

# HAIRER'S RECONSTRUCTION THEOREM WITHOUT REGULARITY STRUCTURES

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ABSTRACT. This survey is devoted to Martin Hairer's Reconstruction Theorem, which is one of the cornerstones of his theory of Regularity Structures [Hai14]. Our aim is to give a new self-contained and elementary proof of this Theorem, together with some applications, including a characterization, based on a single arbitrary test function, of negative Hölder spaces. We present the Reconstruction Theorem as a general result in the theory of distributions that can be understood without any knowledge of Regularity Structures themselves, which we do not even need to define.

## 1. INTRODUCTION

Consider the following problem: if at each point  $x \in \mathbb{R}^d$  we are given a distribution (generalized function)  $F_x$  on  $\mathbb{R}^d$ , is there a distribution  $f$  on  $\mathbb{R}^d$  which is well approximated by  $F_x$  around each point  $x \in \mathbb{R}^d$ ?

A classical example is when  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  is a smooth function and  $F_x$  is the Taylor polynomial of  $f$  based at  $x$ , of some fixed order  $r \in \mathbb{N}$ ; then we know that  $f(y) - F_x(y)$  is of order  $|y - x|^{r+1}$  for  $y \in \mathbb{R}^d$  close to  $x$ . Of course, in this example  $F_x$  is built from  $f$ , which is known in advance. We are rather interested in the reverse problem of finding  $f$  given a (suitable) family of  $F_x$ 's, as in Whitney's Extension Theorem [Whi34]. However if we allow the local descriptions  $F_x$  to be non-smooth and even distributions, then existence and uniqueness of such  $f$  become non-trivial.

Martin Hairer's Reconstruction Theorem [Hai14] provides a complete and elegant solution to this problem. We present here an enhanced version of this result which allows to prove existence and uniqueness of  $f$  under an *optimal* assumption on the family of distributions  $(F_x)_{x \in \mathbb{R}^d}$ , that we call *coherence*. We also present some applications of independent interest, including a characterization of negative Hölder spaces based on a single *arbitrary* test function.

The Reconstruction Theorem was originally formulated in the framework of Hairer's theory of *regularity structures* [Hai14]. In this survey we state and prove this result without any reference to regularity structures, which we do not even define. The original motivation for this theory was stochastic analysis, but here we present the Reconstruction Theorem in a completely analytical and deterministic framework. Our approach contains novel ideas and techniques which may be generalized to other settings, e.g. to distributions on manifolds.

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Although regularity structures have already attracted a lot of attention, we hope that this survey will give the opportunity to an even larger audience to become familiar with some of the ideas of this theory, which may still find applications outside the area which motivated it first.

**A LOOK AT THE LITERATURE.** With his theory of Rough Paths [Lyo98], Terry Lyons introduced the idea of a local description of the solution to a stochastic differential equation as a generalized Taylor expansion, where classical monomials are replaced by iterated integrals of the driving Brownian motion. This idea led Massimiliano Gubinelli to introduce his Sewing Lemma [Gub04], which is a version of the Reconstruction Theorem in  $\mathbb{R}^1$  (the name “Sewing Lemma” is actually due to Feyel and de La Pradelle [FdLP06], who gave the proof which is now commonly used). With his theory of regularity structures [Hai14], Martin Hairer translated these techniques in the context of *stochastic partial differential equations* (SPDEs), whose solutions are defined on  $\mathbb{R}^d$  with  $d > 1$  (see [Zam21] for a history of SPDEs).

The first proof of the Reconstruction Theorem was based on wavelets [Hai14]. Later Otto-Weber [OW19] proposed a self-contained approach based on semigroup methods. The core of our proof is based on elementary multiscale arguments, which allow to characterize the regularity of a distribution via scaling of a *single arbitrary test function*. The second edition of Friz-Hairer’s book [FH20] contains a proof close in spirit to the one presented here. For other proofs of versions of the Reconstruction Theorem, see [GIP15, HL17, MW18, ST18].

**OUTLINE OF THE PAPER.** In Section 2 we set the notation used throughout this survey and in Section 3 we recall basic facts on test functions and distributions.

In Section 4 we define the key notion of *germs of distributions* and the property of *coherence*. This leads directly to the Reconstruction Theorem in Section 5, see Theorem 5.1. We then show in Section 6 that the coherence condition is *optimal*.

The core of the paper, from Section 7 to Section 11, is devoted to the proof of the Reconstruction Theorem (see Section 5.1 for a guide).

The last sections are devoted to applications of the Reconstruction Theorem. In Section 12 we study negative Hölder spaces, providing criteria based on a single arbitrary test function, see Theorem 12.4. In Section 13 we investigate more closely the coherence condition. In Section 14 we construct a suitable product between distributions and non smooth functions, see Theorem 14.1, which is a multi-dimensional analogue of Young integration.

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## 2. NOTATION

We work on the domain  $\mathbb{R}^d$ , equipped with the Euclidean norm  $|\cdot|$ . We denote by  $B(x, r) = \{z \in \mathbb{R}^d : |z - x| \leq r\}$  the closed ball centered at  $x$  of radius  $r$ . The  $R$ -enlargement of a set  $K \subseteq \mathbb{R}^d$  is denoted by

$$\bar{K}_R := K + B(0, R) = \{z \in \mathbb{R}^d : |z - x| \leq R \text{ for some } x \in K\}. \quad (2.1)$$

Partial derivatives of a differentiable function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  are denoted by

$$\partial^k \varphi = \partial_{x_1}^{k_1} \cdots \partial_{x_d}^{k_d} \varphi \quad \text{for a multi-index } k = (k_1, \dots, k_d) \in \mathbb{N}_0^d,$$

where  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$  and we set  $|k| := k_1 + \dots + k_d$ . If  $k_i = 0$  then  $\partial_{x_i}^{k_i} \varphi := \varphi$ .

For functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  we use the standard notation

$$\|\varphi\|_\infty := \sup_{x \in \mathbb{R}^d} |\varphi(x)|.$$

We denote by  $C^r$ , for  $r \in \mathbb{N}_0 \cup \{\infty\}$ , the space of functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  which admit continuous derivatives  $\partial^k \varphi$  for every multi-index  $k$  with  $|k| \leq r$ . We set

$$\|\varphi\|_{C^r} := \max_{|k| \leq r} \|\partial^k \varphi\|_\infty. \quad (2.2)$$

We denote by  $\mathcal{C}^\alpha$ , for  $\alpha > 0$ , the space of *locally  $\alpha$ -Hölder functions*  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ . More explicitly,  $\varphi \in \mathcal{C}^\alpha$  means that:

- (1)  $\varphi$  is of class  $C^r$ , where  $r = \underline{r}(\alpha) := \max\{n \in \mathbb{N}_0 : n < \alpha\}$ ;
- (2) uniformly for  $x, y$  in compact sets we have

$$|\varphi(y) - F_x(y)| \lesssim |y - x|^\alpha \quad (2.3)$$

where  $F_x(\cdot)$  is the Taylor polynomial of  $\varphi$  of order  $r$  based at  $x$ , namely

$$F_x(y) := \sum_{|k| < \alpha} \partial^k \varphi(x) \frac{(y - x)^k}{k!}, \quad y \in \mathbb{R}^d. \quad (2.4)$$

**Remark 2.1.** The meaning of  $\lesssim$  in (2.3) is that for any compact set  $K \subseteq \mathbb{R}^d$  there is a constant  $C = C_K < \infty$  such that  $|\varphi(y) - F_x(y)| \leq C|y - x|^\alpha$  for all  $x, y \in K$ . *This notation will be used extensively throughout the paper.*

**Remark 2.2.** For  $r \in \mathbb{N}$  and  $\alpha < r \leq \alpha'$  we have the (strict) inclusions  $\mathcal{C}^{\alpha'} \subset C^r \subset \mathcal{C}^\alpha$ . We stress that for  $r \in \mathbb{N}$  the space  $\mathcal{C}^r$  is strictly larger than  $C^r$  (for instance,  $\mathcal{C}^1$  is the space of locally Lipschitz functions, and similarly  $\mathcal{C}^r$  is the space of functions in  $C^{r-1}$  whose derivatives of order  $r - 1$  are locally Lipschitz). Incidentally, we note that other definitions of the space  $\mathcal{C}^r$  for  $r \in \mathbb{N}$  are possible, see e.g. [HL17]. The one that we give here is convenient for our goals.

**Remark 2.3.** We will later extend the definition of  $\mathcal{C}^\alpha$  to negative exponents  $\alpha \leq 0$ : this will no longer be a space of functions, but rather of *distributions*.

### 3. TEST FUNCTIONS, DISTRIBUTIONS, AND SCALING

We introduce the fundamental notions of test functions and distributions on  $\mathbb{R}^d$ .

**Definition 3.1 (Test functions).** We denote by  $\mathcal{D} := \mathcal{D}(\mathbb{R}^d)$  the space of  $C^\infty$  functions  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  with compact support, called test functions. We denote by  $\mathcal{D}(K)$  the subspace of functions in  $\mathcal{D}$  supported on a set  $K \subseteq \mathbb{R}^d$ .

**Definition 3.2 (Distributions).** A linear functional  $T : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{R}$  is called a distribution on  $\mathbb{R}^d$  (or simply a distribution, or generalized function), if for every compact set  $K \subseteq \mathbb{R}^d$  there exist  $r = r_K \in \mathbb{N}_0$  and  $C = C_K < \infty$  such that

$$|T(\varphi)| \leq C \|\varphi\|_{C^r}, \quad \forall \varphi \in \mathcal{D}(K). \quad (3.1)$$

The space of distributions on  $\mathbb{R}^d$  is denoted by  $\mathcal{D}' := \mathcal{D}'(\mathbb{R}^d)$ .

Given  $K \subseteq \mathbb{R}^d$ , any linear functional  $T : \mathcal{D}(K) \rightarrow \mathbb{R}$  which satisfies (3.1) for some  $r \in \mathbb{N}$ ,  $C < \infty$  is called a distribution on  $K$ . Their space is denoted by  $\mathcal{D}'(K)$ .

**Remark 3.3.** When relation (3.1) holds, we say that  $T$  is a distribution of order  $r$  on the set  $K$ . If one can choose  $r$  independently of  $K$ , we say that  $T$  is a distribution of finite order  $r$  on  $\mathbb{R}^d$  (the constant  $C$  in (3.1) is allowed to depend on  $K$ ). Note that a distribution of order  $r$  on the set  $K$  is also of order  $r' \geq r$  on  $K$ .

**Remark 3.4.** Here are some basic examples of distributions.

- Any locally integrable function  $f \in L^1_{\text{loc}}$  (hence any continuous function) can be canonically identified with the distribution  $f(\varphi) := \int f(z) \varphi(z) dz$ .
- More generally, any Borel measure  $\mu$  on  $\mathbb{R}^d$  which is finite on compact sets can be identified with the distribution  $\mu(\varphi) := \int \varphi d\mu$ .

Both  $f(\varphi)$  and  $\mu(\varphi)$  are distributions of finite order  $r = 0$  on  $\mathbb{R}^d$ .

**SCALING.** We next introduce the key notion of *scaling*. Given a function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$ , we denote by  $\varphi_x^\lambda : \mathbb{R}^d \rightarrow \mathbb{R}$  the scaled version of  $\varphi$  that is centered at  $x$  and localised at scale  $\lambda > 0$ , defined as follows:

$$\varphi_x^\lambda(z) := \lambda^{-d} \varphi(\lambda^{-1}(z - x)). \quad (3.2)$$

When  $x = 0$  we write  $\varphi^\lambda = \varphi_0^\lambda$ , when  $\lambda = 1$  we write  $\varphi_x = \varphi_x^1$ .

Note that if  $\varphi$  is supported in  $B(0, 1)$ , then  $\varphi_x^\lambda$  is supported in  $B(x, \lambda)$ . The scaling factor  $\lambda^{-d}$  in (3.2) is chosen to preserve the integral:

$$\int \varphi_x^\lambda(z) dz = \int \varphi(z) dz, \quad \|\varphi_x^\lambda\|_{L^1} = \|\varphi\|_{L^1}.$$

**We will use scaled functions  $\varphi_x^\lambda$  extensively.** The basic intuition is that given a distribution  $T \in \mathcal{D}'$  and a test function  $\varphi \in \mathcal{D}$ , the map  $\lambda \mapsto T(\varphi_x^\lambda)$  for small  $\lambda > 0$  tells us something useful about the *behavior of  $T$  close to  $x \in \mathbb{R}^d$* .

**Remark 3.5.** We can bound the  $C^r$  norm of a scaled test function  $\varphi_x^\lambda$  as follows:

$$\|\varphi_x^\lambda\|_{C^r} \leq \lambda^{-d-r} \|\varphi\|_{C^r}, \quad (3.3)$$

simply because  $\|\partial^k \varphi_x^\lambda\|_\infty = \lambda^{-|k|-d} \|\partial^k \varphi\|_\infty$ , see (2.2) and (3.2).

As a consequence, given a distribution  $T \in \mathcal{D}'$ , a compact set  $K \subseteq \mathbb{R}^d$  and a test function  $\varphi \in \mathcal{D}$ , we have the following bound, for a suitable  $r \in \mathbb{N}$ :

$$|T(\varphi_x^\lambda)| \lesssim \lambda^{-r-d}, \quad (3.4)$$

uniformly for  $x \in K$  and  $\lambda \in (0, 1]$ . Indeed, it suffices to take  $r = r_{\bar{K}_1}$  in (3.1) for the compact set  $\bar{K}_1$  (the 1-enlargement of  $K$ , see (2.1)) and to apply (3.3).

In some cases it can be useful to consider *non-Euclidean* scalings (like in the theory of regularity structures for applications to *parabolic* SPDEs, see [Hai14, Section 2]). Our approach could be easily adapted to such scalings, but for simplicity of presentation we refrain from doing so in this survey.

## 4. GERMS OF DISTRIBUTIONS AND COHERENCE

The following definition is crucial to our approach.

**Definition 4.1 (Germs).** We call germ a family  $F = (F_x)_{x \in \mathbb{R}^d}$  of distributions  $F_x \in \mathcal{D}'(\mathbb{R}^d)$  indexed by  $x \in \mathbb{R}^d$ , or equivalently a map  $F : \mathbb{R}^d \rightarrow \mathcal{D}'(\mathbb{R}^d)$ , such that for all  $\psi \in \mathcal{D}$  the map  $x \mapsto F_x(\psi)$  is measurable.

We think of a germ  $F = (F_x)_{x \in \mathbb{R}^d}$  as a collection of candidate local approximations for an unknown distribution. More precisely, the problem is to find a distribution  $f \in \mathcal{D}'(\mathbb{R}^d)$  which in the proximity of any point  $x \in \mathbb{R}^d$  is well-approximated by  $F_x$ , in the sense that “ $f - F_x$  is small close to  $x$ ”. This can be made precise by requiring that for some given test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  we have

$$\lim_{\lambda \downarrow 0} |(f - F_x)(\varphi_x^\lambda)| = 0 \quad \text{uniformly for } x \text{ in compact sets.} \quad (4.1)$$

Remarkably, this property is enough to guarantee *uniqueness*. The simple proof of the next result is given in Section 7 below.

**Lemma 4.2 (Uniqueness).** Given any germ  $F = (F_x)_{x \in \mathbb{R}^d}$  and any test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$ , there is at most one distribution  $f \in \mathcal{D}'$  which satisfies (4.1).

More precisely, given a compact set  $K \subseteq \mathbb{R}^d$  and two distributions  $f_1, f_2 \in \mathcal{D}'$  such that  $\lim_{\lambda \downarrow 0} |(f_i - F_x)(\varphi_x^\lambda)| = 0$  uniformly for  $x \in K$ , then  $f_1$  and  $f_2$  must “coincide on  $K$ ”, in the sense that  $f_1(\psi) = f_2(\psi)$  for any  $\psi \in \mathcal{D}(K)$ .

**COHERENCE.** Given a germ  $F = (F_x)_{x \in \mathbb{R}^d}$ , we now investigate the *existence* of a distribution  $f \in \mathcal{D}'$  which satisfies (4.1). The key to solving this problem is the following condition, that we call *coherence*.

**Definition 4.3 (Coherent germ).** Fix  $\gamma \in \mathbb{R}$ . A germ  $F = (F_x)_{x \in \mathbb{R}^d}$  is called  $\gamma$ -coherent if there is a test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  with the following property: for any compact set  $K \subseteq \mathbb{R}^d$  there is a real number  $\alpha_K \leq \min\{0, \gamma\}$  such that

$$|(F_z - F_y)(\varphi_y^\varepsilon)| \lesssim \varepsilon^{\alpha_K} (|z - y| + \varepsilon)^{\gamma - \alpha_K} \quad (4.2)$$

uniformly for  $z, y \in K$  and for  $\varepsilon \in (0, 1]$ .

If  $\alpha = (\alpha_K)$  is the family of exponents in (4.2), we say that  $F$  is  $(\alpha, \gamma)$ -coherent. If  $\alpha_K = \alpha$  for every  $K$ , we say that  $F$  is  $(\alpha, \gamma)$ -coherent.

We can already state a preliminary version of the Reconstruction Theorem.

**Theorem 4.4 (Reconstruction Theorem, preliminary version).** Let  $\gamma \in \mathbb{R}$  and  $F = (F_x)_{x \in \mathbb{R}^d}$  a  $\gamma$ -coherent germ as in Definition 4.3. Then there exists a distribution  $f = \mathcal{R}F \in \mathcal{D}'(\mathbb{R}^d)$  such that, for any given test function  $\xi \in \mathcal{D}$ , we have

$$|(f - F_x)(\xi_x^\lambda)| \lesssim \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0 \\ 1 + |\log \lambda| & \text{if } \gamma = 0 \end{cases} \quad (4.3)$$

uniformly for  $x$  in compact sets and  $\lambda \in (0, 1]$ .

If  $\gamma > 0$ , the distribution  $f$  is unique and we call it the reconstruction of  $F$ .

The Reconstruction Theorem will be stated in full in Section 5 below, with a strengthened version of relation (4.3) which holds *uniformly* over a suitable class of test functions  $\xi$ . We first need to investigate the notion of coherence.

**Remark 4.5.** The coherence condition (4.2) is a strong constraint on the germ. Indeed, we can equivalently rewrite this condition as follows

$$|(F_z - F_y)(\varphi_y^\varepsilon)| \lesssim \begin{cases} \varepsilon^\gamma & \text{if } 0 \leq |z - y| \leq \varepsilon \\ \varepsilon^{\alpha_K} |z - y|^{\gamma - \alpha_K} & \text{if } |z - y| > \varepsilon. \end{cases}$$

In particular, as  $|z - y|$  decreases from 1 to  $\varepsilon$ , the right hand side *improves* from  $\varepsilon^{\alpha_K}$  to  $\varepsilon^\gamma$ , since  $\alpha_K \leq \gamma$ . In the case  $\alpha_K < 0 < \gamma$  this improvement is particularly dramatic, since, as  $\varepsilon \downarrow 0$ ,  $\varepsilon^{\alpha_K}$  *diverges* while  $\varepsilon^\gamma$  *vanishes*.

**Remark 4.6 (Monotonicity of  $\alpha_K$ ).** Without any real loss of generality, we will always assume that the family of exponents  $\alpha = (\alpha_K)$  in (4.2) is monotone:

$$\forall K \subseteq K' : \quad \alpha_K \geq \alpha_{K'}. \quad (4.4)$$

This is natural, because the right hand side of (4.2) is non-increasing in  $\alpha_K$ . Indeed, starting from an arbitrary family  $\alpha = (\alpha_K)$  for which (4.2) holds, we can easily build a *monotone* family  $\tilde{\alpha} = (\tilde{\alpha}_K)$  for which (4.2) still holds, e.g. as follows:

- for balls  $B(0, n)$  of radius  $n \in \mathbb{N}$  we define  $\tilde{\alpha}_{B(0, n)} := \min\{\alpha_{B(0, i)} : i = 1, \dots, n\}$ ;
- for general compact sets  $K$  we first define  $n_K := \min\{n \in \mathbb{N} : K \subseteq B(0, n)\}$  and then  $\tilde{\alpha}_K := \tilde{\alpha}_{B(0, n_K)}$ .

**Remark 4.7 (Vector space).** We stress that the coherence condition (4.2) is required to hold for a single arbitrary test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$ . We will show in Proposition 13.1 the non obvious fact that  $\varphi$  in (4.2) can be replaced by any test function  $\xi \in \mathcal{D}$ , provided we also replace  $\alpha_K$  by  $\alpha'_K := \alpha_{\bar{K}_2}$ , where  $\bar{K}_R$  denotes the  $R$ -enlargement of the set  $K$ , see (2.1). It follows that, for any given  $\gamma \in \mathbb{R}$ , the family of  $\gamma$ -coherent germs is a vector space.

**Remark 4.8 (Cutoffs).** In the coherence condition (4.2) we could require that the base points  $z, y$  are at bounded distance. Indeed, if (4.2) holds when  $|z - y| \leq R$  for some fixed  $R > 0$ , then the constraint  $|z - y| \leq R$  can be dropped and (4.2) still holds (possibly with a different multiplicative constant). Similarly, the constraint  $\varepsilon \in (0, 1]$  can be replaced by  $\varepsilon \in (0, \eta]$ , for any fixed  $\eta > 0$ . The proof is left as an exercise.

It is convenient to introduce a semi-norm which quantifies the coherence of a germ. Fix a compact set  $K \subseteq \mathbb{R}^d$ , a test function  $\varphi \in \mathcal{D}$  and two real numbers  $\alpha_K \leq 0$ ,  $\gamma \geq \alpha_K$ . Given an arbitrary germ  $F = (F_x)_{x \in \mathbb{R}^d}$ , we denote by  $\|F\|_{K, \varphi, \alpha_K, \gamma}^{\text{coh}}$  the best (possibly infinite) constant for which the inequality (4.2) holds for  $y, z \in K$  with  $|z - y| \leq 2$  (this last restriction is immaterial, by Remark 4.8):

$$\|F\|_{K, \varphi, \alpha_K, \gamma}^{\text{coh}} := \sup_{y, z \in K, |z-y| \leq 2, \varepsilon \in (0, 1]} \frac{|(F_z - F_y)(\varphi_y^\varepsilon)|}{\varepsilon^{\alpha_K} (|z - y| + \varepsilon)^{\gamma - \alpha_K}}. \quad (4.5)$$

Then, given  $\gamma \in \mathbb{R}$  and  $\alpha = (\alpha_K)$ , a germ  $F$  is  $(\alpha, \gamma)$ -coherent if and only if for some  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  we have  $\|F\|_{K, \varphi, \alpha_K, \gamma}^{\text{coh}} < \infty$  for every compact set  $K$ .

**EXAMPLES.** We now present a few concrete examples of germs.

**Example 4.9 (Constant germ).** Let us fix any distribution  $T \in \mathcal{D}'$  and set  $F_x := T$  for all  $x \in \mathbb{R}^d$ . Then  $F = (F_x)_{x \in \mathbb{R}^d}$  is a  $(\alpha, \gamma)$ -coherent germ for any  $(\alpha, \gamma)$ , since  $F_z - F_y = 0$  for all  $z, y \in \mathbb{R}^d$ . Although this example may look trivial, it does occur in regularity structures, in particular for some notable elements of negative homogeneity (see Lemma 4.12 below).

**Example 4.10 (A link with Regularity Structures).** Let  $\varphi \in \mathcal{D}$  be a fixed test function with  $\int \varphi \neq 0$ . Let  $A \subset \mathbb{R}$  be a finite set and set  $\alpha := \min A$ . Let  $F = (F_x)_{x \in \mathbb{R}^d}$  be a germ such that, for some  $\gamma > \alpha$ , we have

$$|(F_z - F_y)(\varphi_y^\varepsilon)| \lesssim \sum_{a \in A: a < \gamma} \varepsilon^a |z - y|^{\gamma - a} \quad (4.6)$$

uniformly for  $z, y$  in compact sets and for  $\varepsilon \in (0, 1]$ .

Then the germ  $F$  is  $(\alpha, \gamma)$ -coherent. Indeed, it suffices to note that for  $\alpha \leq a < \gamma$

$$\varepsilon^a |z - y|^{\gamma - a} = \varepsilon^\alpha (\varepsilon^{a - \alpha} |z - y|^{\gamma - a}) \leq \varepsilon^\alpha (\varepsilon + |z - y|)^{\gamma - \alpha},$$

simply because  $v^\beta w^\delta \leq (v + w)^{\beta + \delta}$  for any  $v, w, \beta, \delta \geq 0$ .

All germs which appear in Regularity Structures satisfy (4.6). For readers who are familiar with this theory, the precise link is the following: given a Regularity

Structure  $(A, \mathcal{T}, G)$ , if  $(\Pi_x, \Gamma_{xy})_{x,y \in \mathbb{R}^d}$  is a model and  $f \in \mathcal{D}^\gamma$  is a modelled distribution, then the germ  $(F_x := \Pi_x f(x))_{x \in \mathbb{R}^d}$  satisfies (4.6) since one can write

$$(\Pi_z f(z) - \Pi_y f(y))(\varphi_y^\varepsilon) = -\Pi_y(f(y) - \Gamma_{yz} f(z))(\varphi_y^\varepsilon) = \sum_{|\tau| < \gamma} g_{zy}^\tau \Pi_y \tau(\varphi_y^\varepsilon)$$

with  $|\Pi_y \tau(\varphi_y^\varepsilon)| \lesssim \varepsilon^{|\tau|}$ , which holds for all models, and  $|g_{zy}^\tau| \lesssim |z - y|^{\gamma - |\tau|}$ , which holds for all modelled distributions.

**Example 4.11 (Taylor polynomials).** Let  $\gamma > 0$  and fix a function  $f \in \mathcal{C}^\gamma(\mathbb{R}^d)$ . We recall that by (2.3) we have  $|f(w) - F_y(w)| \lesssim |w - y|^\gamma$  for  $w, y$  in compact sets, where for all  $y \in \mathbb{R}^d$  the function  $F_y \in C^\infty(\mathbb{R}^d)$  given by

$$F_y(w) := \sum_{|k| < \gamma} \partial^k f(y) \frac{(w - y)^k}{k!}, \quad w \in \mathbb{R}^d,$$

is the Taylor polynomial of  $f$  centered at  $y$  of order  $\underline{r}(\gamma) := \max\{n \in \mathbb{N}_0 : n < \gamma\}$  defined in (2.4). Let us now show that  $F = (F_x)_{x \in \mathbb{R}^d}$  is a  $(0, \gamma)$ -coherent germ.

Fix a compact set  $K \subset \mathbb{R}^d$ . Note that for every  $k \in \mathbb{N}_0^d$  such that  $|k| < \gamma$  we have  $\partial^k f \in \mathcal{C}^{\gamma - |k|}$ . By Taylor expanding  $\partial^k f(y)$  around  $z$ , we obtain

$$F_y(w) = \sum_{|k| < \gamma} \left( \sum_{|\ell| < \gamma - |k|} \partial^{k+\ell} f(z) \frac{(y - z)^\ell}{\ell!} + R^k(y, z) \right) \frac{(w - y)^k}{k!}$$

with  $|R^k(y, z)| \lesssim |y - z|^{\gamma - |k|}$  uniformly for  $y, z \in K$ . We change variable in the inner sum from  $\ell$  to  $k' := k + \ell$  and note that the constraint  $|\ell| < \gamma - |k|$  becomes  $\{|k'| < \gamma\} \cap \{k' \geq k\}$ , where  $k' \geq k$  means  $k'_i \geq k_i \forall i = 1, \dots, d$ . If we interchange the two sums we then get, by the binomial theorem,

$$\begin{aligned} F_y(w) &= \sum_{|k'| < \gamma} \partial^{k'} f(z) \left( \sum_{k \leq k'} \frac{(y - z)^{k' - k}}{(k' - k)!} \frac{(w - y)^k}{k!} \right) + \sum_{|k| < \gamma} R^k(y, z) \frac{(w - y)^k}{k!} \\ &= F_z(w) + \sum_{|k| < \gamma} R^k(y, z) \frac{(w - y)^k}{k!}. \end{aligned}$$

Therefore

$$F_z(w) - F_y(w) = - \sum_{|k| < \gamma} R^k(y, z) \frac{(w - y)^k}{k!} \quad (4.7)$$

and since  $|R^k(z, y)| \lesssim |y - z|^{\gamma - |k|}$  we get

$$|F_z(w) - F_y(w)| \lesssim \sum_{|k| < \gamma} |w - y|^{|k|} |y - z|^{\gamma - |k|}.$$

Therefore, for any  $\varphi \in \mathcal{D}$  we have, uniformly for  $y, z \in K$ ,

$$\left| \int_{\mathbb{R}^d} (F_z(w) - F_y(w)) \varphi_y^\varepsilon(w) dw \right| \lesssim \sum_{n < \gamma} |z - y|^{\gamma - n} \varepsilon^n.$$



This is a particular case of the class studied in Example 4.10, with  $\alpha = 0$  and  $A = \{n \in \mathbb{N}_0 : n < \gamma\}$ , therefore the germ  $F$  is  $(0, \gamma)$ -coherent. General germs are meant to be a generalisation of local Taylor expansions.

**HOMOGENEITY.** For a coherent germ  $F = (F_x)_{x \in \mathbb{R}^d}$ , we can bound  $|F_x(\varphi_x^\varepsilon)|$  as  $\varepsilon \downarrow 0$ .

**Lemma 4.12 (Homogeneity).** *Let  $F = (F_x)_{x \in \mathbb{R}^d}$  be a  $\gamma$ -coherent germ. For any compact set  $K \subseteq \mathbb{R}^d$ , there is a real number  $\beta_K < \gamma$  such that*

$$|F_x(\varphi_x^\varepsilon)| \lesssim \varepsilon^{\beta_K} \quad \text{uniformly for } x \in K \text{ and } \varepsilon \in (0, 1], \quad (4.8)$$

with  $\varphi$  as in Definition 4.3. We say that  $F$  has local homogeneity bounds  $\beta = (\beta_K)$ . If  $\beta_K = \beta$  for all  $K$ , we say that  $F$  has global homogeneity bound  $\beta$ .

The request  $\beta_K < \gamma$  is to rule out trivialities.

*Proof.* Let  $(F_x)_{x \in \mathbb{R}^d}$  be  $\gamma$ -coherent. Given a compact set  $K \subseteq \mathbb{R}^d$ , fix a point  $z \in K$ . By Remark 3.5 applied to  $T = F_z$ , see (3.4), there is  $r \in \mathbb{N}_0$  such that

$$|F_z(\varphi_x^\varepsilon)| \lesssim \varepsilon^{-r-d} \quad \text{uniformly for } x \in K \text{ and } \varepsilon \in (0, 1].$$

If we denote by  $\text{diam}(K) := \sup\{|x - z| : x, z \in K\}$ , by (4.2) we can bound

$$|(F_x - F_z)(\varphi_x^\varepsilon)| \lesssim \varepsilon^{\alpha_K} (|x - z| + \varepsilon)^{\gamma - \alpha_K} \leq \varepsilon^{\alpha_K} (\text{diam}(K) + 1)^{\gamma - \alpha_K} \lesssim \varepsilon^{\alpha_K},$$

always uniformly for  $x \in K$  and  $\varepsilon \in (0, 1]$ . This yields

$$|F_x(\varphi_x^\varepsilon)| \leq |(F_x - F_z)(\varphi_x^\varepsilon)| + |F_z(\varphi_x^\varepsilon)| \lesssim \varepsilon^{\alpha_K} + \varepsilon^{-r-d},$$

hence (4.8) holds with  $\beta_K = \min\{\alpha_K, -r-d\}$  (which, of course, might not be the best value of  $\beta_K$ ). By further decreasing  $\beta_K$ , if needed, we may ensure that  $\beta_K < \gamma$ .  $\square$

**Remark 4.13 (Monotonicity of  $\beta_K$ ).** In analogy with Remark 4.6, we will always assume that the homogeneity bounds  $\beta = (\beta_K)$  in (4.8) are *monotone*:

$$\forall K \subseteq K' : \quad \beta_K \geq \beta_{K'}. \quad (4.9)$$

Note that the right hand side of (4.8) is non-increasing in  $\beta_K$ .

**Remark 4.14 (Vector space).** We will show in Proposition 13.2 that in (4.8) we can replace  $\varphi$  by *any test function*  $\xi \in \mathcal{D}$ , provided we also replace  $\beta_K$  by  $\beta'_K := \beta_{\bar{K}_2}$ . As a consequence (recall also Remark 4.7), for any given  $\alpha \leq 0$  and  $\gamma \geq \alpha$ , *the family of  $(\alpha, \gamma)$ -coherent germs with global homogeneity bound  $\beta$  is a vector space.*

**Remark 4.15 (Positive homogeneity bounds).** In concrete applications, we typically have  $\beta_K \leq 0$  in (4.8), because the case  $\beta_K > 0$  is somewhat trivial. Indeed, we recall that given a  $\gamma$ -coherent germ  $F$ , our problem is to find a distribution  $f \in \mathcal{D}'$  that satisfies (4.1). If  $\beta_K > 0$  for some compact set  $K \subseteq \mathbb{R}^d$ , then  $f = 0$  satisfies (4.1) on  $K$  and, by Lemma 4.2, any solution  $f$  of (4.1) must therefore vanish on  $K$ . In particular, *if  $\beta_K > 0$  for all  $K$ , the only solution to (4.1) is  $f = 0$ .* Using the notation of the Reconstruction Theorem, we can write  $\mathcal{R}F = 0$ .

**Example 4.16.** For a coherent germ there is in general no fixed order between the lower bound  $\beta_K$  of the homogeneity in (4.8) and the exponent  $\alpha_K$  appearing in the coherence definition (4.2).

- In Regularity Structures, see Example 4.10, it is usually assumed that  $\beta_K = \alpha_K = \alpha$  for all  $K$ .
- A constant germ  $F_x = T$  with  $T \in \mathcal{D}'$ , see Example 4.9, is  $(\alpha, \gamma)$ -coherent for any  $\alpha$  and  $\gamma$ . It is possible that  $\beta_K < 0$ , e.g. for the function  $T(y) := |y|^{-1/2}$  we have  $\beta_K = -\frac{1}{2}$  for  $K = B(0, 1)$ . Since we can choose  $\alpha_K = 0$  here, we might have  $\beta_K < \alpha_K$ .
- If  $F$  is a  $(\alpha, \gamma)$ -coherent germ, then for any fixed distribution  $f \in \mathcal{D}'$  the germ  $G = (G_x := f - F_x)_{x \in \mathbb{R}^d}$  is still  $(\alpha, \gamma)$ -coherent. By the Reconstruction Theorem that we are about to state, it is possible to choose  $f = \mathcal{R}F$  such that for the germ  $G$  we have that  $\beta_K \geq \gamma$  (see (4.3) below), hence  $\beta_K \geq \alpha_K$ .

## 5. THE RECONSTRUCTION THEOREM

We are ready to state the full version of Hairer's *Reconstruction Theorem* [Hai14, Th. 3.10] in our context (see also [Hai14, Prop. 3.25]). Recalling the definition (2.2) of  $\|\cdot\|_{C^r}$ , for  $r \in \mathbb{N}_0$  we define the following family of test functions:

$$\mathcal{B}_r := \{\psi \in \mathcal{D}(B(0, 1)) : \|\psi\|_{C^r} \leq 1\}. \quad (5.1)$$

We also recall that  $\bar{K}_R$  denotes the  $R$ -enlargement of the set  $K$ , see (2.1).

**Theorem 5.1 (Reconstruction Theorem).** *Let  $\gamma \in \mathbb{R}$  and  $F = (F_x)_{x \in \mathbb{R}^d}$  be a  $(\alpha, \gamma)$ -coherent germ as in Definition 4.3 with local homogeneity bounds  $\beta$ , see Lemma 4.12. Then there exists a distribution  $f \in \mathcal{D}'(\mathbb{R}^d)$  such that for any compact set  $K \subset \mathbb{R}^d$  and any integer  $r > \max\{-\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}$  we have, for  $\alpha := \alpha_{\bar{K}_2}$ ,*

$$|(f - F_x)(\psi_x^\lambda)| \leq \mathbf{c}_{\alpha, \gamma, r, d, \varphi} \|F\|_{\bar{K}_2, \varphi, \alpha, \gamma}^{\text{coh}} \cdot \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0 \\ (1 + |\log \lambda|) & \text{if } \gamma = 0 \end{cases} \quad (5.2)$$

uniformly for  $\psi \in \mathcal{B}_r$ ,  $x \in K$ ,  $\lambda \in (0, 1]$ ,

where the semi-norm  $\|F\|_{\bar{K}_2, \varphi, \alpha, \gamma}^{\text{coh}}$  is defined in (4.5),  $\varphi$  is as in Definition 4.3, and  $\mathbf{c}_{\alpha, \gamma, r, d, \varphi}$  is an explicit constant, see (10.39)-(11.14)-(11.15).

If  $\gamma > 0$ , such a distribution  $f = \mathcal{R}F$  is unique and we call it the reconstruction of  $F$ . Moreover the map  $F \mapsto \mathcal{R}F$  is linear.

If  $\gamma \leq 0$  the distribution  $f$  is not unique but, for any fixed  $\alpha \leq 0$  and  $\gamma \geq \alpha$ , one can choose  $f$  in such a way that the map  $F \mapsto f = \mathcal{R}F$  is linear on the vector space of  $(\alpha, \gamma)$ -coherent germs with global homogeneity bound  $\beta$ .

The strategy of our proof of the Reconstruction Theorem is close in spirit to the original proof by Hairer: given a germ  $F$ , we “paste together” the distributions  $F_x$  on smaller and smaller scales, in order to build  $\mathcal{R}F$ . The existing proofs exploit test functions possessing special multi-scale properties, such as wavelets (by Hairer

[Hai14]) or the heat kernel (by Otto-Weber [OW19]). Our proof is based on the *single arbitrary test function*  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  which appears in the coherence condition (4.2), that we will suitably tweak in order to perform multi-scale arguments.

**Remark 5.2.** Theorem 4.4 is a special case of Theorem 5.1, because equation (4.3) is a consequence of (5.2). This is obvious if  $\xi \in \mathcal{B}_r$ , while for generic  $\xi \in \mathcal{D}$  it suffices to note that  $\psi := c\xi^\eta \in \mathcal{B}_r$  for suitable  $c, \eta > 0$ , recall the notation (3.2). As a consequence, we can write  $\xi_x^\lambda = c^{-1} \psi_x^{\eta^{-1}\lambda}$  with  $\psi \in \mathcal{B}_r$ , hence (5.2) yields (4.3) for  $\varepsilon > 0$  small enough, which is enough (exercise).

**Example 5.3 (Constant germ, reprise).** If we consider the constant germ  $F_x = T$  of Example 4.9 then for  $f = T$  we have  $f - F_x = 0$  and therefore we can set  $\mathcal{R}F = T$ .

If we view a germ as a generalised local Taylor expansion, the Reconstruction Theorem associates to a *coherent* germ  $F = (F_x)$  a *global* distribution  $f$  which is approximated by the germ  $F_x$  locally around every  $x \in \mathbb{R}^d$ . If the germ is a classical Taylor expansion of a function in  $\mathcal{C}^\gamma$ , as discussed in Example 4.11, then the Reconstruction Theorem yields the function itself, as shown in the next example.

**Example 5.4 (Taylor polynomial, reprise).** Consider the germ given by the Taylor expansion of a function  $f \in \mathcal{C}^\gamma$ , see Example 4.11. Then by the Taylor theorem

$$|f(y) - F_x(y)| \lesssim |y - x|^\gamma$$

uniformly for  $x, y$  in compact sets. If  $\psi$  is supported in  $B(0, 1)$ , then  $\psi_x^\lambda$  is supported in  $B(x, \lambda)$ , therefore uniformly for  $\lambda \in (0, 1]$  we can bound

$$\left| \int_{\mathbb{R}^d} (f(y) - F_x(y)) \psi_x^\lambda(y) dy \right| \lesssim \lambda^\gamma \int |\psi_x^\lambda(y)| dy = \lambda^\gamma \int |\psi(y)| dy. \quad (5.3)$$

This shows that  $f$  satisfies (4.1), therefore by uniqueness we have  $\mathcal{R}F = f$ . As a matter of fact, relation (5.3) holds uniformly for  $\psi \in \mathcal{B}_0$  because  $\int |\psi| \lesssim \|\psi\|_\infty \leq 1$  (recall that  $\psi \in \mathcal{B}_0$  are supported in  $B(0, 1)$ ).

**Example 5.5 (On the case  $\gamma = 0$ ).** If  $F = (F_x)_{x \in \mathbb{R}^d}$  is a  $(\alpha, 0)$ -coherent germ, i.e.  $\gamma = 0$ , the estimate (5.2) in the Reconstruction Theorem reads as follows:

$$|(f - F_x)(\psi_x^\lambda)| \lesssim \log(1 + \frac{1}{\lambda}) \quad (5.4)$$

uniformly for  $x$  in compact sets,  $\psi \in \mathcal{B}_r$  and  $\lambda \in (0, 1]$ . We now show by an example that *the logarithmic rate in the right hand side of (5.4) is optimal*.

Consider the germ of functions  $F = (F_x(y) := \log(1 + \frac{1}{|y-x|}))_{x \in \mathbb{R}^d}$ . If  $\varphi \in \mathcal{D}$  is a non-negative test function supported in  $B(0, 1)$  with  $\int \varphi > 0$ , we can bound

$$|(F_z - F_y)(\varphi_y^\varepsilon)| \leq |F_z(\varphi_y^\varepsilon)| + |F_y(\varphi_y^\varepsilon)| \lesssim \log(1 + \frac{1}{\varepsilon}) \lesssim \varepsilon^\alpha \quad \text{for any given } \alpha < 0.$$

This shows that the germ  $F$  is  $(\alpha, 0)$ -coherent, hence by the Reconstruction Theorem there is  $f \in \mathcal{D}'$  such that (5.4) holds (e.g.  $f \equiv 0$ ). *We claim that this bound cannot be improved, i.e. there is no  $f \in \mathcal{D}'$  such that  $|(f - F_x)(\psi_x^\lambda)| \ll \log(1 + \frac{1}{\lambda})$ .*

By contradiction, assume that such  $f \in \mathcal{D}'$  exists. Given a test function  $\psi \geq 0$  with  $\psi(0) > 0$  and  $\int \psi = 1$ , we can bound  $F_x(\psi_x^\lambda) \gtrsim \log(1 + \frac{1}{\lambda})$  and by triangle inequality

$$f(\psi_x^\lambda) \geq F_x(\psi_x^\lambda) - |(F_x - f)(\psi_x^\lambda)| \gtrsim \log(1 + \frac{1}{\lambda})$$

uniformly for  $x$  in compact sets. In particular, there is a constant  $c > 0$  such that

$$f(\psi_x^\lambda) \geq c \log(1 + \frac{1}{\lambda}) \quad \forall x \in B(0, 2).$$

This is impossible, for the following reason. Since  $(\psi^\lambda)$  are mollifiers as  $\lambda \downarrow 0$  (recall that we have fixed  $\int \psi = 1$ ), for any given test function  $\xi \in \mathcal{D}$  we can write

$$f(\xi) = \lim_{\lambda \downarrow 0} f(\xi * \psi^\lambda) = \lim_{\lambda \downarrow 0} \int_{\mathbb{R}^d} f(\psi_x^\lambda) \xi(x) dx.$$

If we fix  $\xi \geq 0$  supported in  $B(0, 1)$  with  $\int \xi = 1$ , we finally get

$$f(\xi) \geq \lim_{\lambda \downarrow 0} \int_{\mathbb{R}^d} c \log(1 + \frac{1}{\lambda}) \xi(x) dx = \lim_{\lambda \downarrow 0} c \log(1 + \frac{1}{\lambda}) = \infty$$

which is clearly a contradiction.

**Remark 5.6.** In the original formulation of the Reconstruction Theorem [Hai14, Thm. 3.10], the estimate in the right-hand side of (5.2) for  $\gamma = 0$  contains a factor  $\lambda^\gamma$  instead of  $(1 + |\log \lambda|)$ . This is not correct, as we showed in Example 5.5. The mistake in [Hai14] is in the very last display of the proof on page 324: in this formula we have  $\|x - y\|_s \lesssim \delta + 2^{-n}$  and  $2^{-n} > \delta$ , so that the factor  $\delta^{\gamma-\beta}$  in the left-hand side must be replaced by  $2^{-(\gamma-\beta)n}$ . For  $\gamma < 0$  the result does not change, but for  $\gamma = 0$  one obtains  $1 + |\log \delta|$  instead of  $\delta^0$ .

**5.1. GUIDE TO THE PROOF OF THE RECONSTRUCTION THEOREM.** The next sections are devoted to the proof of Theorem 5.1.

- In Section 6 we show the necessity of coherence for the Reconstruction Theorem.
- In Section 7 we recall basic results on test functions (such as convergence, convolutions and mollifiers) and we prove Lemma 4.2.
- In Section 8 we show how to “tweak” an arbitrary test function, in order to ensure that it *annihilates all monomials up to a given degree*. This is a key ingredient in the proof of the Reconstruction Theorem because it will allow us to perform efficiently multi-scale arguments.
- In Section 9 we present some elementary but crucial estimates on convolutions.
- Finally, in Sections 10 and 11 we give the proof of the Reconstruction Theorem, first when  $\gamma > 0$  and then when  $\gamma \leq 0$ .

## 6. NECESSITY OF COHERENCE

If a germ  $F = (F_x)_{x \in \mathbb{R}^d}$  is  $\gamma$ -coherent, by the Reconstruction Theorem there is a distribution  $f \in \mathcal{D}'$  which is locally well approximated by  $F$ , see (5.2). In case  $\gamma \neq 0$ ,

this means the following:

$$\begin{aligned} \forall \text{ compact set } K \subseteq \mathbb{R}^d \exists r = r(K) \in \mathbb{N} \text{ such that} \\ |(f - F_x)(\psi_x^\lambda)| \lesssim \lambda^\gamma \\ \text{uniformly for } x \in K, \lambda \in (0, 1] \text{ and } \psi \in \mathcal{B}_r. \end{aligned} \quad (6.1)$$

Remarkably, coherence is also *necessary* for (6.1), as we now show.

**Theorem 6.1 (Coherence is necessary).** *If a germ  $(F_x)_{x \in \mathbb{R}^d}$  satisfies (6.1) for some  $\gamma \in \mathbb{R}$ , then it is  $\gamma$ -coherent, i.e. it satisfies the coherence condition (4.2), for any function  $\varphi \in \mathcal{D}$  and for a suitable family of exponents  $\alpha = (\alpha_K)$ .*

*If furthermore (6.1) holds with  $r(K) = r$  for every  $K$ , for a fixed  $r \in \mathbb{N}$ , then the germ  $F$  is  $(\alpha, \gamma)$ -coherent for a suitable  $\alpha \leq 0$ , i.e. we can take  $\alpha_K = \alpha$  for all  $K$ .*

This is a direct corollary of the next quantitative result.

**Proposition 6.2.** *Let  $(F_x)_{x \in \mathbb{R}^d}$  be a germ with the following property: there exist a distribution  $f \in \mathcal{D}'$ , numbers  $\gamma \in \mathbb{R}$ ,  $r \in \mathbb{N}$ ,  $C < \infty$  and a set  $K \subseteq \mathbb{R}^d$  such that*

$$\begin{aligned} |(f - F_x)(\psi_x^\lambda)| \leq C \lambda^\gamma \\ \text{for all } x \in K, \lambda \in (0, 1] \text{ and } \psi \in \mathcal{B}_r, \end{aligned} \quad (6.2)$$

*with  $\mathcal{B}_r$  defined in (5.1). Then for  $\alpha := \min\{-r - d, \gamma\}$  we have*

$$\begin{aligned} |(F_z - F_y)(\psi_y^\lambda)| \leq 2C \lambda^\alpha (|z - y| + \lambda)^{\gamma - \alpha} \\ \text{for all } y, z \in K \text{ with } |z - y| \leq \frac{1}{2}, \lambda \in (0, \frac{1}{2}] \text{ and } \psi \in \mathcal{B}_r. \end{aligned} \quad (6.3)$$

*Proof.* For  $y, z \in K$ ,  $\lambda \in (0, 1]$  and  $\psi \in \mathcal{B}_r$ . By (6.2) we can estimate

$$\begin{aligned} |(F_z - F_y)(\psi_y^\lambda)| &= |(f - F_y)(\psi_y^\lambda) - (f - F_z)(\psi_y^\lambda)| \\ &\leq |(f - F_y)(\psi_y^\lambda)| + |(f - F_z)(\psi_y^\lambda)| \\ &\leq C \lambda^\gamma + |(f - F_z)(\psi_y^\lambda)|. \end{aligned}$$

We claim that for  $|z - y| \leq \frac{1}{2}$  and  $\lambda \in (0, \frac{1}{2}]$  we can bound

$$|(f - F_z)(\psi_y^\lambda)| \leq C \left( \frac{\lambda}{|z - y| + \lambda} \right)^{-r - d} (|z - y| + \lambda)^\gamma. \quad (6.4)$$

Note that for any  $\alpha \leq \gamma$  we can estimate  $\lambda^\gamma = \lambda^\alpha \lambda^{\gamma - \alpha} \leq \lambda^\alpha (|z - y| + \lambda)^{\gamma - \alpha}$ , therefore if we set  $\alpha := \min\{-r - d, \gamma\}$  we obtain (6.3).

It remains to prove (6.4). Estimating  $|(f - F_z)(\psi_y^\lambda)|$  is non obvious because  $\psi_y^\lambda$  is centered at  $y$  rather than  $z$ . However, we claim that we can write

$$\psi_y^\lambda = \xi_z^{\lambda_1} \quad \text{where} \quad \xi := \psi_w^{\lambda_2}, \quad (6.5)$$

where  $\lambda_1, \lambda_2 \in (0, 1]$  and  $w \in B(0, 1)$  are defined as follows:

$$\lambda_1 := |z - y| + \lambda, \quad \lambda_2 := \frac{\lambda}{|z - y| + \lambda}, \quad w := \frac{y - z}{|z - y| + \lambda}.$$

To prove (6.5), recall that  $\xi_z^{\lambda_1}(x) = \lambda_1^{-d} \xi(\lambda_1^{-1}(x - z))$ , hence for  $\xi = \psi_w^{\lambda_2}$  we get

$$\begin{aligned} \xi_z^{\lambda_1}(x) &= \lambda_1^{-d} \psi_w^{\lambda_2}(\lambda_1^{-1}(x - z)) = \lambda_1^{-d} \lambda_2^{-d} \psi(\lambda_2^{-1}\{\lambda_1^{-1}(x - z) - w\}) \\ &= (\lambda_1 \lambda_2)^{-d} \psi((\lambda_1 \lambda_2)^{-1}\{(x - z) - \lambda_1 w\}) = \lambda^{-d} \psi(\lambda^{-1}\{x - y\}) = \psi_y^\lambda(x). \end{aligned}$$

Note that  $\xi = \psi_w^{\lambda_2}$  is supported in  $B(w, \lambda_2) \subseteq B(0, 1)$ , because  $|w| + \lambda_2 \leq 1$  and  $\psi$  is supported in  $B(0, 1)$ . Since  $\xi$  is supported in  $B(0, 1)$ , we have  $\xi/\|\xi\|_{C^r} \in \mathcal{B}_r$ , hence we can apply equation (6.2) with the replacements

$$x \rightsquigarrow z, \quad \psi \rightsquigarrow \xi/\|\xi\|_{C^r}, \quad \lambda \rightsquigarrow \lambda_1$$

(note that  $\lambda_1 \in (0, 1]$  if  $|z - y| \leq \frac{1}{2}$  and  $\lambda \in (0, \frac{1}{2}]$ ). This yields

$$|(f - F_z)(\xi_z^{\lambda_1})| \leq C (\lambda_1)^\gamma \|\xi\|_{C^r}. \quad (6.6)$$

It remains to bound

$$\|\xi\|_{C^r} = \|\psi_w^{\lambda_2}\|_{C^r} = \max_{|k| \leq r} \|\partial^k \psi_w^{\lambda_2}\|_\infty = \max_{|k| \leq r} \|\lambda_2^{-|k|-d} \partial^k \psi\|_\infty \leq \lambda_2^{-r-d},$$

because  $\max_{|k| \leq r} \|\partial^k \psi\|_\infty = \|\psi\|_{C^r} \leq 1$  for  $\psi \in \mathcal{B}_r$ . By (6.5) and (6.6), we get (6.4).  $\square$

## 7. CONVERGENCE OF TEST FUNCTIONS, CONVOLUTIONS AND MOLLIFIERS

The space of test functions  $\mathcal{D}$  is equipped with a strong notion of convergence.

**Definition 7.1 (Convergence of test functions).** *We say that  $\varphi_n \rightarrow \varphi$  in  $\mathcal{D}$  if and only if the following two conditions hold:*

- (1) *all  $\varphi_n$ 's are supported in some fixed compact set  $K$ , i.e.  $\varphi_n \in \mathcal{D}(K) \forall n$ ;*
- (2)  *$\varphi_n$  converges to  $\varphi$  uniformly with all derivatives:*

$$\forall r \in \mathbb{N}_0 : \quad \|\varphi_n - \varphi\|_{C^r} \rightarrow 0.$$

*We typically consider sequences indexed by  $n \in \mathbb{N}$ , with convergence as  $n \rightarrow \infty$ , or continuous families indexed by  $n = \lambda \in (0, 1]$ , with convergence as  $\lambda \downarrow 0$ .*

**Remark 7.2.** This notion of convergence is induced by a natural topology on  $\mathcal{D}$ , called *locally convex inductive limit topology*. It is quite subtle – non metrizable, not even first countable – but we will not need to use it directly.

We now show that the ‘‘continuity property’’ (3.1) in the definition of a distribution corresponds to ‘‘sequential continuity’’ with respect to convergence in  $\mathcal{D}$ .<sup>†</sup>

**Lemma 7.3.** *A linear functional  $T : \mathcal{D}(\mathbb{R}^d) \rightarrow \mathbb{R}$  is a distribution if and only if*

$$\varphi_n \rightarrow \varphi \text{ in } \mathcal{D} \quad \text{implies} \quad T(\varphi_n) \rightarrow T(\varphi). \quad (7.1)$$

<sup>†</sup>If a map  $T : \mathcal{D} \rightarrow \mathbb{R}$  is sequentially continuous, i.e. it satisfies (7.1), this does *not* imply that  $T$  is a continuous map, because the topology on  $\mathcal{D}$  is not first countable (recall Remark 7.2). However, if  $T$  is a *linear map*, then sequential continuity implies continuity.

*Proof.* By the definition of convergence in  $\mathcal{D}$ , it is clear that (3.1) implies (7.1). Vice versa, if (3.1) fails for some compact  $K$ , then for every  $r = n \in \mathbb{N}$  and  $C = n \in \mathbb{N}$  we can find  $\varphi_n \in \mathcal{D}(K)$  such that  $|T(\varphi_n)| > n \|\varphi_n\|_{C^n}$ ; if we define  $\psi_n := n^{-1} \varphi_n / \|\varphi_n\|_{C^n}$ , we have  $|T(\psi_n)| > 1$  for every  $n \in \mathbb{N}$ , which contradicts (7.1) because  $\psi_n \rightarrow 0$  in  $\mathcal{D}$  (indeed, for any fixed  $r \in \mathbb{N}$  we have  $\|\psi_n\|_{C^r} \leq n^{-1}$  as soon as  $n \geq r$ ).  $\square$

We recall that the convolution of two measurable functions  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$  is the function  $f * g = g * f : \mathbb{R}^d \rightarrow \mathbb{R}$  defined by

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x - y) g(y) dy = \int_{\mathbb{R}^d} f(z) g(x - z) dz, \quad (7.2)$$

provided the integral makes sense for almost every  $x \in \mathbb{R}^d$ . This holds, in particular, when  $f = \varphi \in \mathcal{D}$  is a test function and  $g$  is *locally integrable and compactly supported*: in this case the convolution  $\varphi * g \in \mathcal{D}$  is a test function too, and we have

$$\partial^k(\varphi * g) = (\partial^k \varphi) * g. \quad (7.3)$$

Given any distribution  $T \in \mathcal{D}'$ , we can compute

$$T(\varphi * g) = \int_{\mathbb{R}^d} T(\varphi(\cdot - y)) g(y) dy,$$

as one can deduce from (7.2) (e.g. by linearity and Riemann sum approximations). If we set  $\varphi_y(x) := \varphi(x - y) = \varphi_y^1(x)$ , recall (3.2), we obtain the basic formula

$$T(\varphi * g) = \int_{\mathbb{R}^d} T(\varphi_y) g(y) dy, \quad (7.4)$$

that will be used repeatedly in the sequel.

We next state a classical result that will be used frequently.

**Lemma 7.4 (Mollifiers).** *Let  $\rho : \mathbb{R}^d \rightarrow \mathbb{R}$ , with  $\int \rho = 1$  be compactly supported and integrable. Then  $\rho^\varepsilon(z) := \rho_0^\varepsilon(z) := \varepsilon^{-d} \rho(\varepsilon^{-1} z)$  are mollifiers as  $\varepsilon \downarrow 0$ , i.e.*

$$\forall \varphi \in \mathcal{D} : \quad \varphi * \rho^\varepsilon \rightarrow \varphi \quad \text{in } \mathcal{D} \text{ as } \varepsilon \downarrow 0.$$

*Proof.* By (7.3) and  $\int \rho^\varepsilon = \int \rho = 1$  we can write, for any multi-index  $k$ ,

$$\partial^k(\varphi * \rho^\varepsilon)(x) - \partial^k \varphi(x) = \int_{\mathbb{R}^d} (\partial^k \varphi(x - y) - \partial^k \varphi(x)) \rho^\varepsilon(y) dy,$$

hence, by the change of variables  $y = \varepsilon z$ ,

$$\begin{aligned} |\partial^k(\varphi * \rho^\varepsilon)(x) - \partial^k \varphi(x)| &\leq \int_{\mathbb{R}^d} |\partial^k \varphi(x - y) - \partial^k \varphi(x)| |\rho^\varepsilon(y)| dy \\ &= \int_{\mathbb{R}^d} |\partial^k \varphi(x - \varepsilon z) - \partial^k \varphi(x)| |\rho(z)| dz. \end{aligned} \quad (7.5)$$

Fix a compact set  $K \subseteq \mathbb{R}^d$  and take  $x \in K$ . Since  $\rho$  is compactly supported, say on the ball  $B(0, R)$ , for  $\varepsilon \in (0, 1)$  the variable  $x - \varepsilon z$  belongs to the compact set  $K_R$ , the  $R$ -neighborhood of  $K$ . Then we can bound  $|\partial^k \varphi(x - \varepsilon z) - \partial^k \varphi(x)| \lesssim \varepsilon |z|$ , because

$\partial^k \varphi$  is of class  $C^1$  (in fact  $C^\infty$ ). Since  $\int |z| |\rho(z)| dz < \infty$ , it follows by (7.5) that  $\sup_{x \in K} |\partial^k(\varphi * \rho^\varepsilon)(x) - \partial^k \varphi(x)| \lesssim \varepsilon \rightarrow 0$ . This shows that  $\varphi * \rho^\varepsilon \rightarrow \varphi$  in  $\mathcal{D}$ .  $\square$

We finally give the easy proof of Lemma 4.2 (Uniqueness).

*Proof of Lemma 4.2.* Let  $\gamma > 0$ . We fix a germ  $(F_x)_{x \in \mathbb{R}^d}$ , a test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$ , a compact set  $K \subseteq \mathbb{R}^d$  and two distributions  $f, g \in \mathcal{D}'$  which satisfy, uniformly for  $x \in K$ ,

$$\lim_{\lambda \downarrow 0} |(f - F_x)(\varphi_x^\lambda)| = \lim_{\lambda \downarrow 0} |(g - F_x)(\varphi_x^\lambda)| = 0. \quad (7.6)$$

Our goal is to show that  $f(\psi) = g(\psi)$  for every test function  $\psi$  supported in  $K$ , i.e.  $\psi \in \mathcal{D}(K)$ . We may assume that  $c := \int \varphi = 1$  (otherwise just replace  $\varphi$  by  $c^{-1} \varphi$ ).

We set  $T := f - g$ , we fix a test function  $\psi \in \mathcal{D}(K)$  and we show that  $T(\psi) = 0$ . We have  $T(\psi) = \lim_{\lambda \downarrow 0} T(\psi * \varphi^\lambda)$  by Lemma 7.3, because  $\lim_{\lambda \downarrow 0} \psi * \varphi^\lambda = \psi$  in  $\mathcal{D}$  by Lemma 7.4. Recalling (7.4), we can write

$$|T(\psi * \varphi^\lambda)| = \left| \int_{\mathbb{R}^d} T(\varphi_x^\lambda) \psi(x) dx \right| \leq \|\psi\|_{L^1} \sup_{x \in K} |T(\varphi_x^\lambda)|,$$

where the last inequality holds for any  $\lambda > 0$  since  $\psi$  is supported in  $K$ . It remains to show that  $\lim_{\lambda \downarrow 0} T(\varphi_x^\lambda) = 0$  uniformly for  $x \in K$ , for which it is enough to observe that

$$|T(\varphi_x^\lambda)| = |f(\varphi_x^\lambda) - g(\varphi_x^\lambda)| \leq |(f - F_x)(\varphi_x^\lambda)| + |(g - F_x)(\varphi_x^\lambda)|$$

and these terms vanish as  $\lambda \downarrow 0$  uniformly for  $x \in K$ , by (7.6).  $\square$

## 8. TWEAKING A TEST FUNCTION

Given an arbitrary test function  $\varphi$  and an integer  $r \in \mathbb{N}$ , we build a “tweaked” test function  $\hat{\varphi}$  which annihilates monomials of degree from 1 to  $r - 1$ . Recall that  $\varphi^\lambda$  denotes the function  $\varphi^\lambda(x) := \lambda^{-d} \varphi(\lambda^{-1}x)$ .

**Lemma 8.1 (Tweaking).** *Fix  $r \in \mathbb{N} = \{1, 2, \dots\}$  and distinct  $\lambda_0, \lambda_1, \dots, \lambda_{r-1} \in (0, \infty)$ . Define the constants  $c_0, c_1, \dots, c_{r-1} \in \mathbb{R}$  as follows:*

$$c_i = \prod_{k \in \{0, \dots, r-1\}: k \neq i} \frac{\lambda_k}{\lambda_k - \lambda_i} \quad (8.1)$$

(when  $r = 1$  we agree that  $c_0 := 1$ ). Then, for any measurable and compactly supported  $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$  and any  $a \in \mathbb{R}$ , the “tweaked” function  $\hat{\varphi}$  defined by

$$\hat{\varphi} := a \sum_{i=0}^{r-1} c_i \varphi^{\lambda_i} \quad (8.2)$$

has integral equal to  $a \int \varphi$  and annihilates monomials of degree from 1 to  $r - 1$ :

$$\int \hat{\varphi} = a \int \varphi \quad \text{and} \quad \int_{\mathbb{R}^d} y^k \hat{\varphi}(y) dy = 0, \quad \forall k \in \mathbb{N}_0^d : 1 \leq |k| \leq r - 1. \quad (8.3)$$



**Remark 8.2.** For fixed  $a \in \mathbb{R}$ , equation (8.3) is a set of conditions, one for each  $k \in (\mathbb{N}_0)^d$  with  $|k| \leq r - 1$  (where  $k = 0$  corresponds to  $\int \hat{\varphi} = a \int \varphi$ ). The number of such conditions equals  $r$  for  $d = 1$ , while *it is strictly larger than  $r$  for  $d \geq 2$* . Nevertheless, we can fulfill these conditions by choosing only  $r$  variables  $c_0, c_1, \dots, c_{r-1}$  as in (8.1). This is due to the scaling properties of monomials.

We now show that in the coherence condition (4.2) we can replace  $\varphi \in \mathcal{D}$  by a suitable  $\hat{\varphi}$  as in Lemma 8.1. Assume that for some  $R_\varphi < \infty$  we have that

$$\int \varphi \neq 0, \quad \varphi \text{ is supported in } B(0, R_\varphi).$$

Then, given  $r \in \mathbb{N}$ , we define  $\hat{\varphi} = \hat{\varphi}^{[r]}$  by (8.2) for  $a = 1/\int \varphi$  and for suitable  $\lambda_i$ 's:

$$\hat{\varphi} := \frac{1}{\int \varphi} \sum_{i=0}^{r-1} c_i \varphi^{\lambda_i} \quad \text{where} \quad \lambda_i := \frac{2^{-i-1}}{1 + R_\varphi} \quad \text{and } c_i \text{ as in (8.1)}. \quad (8.4)$$

**Lemma 8.3.** *Let  $F = (F_x)_{x \in \mathbb{R}^d}$  be a  $(\alpha, \gamma)$ -coherent germ as in Definition 4.3. For any  $r \in \mathbb{N}$ , the coherence condition (4.2) still holds if  $\varphi$  is replaced by  $\hat{\varphi} = \hat{\varphi}^{[r]}$  defined in (8.4). Such a test function  $\hat{\varphi}$  has the following properties:*

$$\hat{\varphi} \text{ is supported in } B(0, \frac{1}{2}), \quad (8.5)$$

$$\int_{\mathbb{R}^d} \hat{\varphi}(y) dy = 1, \quad \int_{\mathbb{R}^d} y^k \hat{\varphi}(y) dy = 0 \quad \text{for } 1 \leq |k| \leq r - 1, \quad (8.6)$$

$$\|\hat{\varphi}\|_{L^1} \leq \frac{e^2 r}{|\int \varphi|} \|\varphi\|_{L^1}. \quad (8.7)$$

*Proof.* The function  $\hat{\varphi}$  is supported in  $B(0, 1/2)$  because  $\lambda_i \leq \frac{1}{2R_\varphi}$ . Relation (8.6) holds by (8.3). To prove (8.7), note that by (8.1) we can bound

$$|c_i| = \prod_{k \in \{0, \dots, r-1\}: k \neq i} \frac{1}{|1 - 2^{k-i}|} \leq \prod_{m=1}^{\infty} \frac{1}{1 - 2^{-m}} \leq \prod_{m=1}^{\infty} (1 + 2^{-m}) \leq e^2, \quad (8.8)$$

because  $|1 - 2^{k-i}| \geq 1$  for  $k > i$  and  $(1 - x)^{-1} \leq 1 + 2x \leq e^{2x}$  for  $0 \leq x \leq \frac{1}{2}$ . This bound proves (8.7), by (8.4) and the fact that  $\|\varphi^{\lambda_i}\|_{L^1} = \|\varphi\|_{L^1}$ .  $\square$

*Proof of Lemma 8.1.* If  $r = 1$  equation (8.3) reduces to  $\int \hat{\varphi} = \int \varphi$ , which holds because  $\hat{\varphi} = \varphi^{\lambda_0}$  (recall that  $c_0 = 1$  when  $r = 1$ ). Henceforth we fix  $r \in \mathbb{N}$  with  $r \geq 2$ .

Fix distinct  $\lambda_0, \lambda_1, \dots, \lambda_{r-1} \in (0, \infty)$  and define  $c_0, c_1, \dots, c_{r-1}$  by (8.1). Define  $\hat{\varphi}$  by (8.2). For any multi-index  $k \in \mathbb{N}_0^d$ , since  $y^k := y_1^{k_1} y_2^{k_2} \dots y_d^{k_d}$ , we can compute

$$\int_{\mathbb{R}^d} y^k \hat{\varphi}(y) dy = \sum_{i=0}^{r-1} c_i \int_{\mathbb{R}^d} y^k \lambda_i^{-d} \varphi(\lambda_i^{-1} y) dy = \left( \sum_{i=0}^{r-1} c_i \lambda_i^{|k|} \right) \int_{\mathbb{R}^d} x^k \varphi(x) dx,$$

using the change of variables  $y = \lambda_i x$ . Therefore  $\hat{\varphi}$  fulfills the conditions in (8.3) if

$$\sum_{i=0}^{r-1} c_i = 1 \quad \text{and} \quad \sum_{i=0}^{r-1} c_i \lambda_i^{|k|} = 0 \quad \text{for } 1 \leq |k| \leq r-1.$$

This is a linear system of  $r$  equations, namely

$$A \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{r-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{where} \quad A := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_0 & \lambda_1 & \dots & \lambda_d \\ \lambda_0^2 & \lambda_1^2 & \dots & \lambda_d^2 \\ \vdots & \vdots & \vdots & \vdots \\ \lambda_0^{r-1} & \lambda_1^{r-1} & \dots & \lambda_d^{r-1} \end{pmatrix}.$$

Note that  $A$  is a Vandermonde matrix with  $\det(A) = \prod_{0 \leq i < j \leq d} (\lambda_j - \lambda_i) \neq 0$ , because  $\lambda_0, \lambda_1, \dots, \lambda_{r-1}$  are all distinct. The inverse matrix  $A^{-1}$  is explicit, see equation (7) (where a transpose is missing) in [Kli67]:<sup>†</sup>

$$(A^{-1})_{ij} = (-1)^j \frac{\sum_{\substack{K \subseteq \{0, \dots, r-1\} \setminus \{i\} \\ |K|=r-1-j}} \prod_{k \in K} \lambda_k}{\prod_{k \in \{0, \dots, r-1\} \setminus \{i\}} (\lambda_k - \lambda_i)} \quad \forall 0 \leq i, j \leq r-1.$$

In particular, if we set  $j = 0$ , we see that  $c_i = (A^{-1})_{i0}$  is given by

$$c_i = \frac{\prod_{k \in \{0, \dots, r-1\} \setminus \{i\}} \lambda_k}{\prod_{k \in \{0, \dots, r-1\} \setminus \{i\}} (\lambda_k - \lambda_i)} = \prod_{k \in \{0, \dots, r-1\}: k \neq i} \frac{\lambda_k}{\lambda_k - \lambda_i},$$

which matches (8.1). □

## 9. BASIC ESTIMATES ON CONVOLUTIONS

In this section we give two elementary but important Lemmas on convolutions. We fix  $r \in \mathbb{N} = \{1, 2, \dots\}$  and a test function  $\hat{\varphi} = \hat{\varphi}^{[r]} \in \mathcal{D}$  with the following properties:

$$\hat{\varphi} \text{ is supported in } B(0, \tfrac{1}{2}), \tag{9.1}$$

$$\int_{\mathbb{R}^d} y^k \hat{\varphi}(y) dy = 0 \quad \text{for } 1 \leq |k| \leq r-1. \tag{9.2}$$

We stress that (9.2) is *not* required for  $k = 0$  (indeed, we typically want  $\int \hat{\varphi} = 1$ ).

**Remark 9.1.** Starting from an *arbitrary* test function  $\varphi \in \mathcal{D}$ , we can define  $\hat{\varphi}$  as in Lemma 8.1, for *any* choice of distinct  $(\lambda_i)_{i=0, \dots, r-1}$  and  $a \in \mathbb{R}$ . Then (9.2) holds by (8.3), while (9.1) holds provided we choose the  $\lambda_i$ 's small enough.

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<sup>†</sup>See also [https://proofwiki.org/wiki/Inverse\\_of\\_Vandermonde\\_Matrix](https://proofwiki.org/wiki/Inverse_of_Vandermonde_Matrix)

Next we define

$$\check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2, \quad (9.3)$$

where by  $\hat{\varphi}^{\frac{1}{2}}, \hat{\varphi}^2$  we mean  $\hat{\varphi}^\lambda(z) = \lambda^{-d} \hat{\varphi}(\lambda^{-1}z)$  for  $\lambda = \frac{1}{2}, 2$ , respectively. The function  $\check{\varphi}$  will play an important role in the sequel. It follows by (9.1) and (9.2) that

$$\check{\varphi} \text{ is supported in } B(0, 1), \quad (9.4)$$

$$\int_{\mathbb{R}^d} y^k \check{\varphi}(y) dy = 0 \quad \text{for } 0 \leq |k| \leq r-1. \quad (9.5)$$

We stress that (9.5) holds also for  $k = 0$ , because  $\int \hat{\varphi}^{\frac{1}{2}} = \int \hat{\varphi}^2 = \int \hat{\varphi}^\lambda$  for any  $\lambda$ .

Our first Lemma concerns the convolution of a test function  $\eta$  with  $\check{\varphi}$ .

**Lemma 9.2.** *Fix a test function  $\eta \in \mathcal{D}(H)$  supported in a compact set  $H \subseteq \mathbb{R}^d$ . Let  $\check{\varphi} \in \mathcal{D}$  satisfy (9.4) and (9.5). For any  $\varepsilon > 0$ , the function  $\check{\varphi}^\varepsilon * \eta$  is supported in the  $\varepsilon$ -enlargement  $\bar{H}_\varepsilon$  of  $H$ , see (2.1), and*

$$\|\check{\varphi}^\varepsilon * \eta\|_{L^1} \leq \text{Vol}(\bar{H}_\varepsilon) \|\eta\|_{C^r} \|\check{\varphi}\|_{L^1} \varepsilon^r. \quad (9.6)$$

*Proof.* Since  $\eta$  is supported in  $H$  and  $\check{\varphi}$  is supported in  $B(0, 1)$ , then  $\check{\varphi}^\varepsilon * \eta$  is supported in  $\bar{H}_\varepsilon$ . Fix  $y \in \bar{H}_\varepsilon$  and denote by  $p_y(\cdot) := \sum_{|k| \leq r-1} \frac{\partial^k \eta(y)}{k!} (\cdot - y)^k$  the Taylor polynomial of  $\eta$  of order  $r-1$  based at  $y$ , which satisfies for all  $z \in \mathbb{R}^d$

$$|\eta(z) - p_y(z)| \leq \|\eta\|_{C^r} |z - y|^r. \quad (9.7)$$

It follows by (9.5) that  $\int_{\mathbb{R}^d} \check{\varphi}^\varepsilon(y-z) p_y(z) dz = 0$ , hence we can write

$$(\check{\varphi}^\varepsilon * \eta)(y) = \int_{\mathbb{R}^d} \check{\varphi}^\varepsilon(y-z) \{\eta(z) - p_y(z)\} dz.$$

Since  $\check{\varphi}^\varepsilon$  is supported in  $B(0, \varepsilon)$ , by (9.7)

$$|(\check{\varphi}^\varepsilon * \eta)(y)| \leq \|\eta\|_{C^r} \int_{\mathbb{R}^d} |\check{\varphi}^\varepsilon(y-z)| |z - y|^r dz \leq \|\eta\|_{C^r} \|\check{\varphi}\|_{L^1} \varepsilon^r.$$

This completes the proof of (9.6).  $\square$

Our second Lemmas concerns convolutions of (scaled versions of) a test function  $\psi$  with either  $\hat{\varphi}$  or  $\check{\varphi}$ , integrated against an arbitrary function  $G$ .

**Lemma 9.3.** *Let  $\lambda, \varepsilon > 0$ ,  $K \subset \mathbb{R}^d$  a compact set and  $G : \mathbb{R}^d \rightarrow \mathbb{R}$  a measurable function. Let  $\hat{\varphi}, \check{\varphi} \in \mathcal{D}$  satisfy (9.1), (9.2) and (9.4), (9.5), respectively. Then for all  $x \in K$  and  $\psi \in \mathcal{B}_r$ , see (5.1),*

$$\left| \int_{\mathbb{R}^d} G(y) (\hat{\varphi}^{2\varepsilon} * \psi_x^\lambda)(y) dy \right| \leq 2^d \|\hat{\varphi}\|_{L^1} \sup_{B(x, \lambda+\varepsilon)} |G|, \quad (9.8)$$

$$\left| \int_{\mathbb{R}^d} G(y) (\check{\varphi}^\varepsilon * \psi_x^\lambda)(y) dy \right| \leq 4^d \|\check{\varphi}\|_{L^1} \min\{\varepsilon/\lambda, 1\}^r \sup_{B(x, \lambda+\varepsilon)} |G|. \quad (9.9)$$

*Proof.* Since  $\hat{\varphi}$  and  $\psi$  are supported in  $B(0, 1/2)$  and  $B(0, 1)$  respectively, the function  $\hat{\varphi}^{2\varepsilon} * \psi_x^\lambda$  is supported in  $B(x, \lambda + \varepsilon)$ . Then we can bound

$$\left| \int_{\mathbb{R}^d} G(y) (\hat{\varphi}^{2\varepsilon} * \psi_x^\lambda)(y) dy \right| \leq \|\hat{\varphi}^{2\varepsilon} * \psi_x^\lambda\|_{L^1} \sup_{B(x, \lambda + \varepsilon)} |G|.$$

Now

$$\|\hat{\varphi}^{2\varepsilon} * \psi_x^\lambda\|_{L^1} \leq \|\hat{\varphi}^{2\varepsilon}\|_{L^1} \|\psi_x^\lambda\|_{L^1} \leq 2^d \|\hat{\varphi}\|_{L^1},$$

because  $\|\hat{\varphi}^{2\varepsilon}\|_{L^1} = \|\hat{\varphi}\|_{L^1}$  and (9.8) is proved, because

$$\sup_{\psi \in \mathcal{B}_r} \|\psi_x^\lambda\|_{L^1} = \sup_{\psi \in \mathcal{B}_r} \|\psi\|_{L^1} \leq 2^d \sup_{\psi \in \mathcal{B}_r} \|\psi\|_\infty \leq 2^d, \quad (9.10)$$

since the volume of the unit ball in  $\mathbb{R}^d$  is bounded above by  $2^d$ . Analogously

$$\left| \int_{\mathbb{R}^d} G(y) (\check{\varphi}^{2\varepsilon} * \psi_x^\lambda)(y) dy \right| \leq \|\check{\varphi}^{2\varepsilon} * \psi_x^\lambda\|_{L^1} \sup_{B(x, \lambda + \varepsilon)} |G|.$$

As in (9.10) we can bound

$$\|\check{\varphi}^{2\varepsilon} * \psi_x^\lambda\|_{L^1} \leq \|\check{\varphi}^{2\varepsilon}\|_{L^1} \|\psi_x^\lambda\|_{L^1} = \|\check{\varphi}\|_{L^1} \|\psi\|_{L^1} \leq 2^d \|\check{\varphi}\|_{L^1},$$

which proves (9.9) for  $\lambda \leq \varepsilon$ . When  $\lambda > \varepsilon$ , we apply (9.6) to get

$$\|\check{\varphi}^{2\varepsilon} * \psi_x^\lambda\|_{L^1} \leq \text{Vol}(B(x, \lambda + \varepsilon)) \|\psi_x^\lambda\|_{C^r} \varepsilon^r \|\check{\varphi}\|_{L^1}.$$

Note that  $\text{Vol}(B(x, \lambda + \varepsilon)) \leq (2(\lambda + \varepsilon))^d \leq 4^d \lambda^d$  for  $\lambda > \varepsilon$ . Since  $\psi \in \mathcal{B}_r$ , see (5.1), we can easily bound  $\|\psi_x^\lambda\|_{C^r}$  by (3.2):

$$\|\psi_x^\lambda\|_{C^r} = \max_{|k| \leq r} \|\partial^k(\psi_x^\lambda)\|_\infty = \max_{|k| \leq r} \|\lambda^{-|k|-d}(\partial^k \psi)\|_\infty \leq \lambda^{-r-d}.$$

The proof of (9.9) is complete.  $\square$

## 10. PROOF OF THE RECONSTRUCTION THEOREM FOR $\gamma > 0$

In this section we prove Theorem 5.1 when  $\gamma > 0$ . Given any  $\gamma$ -coherent germ  $F = (F_x)_{x \in \mathbb{R}^d}$ , we show the existence of a distribution  $f \in \mathcal{D}'$  which satisfies (5.2). Uniqueness of  $f$  follows by Lemma 4.2, because the right hand side of (5.2) vanishes for  $\gamma > 0$ . Then linearity of the map  $F \mapsto \mathcal{R}F$  is a consequence of uniqueness.

We now turn to existence. A large part of the proof actually holds for any  $\gamma \in \mathbb{R}$ , only in the last steps we specialize to  $\gamma > 0$ .

**STEP 0. SETUP.** We fix a  $(\alpha, \gamma)$ -coherent germ  $(F_x)_{x \in \mathbb{R}^d}$  as in Definition 4.3, for some  $\alpha = (\alpha_K)$ , with local homogeneity bounds  $\beta = (\beta_K)$  as in Lemma 4.12. Without loss of generality, we suppose that with  $K \mapsto \alpha_K$  and  $K \mapsto \beta_K$  are monotone as in (4.4) and (4.9). We will specify when we need to assume  $\gamma > 0$ .

We fix a compact set  $K \subset \mathbb{R}^d$  and define its 3/2-fattening  $\bar{K}_{3/2}$  as in (2.1). Throughout the proof we set

$$\alpha := \alpha_{\bar{K}_{3/2}}, \quad \beta := \beta_{\bar{K}_{3/2}}, \quad (10.1)$$

so that (4.2) and (4.8) hold on the compact set  $\bar{K}_{3/2}$ . More explicitly, there are finite constants  $C_1, C_2$  such that for all  $y, z \in \bar{K}_{3/2}$  with  $|z - y| \leq 2$  and  $\varepsilon \in (0, 1]$  we have

$$|(F_z - F_y)(\varphi_y^\varepsilon)| \leq C_1 \varepsilon^\alpha (|z - y| + \varepsilon)^{\gamma - \alpha}, \quad |F_y(\varphi_y^\varepsilon)| \leq C_2 \varepsilon^\beta, \quad (10.2)$$

and in fact we can choose  $C_1 := \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}}$ . We also fix an integer  $r \in \mathbb{N}$  such that

$$r = r_{\bar{K}_{3/2}} > \max\{-\alpha, -\beta\}. \quad (10.3)$$

By Lemma 8.3, we can build a “tweaked” test function  $\hat{\varphi} = \hat{\varphi}^{[r]}$  which fulfills properties (8.5) and (8.6), namely the support of  $\hat{\varphi}$  is included in  $B(0, 1/2)$  and

$$\int_{\mathbb{R}^d} \hat{\varphi}(y) dy = 1, \quad \int_{\mathbb{R}^d} y^k \hat{\varphi}(y) dy = 0 \quad \text{for } 1 \leq |k| \leq r - 1.$$

We claim that we can replace  $\varphi$  by  $\hat{\varphi}$  in (10.2) and obtain, for all  $y, z \in \bar{K}_{3/2}$  with  $|z - y| \leq 2$  and  $\varepsilon \in (0, 1]$ ,

$$|(F_z - F_y)(\hat{\varphi}_y^\varepsilon)| \leq \hat{C}_1 \varepsilon^\alpha (|z - y| + \varepsilon)^{\gamma - \alpha}, \quad (10.4)$$

$$|F_y(\hat{\varphi}_y^\varepsilon)| \leq \hat{C}_2 \varepsilon^\beta, \quad (10.5)$$

where the constants  $\hat{C}_1, \hat{C}_2$  are given by

$$\hat{C}_1 := \frac{e^2}{|\int \varphi|} r \left(\frac{2^{-r-1}}{1+R_\varphi}\right)^\alpha \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}}, \quad \hat{C}_2 := \frac{e^2}{|\int \varphi|} r \left(\frac{2^{-r-1}}{1+R_\varphi}\right)^{\beta \wedge 0} C_2, \quad (10.6)$$

and  $R_\varphi$  is such that  $\varphi$  is supported in  $B(0, R_\varphi)$ .

Indeed, for every  $\varepsilon \in (0, 1]$  and  $i = 0, \dots, r - 1$  we can estimate by (8.4)

$$(\varepsilon \lambda_i)^\alpha (|z - y| + \varepsilon \lambda_i)^{\gamma - \alpha} \leq \left(\frac{2^{-r-1}}{1+R_\varphi}\right)^\alpha \varepsilon^\alpha (|z - y| + \varepsilon)^{\gamma - \alpha},$$

because  $\frac{2^{-r-1}}{1+R_\varphi} < \lambda_i \leq 1$  (recall that  $\alpha \leq 0$  and  $\gamma \geq \alpha$ , see Definition 4.3). Similarly

$$(\varepsilon \lambda_i)^\beta \leq \left(\frac{2^{-r-1}}{1+R_\varphi}\right)^{\beta \wedge 0} \varepsilon^\beta.$$

Plugging these bounds into (10.2), by (8.4) and (8.8) we obtain (10.4)-(10.5)-(10.6).

STEP 1. STRATEGY. We can now outline our strategy. We use the mollifiers

$$\rho^\varepsilon(z) = \varepsilon^{-d} \rho(\varepsilon^{-1}z)$$

where  $\rho$  is defined as follows (recall that  $\hat{\varphi}^2$  means  $\hat{\varphi}^\lambda(z) = \lambda^{-d} \hat{\varphi}(\lambda^{-1}z)$  for  $\lambda = 2$ ):

$$\rho := \hat{\varphi}^2 * \hat{\varphi} \quad \text{and} \quad \varepsilon = \varepsilon_n := 2^{-n}, \quad n \in \mathbb{N}_0. \quad (10.7)$$

Note that  $\int \rho = \int \hat{\varphi}^2 \int \hat{\varphi} = 1$ .

This peculiar choice of  $\rho$  ensures that *the difference  $\rho^{\frac{1}{2}} - \rho$  is a convolution*:

$$\rho^{\frac{1}{2}} - \rho = \hat{\varphi} * \check{\varphi} \quad \text{where we define} \quad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2, \quad (10.8)$$

because  $(f^\lambda)^{\lambda'} = f^{\lambda\lambda'}$  and  $(f * g)^\lambda = f^\lambda * g^\lambda$ , see (3.2) and (9.3). It follows that

$$\rho^{\varepsilon_{n+1}} - \rho^{\varepsilon_n} = (\rho^{\frac{1}{2}} - \rho)^{\varepsilon_n} = \hat{\varphi}^{\varepsilon_n} * \check{\varphi}^{\varepsilon_n}. \quad (10.9)$$

This will allow us to compare efficiently convolutions with  $\rho^{\varepsilon_{n+1}}$  and  $\rho^{\varepsilon_n}$ .

We are ready to define a sequence of distributions that will be shown to converge to a limiting distribution  $f \in \mathcal{D}'$  which fulfills (5.2). To motivate the definition, note that for any distribution  $\xi \in \mathcal{D}'$  and test function  $\psi \in \mathcal{D}$ , by Lemma 7.3, we have

$$\xi(\psi) = \lim_{n \rightarrow \infty} \xi(\rho^{\varepsilon_n} * \psi) = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \xi(\rho_z^{\varepsilon_n}) \psi(z) dz,$$

where we applied (7.4). When we have a germ  $F = (F_x)_{x \in \mathbb{R}^d}$  instead of a fixed distribution  $\xi$ , a natural idea is to replace  $\xi(\rho_z^{\varepsilon_n})$  by  $F_z(\rho_z^{\varepsilon_n})$ . This leads to:

**Definition 10.1 (Approximating distributions).** *Given a germ  $F = (F_x)_{x \in \mathbb{R}^d}$ , for  $n \in \mathbb{N}$  we define  $f_n \in \mathcal{D}'$  as follows:*

$$f_n(\psi) := \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_n}) \psi(z) dz, \quad \psi \in \mathcal{D}. \quad (10.10)$$

**Remark 10.2.** We recall that, by Definition 4.1, the map  $z \mapsto F_z(\rho^\varepsilon)$  is measurable. Since the map  $z \mapsto \rho_z^\varepsilon \in \mathcal{D}$  is continuous, it follows that the map  $(z, y) \mapsto F_z(\rho_y^\varepsilon)$  is jointly measurable as pointwise limit of measurable maps:  $F_z(\rho_y^\varepsilon) = \lim_{n \rightarrow \infty} F_z(\rho_{[ny]/n}^\varepsilon)$ , where  $[x] := ([x_1], \dots, [x_d])$  and  $[a] := \max\{n \in \mathbb{Z} : z \leq a\}$  is the integer part of  $a \in \mathbb{R}$ . In particular,  $z \mapsto F_z(\rho_z^{\varepsilon_n})$  is measurable.

STEP 2. DECOMPOSITION. Let us look closer at  $f_n(\psi)$  in (10.10). We start with a telescopic sum:

$$f_n(\psi) = f_1(\psi) + \sum_{k=1}^{n-1} g_k(\psi) \quad \text{where} \quad g_k(\psi) := f_{k+1}(\psi) - f_k(\psi). \quad (10.11)$$

We can write  $g_k(\psi) = \int_{\mathbb{R}^d} F_z(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \psi(z) dz$  by (10.10) and then  $F_z(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) = \int_{\mathbb{R}^d} F_z(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}_z^{\varepsilon_k}(y) dy$ , by (10.9) and (7.4), which leads to the fundamental expression

$$g_k(\psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_z(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \psi(z) dy dz.$$

If we write  $F_z = F_y + (F_z - F_y)$  inside the last integral, we can decompose

$$\begin{aligned} g_k(\psi) &= \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_y(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \psi(z) dy dz}_{g'_k(\psi)} \\ &\quad + \underbrace{\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \psi(z) dy dz}_{g''_k(\psi)}. \end{aligned} \quad (10.12)$$

When we plug this into (10.11), we can write

$$f_n(\psi) = f_1(\psi) + f'_n(\psi) + f''_n(\psi), \quad (10.13)$$

$$\text{where} \quad f'_n(\psi) := \sum_{k=1}^{n-1} g'_k(\psi), \quad f''_n(\psi) := \sum_{k=1}^{n-1} g''_k(\psi). \quad (10.14)$$

In the next steps we proceed as follows. Recall that we fixed a compact set  $K \subseteq \mathbb{R}^d$ .

- In Step 3 we show that

$$\forall \gamma \in \mathbb{R} : \quad f'(\psi) := \lim_{n \rightarrow \infty} f'_n(\psi) \quad \text{exists} \quad \forall \psi \in \mathcal{D}(\bar{K}_1). \quad (10.15)$$

- In Step 4 we show that

$$\forall \gamma > 0 : \quad f''(\psi) := \lim_{n \rightarrow \infty} f''_n(\psi) \quad \text{exists} \quad \forall \psi \in \mathcal{D}(\bar{K}_1). \quad (10.16)$$

Then if  $\gamma > 0$  the limit  $f^K(\psi) := \lim_{n \rightarrow \infty} f_n(\psi)$  exists for  $\psi \in \mathcal{D}(\bar{K}_1)$  and equals

$$f^K(\psi) = f_1(\psi) + f'(\psi) + f''(\psi), \quad \psi \in \mathcal{D}(\bar{K}_1). \quad (10.17)$$

- In Step 5 we show that  $f^K$  is a distribution on  $\bar{K}_1$  which satisfies

$$\begin{aligned} \forall \gamma > 0 : \quad |(f^K - F_x)(\psi_x^\lambda)| &\leq \mathbf{c} \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}} \lambda^\gamma \\ &\text{uniformly for } \psi \in \mathcal{B}_r, \quad x \in K, \quad \lambda \in (0, 1], \end{aligned} \quad (10.18)$$

where the constant  $\mathbf{c} = \mathbf{c}_{\alpha, \gamma, r, d, \varphi}$  is given in (10.39) below.

We stress that *in principle*  $f^K(\psi)$  depends on the chosen compact set  $K$ , because  $f_n(\psi)$  depends on  $\hat{\varphi} = \hat{\varphi}^{[r]}$ , see (10.10) and (10.7), and *the value of  $r$  depends on  $K$  through  $\alpha = \alpha_{\bar{K}_{3/2}}$ ,  $\beta = \beta_{\bar{K}_{3/2}}$* , see (10.3) and (10.1). In the special case when  $\alpha_K = \alpha$  and  $\beta_K = \beta$  for every  $K$  (i.e. the germ  $F$  is  $(\alpha, \gamma)$ -coherent with global homogeneity bound  $\beta$ ), then  $f^K(\psi) = f(\psi)$  does not depend on  $K$  and the proof is completed, because  $f$  satisfies (5.2) in virtue of (10.18). In the general case, a small extra step is needed to complete the proof.

- In Step 6 we show that for  $\gamma > 0$  the distributions  $f^K$  are consistent, i.e.

$$\text{for } K \subseteq K' : \quad f^K(\psi) = f^{K'}(\psi) \quad \forall \psi \in \mathcal{D}(\bar{K}_1). \quad (10.19)$$

This property lets us define a *global* distribution  $f \in \mathcal{D}'$  which satisfies (5.2), thanks to (10.18). This concludes the proof for  $\gamma > 0$ .

STEP 3. PROOF OF (10.15) FOR  $\gamma \in \mathbb{R}$ . By (10.14), to prove (10.15) it suffices to show that

$$\text{for all } \gamma \in \mathbb{R} : \quad \sum_{k=1}^{\infty} |g'_k(\psi)| < \infty, \quad \forall \psi \in \mathcal{D}(\bar{K}_1). \quad (10.20)$$

Recall that

$$g'_k(\psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_y(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y-z) \psi(z) \, dy \, dz = \int_{\mathbb{R}^d} F_y(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k} * \psi(y) \, dy.$$

Note that  $\check{\varphi} = \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2$  is supported in  $B(0, 1)$ , because  $\hat{\varphi}$  is supported in  $B(0, \frac{1}{2})$ . Since  $\psi$  is supported by  $\bar{K}_1$  and  $\check{\varphi}^{\varepsilon_k}$  by  $B(0, \varepsilon_k)$  with  $\varepsilon_k \leq 1/2$ , then  $\check{\varphi}^{\varepsilon_k} * \psi$  is supported by  $\bar{K}_{3/2}$ . Then

$$|g'_k(\psi)| \leq \|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \sup_{y \in \bar{K}_{3/2}} |F_y(\hat{\varphi}_y^{\varepsilon_k})|.$$

By (9.6) we have the bound

$$\|\check{\varphi}^{\varepsilon_k} * \psi\|_{L^1} \leq \text{Vol}(\bar{K}_{3/2}) \|\psi\|_{C^r} \varepsilon_k^r \|\check{\varphi}\|_{L^1}.$$

By (10.5), for all  $y \in \bar{K}_{3/2}$  we have the bound  $|F_y(\hat{\varphi}_y^\varepsilon)| \leq \hat{C}_2 \varepsilon^\beta$ . Then we obtain

$$|g'_k(\psi)| \leq \{\hat{C}_2 \text{Vol}(\bar{K}_{3/2}) \|\check{\varphi}\|_{L^1} \|\psi\|_{C^r}\} \varepsilon_k^{\beta+r}. \quad (10.21)$$

Since  $\varepsilon_k = 2^{-k}$  and  $\beta + r > 0$  by assumption, see (10.3), we have  $\sum_{k=1}^{\infty} |g'_k(\psi)| < \infty$  which completes the proof of (10.20).

STEP 4. PROOF OF (10.16) FOR  $\gamma > 0$ . By (10.14), to prove (10.16) it suffices to show that

$$\text{if } \gamma > 0 : \quad \sum_{k=1}^{\infty} |g''_k(\psi)| < \infty, \quad \forall \psi \in \mathcal{D}(\bar{K}_1). \quad (10.22)$$

Recall that

$$g''_k(\psi) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \psi(z) dy dz. \quad (10.23)$$

We recall that  $\check{\varphi}^{\varepsilon_k}$  is supported in  $B(0, \varepsilon_k)$ , so that

$$|g''_k(\psi)| \leq \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \|\psi\|_{L^1} \sup_{z \in \bar{K}_1, |y-z| \leq \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})|,$$

with  $\varepsilon_k \leq 1/2$  since  $k \geq 1$ . Then (10.4) gives

$$\sup_{z \in \bar{K}_1, |y-z| \leq \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \leq \hat{C}_1 \varepsilon_k^\alpha (2\varepsilon_k)^{\gamma-\alpha},$$

hence from (10.23) we obtain  $|g''_k(\psi)| \leq 2^{\gamma-\alpha} \hat{C}_1 \varepsilon_k^\gamma \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \|\psi\|_{L^1}$ . We finally observe that  $\|\check{\varphi}^{\varepsilon_k}\|_{L^1} = \|\check{\varphi}\|_{L^1}$  by (3.2). This gives the bound

$$|g''_k(\psi)| \leq \{2^{\gamma-\alpha} \hat{C}_1 \|\check{\varphi}\|_{L^1} \|\psi\|_{L^1}\} \varepsilon_k^\gamma. \quad (10.24)$$

Since  $\gamma > 0$  and  $\varepsilon_k = 2^{-k}$ , we obtain  $\sum_{k=1}^{\infty} |g''_k(\psi)| < \infty$ , proving (10.22).

STEP 5. PROOF OF (10.18). We showed in the previous steps that both  $f'_n(\psi)$  and  $f''_n(\psi)$  converge for  $\gamma > 0$ . Recalling (10.13), we have that  $f_n(\psi)$  converges to  $f^K(\psi)$  given by (10.17), i.e.

$$f^K(\psi) = f_1(\psi) + \sum_{k=1}^{\infty} g'_k(\psi) + \sum_{k=1}^{\infty} g''_k(\psi).$$

**Remark 10.3.** By (10.21) and (10.24) there is  $C = C_{K,\gamma,\beta,r,\check{\varphi}} < \infty$  such that

$$|f^K(\psi)| \leq C \{\|\psi\|_{L^1} + \|\psi\|_{C^r}\} \leq C \{\text{Vol}(\bar{K}_{3/2}) + 1\} \|\psi\|_{C^r} \quad \text{for } \psi \in \mathcal{D}(\bar{K}_1).$$

This shows that  $f^K \in \mathcal{D}'(\bar{K}_1)$  is indeed a distribution on  $\bar{K}_1$ , see (3.1).

We now prove that  $f^K(\cdot)$  satisfies (10.18). We fix a point  $x \in K$  and define

$$\tilde{f}(\psi) := f^K(\psi) - F_x(\psi), \quad \psi \in \mathcal{D}(\bar{K}_1).$$



We also define  $\tilde{f}_n(\psi)$  similarly to  $f_n(\psi)$  in (10.10), just replacing  $F_z$  by  $F_z - F_x$ :

$$\tilde{f}_n(\psi) := \int_{\mathbb{R}^d} (F_z - F_x)(\rho_z^{\varepsilon_n}) \psi(z) dz = f_n(\psi) - F_x(\rho^{\varepsilon_n} * \psi), \quad (10.25)$$

having used (7.4). Since  $F_x(\rho^{\varepsilon_n} * \psi) \rightarrow F_x(\psi)$  by Lemma 7.4 and Lemma 7.3, we have

$$\tilde{f}(\psi) = \lim_{n \rightarrow \infty} \tilde{f}_n(\psi). \quad (10.26)$$

We now fix  $\lambda \in (0, 1]$  and define

$$N = N_\lambda := \min\{k \in \mathbb{N} : \varepsilon_k \leq \lambda\}, \quad (10.27)$$

so that  $N \geq 1$  and in particular

$$\varepsilon_N \leq \lambda < \varepsilon_{N-1} = 2\varepsilon_N. \quad (10.28)$$

Let us now fix  $\psi \in \mathcal{B}_r$ , see (5.1). By the triangle inequality we can bound

$$|\tilde{f}(\psi_x^\lambda)| \leq |\tilde{f}_N(\psi_x^\lambda)| + |(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)|. \quad (10.29)$$

We will estimate separately the two terms in the right-hand side.

*First term in (10.29).* By (10.25), recalling (10.7) and (7.4), we can write

$$\tilde{f}_N(\psi_x^\lambda) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_N}) \hat{\varphi}^{2\varepsilon_N}(y - z) \psi_x^\lambda(z) dy dz. \quad (10.30)$$

This integral is similar to (10.23) and we argue as in the proof of (10.24). Recall that  $\hat{\varphi}$  has support in  $B(0, \frac{1}{2})$ . Then  $\hat{\varphi}^{2\varepsilon_N}$  has support in  $B(0, \varepsilon_N)$  and we may assume that  $|y - z| \leq \varepsilon_N \leq \frac{1}{2}$  in the right-hand side of (10.30). Since  $\psi_x^\lambda$  is supported in  $B(x, \lambda) \subset \bar{K}_1$ , we can assume that  $|z - x| \leq \lambda$ , hence  $z \in \bar{K}_1$  and  $y \in \bar{K}_{3/2}$ . Then

$$|\tilde{f}_N(\psi_x^\lambda)| \leq \|\hat{\varphi}^{2\varepsilon_N}\|_{L^1} \|\psi_x^\lambda\|_{L^1} \sup_{z \in B(x, \lambda), |y - z| \leq \varepsilon_N} |(F_z - F_x)(\hat{\varphi}_y^{\varepsilon_N})|.$$

By the triangle inequality  $|(F_z - F_x)(\hat{\varphi}_y^{\varepsilon_N})| \leq |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_N})| + |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_N})|$ , and since (10.4) and (10.28) give

$$\begin{aligned} \sup_{z \in B(x, \lambda), |y - z| \leq \varepsilon_N} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_N})| &\leq \hat{C}_1 \varepsilon_N^\alpha (2\varepsilon_N)^{\gamma - \alpha} \leq \hat{C}_1 2^{\gamma - \alpha} \lambda^\gamma, \\ \sup_{z \in B(x, \lambda), |y - z| \leq \varepsilon_N} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_N})| &\leq \hat{C}_1 \varepsilon_N^\alpha (\lambda + 2\varepsilon_N)^{\gamma - \alpha} \leq \hat{C}_1 4^{\gamma - \alpha} \lambda^\gamma, \end{aligned}$$

we obtain

$$|\tilde{f}_N(\psi_x^\lambda)| \leq 2 \cdot 4^{\gamma - \alpha} \hat{C}_1 \lambda^\gamma \|\hat{\varphi}^{2\varepsilon_N}\|_{L^1} \|\psi_x^\lambda\|_{L^1}.$$

We can easily bound  $\|\psi_x^\lambda\|_{L^1} \leq 2^d$  for  $\psi \in \mathcal{B}_r$ , see (9.10), and  $\|\hat{\varphi}^{2\varepsilon_N}\|_{L^1} = \|\hat{\varphi}\|_{L^1}$ . All this yields the following estimate for the first term  $|\tilde{f}_N(\psi_x^\lambda)|$  in (10.29)

$$|\tilde{f}_N(\psi_x^\lambda)| \leq \{4^{\gamma - \alpha} 2^{d+1}\} \|\hat{\varphi}\|_{L^1} \hat{C}_1 \lambda^\gamma. \quad (10.31)$$

*Second term in (10.29).* Next we bound, by (10.26),

$$|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| \leq \sum_{k \geq N} |(\tilde{f}_{k+1} - \tilde{f}_k)(\psi_x^\lambda)|. \quad (10.32)$$

Recalling (10.25) and (10.13)-(10.14), we can write

$$\begin{aligned} (\tilde{f}_{k+1} - \tilde{f}_k)(\psi_x^\lambda) &= (f_{k+1} - f_k)(\psi_x^\lambda) - F_x((\rho^{\varepsilon_{k+1}} - \rho^{\varepsilon_k}) * \psi_x^\lambda) \\ &= \underbrace{g'_k(\psi_x^\lambda) - F_x((\rho^{\varepsilon_{k+1}} - \rho^{\varepsilon_k}) * \psi_x^\lambda)}_{A_k^\lambda} + \underbrace{g''_k(\psi_x^\lambda)}_{B_k^\lambda}. \end{aligned} \quad (10.33)$$

We now look at  $A_k^\lambda$  and  $B_k^\lambda$ . The estimates for  $A_k^\lambda$  hold for any  $\gamma \in \mathbb{R}$  and will be useful in Section 11 for the case  $\gamma \leq 0$ , hence we state them as a separate result.

**Lemma 10.4.** *Define  $A_k^\lambda$  as in (10.33). For any  $\gamma \in \mathbb{R}$  we have*

$$|A_k^\lambda| \leq 4^{d+\gamma-\alpha} \hat{C}_1 \|\check{\varphi}\|_{L^1} \cdot \begin{cases} \lambda^{\gamma-\alpha-r} \varepsilon_k^{\alpha+r} & \text{if } \varepsilon_k < \lambda \\ \varepsilon_k^\gamma & \text{if } \varepsilon_k \geq \lambda \end{cases}, \quad (10.34)$$

and for  $N = N_\lambda$  in (10.27) we have

$$\sum_{k \geq N} |A_k^\lambda| \leq \frac{4^{d+\gamma-\alpha}}{1-2^{-\alpha-r}} \hat{C}_1 \|\check{\varphi}\|_{L^1} \lambda^\gamma. \quad (10.35)$$

*Proof.* By (7.4), together with the crucial property (10.9) of  $\rho^{\varepsilon_{k+1}} - \rho^{\varepsilon_k}$ , we can write

$$\begin{aligned} F_x((\rho^{\varepsilon_{k+1}} - \rho^{\varepsilon_k}) * \psi_x^\lambda) &= \int_{\mathbb{R}^d} F_x(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \psi_x^\lambda(z) dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F_x(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y-z) \psi_x^\lambda(z) dy dz. \end{aligned}$$

Recalling the definition (10.12) of  $g'_k$ , we obtain

$$\begin{aligned} A_k^\lambda &:= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y-z) \psi_x^\lambda(z) dy dz \\ &= \int_{\mathbb{R}^d} (F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k}) (\check{\varphi}^{\varepsilon_k} * \psi_x^\lambda)(y) dy. \end{aligned}$$

If we define  $G(y) := (F_y - F_x)(\hat{\varphi}_y^\varepsilon)$ , we see that  $|A_k^\lambda|$  can be estimated as in (9.9), which yields

$$|A_k^\lambda| \leq 4^d \|\check{\varphi}\|_{L^1} \min\{\varepsilon_k/\lambda, 1\}^r \sup_{y \in B(x, \lambda + \varepsilon_k)} |(F_y - F_x)(\hat{\varphi}_y^{\varepsilon_k})|.$$

For  $y \in B(x, \lambda + \varepsilon_k)$ , by (10.4) we have

$$|(F_x - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \leq \hat{C}_1 \varepsilon_k^\alpha (|x - y| + \varepsilon_k)^{\gamma-\alpha} \leq \hat{C}_1 (\lambda + 2\varepsilon_k)^{\gamma-\alpha} \varepsilon_k^\alpha,$$

which proves (10.34) because  $(\lambda + 2\varepsilon_k)^{\gamma-\alpha} \leq 3^{\gamma-\alpha} \max\{\varepsilon_k, \lambda\}^{\gamma-\alpha}$ .

We next turn to (10.35). For  $k \geq N$  we have  $\varepsilon_k \leq \lambda$ , see (10.28), hence we can apply the first line in (10.34). Since  $\alpha + r > 0$  by assumption, we have

$$\sum_{k \geq N} \varepsilon_k^{\alpha+r} = \frac{\varepsilon_N^{\alpha+r}}{1-2^{-\alpha-r}} \leq \frac{\lambda^{\alpha+r}}{1-2^{-\alpha-r}},$$

therefore from (10.34) we obtain (10.35).  $\square$

We next focus on  $B_k^\lambda = g_k''(\psi_x^\lambda)$  in (10.33). Recalling the definition (10.12) of  $g_k''$ , we can write

$$B_k^\lambda := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y - z) \psi_x^\lambda(z) \, dy \, dz.$$

Since  $\psi$  and  $\check{\varphi}$  are both supported in  $B(0, 1)$ , we can suppose that  $|z - x| \leq \lambda$  and  $|y - z| \leq \varepsilon_k \leq 1/2$  and therefore  $z \in \bar{K}_1$ ,  $y \in \bar{K}_{3/2}$ . Then

$$|B_k^\lambda| \leq \|\check{\varphi}^{\varepsilon_k}\|_{L^1} \|\psi_x^\lambda\|_{L^1} \sup_{z \in \bar{K}_1, |y-z| \leq \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})|.$$

By (10.4) we have the bound

$$\sup_{z \in \bar{K}_1, |y-z| \leq \varepsilon_k} |(F_z - F_y)(\hat{\varphi}_y^{\varepsilon_k})| \leq \hat{C}_1 \varepsilon_k^\alpha (2\varepsilon_k)^{\gamma-\alpha},$$

and therefore, since  $\|\psi_x^\lambda\|_{L^1} \leq 2^d$  for  $\psi \in \mathcal{B}_r$  by (9.10),

$$|B_k^\lambda| \leq 2^{\gamma-\alpha} 2^d \hat{C}_1 \|\check{\varphi}\|_{L^1} \varepsilon_k^\gamma.$$

Note now that  $\gamma > 0$  here, so that

$$\sum_{k \geq N} \varepsilon_k^\gamma = \frac{\varepsilon_N^\gamma}{1 - 2^{-\gamma}} \leq \frac{\lambda^\gamma}{1 - 2^{-\gamma}},$$

which yields

$$\sum_{k \geq N} |B_k^\lambda| \leq \frac{2^{\gamma-\alpha} 2^d}{1 - 2^{-\gamma}} \hat{C}_1 \|\check{\varphi}\|_{L^1} \lambda^\gamma. \quad (10.36)$$

Recalling (10.32) and (10.33), we obtain from (10.35) and (10.36) the desired estimate for the second term in (10.29):

$$|(\tilde{f} - \tilde{f}_N)(\psi_x^\lambda)| \leq \frac{2 \cdot 4^{d+\gamma-\alpha}}{1 - 2^{-\gamma \wedge (\alpha+r)}} \hat{C}_1 \|\check{\varphi}\|_{L^1} \lambda^\gamma. \quad (10.37)$$

*Conclusion.* At last, we can gather (10.29), (10.31) and (10.37). We estimate  $\|\check{\varphi}\|_{L^1} \leq 2 \|\hat{\varphi}\|_{L^1}$  by (10.8), to get

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \frac{2 \cdot 4^{d+1+\gamma-\alpha}}{1 - 2^{-\gamma \wedge (\alpha+r)}} \hat{C}_1 \|\hat{\varphi}\|_{L^1} \lambda^\gamma.$$

If we estimate  $\|\hat{\varphi}\|_{L^1}$  using (8.7) and  $\hat{C}_1$  using (10.6), we obtain

$$\hat{C}_1 \|\hat{\varphi}\|_{L^1} \leq \left(\frac{e^2}{|\int \varphi|} r\right)^2 \left(\frac{2^{-r-1}}{1+R_\varphi}\right)^\alpha \|\varphi\|_{L^1} \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}}. \quad (10.38)$$

If we bound  $e \leq 4$  for simplicity, we obtain finally (10.18) with

$$\mathbf{c} = \mathbf{c}_{\alpha, \gamma, r, d, \varphi} = \frac{r^2 2^{(r+1)\alpha} 4^{d+\gamma-\alpha+6}}{1 - 2^{-\gamma \wedge (\alpha+r)}} \frac{\|\varphi\|_{L^1} (1 + R_\varphi)^{-\alpha}}{|\int \varphi|^2} \quad (\text{for } \gamma > 0) \quad (10.39)$$

where  $R_\varphi$  is the radius of a ball  $B(0, R_\varphi)$  which contains the support of  $\varphi$ .

STEP 6. PROOF OF (10.19). We finally show that the distributions  $f^K$  built in the previous steps are consistent, namely for  $K \subseteq K'$  and for all test functions  $\psi \in \mathcal{D}(\bar{K}_1)$  that are supported in  $\bar{K}_1$  we have  $f^{K'}(\psi) = f^K(\psi)$ . This is an immediate consequence of Lemma 4.2, because if we fix any  $\xi \in \mathcal{D}(\bar{K}_1)$  with  $\int \xi \neq 0$  it follows by (10.18) with  $\psi = \xi$  that both  $f^K$  and  $f^{K'}$  satisfy (4.1) with  $\varphi = \xi$  on the compact set  $\bar{K}_1$ .

We can finally define a *global* distribution  $f \in \mathcal{D}'$ : given any test function  $\psi \in \mathcal{D}$ , we pick a compact set  $K$  large enough so that  $\psi \in \mathcal{D}(\bar{K}_1)$  and we define  $f(\psi) := f^K(\psi)$  (this is well-posed thanks to the consistency relation (10.19) that we have just proved). Then, for any compact set  $K$ , we can replace  $f^K$  by  $f$  in (10.18), which shows that  $f$  satisfies (5.2). This completes the proof of Theorem 5.1 for  $\gamma > 0$ .  $\square$

## 11. PROOF OF THE RECONSTRUCTION THEOREM FOR $\gamma \leq 0$

In this section we prove Theorem 5.1 when  $\gamma \leq 0$ . We stress that we do not have a unique choice for the reconstruction  $\mathcal{R}F$ , because relation (5.2) for  $\gamma \leq 0$  does not characterize  $f$  uniquely, see Lemma 4.2 above and Remark 12.9 below.

Henceforth we fix a germ  $F = (F_x)_{x \in \mathbb{R}^d}$  which is  $\gamma$ -coherent with  $\gamma \leq 0$ . In order to find a correct choice of  $\mathcal{R}F$ , we start following the proof of the case  $\gamma > 0$ , see Section 10. We fix a compact set  $K \subseteq \mathbb{R}^d$  and we fix  $\alpha, \beta, r$  as in (10.1)-(10.3). The key problem when  $\gamma \leq 0$  is that the sequence of approximating distributions  $f_n$  that we defined in (10.10) will typically *not* converge, hence we can no longer define  $f^K := \lim_{n \rightarrow \infty} f_n$ . More precisely, if we recall the decomposition

$$f_n(\psi) = f_1(\psi) + f'_n(\psi) + f''_n(\psi), \quad \psi \in \mathcal{D}(\bar{K}_1),$$

see (10.11)-(10.14), then it is the term  $f''_n(\psi)$  which can fail to converge for  $\gamma \leq 0$ , since the proof in Step 4 was based on (10.24) and exploited  $\gamma > 0$ . On the other hand, we showed that  $f'(\psi) := \lim_{n \rightarrow \infty} f'_n(\psi)$  exists for every  $\gamma \in \mathbb{R}$ , see (10.15).

Therefore, for  $\gamma \leq 0$  the idea is to suppress  $f''_n(\psi)$ . Recalling (10.14), we thus set

$$f^K(\psi) = f_1(\psi) + f'_1(\psi) = f_1(\psi) + \sum_{k=1}^{\infty} g'_k(\psi), \quad \psi \in \mathcal{D}(\bar{K}_1). \quad (11.1)$$

We complete the proof in two steps, that we now describe.

- In Step I we show that  $f^K \in \mathcal{D}'(\bar{K}_1)$  is a distribution on  $\bar{K}_1$  which satisfies

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \mathfrak{C} \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}} \cdot \begin{cases} \lambda^\gamma & \text{if } \gamma < 0 \\ (1 + |\log \lambda|) & \text{if } \gamma = 0 \end{cases} \quad (11.2)$$

uniformly for  $\psi \in \mathcal{B}_r$ ,  $x \in K$ ,  $\lambda \in (0, 1]$ ,

for a suitable  $\mathfrak{C}$  given below, see (11.11) for  $\gamma < 0$  and (11.12) for  $\gamma = 0$ .

**Remark 11.1.** We stress that in general  $f^K(\psi)$  depends on the compact set  $K$ . Indeed,  $f_1(\psi)$  and  $g'_k(\psi)$  depend on  $\hat{\varphi} = \hat{\varphi}^{[r]}$  and the value of  $r > \max\{-\alpha, -\beta\} := \max\{-\alpha_{\bar{K}_{3/2}}, -\beta_{\bar{K}_{3/2}}\}$  is a function of  $K$ , see (10.3) and (10.1).

We first consider the special case when *the germ  $F$  is  $(\alpha, \gamma)$ -coherent with global homogeneity bound  $\beta$* , that is when  $\alpha_K = \alpha$  and  $\beta_K = \beta$  for every compact set  $K$ . Then we can choose a fixed  $r > \max\{-\alpha, -\beta\}$  and  $f^K(\psi) = f(\psi)$  does not depend on  $K$ , hence replacing  $f^K$  by  $f$  in (11.2) we obtain precisely (5.2).

It remains to show that the map  $F \mapsto f =: \mathcal{R}F$  is linear (we recall that the family of  $(\alpha, \gamma)$ -coherent germs with global homogeneity bound  $\beta$  is a vector space, see Remark 4.14). This follows easily by the definition (11.1) of  $f^K = f$ , because both  $f_1$  and  $g'_k$  are linear functions of  $F$ , see (10.10) and (10.12). We have thus completed the proof of Theorem 5.1 for  $\gamma \leq 0$  in this special case.

We finally go back to the general case when  $\alpha_K$  and  $\beta_K$  may depend on  $K$ , hence  $f^K$  also depends on  $K$ . We complete the proof of Theorem 5.1 for  $\gamma \leq 0$  as follows.

- In Step II we build a global distribution  $f \in \mathcal{D}'$  out of the  $f^K$ 's, by a localisation argument based on a partition of unity, and we show that  $f$  satisfies (5.2).

It only remains to prove Steps I and II.

STEP I. PROOF OF (11.2). Let us outline the strategy we are going to follow.

We have fixed a compact set  $K \subseteq \mathbb{R}^d$ . We now fix a point  $x \in K$ . By Lemma 7.3 and Lemma 7.4 we have  $F_x(\psi) = \lim_{n \rightarrow \infty} F_x(\rho^{\varepsilon_n} * \psi)$ . In view of (11.1), we define

$$\bar{f}_n(\psi) := \left( f_1(\psi) + \sum_{k=1}^{n-1} g'_k(\psi) \right) - F_x(\rho^{\varepsilon_n} * \psi) \quad (11.3)$$

so that we can write

$$\bar{f}(\psi) := (f^K - F_x)(\psi) = \lim_{n \rightarrow \infty} \bar{f}_n(\psi).$$

We now fix  $\lambda \in (0, 1]$  and replace  $\psi$  by  $\psi_x^\lambda$ . By the triangle inequality, we get

$$|\bar{f}(\psi_x^\lambda)| \leq |(\bar{f} - \bar{f}_N)(\psi_x^\lambda)| + |\bar{f}_N(\psi_x^\lambda)|, \quad (11.4)$$

where  $N \geq 1$ , defined in (10.27), is such that (we recall that  $\varepsilon_k = 2^{-k}$ )

$$\varepsilon_N \leq \lambda < \varepsilon_{N-1} = 2\varepsilon_N.$$

We estimate the two terms in the right hand side of (11.4) separately.

*First term in (11.4).* We bound

$$|(\bar{f} - \bar{f}_N)(\psi_x^\lambda)| \leq \sum_{k \geq N} |(\bar{f}_{k+1} - \bar{f}_k)(\psi_x^\lambda)|.$$

By (11.3) we can write

$$(\bar{f}_{k+1} - \bar{f}_k)(\psi_x^\lambda) = g'_k(\psi_x^\lambda) - F_x((\rho^{\varepsilon_{k+1}} - \rho^{\varepsilon_k}) * \psi_x^\lambda) = A_k^\lambda,$$

where the term  $A_k^\lambda$  was defined in (10.33). We can then apply Lemma 10.4, which holds also for  $\gamma \leq 0$ : in particular, by relation (10.35) we obtain

$$|(\bar{f} - \bar{f}_N)(\psi_x^\lambda)| \leq \frac{4^{d+\gamma-\alpha}}{1-2^{-\alpha-r}} \hat{C}_1 \|\check{\varphi}\|_{L^1} \lambda^\gamma. \quad (11.5)$$

Second term in (11.4). Since  $N \geq 1$ , we can bound

$$|\bar{f}_N(\psi_x^\lambda)| \leq |\bar{f}_1(\psi_x^\lambda)| + \sum_{k=1}^{N-1} |(\bar{f}_{k+1} - \bar{f}_k)(\psi_x^\lambda)|. \quad (11.6)$$

For  $k \leq N-1$  we have  $\varepsilon_k \geq \varepsilon_{N-1} \geq \lambda$ , therefore by the second line in (10.34)

$$|(\bar{f}_{k+1} - \bar{f}_k)(\psi_x^\lambda)| \leq 4^{d+\gamma-\alpha} \hat{C}_1 \|\check{\varphi}\|_{L^1} \varepsilon_k^\gamma. \quad (11.7)$$

Next we estimate  $|\bar{f}_1(\psi_x^\lambda)|$ . By (7.4) we have  $F_x(\rho^{\varepsilon_1} * \psi) = \int_{\mathbb{R}^d} F_x(\rho_z^{\varepsilon_1}) \psi(z) dz$ . Recalling (11.3) and the definitions (10.10), (10.7) of  $f_1$  and  $\rho$ , we obtain

$$\begin{aligned} \bar{f}_1(\psi_x^\lambda) &= f_1(\psi) - F_x(\rho^{\varepsilon_1} * \psi) = \int_{\mathbb{R}^d} (F_y - F_x)(\rho_y^{\varepsilon_1}) \psi_x^\lambda(y) dy \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (F_z - F_x)(\hat{\varphi}_y^{\varepsilon_1}) \hat{\varphi}^{2\varepsilon_1}(y-z) \psi_x^\lambda(z) dy dz. \end{aligned}$$

Since  $\varepsilon_k = 2^{-k}$  and  $\hat{\varphi}$  has support in  $B(0, \frac{1}{2})$ , then  $\hat{\varphi}^{2\varepsilon_1}$  has support in  $B(0, \varepsilon_1) = B(0, \frac{1}{2})$  and we may assume that  $|y-z| \leq \frac{1}{2}$ . Since  $\psi_x^\lambda$  is supported in  $B(x, \lambda) \subset \bar{K}_1$ , we can assume that  $|z-x| \leq \lambda$  and  $z \in \bar{K}_1$ ,  $y \in \bar{K}_{3/2}$ . Then

$$|\bar{f}_1(\psi_x^\lambda)| \leq \|\hat{\varphi}^{2\varepsilon_1}\|_{L^1} \|\psi_x^\lambda\|_{L^1} \sup_{z \in B(x, \lambda), |y-z| \leq \frac{1}{2}} |(F_z - F_x)(\hat{\varphi}_y)|.$$

Moreover (10.4) for  $\varepsilon = 1$  gives

$$\begin{aligned} \sup_{z \in B(x, \lambda), |y-z| \leq \frac{1}{2}} |(F_z - F_y)(\hat{\varphi}_y)| &\leq \hat{C}_1 (|y-z| + 1)^{\gamma-\alpha} \leq \hat{C}_1 2^{\gamma-\alpha}, \\ \sup_{z \in B(x, \lambda), |y-z| \leq \frac{1}{2}} |(F_y - F_x)(\hat{\varphi}_y)| &\leq \hat{C}_1 (|y-x| + 1)^{\gamma-\alpha} \leq \hat{C}_1 3^{\gamma-\alpha}, \end{aligned}$$

therefore by the triangular inequality

$$|\bar{f}_1(\psi_x^\lambda)| \leq 2 \hat{C}_1 3^{\gamma-\alpha} \|\psi_x^\lambda\|_{L^1} \|\hat{\varphi}^{2\varepsilon_1}\|_{L^1} \leq \hat{C}_1 3^{\gamma-\alpha} 2^{d+1} \|\hat{\varphi}\|_{L^1}, \quad (11.8)$$

since  $\|\psi_x^\lambda\|_{L^1} \leq 2^d$  for  $\psi \in \mathcal{B}_r$ , by (9.10), and  $\|\hat{\varphi}^{2\varepsilon_1}\|_{L^1} = \|\hat{\varphi}\|_{L^1}$ . We can finally estimate  $|\bar{f}_N(\psi_x^\lambda)|$  by (11.6). We get by (11.7) and (11.8)

$$|\bar{f}_N(\psi_x^\lambda)| \leq 4^{d+\gamma-\alpha} \hat{C}_1 \|\check{\varphi}\|_{L^1} \sum_{k=0}^{N-1} \varepsilon_k^\gamma. \quad (11.9)$$

Recalling that  $\varepsilon_k = 2^{-k}$  and  $\varepsilon_N = 2^{-N} \geq \lambda/2$ , we obtain for  $\gamma \leq 0$

$$\sum_{k=0}^{N-1} \varepsilon_k^\gamma = \sum_{k=0}^{N-1} 2^{-\gamma k} \leq \begin{cases} \frac{(\lambda/2)^\gamma - 1}{2^{-\gamma} - 1} \leq \frac{\lambda^\gamma}{1 - 2^\gamma} & \text{if } \gamma < 0 \\ \frac{\log \frac{2}{\lambda}}{\log 2} & \text{if } \gamma = 0. \end{cases} \quad (11.10)$$

*Conclusion.* At last, we can gather (11.4), (11.5) and (11.9)-(11.10). For  $\gamma < 0$ , since  $\|\check{\varphi}\|_{L^1} \leq 2 \|\hat{\varphi}\|_{L^1}$  by (10.8), we obtain

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \frac{4^{d+\gamma-\alpha+1}}{1 - 2^{-(\alpha+r)\wedge(-\gamma)}} \|\hat{\varphi}\|_{L^1} \hat{C}_1 \lambda^\gamma.$$

By (10.38), if we bound  $e \leq 4$  for simplicity, we obtain for all  $\lambda \in (0, 1]$

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \underbrace{\frac{r^2 2^{-(r+1)\alpha} 4^{d+\gamma-\alpha+6}}{1 - 2^{-(\alpha+r)\wedge(-\gamma)}} \frac{\|\varphi\|_{L^1} (1 + R_\varphi)^{-\alpha}}{|\int \varphi|^2}}_e \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}} \lambda^\gamma. \quad (11.11)$$

For  $\gamma = 0$ , since  $\log(2/\lambda)/\log 2 \leq 2(1 + |\log \lambda|)$ , we obtain by (11.9)

$$|(f^K - F_x)(\psi_x^\lambda)| \leq \underbrace{\frac{r^2 2^{-(r+1)\alpha} 4^{d-\alpha+6}}{1 - 2^{-\alpha-r}} \frac{\|\varphi\|_{L^1} (1 + R_\varphi)^{-\alpha}}{|\int \varphi|^2}}_e \|F\|_{\bar{K}_{3/2}, \varphi, \alpha, \gamma}^{\text{coh}} (1 + |\log \lambda|). \quad (11.12)$$

This completes the proof of (11.2).

**STEP II. LOCALIZATION.** In Step I we constructed for every compact set  $K \subset \mathbb{R}^d$  a distribution  $f^K \in \mathcal{D}'(\bar{K}_1)$  which satisfies (11.2). We now exploit this construction only when  $K$  is a ball. Indeed, we use a partition of unity subordinated to a cover made by balls, to construct a global distribution  $f \in \mathcal{D}'$  which satisfies (5.2).

Fix  $\eta \in \mathcal{D}(B(0, \frac{1}{4}))$  such that  $\eta \geq 0$  on  $B(0, \frac{1}{4})$  and  $\eta \geq 1$  on  $B(0, \frac{1}{8})$  and set

$$E := \frac{1}{4\sqrt{d}} \mathbb{Z}^d, \quad \xi(x) := \frac{\eta(x)}{\sum_{z \in E} \eta(x-z)}, \quad x \in \mathbb{R}^d.$$

For every  $y \in E$  we set  $\xi_y(x) := \xi(x-y)$ ,  $x \in \mathbb{R}^d$ . Then  $\xi_y$  is supported in  $B(y, \frac{1}{4})$  and note that  $\sum_{y \in E} \xi_y(x) = 1$ , for all  $x \in \mathbb{R}^d$ , that is  $(\xi_y)_{y \in E}$  is a partition of unity subordinated to the cover  $(B(y, \frac{1}{4}))_{y \in E}$ . We finally define a distribution  $f \in \mathcal{D}'(\mathbb{R}^d)$  by

$$f := \sum_{y \in E} f^{B_y} \cdot \xi_y, \quad \text{where } B_y := B(y, \frac{1}{4}),$$

or more explicitly

$$f(\psi) = \sum_{y \in E} f^{B_y}(\xi_y \psi), \quad \forall \psi \in \mathcal{D}(\mathbb{R}^d).$$

We fix an arbitrary compact set  $K \subset \mathbb{R}^d$  and we redefine  $\alpha, \beta$  and  $r$  as follows:

$$\alpha := \alpha_{\bar{K}_2}, \quad \beta := \beta_{\bar{K}_2}, \quad r > \max\{-\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}, \quad (11.13)$$

i.e. we replace  $\bar{K}_{3/2}$  by  $\bar{K}_2$ , as in the statement of Theorem 5.1. It remains to show that  $f$  satisfies (5.2) on  $K$  with these values of  $r$  and  $\alpha$ .

We select the finite family of points  $y_1, \dots, y_n \in E$  for which the balls  $B_{y_i}$  have non-empty intersection with  $K$ . Since each ball  $B_{y_i}$  has diameter  $\frac{1}{2}$ , we have

$$K \subseteq \bigcup_{i=1, \dots, n} B_{y_i} \subseteq \bar{K}_{1/2}.$$

Note that the 3/2-enlargement of each  $B_{y_i}$  is contained in  $\bar{K}_2$ , the 2-enlargement of  $K$ . Then, by Step I and by the monotonicity properties (4.4)-(4.9) of  $K \mapsto \alpha_K$  and  $K \mapsto \beta_K$ , each distribution  $f^{B_{y_i}}$  satisfies (11.2) for  $K = B_{y_i}$  and for  $r$  and  $\alpha$  chosen as in (11.13). For any test function  $\psi$  supported in  $B(0, 1)$  we can write

$$\xi_y(z) \psi_x^\lambda(z) = \zeta_x^\lambda(z) \quad \text{where} \quad \zeta(z) = \zeta^{[x, y, \lambda]}(z) := \xi_y(x + \lambda z) \psi(z).$$

If we apply (11.2) for  $K = B_y$  to  $\zeta/\|\zeta\|_{C^r} \in \mathcal{B}_r$ , we obtain for  $\gamma < 0$

$$|(f^{B_y} - F_x)(\xi_y \psi_x^\lambda)| = |(f^{B_y} - F_x)(\zeta_x^\lambda)| \leq \mathfrak{C} \|\zeta\|_{C^r} \|F\|_{\bar{K}_{2,\varphi,\alpha,\gamma}}^{\text{coh}} \lambda^\gamma,$$

where  $\mathfrak{C}$  is as in (11.11). Note that, by Leibniz's rule,

$$\|\zeta\|_{C^r} \leq 2^r \|\xi\|_{C^r} \|\psi\|_{C^r}.$$

Then, by definition of  $f$  and by  $\sum_{y \in E} \xi_y \equiv 1$ ,

$$\begin{aligned} |(f - F_x)(\psi_x^\lambda)| &= \left| \sum_{y \in E} (f^{B_y} - F_x)(\xi_y \psi_x^\lambda) \right| \leq \sum_{y \in E} |(f^{B_y} - F_x)(\xi_y \psi_x^\lambda)| \\ &\leq (11\sqrt{d})^d \mathfrak{C} 2^r \|\xi\|_{C^r} \|\psi\|_{C^r} \|F\|_{\bar{K}_{2,\varphi,\alpha,\gamma}}^{\text{coh}} \lambda^\gamma. \end{aligned}$$

The last inequality holds because  $\xi_y \psi_x^\lambda \equiv 0$  unless  $|x - y| \leq \frac{1}{4} + \lambda \leq \frac{5}{4}$  and this can be satisfied by at most  $(2 \cdot \frac{5}{4} \cdot 4\sqrt{d} + 1)^d \leq (11\sqrt{d})^d$  many  $y \in E$ . Therefore  $f \in \mathcal{D}'$  satisfies (5.2) for  $\gamma < 0$ , with  $\mathfrak{c} = \mathfrak{c}_{\alpha,\gamma,r,d,\varphi}$  given as follows (recall (11.11)):

$$\mathfrak{c}_{\alpha,\gamma,r,d,\varphi} = 2^r \|\xi\|_{C^r} (11\sqrt{d})^d \frac{r^2 2^{-(r+1)\alpha} 4^{d+\gamma-\alpha+6}}{1 - 2^{-(\alpha+r)\wedge(-\gamma)}} \frac{\|\varphi\|_{L^1} (1 + R_\varphi)^{-\alpha}}{|\int \varphi|^2} \quad (\text{for } \gamma < 0) \quad (11.14)$$

where  $R_\varphi$  is the radius of a ball  $B(0, R_\varphi)$  which contains the support of  $\varphi$ . With similar arguments, using (11.12), for  $\gamma = 0$  we obtain that  $f \in \mathcal{D}'$  satisfies (5.2), with

$$\mathfrak{c}_{\alpha,\gamma,r,d,\varphi} = 2^r \|\xi\|_{C^r} (11\sqrt{d})^d \frac{r^2 2^{-(r+1)\alpha} 4^{d-\alpha+6}}{1 - 2^{-\alpha-r}} \frac{\|\varphi\|_{L^1} (1 + R_\varphi)^{-\alpha}}{|\int \varphi|^2} \quad (\text{for } \gamma = 0) \quad (11.15)$$

The proof is complete.  $\square$

In the next sections we introduce the spaces of distributions with negative Hölder regularity and we discuss some consequences of the Reconstruction Theorem.

## 12. NEGATIVE HÖLDER SPACES

We generalize the classical Hölder spaces  $\mathcal{C}^\alpha$ , by allowing the index  $\alpha$  to be negative. We recall that the family  $\mathcal{B}_r$  of test functions was defined in (5.1).

**Definition 12.1 (Negative Hölder spaces).** *Given  $\alpha \in (-\infty, 0]$ , we define  $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\mathbb{R}^d)$  as the space of distributions  $T \in \mathcal{D}'$  such that*

$$|T(\psi_x^\varepsilon)| \lesssim \varepsilon^\alpha \quad (12.1)$$

*uniformly for  $x$  in compact sets, for  $\varepsilon \in (0, 1]$  and for  $\psi \in \mathcal{B}_{r_\alpha}$ ,*

*where we define  $r_\alpha$  as the smallest integer  $r \in \mathbb{N}$  such that  $r > -\alpha$ .*

**Remark 12.2.** Other definitions of the space  $\mathcal{C}^0$  are possible, see e.g. [HL17]. The one that we give here is convenient for our goals.



For any distribution  $T \in \mathcal{D}'$  and  $\alpha \leq 0$ , we define  $\|T\|_{\mathcal{C}^\alpha(K)}$  as the best constant in (12.1):

$$\|T\|_{\mathcal{C}^\alpha(K)} := \sup_{x \in K, \lambda \in (0,1], \psi \in \mathcal{B}_{r_\alpha}} \frac{|T(\psi_x^\lambda)|}{\lambda^\alpha}. \quad (12.2)$$

Then  $T \in \mathcal{C}^\alpha$  if and only if  $\|T\|_{\mathcal{C}^\alpha(K)} < \infty$ , for all compact sets  $K \subseteq \mathbb{R}^d$ .

**Remark 12.3.** The quantity  $\|\cdot\|_{\mathcal{C}^\alpha(K)}$  is a semi-norm on  $\mathcal{C}^\alpha$ . It is actually a true norm for distributions  $T$  which are *supported in  $K$* , i.e. such that  $T(\xi) = 0$  for all test functions  $\xi \in \mathcal{D}$  which are supported in  $K^c$ .

Remarkably, in order for a distribution  $T \in \mathcal{D}'$  to belong to  $\mathcal{C}^\alpha$ , it is enough that (12.1) holds for a single, arbitrary test function  $\psi = \varphi$  with  $\int \varphi \neq 0$ , rather than uniformly for  $\psi \in \mathcal{B}_{r_\alpha}$ . This is ensured by our next results, that we prove below using the same ideas as in the proof of the Reconstruction Theorem.

**Theorem 12.4 (Characterization of negative Hölder spaces).** *Given a distribution  $T \in \mathcal{D}'$  and  $\alpha \in (-\infty, 0]$ , the following conditions are equivalent.*

- (1)  $T$  is in  $\mathcal{C}^\alpha$
- (2) There is an integer  $r > -\alpha$  such that (12.1) holds with  $\mathcal{B}_{r_\alpha}$  replaced by  $\mathcal{B}_r$ .
- (3) There is a test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  such that

$$|T(\varphi_x^\varepsilon)| \lesssim \varepsilon^\alpha$$

uniformly for  $x$  in compact sets and for  $\varepsilon \in \{2^{-k}\}_{k \in \mathbb{N}} \subseteq (0, 1]$ .

Moreover, the semi-norm  $\|T\|_{\mathcal{C}^\alpha(K)}$  defined in (12.2) can be estimated explicitly using an arbitrary test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$ :

$$\|T\|_{\mathcal{C}^\alpha(K)} \leq \mathfrak{b}_{\varphi, \alpha, r_\alpha, d} \sup_{x \in \bar{K}_2, \varepsilon \in (0,1]} \frac{|T(\varphi_x^\varepsilon)|}{\varepsilon^\alpha} \quad (12.3)$$

where  $\mathfrak{b}_{\varphi, \alpha, r, d}$  is an explicit constant, defined in (12.19) below.

We deduce a simple *countable* criterion for a distribution  $T \in \mathcal{D}'$  to belong to  $\mathcal{C}^\alpha$ .

**Theorem 12.5 (Countable criterion for negative Hölder spaces).** *Let  $\alpha \leq 0$  and  $T \in \mathcal{D}'$ . Then  $T \in \mathcal{C}^\alpha$  if (and only if) there is a test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  such that, for every fixed  $n \in \mathbb{N}$ , we have*

$$|T(\varphi_x^\varepsilon)| \lesssim \varepsilon^\alpha \quad (12.4)$$

uniformly for  $x \in \mathbb{Q}^d \cap B(0, n)$  and  $\varepsilon \in \{2^{-k}\}_{k \in \mathbb{N}}$ .

*Proof.* The map  $x \mapsto \varphi_x^\varepsilon \in \mathcal{D}$  is continuous, hence  $x \mapsto T(\varphi_x^\varepsilon)$  is a continuous function. It follows that (12.4) holds for all  $x \in B(0, n)$ , so Theorem 12.4 applies.  $\square$

We finally turn to the proof of Theorem 12.4, that we obtain as a corollary of the following more general result, proved at the end of this section.

**Proposition 12.6.** *Let  $T \in \mathcal{D}'(\mathbb{R}^d)$  be a distribution with the following property: there are a subset  $K \subseteq \mathbb{R}^d$  and a test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  such that*

$$\forall x \in \bar{K}_2, \quad \forall \varepsilon \in \{2^{-k}\}_{k \in \mathbb{N}} : \quad |T(\varphi_x^\varepsilon)| \leq \varepsilon^\alpha f(\varepsilon, x), \quad (12.5)$$

for some exponent  $\alpha \leq 0$  and some arbitrary function  $f : (0, 1] \times \bar{K}_2 \rightarrow [0, \infty)$ .

Then we can upgrade relation (12.5) as follows: for any integer  $r > -\alpha$ ,

$$\forall x \in K, \quad \forall \lambda \in (0, 1], \quad \forall \psi \in \mathcal{B}_r : \quad |T(\psi_x^\lambda)| \leq \mathfrak{b}_{\varphi, \alpha, r, d} \lambda^\alpha \bar{f}(\lambda, x), \quad (12.6)$$

where  $\mathfrak{b}_{\varphi, \alpha, r, d}$  is the constant in (12.19) below, and  $\bar{f} : (0, 1] \times K \rightarrow [0, \infty)$  equals

$$\bar{f}(\lambda, x) := \sup_{\lambda' \in (0, \lambda], \quad x' \in B(x, 2\lambda)} f(\lambda', x'). \quad (12.7)$$

*Proof of Theorem 12.4.* Clearly 1. implies 2., because  $\mathcal{B}_r \subseteq \mathcal{B}_{r_\alpha}$  for  $r \geq r_\alpha$ , and 2. implies 3., because we can choose any  $\varphi = \psi \in \mathcal{B}_r$  with  $\int \psi \neq 0$ .

To prove that 3. implies 1., it suffices to apply Proposition 12.6 on every compact set with a constant function  $f(\lambda, x) \equiv C$ . Equation (12.3) then follows by (12.6).  $\square$

We next show that the reconstruction  $f = \mathcal{R}F$  of a coherent germ  $F$  provided by the Reconstruction Theorem belongs to a negative Hölder space and it is a continuous function of the germ, in a suitable sense.

We recall that the *coherence* of a germ is quantified by the semi-norm  $\|F\|_{\bar{K}_1, \varphi, \alpha, \gamma}^{\text{coh}}$  defined in (4.5). We introduce a second semi-norm which quantifies the *homogeneity* of a coherent germ: for any compact set  $K \subset \mathbb{R}^d$  we define, recalling Lemma 4.12,

$$\|F\|_{K, \varphi, \beta}^{\text{hom}} := \sup_{x \in K, \quad \varepsilon \in (0, 1]} \frac{|F_x(\varphi_x^\varepsilon)|}{\varepsilon^\beta}, \quad (12.8)$$

where  $\varphi$  is as in Definition 4.3. We can now state the following result.

**Theorem 12.7 (Reconstruction Theorem and Hölder spaces).** *Let  $(F_x)_{x \in \mathbb{R}^d}$  be a  $(\alpha, \gamma)$ -coherent germ with local homogeneity bound  $\beta < \gamma$ . If  $\beta > 0$ , then  $\mathcal{R}F = 0$ . If  $\beta \leq 0$ , then  $\mathcal{R}F$  belongs to  $\mathcal{C}^\beta$  and for every compact set  $K \subseteq \mathbb{R}^d$*

$$\|\mathcal{R}F\|_{\mathcal{C}^\beta(K)} \leq \mathfrak{C} \left( \|F\|_{\bar{K}_4, \varphi, \alpha, \gamma}^{\text{coh}} + \|F\|_{\bar{K}_2, \varphi, \beta}^{\text{hom}} \right), \quad (12.9)$$

where  $\varphi$  be the test function in the coherence condition (4.2) and  $\mathfrak{C} = \mathfrak{C}_{\alpha, \gamma, \beta, d, \varphi} < \infty$  is a constant which depends neither on  $F$  nor on  $K$ .

**Remark 12.8.** The bound (12.9) holds for any test function  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$ , as for the coherence condition (4.2). This will be shown in Proposition 13.1.

*Proof.* When  $\beta > 0$  we already observed in Remark 4.15 that  $\mathcal{R}F = 0$ . Henceforth we fix  $\beta \leq 0$ . Let  $\varphi$  be the test function in the coherence condition (4.2). Let  $f = \mathcal{R}F$  by a reconstruction of  $F$ . Fix a compact set  $K$ : if we show that

$$\sup_{x \in \bar{K}_2, \quad \lambda \in (0, 1]} \frac{|f(\varphi_x^\lambda)|}{\lambda^\beta} \leq \mathfrak{C}' \left( \|F\|_{\bar{K}_4, \varphi, \alpha, \gamma}^{\text{coh}} + \|F\|_{\bar{K}_2, \varphi, \beta}^{\text{hom}} \right) \quad (12.10)$$

for some  $\mathfrak{C}' = \mathfrak{C}'_{\alpha,\gamma,\beta,d,\varphi} < \infty$ , then we obtain (12.9) by (12.3) with  $\mathfrak{C} = \mathfrak{b}_{\varphi,\beta,r,\beta,d} \mathfrak{C}'$ .

It remains to prove (12.10). Let us set  $\bar{r} := \min\{r \in \mathbb{N} : r > \max\{-\alpha, -\beta\}\}$ . We observed in Remark 5.2 that  $\xi := c\varphi^\eta \in \mathcal{B}_{\bar{r}}$  for suitable  $c, \eta > 0$  (which depend on  $\varphi$  and  $\bar{r}$ ). Then by (5.2) for  $r = \bar{r}$  we have, uniformly for  $x \in \bar{K}_2$  and  $\lambda \in (0, 1]$ ,

$$|(f - F_x)(\varphi_x^\lambda)| = c^{-1} |(f - F_x)(\psi_x^{\eta^{-1}\lambda})| \leq \mathfrak{C}' \|F\|_{\bar{K}_4, \varphi, \alpha, \gamma}^{\text{coh}} \cdot \begin{cases} \lambda^\gamma & \text{if } \gamma \neq 0 \\ (1 + |\log \lambda|) & \text{if } \gamma = 0 \end{cases}$$

for a suitable  $\mathfrak{C}' = \mathfrak{C}'_{\alpha,\gamma,\beta,d,\varphi}$ . Since  $\beta < \gamma$ , we bound  $\lambda^\gamma \leq \lambda^\beta$  for  $\gamma \neq 0$  and  $1 + |\log \lambda| \leq c_\beta \lambda^\beta$  for  $\gamma = 0$ , for all  $\lambda \in (0, 1]$  (by direct computation  $c_\beta = -\beta^{-1} e^{-1-\beta}$ ). Recalling (12.8), by the triangle inequality we obtain

$$\begin{aligned} \sup_{x \in \bar{K}_2, \lambda \in (0,1]} \frac{|f(\varphi_x^\lambda)|}{\lambda^\beta} &\leq \sup_{x \in \bar{K}_2, \lambda \in (0,1]} \frac{|(f - F_x)(\varphi_x^\lambda)| + |F_x(\varphi_x^\lambda)|}{\lambda^\beta} \\ &\leq (1 + c_\beta) \mathfrak{C}' \|F\|_{\bar{K}_4, \varphi, \alpha, \gamma}^{\text{coh}} + \|F\|_{\bar{K}_2, \varphi, \beta}^{\text{hom}}, \end{aligned}$$

which completes the proof of (12.10).  $\square$

**Remark 12.9 (Non uniqueness).** Let  $(F_x)_{x \in \mathbb{R}^d}$  be a  $(\alpha, \gamma)$ -coherent germ with  $\gamma < 0$  and let  $f_1$  and  $f_2$  be two distributions which both satisfy (5.2). Then

$$|(f_1 - f_2)(\psi_x^\lambda)| \leq |(f_1 - F_x)(\psi_x^\lambda)| + |(f_2 - F_x)(\psi_x^\lambda)| \lesssim \lambda^\gamma$$

uniformly for  $x$  in compact sets and  $\lambda \in (0, 1]$  and therefore  $f_1 - f_2 \in \mathcal{C}^\gamma$ , by Theorem 12.4. Viceversa, if  $f \in \mathcal{D}'$  satisfies (5.2) and  $D \in \mathcal{C}^\gamma$ , then  $f + D$  also satisfies (5.2). Therefore, *the reconstruction  $f = \mathcal{R}F$  of a  $(\alpha, \gamma)$ -coherent germ  $F$  with  $\gamma < 0$  is not unique, but it is well-defined up to an element of  $\mathcal{C}^\gamma$ .*

We conclude this section with the proof of Proposition 12.6.

*Proof of Proposition 12.6.* Fix  $\varphi \in \mathcal{D}$  with  $\int \varphi \neq 0$  which satisfies (12.5) and  $r \in \mathbb{N}$  with  $r > -\alpha$ . We define the test function  $\hat{\varphi} = \hat{\varphi}^{[r]}$  by (8.4) and we claim that

$$\forall x \in \bar{K}_2, \quad \forall \varepsilon \in \{2^{-k}\}_{k \in \mathbb{N}} : \quad |T(\hat{\varphi}_x^\varepsilon)| \leq C \varepsilon^\alpha \tilde{f}(\varepsilon, x), \quad (12.11)$$

where

$$\tilde{f}(\varepsilon, x) := \sup_{\varepsilon' \in (0, \varepsilon]} f(\varepsilon', x), \quad C := \frac{\varepsilon^2 r}{|\int \varphi|} \left( \frac{2^{-r-1}}{1+R_\varphi} \right)^\alpha. \quad (12.12)$$

To prove this claim, it suffices to write  $T(\hat{\varphi}_x^\varepsilon) = \frac{1}{\int \varphi} \sum_{i=0}^{r-1} c_i T(\varphi_x^{\varepsilon \lambda_i})$  and to apply (12.5) to  $T(\varphi_x^{\varepsilon \lambda_i})$ , noting that  $\frac{2^{-r-1}}{1+R_\varphi} < \lambda_i \leq 1$  by (8.4) and  $|c_i| \leq e^2$  by (8.8).

We recall that  $\hat{\varphi}$  satisfies (8.5)-(8.6) as well as (8.7). Next we define

$$\rho := \hat{\varphi}^2 * \hat{\varphi}, \quad \varepsilon_k = 2^{-k},$$

as in (10.7) above. Then, see (10.9),

$$\rho^{\varepsilon_{k+1}} - \rho^{\varepsilon_k} = \hat{\varphi}^{\varepsilon_{k+1}} * \check{\varphi}^{\varepsilon_k} \quad \text{where} \quad \check{\varphi} := \hat{\varphi}^{\frac{1}{2}} - \hat{\varphi}^2. \quad (12.13)$$

Note that  $(\rho^{\varepsilon_n})_{n \in \mathbb{N}}$  are mollifiers, because  $\int \rho = \int \hat{\varphi} \cdot \int \hat{\varphi}^2 = 1$  (recall that  $\int \hat{\varphi} = 1$ ), therefore for any test function  $\psi$  we have

$$T(\psi_x^\lambda) = \lim_{n \rightarrow \infty} T(\rho^{\varepsilon_n} * \psi_x^\lambda) \quad (12.14)$$

hence for every  $N \in \mathbb{N}$  we can write

$$T(\psi_x^\lambda) = \underbrace{T(\rho^{\varepsilon_N} * \psi_x^\lambda)}_A + \underbrace{(T(\psi_x^\lambda) - T(\rho^{\varepsilon_N} * \psi_x^\lambda))}_B. \quad (12.15)$$

Henceforth we fix  $\psi \in \mathcal{B}_r$  and we set  $N := \min\{k \in \mathbb{N} : \varepsilon_k \leq \lambda\}$  so that  $N \geq 1$  and

$$\frac{1}{2}\lambda < \varepsilon_N \leq \lambda. \quad (12.16)$$

We estimate separately the two terms  $A$  and  $B$  in (12.15).

*Estimate of A.* We can write

$$\begin{aligned} A &= T(\rho^{\varepsilon_N} * \psi_x^\lambda) = \int_{\mathbb{R}^d} T(\rho_z^{\varepsilon_N}) \psi_x^\lambda(z) \, dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T(\hat{\varphi}_y^{\varepsilon_N}) \hat{\varphi}^{2\varepsilon_N}(y-z) \psi_x^\lambda(z) \, dy \, dz \\ &= \int_{\mathbb{R}^d} T(\hat{\varphi}_y^{\varepsilon_N}) (\hat{\varphi}^{2\varepsilon_N} * \psi_x^\lambda)(y) \, dy. \end{aligned}$$

We now apply Lemma 9.3 with  $G(y) := T(\hat{\varphi}_y^\varepsilon)$ : by (9.8) we obtain

$$|A| \leq 2^d \|\hat{\varphi}\|_{L^1} \sup_{y \in B(x, \lambda + \varepsilon_N)} |T(\hat{\varphi}_y^{\varepsilon_N})|.$$

By (12.16) we have  $\lambda + \varepsilon_N \leq 2\lambda$  and  $\varepsilon_N \geq \lambda/2$ . Since  $\alpha \leq 0$ , we obtain by (12.11)

$$\sup_{y \in B(x, \lambda + \varepsilon_N)} |T(\hat{\varphi}_y^{\varepsilon_N})| \leq C \varepsilon_N^\alpha \sup_{y \in B(x, 2\lambda)} \tilde{f}(\varepsilon_N, y) \leq C (\lambda/2)^\alpha \sup_{\lambda' \in (0, \lambda], x' \in B(x, 2\lambda)} f(\lambda', x'),$$

and finally, recalling (12.7),

$$|A| \leq \{2^{d-\alpha} C \|\hat{\varphi}\|_{L^1}\} \lambda^\alpha \bar{f}(\lambda, x). \quad (12.17)$$

*Estimate of B.* Let us fix  $k \in \mathbb{N}$  with  $k \geq N$ . We can write, by (12.13),

$$\begin{aligned} b_k &:= T(\rho^{\varepsilon_{k+1}} * \psi_x^\lambda) - T(\rho^{\varepsilon_k} * \psi_x^\lambda) = \int_{\mathbb{R}^d} T(\rho_z^{\varepsilon_{k+1}} - \rho_z^{\varepsilon_k}) \psi_x^\lambda(z) \, dz \\ &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} T(\hat{\varphi}_y^{\varepsilon_k}) \check{\varphi}^{\varepsilon_k}(y-z) \psi_x^\lambda(z) \, dy \, dz \\ &= \int_{\mathbb{R}^d} T(\hat{\varphi}_y^{\varepsilon_k}) (\check{\varphi}^{\varepsilon_k} * \psi_x^\lambda)(y) \, dy. \end{aligned}$$

Note that  $\check{\varphi}$  is supported in  $B(0, 1)$  (because  $\hat{\varphi}$  is supported in  $B(0, \frac{1}{2})$ , recall (8.5)) and  $\varepsilon_k \leq \varepsilon_N \leq \lambda$  for  $k \geq N$ . Then  $\check{\varphi}^{\varepsilon_k} * \psi_x^\lambda$  is supported in  $B(w, \lambda + \varepsilon_k) \subseteq B(w, 2\lambda)$ . We apply again Lemma 9.3 with  $G(y) := T(\hat{\varphi}_y^{\varepsilon_k})$ : by (12.11) and (12.7) we can bound  $\sup_{y \in B(w, \lambda + \varepsilon_k)} |G(y)| \leq C \varepsilon_k^\alpha \bar{f}(\varepsilon_k, w) \leq C \varepsilon_k^\alpha f(\lambda, w)$  which yields, by (9.9),

$$|b_k| \leq C 4^d \|\check{\varphi}\|_{L^1} \lambda^{-r} \varepsilon_k^{\alpha+r} \bar{f}(\lambda, w).$$

Since  $\alpha + r > 0$  by assumption, we obtain  $\sum_{k \geq N} |b_k| < +\infty$  and, recalling (12.14), we can write  $B = T(\psi_x^\lambda) - T(\rho^{\varepsilon_N} * \psi_x^\lambda)$  as the converging sequence  $B = \sum_{k=N}^{\infty} b_k$ . Since  $\sum_{k=N}^{\infty} \varepsilon_k^{\alpha+r} = (1 - 2^{-\alpha-r})^{-1} \varepsilon_N^{\alpha+r}$ , this yields

$$|B| \leq \sum_{k=N}^{\infty} |b_k| \leq \frac{C 4^d \|\check{\varphi}\|_{L^1}}{1 - 2^{-\alpha-r}} \lambda^{-r} \varepsilon_N^{\alpha+r} \bar{f}(\lambda, w). \quad (12.18)$$

*Conclusion.* By (12.15), (12.17) and (12.18), since  $\|\check{\varphi}\|_{L^1} \leq 2\|\hat{\varphi}\|_{L^1}$  and  $\varepsilon_N \leq \lambda$ , we get

$$|T(\psi_x^\lambda)| \leq \frac{4^{d-\alpha+1}}{1 - 2^{-\alpha-r}} \|\hat{\varphi}\|_{L^1} C \lambda^\alpha \bar{f}(\lambda, w).$$

If we plug the bound (8.7) and the definition (12.12) of  $C$ , we get

$$|T(\psi_x^\lambda)| \leq \left\{ \frac{4^{d-\alpha+1}}{1 - 2^{-\alpha-r}} \left( \frac{e^2 r}{|\int \varphi|} \right)^2 \left( \frac{2^{-r-1}}{1 + R_\varphi} \right)^\alpha \|\varphi\|_{L^1} \right\} \lambda^\alpha \bar{f}(\lambda, w).$$

Therefore we have proved (12.6), with the explicit constant

$$\mathfrak{b}_{\varphi, \alpha, r, d} := \frac{4^{d-\alpha+1} e^4 2^{-\alpha(r+1)} r^2 (1 + R_\varphi)^{-\alpha} \|\varphi\|_{L^1}}{1 - 2^{-\alpha-r} |\int \varphi|} \quad (12.19)$$

The proof is complete.  $\square$

### 13. MORE ON COHERENT GERMS

As an application of Proposition 12.6, we show that the coherence condition (4.2) can be strengthened, replacing the test function  $\varphi$  by an arbitrary test function, provided we slightly adjust the exponent  $\alpha_K$ .

**Proposition 13.1 (Enhanced coherence).** *Let  $F = (F_x)_{x \in \mathbb{R}^d}$  be a  $\gamma$ -coherent germ, i.e. (4.2) holds for some  $\varphi \in \mathcal{D}$  and some family  $\alpha = (\alpha_K)$ . If we define*

$$\alpha' = (\alpha'_K) \quad \text{where} \quad \alpha'_K := \alpha_{\bar{K}_2}, \quad (13.1)$$

*then we can replace  $\varphi$  in (4.2) by an arbitrary test function, provided we replace  $\alpha_K$  by  $\alpha'_K$ . More precisely, for any compact set  $K \subseteq \mathbb{R}^d$  and any  $r > -\alpha'_K$  we have*

$$|(F_z - F_y)(\psi_y^\varepsilon)| \lesssim \varepsilon^{\alpha'_K} (|z - y| + \varepsilon)^{\gamma - \alpha'_K} \quad (13.2)$$

*uniformly for  $z, y \in K$ ,  $\varepsilon \in (0, 1]$  and  $\psi \in \mathcal{B}_r$ .*

*It follows that the family of  $\gamma$ -coherent germs is a vector space.*

*Proof.* Assume that (13.2) has been proved. Given an arbitrary test function  $\xi \in \mathcal{D}$ , we can write  $\xi = c\psi^\lambda$  for suitable  $c \in \mathbb{R}$ ,  $\lambda \in (0, 1]$  and  $\psi \in \mathcal{B}_r$  (exercise), hence  $\xi_y^\varepsilon = c\psi_y^{\lambda\varepsilon}$ . Then it follows by (13.2) that we can replace  $\varphi$  by  $\xi$  in (4.2).

It remains to prove (13.2). It is convenient to center the test function at a third point  $x$ , i.e. to replace  $\psi_y^\varepsilon$  by  $\psi_x^\varepsilon$ . By the triangle inequality we can bound

$$|(F_z - F_y)(\varphi_x^\varepsilon)| \leq |(F_z - F_x)(\varphi_x^\varepsilon)| + |(F_y - F_x)(\varphi_x^\varepsilon)|. \quad (13.3)$$

Let us fix a compact set  $K \subseteq \mathbb{R}^d$ . Both terms in the right hand side of (13.3) can be estimated by the coherence condition (4.2) for the enlarged set  $\bar{K}_2$ . Recalling (13.1), we see that there is  $c_K < \infty$  such that

$$\begin{aligned} & \forall z, y \in K, \forall x \in \bar{K}_2, \forall \varepsilon \in (0, 1] : \\ & |(F_z - F_y)(\varphi_x^\varepsilon)| \leq c_K \varepsilon^{\alpha'_K} (|z - x| + |y - x| + \varepsilon)^{\gamma - \alpha'_K}. \end{aligned}$$

For fixed  $y, z \in K$  we can apply Proposition 12.6, with  $T = F_z - F_y$  and  $f(\varepsilon, x) = (|z - x| + |y - x| + \varepsilon)^{\gamma - \alpha'_K}$ . Given any  $r \in \mathbb{N}$  with  $r > -\alpha'_K$ , relation (12.6) yields

$$\begin{aligned} & \forall z, y, x \in K, \forall \lambda \in (0, 1], \forall \psi \in \mathcal{B}_r : \\ & |(F_z - F_y)(\psi_x^\lambda)| \leq \mathfrak{b}_{\varphi, \alpha'_K, r, d} \lambda^{\alpha'_K} (|z - x| + |y - x| + 5\lambda)^{\gamma - \alpha'_K} \\ & \lesssim \lambda^{\alpha'_K} (|z - x| + |y - x| + \lambda)^{\gamma - \alpha'_K}. \end{aligned}$$

If we plug  $x = y$  we obtain (13.2).  $\square$

We now show that also the local homogeneity relation (4.8) can be strengthened, replacing  $\varphi$  by an arbitrary test function, provided we slightly adjust  $\beta_K$ .

**Proposition 13.2 (Enhanced local homogeneity).** *Let  $F = (F_x)_{x \in \mathbb{R}^d}$  be a  $\gamma$ -coherent germ with local homogeneity bounds  $\beta = (\beta_K)$ , see (4.8). If we set*

$$\beta' = (\beta'_K) \quad \text{where} \quad \beta'_K := \beta_{\bar{K}_2},$$

*then we can replace  $\varphi$  in (4.8) by an arbitrary test function, provided we replace  $\beta_K$  by  $\beta'_K$ . More precisely, for any compact set  $K \subseteq \mathbb{R}^d$  and any  $r > \max\{-\alpha'_K, -\beta'_K\}$ , with  $\alpha'_K$  defined in (13.1), we have*

$$\begin{aligned} & |F_x(\psi_x^\varepsilon)| \lesssim \varepsilon^{\beta'_K} \\ & \text{uniformly for } x \in K, \varepsilon \in (0, 1] \text{ and } \psi \in \mathcal{B}_r. \end{aligned} \tag{13.4}$$

*Proof.* We apply the Reconstruction Theorem: let  $f = \mathcal{R}F$  is a reconstruction of  $F$ . Fix a compact set  $K \subseteq \mathbb{R}^d$  and  $r > \max\{-\alpha_{\bar{K}_2}, -\beta_{\bar{K}_2}\}$ . Then, by (5.2),

$$|(f - F_x)(\psi_x^\varepsilon)| \lesssim \begin{cases} \varepsilon^\gamma & \text{if } \gamma \neq 0 \\ (1 + |\log \varepsilon|) & \text{if } \gamma = 0 \end{cases}$$

uniformly for  $x \in K, \varepsilon \in (0, 1]$  and  $\psi \in \mathcal{B}_r$ . Since  $f \in \mathcal{C}^\beta$  by Theorem 12.7, we have

$$|f(\psi_x^\varepsilon)| \lesssim \varepsilon^\beta$$

uniformly for  $x \in K, \varepsilon \in (0, 1]$  and  $\psi \in \mathcal{B}_r$ . Since  $\beta < \gamma$ , we finally get

$$|F_x(\psi_x^\varepsilon)| \leq |(F_x - f)(\psi_x^\varepsilon)| + |f(\psi_x^\varepsilon)| \lesssim \varepsilon^\beta,$$

uniformly for  $x \in K, \varepsilon \in (0, 1]$  and  $\psi \in \mathcal{B}_r$ . This proves (13.4).  $\square$

## 14. YOUNG PRODUCT OF FUNCTIONS AND DISTRIBUTIONS

As an application of the Reconstruction Theorem, we prove that there is a canonical definition of product between a Hölder function  $f \in \mathcal{C}^\alpha$ , with  $\alpha > 0$ , and a Hölder distribution  $g \in \mathcal{C}^\beta$ , with  $\beta \leq 0$ , provided  $\alpha + \beta > 0$ . This classical result has been obtained with wavelets analysis or Bony's paraproducts, see e.g. [RS96, Theorem 1 in Section 4.4.3], [BCD11, Theorem 2.52] and [Hai14, Proposition 4.14]. Our proof of the Reconstruction Theorem provides a new approach to this result, which bypasses Fourier analysis and applies to general (non tempered) distributions. In the case  $\alpha + \beta \leq 0$ , a non-unique and non-canonical “product” can still be constructed.

We start with some general considerations. Given any distribution  $g \in \mathcal{D}'$  and any smooth function  $f \in C^\infty$ , their product  $P = g \cdot f$  is canonically defined by

$$P(\varphi) = (g \cdot f)(\varphi) := g(\varphi f), \quad \forall \varphi \in \mathcal{D}.$$

If  $f \in \mathcal{C}^\alpha$  with  $\alpha > 0$  this no longer makes sense, as  $\varphi f$  might not be a test function. However we can still give a *local definition of  $g \cdot f$  close to a point  $x \in \mathbb{R}^d$* , replacing  $f$  by its Taylor polynomial  $F_x$  of order  $\underline{r}(\alpha) := \max\{n \in \mathbb{N}_0 : n < \alpha\}$  based at  $x$ :

$$F_x(\cdot) := \sum_{0 \leq |k| < \alpha} \partial^k f(x) \frac{(\cdot - x)^k}{k!}. \quad (14.1)$$

This leads us to define the germ  $P = (P_x := g \cdot F_x)_{x \in \mathbb{R}^d}$ , that is

$$P_x(\varphi) = (g \cdot F_x)(\varphi) := g(\varphi F_x), \quad \varphi \in \mathcal{D}. \quad (14.2)$$

We can now state the following result.

**Theorem 14.1 (Young product).** *Fix  $\alpha > 0$  and  $\beta \leq 0$ .*

- *If  $\alpha + \beta > 0$ , there exists a bilinear continuous map  $\mathcal{M} : \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^\beta$  which extends the usual product  $\mathcal{M}(f, g) = f \cdot g$  when  $f \in C^\infty$ . This map is characterized by the following property: for any  $r \in \mathbb{N}$  with  $r > -\beta$*

$$|(\mathcal{M}(f, g) - g \cdot F_x)(\psi_x^\lambda)| \lesssim \begin{cases} \lambda^{\alpha+\beta} & \text{if } \alpha + \beta \neq 0 \\ (1 + |\log \lambda|) & \text{if } \alpha + \beta = 0 \end{cases} \quad (14.3)$$

*uniformly for  $x$  in compact sets,  $\lambda \in (0, 1]$  and  $\psi \in \mathcal{B}_r$ ,*

*where  $F_x$  is the Taylor polynomial of  $f$  based at  $x$ , see (14.1).*

- *If  $\alpha + \beta \leq 0$ , there exists a bilinear continuous map  $\mathcal{M} : \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^\beta$  which satisfies property (14.3). This map is neither unique nor canonical. However, for  $\alpha + \beta < 0$  any two maps  $\mathcal{M}, \mathcal{M}'$  which satisfy property (14.3) must differ by a map in  $\mathcal{C}^{\alpha+\beta}$ , i.e. we must have  $\mathcal{M} - \mathcal{M}' : \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^{\alpha+\beta}$ .*

**Remark 14.2.** For fixed  $\alpha > 0$  and  $\beta \leq 0$  with  $\alpha + \beta > 0$ , we cannot claim that  $\mathcal{M} : \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^\beta$  is the unique continuous map which extends the usual product  $\mathcal{M}(f, g) = f \cdot g$  when  $f \in C^\infty$ , simply because  $C^\infty$  is not dense in  $\mathcal{C}^\alpha$ . On the other hand, given any  $\beta \leq 0$ , we can state that  $\mathcal{M} : \bigcup_{\alpha > -\beta} \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^\beta$  is indeed the

unique continuous map which extends the usual product, because  $C^\infty$  is dense in  $C^\alpha$  with respect to the topology of  $C^{\alpha'}$ , for any  $\alpha' < \alpha$ .

**Remark 14.3.** For  $\alpha + \beta \leq 0$  the “product”  $\mathcal{M}$  that we construct is *non-local*, as can be inferred from the proof of the Reconstruction Theorem. This is reminiscent of the para-products studied by Gubinelli-Imkeller-Perkowski [GIP15].

Before proving Theorem 14.1 we need some preparation. We recall that the negative Hölder space  $\mathcal{C}^\beta$  with  $\beta \leq 0$  is equipped with the family of semi-norms  $\|\cdot\|_{\mathcal{C}^\beta(K)}$  defined in (12.2), for compact sets  $K \subseteq \mathbb{R}^d$ . We now introduce a corresponding family of semi-norms  $\|\varphi\|_{\mathcal{C}^\alpha(K)}$  for *positive* Hölder spaces  $\mathcal{C}^\alpha$  with  $\alpha > 0$ . Recall that

$$\underline{r}(\alpha) := \max\{n \in \mathbb{N}_0 : n < \alpha\}.$$

Then, given a compact set  $K \subseteq \mathbb{R}^d$ , we define  $\|\cdot\|_{\mathcal{C}^\alpha(K)}$  by taking the maximum between  $\|f\|_{\mathcal{C}^{\underline{r}(K)}}$  and the best implicit constant in (2.3) when  $x, y \in K$ :

$$\|f\|_{\mathcal{C}^\alpha(K)} := \max \left\{ \|f\|_{\mathcal{C}^{\underline{r}(K)}}, \sup_{x, y \in K} \frac{|f(y) - F_x(y)|}{|y - x|^\alpha} \right\}. \quad (14.4)$$

We can now formulate more precisely the continuity of  $\mathcal{M}$  stated in Theorem 14.1: we are going to prove that for every compact set  $K \subseteq \mathbb{R}^d$

$$\|\mathcal{M}(f, g)\|_{\mathcal{C}^\beta(K)} \lesssim \|f\|_{\mathcal{C}^\alpha(\bar{K}_4)} \|g\|_{\mathcal{C}^\beta(\bar{K}_4)}. \quad (14.5)$$

To prove Theorem 14.1, we first quantify the coherence of the germ  $P$  in (14.2).

**Proposition 14.4.** *If  $f \in \mathcal{C}^\alpha$  and  $g \in \mathcal{C}^\beta$ , with  $\alpha > 0$  and  $\beta \leq 0$ , then the germ  $P = (P_x)_{x \in \mathbb{R}^d}$  is  $(\beta, \alpha + \beta)$ -coherent and has homogeneity bounded below by  $\beta$ .*

*Proof.* We are going to show that there is a test function  $\varphi \in \mathcal{D}(B(0, 1))$  with  $\int \varphi \neq 0$  such that, for every compact set  $K \subset \mathbb{R}^d$ , the following relations hold:

$$|(P_z - P_y)(\varphi_y^\varepsilon)| \lesssim \|f\|_{\mathcal{C}^\alpha(K)} \|g\|_{\mathcal{C}^\beta(K)} \varepsilon^\beta (|z - y| + \varepsilon)^\alpha, \quad (14.6)$$

$$|P_x(\varphi_x^\varepsilon)| \lesssim \|f\|_{\mathcal{C}^\alpha(K)} \|g\|_{\mathcal{C}^\beta(K)} \varepsilon^\beta, \quad (14.7)$$

uniformly for  $x, y, z \in K$  and  $\varepsilon \in (0, 1]$ . Throughout this proof, all implicit constants hidden in the notation  $\lesssim$  may depend on the parameters  $\alpha, \beta$ , but not on  $K, f, g$ .

We first prove (14.6). Let us fix a compact set  $K \subset \mathbb{R}^d$  and we set  $r = r_\beta := \min\{r \in \mathbb{N} : r > -\beta\}$ . By (12.1) applied to  $\psi/\|\psi\|_{\mathcal{C}^r}$  we can bound, recalling (12.2),

$$|g(\psi_y^\varepsilon)| \leq \|g\|_{\mathcal{C}^\beta(K)} \|\psi\|_{\mathcal{C}^r} \varepsilon^\beta \quad \text{for all } \varepsilon \in (0, 1], \psi \in \mathcal{D}(B(0, 1)), y \in K. \quad (14.8)$$

Fix now any  $\varphi \in \mathcal{D}(B(0, 1))$  with  $\int \varphi \neq 0$  and  $\|\varphi\|_{\mathcal{C}^r} \leq 1$ . By (4.7), for any  $y, z \in K$

$$(P_z - P_y)(\varphi_y^\varepsilon) = - \sum_{0 \leq |k| < \alpha} g((\cdot - y)^k \varphi_y^\varepsilon) \frac{R^k(y, z)}{k!}$$

where  $|R^k(y, z)| \lesssim \|f\|_{\mathcal{C}^\alpha(K)} |z - y|^{\alpha - |k|}$ . We have for fixed  $y \in \mathbb{R}^d$ ,  $k \in \mathbb{N}_0^d$  and  $\varepsilon > 0$

$$(w - y)^k \varphi_y^\varepsilon(w) = \varepsilon^{|k|} \psi_y^\varepsilon(w), \quad \text{where } \psi(w) := w^k \varphi(w).$$



Then  $\psi \in \mathcal{D}(B(0, 1))$  and  $\|\psi\|_{C^r} \lesssim \|\varphi\|_{C^r} \leq 1$ , hence it follows by (14.8) that

$$|g((\cdot - y)^k \varphi_y^\varepsilon)| = \varepsilon^{|k|} g(\psi_y^\varepsilon) \lesssim \|g\|_{C^\beta(K)} \varepsilon^{\beta+|k|}. \quad (14.9)$$

We thus obtain, uniformly for  $z, y \in K$  and  $\varepsilon \in (0, 1]$ ,

$$\begin{aligned} |(P_z - P_y)(\varphi_y^\varepsilon)| &\lesssim \|f\|_{C^\alpha(K)} \|g\|_{C^\beta(K)} \sum_{0 \leq |k| < \alpha} \varepsilon^{\beta+|k|} |z - y|^{\alpha-|k|} \\ &\lesssim \|f\|_{C^\alpha(K)} \|g\|_{C^\beta(K)} \varepsilon^\beta (|z - y| + \varepsilon)^\alpha, \end{aligned}$$

which completes the proof of (14.6).

We next prove (14.7). By (14.1) and (14.2), recalling (14.4) and (14.9), we obtain

$$\begin{aligned} |P_x(\varphi_x^\varepsilon)| &\leq \sum_{0 \leq |k| < \gamma} |g((\cdot - x)^k \varphi_x^\varepsilon)| \left| \frac{\partial^k f(x)}{k!} \right| \lesssim \|g\|_{C^\beta(K)} \sum_{0 \leq |k| < \gamma} \varepsilon^{\beta+|k|} \left| \frac{\partial^k f(x)}{k!} \right| \\ &\lesssim \|f\|_{C^\alpha(K)} \|g\|_{C^\beta(K)} \sum_{0 \leq |k| < \gamma} \varepsilon^{\beta+|k|} \lesssim \|f\|_{C^\alpha(K)} \|g\|_{C^\beta(K)} \varepsilon^\beta, \end{aligned}$$

uniformly for  $x$  in compact sets and  $\varepsilon \in (0, 1]$ . This completes the proof.  $\square$

We can finally give the proof of Theorem 14.1.

*Proof of Theorem 14.1.* We know that the germ  $P$  in (14.2) is  $(\alpha, \alpha + \beta)$ -coherent and has local homogeneity bound  $\beta$ , by Proposition 14.4. We also know by Theorem 12.7 that  $\mathcal{R}P$  belongs to  $\mathcal{C}^\beta$  (note that  $\beta < \alpha + \beta$ ). Since the map  $P \mapsto \mathcal{R}P$  is linear, and since  $P$  is a bilinear function of  $(f, g)$ , it follows that we can define a bilinear map

$$\mathcal{M} : \mathcal{C}^\alpha \times \mathcal{C}^\beta \rightarrow \mathcal{C}^\beta, \quad \mathcal{M}(f, g) := \mathcal{R}P.$$

Property (14.3) is a translation of (5.2), which characterizes  $\mathcal{M}$  if and only if  $\alpha + \beta > 0$ .

Note that by (12.9)

$$\|\mathcal{M}(f, g)\|_{C^\beta(K)} \lesssim \left( \|P\|_{\bar{K}_4, \varphi, \alpha, \gamma}^{\text{coh}} + \|P\|_{\bar{K}_2, \varphi, \beta}^{\text{hom}} \right).$$

It follows by the estimates (14.6)-(14.7) in the proof of Proposition 14.4 that

$$\|P\|_{\bar{K}_4, \varphi, \alpha, \gamma}^{\text{coh}} + \|P\|_{\bar{K}_2, \varphi, \beta}^{\text{hom}} \lesssim \|g\|_{C^\beta(\bar{K}_4)} \|f\|_{C^\alpha(\bar{K}_4)},$$

which proves (14.5), hence  $\mathcal{M}$  is a *continuous map*.  $\square$

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