Auxiliary problem principles for equilibria*

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Abstract: The auxiliary problem principle allows solving a given equilibrium problem (EP) through an equivalent auxiliary problem with better properties. The paper investigates two families of auxiliary EPs: the classical auxiliary problems, in which a regularizing term is added to the equilibrium bifunction, and the regularized Minty EPs. The conditions that ensure the equivalence of a given EP with each of these auxiliary problems are investigated exploiting parametric definitions of different kinds of convexity and monotonicity. This analysis leads to extending some known results for variational inequalities and linear EPs to the general case together with new equivalences. Stationarity and convexity properties of gap functions are investigated as well in this framework. Moreover, both new results on the existence of a unique of solution and new error bounds based on gap functions with good convexity properties are obtained under weak quasimonotonicity or weak concavity assumptions.

Keywords: Equilibrium problem; auxiliary problem; Minty equilibrium problem; gap function; error bound.

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1 Introduction

In computational optimization solving a given problem through a family of auxiliary problems, which enjoy better properties and are therefore easier to be solved, is a widespread approach. Among others, penalty and proximal point algorithms for nonlinear programs (see, for instance, [1, 2]) fall within this scheme. A quite general auxiliary principle has been developed in [3, 4] within the framework of decomposition and coordination algorithms, and it provides sufficient conditions for an optimal solution of one suitable auxiliary problem to solve the given problem. This principle was later extended to variational inequalities [5], while the full equivalence with the auxiliary variational inequality was investigated in [6] and exploited to develop solution methods through optimization techniques, for instance, in [6–8]. Quite recently this kind of approach has been considered also for more general equilibrium problems (see [9–12]), which include optimization, variational inequalities, saddle point problems and Nash equilibria in noncooperative games as particular cases.

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In this paper we aim at deepening the analysis of auxiliary principles for equilibria and we focus on the well-known format (see [13–15]) of an equilibrium problem

$$\text{find } \; x^* \in C \;\text{ s.t. }\; f(x^*, y) \geq 0, \qquad \forall\, y \in C, \tag{EP}$$

where $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set and the equilibrium bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$. Given any $\alpha \in \mathbb{R}$ and the corresponding bifunction

$$f_\alpha(x, y) := f(x, y) + \alpha \, \|y - x\|^2/2,$$

the "classical" auxiliary equilibrium problem

$$\text{find } \; x^* \in C \;\text{ s.t. }\; f_\alpha(x^*, y) \geq 0, \qquad \forall\, y \in C, \tag{EP_α}$$

and the so-called Minty equilibrium problem

$$\text{find } \; x^* \in C \;\text{ s.t. }\; f_\alpha(y, x^*) \leq 0, \qquad \forall\, y \in C \tag{MEP_α}$$

will be considered.

It is well-known that (EP_α) is equivalent to (EP), in the sense that their solution sets coincide, whenever $\alpha > 0$ and $f(x, \cdot)$ is convex [12]. Auxiliary problems (EP_α) with $\alpha > 0$ have been exploited to reformulate (EP) as an equivalent optimization problem through smooth gap functions and to develop corresponding descent methods (see, for instance [7,8,10,11,16]).

On the other hand, the Minty equilibrium problem (MEP_α) with $\alpha = 0$ has been extensively used as an auxiliary problem for (EP) both to obtain existence results (see, for instance, [17–20]) and within algorithmic frameworks (see, for instance, [21–25]). On the contrary, the whole class of problems (MEP_α) has been considered only for variational inequalities in [6, 26–28] and very recently, with $\alpha < 0$, in [29] to refine some existence results for equilibrium problems.

The goal of the paper is to analyse in details the conditions that guarantee the equivalence between (EP) and each of the above auxiliary problems together with the properties and advantages that each equivalence brings. A rather novel feature is the analysis of (EP_α) with $\alpha < 0$ which, up to now, has been considered only for the so-called linear equilibrium problem in [27]. Furthermore, a systematic analysis of (MEP_α) allows achieving new equivalence results also for $\alpha > 0$.

Section 2 explores the connections between the convexity and monotonicity properties of the bifunctions f and f_α. Section 3 and Section 4 investigate the relationships of (EP) with (EP_α) and (MEP_α), respectively, and the properties of the corresponding gap functions, which allow reformulating the equilibrium problems as optimization programs. Exploiting suitable values for α, also new results on the existence of a unique solution, stationarity properties of gap functions and error bounds are achieved under weak monotonicity or weak concavity assumptions on f.

2 Convexity and monotonicity

Both convexity and monotonicity play an important role in the study of equilibrium problems and in the development of solution methods. In order to analyse strong and weak concepts in a unified way, suitable parametric definitions can be introduced.

Definition 2.1 Given $\gamma \in \mathbb{R}$, a function $g : \mathbb{R}^n \to \mathbb{R}$ is called γ-convex on C if any $u, v \in C$ and $t \in [0, 1]$ satisfy

$$g(t\,u + (1-t)\,v) \leq t\,g(u) + (1-t)\,g(v) - \frac{\gamma}{2}\,t\,(1-t)\,\|u-v\|^2.$$

If $\gamma = 0$, the above inequality provides the usual definition of a convex function. If $\gamma > 0$, the inequality is strengthened and f is also called *strongly convex*; similarly, f is called *weakly convex* if $\gamma < 0$. Indeed, g is γ-convex on C if and only if $g(x) - \gamma\,\|x\|^2/2$ is convex on C (see [30]).

In the paper the following concepts of generalized convexity notions will be exploited as well.

Definition 2.2 A function $g : \mathbb{R}^n \to \mathbb{R}$ is called quasiconvex on C if any $u, v \in C$ and $t \in [0, 1]$ satisfy

$$g(t\,u + (1-t)\,v) \leq \max\{g(u), g(v)\}.$$

Definition 2.3 A function $g : \mathbb{R}^n \to \mathbb{R}$ is called explicitly quasiconvex on C if it is quasiconvex on C and

$$g(t\,u + (1-t)\,v) < \max\{g(u), g(v)\}$$

holds for any $u, v \in C$ with $g(u) \neq g(v)$ and any $t \in (0, 1)$.

A function $g : \mathbb{R}^n \to \mathbb{R}$ is called γ-concave on C if $-g$ is γ-convex on C. Quasiconcavity and explicit quasiconcavity are defined accordingly.

Notice that γ-convexity is stronger than explicit quasiconvexity if $\gamma \geq 0$, while they are not related if $\gamma < 0$ as the following example shows.

Example 2.1 Let $n = 1$ and $\gamma < 0$. The function $g(x) = \gamma\,x^2/2$ is γ-convex but it is not (explicitly) quasiconvex on $\mathbb{R}$. Furthermore, $g(x) = x^3$ is explicitly quasiconvex on $\mathbb{R}$ since it is monotone increasing, but it is not γ-convex on $\mathbb{R}$ since the function $x^3 - \gamma\,x^2/2$ is not convex on $(-\infty, \gamma/6)$.

Any γ-convex function on $\mathbb{R}^n$ is continuous for any $\gamma \in \mathbb{R}$, while it is well-known that a quasiconvex function is not necessarily continuous. In the paper the following weaker concept of continuity will be used as well.

Definition 2.4 A function $g : \mathbb{R}^n \to \mathbb{R}$ is called lower hemicontinuous on C if any $u, v \in C$ satisfy

$$g(v) \leq \liminf_{t \to 0^+} g(t\,u + (1-t)\,v);$$

g is called upper hemicontinuous on C if $-g$ is lower hemicontinuous on C.

Notice that a quasiconvex function is not necessarily lower hemicontinuous: for instance, take $n = 1$ and consider

$$g(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0. \end{cases}$$

Clearly, g is quasiconvex on $\mathbb{R}$ but is not lower hemicontinuous at $x = 0$.

In this paper the concepts of lower and upper hemicontinuity will be used for the sake of simplicity. However, some results which will be shown in the following sections could be proved also under slightly weaker assumptions as upper sign-continuity (see, for instance, [18]) or upper sign-property (see [29]).

Definition 2.5 Given $\mu \in \mathbb{R}$, the bifunction f is called

- μ-*monotone* on C if any $x, y \in C$ satisfy the inequality

$$f(x,y) + f(y,x) \leq -\mu \, \|y - x\|^2;$$

- μ-*pseudomonotone* on C if any $x, y \in C$ satisfy the implication

$$f(x,y) \geq 0 \quad \Longrightarrow \quad f(y,x) \leq -\mu \, \|y - x\|^2;$$

- μ-*quasimonotone* on C if any $x, y \in C$ satisfy the implication

$$f(x,y) > 0 \quad \Longrightarrow \quad f(y,x) \leq -\mu \, \|y - x\|^2.$$

Clearly, μ-monotonicity implies μ-pseudomonotonicity, which in turn implies μ-quasimonotonicity. If $\mu = 0$, the well-known concepts of monotonicity, pseudomonotonicity and quasimonotonicity are recovered (see [14, 18, 31, 32]). If $\mu > 0$, the requirements are strengthened and the strong counterparts of the above monotonicity concepts defined: strong monotonicity has been often exploited in algorithmic frameworks (see [13]) while strong pseudomonotonicity has been considered mainly for variational inequalities (see [33, 34]) and only very recently for more general equilibrium problems [35]. Similarly, if $\mu < 0$ weaker concepts are introduced: weak monotonicity has been exploited in a few papers [36–39], while, to the best of our knowledge, weak quasimonotonicity has been used only in [29] to prove existence results for (EP).

The auxiliary bifunction f_α inherits convexity and monotonicity properties from f in the following way.

Proposition 2.1 *(Convexity and monotonicity properties of f_α).*
Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x,x) = 0$ for all $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Let $x, y \in C$ and $\tau, \gamma, \mu \in \mathbb{R}$.

a) If $f(x,\cdot)$ is τ-convex, then $f_\alpha(x,\cdot)$ is $(\tau + \alpha)$-convex for any $\alpha \in \mathbb{R}$.

b) If $f(\cdot,y)$ is γ-concave, then $f_\alpha(\cdot,y)$ is $(\gamma - \alpha)$-concave for any $\alpha \in \mathbb{R}$.

c) If f is μ-monotone on C, then f_α is $(\mu - \alpha)$-monotone on C for any $\alpha \in \mathbb{R}$.

d) If f is μ-quasimonotone on C, then f_α is $(-\alpha/2)$-pseudomonotone on C for any $\alpha < 2\mu$.

Proof.
a),b) Since $\alpha \, \|y - x\|^2/2$ is α-convex, they follow directly from [30, Proposition 4.1].
c) $f_\alpha(x,y) + f_\alpha(y,x) = f(x,y) + f(y,x) + \alpha \|y - x\|^2 \leq (\alpha - \mu)\|y - x\|^2$.

d) Take any $x, y \in C$ with $x \neq y$ such that $f_\alpha(x,y) \geq 0$. Then,

$$f(x,y) \geq -\alpha \|y-x\|^2/2 > -\mu \|y-x\|^2$$

implies $f(y,x) \leq 0$ since f is μ-quasimonotone on C. Hence, the thesis follows immediately since $f_\alpha(y,x) = f(y,x) + \alpha \|y-x\|^2/2 \leq \alpha \|y-x\|^2/2$. □

The choice of α should be aimed at getting a function f_α which satisfies better properties than f. Anyway, there is some kind of tradeoff between convexity and monotonicity/concavity. Indeed, if $\alpha > 0$, then $f_\alpha(x,\cdot)$ satisfies a stronger convexity condition than $f(x,\cdot)$ but f has stronger monotonicity properties than f_α and $f(\cdot,y)$ stronger concavity properties than $f_\alpha(\cdot,y)$. Vice versa, if $\alpha < 0$, then $f(x,\cdot)$ satisfies a stronger convexity condition than $f_\alpha(x,\cdot)$ while f_α has stronger monotonicity properties than f and $f_\alpha(\cdot,y)$ stronger concavity properties than $f(\cdot,y)$. For example, consider the case in which f is monotone and $f(x,\cdot)$ is convex: f_α is just weakly monotone but $f_\alpha(x,\cdot)$ is strongly convex if $\alpha > 0$, while f_α is strongly monotone but $f_\alpha(x,\cdot)$ is just weakly convex if $\alpha < 0$. This tradeoff between convexity and monotonicity/concavity properties is summarised in Table 1.

Table 1: Tradeoff between convexity and monotonicity/concavity properties of f_α with respect to f.

	$\alpha > 0$	$\alpha < 0$
convexity of $f_\alpha(x,\cdot)$	better	worse
concavity of $f_\alpha(\cdot,y)$	worse	better
(quasi)monotonicity of f_α	worse	better

Notice that if f is quasimonotone, then Proposition 2.1 guarantees that f_α is strongly pseudomonotone if $\alpha < 0$, but it is not necessarily weakly monotone, as the following example shows.

Example 2.2 Consider $n = 1$, $f(x,y) = (x^2+1)(y-x)$ and $C = (-\infty, 0]$. Clearly, f is 0-quasimonotone since $f(x,y) > 0$ implies $y > x$ and thus $f(y,x) < 0$. However, both f and f_α are not μ-monotone for any $\mu \in \mathbb{R}$ since

$$[f_\alpha(x,y) + f_\alpha(y,x)]/(y-x)^2 = -(x+y-\alpha) \to +\infty$$

as $x \to -\infty$ and $y \to -\infty$ with $x \neq y$.

Furthermore, notice that if $f(x,\cdot)$ is $(-\alpha)$-convex, then $f_\alpha(x,\cdot)$ is convex and thus explicitly quasiconvex, while the (explicit) quasiconvexity of $f(x,\cdot)$ does not imply necessarily the (explicit) quasiconvexity of $f_\alpha(x,\cdot)$ even if $\alpha > 0$, as the following example shows.

Example 2.3 Let $n = 1$, $f(x,y) = (y-x)^3/3$ and $C = \mathbb{R}$. The function $f(x,\cdot)$ is explicitly quasiconvex for any $x \in \mathbb{R}$ since it is monotone increasing. However, taking $\alpha > 0$, $f_\alpha(x,\cdot)$ admits a local maximum at $x - \alpha$, a local minimum at x, and $\lim_{y\to\pm\infty} f_\alpha(x,y) = \pm\infty$. Hence, $f_\alpha(x,\cdot)$ is not quasiconvex since its sublevel sets are not convex.

3 Classical auxiliary problems

The exploitation of (EP_α) as an auxiliary problem is rooted in proximal point algorithms for nonlinear optimization. At first it has been often used in the framework of variational inequalities and afterwards also for other equilibrium problems (see, for instance, Section 3.1 in [13] and the references therein).

The relationships between the solution sets $S(f)$ and $S_\alpha(f)$ of (EP) and (EP_α) are analysed considering the cases $\alpha > 0$ and $\alpha < 0$ separately. The analysis involves the solution set $M_\alpha(f)$ of the Minty equilibrium problem (MEP_α) as a tool.

Theorem 3.1 *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x,x) = 0$ for all $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Let $\alpha > 0$.*

a) $S(f) \subseteq S_\alpha(f)$;

b) $S_\alpha(f) \subseteq S(f)$ if any of the following conditions hold:

i) $f(x,\cdot)$ is convex for any $x \in C$;

ii) $f(x,\cdot)$ is explicitly quasiconvex and lower hemicontinuous for any $x \in C$ and $f(\cdot,y)$ is concave for any $y \in C$.

Proof.
a) It is obvious since $f_\alpha \geq f$.
b) Suppose i) holds. Since $f(x,\cdot)$ and $f_\alpha(x,\cdot)$ are convex for any $x \in C$, the following equivalences hold:

$$\begin{aligned}
x^* \text{ solves } (EP) \quad &\Longleftrightarrow \quad x^* \in \arg\min\{\ f(x^*,y)\ :\ y \in C\ \} \\
&\Longleftrightarrow \quad \exists\ g^* \in \partial_y f(x^*,x^*) \text{ s.t. } \langle g^*, y - x^* \rangle \geq 0,\ \forall\ y \in C \\
&\Longleftrightarrow \quad \exists\ g^* \in \partial_y f_\alpha(x^*,x^*) \text{ s.t. } \langle g^*, y - x^* \rangle \geq 0,\ \forall\ y \in C \\
&\Longleftrightarrow \quad x^* \in \arg\min\{\ f_\alpha(x^*,y)\ :\ y \in C\ \} \\
&\Longleftrightarrow \quad x^* \text{ solves } (EP_\alpha),
\end{aligned}$$

where $\partial_y f(x^*,x^*)$ denotes the subdifferential of the convex function $f(x^*,\cdot)$ at x^*. The first and the last equivalence are obvious consequences of the definition of (EP) as $f(x,x) = 0$ for any $x \in C$, while the second and the fourth are the optimality conditions of convex programming. Finally, the third equivalence is due to the equality $\partial_y f_\alpha(x,y) = \partial_y f(x,y) + \alpha(y-x)$ (see [41, Theorem 4.1.1]).

Now suppose ii) holds. Considering the bifunction $\hat{f}(x,y) := -f(y,x)$, clearly $S_\alpha(f) = M_{-\alpha}(\hat{f})$ holds by definition. Moreover, $\hat{f}$ satisfies the assumptions of Theorem 2 in [29], hence $M_{-\alpha}(\hat{f}) \subseteq S(\hat{f})$. On the other hand, $S(\hat{f}) = M_0(f)$ holds as well. Finally, $M_0(f) \subseteq S(f)$ follows since f satisfies the assumptions of Theorem 1 in [29]. □

Notice that the second inclusion of Theorem 3.1, i.e., $S_\alpha(f) \subseteq S(f)$, does not hold under the quasiconvexity of $f(x,\cdot)$ only as the following example shows.

Example 3.1 Let $n = 1$, $f(x, y) = y^3 - x^3$, $C = [-\varepsilon, +\infty)$ with $\varepsilon > 0$ and $\alpha \geq 2\,\varepsilon$. The function $f(x, \cdot)$ is quasiconvex but not convex for any $x \in C$. Furthermore, $x = 0$ solves (EP_α) since

$$f_\alpha(0, y) = y^3 + \alpha\, y^2/2 = y^2\,(y + \alpha/2) \geq 0, \qquad \forall\; y \in C,$$

but it does not solve (EP) since $f(0, y) < 0$ for any $y < 0$. Therefore, the inclusion $S_\alpha(f) \subseteq S(f)$ does not hold. Notice that $S(f)$ is nonempty since $x = -\varepsilon$ solves (EP):

$$f(-\varepsilon, y) = y^3 + \varepsilon^3 \geq 0, \qquad \forall\; y \in C.$$

Theorem 3.2 *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Let $\alpha < 0$.*

a) $S_\alpha(f) \subseteq S(f)$;

b) $S(f) \subseteq S_\alpha(f)$ if any of the following conditions hold:

  i) $f(x, \cdot)$ is $(-\alpha)$-convex for any $x \in C$;

  ii) $f_\alpha(x, \cdot)$ is explicitly quasiconvex and lower hemicontinuous for any $x \in C$ and $f(\cdot, y)$ is explicitly quasiconcave and upper hemicontinuous for any $y \in C$.

Proof.
a) It is obvious since $f \geq f_\alpha$.
b) If i) holds, then the proof is the same as of the first part of Theorem 3.1 b).

Now suppose ii) holds. Notice that the bifunction $\hat{f}(x, y) := -f(y, x)$ and f_α satisfy the assumptions of Theorem 1 in [29], hence the following chain of inclusions

$$S(f) = M_0(\hat{f}) \subseteq S(\hat{f}) = M_0(f) \subseteq M_\alpha(f) \subseteq S_\alpha(f).$$

follows □

The second inclusion of Theorem 3.2, i.e., $S(f) \subseteq S_\alpha(f)$, does not hold supposing $f(x, \cdot)$ to be affine but not strongly convex as the following example shows.

Example 3.2 Let $f(x, y) = \varepsilon\, x\,(x - y)$ with $\varepsilon > 0$, $C = [0, 1]$ and $\alpha \in [-2\,\varepsilon, 0)$. For any $x \in C$ the function $f(x, \cdot)$ is affine but not strongly convex. Furthermore, it is easy to check that $x = 0$ solves (EP); however it does not solve (EP_α) since

$$f_\alpha(0, y) = \alpha\, y^2/2 < 0, \qquad \forall\; y \in (0, 1].$$

Therefore, the inclusion $S(f) \subseteq S_\alpha(f)$ does not hold. Notice that $S_\alpha(f)$ is nonempty because $x = 1$ solves (EP_α):

$$f_\alpha(1, y) = \varepsilon\,(1 - y) + \alpha\,(1 - y)^2/2 = (1 - y)\,[\varepsilon + \alpha\,(1 - y)/2] \geq 0, \qquad \forall\; y \in [0, 1].$$

Theorems 3.1 and 3.2 allow achieving the following auxiliary problem principle.

Corollary 3.1 *(Classical auxiliary problem principle).*
Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. If there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ-convex for any $x \in C$, then $S(f) = S_\alpha(f)$ for any $\alpha \geq -\tau$.

If $\tau = 0$, then the above principle collapses to the well-known one given by Lemma 3.1 in [12], while it has been considered with $\tau > 0$ only for linear equilibrium problems [27, Lemma 5].

Notice that variational inequalities, that is (EP) with

$$f(x, y) = \langle F(x), y - x \rangle$$

for some mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, satisfy the assumption of Corollary 3.1 only for $\tau = 0$, while the so-called linear equilibrium problems (see [27]), that is (EP) with

$$f(x, y) = \langle Px + Qy + r, y - x \rangle \tag{1}$$

for some $r \in \mathbb{R}^n$ and some $P, Q \in \mathbb{R}^{n \times n}$, with Q positive semidefinite, satisfy the assumption of Corollary 3.1 with τ equal to the minimum eigenvalue of $Q + Q^T$. Nash equilibrium problems, in which each player i selects one strategy from the set $C_i \subseteq \mathbb{R}^{n_i}$ to minimize a cost function $c_i : C_1 \times \cdots \times C_N \to \mathbb{R}$, that is (EP) with $C = C_1 \times \cdots \times C_N$ and

$$f(x, y) = \sum_{i=1}^{N} [c_i(x_{-i}, y_i) - c_i(x))]$$

where $y_i \in C_i$ and $(x_{-i}, y_i) \in C$, satisfy the assumption of Corollary 3.1 with $\tau = \min\{\tau_i : i = 1, \ldots, N\}$ for τ_i's such that $c_i(x_{-i}, \cdot)$ is a τ_i-convex function on C_i.

The auxiliary problem principle of Corollary 3.1 can be exploited to guarantee the existence of a unique solution for (EP), relying on appropriate values of α.

Corollary 3.2 *(Existence of a unique solution).*
Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$, $f(\cdot, y)$ is upper hemicontinuous on C for any $y \in C$ and there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ-convex for any $x \in C$. If any of the following conditions holds:

a) f is μ-monotone on C with $\mu > -\tau$,

b) f is μ-quasimonotone on C with $\mu > -\tau/2$ and $\tau > 0$,

then there exists a unique solution of (EP).

Proof. By Corollary 3.1 it is enough to prove the thesis for $(EP_{-\tau})$. Proposition 2.1 guarantees that $f_{-\tau}(x, \cdot)$ is convex. Furthermore, $f_{-\tau}$ is strongly pseudomonotone on C since it is $(\mu + \tau)$-monotone (in case a)) or $(\tau/2)$-pseudomonotone (in case b)). Therefore, $(EP_{-\tau})$ has a unique solution by Proposition 1 in [35]. □

If $\tau = 0$, then case a) collapses into the well-known existence and uniqueness result that exploits the strong monotonicity of f (see, for instance, [13]). It is worth noting that the weak quasimonotonicity of f is enough to guarantee the existence of a unique solution of (EP) if $\tau > 0$. Notice that the weak quasimonotonicity of f, together with additional assumptions, has been exploited also in [29] to obtain the existence of solutions of (EP) but the results of [29] do not guarantee uniqueness.

Notice that part a) of Corollary 3.2, involving μ-monotonicity of f (with $\tau > 0$ and $\mu < 0$), is not a special case of part b) which deals with μ-quasimonotonicity (with $\mu < 0$). Indeed, there are cases in which f is μ-monotone but not $(\mu/2)$-quasimonotone just like the following example.

Example 3.3 Consider $n = 1$, $f(x, y) = y^2 - x\,y$ and $C = \mathbb{R}$. The function $f(x, \cdot)$ is τ-convex with $\tau = 2$ for any $x \in C$ and f is μ-monotone with $\mu = -1$ since

$$f(x, y) + f(y, x) = (y - x)^2.$$

Hence, Corollary 3.2 a) guarantees (EP) to have a unique solution. However, f is not μ-quasimonotone for any $\mu > -1$. In fact, choosing $x < \mu$ and $y = \mu + 1$ we get

$$f(x, y) = y\,(y - x) = (\mu + 1)\,(\mu + 1 - x) > 0,$$

but

$$f(y, x) + \mu\,(y - x)^2 = (y - x)\,[\mu\,(y - x) - x] = (\mu + 1 - x)\,(\mu + 1)\,(\mu - x) > 0.$$

Therefore, Corollary 3.2 b) cannot be applied.

3.1 Gap functions

Equilibrium problems can be reformulated as optimization programs through suitable gap functions. Indeed, whenever $f(x, \cdot)$ is convex, the value function

$$\varphi_\alpha(x) := \sup\{\ -f_\alpha(x, y)\ :\ y \in C\ \}$$

is a *gap function* for (EP) for any given $\alpha \geq 0$, i.e., φ_α is non-negative on C and x^* solves (EP) if and only if $x^* \in C$ and $\varphi_\alpha(x^*) = 0$ (see, for instance, [10, 11, 16]).

The auxiliary problem principle given by Corollary 3.1 allows extending this reformulation of equilibria for suitable negative values of α if $f(x, \cdot)$ is τ-convex with $\tau > 0$.

Theorem 3.3 *(Properties of φ_α).*
Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ and there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ-convex for any $x \in C$.

a) φ_α is a gap function for (EP) for any $\alpha \geq -\tau$.

b) $\varphi_\alpha(x) < +\infty$ for any $x \in C$ and $\alpha > -\tau$.

c) If f is continuously differentiable on $C \times C$, then φ_α is continuously differentiable on C for any $\alpha > -\tau$ and

$$\nabla\varphi_\alpha(x) = -\nabla_x f(x, y_\alpha(x)) - \alpha\,(x - y_\alpha(x)),$$

where $y_\alpha(x) = \arg\max\{\ -f_\alpha(x, y)\ :\ y \in C\ \}$.

d) If there exists $\gamma \in \mathbb{R}$ such that $f(\cdot, y)$ is γ-concave on C for any $y \in C$, then φ_α is $(\gamma - \alpha)$-convex on C for any $\alpha \in \mathbb{R}$.

Proof.
a) It follows directly from Corollary 3.1.

b) Since $\alpha > -\tau$, $-f_\alpha(x,\cdot)$ is strongly concave by Proposition 2.1. Hence, its maximum value over C, that is $\varphi_\alpha(x)$, is finite.

c) Since $-f_\alpha(x,\cdot)$ is strongly concave, it admits a unique maximizer $y_\alpha(x)$ over C. The thesis follows from the Danskin's Theorem [40].

d) Since $\alpha\,\|y-x\|^2/2$ is $(-\alpha)$-concave on C for any $y \in C$, $f_\alpha(\cdot,y)$ is $(\gamma-\alpha)$-concave on C by Proposition 2.1. As it is the pointwise supremum of a family of $(\gamma-\alpha)$-convex functions, the gap function φ_α is $(\gamma-\alpha)$-convex on C as well (see [30, Proposition 4.1]). □

Notice that it is more likely for φ_α to be convex if $\alpha < 0$: for instance, whenever $f(\cdot,y)$ is weakly concave or concave on C, that is $\gamma \le 0$, φ_α is convex if $\alpha \le \gamma$, which does not hold when $\alpha > 0$. The following example shows a case in which φ_α is not convex for any $\alpha > 0$, while it is convex for some negative values of α.

Example 3.4 Consider $n = 1$, $f(x,y) = (-x+2y+1)(y-x)$ and $C = [0,+\infty)$: $f(x,\cdot)$ is τ-convex with $\tau = 4$, thus φ_α is a gap function for any $\alpha \ge -4$. Moreover, it holds

$$\varphi_\alpha(x) = (-1-\alpha/2)\,x^2 + x$$

for $\alpha \in [-4,-3]$, while

$$\varphi_\alpha(x) = \begin{cases} (-1-\alpha/2)\,x^2 + x, & \text{if } x \in \left[0,(3+\alpha)^{-1}\right), \\[2ex] (x+1)^2/(8+2\alpha), & \text{if } x \in \left[(3+\alpha)^{-1},+\infty\right), \end{cases}$$

for $\alpha > -3$. Therefore, φ_α is convex on C for any $\alpha \in [-4,-2]$, while it is not convex on C if $\alpha > -2$. Indeed, $f(\cdot,y)$ is γ-concave with $\gamma = -2$ for any $y \in C$.

Under suitable convexity/concavity assumptions, any stationary point of the gap function φ_α is not just a global minimum, which is already guaranteed by Theorem 3.3 d), but also a solution of (EP).

Theorem 3.4 *Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x,x) = 0$ for all $x \in \mathbb{R}^n$, there exist $\tau \ge 0$ and $\gamma > -\tau$ such that $f(x,\cdot)$ is τ-convex for any $x \in C$ and $f(\cdot,y)$ is γ-concave for any $y \in C$. Then the following statements hold for any $\alpha \in (-\tau,\gamma]$:*

a) if $x \in C$ does not solve (EP), then

$$\varphi'_\alpha(x;\ y_\alpha(x)-x) \le -(\tau+\gamma)\,\|y_\alpha(x)-x\|^2/2 < 0,$$

where $\varphi'_\alpha(x;\ y_\alpha(x)-x)$ denotes the directional derivative of φ_α at x in the direction $y_\alpha(x)-x$;

b) $x^* \in C$ *solves* (EP) *if and only if* x^* *is a stationary point of* φ_α *over* C*, i.e.,*

$$\varphi'_\alpha(x^*;\ y - x^*) \geq 0, \qquad \forall\, y \in C.$$

Proof.
a) If $x \in C$ does not solve (EP), then $y_\alpha(x) \neq x$ (see, for instance, [11, Proposition 3.1]). Proposition 2.1 guarantees that $f_\alpha(x,\cdot)$ is $(\tau+\alpha)$-convex for any $x \in C$ and $f_\alpha(\cdot,y)$ is $(\gamma-\alpha)$-concave for any $y \in C$. Then, the following chain of inequalities and equalities holds:

$$\begin{aligned}
\varphi'_\alpha(x; y_\alpha(x) - x) \quad &= (-f_\alpha(\cdot, y_\alpha(x)))'(x; y_\alpha(x) - x) \\
&\leq -f_\alpha(y_\alpha(x), y_\alpha(x)) + f_\alpha(x, y_\alpha(x)) - (\gamma - \alpha)\, \|y_\alpha(x) - x\|^2/2 \\
&= f(x, y_\alpha(x)) - (\gamma - 2\,\alpha)\, \|y_\alpha(x) - x\|^2/2 \\
&\leq f(x,x) - (f(x,\cdot))'(y_\alpha(x); x - y_\alpha(x)) - \tau\, \|y_\alpha(x) - x\|^2/2 \\
&\qquad -(\gamma - 2\,\alpha)\, \|y_\alpha(x) - x\|^2/2 \\
&= -(f(x,\cdot))'(y_\alpha(x); x - y_\alpha(x)) + \alpha\, \|y_\alpha(x) - x\|^2 \\
&\qquad -(\tau + \gamma)\, \|y_\alpha(x) - x\|^2/2 \\
&\leq -(\tau + \gamma)\, \|y_\alpha(x) - x\|^2/2,
\end{aligned}$$

where the first equality follows from the Danskin's theorem [40], the first two inequalities from Theorem 6.1.2 in [41] since $-f_\alpha(\cdot, y_\alpha(x))$ is $(\gamma - \alpha)$-convex and $f(x,\cdot)$ is τ-convex, and the last inequality is due to the first-order optimality conditions for $y_\alpha(x)$.
b) If $x^* \in C$ solves (EP), then it is a minimizer of φ_α over C by Theorem 3.3 a): hence, it is a stationary point of φ_α over C. Conversely, if x^* is a stationary point of φ_α over C, then $\varphi'_\alpha(x^*;\ y_\alpha(x^*) - x^*) \geq 0$ and a) implies that x^* solves (EP). □

Theorem 3.4 suggests the basic idea for a descent method for solving (EP) through the minimization of the gap function φ_α: at each iteration a line search along the descent direction $y_\alpha(x) - x$ can be performed in order to decrease the value of φ_α. While methods of this kind have been extensively developed under monotonicity assumptions on f (see, for instance, [13]), the case of convex-concave bifunctions has not received comparable attention yet.

Theorems 3.3 and 3.4 allow achieving the existence of a unique solution of (EP) under suitable convexity-concavity assumptions.

Corollary 3.3 *(Existence of a unique solution).*
Suppose that $C \subseteq \mathbb{R}^n$ *is a nonempty, closed and convex set,* $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *satisfies* $f(x,x) = 0$ *for all* $x \in \mathbb{R}^n$*, there exist* $\tau \geq 0$ *and* $\gamma > -\tau$ *such that* $f(x,\cdot)$ *is* τ*-convex for any* $x \in C$ *and* $f(\cdot,y)$ *is* γ*-concave for any* $y \in C$*. Then, there exists a unique solution of* (EP)*.*

Proof. If $\alpha \in (-\tau, \gamma)$, then Theorem 3.3 guarantees that φ_α is a strongly convex gap function, hence it has a unique stationary point over C. Therefore, there exists a unique solution of (EP) by Theorem 3.4 b). □

If f is strongly monotone on C, the gap function φ_α provides an error bound for (EP) whenever $\alpha > 0$ is small enough (see [11, Proposition 4.2]). If $f(x,\cdot)$ is τ-convex for any

$x \in C$ for some $\tau > 0$, then the use of negative values for α brings further improvements: error bounds can be established even under weak monotonicity or weak concavity assumptions.

Theorem 3.5 *(Error bound).*
Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x,x) = 0$ for all $x \in \mathbb{R}^n$ and there exists $\tau \geq 0$ such that $f(x,\cdot)$ is τ-convex for any $x \in C$.

a) If f is μ-monotone on C with $\mu > -\tau$ and $\alpha \in [-\tau, \mu)$, then

$$\varphi_\alpha(x) \geq (\mu - \alpha)\, \|x - x^*\|^2, \qquad \forall\, x \in C,$$

where x^* is the unique solution of (EP).

b) If $f(\cdot, y)$ is γ-concave on C for any $y \in C$ with $\gamma > -\tau$ and $\alpha \in [-\tau, \gamma)$, then

$$\varphi_\alpha(x) \geq (\gamma - \alpha)\|x - x^*\|^2/2, \qquad \forall\, x \in C,$$

where x^* is the unique solution of (EP).

Proof.
a) By Corollary 3.2 a) there exists a unique solution x^* of (EP). Since Corollary 3.1 guarantees the equivalence between (EP) and (EP_α), x^* solves (EP_α) as well. Therefore, any $x \in C$ satisfies

$$\begin{aligned}
\varphi_\alpha(x) &\geq -f_\alpha(x, x^*) \\
&= -f_\alpha(x, x^*) - f_\alpha(x^*, x) + f_\alpha(x^*, x) \\
&\geq -f_\alpha(x, x^*) - f_\alpha(x^*, x) \\
&\geq (\mu - \alpha)\, \|x - x^*\|^2,
\end{aligned}$$

where the last inequality holds since f_α is $(\mu - \alpha)$-monotone on C by Proposition 2.1.

b) By Corollary 3.3 there exists a unique solution x^* of (EP) and it minimizes φ_α over C by Theorem 3.3 a). Furthermore, φ_α is strongly convex on C by Theorem 3.3 d). Therefore, Theorem 6.1.2 in [41] guarantees that the inequality

$$\varphi_\alpha(x) \geq \varphi_\alpha(x^*) + \langle g^*, x - x^* \rangle + (\gamma - \alpha)\|x - x^*\|^2/2$$

holds for any $x \in C$ and any $g^* \in \partial\varphi_\alpha(x^*)$. The optimality of x^* implies $\varphi_\alpha(x^*) = 0$ and the existence of some $g^* \in \partial\varphi_\alpha(x^*)$ such that $\langle g^*, x - x^* \rangle \geq 0$ for any $x \in C$. Therefore, the conclusion follows. □

Theorem 3.5 a) with $\tau = 0$ and $\mu > 0$ has been proved for variational inequalities in [42] and extended to equilibrium problems in [11]. If $\tau > 0$, then the above result provides new error bounds under weaker assumptions than the usual ones.

In the case of linear equilibrium problems, that is when f is given by (1), f is μ-monotone and $f(\cdot, y)$ is γ-concave and the moduli of monotonicity and concavity are explicitly known: μ and γ are the minimum eigenvalues of $(P - Q + (P - Q)^T)/2$ and $P + P^T$, respectively. As τ is the minimum eigenvalue of $Q + Q^T$, the inequality $\gamma \geq \tau + 2\mu$ always holds as well.

4 Minty auxiliary problems

The Minty equilibrium problem (MEP_α) with $\alpha = 0$ has been extensively used as an auxiliary problem for (EP) both in order to obtain existence results (see, for instance, [17–20]) and within algorithmic frameworks (see, for instance, [21–25]). On the contrary, the class of problems (MEP_α) has been explicitly considered only for variational inequalities with $\alpha < 0$ in [6, 26], indirectly through gap functions with $\alpha > 0$ in [27, 28] and very recently in [29] to refine some existence results for equilibrium problems.

The next two results provide the relationships between $S(f)$ and the solution set $M_\alpha(f)$ of (MEP_α) for any value of α.

Theorem 4.1 *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Then $M_\alpha(f) \subseteq S(f)$ if any of the following sets of conditions holds:*

a) $f(x, \cdot)$ is explicitly quasiconvex for any $x \in C$, $f(\cdot, y)$ is upper hemicontinuous for any $y \in C$ and $\alpha \geq 0$;

b) $f(x, \cdot)$ is convex for any $x \in C$, $f(\cdot, y)$ is upper hemicontinuous for any $y \in C$ and $\alpha \in \mathbb{R}$.

Proof.
a) Theorem 1 in [29] guarantees $M_0(f) \subseteq S(f)$. Since $M_\alpha(f) \subseteq M_0(f)$ holds for any $\alpha \geq 0$, the thesis follows.
b) Let $x^* \in M_\alpha(f)$. Given any $y \in C$, consider the point $y_t = t\, y + (1 - t)\, x^*$ for $t \in (0, 1)$ so that $y_t \in C$. The following chain of inequalities holds

$$\begin{aligned} 0 &= f(y_t, y_t) \\ &\leq t\, f(y_t, y) + (1 - t)\, f(y_t, x^*) \\ &\leq t\, f(y_t, y) - \alpha\, (1 - t)\, \|y_t - x^*\|^2/2 \\ &= t\, f(y_t, y) - \alpha\, t^2(1 - t)\, \|y - x^*\|^2/2. \end{aligned}$$

The first is due to the convexity of $f(y_t, \cdot)$ while the second holds since $x^* \in M_\alpha(f)$. As a consequence, the inequality

$$f(y_t, y) \geq \alpha\, t(1 - t)\, \|y - x^*\|^2/2$$

holds as well. Taking the limit as $t \downarrow 0$, the upper hemicontinuity of $f(\cdot, y)$ guarantees $f(x^*, y) \geq 0$. Hence, $x^* \in S(f)$. □

If $\alpha = 0$, then Theorem 4.1 collapses into Lemma 2.1 in [18]. Moreover, the assumptions of Theorem 4.1 b) could be slightly weakened supposing f to be explicitly quasiconvex with respect to the second variable together with the α-upper sign property with respect to the first variable (see [29]). However, notice that explicit quasiconvexity of $f(x, \cdot)$ alone is not sufficient to guarantee that $M_\alpha(f) \subseteq S(f)$ holds for negative values of α, as the following example shows.

Example 4.1 Consider $n = 1$, $f(x, y) = y^3 - x^3$, $C = [-\varepsilon, +\infty)$ with $\varepsilon > 0$ and $\alpha \leq -2\,\varepsilon$. Given any $x \in C$, the function $f(x, \cdot)$ is explicitly quasiconvex but not convex. Furthermore, $x = 0$ solves (MEP_α) since

$$f_\alpha(x, 0) = -x^3 + \alpha\, x^2/2 \leq 0, \qquad \forall\, x \in C.$$

However, it does not solve (EP) since

$$f(0, y) = y^3 < 0, \qquad \forall\, y < 0.$$

Therefore, the inclusion $M_\alpha(f) \subseteq S(f)$ does not hold. Notice that $S(f)$ is nonempty since $x = -\varepsilon$ solves (EP):

$$f(-\varepsilon, y) = y^3 + \varepsilon^3 \geq 0, \qquad \forall\, y \in C.$$

The opposite inclusion of Theorem 4.1 requires suitable monotonicity or concavity assumptions.

Theorem 4.2 *Suppose that $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set. Then $S(f) \subseteq M_\alpha(f)$ if any of the following sets of conditions holds:*

a) f is μ-pseudomonotone on C with $2\mu \geq \alpha$;

b) f is μ-monotone on C, $f(x, \cdot)$ is τ-convex for any $x \in C$ with $\tau \geq 0$ and $2\mu + \tau \geq \alpha$;

c) $f(x, \cdot)$ is lower hemicontinuous for any $x \in C$ and $f_\gamma(\cdot, y)$ is explicitly quasiconcave for any $y \in C$ with $\gamma \geq 0$ and $\gamma \geq \alpha$;

d) $f(x, \cdot)$ is τ-convex for any $x \in C$ with $\tau \geq 0$ and $f_\gamma(\cdot, y)$ is explicitly quasiconcave for any $y \in C$ with $\gamma \geq -\tau$ and $\gamma \geq \alpha$.

Proof. Consider any $x^* \in S(f)$.
a) Any $y \in C$ satisfies

$$f_\alpha(y, x^*) = f(y, x^*) + \alpha\|y - x^*\|^2/2 \leq (\alpha/2 - \mu)\, \|y - x^*\|^2 \leq 0$$

where the first inequality follows from the μ-pseudomonotonicity of f.
b) Any $y \in C$ satisfies

$$\begin{aligned} f_\alpha(y, x^*) &= f(y, x^*) + \alpha\|y - x^*\|^2/2 \\ &\leq -f(x^*, y) + (\alpha/2 - \mu)\, \|y - x^*\|^2 \\ &\leq (\alpha/2 - \mu - \tau/2)\, \|y - x^*\|^2 \\ &\leq 0, \end{aligned}$$

where the first inequality is due to the μ-monotonicity of f while the second holds since x^* solves $(EP_{-\tau})$ by Theorem 3.1 and therefore $f(x^*, y) \geq \tau\|y - x^*\|^2/2$.
c) The bifunction $\hat{g}(x, y) := -f_\gamma(y, x)$ satisfies the assumptions of Theorem 1 in [29]. Hence, the following chain of inclusions holds:

$$S(f) \subseteq S_\gamma(f) = M_0(\hat{g}) \subseteq S(\hat{g}) = M_\gamma(f) \subseteq M_\alpha(f).$$

d) Same as c) just noting that $S(f) = S_\gamma(f)$ by Corollary 3.1. $\square$

Theorem 4.2 subsumes some known particular cases together with new results. Case a) with $\mu = 0$ and $\alpha = 0$ together with Theorem 4.1 is the well-known Minty Lemma about the equivalence between (EP) e (MEP_0) (see, for instance, [17,20]). Case b) with $\tau = 0$, $\mu > 0$ and $\alpha = 0$ or $\alpha = \mu$ was considered in [27]. Considering just variational inequalities, case a) with $\mu = 0$ and $\alpha \leq 0$ has been proved in [6,26], while case b) with $\mu > 0$ and $\alpha \geq 0$ in [28, Lemma 2.3]. To the best of our knowledge, up to now cases c) and d) have not been considered in any framework. Notice that the explicit quasiconcavity of $f_\gamma(\cdot, y)$ is satisfied if $f(\cdot, y)$ is γ-concave. Since parameters μ and γ can be negative, Theorem 4.2 provides results that may just require weak pseudomonotonicity or weak concavity assumptions.

A Minty auxiliary problem principle, i.e. $S(f) = M_\alpha(f)$, can be obtained combining Theorems 4.1 and 4.2. Notice that this auxiliary principle and Corollary 3.1 are somehow symmetric: the former requires α to be bounded by above and some monotonicity conditions while the latter needs α to be bounded by below.

As (MEP_0) has been widely used to obtain existence results for (EP), the auxiliary problem (MEP_α) can be exploited in the same fashion of [29]. Indeed, the Minty auxiliary problem principle given by Theorems 4.1 and 4.2 can be exploited to guarantee the existence of a unique solution for (EP).

Corollary 4.1 *(Existence of a unique solution).*
Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x, x) = 0$ for all $x \in \mathbb{R}^n$, there exists $\tau \geq 0$ such that $f(x, \cdot)$ is τ-convex for any $x \in C$ and there exists $\gamma > -\tau$ such that $f_\gamma(\cdot, y)$ is explicitly quasiconcave and upper hemicontinuous for any $y \in C$. Then there exists a unique solution of (EP).

Proof. Exploiting Theorem 4.1, Theorem 4.2 d), $S(f) = M_\gamma(f)$ follows. Hence, it is enough to prove that there exists a unique solution of (MEP_γ).

Suppose x^* solves (MEP_γ). By Theorem 4.1 x^* solves also (EP_γ), that is $f_\gamma(x^*, y) \geq 0$ for any $y \in C$. Since $f_\gamma(x^*, \cdot)$ is strongly convex (see Proposition 2.1) and $f_\gamma(x^*, x^*) = 0$, any $y \in C$ with $y \neq x^*$ satisfies $f_\gamma(x^*, y) > 0$ and thus it does not solve (MEP_γ). Therefore, the solution of (MEP_γ), if any exists, is unique.

Considering the set-valued map $M(x) := \{y \in C : f_\gamma(x, y) \leq 0\}$, $x^* \in C$ solves (MEP_γ) if and only if $x^* \in M(x)$ for any $x \in C$. Therefore, it is enough to prove that the intersection of the sets $M(x)$ over all $x \in C$ is nonempty.

Given any $x \in C$, $M(x)$ is nonempty, closed and bounded since $x \in M(x)$ and $f_\gamma(x, \cdot)$ is strongly convex. Therefore, the Knaster-Kuratowski-Mazurkiewicz Theorem [43] guarantees the desired nonemptiness if

$$z := \sum_{i=1}^{k} \beta_i x^i \in \bigcup_{i=1}^{k} M(x^i) \tag{2}$$

holds for any $x^1, \ldots, x^k \in C$ and any $\beta_1, \ldots, \beta_k \geq 0$ such that $\beta_1 + \cdots + \beta_k = 1$. By contradiction, suppose $z \notin M(x^i)$ for any i. Then, $f_\gamma(x^i, z) > 0$ holds for any i, and hence the quasiconcavity of $f_\gamma(\cdot, z)$ implies the contradiction $0 = f_\gamma(z, z) > 0$. $\square$

Notice that the exploitation of the Minty auxiliary problem (MEP_γ) in the proof of Corollary 4.1 allows achieving the existence of a unique solution for (EP) under weaker assumptions than Corollary 3.3, which is based on the classical auxiliary problem principle.

4.1 Minty gap functions

In the same way gap functions for (EP) can be introduced through (EP_α), the auxiliary problem (MEP_α) can be exploited to introduce another family of gap functions. Indeed, the parallel treatment of gap functions for Minty formulations provides new insights on the classical equilibrium problem (EP) as well: further solution methods and error bounds can be developed under suitable assumptions that guarantee that Minty equilibria are solution of (EP) (see Theorem 4.1). In details, the function

$$\psi_\alpha(x) := \sup\{\ f_\alpha(y,x)\ :\ y \in C\ \}$$

is non-negative on C and x^* solves (MEP_α) if and only if $x^* \in C$ and $\psi_\alpha(x^*) = 0$. Therefore, ψ_α is a gap function for (EP) whenever $S(f) = M_\alpha(f)$ (see Theorems 4.1 and 4.2). Moreover, unlike the gap function φ_α of the previous section, it is always convex provided that $f_\alpha(x,\cdot)$ is convex and this property makes it attractive: in fact, it has been exploited in algorithmic frameworks for variational inequalities [6, 21, 23, 26] and for more general equilibrium problems [24, 27].

The following result groups its main properties.

Theorem 4.3 *(Properties of ψ_α).*
Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x,x) = 0$ for all $x \in \mathbb{R}^n$.

- *a) ψ_α is a gap function for (EP) if $S(f) = M_\alpha(f)$;*
- *b) $\psi_\alpha(x) < +\infty$ for any $x \in C$ if any of the following conditions holds:*
  - *b1) $f(\cdot,y)$ is γ-concave on C for any $y \in C$ and $\alpha < \gamma$,*
  - *b2) there exists $\tau \in \mathbb{R}$ such that $f(x,\cdot)$ is τ-convex on C for any $x \in C$, f is μ-monotone on C and $\alpha < 2\mu + \tau$.*
- *c) If there exists $\tau \in \mathbb{R}$ such that $f(x,\cdot)$ is τ-convex on C for any $x \in C$, then ψ_α is $(\tau+\alpha)$-convex on C for any $\alpha \in \mathbb{R}$.*

Proof.
a) Obvious.

b1) Proposition 2.1 guarantees that $f_\alpha(\cdot,x)$ is $(\gamma-\alpha)$-concave for any $x \in C$. Since $\gamma > \alpha$, $f_\alpha(\cdot,x)$ is strongly concave, thus its maximum value over C, that is $\psi_\alpha(x)$, is finite.

b2) The μ-monotonicity of f implies

$$f_\alpha(y,x) \leq -f(x,y) + (\alpha/2-\mu)\,\|y-x\|^2 = -f_{2\mu-\alpha}(x,y).$$

Proposition 2.1 guarantees that $f_{2\mu-\alpha}(x,\cdot)$ is $(\tau+2\mu-\alpha)$-convex, hence $-f_{2\mu-\alpha}(x,\cdot)$ is $(\tau+2\mu-\alpha)$-concave. Since $\tau+2\mu>\alpha$, it is actually strongly concave and thus its maximum value over C is finite, which implies that $\psi_\alpha(x)$ is finite as well.

c) Since $\alpha\,\|y-x\|^2/2$ is α-convex with respect to x for any $y\in C$, $f_\alpha(y,\cdot)$ is $(\tau+\alpha)$-convex by Proposition 2.1. As it is the pointwise supremum of a family of $(\tau+\alpha)$-convex functions, the gap function ψ_α is $(\tau+\alpha)$-convex as well (see [30, Proposition 4.1]). □

Theorem 4.3 a) has been already given in [6, 26–28] for the same particular cases of Theorem 4.2 that have been previously recalled.

Notice that Theorem 3.3 and Theorem 4.3 are somehow symmetric: considering any $\alpha>-\tau$, φ_α is a finite gap function for (EP) but it is convex only under additional assumptions, while ψ_α is always convex but it is a finite gap function for (EP) only under additional assumptions; moreover, γ-concavity provides a common assumption to get convexity in Theorem 3.3 and finiteness in Theorem 4.3.

A result similar to Theorem 3.4 can be proved for the gap function ψ_α as well.

Theorem 4.4 *Suppose that $C\subseteq\mathbb{R}^n$ is a nonempty, closed and convex set, $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ satisfies $f(x,x)=0$ for all $x\in\mathbb{R}^n$ and there exist $\tau\geq 0$ and $\gamma>-\tau$ such that $f(x,\cdot)$ is τ-convex for any $x\in C$ and $f(\cdot,y)$ is γ-concave for any $y\in C$. Then, the following statements hold for any $\alpha\in[-\tau,\gamma)$:*

a) if $x\in C$ does not solve (EP), then

$$\psi'_\alpha(x;\ z_\alpha(x)-x)\leq-(\tau+\gamma)\,\|z_\alpha(x)-x\|^2/2<0,$$

where $z_\alpha(x):=\arg\max\{\ f_\alpha(y,x)\ :\ y\in C\ \}$;

b) $x^*\in C$ solves (EP) if and only if x^* is a stationary point of ψ_α over C, i.e.,

$$\psi'_\alpha(x^*;\ y-x^*)\geq 0,\qquad \forall\, y\in C.$$

Proof.
a) Proposition 2.1 guarantees that $f_\alpha(y,\cdot)$ is $(\tau+\alpha)$-convex for any $y\in C$ and $f_\alpha(\cdot,x)$ is $(\gamma-\alpha)$-concave for any $x\in C$. Since $\alpha<\gamma$, there exists a unique maximizer of $f_\alpha(\cdot,x)$ over C, and thus $z_\alpha(x)$ is well defined for any $x\in C$.

Theorem 4.3 implies that ψ_α is a gap function for (EP). As a consequence, $z_\alpha(x)\neq x$

since x does not solve (EP). Finally, the following chain of inequalities and equalities holds:

$$\begin{aligned}
\psi'_\alpha(x; z_\alpha(x)-x) \quad &= (f_\alpha(z_\alpha(x),\cdot))'(x; z_\alpha(x)-x) \\
&\le f_\alpha(z_\alpha(x), z_\alpha(x)) - f_\alpha(z_\alpha(x), x) - (\tau+\alpha)\,\|z_\alpha(x)-x\|^2/2 \\
&= -f(z_\alpha(x), x) - (\tau + 2\,\alpha)\,\|z_\alpha(x)-x\|^2/2 \\
&\le -f(x,x) + (f(\cdot,x))'(z_\alpha(x); x - z_\alpha(x)) - \gamma\,\|z_\alpha(x)-x\|^2/2 \\
&\qquad -(\tau+2\,\alpha)\,\|z_\alpha(x)-x\|^2/2 \\
&= (f(\cdot,x))'(z_\alpha(x); x - z_\alpha(x)) - \alpha\,\|z_\alpha(x)-x\|^2 \\
&\qquad -(\tau+\gamma)\,\|z_\alpha(x)-x\|^2/2 \\
&\le -(\tau+\gamma)\,\|z_\alpha(x)-x\|^2/2,
\end{aligned}$$

where the first equality follows from the Danskin's theorem [40], the first two inequalities from Theorem 6.1.2 in [41] since $f_\alpha(z_\alpha(x),\cdot)$ is $(\tau+\alpha)$-convex and $f(\cdot,x)$ is γ-concave, and the last inequality is due to the first-order optimality conditions for $z_\alpha(x)$.
b) If $x^* \in C$ solves (EP), then it is a minimizer of ψ_α over C by Theorem 4.3 a): hence, it is a stationary point of ψ_α over C. Conversely, if x^* is a stationary point of ψ_α over C, then $\psi'_\alpha(x^*;\ z_\alpha(x^*)-x^*) \ge 0$ and a) implies that x^* solves (EP). $\square$

Also the gap function ψ_α can be exploited to obtain error bounds for (EP). Since it is always convex whenever $f_\alpha(x,\cdot)$ is convex, some bounds can be achieved also under pseudomonotonicity assumptions.

Theorem 4.5 *(Error bound).*
Suppose that $C \subseteq \mathbb{R}^n$ is a nonempty, closed and convex set, $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfies $f(x,x) = 0$ for all $x \in \mathbb{R}^n$ and there exists $\tau \ge 0$ such that $f(x,\cdot)$ is τ-convex on C for any $x \in C$. If any of the following conditions holds:

- *a) f is μ-pseudomonotone on C with $\mu > -\tau/2$ and $\alpha \in (-\tau, 2\mu]$,*
- *b) f is μ-monotone on C with $\mu > -\tau$ and $\alpha \in (-\tau, 2\mu+\tau]$,*
- *c) $f_\gamma(\cdot,y)$ is explicitly quasiconcave and upper hemicontinuous on C for any $y \in C$ with $\gamma > -\tau$ and $\alpha \in (-\tau,\gamma]$,*

then

$$\psi_\alpha(x) \ge (\tau+\alpha)\|x-x^*\|^2/2, \qquad \forall\ x \in C,$$

where x^* is the unique solution of (EP).

Proof. In case c) the existence of a unique solution x^* of (EP) is guaranteed by Corollary 4.1. In the cases a) and b) it is guaranteed by Proposition 2.1 in [35] if $\tau = 0$ and by Corollary 3.2 if $\tau > 0$.

Any $x \in C$ satisfies

$$\psi_\alpha(x) \ge f_\alpha(x^*, x) = f(x^*, x) + \alpha\|x-x^*\|^2/2 \ge (\tau+\alpha)\|x-x^*\|^2/2,$$

where the second inequality holds since x^* solves $(EP_{-\tau})$ by Theorem 3.2 and therefore $f(x^*, x) \geq \tau\|x - x^*\|^2/2$. □

Theorem 4.5 b) with $\tau = 0$ and $\mu > 0$ has been proved for variational inequalities in [28, Lemma 4.2], while all the other cases are, to the best of our knowledge, new. Notice that the use of negative values for α (which is possible if $\tau > 0$) provides error bounds also under weak (pseudo)monotonicity or weak concavity assumptions.

5 Conclusions

The paper investigates in depth the relations of the equilibrium problem (EP) with two different families of auxiliary problems: classical auxiliary problems (EP_α) and Minty auxiliary problems (MEP_α), which both depend upon a regularization parameter α.

Exploiting parametric definitions of strong/weak convexity and monotonicity, results are presented in a unified form that allows subsuming known particular cases together with new results. Indeed, the results require precise relations between α and the moduli of monotonicity and convexity/concavity.

This kind of analysis has been the key tool for improvements by investigating auxiliary problem principles both for positive and negative values of α. The equivalence between (EP) and (EP_α) was already well-known for $\alpha > 0$, yet the analysis for negative values is new, while the equivalence between (EP) and (MEP_α) was already well-known just for $\alpha = 0$ in the pseudomonotone case.

These principles lead to new results on the existence of a unique solution for (EP) under weak quasimonotonicity or weak concavity assumptions. Furthermore, the auxiliary problems (EP_α) and (MEP_α) with suitable choices of α bring in gap functions φ_α and ψ_α with good convexity properties. New error bounds follow too, which could be exploited in the analysis of quantitative stability of equilibria under perturbations (see, for instance, [44, 45]).

Table 2 provides an overview of the monotonicity or concavity conditions that guarantee the properties of the gap functions and the corresponding error bounds.

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Table 2: Conditions that guarantee properties of the gap functions φ_α and ψ_α supposing $f(x,\cdot)$ τ-convex with $\tau \geq 0$ and $f(\cdot,y)$ upper hemicontinuous

	φ_α	ψ_α
gap function	$\alpha \geq -\tau$	f μ-pseudomonotone, $\alpha \leq 2\mu$ or f μ-monotone, $\alpha \leq 2\mu+\tau$ or $f_\gamma(\cdot,y)$ explicitly quasiconcave, $\gamma \geq \max\{-\tau,\alpha\}$
convex	$f(\cdot,y)$ γ-concave, $\alpha \leq \gamma$	$\alpha \geq -\tau$
strongly convex	$f(\cdot,y)$ γ-concave, $\alpha < \gamma$	$\alpha > -\tau$
error bound	f is μ-monotone, $\alpha \in [-\tau,\mu)$ or $f(\cdot,y)$ γ-concave, $\alpha \in [-\tau,\gamma)$	f μ-pseudomonotone, $\alpha \in (-\tau,2\mu]$ or f μ-monotone, $\alpha \in (-\tau,2\mu+\tau]$ or $f_\gamma(\cdot,y)$ explicitly quasiconcave, $\alpha \in (-\tau,\gamma]$

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