)) = \delta' A_{n-1}(\mathbb{P}(Q))$. In particular $(\delta'/q'_j)D_j \in |\mathcal{O}_{\mathbb{P}(Q)}(1)|$. This also suffices to prove that $(\delta'/q'_j)D_j$ is ample. □

Set Δ_j be the integral polytope associated with the divisor $H = (\delta'/q'_j)D_j$, as in (4). One can easily check that there exist n points $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$, depending on the choice of D_j, such that Δ_j is the convex hull $\mathrm{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$: in particular *the ampleness of* $(\delta'/q'_j)D_j$ *implies that* $\{\mathbf{w}_1, \dots, \mathbf{w}_n\}$ *is a set of n linearly independent integral vectors* ([17] Corollary 2.14).

Let $\mathfrak{P}_n$ be the set of integral polytopes in $M_{\mathbb{R}}$ obtained as the convex hull of the origin and n linearly independent integral vectors and $\mathfrak{F}(Q)$ be the set of fans in $N_{\mathbb{R}}$ defining $\mathbb{P}(Q)$. Then we have established maps

$$\begin{array}{rccc} \forall\, 0 \leq j \leq n\ , \quad \Delta^j_Q : & \mathfrak{F}(Q) & \longrightarrow & \mathfrak{P}_n \\ & \Sigma & \longmapsto & \Delta^j_Q(\Sigma) := \Delta_j \end{array} \tag{19}$$

Let $W = (w_{ik})$ be the $n \times n$ matrix of the components of vectors $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$ over the dual basis: namely

$$\forall\, k = 1, \dots, n \quad \mathbf{w}_k = \sum_{i=1}^{n} w_{ik} \mathbf{e}_i^{\vee}$$

where $\{\mathbf{e}_1^{\vee}, \dots, \mathbf{e}_n^{\vee}\}$ is the dual basis of $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then we get the following representation of the map Δ^0_Q:

THEOREM 4. *Given the fan* $\Sigma := \mathrm{fan}(\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathfrak{F}(Q)$*, the image* $\Delta^0_Q(\Sigma)$ *defined in (19) is the convex hull* $\mathrm{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$ *of the origin with the n linearly independent integral vectors* $\mathbf{w}_1, \dots, \mathbf{w}_n \in M_{\mathbb{R}}$ *giving the columns of the* $(0, Q)$*-weighted transverse matrix of* $V = (\mathbf{v}_0, \dots, \mathbf{v}_n)$*, i.e.*

$$W = (V^0)^*_Q \ ,$$

where $Q = (|V_0|, \dots, |V_n|)$*. Namely the entries of W are given by*

$$\forall\, 1 \leq i \leq n\ ,\ 1 \leq k \leq n \quad w_{ik} = \frac{\delta V^0_{ik}}{q_k V_0}$$

where V^0_{ik} *is the cofactor of* v_{ik} *in* V^0 *and* $V_0 = \det(V^0) = \pm q_0$ *(by either* (3) *or* (4) *in Theorem 3).*

Proof. Recalling (4), to define $\Delta_Q^0(\Sigma) = \Delta_0$ one has to write down the hyperplanes of $M_{\mathbb{R}}$

$$\forall \rho \in \Sigma(1) \quad \langle \mathbf{u}, \mathbf{n}_\rho \rangle = -a_\rho \ , \quad \text{where } \mathbf{n}_\rho \text{ generates } \rho \cap N \ , \tag{20}$$

for the divisor $H = (\delta'/q_0')D_0$. Since $\Sigma(1) = \{\langle \mathbf{v}_j \rangle \subset N_{\mathbb{R}} |\ j = 0, \ldots, n\}$ the hyperplanes (20) are then given by

$$\begin{array}{rrcl} & \sum_{i=1}^{n} n_{i0} u_i & = & -\delta'/q_0' \\ \forall\ k = 1, \ldots, n & \sum_{i=1}^{n} n_{ik} u_i & = & 0 \end{array} \tag{21}$$

where $\mathbf{n}_j = \sum_{i=1}^n n_{ij} \mathbf{e}_i$ generates the 1-dimensional cone $\langle \mathbf{v}_j \rangle \cap N$. In the part (c) of Lemma 1 it has been observed that $q_0' \mathbf{n}_0 = -\sum_{k=1}^n q_k' \mathbf{n}_k$. Then the first equation in (21) can be rewritten as follows

$$\sum_{i=1}^{n} \left(\sum_{k=1}^{n} q_k' n_{ik} \right) u_i = \delta' \ .$$

Let us represent equations in (21) by the following $(n+1) \times (n+1)$-matrix

$$M = \left(\begin{array}{ccc|c} \sum_{k=1}^n q_k' n_{1k} & \cdots & \sum_{k=1}^n q_k' n_{nk} & \delta' \\ n_{11} & \cdots & n_{n1} & 0 \\ & \vdots & & \vdots \\ n_{1n} & \cdots & n_{nn} & 0 \end{array} \right) .$$

For $j = 0, 1, \ldots, n$, the vertex $\mathbf{w}_j$ of $\Delta_Q^0(\Sigma)$ is then given by the (unique, for (3) in Theorem 3 and recalling that $v_{ij} = d_j n_{ij}$) solution of the linear system associated with the matrix M^{j+1}, obtained removing the $(j+1)$-st row in M. Clearly $\mathbf{w}_0 = 0$. For $j = k = 1, \ldots, n$ we get

$$w_{ik} = M_{k+1,i} / M_{k+1,n+1}$$

where $M_{a,b}$ is the (a,b)-cofactor in M. Observe that $M_{k+1,n+1} = (-1)^{k-1} q_k' \, V_0 \big/ a_0$ and $M_{k+1,i} = (-1)^{k+1} \delta' d_k \, V_{ik}^0 \big/ a_0$. Then

$$w_{ik} = \frac{\delta' d_k}{q_k'} \frac{V_{ik}^0}{V_0} = \frac{\delta' a_k d_k}{q_k} v_{ik}^* = \frac{\delta}{q_k} v_{ik}^*$$

where $v_{ik}^* = V_{ik}^0 / V_0$ is the (i,k)-entry of $V^{0*} := ((V^0)^{-1})^T$. The last equality on the right is obtained by recalling Proposition 3(5) and Proposition 4. □

REMARK 7. Clearly same conclusions as in Theorem 4 can be obtained by exchanging 0 with any other value j such that $0 \leq j \leq n$.

REMARK 8. Let Q be a weight vector whose reduction is given by Q'. Consider $\Sigma = \mathrm{fan}(\mathbf{v}_0,\ldots,\mathbf{v}_n) \in \mathfrak{F}(Q)$ and, for any $0 \le j \le n$, consider the generator $\mathbf{n}_j$ of the semigroup $\langle \mathbf{v}_j \rangle \cap N$, where N is the lattice generated by $\mathbf{v}_0,\ldots,\mathbf{v}_n$. Then Lemma 1(c) and Theorem 3 ensure that $\Sigma := \mathrm{fan}(\mathbf{n}_0,\ldots,\mathbf{n}_n) \in \mathfrak{F}(Q')$. Then the previous Theorem 4 gives that

$$\Delta^0_Q(\Sigma) = \Delta^0_{Q'}(\Sigma)$$

since, recalling once again Propositions 3 and 4,

$$w_{ik} = (\delta/q_k)(V^0_{ik}/V_0) = (\delta' a/q'_k a_k)(N^0_{ik}/d_k N_0) = (\delta'/q'_k)(N^0_{ik}/N_0)$$

(here N denotes the matrix $N = (\mathbf{n}_0,\ldots,\mathbf{n}_n)$).

EXAMPLE 2. Let us still consider Example 1 to apply the weighted transversion and Theorem 4 for producing by hand a polytope of a given wps $\mathbb{P}(Q)$ with the minimal polarization.
Recall that $Q = (2,3,4,15,25)$ and the matrix fan obtained in the Example 1 is

$$V = \begin{pmatrix} -14 & 1 & 0 & 0 & 1 \\ -2 & 0 & 1 & 0 & 0 \\ -20 & 0 & 0 & 1 & 1 \\ -25 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \Longrightarrow \quad (V^0)^* = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix}.$$

Since $\delta = \mathrm{lcm}(2,3,4,15,25) = 300$, we get

$$W = (V^0)^*_Q = (V^0)^* \cdot \delta I_Q = 150 \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ -1 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/15 & 0 \\ 0 & 0 & 0 & 1/25 \end{pmatrix}$$

giving $W = \begin{pmatrix} 100 & 0 & 0 & 0 \\ 0 & 75 & 0 & 0 \\ 0 & 0 & 20 & 0 \\ -50 & 0 & -10 & 6 \end{pmatrix}$. Then the polytope we are looking for is

$$\Delta = \mathrm{Conv}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 100 \\ 0 \\ 0 \\ -50 \end{pmatrix}, \begin{pmatrix} 0 \\ 75 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 20 \\ -10 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 6 \end{pmatrix} \right).$$

More precisely $(\mathbb{P}_\Delta, O(1)) \cong (\mathbb{P}(Q), \delta/q_0\, D_0) = (\mathbb{P}^4(2,3,4,15,25), 150\, D_0)$.

DEFINITION 6 (P–admissible matrices). *A square matrix $W \in \mathrm{Mat}(n,n,\mathbb{Z})$ is called P–*admissible *if there exists an F-admissible matrix $V \in \mathfrak{V}_n$ such that W is the weighted transverse matrix of V, which is*

$$W = (V^0)^*_Q \quad \textit{with} \quad Q = (|V_0|,\ldots,|V_n|)\ .$$

In other words $W = (\mathbf{w}_1, \dots, \mathbf{w}_n)$ is admissible if and only if the polytope

$$\mathrm{Conv}(\mathbf{0}, \mathbf{w}_1, \dots, \mathbf{w}_n)$$

belongs to the image of the map Δ^0_Q, as defined in (19). In this case we say that Q, W, V are associated *to each other.*

Let us denote $\mathfrak{W}_n \subset \mathrm{GL}(n,\mathbb{Q}) \cap \mathrm{Mat}(n,n,\mathbb{Z})$ the subset of P–admissible matrices: notice that any such matrix has integer entries by either Theorem 4 or the following Proposition 7.

REMARK 9. Remark 8 guarantees that weight vectors Q_1 and Q_2 admitting the same reduction Q' are associated with the same P–admissible matrix W, which is the P–admissible matrix associated with the reduced weight vector Q'. Conversely, there exists a unique reduced weight vector Q' to which W is associated. Proposition 9(c) will prove this fact in a purely algebraic setting; moreover Proposition 9(b) will exhibit a constructive method for finding Q'.

DEFINITION 7. *Consider a matrix $W \in \mathrm{GL}(n,\mathbb{Q}) \cap \mathrm{Mat}(n,n,\mathbb{Z})$. Let s_i be the* gcd *of entries in the i-th row of* $\mathrm{Adj}(W)$. *Then we define the* reduced adjoint *of W as follows*

$$\begin{aligned} \widehat{W} &:= \frac{|\det(W)|}{\det(W)} \operatorname{diag}\left(\frac{1}{s_1}, \dots, \frac{1}{s_n}\right) \cdot \mathrm{Adj}(W) \\ &= \operatorname{diag}\left(\frac{|\det(W)|}{s_1}, \dots, \frac{|\det(W)|}{s_n}\right) \cdot W^{-1} \end{aligned}$$

Notice that if V is a square matrix in $\mathrm{Mat}(n,n,\mathbb{Z})$ such that $V \cdot W$ is a diagonal matrix with positive entries then

$$V = \operatorname{diag}(r_1, \dots, r_n) \cdot \widehat{W} \tag{22}$$

for some $r_1, \dots, r_n \in \mathbb{N}$.

PROPOSITION 9. *Let W be a P–admissible matrix and let $Q = (q_0, \dots, q_n)$ be a* reduced *weight vector associated to W. Then*

(a) $\left(\widehat{W}^T\right)^*_Q = W$*;*

(b) if $s := \gcd(s_1, \dots, s_n)$ is the greatest common divisor of the terms in $\mathrm{Adj}(W)$ *then*

$$q_0 = |\det(\widehat{W})| \quad , \quad \forall\, 1 \le i \le n \quad q_i = \frac{s_i}{s} \quad , \quad \mathrm{lcm}(Q) = \frac{|\det(W)|}{s}$$

(c) if Q_1 and Q_2 are reduced weight vectors associated with the same P–admissible matrix W, then $Q_1 = Q_2$;

(d) there exists a unique F–admissible matrix V associated with W and Q i.e. such that $W = \left(V^0\right)^*_Q$ with $Q = (|V_0|, \dots, |V_n|)$.

Proof. (a). W is a P–admissible matrix. Then there exists a F–admissible matrix V such that $W = ((V^0)^T)^{-1}\delta I_Q$ and $Q = (|V_0|,\dots,|V_n|)$, meaning that $(V^0)^T W = \delta I_Q^0$ is diagonal with positive entries. Recalling (22) we get that $(V^0)^T = \mathrm{diag}(r_1,\dots,r_n)\cdot \widehat{W}$ for some $r_1,\dots,r_n \in \mathbb{N}$. But Q is reduced, which implies that the columns of V^0 have coprime entries. Therefore $r_1 = \cdots = r_n = 1$ and $(V^0)^T = \widehat{W}$. (a) follows immediately.
(b) On the one hand $\widehat{W}\cdot W = \mathrm{diag}\left(|\det(W)|/s_1,\dots,|\det(W)|/s_n\right)$. On the other hand, by (a), $\widehat{W} = (V^0)^T$ and $\widehat{W}\cdot W = \mathrm{diag}\left(\delta/q_1,\dots,\delta/q_n\right)$, where $\delta := \mathrm{lcm}(Q)$. Therefore

$$\forall\, 1\le i\le n \quad \frac{\delta}{q_i} = \frac{|\det(W)|}{s_i}\,. \tag{23}$$

Observe now that

$$\begin{aligned}
\mathrm{lcm}\left(\frac{\delta}{q_1},\dots,\frac{\delta}{q_n}\right) &= \frac{\delta}{\gcd(q_1,\dots,q_n)} = \delta \\
\mathrm{lcm}\left(\frac{|\det(W)|}{s_1},\dots,\frac{|\det(W)|}{s_n}\right) &= \frac{|\det(W)|}{s}
\end{aligned}$$

Then (23) gives that $\delta = |\det(W)|/s$ and, for any $1\le i\le n$, $q_i = s_i/s$. Finally (a) gives that $q_0 = |V_0| = |\det(\widehat{W})|$.
(c) follows immediately by the previous part (b).
(d). If there exist two F–admissible matrix U,V such that they are both associated with W and Q, then

$$(\mathbf{v}_1,\dots,\mathbf{v}_n) = V^0 = U^0 = (\mathbf{u}_1,\dots,\mathbf{u}_n) \;\Rightarrow\; \mathbf{v}_0 = -\frac{1}{q_0}\sum_{i=1}^n q_i\mathbf{v}_i = -\frac{1}{q_0}\sum_{i=1}^n q_i\mathbf{u}_i = \mathbf{u}_0$$

implying that $V = U$. □

REMARK 10. In a sense the previous Proposition 9 states that, when restricted to wps fans associated with *reduced* weight vector, the weighted transversion process giving a polytope starting from a fan, can be inverted by considering the *transposed reduced adjoint* of the polytope matrix. Namely if W is a polytope matrix of $(\mathbb{P}(Q),O(1))$, with Q reduced, then $V := \left(\begin{array}{c|c} \mathbf{v}_0 & \widehat{W}^T \end{array}\right)$ is a fan matrix of $\mathbb{P}(Q)$ when $\mathbf{v}_0$ is defined by setting $\mathbf{v}_0 = -(\sum_{i=1}^n q_i\mathbf{v}_i)/q_0$, where $(\mathbf{v}_1,\dots,\mathbf{v}_n) = \widehat{W}^T$.

The following Proposition 10 shows criteria for a matrix W to be P-admissible.

PROPOSITION 10. *Let* $W = (w_{ij}) \in \mathrm{GL}(n,\mathbb{Q})\cap \mathrm{Mat}(n,n,\mathbb{Z})$ *be a matrix such that* $\gcd(w_{ij}) = 1$. *Let s be the greatest common divisor of the entries in* $\mathrm{Adj}(W)$ *and* $\mathbf{v}$ *be the sum of the rows of* $\mathrm{Adj}(W)$. *Define* $q_0 = |\det(\widehat{W})|$, $\delta = \frac{|\det(W)|}{s}$. *The following statements are equivalent:*

(a) W is P–admissible;

(b) the vector $\mathbf{v}$ *is divisible by* $q_0 s$*;*

(c) *q_0 divides δ and the vector $\frac{\delta}{q_0}(1,\ldots,1)$ is in the lattice generated by the rows of W.*

Proof. (a) $\Rightarrow$(b): If W is P–admissible then there exist a unique reduced weight vector Q and a unique F–admissible matrix V, associated with W like in Definition 6. By Proposition 9, $\widehat{W} = (V^0)^T$ and $Q = (q_0,\ldots,q_n)$ with $q_i = s_i/s$, for $i = 1,\ldots,n$. Let $\mathbf{v}_i$ be the i-th row of $\widehat{W}$; then $\sum_{i=1}^n q_i\mathbf{v}_i$ is divisible by q_0 since $\widehat{W}^T = V^0$ is F–admissible, meaning that its columns satisfy the relation $\sum_{i=1}^n q_i\mathbf{v}_i = -q_0\mathbf{v}_0$. Then $\sum_{i=1}^n s_i\mathbf{v}_i$ is divisible by $q_0 s$ and $s_i\mathbf{v}_i$ is the i-th row of $\mathrm{Adj}(W)$.

(b) $\Rightarrow$(a): Assume that $q_0 s$ divides any entry in $\mathbf{v}$. For $1 \le i \le n$, let $\mathbf{v}_i$ be the i-th row of $\widehat{W}$ and $q_i = s_i/s$ be defined as in Proposition 9(b); then $\sum_{i=1}^n q_i\mathbf{v}_i$ is divisible by q_0. Put $\mathbf{v}_0 = -\frac{1}{q_0}\sum_{i=1}^n q_i\mathbf{v}_i$. Then the matrix

$$V := \left(\begin{array}{c|c} \mathbf{v}_0 & \widehat{W}^T \end{array} \right) = (\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_n)$$

turns out to be F–admissible with respect to Q by Theorem 3(4). Then $W = \left(V^0\right)^*_Q$ is P–admissible.

(b) $\Leftrightarrow$(c): the sum of the rows of $\mathrm{Adj}(W)$ is the row vector $(1,\ldots,1)\cdot\mathrm{Adj}(W) = (1,\ldots,1)\cdot\det(W)W^{-1}$. Thus it is divisible by $q_0 s$ if and only if there exists $(x_1,\ldots,x_n) \in \mathbb{Z}^n$ such that $(x_1,\ldots,x_n)\cdot W = \frac{\delta}{q_0}(1,\ldots,1)$, that is if and only if (c) holds. □

3.2. Characterizing the polytope of a polarized wps

Given an integral polytope $\Delta = \mathrm{Conv}(\mathbf{0},\mathbf{w}_1,\ldots,\mathbf{w}_n)$, for a suitable subset $\{\mathbf{w}_1,\ldots,\mathbf{w}_n\} \subset M$, let $W := (\mathbf{w}_1,\ldots,\mathbf{w}_n)$ be the associated polytope matrix. Then the following result is a consequence of Propositions 9 and 10.

THEOREM 5. *Let $\Delta = \mathrm{Conv}(\mathbf{0},\mathbf{w}_1,\ldots,\mathbf{w}_n) \subset M_{\mathbb{R}}$ be a n-dimensional integral polytope. Set $m := \gcd(w_{ij})$ and define $W' := \frac{1}{m}W$. Let $Q = (q_0,\ldots,q_n)$ be the reduction of the weight vector defined as in Proposition 9(b). Then the following facts are equivalent:*

1. *W' is a P-admissible matrix, hence it satisfies one of the equivalent conditions in Proposition 10,*
2. *$(\mathbb{P}_\Delta, \mathcal{O}(1)) \cong (\mathbb{P}(Q), \mathcal{O}(m))$.*

Proof. (1)$\Rightarrow$(2): By definition if W' is P-admissible then there exists a F-admissible matrix V such that $W' = (V^0)^*_{\tilde{Q}}$, where $\tilde{Q} = (|V_0|,\ldots,|V_n|)$. By Proposition 9(b) $\tilde{Q} = Q$. Then the polytope

$$\mathrm{Conv}\left(\mathbf{0},\frac{\mathbf{w}_1}{m},\ldots,\frac{\mathbf{w}_n}{m}\right)$$

belongs to the image of the map Δ^0_Q, as defined in (19). Then W' is the polytope matrix of $\Delta_{D'}$ for some divisor $D' \in \mathcal{O}_{\mathbb{P}(Q)}(1)$ and $W = mW'$ is the polytope matrix of $\Delta := \Delta_{mD'}$ giving (2).

(2)⇒(1): There exists a divisor D of $\mathbb{P}(Q)$, belonging to the linear system $|\mathcal{O}(m)|$, such that $\Delta = \Delta_D$. Moreover there exists a divisor $D' \in |\mathcal{O}(1)|$ such that $D = mD'$ and in particular $\Delta = \Delta_D = m\Delta_{D'}$. This means that $\Delta_{D'} = \mathrm{Conv}(\mathbf{0}, \mathbf{w}'_1, \ldots, \mathbf{w}'_n)$ and $W' := (\mathbf{w}'_1, \ldots, \mathbf{w}'_n) = \frac{1}{m}W$ is a P–admissible matrix associated with Q. □

Acknowledgment. The authors would like to thank the referee for his careful reading of the manuscript and his pertinent remarks and corrections. In particular Theorem 5 and the proofs of Theorem 3 and Proposition 5 have been significantly improved by his observations.

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AMS Subject Classification: 14M25, 52B20

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Lavoro pervenuto in redazione il MM.GG.AAAA.