

Interpolation and stability properties of low-order face and edge virtual element spaces

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[Received on 22 June 2021; revised on 17 November 2021]

We analyse the interpolation properties of two-dimensional and three-dimensional low-order virtual element (VE) face and edge spaces, which generalize Nédélec and Raviart–Thomas polynomials to polygonal-polyhedral meshes. Moreover, we investigate the stability properties of the associated L^2 -discrete bilinear forms, which typically appear in the VE discretization of problems in electromagnetism.

Keywords: virtual element methods; polytopal meshes; face and edge spaces; stability.

1. Introduction

Conforming $H(\text{div})$ and $H(\text{curl})$ elements (Raviart & Thomas, 1977; Nédélec, 1986; Boffi *et al.*, 2013) are fundamental in the discretization of a variety of problems, such as diffusion problems in mixed form and electromagnetic equations. Virtual element (VE) face and edge spaces generalize Nédélec and Raviart–Thomas polynomials to polygonal–polyhedral meshes. When coupled with nodal VE spaces and piecewise polynomials for the discretization of H^1 and L^2 they constitute a discrete exact complex. Such VE spaces were first introduced in Beirão Da Veiga *et al.* (2016) and later improved in a series of papers dealing with magnetostatic problems: Beirão da Veiga *et al.* (2017a) for the two-dimensional case; Beirão da Veiga *et al.* (2018a,b) for the three-dimensional case.

The definition of VE spaces in the above papers goes hand in hand with the introduction of discrete scalar products mimicking the L^2 scalar product used to discretize the problem of interest. Such scalar products and in general discrete bilinear forms are constructed upon two main ingredients: a projection that maps VE functions into a polynomial (sub)space and a computable ‘stabilization’.

For standard H^1 -conforming VE spaces (Beirão da Veiga *et al.*, 2013) there is a vast literature dealing with the interpolation properties of the spaces and the stability properties of the associated discrete bilinear forms; see, e.g., Mora *et al.* (2015); Brenner *et al.* (2017); Beirão da Veiga *et al.* (2017b); Cangiani *et al.* (2017); Brenner & Sung (2018); Cao & Chen (2018). Notwithstanding, only limited results regarding the approximation properties of face and edge VEs are available. In Beirão Da Veiga *et al.* (2017) face elements in two-dimensional are analyzed in a non-enhanced setting, simpler than that

in [Beirão da Veiga *et al.* \(2017a\)](#), and no result on the stability properties of the discrete scalar products is provided.

The goal of this contribution is to rigorously present (i) the interpolation properties of three-dimensional face and edge VE spaces and (ii) the stability properties of the associated discrete scalar products. In order to better underline the challenges and the interesting aspects of the analysis we focus on the lowest order case introduced in [Beirão da Veiga *et al.* \(2018b\)](#). Although the extension to the higher order case hides several additional technical difficulties the path to follow is essentially the same drawn in this paper. As outlined in the appendix the two-dimensional analysis is simpler and follows along the same lines as the three-dimensional one. The main challenges we have to cope with are related to the virtual nature of the involved spaces, which contain polynomials of a suitable maximum degree, but also other functions that are defined through a partial differential equation problem on the element. Furthermore, the lack of a reference element, which is an important tool in the finite element method (FEM) analysis, introduces several complications, such as rendering even standard inverse estimates in VE spaces quite challenging. We indeed believe that some proving techniques introduced in the present article can turn out to have a general value, representing useful tools and ideas for the VEM community. For instance, in order to derive the stability of the interpolation operator, we need to combine a Helmholtz decomposition with additional nonstandard manipulations in order to handle the serendipity constraint, and express the ensuing bound in terms of evaluations that represent the degrees of freedom. The stability of the scalar product hinges on a critical result that requires to split and re-assemble the error with an *ad-hoc* argument that we find quite interesting.

The paper is organized as follows. In Section 2 we introduce regular meshes and the related notation. Moreover, we recall a number of technical results in Sobolev spaces. We address the interpolation properties in three-dimensional face and edge VE spaces in Sections 3 and 4, respectively. We analyze the properties of explicit stabilizations for face and edge VE spaces in Section 5. Appendix A deals with the two-dimensional counterpart of the results presented in the paper.

We conclude this introduction setting the necessary notation and functional setting.

Notation and functional spaces

Given two positive quantities a and b , we use the short-hand notation ' $a \lesssim b$ ' if there exists a positive constant c independent of the discretization parameters such that ' $a \leq cb$ '. Moreover, we write ' $a \approx b$ ' if and only if ' $a \lesssim b$ and $b \lesssim a$ '.

We recall the definition of some differential operators. Given a face F parallel to the xy -plane and $\mathbf{v} = (v_1, v_2) : F \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $v : F \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$, define

$$\operatorname{div}_F \mathbf{v} := \partial_x v_1 + \partial_y v_2, \quad \operatorname{rot}_F \mathbf{v} := -\partial_y v_1 + \partial_x v_2, \quad \operatorname{curl}_F v := (\partial_y v, -\partial_x v)^T.$$

The above definitions trivially extend to the case of all faces F .

Furthermore, given $\mathbf{v} = (v_1, v_2, v_3) : K \subseteq \mathbb{R}^3 \rightarrow \mathbb{R}^3$, we define

$$\operatorname{div} \mathbf{v} := \partial_x v_1 + \partial_y v_2 + \partial_z v_3, \quad \operatorname{curl} \mathbf{v} := (\partial_y v_3 - \partial_z v_2, \partial_z v_1 - \partial_x v_3, \partial_x v_2 - \partial_y v_1)^T.$$

Next, we introduce several Sobolev spaces. Given a domain $D \subset \mathbb{R}^d$, $d = 2, 3$ we begin by defining the Sobolev space of order $s > 0$ by $H^s(D)$, which we endow with the norm and seminorm $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$. The case $s = 0$ boils down to the Lebesgue space $L^2(D)$. Negative Sobolev spaces are defined

via duality. In particular, for $\mathbf{v} : D \subset \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$, $d_1, d_2 \in \mathbb{N}$, we write

$$\|\mathbf{v}\|_{-1,D} := \sup_{\boldsymbol{\Phi} \in [H_0^1(D)]^{d_2}} \frac{(\mathbf{v}, \boldsymbol{\Phi})_{0,D}}{|\boldsymbol{\Phi}|_{1,D}}.$$

We introduce $H(\operatorname{div}, D)$ as the space of $L^2(D)$ functions having weak divergence in $L^2(D)$. Analogously, in the $d = 2$ and $d = 3$ cases, we introduce $H(\operatorname{rot}_F, D)$, $H(\mathbf{curl}, D)$ and $H(\mathbf{curl curl}, D)$ as the spaces of $[L^2(D)]^2$ and $[L^2(D)]^3$ functions having weak rotor, \mathbf{curl} and $\mathbf{curl curl}$ in $L^2(D)$, $[L^2(D)]^3$ and $[L^2(D)]^3$, respectively. Finally, for $D \subset \mathbb{R}^3$ and $s > 0$, we define $H^s(\mathbf{curl}, D)$ as the subspace of functions $\mathbf{v} \in [H^s(D)]^3$ such that $\mathbf{curl v} \in [H^s(D)]^3$.

2. Meshes and preliminaries

Henceforth, let $\Omega \subset \mathbb{R}^3$ denote a given polyhedral domain and $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$ be a sequence of polyhedral meshes. For all $n \in \mathbb{N}$ we denote the set of faces and edges by \mathcal{F}_n and \mathcal{E}_n , respectively. Moreover, given $K \in \mathcal{T}_n$, its set of faces is \mathcal{F}^K , whereas, given a face $F \in \mathcal{F}_n$, its set of edges is \mathcal{E}^F . The diameter of an element $K \in \mathcal{T}_n$ and a face $F \in \mathcal{F}_n$ is h_K and h_F , whereas the length of an edge $e \in \mathcal{E}_n$ is h_e .

We denote the outward unit normal vector to the boundary of an element K by \mathbf{n}_K . For each face F in ∂K we fix \mathbf{n}_F as $\mathbf{n}_{K|_F}$. For future convenience, for every face F , we also introduce $\mathbf{n}_{\partial F}$ defined as the outward unit normal vector to ∂F in the plane containing F . Then, for each edge e in ∂F , we fix \mathbf{n}_e as $\mathbf{n}_{\partial F|_e}$ in the plane containing F . Further, given an edge e of ∂F , $\boldsymbol{\tau}_e$ represents the tangent unit vector of e pointed in counter-clockwise sense with respect to the couple (F, \mathbf{n}_F) . We introduce $\boldsymbol{\tau}_{\partial F}$ such that $\boldsymbol{\tau}_{\partial F|_e} = \boldsymbol{\tau}_e$ for every edge $e \in \mathcal{E}^F$.

In what follows, (x_K, y_K, z_K) is the barycenter of an element K , whereas (x_F, y_F) represents the barycenter of a face F .

Henceforth, we demand that $\{\mathcal{T}_n\}_{n \in \mathbb{N}}$ satisfies the following assumptions: for all $n \in \mathbb{N}$,

- every element $K \in \mathcal{T}_n$ and face $F \in \mathcal{F}_n$ is star-shaped with respect to a ball of diameter γh_K and γh_F , respectively;
- given an element $K \in \mathcal{T}_n$, for all its faces $F \in \mathcal{F}^K$, $h_K \approx h_F$; moreover, given a face $F \in \mathcal{F}_n$, for all its edges $e \in \mathcal{E}^F$, $h_F \approx h_e$;
- every element $K \in \mathcal{T}_n$ is Lipschitz.

The above couple of assumptions is the accepted standard in the VE literature (taking the role of the classical shape-regularity for tetrahedral meshes in finite elements). Such conditions have been relaxed only in a few papers, see, e.g., [Beirão da Veiga *et al.* \(2017b\)](#); [Brenner & Sung \(2018\)](#); [Cao & Chen \(2018\)](#), all dealing with simpler H^1 -elliptic scalar problems. Note that, as a simple consequence of the conditions above, the number of edges per face and the number of faces per element are uniformly bounded in the mesh family.

We state here some technical results that we shall employ in the forthcoming analysis. The first one deals with several trace type inequalities.

LEMMA 2.1 Given $K \in \mathcal{T}_n$ with size h_K , the following inequality is valid: for all $v \in H^s(K)$, $s > 1/2$,

$$\|v\|_{0,\partial K} \lesssim h_K^{-\frac{1}{2}} \|v\|_{0,K} + h_K^{s-\frac{1}{2}} |v|_{s,K}. \quad (2.1)$$

Moreover, if additionally $1/2 < s \leq 1$ and v has zero average on either ∂K or K , then the following bound is valid as well:

$$\|v\|_{0,\partial K} \lesssim h_K^{s-\frac{1}{2}} |v|_{s,K}. \tag{2.2}$$

Finally, if $v \in H^1(K)$ and has zero average on either ∂K or K , we also have

$$\|v\|_{0,\partial K} \lesssim \|v\|_{0,K}^{\frac{1}{2}} |v|_{1,K}^{\frac{1}{2}}. \tag{2.3}$$

Proof. Bound (2.3) is the standard multiplicative trace; see, e.g., in Brenner & Scott (2008, Section 1.6). Bound (2.1) is a standard (scaled) trace inequality, whereas we obtain (2.2) by using (2.1) and the Poincaré inequality. \square

Next, we recall the Friedrichs' inequality.

LEMMA 2.2 Let $K \in \mathcal{T}_n$ and \mathbf{v} be a divergence free function satisfying $\mathbf{v}|_{\partial K} \times \mathbf{n}_K \in [L^2(\partial K)]^2$. Then, the following bound is valid:

$$\|\mathbf{v}\|_{0,K} \lesssim h_K \|\mathbf{curl} \mathbf{v}\|_{0,K} + h_K^{\frac{1}{2}} \|\mathbf{v} \times \mathbf{n}_K\|_{0,\partial K}. \tag{2.4}$$

Proof. This bound is a scaled version of (Monk, 2003, Corollary 3.51). \square

Further, we recall two polynomial inverse inequalities in polyhedral domains.

LEMMA 2.3 Given $K \in \mathcal{T}_n$ the following polynomial inverse estimate is valid: for all scalar and vector polynomials \mathbf{q} of bounded degree in K ,

$$|\mathbf{q}|_{1,K} \lesssim h_K^{-1} \|\mathbf{q}\|_{0,K}. \tag{2.5}$$

Set the scaled norm

$$\|\cdot\|_{-\frac{1}{2},\partial K} := \sup_{\varphi \in H^{\frac{1}{2}}(\partial K)} \frac{(\varphi, \cdot)_{0,\partial K}}{|\varphi|_{\frac{1}{2},\partial K} + h_K^{-\frac{1}{2}} \|\varphi\|_{0,\partial K}}. \tag{2.6}$$

Then, for every piecewise, i.e., face by face scalar polynomials q of bounded degree over ∂K , we also have

$$\|q\|_{0,\partial K} \lesssim h_K^{-\frac{1}{2}} \|q\|_{-\frac{1}{2},\partial K}. \tag{2.7}$$

The estimates above are valid substituting the element K with a face F as well.

Proof. Thanks to the regularity assumptions on the mesh in Section 2, each element $K \in \mathcal{T}_n$ can be split into a sub-tessellation $\tilde{\mathcal{T}}_n(K)$ consisting of a uniformly bounded set of shape-regular tetrahedra. Clearly, $\tilde{\mathcal{T}}_n(K)$ induces a shape-regular sub-triangulation $\tilde{\mathcal{F}}_n(F)$ on each face F of K .

As for (2.5) it suffices to apply a standard polynomial inverse estimate over each tetrahedron of $\tilde{\mathcal{T}}_n(K)$; see, e.g., Verfürth (2013, Proposition 3.37) and the references therein.

As for (2.7)₂, on each face $F \in \mathcal{F}^K$, we introduce b_F , which is defined as the piecewise cubic bubble function over $\tilde{\mathcal{F}}_n(F)$ such that $\|b_F\|_{\infty,T} = 1$ for all T in $\tilde{\mathcal{F}}_n(F)$. Also define a function $b_{\partial K}$ over ∂K satisfying $b_{\partial K}|_F = b_F$. A standard bubble argument triangle by triangle, recalling that q is a polynomial for all faces $F \in \mathcal{F}^K$, yields

$$\|b_F q\|_{0,F} \approx \|b_F^{1/2} q\|_{0,F} \approx \|q\|_{0,F}.$$

For the same reason a polynomial inverse estimate gives

$$|b_F q|_{1,F} \lesssim h_K^{-1} \|b_F q\|_{0,F}.$$

The above inequalities trivially imply

$$\|b_{\partial K} q\|_{0,\partial K} \approx \|b_{\partial K}^{1/2} q\|_{0,\partial K} \approx \|q\|_{0,\partial K} \quad |b_{\partial K} q|_{1,\partial K} \lesssim h_K^{-1} \|b_{\partial K} q\|_{0,\partial K}.$$

In turn interpolation theory (Triebel, 1978) yields

$$|b_{\partial K} q|_{\frac{1}{2},\partial K} \lesssim h_K^{-\frac{1}{2}} \|b_{\partial K} q\|_{0,\partial K}. \tag{2.8}$$

With this at hand we can write

$$\|q\|_{-\frac{1}{2},\partial K} = \sup_{\varphi \in H^{\frac{1}{2}}(\partial K)} \frac{(\varphi, q)_{0,\partial K}}{|\varphi|_{\frac{1}{2},\partial K} + h_K^{-\frac{1}{2}} \|\varphi\|_{0,\partial K}} \geq \frac{\|b_{\partial K}^{1/2} q\|_{0,\partial K}^2}{|b_{\partial K} q|_{\frac{1}{2},\partial K} + h_K^{-\frac{1}{2}} \|b_{\partial K} q\|_{0,\partial K}} \stackrel{(2.8)}{\gtrsim} h_K^{\frac{1}{2}} \|q\|_{0,\partial K},$$

which is the assertion. □

Finally, we recall the div- and **curl**-trace inequalities.

LEMMA 2.4 Given $K \in \mathcal{T}_n$ let $\boldsymbol{\psi} \in H(\text{div}, K)$ and $\mathbf{v} \in H(\mathbf{curl}, K)$. Then, the two following trace inequalities are valid:

$$\|\boldsymbol{\psi} \cdot \mathbf{n}_K\|_{-\frac{1}{2},\partial K} \lesssim \|\boldsymbol{\psi}\|_{0,K} + h_K \|\text{div} \boldsymbol{\psi}\|_{0,K} \tag{2.9}$$

and

$$\|\mathbf{v} \times \mathbf{n}_K\|_{-\frac{1}{2},\partial K} \lesssim \|\mathbf{v}\|_{0,K} + h_K \|\mathbf{curl} \mathbf{v}\|_{0,K}. \tag{2.10}$$

Moreover, given a face F , let $\mathbf{v} \in H(\text{rot}_F, F)$ and $\mathbf{w} \in H(\text{div}_F, F)$. Then, the following trace inequalities are valid as well:

$$\|\mathbf{v} \cdot \boldsymbol{\tau}_{\partial F}\|_{-\frac{1}{2},\partial F} \lesssim \|\mathbf{v}\|_{0,F} + h_F \|\text{rot}_F \mathbf{v}\|_{0,F}, \tag{2.11}$$

$$\|\mathbf{w} \cdot \mathbf{n}_{\partial F}\|_{-\frac{1}{2},\partial F} \lesssim \|\mathbf{w}\|_{0,F} + h_F \|\text{div}_F \mathbf{w}\|_{0,F}. \tag{2.12}$$

Proof. See, e.g., Monk (2003, Section 3.5). □

REMARK 2.5 All the forthcoming interpolation estimates are local to the element and immediately yield the associated global results by summation on all mesh elements. Analogously, we do not detail here the (conforming) global spaces associated to the local ones described here; the definition of such global spaces, which trivially follows by a standard FEM/VEM gluing procedure, can be found in Beirão da Veiga *et al.* (2018b).

3. Interpolation properties of three-dimensional face spaces

Let $K \in \mathcal{T}_h$ be an element. Following Beirão da Veiga *et al.* (2018b), we define the low-order local face VE space in three dimensions as follows:

$$\mathbf{V}_0^{\text{face}}(K) := \left\{ \boldsymbol{\psi}_h \in [L^2(K)]^3 \mid \operatorname{div} \boldsymbol{\psi}_h \in \mathbb{P}_0(K), \operatorname{curl} \boldsymbol{\psi}_h \in [\mathbb{P}_0(K)]^3, \right. \\ \left. \boldsymbol{\psi}_{h|F} \cdot \mathbf{n}_F \in \mathbb{P}_0(F) \forall F \in \mathcal{F}^K, \int_K \boldsymbol{\psi}_h \cdot (\mathbf{x}_K \times \mathbf{q}_0) = 0 \quad \forall \mathbf{q}_0 \in [\mathbb{P}_0(K)]^3 \right\}, \tag{3.1}$$

where we have set

$$\mathbf{x}_K := (x - x_K, y - y_K, z - z_K)^T. \tag{3.2}$$

In what follows, *enhancing constraint* refers to the constraint

$$\int_K \boldsymbol{\psi}_h \cdot (\mathbf{x}_K \times \mathbf{q}_0) = 0 \quad \forall \mathbf{q}_0 \in [\mathbb{P}_0(K)]^3. \tag{3.3}$$

We endow the local space $\mathbf{V}_0^{\text{face}}(K)$ with a set of unisolvent degrees of freedom, which is provided by the single value of $\boldsymbol{\psi}_h \cdot \mathbf{n}_F$ over each face $F \in \mathcal{F}^K$. In the following lemma we show that such normal component values uniformly control the L^2 norm of functions in $\mathbf{V}_0^{\text{face}}(K)$.

PROPOSITION 3.1 The following *a priori* bound is valid for any $\boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(K)$:

$$\|\boldsymbol{\psi}_h\|_{0,K} \lesssim h_K^{\frac{1}{2}} \|\boldsymbol{\psi}_h \cdot \mathbf{n}_K\|_{0,\partial K}. \tag{3.4}$$

Proof. There exist $\boldsymbol{\rho} \in H(\operatorname{curl} \operatorname{curl}, K) \cap H(\operatorname{div}, K)$ and $\Psi \in H^1(K)/\mathbb{R}$ such that the following Helmholtz decomposition is valid:

$$\boldsymbol{\psi}_h = \operatorname{curl} \boldsymbol{\rho} + \nabla \Psi. \tag{3.5}$$

To prove (3.5) we first define a function $\Psi \in H^1(\Omega)/\mathbb{R}$, satisfying weakly

$$\begin{cases} \Delta \Psi = \operatorname{div} \boldsymbol{\psi}_h & \text{in } K \\ \mathbf{n}_K \cdot \nabla \Psi = \mathbf{n}_K \cdot \boldsymbol{\psi}_h & \text{on } \partial K. \end{cases} \tag{3.6}$$

Next, we define a function $\boldsymbol{\rho} \in H(\mathbf{curl}, \Omega)$ satisfying weakly, see, e.g., [Kanayama et al. \(1990\)](#),

$$\begin{cases} \mathbf{curl} \mathbf{curl} \boldsymbol{\rho} = \mathbf{curl} \boldsymbol{\psi}_h & \text{in } K \\ \operatorname{div} \boldsymbol{\rho} = 0 & \text{in } K \\ \mathbf{n}_K \times \boldsymbol{\rho} = 0 & \text{on } \partial K. \end{cases} \quad (3.7)$$

It can be readily checked that $\boldsymbol{\rho}$ and Ψ are defined so that (3.5) is valid. Moreover, we also have the orthogonality condition $(\mathbf{curl} \boldsymbol{\rho}, \nabla \Psi)_{0,K} = 0$, whence we get

$$\|\boldsymbol{\psi}_h\|_{0,K}^2 = \|\nabla \Psi\|_{0,K}^2 + \|\mathbf{curl} \boldsymbol{\rho}\|_{0,K}^2. \quad (3.8)$$

We bound the two terms on the right-hand side of (3.8) separately. We begin with that involving the gradient. Using an integration by parts, the definition (3.6), the fact that Ψ has zero average over K and $\operatorname{div} \boldsymbol{\psi}_h \in \mathbb{P}_0(K)$, we get

$$\|\nabla \Psi\|_{0,K}^2 = - \int_K \operatorname{div} \boldsymbol{\psi}_h \Psi + \int_{\partial K} \mathbf{n}_K \cdot \boldsymbol{\psi}_h \Psi = \int_{\partial K} \mathbf{n}_K \cdot \boldsymbol{\psi}_h \Psi \leq \|\boldsymbol{\psi}_h \cdot \mathbf{n}_K\|_{0,\partial K} \|\Psi\|_{0,\partial K}.$$

Further using inequality (2.2) with $s = 1$, we arrive at

$$|\Psi|_{1,K} \lesssim h_K^{\frac{1}{2}} \|\boldsymbol{\psi}_h \cdot \mathbf{n}_K\|_{0,\partial K}. \quad (3.9)$$

Next, we deal with the second term on the right-hand side of (3.8). Integrating by parts and using (3.7), we obtain

$$\|\mathbf{curl} \boldsymbol{\rho}\|_{0,K}^2 = \int_K \mathbf{curl} \boldsymbol{\rho} \cdot \mathbf{curl} \boldsymbol{\rho} = \int_K \boldsymbol{\rho} \cdot \mathbf{curl} \mathbf{curl} \boldsymbol{\rho} = \int_K \boldsymbol{\rho} \cdot \mathbf{curl} \boldsymbol{\psi}_h.$$

Set $\mathbf{q}_0 := \mathbf{curl} \boldsymbol{\psi}_h$. Since $\mathbf{q}_0 \in [\mathbb{P}_0(K)]^3$ a direct calculation shows that

$$\mathbf{q}_0 = \frac{1}{2} \mathbf{curl}(\mathbf{q}_0 \times \mathbf{x}_K).$$

Together with an integration by parts and the property $\mathbf{n}_K \times \boldsymbol{\rho}|_{\partial K} = 0$, see (3.7), this entails

$$\begin{aligned} \|\mathbf{curl} \boldsymbol{\rho}\|_{0,K}^2 &= \frac{1}{2} \int_K \boldsymbol{\rho} \cdot \mathbf{curl}(\mathbf{q}_0 \times \mathbf{x}_K) = \frac{1}{2} \int_K \mathbf{curl} \boldsymbol{\rho} \cdot (\mathbf{q}_0 \times \mathbf{x}_K) + \frac{1}{2} \int_{\partial K} (\mathbf{n}_K \times \boldsymbol{\rho}) \cdot (\mathbf{q}_0 \times \mathbf{x}_K) \\ &= \frac{1}{2} \int_K \mathbf{curl} \boldsymbol{\rho} \cdot (\mathbf{q}_0 \times \mathbf{x}_K). \end{aligned}$$

Next, we use (3.5), the enhancing constraint (3.3) and a Cauchy–Schwarz inequality:

$$\|\mathbf{curl} \boldsymbol{\rho}\|_{0,K}^2 = \frac{1}{2} \int_K (\boldsymbol{\psi}_h - \nabla \Psi) \cdot (\mathbf{q}_0 \times \mathbf{x}_K) = -\frac{1}{2} \int_K \nabla \Psi \cdot (\mathbf{q}_0 \times \mathbf{x}_K) \leq \frac{1}{2} |\Psi|_{1,K} \|\mathbf{q}_0\|_{0,K} \|\mathbf{x}_K\|_{\infty,K}.$$

Recalling that $\mathbf{q}_0 := \mathbf{curl} \boldsymbol{\psi}_h = \mathbf{curl} \mathbf{curl} \boldsymbol{\rho}$ and using $\|\mathbf{x}_K\|_{\infty, K} \leq h_K$, we end up with

$$\|\mathbf{curl} \boldsymbol{\rho}\|_{0, K}^2 \lesssim h_K |\Psi|_{1, K} \|\mathbf{curl} \mathbf{curl} \boldsymbol{\rho}\|_{0, K}. \tag{3.10}$$

We claim that the following inverse inequality is valid:

$$h_K \|\mathbf{curl} \mathbf{curl} \boldsymbol{\rho}\|_{0, K} \lesssim \|\mathbf{curl} \boldsymbol{\rho}\|_{0, K}. \tag{3.11}$$

To prove (3.11) define b_K as the piecewise quartic bubble function over the shape-regular decomposition of K into tetrahedra T ; see the proof of Lemma 2.3. More precisely, b_K is the positive quartic polynomial over each tetrahedron T given by the product of the four barycentric coordinates of T , scaled such that $\|b_K\|_{\infty, T} = 1$.

Since $\mathbf{curl} \mathbf{curl} \boldsymbol{\rho} \in [\mathbb{P}_0(K)]^3$ we can use the equivalence of norms in finite-dimensional spaces on each tetrahedron T , and an integration by parts with the zero boundary conditions provided by the bubble function b_K , and write

$$\|\mathbf{curl} \mathbf{curl} \boldsymbol{\rho}\|_{0, K}^2 \approx \int_K b_K (\mathbf{curl} \mathbf{curl} \boldsymbol{\rho})^2 = \int_K \mathbf{curl} (b_K \mathbf{curl} \mathbf{curl} \boldsymbol{\rho}) \cdot \mathbf{curl} \boldsymbol{\rho}.$$

Next, a Cauchy–Schwarz inequality, the polynomial inverse inequality (2.5), and a trivial bound on the L^∞ norm of the bubble function b_K yield

$$\|\mathbf{curl} \mathbf{curl} \boldsymbol{\rho}\|_{0, K}^2 \lesssim \|\mathbf{curl} (b_K \mathbf{curl} \mathbf{curl} \boldsymbol{\rho})\|_{0, K} \|\mathbf{curl} \boldsymbol{\rho}\|_{0, K} \lesssim h_K^{-1} \|\mathbf{curl} \mathbf{curl} \boldsymbol{\rho}\|_{0, K} \|\mathbf{curl} \boldsymbol{\rho}\|_{0, K},$$

which entails (3.11).

Inserting (3.11) in (3.10) and using bound on Ψ in (3.9) give

$$\|\mathbf{curl} \boldsymbol{\rho}\|_{0, K} \lesssim |\Psi|_{1, K} \lesssim h_K^{\frac{1}{2}} \|\boldsymbol{\psi}_h \cdot \mathbf{n}_K\|_{0, \partial K}. \tag{3.12}$$

Combining (3.9) and (3.12) into (3.8) yields the assertion. □

Thanks to the *a priori* estimate (3.4) we can prove the following interpolation result.

PROPOSITION 3.2 For all $K \in \mathcal{T}_n$ let $\boldsymbol{\psi} \in [H^s(K)]^3$, $1/2 < s \leq 1$. Then, there exists $\boldsymbol{\psi}_I \in \mathbf{V}_0^{\text{face}}(K)$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0, K} \lesssim h_K^s |\boldsymbol{\psi}|_{s, K}.$$

Proof. Given $K \in \mathcal{T}_n$, let $\boldsymbol{\psi}_T$ be the vector given by the average of each component of $\boldsymbol{\psi}$ over K . We define the interpolant $\boldsymbol{\psi}_I \in \mathbf{V}_0^{\text{face}}(K)$ through its degrees of freedom as follows:

$$\int_F (\boldsymbol{\psi} - \boldsymbol{\psi}_I) \cdot \mathbf{n}_F = 0 \quad \forall F \in \mathcal{F}^K. \tag{3.13}$$

Note that (3.13) is well defined since $\boldsymbol{\psi}$ belongs to $[H^s(K)]^3$ with $s > 1/2$.

The triangle inequality, the definition of $\boldsymbol{\psi}_\pi$ and the Poincaré inequality together with interpolation theory yield

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0,K} \leq \|\boldsymbol{\psi} - \boldsymbol{\psi}_\pi\|_{0,K} + \|\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi\|_{0,K} \lesssim h_K^s |\boldsymbol{\psi}|_{s,K} + \|\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi\|_{0,K}.$$

We still have to bound the second term on the right-hand side. Since vector constant functions are contained in the space $\mathbf{V}_0^{\text{face}}(K)$ the term $\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi$ belongs to $\mathbf{V}_0^{\text{face}}(K)$. To get the assertion we apply (3.4), the triangle inequality, the definition of $\boldsymbol{\psi}_I$ in (3.13) and inequality (2.2):

$$\begin{aligned} \|\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi\|_{0,K} &\lesssim h_K^{\frac{1}{2}} \|(\boldsymbol{\psi}_I - \boldsymbol{\psi}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \leq h_K^{\frac{1}{2}} \|(\boldsymbol{\psi} - \boldsymbol{\psi}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} + h_K^{\frac{1}{2}} \|(\boldsymbol{\psi} - \boldsymbol{\psi}_I) \cdot \mathbf{n}_K\|_{0,\partial K} \\ &\leq 2h_K^{\frac{1}{2}} \|(\boldsymbol{\psi} - \boldsymbol{\psi}_\pi) \cdot \mathbf{n}_K\|_{0,\partial K} \lesssim h_K^s |\boldsymbol{\psi}|_{s,K}, \end{aligned}$$

whence the assertion follows. □

For future convenience we also define face VE spaces on faces. More precisely let $F \in \mathcal{E}_n$ be a face. We define the low-order local two-dimensional face VE space as follows:

$$\begin{aligned} \mathbf{V}_0^{\text{face}}(F) := \left\{ \boldsymbol{\psi}_h \in [L^2(F)]^2 \mid \operatorname{div}_F \boldsymbol{\psi}_h \in \mathbb{P}_0(K), \operatorname{rot}_F \boldsymbol{\psi}_h \in \mathbb{P}_0(K), \right. \\ \left. \boldsymbol{\psi}_{h|e} \cdot \mathbf{n}_e \in \mathbb{P}_0(e) \forall e \in \mathcal{E}^F, \int_K \boldsymbol{\psi}_h \cdot \mathbf{x}_F^\perp = 0 \right\}, \end{aligned} \tag{3.14}$$

where we have set, in the local coordinates of F ,

$$\mathbf{x}_F := (x - x_F, y - y_F)^T \quad \mathbf{x}_F^\perp := (y_F - y, x - x_F)^T. \tag{3.15}$$

COROLLARY 3.3 For all $K \in \mathcal{T}_n$ let $\boldsymbol{\psi} \in [H^\varepsilon(K)]^3$, $\varepsilon > 0$, and $\operatorname{div} \boldsymbol{\psi} \in H^s(K)$, $0 \leq s \leq 1$. Then, the following bound is valid:

$$\|\operatorname{div}(\boldsymbol{\psi} - \boldsymbol{\psi}_I)\|_{0,K} \lesssim h_K^s |\operatorname{div} \boldsymbol{\psi}|_{s,K}.$$

Such result, combined with Proposition 3.2, yields an optimal $O(h_K^s)$ interpolation estimate in the H_{div} norm.

Proof. It is trivial to check that $\operatorname{div} \boldsymbol{\psi}_I \in \mathbb{P}_0(K)$ corresponds to the average of $\operatorname{div} \boldsymbol{\psi}$ on K . The assertion follows from the Poincaré inequality. The regularity condition above on $\boldsymbol{\psi}$ is stated also to guarantee the well-posedness of the interpolant. □

4. Interpolation properties of three-dimensional edge spaces

4.1 *A useful result for two-dimensional edge spaces*

Let $K \in \mathcal{T}_n$ be an element and $F \in \mathcal{F}^K$ be any of its faces. We define the low-order local two-dimensional edge VE space as follows:

$$\mathbf{V}_0^{\text{edge}}(F) := \left\{ \mathbf{v}_h \in [L^2(F)]^2 \mid \begin{aligned} &\text{div}_F \mathbf{v}_h \in \mathbb{P}_0(F), \text{rot}_F \mathbf{v}_h \in \mathbb{P}_0(F), \\ &\mathbf{v}_h|_e \cdot \boldsymbol{\tau}_e \in \mathbb{P}_0(e) \forall e \in \mathcal{E}^F, \int_F \mathbf{v}_h \cdot \mathbf{x}_F = 0 \end{aligned} \right\}, \tag{4.1}$$

where \mathbf{x}_F is defined in (3.15).

In what follows, *enhancing constraint* refers to the constraint

$$\int_F \mathbf{v}_h \cdot \mathbf{x}_F = 0. \tag{4.2}$$

We endow the local space $\mathbf{V}_0^{\text{edge}}(F)$ with a set of unisolvent degrees of freedom, which is provided by the single value of $\mathbf{v}_h \cdot \boldsymbol{\tau}_e$ over each edge $e \in \mathcal{E}^F$. Moreover, it contains two-dimensional Nédélec polynomials of degree 1, i.e., the space $[\mathbb{P}_0(K)]^2 \oplus (\mathbf{x}^\perp \mathbb{P}_0(K))$.

The main result of the section is the following *a priori* estimate bound. Although its proof has some similarities with that of Proposition 3.1, it is worth reporting the details here for completeness.

PROPOSITION 4.1 Given $\mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(F)$ the following *a priori* bound is valid:

$$\|\mathbf{v}_h\|_{0,F} \lesssim h_F^{\frac{1}{2}} \|\mathbf{v}_h \cdot \boldsymbol{\tau}_{\partial F}\|_{0,\partial F}. \tag{4.3}$$

Proof. We first prove that there exist $\rho \in H^1(F)/\mathbb{R}$ and $\Psi \in H^1(F)$ such that the following Helmholtz decomposition is valid:

$$\mathbf{v}_h = \mathbf{curl}_F \rho + \nabla_F \Psi. \tag{4.4}$$

Define $\rho \in H^1(F)/\mathbb{R}$ and $\Psi \in H^1(F)$ as the weak solutions to the problems

$$\begin{cases} \Delta_F \rho = \text{rot}_F \mathbf{v}_h & \text{in } F \\ \boldsymbol{\tau}_{\partial F} \cdot \mathbf{curl}_F \rho (= \mathbf{n}_{\partial F} \cdot \nabla_F \rho) = \boldsymbol{\tau}_{\partial F} \cdot \mathbf{v}_h & \text{on } \partial F \end{cases} \tag{4.5}$$

and

$$\begin{cases} \Delta_F \Psi = \text{div}_F \mathbf{v}_h & \text{in } F \\ \Psi = 0 & \text{on } \partial F. \end{cases}$$

It can be readily checked that ρ and Ψ are defined so that (4.4) is valid. Moreover, we also have the orthogonality condition $(\mathbf{curl}_F \rho, \nabla_F \Psi)_{0,F} = 0$, whence we get

$$\|\mathbf{v}_h\|_{0,F}^2 = \|\mathbf{curl}_F \rho\|_{0,K}^2 + \|\nabla_F \Psi\|_{0,K}^2. \quad (4.6)$$

We bound the two terms on the right-hand side of (4.6) separately. We begin with that involving the \mathbf{curl}_F operator. To this aim we preliminarily recall the identity

$$\operatorname{rot}_F \mathbf{curl}_F = \Delta_F. \quad (4.7)$$

Integrating by parts and using differential identity (4.7) together with the fact that ρ has constant Laplacian and zero average over F , we obtain

$$\|\mathbf{curl}_F \rho\|_{0,F}^2 = - \int_F \rho \Delta_F \rho + \int_{\partial F} \rho (\boldsymbol{\tau}_{\partial F} \cdot \mathbf{curl}_F \rho) = \int_{\partial F} \rho (\boldsymbol{\tau}_{\partial F} \cdot \mathbf{curl}_F \rho),$$

which, recalling (4.5), applying a Cauchy–Schwarz inequality and using (2.2) with $s = 1$, yields

$$\|\mathbf{curl}_F \rho\|_{0,F}^2 \leq \|\rho\|_{0,\partial F} \|\boldsymbol{\tau}_{\partial F} \cdot \mathbf{v}_h\|_{0,\partial F} \lesssim h_K^{\frac{1}{2}} \|\nabla_F \rho\|_{0,F} \|\boldsymbol{\tau}_{\partial F} \cdot \mathbf{v}_h\|_{0,\partial F}.$$

Observing that $\|\nabla_F \rho\|_{0,F} = \|\mathbf{curl}_F \rho\|_{0,F}$, we get

$$\|\mathbf{curl}_F \rho\|_{0,F} \lesssim h_F^{\frac{1}{2}} \|\boldsymbol{\tau}_{\partial F} \cdot \mathbf{v}_h\|_{0,\partial F}. \quad (4.8)$$

Next, we deal with the second term on the right-hand side of (4.6). To this aim we use an integration by parts, $\Psi|_{\partial F} = 0$, the fact that $\Delta_F \Psi$ is constant over F and the identity $\operatorname{div}_F \mathbf{x}_F = 2$:

$$\|\nabla_F \Psi\|_{0,F}^2 = \int_F (-\Delta_F \Psi) \Psi + \int_{\partial F} \Psi (\nabla_F \Psi \cdot \mathbf{n}_{\partial F}) = \frac{1}{2} \int_F (-\Delta_F \Psi) \Psi \operatorname{div}_F \mathbf{x}_F.$$

First an integration by parts, then equation (4.4) and the enhancing constraint (4.2), finally a Cauchy–Schwarz inequality lead to

$$\begin{aligned} \|\nabla_F \Psi\|_{0,F}^2 &= \frac{1}{2} \int_F (\Delta_F \Psi) \nabla_F \Psi \cdot \mathbf{x}_F = \frac{1}{2} \int_F (\Delta_F \Psi) (\mathbf{v}_h - \mathbf{curl}_F \rho) \cdot \mathbf{x}_F \\ &= -\frac{1}{2} \int_F (\Delta_F \Psi) \mathbf{curl}_F \rho \cdot \mathbf{x}_F \lesssim \|\Delta_F \Psi\|_{0,F} \|\mathbf{curl}_F \rho\|_{0,F} \|\mathbf{x}_F\|_{\infty,F}. \end{aligned} \quad (4.9)$$

The following inverse estimate is valid:

$$\|\Delta_F \Psi\|_{0,F} \lesssim h_F^{-1} \|\nabla_F \Psi\|_{0,F}. \quad (4.10)$$

To prove (4.10) introduce b_F the piecewise cubic bubble function over the decomposition of F into triangles T ; see the proof of Lemma 2.3. Using that $\Delta_F \Psi$ is constant over F and the equivalence of

norms in finite-dimensional spaces on each triangle T yields

$$\|\Delta_F \Psi\|_{0,F}^2 \approx \int_F b_F (\Delta_F \Psi)^2.$$

Integrating by parts, using that $b_F|_{\partial F} = 0$, applying the polynomial inverse inequality (2.5), recalling that $\|b_K\|_{\infty,K} = 1$ and using again the fact that $\Delta_F \Psi$ is constant over F , allow us to write

$$\begin{aligned} \|\Delta_F \Psi\|_{0,F}^2 &\approx \int_F \operatorname{div}_F(b_F \Delta_F \Psi) \cdot \nabla_F \Psi \leq \|\operatorname{div}_F(b_F \Delta_F \Psi)\|_{0,F} |\Psi|_{1,F} \\ &\lesssim h_F^{-1} \|b_F \Delta_F \Psi\|_{0,F} |\Psi|_{1,F} \leq h_F^{-1} \|\Delta_F \Psi\|_{0,F} |\Psi|_{1,F}, \end{aligned}$$

whence (4.10) follows.

Inserting the inverse inequality (4.10), the bound $\|\mathbf{x}_F\|_{\infty,F} \lesssim h_F$ and (4.8) in (4.9) yields

$$\|\nabla \Psi\|_{0,F} \lesssim \|\mathbf{curl}_F \rho\|_{0,F} \lesssim h_F^{\frac{1}{2}} \|\boldsymbol{\tau}_{\partial F} \cdot \mathbf{v}_h\|_{0,\partial F}. \tag{4.11}$$

Inserting (4.8) and (4.11) in (4.6) provides (4.3), which concludes the proof. □

4.2 The main interpolation result for edge spaces

Henceforth, given a vector function $\mathbf{v} : K \rightarrow \mathbb{R}^3$ and any face $F \in \mathcal{F}^K$, the symbol \mathbf{v}^F denotes the vector field given by the tangent component of \mathbf{v}^F on F , i.e., the restriction to F of $\mathbf{v} - \mathbf{n}_F(\mathbf{v} \cdot \mathbf{n}_F)$. We can interpret such a function as a three-component vector field, but also as a two-component vector field living in the tangent plane to the face. With an abuse of notation we shall use both interpretations in the following. Finally, observe that \mathbf{v}^F corresponds to a 90° rotation of $\mathbf{v}^F \times \mathbf{n}_F$. We also introduce a function $\mathbf{v}^{\partial K}$ over ∂K defined as $\mathbf{v}^{\partial K}|_F := \mathbf{v}^F$ on each face F .

Given $K \in \mathcal{T}_h$, following *Beirão da Veiga et al. (2018b)* we define the low-order local three-dimensional edge VE space in K as follows:

$$\begin{aligned} \mathbf{V}_0^{\text{edge}}(K) := & \left\{ \mathbf{v}_h \in [L^2(K)]^3 \mid \operatorname{div} \mathbf{v}_h = 0, \mathbf{curl} \operatorname{curl} \mathbf{v}_h \in [\mathbb{P}_0(K)]^3, \right. \\ & \mathbf{v}_h^F \in \mathbf{V}_0^{\text{edge}}(F) \forall F \in \mathcal{F}^K, \mathbf{v}_h \cdot \tilde{\boldsymbol{\tau}}_e \text{ continuous at each edge } e \text{ of } K, \\ & \left. \int_K \mathbf{curl} \mathbf{v}_h \cdot (\mathbf{x}_K \times \mathbf{q}_0) = 0 \forall \mathbf{q}_0 \in [\mathbb{P}_0(K)]^3 \right\}, \end{aligned}$$

where \mathbf{x}_K is defined as in (3.2), and $\tilde{\boldsymbol{\tau}}_e$ denotes a single valued unit tangent vector to edge e chosen once and for all.

We endow the local space $\mathbf{V}_0^{\text{edge}}(K)$ with a set of unisolvent degrees of freedom, which is provided by the single value of $\mathbf{v}_h \cdot \boldsymbol{\tau}_e$ over each edge e of K . Moreover, it contains three-dimensional Nédélec polynomials of degree 1, i.e., the space $[\mathbb{P}_0(K)]^3 \oplus (\mathbf{x} \times \mathbb{P}_0(K))$.

In what follows, we use the following exact sequence property, which was proven in (Beirão da Veiga *et al.*, 2018b, Proposition 4.2):

$$\mathbf{curl}(\mathbf{V}_0^{\text{edge}}(K)) = \{\boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(K) \mid \text{div} \boldsymbol{\psi}_h = 0\} \subset \mathbf{V}_0^{\text{face}}(K). \tag{4.12}$$

With this result at hand, we are able to prove the following bound for functions in three-dimensional edge VE spaces.

PROPOSITION 4.2 The following bound is valid for any $\mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(K)$:

$$\|\mathbf{v}_h\|_{0,K} \lesssim h_K^{\frac{1}{2}} \|\mathbf{v}_h^F\|_{0,\partial K}. \tag{4.13}$$

Proof. Recall the differential identity $\mathbf{curl} \mathbf{v}_h \cdot \mathbf{n}_{F|F} = \text{rot}_F \mathbf{v}_h^F$ on each face $F \in \mathcal{F}^K$.

Using (4.12) and (3.4) for three-dimensional face VE spaces, we readily get

$$\|\mathbf{curl} \mathbf{v}_h\|_{0,K} \lesssim h_K^{\frac{1}{2}} \sum_{F \in \mathcal{F}^K} \|\mathbf{curl} \mathbf{v}_h \cdot \mathbf{n}_F\|_{0,F} = h_K^{\frac{1}{2}} \sum_{F \in \mathcal{F}^K} \|\text{rot}_F \mathbf{v}_h^F\|_{0,F}.$$

The following inverse estimate is valid:

$$\|\text{rot}_F \mathbf{v}_h^F\|_{0,F} \lesssim h_K^{-1} \|\mathbf{v}_h^F\|_{0,F}. \tag{4.14}$$

Bound (4.14) can be shown using the same arguments employed to prove (3.11) and (4.10), the essential ingredient being the fact that \mathbf{v}_h^F is a two-dimensional edge VE function, which in turn implies that $\text{rot}_F \mathbf{v}_h^F$ is constant over F .

We deduce

$$\|\mathbf{curl} \mathbf{v}_h\|_{0,K} \lesssim h_K^{-\frac{1}{2}} \sum_{F \in \mathcal{F}^K} \|\mathbf{v}_h^F\|_{0,F}.$$

Assertion (4.13) follows by inserting the above inequality on the right-hand side of the Friedrichs' inequality (2.4) and summing over the faces of K . □

We collect estimates (4.3) and (4.13) in the following result, stating that the opportunely scaled degrees of freedom uniformly control the L^2 norm of functions in $\mathbf{V}_0^{\text{edge}}(K)$.

COROLLARY 4.3 The following *a priori* bound is valid for any $\mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(K)$:

$$\|\mathbf{v}_h\|_{0,K}^2 \lesssim h_K^2 \sum_{F \in \mathcal{F}^K} \sum_{e \in \mathcal{E}^F} \|\mathbf{v}_h \cdot \boldsymbol{\tau}_e\|_{0,e}^2. \tag{4.15}$$

Next, we show interpolation properties for three-dimensional edge VE spaces. To this aim we need a preliminary technical result.

LEMMA 4.4 Let $F \in \mathcal{F}_n$. Then, for all $\mathbf{v} \in [H^\varepsilon(F)]^2 \cap H(\text{rot}_F, F)$, with $\varepsilon \in (0, 1/2]$, such that its tangent component on each edge is integrable, the following bound is valid:

$$\sum_{e \in \mathcal{E}^F} \left| \int_e \mathbf{v} \cdot \boldsymbol{\tau}_{\partial F} \right|^2 \lesssim \|\mathbf{v}\|_{0,F}^2 + h_F^{2\varepsilon} |\mathbf{v}|_{\varepsilon,F}^2 + h_F^2 \|\text{rot}_F \mathbf{v}\|_{0,F}^2. \tag{4.16}$$

Proof. The first and main step in the proof is showing the following trace-type inequality: for all fixed $p > 2$,

$$\sum_{e \in \mathcal{E}^F} \left| \int_e \mathbf{v} \cdot \boldsymbol{\tau}_{\partial F} \right| \lesssim h_F^{1-\frac{2}{p}} \|\mathbf{v}\|_{L^p(F)} + h_F \|\text{rot}_F \mathbf{v}\|_{0,F}. \tag{4.17}$$

We split the face F into a shape-regular triangulation $\tilde{\mathcal{T}}_n(F)$, e.g., as in the proof of Lemma 2.3. It suffices to prove, for all $e \in \mathcal{E}^F$,

$$\left| \int_e \mathbf{v} \cdot \boldsymbol{\tau}_e \right| \lesssim h_F^{1-\frac{2}{p}} \|\mathbf{v}\|_{L^p(T)} + h_F \|\text{rot}_F \mathbf{v}\|_{0,T}, \tag{4.18}$$

where T is the only triangle in $\tilde{\mathcal{T}}_n(F)$ such that $e \subset \partial T$ (and we recall $h_T \sim h_F$).

Let \hat{T} be the reference triangle, T be mapped to \hat{T} through the (rotated) Piola transform, and \hat{e} be the edge of \hat{T} corresponding to the edge e through the (rotated) Piola transform. The trace theorem on Lipschitz domains states that the trace operator is surjective from $W^{1,p'}(\hat{T})$ to $W^{\frac{1}{p},p'}(\partial\hat{T})$. Further, the space $W^{\frac{1}{p},p'}(\partial\hat{T})$ contains piecewise discontinuous functions over $\partial\hat{T}$ for $p > 2$. In particular, there exists a function \hat{w} such that $\hat{w} = 1$ on \hat{e} , $\hat{w} = 0$ on $\partial\hat{T} \setminus \hat{e}$ and $\|\hat{w}\|_{W^{1,p'}(\hat{T})} < \infty$ (we pick one among the many).

Next, we use a scaling argument (the $\hat{\cdot}$ denoting the usual pull-back), an integration by parts, the Hölder inequality and simple manipulations, and get

$$\begin{aligned} \left| \int_e \mathbf{v} \cdot \boldsymbol{\tau}_e \right| &\lesssim h_F \left| \int_{\hat{e}} \hat{\mathbf{v}} \cdot \hat{\boldsymbol{\tau}}_{\hat{e}} \right| = h_F \left| \int_{\partial\hat{T}} (\hat{\mathbf{v}} \cdot \hat{\boldsymbol{\tau}}_{\hat{e}}) \hat{w} \right| = h_F \left| \int_{\hat{T}} \widehat{\text{rot}}_{\hat{F}} \hat{\mathbf{v}} \hat{w} + \int_{\hat{T}} \hat{\mathbf{v}} \cdot \widehat{\text{curl}}_{\hat{F}} \hat{w} \right| \\ &\lesssim h_F \left(\|\widehat{\text{rot}}_{\hat{F}} \hat{\mathbf{v}}\|_{0,\hat{T}} \|\hat{w}\|_{0,\hat{T}} + \|\hat{\mathbf{v}}\|_{L^p(\hat{T})} \|\hat{w}\|_{W^{1,p'}(\hat{T})} \right) \\ &\lesssim h_F \left(\|\hat{\mathbf{v}}\|_{L^p(\hat{T})} + \|\widehat{\text{rot}}_{\hat{F}} \hat{\mathbf{v}}\|_{0,\hat{T}} \right) \|\hat{w}\|_{W^{1,p'}(\hat{T})} \lesssim h_F^{1-\frac{2}{p}} \|\mathbf{v}\|_{L^p(T)} + h_F \|\text{rot}_F \mathbf{v}\|_{0,T}. \end{aligned}$$

We have shown (4.17). The assertion finally follows by picking $p = 2/(1 - \varepsilon) > 2$ in (4.17), using the embedding $H^\varepsilon(F) \hookrightarrow L^p(F)$, and performing standard manipulations. \square

Introduce the following scaled Sobolev norms on faces: given $\varepsilon > 0$,

$$\|\cdot\|_{\varepsilon,F}^2 := \|\cdot\|_{0,F}^2 + h_F^{2\varepsilon} |\cdot|_{\varepsilon,F}^2 \quad \forall F \in \mathcal{F}_n. \tag{4.19}$$

The norm in (4.19) is defined so that it scales like an L^2 norm on faces.

We are now in the position of showing the following interpolation result.

PROPOSITION 4.5 For all $K \in \mathcal{T}_n$ let $\mathbf{v} \in H^s(\mathbf{curl}, K)$, $1/2 < s \leq 1$, such that its tangent component on each edge is integrable. Then, there exists $\mathbf{v}_I \in \mathbf{V}_0^{\text{edge}}(K)$ such that

$$\|\mathbf{v} - \mathbf{v}_I\|_{0,K} \lesssim h_K^s |\mathbf{v}|_{s,K} + h_K \|\mathbf{curl} \mathbf{v}\|_{0,K} + h_K^{s+1} |\mathbf{curl} \mathbf{v}|_{s,K} \quad \forall K \in \mathcal{T}_n.$$

Proof. Define \mathbf{v}_I as the interpolant in $\mathbf{V}_0^{\text{edge}}(K)$ of \mathbf{v} , i.e.,

$$\int_e (\mathbf{v} - \mathbf{v}_I) \cdot \tilde{\boldsymbol{\tau}}_e = 0 \quad \forall \text{ edges } e \text{ of } K. \tag{4.20}$$

It is easy to check that the integral in (4.20) is well defined for $\mathbf{v} \in H^s(\mathbf{curl}, K)$, $s > 1/2$. Let $\mathbf{v}_\pi \in [\mathbb{P}_0(K)]^3$ be the vector average of \mathbf{v} over K , which is contained in $\mathbf{V}_0^{\text{edge}}(K)$. Applying the triangle inequality and (4.15), we get

$$\|\mathbf{v} - \mathbf{v}_I\|_{0,K}^2 \lesssim \|\mathbf{v} - \mathbf{v}_\pi\|_{0,K}^2 + h_K^2 \sum_{F \in \mathcal{F}^K} \sum_{e \in \mathcal{E}^F} \|(\mathbf{v}_I - \mathbf{v}_\pi) \cdot \boldsymbol{\tau}_e\|_{0,e}^2.$$

Using that $\mathbf{v}_I \cdot \boldsymbol{\tau}_e$ is constant on each edge e and (4.20), we write

$$\|(\mathbf{v}_I - \mathbf{v}_\pi) \cdot \boldsymbol{\tau}_e\|_{0,e}^2 = \frac{1}{h_e} \left| \int_e (\mathbf{v}_I - \mathbf{v}_\pi) \cdot \boldsymbol{\tau}_e \right|^2 = \frac{1}{h_e} \left| \int_e (\mathbf{v} - \mathbf{v}_\pi) \cdot \boldsymbol{\tau}_e \right|^2.$$

Combining the two above equations and using $h_e \sim h_K$ entail

$$\|\mathbf{v} - \mathbf{v}_I\|_{0,K}^2 \lesssim \|\mathbf{v} - \mathbf{v}_\pi\|_{0,K}^2 + h_K \sum_{F \in \mathcal{F}^K} \sum_{e \in \mathcal{E}^F} \left| \int_e (\mathbf{v} - \mathbf{v}_\pi) \cdot \boldsymbol{\tau}_e \right|^2 =: T_1 + T_2. \tag{4.21}$$

We start by showing an upper bound for the term T_2 . Given $\varepsilon > 0$, applying (4.16), a (scaled) trace inequality and the Poincaré inequality together with interpolation theory, we get

$$T_2 \lesssim h_K \sum_{F \in \mathcal{F}^K} \left(\|\mathbf{v} - \mathbf{v}_\pi\|_{\varepsilon,F}^2 + h_K^2 \|\text{rot}_F \mathbf{v}_h^F\|_{0,F}^2 \right) \lesssim h_K^{1+2\varepsilon} |\mathbf{v}|_{1/2+\varepsilon,K}^2 + h_K^3 \sum_{F \in \mathcal{F}^K} \|\text{rot}_F \mathbf{v}_h^F\|_{0,F}^2, \tag{4.22}$$

where the scaled norm $\|\cdot\|_{\varepsilon,F}$ is defined in (4.19).

We bound the second term on the right-hand side of (4.22). Upon rewriting the rot_F operator on each face as the normal component of the \mathbf{curl} operator, we apply (2.1) and deduce

$$h_K^3 \sum_{F \in \mathcal{F}^K} \|\text{rot}_F \mathbf{v}_h^F\|_{0,F}^2 \lesssim h_K^2 \|\mathbf{curl} \mathbf{v}\|_{0,K}^2 + h_K^{2\varepsilon+3} |\mathbf{curl} \mathbf{v}|_{\varepsilon+1/2,K}^2. \tag{4.23}$$

Inserting (4.23) into (4.22) and choosing $s = \varepsilon + 1/2$ yield

$$T_2 \lesssim h_K^{2s} |\mathbf{v}|_{s,K}^2 + h_K^2 \|\mathbf{curl} \mathbf{v}\|_{0,K}^2 + h_K^{2+2s} |\mathbf{curl} \mathbf{v}|_{s,K}^2. \tag{4.24}$$

The assertion follows by collecting (4.24) in (4.21) and applying the Poincaré inequality on the term T_1 . \square

Although the result in Proposition 4.5 requires some ‘additional regularity’ for the objective function, this is essentially needed in order to make the interpolation operator well defined. Combining such bound with an approximation estimate for the **curl** yields estimates in the $H_{\mathbf{curl}}$ norm that are optimal also with respect to the required regularity, as shown in the corollary below.

COROLLARY 4.6 For all $K \in \mathcal{T}_h$ let $\mathbf{v} \in H^s(\mathbf{curl}, K)$, $1/2 < s \leq 1$ such that its tangent component on each edge is integrable. Then, the following bound is valid:

$$\|\mathbf{curl}(\mathbf{v} - \mathbf{v}_I)\|_{0,K} \lesssim h_K^s |\mathbf{curl} \mathbf{v}|_{s,K}.$$

Such result, combined with Proposition 4.5, yields an optimal $O(h_K^s)$ interpolation estimate in the $H_{\mathbf{curl}}$ norm.

Proof. Thanks to the exact sequence property (4.12) and the definition of the interpolation operators, it is easy to check that the following commuting diagram property is valid. For all $\mathbf{v} \in H^s(\mathbf{curl}, K)$, $1/2 < s \leq 1$,

$$\mathbf{curl} \mathbf{v}_I = (\mathbf{curl} \mathbf{v})_I,$$

where the interpolation on the left-hand side is into $\mathbf{V}_0^{\text{edge}}(K)$ and the interpolation on the right-hand side is into $\mathbf{V}_0^{\text{face}}(K)$. The result follows from the above identity and Proposition 3.2 with $\boldsymbol{\psi} = \mathbf{curl} \mathbf{v}$. \square

REMARK 4.7 The condition that the tangent component of \mathbf{v} is integrable on each edge, appearing in Lemma 4.4, Proposition 4.5 and Corollary 4.6, is the minimal one in order to define the interpolant \mathbf{v}_I in (4.20) by simple edge integrals. We could avoid such a condition by interpreting $\int_e \mathbf{v} \cdot \boldsymbol{\tau}_e$ in the dual sense rather than as an integral, i.e., $\langle \mathbf{v} \cdot \boldsymbol{\tau}_e, \chi_e \rangle_{W^{1/p,p'}(\partial F)}$ with $\chi_e : \partial F \rightarrow \mathbb{R}$ equal to 1 on e and vanishing on $\partial F \setminus e$.

5. Stability properties of discrete scalar products

Face and edge VE functions are typically associated with the computation of L^2 inner products over elements. For instance, this is the case for the discretization of magnetostatic problems Beirão da Veiga *et al.* (2018a,b) and the Maxwell’s equations Beirão da Veiga *et al.* (2021). Since functions in $\mathbf{V}_0^{\text{face}}(K)$ and $\mathbf{V}_0^{\text{edge}}(K)$ are not available in closed-form, we address the computation of L^2 bilinear forms following the VEM gospel Beirão da Veiga *et al.* (2013). The standard VEM construction of L^2 scalar products is as follows: given any two (face or edge) VE functions \mathbf{u}_h and \mathbf{v}_h ,

$$(\mathbf{u}_h, \mathbf{v}_h)_{0,K} \approx (\boldsymbol{\Pi}_0 \mathbf{u}_h, \boldsymbol{\Pi}_0 \mathbf{v}_h)_{0,K} + S^K ((\mathbf{I} - \boldsymbol{\Pi}_0) \mathbf{u}_h, (\mathbf{I} - \boldsymbol{\Pi}_0) \mathbf{v}_h),$$

where $\boldsymbol{\Pi}_0$ is an L^2 -orthogonal projector into a polynomial space, \mathbf{I} is the vector identity operator and S^K is a bilinear form scaling like the L^2 norm on the VE functions. More precisely, it is required the

existence of two positive constants $\alpha_* \leq \alpha^*$ such that, for all functions \mathbf{v}_h in the VE space under consideration,

$$\alpha_* \|\mathbf{v}_h\|_{0,K}^2 \leq S^K(\mathbf{v}_h, \mathbf{v}_h) \leq \alpha^* \|\mathbf{v}_h\|_{0,K}^2. \tag{5.1}$$

We require the projections Π_0 and the stabilizations $S^K(\cdot, \cdot)$ to be computable via the degrees of freedom of the VE space under consideration.

In the literature of the VEM bounds of the form (5.1) have been analyzed for nodal elements [Beirão da Veiga et al. \(2017b\)](#); [Brenner et al. \(2017\)](#); [Brenner & Sung \(2018\)](#); [Cao & Chen \(2018, 2019\)](#), but no result exists for face and edge elements. Here, we derive an *explicit* analysis of the stability bounds (5.1) for VE face and edge spaces. We cope with face and edge VE spaces in Sections 5.1 and 5.2 below, respectively.

For future convenience we state here a preliminary, technical result.

LEMMA 5.1 Given a face F let $\mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(F)$. Then, the following inverse estimate is valid:

$$\|\mathbf{v}_h \cdot \boldsymbol{\tau}_{\partial F}\|_{0,\partial F} \lesssim h_F^{-\frac{1}{2}} \|\mathbf{v}_h \cdot \boldsymbol{\tau}_{\partial F}\|_{-\frac{1}{2},\partial F}, \tag{5.2}$$

where as usual $\boldsymbol{\tau}_{\partial F|e} := \boldsymbol{\tau}_e$ and $\|\cdot\|_{-\frac{1}{2},\partial F}$ is defined in (2.6).

Proof. It suffices to recall that $\mathbf{v}_h \cdot \boldsymbol{\tau}_{\partial F}$ is piecewise constant over each edge $e \in \mathcal{E}^F$ and apply essentially the same techniques as in Lemma 2.3, in two dimensions instead of in three dimensions. \square

5.1 Stability bounds for face virtual elements

As for face VEs, consider the L^2 projector $\Pi_{\text{face}}^0 : \mathbf{V}_0^{\text{face}}(K) \rightarrow [\mathbb{P}_0(K)]^3$, which is computable via the degrees of freedom, i.e., the value of the normal component over faces. Consider the stabilization, cf. ([Beirão da Veiga et al., 2018b](#), equation (4.17)),

$$S_{\text{face}}^K(\boldsymbol{\psi}_h, \boldsymbol{\phi}_h) := h_K \sum_{F \in \mathcal{F}^K} (\boldsymbol{\psi}_h \cdot \mathbf{n}_F, \boldsymbol{\phi}_h \cdot \mathbf{n}_F)_{0,F} \quad \forall \boldsymbol{\psi}_h, \boldsymbol{\phi}_h \in \mathbf{V}_0^{\text{face}}(K). \tag{5.3}$$

The stabilization $S_{\text{face}}^K(\cdot, \cdot)$ is explicitly computable via the degrees of freedom since $\boldsymbol{\psi}_h|_F \cdot \mathbf{n}_F$ and $\boldsymbol{\phi}_h|_F \cdot \mathbf{n}_F$ belong to $\mathbb{P}_0(F)$ for all faces F .

We prove the following stability result.

PROPOSITION 5.2 Given S_{face}^K the stabilization in (5.3) the following bounds are valid:

$$\alpha_* \|\boldsymbol{\psi}_h\|_{0,K}^2 \leq S_{\text{face}}^K(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) \leq \alpha^* \|\boldsymbol{\psi}_h\|_{0,K}^2 \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(K), \tag{5.4}$$

where α_* and α^* are two positive constants independent of h_K .

Proof. The lower bound in (5.4) follows immediately from (3.4).

As for the upper bound in (5.4), we apply the polynomial inverse inequality (2.7) and the div-trace inequality (2.9), and get

$$\|\boldsymbol{\psi}_h \cdot \mathbf{n}_K\|_{0,\partial K} \lesssim h_K^{-\frac{1}{2}} \|\boldsymbol{\psi}_h \cdot \mathbf{n}_K\|_{-\frac{1}{2},\partial K} \lesssim h_K^{-\frac{1}{2}} \|\boldsymbol{\psi}_h\|_{0,K} + h_K^{\frac{1}{2}} \|\operatorname{div} \boldsymbol{\psi}_h\|_{0,K}. \quad (5.5)$$

The following inverse inequality is valid:

$$\|\operatorname{div} \boldsymbol{\psi}_h\|_{0,K} \lesssim h_K^{-1} \|\boldsymbol{\psi}_h\|_{0,K}. \quad (5.6)$$

To prove (5.6) it suffices to recall that $\operatorname{div} \boldsymbol{\psi}_h \in \mathbb{P}_0(K)$ and proceed as in the proof of inverse estimates (3.11), (4.10) and (4.14). Inserting (5.6) in (5.5) we get the assertion. \square

5.2 Stability bounds for edge virtual elements

As for edge VEs consider the L^2 projector $\Pi_{\text{edge}}^0 : \mathbf{V}_0^{\text{edge}}(K) \rightarrow [\mathbb{P}_0(K)]^3$, which is computable via the degrees of freedom, i.e., the value of the tangential component over edges, as shown in [Beirão da Veiga et al. \(2018b\)](#). Consider the stabilization, cf. [Beirão da Veiga et al. \(2018b\)](#), equation (4.8))

$$S_{\text{edge}}^K(\mathbf{u}_h, \mathbf{v}_h) := h_K^2 \sum_{F \in \mathcal{F}^K} \sum_{e \in \mathcal{E}^F} (\mathbf{u}_h \cdot \boldsymbol{\tau}_e, \mathbf{v}_h \cdot \boldsymbol{\tau}_e)_{0,e} \quad \forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(K). \quad (5.7)$$

The stabilization $S_{\text{edge}}^K(\cdot, \cdot)$ is explicitly computable via the degrees of freedom since $\mathbf{u}_h \cdot \boldsymbol{\tau}_e$ and $\mathbf{v}_h \cdot \boldsymbol{\tau}_e$ belong to $\mathbb{P}_0(e)$ for all edges e .

Before proving the stability result for the stabilization in (5.7), we show two critical preliminary results dealing with inverse estimates for functions in face and edge VE spaces.

LEMMA 5.3 The following inverse estimates in two-dimensional face VE spaces are valid: for all faces $F \in \mathcal{F}_n$,

$$\|\boldsymbol{\psi}_h\|_{0,F} \lesssim h_F^{-1} \|\boldsymbol{\psi}_h\|_{-1,F} \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(F). \quad (5.8)$$

Moreover, the following inverse estimates in three-dimensional face VE spaces is valid as well: for all elements $K \in \mathcal{T}_n$,

$$\|\boldsymbol{\psi}_h\|_{0,K} \lesssim h_K^{-1} \|\boldsymbol{\psi}_h\|_{-1,K} \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(K). \quad (5.9)$$

Proof. Since the proof of the two bounds is essentially identical we focus on the first one only. Therefore, let $\boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(F)$; we recall that $\operatorname{rot}_F \boldsymbol{\psi}_h \in \mathbb{P}_0(F)$.

We start by introducing a vector polynomial $\mathbf{q}_1 \in [\mathbb{P}_1(F)]^2$ such that $\operatorname{rot}_F(\boldsymbol{\psi}_h - \mathbf{q}_1) = 0$, where \mathbf{q}_1 is given in the local coordinates (x, y) of F by

$$\mathbf{q}_1 = \beta(y_F - y, x - x_F), \quad \beta \in \mathbb{R}.$$

Then, we define a function $\Psi \in H^1(K) \setminus \mathbb{R}$ as

$$\begin{cases} \Delta_F \Psi = \operatorname{div}_F(\boldsymbol{\psi}_h - \mathbf{q}_1) & \text{in } F \\ \nabla_F \Psi \cdot \mathbf{n}_{\partial F} = (\boldsymbol{\psi}_h - \mathbf{q}_1) \cdot \mathbf{n}_{\partial F} & \text{on } \partial F. \end{cases}$$

It is easy to check that

$$\boldsymbol{\psi}_h - \mathbf{q}_1 = \nabla_F \Psi, \tag{5.10}$$

since the functions on the left- and right-hand sides of (5.10) have the same divergence, rotor and boundary normal components. As a first step, we prove the following inverse estimate:

$$\|\nabla_F \Psi\|_{0,F} \lesssim h_F^{-1} \|\nabla_F \Psi\|_{-1,F}. \tag{5.11}$$

Since $\Delta_F \Psi \in \mathbb{P}_0(F)$ employing a polynomial inverse estimate yields the bound

$$\|\Delta_F \Psi\|_{0,F} \lesssim h_F^{-1} \|\Delta_F \Psi\|_{-1,F} = h_F^{-1} \sup_{\Phi \in H_0^1(F)} \frac{(\Delta_F \Psi, \Phi)_{0,F}}{|\Phi|_{1,F}} \leq h_F^{-1} \|\nabla_F \Psi\|_{0,F}. \tag{5.12}$$

In order to show (5.11) we first use an integration by parts and use equation (5.12):

$$\|\nabla_F \Psi\|_{0,F}^2 = - \int_F \Psi \Delta_F \Psi + \int_{\partial F} (\mathbf{n}_{\partial F} \cdot \nabla_F \Psi) \Psi \lesssim h_F^{-1} \|\Psi\|_{0,F} \|\nabla_F \Psi\|_{0,F} + \|\mathbf{n}_{\partial F} \cdot \nabla_F \Psi\|_{0,\partial F} \|\Psi\|_{0,\partial F}.$$

Recall that $\mathbf{n}_{\partial F} \cdot \nabla_F \Psi$ is a piecewise polynomial over ∂F , and apply the polynomial inverse inequality (2.7) on the boundary of the face F . We deduce

$$\|\nabla_F \Psi\|_{0,F}^2 \lesssim h_F^{-1} \|\Psi\|_{0,F} \|\nabla_F \Psi\|_{0,F} + h_K^{-\frac{1}{2}} \|\mathbf{n}_{\partial F} \cdot \nabla_F \Psi\|_{-\frac{1}{2},\partial F} \|\Psi\|_{0,\partial F}.$$

Next, we apply the div-trace inequality (2.12) combined with the inverse estimate (5.12), and arrive at

$$\|\nabla_F \Psi\|_{0,F}^2 \lesssim h_F^{-1} \|\Psi\|_{0,F} \|\nabla_F \Psi\|_{0,F} + h_K^{-\frac{1}{2}} \|\nabla_F \Psi\|_{0,F} \|\Psi\|_{0,\partial F}.$$

Subsequently, we use the multiplicative trace inequality (2.3) and get

$$\|\nabla_F \Psi\|_{0,F} \lesssim h_F^{-1} \|\Psi\|_{0,F} + h_F^{-\frac{1}{2}} \|\Psi\|_{0,F}^{\frac{1}{2}} \|\nabla_F \Psi\|_{0,F}^{\frac{1}{2}} \implies \|\nabla_F \Psi\|_{0,F} \lesssim h_F^{-1} \|\Psi\|_{0,F}.$$

The inverse inequality (5.11) follows from the above bound and the inf-sup property of the Stokes problem, see [Boffi et al. \(2013\)](#):

$$\|\Psi\|_{0,F} \lesssim \sup_{\mathbf{v} \in [H_0^1(F)]^2} \frac{(\Psi, \operatorname{div}_F \mathbf{v})_{0,F}}{|\mathbf{v}|_{1,F}} = \sup_{\mathbf{v} \in [H_0^1(F)]^2} \frac{(\nabla_F \Psi, \mathbf{v})_{0,F}}{|\mathbf{v}|_{1,F}} = \|\nabla_F \Psi\|_{-1,F}.$$

Recalling (5.10) and (5.11), we obtain

$$\|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{0,F} \lesssim h_F^{-1} \|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{-1,F}. \tag{5.13}$$

With this at hand we proceed with the proof of the assertion. We apply the triangle inequality, the inverse estimate (5.13) and a polynomial inverse estimate, to obtain

$$\|\boldsymbol{\psi}_h\|_{0,F} \leq \|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{0,F} + \|\mathbf{q}_1\|_{0,F} \lesssim h_F^{-1} \|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{-1,F} + h_F^{-1} \|\mathbf{q}_1\|_{-1,F}. \tag{5.14}$$

We need to show that the two terms on the right-hand side of (5.14) can be somewhat re-assembled together in order to obtain (5.8). To this aim we proceed as follows. Introduce the function $\mathbf{z} \in [H_0^1(F)]^2$ ‘realizing’ the sup in the definition of $\|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{-1,F}$. In other words, for a positive constant c_1 , \mathbf{z} is such that

$$(\boldsymbol{\psi}_h - \mathbf{q}_1, \mathbf{z})_{0,F} \geq c_1 \|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{-1,F}, \quad |\mathbf{z}|_{1,F} = 1. \tag{5.15}$$

Additionally, define

$$w \in H_0^2(F) := \left\{ w \in H^2(F) \mid w|_{\partial F} = 0, \nabla w|_{\partial F} = \mathbf{0} \right\}$$

as the square of the piecewise cubic bubble function over the regular sub-triangulation $\tilde{\mathcal{F}}_n(F)$ of the face F , see the proof of Lemma 2.3, multiplied by $\text{sign}(\text{rot}_F \mathbf{q}_1)$. We scale w so that $|\mathbf{curl}_F w|_{1,F} = 1$ and observe that $\mathbf{curl}_F w \in [H_0^1(F)]^2$.

Using an integration by parts, the definition of w and several polynomial inverse estimates involving \mathbf{q}_1 and w , we get the following bound:

$$(\mathbf{q}_1, \mathbf{curl}_F w)_{0,F} = (\text{rot}_F \mathbf{q}_1, w)_{0,F} = |\text{rot}_F \mathbf{q}_1| \|w\|_{L^1(F)} \gtrsim h_F^{-2} \|\mathbf{q}_1\|_{0,F} h_F^3 |\mathbf{curl}_F w|_{1,F} = h_F \|\mathbf{q}_1\|_{0,F}.$$

An additional polynomial inverse inequality gives, for a positive constant c_2 ,

$$(\mathbf{q}_1, \mathbf{curl}_F w)_{0,F} \geq c_2 \|\mathbf{q}_1\|_{-1,F}. \tag{5.16}$$

We further observe the orthogonality property

$$(\boldsymbol{\psi}_h - \mathbf{q}_1, \mathbf{curl}_F w)_{0,F} = (\nabla \boldsymbol{\psi}, \mathbf{curl}_F w)_{0,F} = 0 \quad \forall w \in H_0^2(F). \tag{5.17}$$

Therefore, we can write

$$\begin{aligned} \|\boldsymbol{\psi}_h\|_{-1,F} &:= \sup_{\mathbf{v} \in [H_0^1(F)]^2} \frac{(\boldsymbol{\psi}_h, \mathbf{v})_{0,F}}{|\mathbf{v}|_{1,F}} = \sup_{\mathbf{v} \in [H_0^1(F)]^2} \frac{(\boldsymbol{\psi}_h - \mathbf{q}_1, \mathbf{v})_{0,F} + (\mathbf{q}_1, \mathbf{v})_{0,F}}{|\mathbf{v}|_{1,F}} \\ &\geq \frac{(\boldsymbol{\psi}_h - \mathbf{q}_1, \mathbf{z} + \alpha \mathbf{curl}_F w)_{0,F} + (\mathbf{q}_1, \mathbf{z} + \alpha \mathbf{curl}_F w)_{0,F}}{|\mathbf{z} + \alpha \mathbf{curl}_F w|_{1,F}}, \end{aligned}$$

where $\alpha > 0$ is a sufficiently large constant to be assigned later.

Using (5.15), (5.16) and (5.17), we deduce:

$$\begin{aligned} \|\boldsymbol{\psi}_h\|_{-1,F} &\geq \frac{(\boldsymbol{\psi}_h - \mathbf{q}_1, \mathbf{z})_{0,F} + (\mathbf{q}_1, \mathbf{z})_{0,F} + \alpha(\mathbf{q}_1, \mathbf{curl}_F w)_{0,F}}{1 + \alpha|\mathbf{curl} w|_{1,F}} \\ &\geq \frac{c_1\|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{-1,F} - \|\mathbf{q}_1\|_{-1,F} + \alpha c_2\|\mathbf{q}_1\|_{-1,F}}{1 + \alpha} \\ &= \left(\frac{c_1}{1 + \alpha}\right)\|\boldsymbol{\psi}_h - \mathbf{q}_1\|_{-1,F} + \left(\frac{c_2\alpha - 1}{1 + \alpha}\right)\|\mathbf{q}_1\|_{-1,K}. \end{aligned} \tag{5.18}$$

The assertion follows by taking $\alpha = 2/c_2$ and combining (5.14) with (5.18). □

COROLLARY 5.4 The following inverse estimates in two-dimensional edge VE spaces is valid: for all faces $F \in \mathcal{F}_n$,

$$\|\mathbf{v}_h\|_{0,F} \lesssim h_F^{-1}\|\mathbf{v}_h\|_{-1,F} \quad \forall \mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(F). \tag{5.19}$$

Moreover, summing over the faces of K and using interpolation theory, we also have that

$$\|\mathbf{v}_h^{\partial K}\|_{0,\partial K} \lesssim h_K^{-\frac{1}{2}}\|\mathbf{v}_h^{\partial K}\|_{-1/2,\partial K} \quad \forall \mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(K), \tag{5.20}$$

where $\mathbf{v}_h^{\partial K}$ denotes the tangential component on ∂K of the function \mathbf{v}_h .

Proof. The proof follows from the ‘rotated’ version of (5.8), since in two dimensions the space $\mathbf{V}_0^{\text{edge}}(F)$ can be obtained by a 90° rotation of the space $\mathbf{V}_0^{\text{face}}(F)$, cf. (3.14) and (4.1). In other words, every function \mathbf{v}_h in $\mathbf{V}_0^{\text{edge}}(F)$ can be written as $\mathcal{R}_{\pi/2}\boldsymbol{\psi}_h$, with $\boldsymbol{\psi}_h$ in $\mathbf{V}_0^{\text{face}}(F)$ and the operator $\mathcal{R}_{\pi/2}$ denoting a rotation of 90° for two-component vector fields living on F .

We briefly comment on the proof of (5.20). Using the definition of the $H^{-1}(F)$ norm and (5.19)

$$\|\mathbf{v}_h^{\partial K}\|_{0,\partial K} \lesssim \sum_{F \in \mathcal{E}^F} \|\mathbf{v}_h^{\partial K}\|_{0,F} \lesssim \sum_{F \in \mathcal{E}^F} h_F^{-1}\|\mathbf{v}_h^{\partial K}\|_{-1,F} \lesssim h_K^{-1}\|\mathbf{v}_h^{\partial K}\|_{-1,\partial K}.$$

The hidden constant depends on the number of faces of the element K . The assertion follows by using interpolation theory. □

We are in the position of proving the stability result for the stabilization in (5.7).

PROPOSITION 5.5 Given $\mathbf{S}_{\text{edge}}^K$ the stabilization defined in (5.7), the following bounds are valid:

$$\beta_*\|\mathbf{v}_h\|_{0,K}^2 \leq \mathbf{S}_{\text{edge}}^K(\mathbf{v}_h, \mathbf{v}_h) \leq \beta^*\|\mathbf{v}_h\|_{0,K}^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(K), \tag{5.21}$$

where β_* and β^* are two positive constants independent of h_K .

Proof. The lower bound in (5.21) follows directly from (4.15).

As for the upper bound in (5.21), we apply the polynomial inverse inequality (5.2) and the two-dimensional rot-trace inequality (2.11), and arrive at

$$\begin{aligned} S_{\text{edge}}^K(\mathbf{v}_h, \mathbf{v}_h) &= h_K^2 \sum_{F \in \mathcal{F}^K} \sum_{e \in \mathcal{E}^F} \|\mathbf{v}_h \cdot \boldsymbol{\tau}_e\|_{0,e}^2 = h_K^2 \sum_{F \in \mathcal{F}^K} \|\mathbf{v}_h \cdot \boldsymbol{\tau}_e\|_{0,\partial F}^2 \lesssim h_K \sum_{F \in \mathcal{F}^K} \|\mathbf{v}_h \cdot \boldsymbol{\tau}_e\|_{-\frac{1}{2},\partial F}^2 \\ &\lesssim h_K \sum_{F \in \mathcal{F}^K} \|\mathbf{v}_h^F\|_{0,F}^2 + h_K^3 \sum_{F \in \mathcal{F}^K} \|\text{rot}_F \mathbf{v}_h^F\|_{0,F}^2. \end{aligned}$$

As for the second term on the right-hand side, we know that $\text{rot}_F \mathbf{v}_h^F$ is constant over F for all faces $F \in \mathcal{F}^K$. Therefore, using tools analogous to those employed to prove inverse estimates (3.11), (4.10) and (4.14), we have

$$\|\text{rot}_F \mathbf{v}_h^F\|_{0,F} \lesssim h_K^{-1} \|\mathbf{v}_h^F\|_{0,F}.$$

Upon combining the two above bounds, we can write

$$S_{\text{edge}}^K(\mathbf{v}_h, \mathbf{v}_h) \lesssim h_K \|\mathbf{v}_h^{\partial K}\|_{0,\partial K}^2.$$

We now employ the inverse estimate (5.20) and the curl-trace inequality (2.10) to arrive at

$$S_{\text{edge}}^K(\mathbf{v}_h, \mathbf{v}_h) \lesssim \|\mathbf{v}_h^{\partial K}\|_{-\frac{1}{2},\partial K}^2 = \|\mathbf{v}_h \times \mathbf{n}_K\|_{-\frac{1}{2},\partial K}^2 \lesssim \|\mathbf{v}_h\|_{0,K}^2 + h_K^2 \|\mathbf{curl} \mathbf{v}_h\|_{0,K}^2.$$

Recalling that $\mathbf{curl} \mathbf{v}_h \in \mathbf{V}_0^{\text{face}}(K)$, we apply the inverse estimate (5.9), integrate by parts in the dual norm definition and deduce

$$S_{\text{edge}}^K(\mathbf{v}_h, \mathbf{v}_h) \lesssim \|\mathbf{v}_h\|_{0,K}^2 + \|\mathbf{curl} \mathbf{v}_h\|_{-1,K}^2 = \|\mathbf{v}_h\|_{0,K}^2 + \sup_{\boldsymbol{\Phi} \in [H_0^1(K)]^2} \frac{(\mathbf{curl} \mathbf{v}_h, \boldsymbol{\Phi})_{0,K}}{|\boldsymbol{\Phi}|_{1,K}} \lesssim \|\mathbf{v}_h\|_{0,K}^2.$$

□

Acknowledgements

The authors are grateful to Gianni Gilardi for an interesting email exchange regarding Lemma 4.4 and Remark 4.7.

Funding

L. Beirão da Veiga was partially supported by the European Research Council through the H2020 Consolidator Grant (grant no. 681162) CAVE, ‘Challenges and Advancements in Virtual Elements’, and by the Ministero dell’Università e della Ricerca through the Italian PRIN 2017 grant ‘Virtual Element Methods: Analysis and Applications’. Both these supports are gratefully acknowledged. L. Mascotto acknowledges support from the Austrian Science Fund (FWF) project P33477.

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A. Interpolation and stability properties in two-dimensional face and edge virtual element spaces

Using tools analogous to those developed in Sections 3–5 for the three-dimensional case, we can easily derive also the analogous two-dimensional counterparts. For completeness, we state the interpolation properties of functions in two-dimensional face and edge VE spaces. Recall that such spaces are defined in (3.14) and (4.1), respectively.

THEOREM A.1 For any generic polygon F of a regular two-dimensional regular mesh in the sense of Section 2, let $\boldsymbol{\psi}$ and $\mathbf{v} \in [H^s(F)]^2$, $1/2 < s \leq 1$. Then, there exist $\boldsymbol{\psi}_I \in \mathbf{V}_0^{\text{face}}(F)$ and $\mathbf{v}_I \in \mathbf{V}_0^{\text{edge}}(F)$ such that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_I\|_{0,F} \lesssim h_K^s |\boldsymbol{\psi}|_{s,F}, \quad \|\mathbf{v} - \mathbf{v}_I\|_{0,F} \lesssim h_K^s |\mathbf{v}|_{s,F}.$$

Proof. We can prove the first bound as in Proposition 3.2. The second bound is the ‘rotated’ version of the first one. \square

We also inherit the two-dimensional counterpart of the stabilization properties detailed in Section 5. More precisely, for all polygons F of the mesh, introduce the bilinear forms $S_{\text{face}}^K : \mathbf{V}_0^{\text{face}}(F) \times \mathbf{V}_0^{\text{face}}(F) \rightarrow \mathbb{R}$

$$S_{\text{face}}^F(\boldsymbol{\psi}_h, \boldsymbol{\phi}_h) := h_K \sum_{e \in \mathcal{E}^F} (\boldsymbol{\psi}_h \cdot \mathbf{n}_e, \boldsymbol{\phi}_h \cdot \mathbf{n}_e)_{0,e} \tag{A.1}$$

and $S_{\text{edge}}^F : \mathbf{V}_0^{\text{edge}}(F) \times \mathbf{V}_0^{\text{edge}}(F) \rightarrow \mathbb{R}$

$$S_{\text{edge}}^F(\mathbf{v}_h, \mathbf{w}_h) := h_K \sum_{e \in \mathcal{E}^F} (\mathbf{v}_h \cdot \boldsymbol{\tau}_e, \mathbf{w}_h \cdot \boldsymbol{\tau}_e)_{0,e}. \tag{A.2}$$

THEOREM A.2 For any generic polygon F of a regular two-dimensional regular mesh in the sense of Section 2, let $S_{\text{face}}^F(\cdot, \cdot)$ be the stabilization defined in (A.1). Then, the following bounds are valid:

$$\alpha_* \|\boldsymbol{\psi}_h\|_{0,F}^2 \leq S_{\text{face}}^F(\boldsymbol{\psi}_h, \boldsymbol{\psi}_h) \leq \alpha^* \|\boldsymbol{\psi}_h\|_{0,F}^2 \quad \forall \boldsymbol{\psi}_h \in \mathbf{V}_0^{\text{face}}(F),$$

where α_* and α^* are two positive constants independent of h_K .

Further, given $S_{\text{edge}}^F(\cdot, \cdot)$ the stabilization defined in (A.2), the following bounds are valid:

$$\beta_* \|\mathbf{v}_h\|_{0,F}^2 \leq S_{\text{edge}}^F(\mathbf{v}_h, \mathbf{v}_h) \leq \beta^* \|\mathbf{v}_h\|_{0,F}^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_0^{\text{edge}}(F),$$

where β_* and β^* are two positive constants independent of h_K .

Proof. We can prove the first bounds as in Proposition 5.2. Those for edge elements are the ‘rotated’ versions of the first ones. \square