

# On some efficiency conditions for vector optimization problems with uncertain cone constraints: a robust approach via set-valued inclusions

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## ABSTRACT

In the present paper, several types of efficiency conditions are established for vector optimization problems with cone constraints affected by uncertainty, but with no information of stochastic nature about the uncertain data. Following a robust optimization approach, data uncertainty is faced by handling set-valued inclusion problems. The employment of recent advances about error bounds and tangential approximations of the solution set to the latter enables one to achieve necessary conditions for weak efficiency via a penalization method as well as via the modern revisitation of the Euler-Lagrange method, with or without generalized convexity assumptions. The presented conditions are formulated in terms of various nonsmooth analysis constructions, expressing first-order approximations of mappings and sets, while the metric increase property plays the role of a constraint qualification.

## KEYWORDS

Vector optimization problem; data uncertainty; robust approach; weak efficiency condition; generalized derivative; generalized convexity.

## 1. Introduction

Consider a vector optimization problem

$$(P) \quad \text{Min}_K f(x) \quad \text{subject to } x \in \mathcal{R},$$

where  $\mathcal{R} \subseteq \mathbb{X}$  is a decision set defining the feasible region of the problem,  $f : \mathbb{X} \rightarrow \mathbb{Y}$  represents the criterion with respect to which decisions in  $\mathcal{R}$  are to be optimized, and  $K \subseteq \mathbb{Y}$  is a convex cone defining the partial order, according to which the outcomes of decisions are compared in the criteria space. Throughout the paper  $(\mathbb{X}, \|\cdot\|)$  and  $(\mathbb{Y}, \|\cdot\|)$  denote real Banach spaces and it will be assumed that  $\text{int } K \neq \emptyset$ . For vector optimization problems the concept of solution is not uniquely defined but several notions, reflecting different aspects of the issue, can be considered. Among the others, the notions of efficient and weakly efficient solution are well recognized and largely investigated in the literature devoted to vector optimization (see [3,10,15–17,19,20,27]). Recall that an element  $\bar{x} \in \mathcal{R}$  is said to be a locally weakly efficient (for short, w-eff.)

solution to  $(\mathcal{P})$  if there exists  $\delta > 0$  such that

$$f(\mathcal{R} \cap B(\bar{x}, \delta)) \cap [f(\bar{x}) - \text{int } K] = \emptyset;$$

an element  $\bar{x} \in \mathcal{R}$  is said to be a locally efficient (for short, eff.) solution to  $(\mathcal{P})$  if there exists  $\delta > 0$  such that

$$f(\mathcal{R} \cap B(\bar{x}, \delta)) \cap [f(\bar{x}) - K] = \{f(\bar{x})\}.$$

Clearly, any locally eff. solution to  $(\mathcal{P})$  is also a locally w-eff. one. The present paper deals with conditions of local weak efficiency for vector optimization problems, whose decision set  $\mathcal{R}$  is formalized by uncertain cone constraints, namely problems of the form

$$(\mathcal{P}_\omega) \quad \text{Min}_K f(x) \quad \text{with } x \in S \text{ subject to } g(\omega, x) \in C,$$

where  $C \subseteq \mathbb{Z}$  is a (proper) closed, convex cone in a real Banach space  $(\mathbb{Z}, \|\cdot\|)$ , with  $C \neq \mathbf{0}$ ,  $S \subseteq \mathbb{X}$  is a closed set expressing a geometric constraint free from uncertainty, and  $g : \Omega \times \mathbb{X} \rightarrow \mathbb{Z}$  is a given mapping. Here  $\Omega$  represents a given uncertainty set, which allows one to describe a decision environment characterized by a crude knowledge of the data. This means that the constraining mapping  $g$ , as a structural element of the problem, is affected by uncertainty, but this uncertainty can not be tackled by handling probability distributions as in stochastic optimization, because such an information is not at disposal. The only information about the data element  $\omega$  is that  $\omega \in \Omega$ . The paper often credited as a first reference in undertaking an aware and systematic study of optimization problems, whose data are affected by this form of uncertainty, is [4]<sup>1</sup>. There, reasons for such a crude knowledge of the data are widely discussed. In this circumstance, situations quite common in reality may require that the cone constraint  $g(\omega, x) \in C$  is satisfied, whatever the actual realization of  $\omega \in \Omega$  is. In other terms, the decision maker is forced to regard as feasible only those elements of  $S$  such that  $g(\omega, x) \in C$  for every  $\omega \in \Omega$ . Examples of such situations, emerging especially in engineering applications, are described in [4]. On this basis the authors developed an approach hedging the decision maker against the worst cases that may occur, called robust approach to uncertain optimization, in analogy with robust control. This ‘pessimistic’ (or ‘ultraconservative’, in the Soyster’s words) approach to uncertainty opened a flourishing line of research, in scalar as well as in vector optimization, known as robust optimization (see [4–6] and references therein).

In the case of vector optimization problems such as  $(\mathcal{P}_\omega)$ , where the objective function is not affected by uncertainty, this approach reduces to consider as a feasible region the set

$$\mathcal{R} = \{x \in S : g(\omega, x) \in C, \forall \omega \in \Omega\}.$$

Thus, by introducing the set-valued mapping  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$ , defined as being

$$G(x) = g(\Omega, x) = \{z = g(\omega, x) \in \mathbb{Z} : \omega \in \Omega\}, \quad (1)$$

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<sup>1</sup>To be more precise, in [4] the authors do indicate as a forerunner of their approach A.L. Soyster, who in [29] introduced a similar point of view, in dealing with uncertainly constrained problems in mathematical programming.

the robust counterpart of the feasible region of  $(\mathcal{P}_\omega)$  leads naturally to consider the so-called set-valued inclusion problem: given a (nonempty) closed set  $S \subseteq \mathbb{X}$ , a proper, closed and convex cone  $C \subseteq \mathbb{Z}$  and a set-valued mapping  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$

$$(\mathcal{SVI}) \quad \text{find } x \in S \text{ such that } G(x) \subseteq C.$$

In fact, recalling that the upper inverse image of  $C$  through the set-valued mapping  $G$  is the set  $G^{+1}(C) = \{x \in \mathbb{X} : G(x) \subseteq C\}$ , one has

$$\mathcal{R} = S \cap G^{+1}(C).$$

To the best of the author's knowledge, problem  $(\mathcal{SVI})$  began to be investigated independently of robust optimization in [7], which focuses on error bound estimates. Solvability and solution stability issues for  $(\mathcal{SVI})$  have been studied more recently in [30,31]. In the light of the role played by  $(\mathcal{SVI})$  in the robust approach to optimization problems with uncertain constraints, it seems to be natural to assess an impact evaluation of the recent achievements about the solution set to  $(\mathcal{SVI})$  and its approximations within the theory of optimality/efficiency conditions. Some initial results along this line of research have been obtained in the case of scalar optimization in [30,32]. So, the present analysis can be regarded as a development of ideas and techniques, presented especially in [32], towards the specific context of vector optimization, in considering problems of the form

$$(\mathcal{P}_G) \quad \text{Min}_K f(x) \text{ with } x \in S \text{ subject to } G(x) \subseteq C.$$

This analysis will be performed here by well-known techniques: in fact, some first-order efficiency conditions are obtained by means of the Clarke penalization principle, through its vector counterpart due to J.J. Ye (see [33]). Some other first-order efficiency conditions are achieved by exploiting tangential approximations of the solution set to  $(\mathcal{SVI})$ , following a modern revisitation of the celebrated Euler-Lagrange method. In both the cases, the main tools employed come from nonsmooth and variational analysis as well as from generalized convexity.

Optimality conditions for vector optimization problems with uncertain constraints are a subject intensively investigated in the last years, in particular through the robust approach (see, among others, [8,9,18] and references therein). A feature distinguishing the analysis here proposed is the great generality kept on  $\Omega$ , in the very spirit of robust optimization, owing to the introduction of the set-valued mapping  $G$ .

The presentation of the contents is organized according to the following arrangement. Section 2 collects some basic technical preliminaries of large employment in optimization and related fields. Some more specific constructions needed in the subsequent analysis will be recalled contextually to their use. In Section 3 first-order necessary conditions for the local weak efficiency of solutions to  $(\mathcal{P}_G)$  are established via a penalization method, with and without generalized convexity assumptions. In Section 4 different Lagrangian-type necessary conditions for local weak efficiency, leading to multiplier rules, are formulated in terms outer prederivatives of  $G$ , with or without smoothness assumption on  $f$ . Needless to say that, since any eff. solution to  $(\mathcal{P}_G)$  is also a w-eff. solution to  $(\mathcal{P}_G)$ , all the mentioned conditions are necessary for local efficiency too.

The notations in use throughout the paper are mainly standard. Quite often, capital letters in bold will denote real Banach spaces, with  $\|\cdot\|$  denoting their norm. The null

vector in a Banach space is denoted by  $\mathbf{0}$ . In a metric space setting, the closed ball centered at an element  $x$ , with radius  $r \geq 0$ , is indicated with  $B(x, r)$ . In particular, in a Banach space,  $\mathbb{B} = B(\mathbf{0}, 1)$ . Whenever  $A$  is a subset of a metric space,  $B(A, r)$  indicates the  $r$ -enlargement of  $A$ , whereas the distance of a point  $x$  from  $A$  is denoted by  $\text{dist}(x, A)$ . If  $W$  is a subset of the same metric space,  $\text{exc}(A, W) = \sup_{a \in A} \text{dist}(a, W)$  indicates the excess of  $A$  over  $W$ . Symbols  $\text{cl } A$  and  $\text{int } A$  denote the topological closure and the interior of  $A$ , respectively. If  $A$  is a subset of a Banach space, its convex hull is denoted by  $\text{conv } A$  and, when  $A$  is convex, its relative interior is denoted by  $\text{ri } A$ . All finite-dimensional Banach spaces will be supposed to be endowed with their Euclidean space structure.  $\mathbb{R}_+^n$  stands for the cone of all vectors with nonnegative components.

By  $\mathcal{L}(\mathbb{X}, \mathbb{Y})$  the Banach space of all bounded linear operators acting between  $\mathbb{X}$  and  $\mathbb{Y}$  is denoted, equipped with the operator norm  $\|\cdot\|_{\mathcal{L}}$ . In particular,  $\mathbb{X}^* = \mathcal{L}(\mathbb{X}, \mathbb{R})$  stands for the dual space of  $\mathbb{X}$ , in which case  $\|\cdot\|_{\mathcal{L}}$  is simply marked by  $\|\cdot\|$ . The null vector of a dual space will be marked by  $\mathbf{0}^*$ . The duality pairing a Banach space with its dual will be denoted by  $\langle \cdot, \cdot \rangle$ . Given a function  $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{\mp\infty\}$ , by  $[\varphi \leq 0] = \varphi^{-1}([-\infty, 0])$  its sublevel set is denoted, whereas  $[\varphi > 0] = \varphi^{-1}((0, +\infty])$  denotes the strict superlevel set of  $\varphi$ . The acronyms l.s.c., u.s.c. and p.h. stand for lower semicontinuous, upper semicontinuous and positively homogeneous, respectively. The symbol  $\text{dom } \varphi = \varphi^{-1}(\mathbb{R})$  indicates the domain of  $\varphi$ , whenever  $\varphi$  is a functional, whereas if  $F : \mathbb{X} \rightrightarrows \mathbb{Y}$  is a set-valued mapping,  $\text{dom } F = \{x \in \mathbb{X} : F(x) \neq \emptyset\}$ . Throughout the paper, the set of all locally w-eff. solutions of a problem  $(\mathcal{P}_G)$  will be denoted by  $\mathcal{WE}(\mathcal{P}_G)$ .

## 2. Basic tools of analysis

Let  $A \subseteq \mathbb{X}$  be a nonempty closed subset of a Banach space and let  $\bar{x} \in A$ . Nonsmooth analysis provides a large variety of concepts for the local, first-order conic approximation of  $A$  near  $\bar{x}$ . For the purposes of the present analysis, the following ones are to be mentioned:

$$\text{T}(A; \bar{x}) = \{v \in \mathbb{X} : \exists (v_n)_n \text{ with } v_n \rightarrow v, \exists (t_n)_n \text{ with } t_n \downarrow 0 : \bar{x} + t_n v_n \in A, \forall n \in \mathbb{N}\},$$

$$\text{I}(A; \bar{x}) = \{v \in \mathbb{X} : \exists \delta > 0 : \bar{x} + tv \in A, \forall t \in (0, \delta)\},$$

and

$$\text{I}_w(A; \bar{x}) = \{v \in \mathbb{X} : \forall \epsilon > 0, \exists t_\epsilon \in (0, \epsilon) : \bar{x} + t_\epsilon v \in A\},$$

called the contingent (or Bouligand tangent) cone, the feasible direction cone and the weak feasible direction cone to  $A$  at  $\bar{x}$ , respectively. They are known to be linked by the inclusion relation of general validity

$$\text{I}(A; \bar{x}) \subseteq \text{I}_w(A; \bar{x}) \subseteq \text{T}(A; \bar{x}),$$

where strict inclusion may hold (see [28]). Whenever  $A$  is locally convex around  $\bar{x}$ , i.e. there exists  $r > 0$  such that  $A \cap B(\bar{x}, r)$  is a convex set, the above inclusion relation

collapses to

$$\text{cl}I(A; \bar{x}) = \text{cl}I_w(A; \bar{x}) = T(A; \bar{x})$$

(see [28, Proposition 11.1.2(d)]). In such an event,  $T(A; \bar{x})$  is a closed convex cone, while  $I(A; \bar{x})$  is a convex cone.

Let  $Q \subseteq \mathbb{Y}$  be a cone. The sets

$$Q^\oplus = \{y^* \in \mathbb{Y}^* : \langle y^*, y \rangle \geq 0, \quad \forall y \in Q\} \quad \text{and} \quad Q^\ominus = -Q^\oplus$$

are called the positive and the negative dual cone of  $Q$ , respectively.

**Remark 1.** (i) Note that, whenever a set  $A$  is locally convex around  $\bar{x}$  (so  $T(A; \bar{x})$  is convex) the negative dual cone operator allows one to represent the normal cone to  $A$  in the sense of convex analysis at some element  $\bar{x} \in A$  in terms of contingent cone as follows

$$N(A; \bar{x}) = \{x^* \in \mathbb{X}^* : \langle x^*, x - \bar{x} \rangle \leq 0, \quad \forall x \in A\} = T(A; \bar{x})^\ominus.$$

(ii) The interaction of the negative dual cone operator with some set operations is described by the following formula: given  $\Lambda \in \mathcal{L}(\mathbb{X}, \mathbb{Y})$  and two closed convex cones  $Q \subseteq \mathbb{Y}$  and  $P \subseteq \mathbb{X}$ , it holds

$$(P \cap \Lambda^{-1}(Q))^\ominus = \text{cl}(P^\ominus + \Lambda^*(Q^\ominus)),$$

where  $\Lambda^* \in \mathcal{L}(\mathbb{Y}^*, \mathbb{X}^*)$  denotes the adjoint operator to  $\Lambda$  (see [28, Lemma 2.4.1]). From this formula one can derive, as a special case, the equality

$$[\Lambda^{-1}(Q)]^\ominus = \text{cl} \Lambda^*(Q^\ominus), \tag{2}$$

and, under the qualification condition  $\text{int} P_1 \cap \text{int} P_2 \neq \emptyset$ ,

$$(P_1 \cap P_2)^\ominus = P_1^\ominus + P_2^\ominus, \tag{3}$$

with  $P_1$  and  $P_2$  being closed convex cones in  $\mathbb{X}$  (see [1, Table 4.3 (5)b]). Note that in the equality (2) the closure operation can be omitted if  $\Lambda^{-1}(\text{ri} Q) \neq \emptyset$ . Such a condition is evidently satisfied if  $\Lambda(\mathbb{X}) \supseteq Q$  and  $Q \neq \{\mathbf{0}\}$  (see, for instance, [26, Corollary 16.3.2]).

Let  $K \subseteq \mathbb{Y}$  be a (proper) convex cone inducing a partial order  $\leq_K$  on  $\mathbb{Y}$  and let  $f : \mathbb{X} \rightarrow \mathbb{Y}$  be a mapping between Banach spaces. Then  $f$  is said to be  $K$ -convex on the convex set  $A \subseteq \mathbb{X}$  if the set

$$\text{epi}_K(f) = \{(x, y) \in \mathbb{X} \times \mathbb{Y} : x \in A, f(x) \leq_K y\}$$

is convex. If, in addition,  $A$  is a cone and  $f$  is also positively homogeneous, then  $f$  is said to be  $K$ -sublinear on  $A$ . It is well known that if  $f$  is  $K$ -convex on  $A$ , then  $f(A) + K$  is convex, while if  $A$  is a cone and  $f$  is  $K$ -sublinear, then  $f(A) + K$  is a convex cone.

Following [12, Definition 2.3], a mapping  $f$  is said to be  $K$ -convexlike on a set (not necessarily convex)  $A$  if the set  $f(A) + K$  is convex.

**Remark 2.** In Section 3 it will be used the fact, which is readily proved by handling the related definitions, that if  $\nu : \mathbb{X} \rightarrow \mathbb{R}$  is a sublinear function on  $\mathbb{X}$  and  $e \in K$ , then the mapping  $\nu e : \mathbb{X} \rightarrow \mathbb{Y}$ , defined by  $x \mapsto \nu(x)e$  is  $K$ -sublinear on  $\mathbb{X}$ .

Generalized convexity notions apply also to set-valued mappings. Following [7], a set-valued mapping  $F : \mathbb{X} \rightrightarrows \mathbb{Z}$  between Banach spaces is said to be  $C$ -concave on  $\mathbb{X}$ , where  $C \subseteq \mathbb{Z}$  is a (proper) convex cone, if

$$F(tx_1 + (1-t)x_2) \subseteq tF(x_1) + (1-t)F(x_2) + C, \quad \forall x_1, x_2 \in \mathbb{X}.$$

Some examples of  $C$ -concave set-valued mappings of interest in optimization can be found in [31]. For the purposes of the present analysis, the special class of  $C$ -concave set-valued mappings known as fans is to be mentioned. Recall that, after [14], a set-valued mapping  $H : \mathbb{X} \rightrightarrows \mathbb{Z}$  is called fan if it fulfils all the following conditions:

- (i) it is p.h.;
- (ii)  $\mathbf{0} \in H(\mathbf{0})$ ;
- (iii) it is convex-valued;
- (iv)  $H(x_1 + x_2) \subseteq H(x_1) + H(x_2), \quad \forall x_1, x_2 \in \mathbb{X}$ .

Fans may appear in a variety of forms. In Section 4, only fans which are generated by bundles of linear mappings will be actually employed, i.e. fans  $H_{\mathcal{G}} : \mathbb{X} \rightrightarrows \mathbb{Z}$  that can be represented as

$$H_{\mathcal{G}}(x) = \{\Lambda x : \Lambda \in \mathcal{G}\},$$

where  $\mathcal{G} \subseteq \mathcal{L}(\mathbb{X}, \mathbb{Z})$  is a (nonempty) convex and weakly closed set.

**Remark 3.** Whenever a fan  $H_{\mathcal{G}}$  is generated by a bounded set  $\mathcal{G}$ , it turns out to be a Lipschitz set-valued mapping, i.e. it holds

$$\text{haus}(H_{\mathcal{G}}(x_1), H_{\mathcal{G}}(x_2)) \leq l \|x_1 - x_2\|, \quad \forall x_1, x_2 \in \mathbb{X},$$

with  $l \geq \sup\{\|\Lambda\|_{\mathcal{L}} : \Lambda \in \mathcal{G}\}$ , where  $\text{haus}(A, W) = \max\{\text{exc}(A, W), \text{exc}(W, A)\}$  denotes the Hausdorff distance between two sets  $A$  and  $W$  (see [32, Remark 2.14(iii)]). This happens, in particular, when  $\mathcal{G}$  is the convex hull of finitely many elements of  $\mathcal{L}(\mathbb{X}, \mathbb{Z})$ , in which case  $H_{\mathcal{G}}$  is said to be finitely generated.

**Remark 4.** Since in all the efficiency conditions established in the paper the set-valued mapping  $G$ , defined as in (1), will be assumed to be l.s.c. around a reference point, it is proper to mention that, if each mapping  $g(\omega, \cdot) : \mathbb{X} \rightarrow \mathbb{Z}$  is continuous at a point  $x_0 \in \mathbb{X}$ , for every  $\omega \in \Omega$ , then  $G$  turns out to be l.s.c. at the same point. Indeed, let  $O \subseteq \mathbb{Z}$  be an arbitrary open set such that  $G(x_0) \cap O \neq \emptyset$ . Then, according to the definition of  $G$ , there exists  $\omega_O \in \Omega$  such that  $g(\omega_O, x_0) \in O$ . Since  $g(\omega_O, \cdot)$  is continuous at  $x_0$  and  $O$  is open, there exists  $r_O > 0$  such that  $g(\omega_O, x) \in O$  for every  $x \in B(x_0, r_O)$ . Consequently, one finds

$$g(\omega_O, x) \in G(x) \cap O \neq \emptyset, \quad \forall x \in B(x_0, r_O),$$

which shows that  $G$  is l.s.c. at  $x_0$ .

Nevertheless, the following example shows that the aforementioned condition is only sufficient, so the lower semicontinuity of  $G$  may take place in many other circumstances, sometimes rather pathological. Let  $\Omega = \mathbb{X} = \mathbb{Z} = \mathbb{R}$ , let  $\chi_{\mathbb{Q}}$  denote the char-

acteristic function associated to the set of all rational numbers and let  $g : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$g(\omega, x) = \omega[\chi_{\mathbb{Q}}(x) + \chi_{\mathbb{Q}}(\omega)].$$

It is clear that  $g(\omega, \cdot)$  is nowhere continuous, for every  $\omega \in \mathbb{R} \setminus \{0\}$ . In spite of this, with the above elements it results in

$$G(x) = g(\mathbb{R}, x) = \begin{cases} \mathbb{R}, & \text{if } x \in \mathbb{Q}, \\ \mathbb{Q}, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

which is l.s.c. everywhere in  $\mathbb{R}$ .

### 3. Weak efficiency conditions via penalization

**Definition 3.1** (*K-Lipschitz continuity*). Let  $f : \mathbb{X} \rightarrow \mathbb{Y}$  be a mapping between normed spaces and let  $K \subseteq \mathbb{Y}$  be a convex cone, with  $\text{int } K \neq \emptyset$ .  $f$  is said to be *K-Lipschitz* on the set  $D \subseteq \mathbb{X}$  if there exist a constant  $\ell_f > 0$  and a vector  $e \in \text{int } K \cap \mathbb{B}$  such that

$$f(x_1) \in f(x_2) - \ell_f \|x_1 - x_2\|e + K, \quad \forall x_1, x_2 \in D.$$

If  $\bar{x} \in \mathbb{X}$  and  $f$  is *K-Lipschitz* on a set  $D = \text{B}(\bar{x}, \delta)$  for some  $\delta > 0$ , then  $f$  is said to be *K-Lipschitz near  $\bar{x}$* .

The above notion has been used in [33] as a key concept to extend the Clarke penalization principle from the scalar case to vector optimization problems. This is done here directly through a local error bound function, whose definition is recalled below.

**Definition 3.2** (*Local error bound function*). Let  $\bar{x} \in \mathcal{R} \subseteq S \subseteq \mathbb{X}$ . A function  $\psi : \mathbb{X} \rightarrow [0, +\infty]$  is said to be a *local error bound function* for  $\mathcal{R}$  near  $\bar{x}$  if there exists  $\delta > 0$  such that both the following conditions are satisfied:

- (i)  $\text{dist}(x, \mathcal{R}) \leq \psi(x), \quad \forall x \in \text{B}(\bar{x}, \delta) \cap S;$
- (ii)  $\text{dist}(x, \mathcal{R}) = \psi(x), \quad \forall x \in \mathcal{R}.$

**Proposition 3.3.** ([33, Theorem 4.2(i)]) *With reference to a problem  $(\mathcal{P})$ , let  $\bar{x} \in \mathcal{R}$  and suppose that:*

- (i)  *$f$  is  $K$ -Lipschitz near  $\bar{x}$ , with constant  $\ell_f$  and vector  $e \in \text{int } K$ ;*
- (ii)  *$\psi : \mathbb{X} \rightarrow [0, +\infty]$  is an error bound function near  $\bar{x}$ .*

*Then, for any  $\ell \geq \ell_f$ , every local  $w$ -eff. solution to  $(\mathcal{P})$  is also a local  $w$ -eff. solution of the problem*

$$(\mathcal{P}_\ell) \quad \text{Min}_K[f(x) + \ell\psi(x)e].$$

*Furthermore, if  $\mathcal{R}$  is closed, for any  $\ell > \ell_f$ , every local eff. solution to  $(\mathcal{P})$  is also a local eff. solution of the problem  $(\mathcal{P}_\ell)$ .*

Proposition 3.3 enables one to free the original problem from its constraints. Notice indeed that problem  $(\mathcal{P}_\ell)$  is unconstrained. For problems such as  $(\mathcal{P}_G)$ , where the

feasible region is structured as a solution set to  $(\mathcal{SVI})$ , one has to adequate the local error bound function to the the constraint definition. In the present analysis, the following merit function  $\nu_{G,C} : \mathbb{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  for problems  $(\mathcal{SVI})$  is exploited to treat the data uncertainty in the constraints:

$$\nu_{G,C}(x) = \sup_{z \in G(x)} \text{dist}(z, C) = \text{exc}(G(x), C).$$

Henceforth, as a standing assumption it is assumed that  $\text{dom } G = \mathbb{X}$ . As a consequence, one has  $\nu_{G,C} : \mathbb{X} \rightarrow [0, +\infty]$  and therefore the following characterization of the feasible region of  $(\mathcal{P}_G)$  holds true:

$$\mathcal{R} = S \cap [\nu_{G,C} \leq 0].$$

The next lemma singles out a constraint qualification, under which the merit function  $\nu_{G,C}$  is shown to actually work as a local error bound function. In order to formulate it, let us recall that, after [11], given a function  $\varphi : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$  defined on a metric space  $(X, d)$  and  $\bar{x} \in \varphi^{-1}(\mathbb{R})$ , the strong slope of  $\varphi$  at  $\bar{x}$  is defined as the quantity

$$|\nabla\varphi|(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} \text{ is a local minimizer of } \varphi, \\ \limsup_{x \rightarrow \bar{x}} \frac{\varphi(\bar{x}) - \varphi(x)}{d(x, \bar{x})}, & \text{otherwise.} \end{cases}$$

In view of the formulation of the next lemma, it is useful to observe that, if as a metric space  $X$  one takes a closed subset  $S \subseteq \mathbb{X}$  containing  $\bar{x}$  and as a distance  $d$  one takes the distance induced by  $\|\cdot\|$ , the above definition becomes

$$|\nabla_S\varphi|(\bar{x}) = \begin{cases} 0, & \text{if } \bar{x} \text{ is a local minimizer} \\ & \text{of } \varphi \text{ over } S, \\ \inf_{r>0} \sup_{x \in B(\bar{x}, r) \cap S \setminus \{\bar{x}\}} \frac{\varphi(\bar{x}) - \varphi(x)}{\|x - \bar{x}\|}, & \text{otherwise.} \end{cases}$$

**Lemma 3.4.** *Let  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$ ,  $S$  and  $C$  as in problem  $(\mathcal{SVI})$ , and let  $\bar{x} \in \mathcal{R}$ . Suppose that:*

- (i)  $G$  is l.s.c. in a neighbourhood of  $\bar{x}$ ;
- (ii) there exist positive  $\sigma$  and  $r$  such that

$$(\mathcal{CQ}) \quad |\nabla_S\nu_{G,C}|(x) \geq \sigma, \quad \forall x \in B(\bar{x}, r) \cap S \cap [\nu_{G,C} > 0].$$

Then function  $\psi = \sigma^{-1}\nu_{G,C}$  is a local error bound function for  $\mathcal{R}$ .

**Proof.** From [30, Lemma 2.3(i)] it is known that the lower semicontinuity of  $G$  (in the sense of set-valued mappings) implies the lower semicontinuity for the functional  $\nu_{G,C}$ . Thus, by hypothesis (i), for some  $\delta_0 > 0$  it is true that  $\nu_{G,C}$  is l.s.c. on  $B(\bar{x}, \delta_0) \cap S$ . Notice that, as a closed subset of a Banach space,  $B(\bar{x}, \delta_0) \cap S$  is a complete metric space, if equipped with the induced metric. Besides, without any loss of generality, it is possible to assume that, if  $r > 0$  is as in hypothesis (ii), it is  $r < \delta_0$ . Notice that the case  $B(\bar{x}, r) \cap S \cap [\nu_{G,C} > 0] = \emptyset$  means  $B(\bar{x}, r) \cap S \subseteq G^{+1}(C) \cap S$ , so it holds  $\text{dist}(x, \mathcal{R}) = 0 \leq \psi(x)$  for every  $x \in B(\bar{x}, r) \cap S$  and any  $\psi : \mathbb{X} \rightarrow [0, +\infty]$ . Otherwise,



it is possible to apply [2, Corollary 3.1] with  $X = B(\bar{x}, r) \cap S$ , according to which

$$\text{dist}(x, \mathcal{R}) = \text{dist}(x, S \cap [\nu_{G,C} \leq 0]) \leq \frac{\nu_{G,C}(x)}{\sigma}, \quad \forall x \in B(\bar{x}, r/2) \cap S.$$

Thus, setting  $\delta = r/2$  and  $\psi(x) = \sigma^{-1}\nu_{G,C}$ , the condition (i) in Definition 3.2 is fulfilled. Since under the above assumptions  $\mathcal{R}$  is closed, one has

$$\text{dist}(x, \mathcal{R}) = 0 = \psi(x), \quad \forall x \in \mathcal{R},$$

so also the condition (ii) in Definition 3.2 is readily satisfied. This completes the proof.  $\square$

With the specialization of  $\psi$  above introduced, upon the constraint qualification  $(\mathcal{CQ})$ , the penalization principle for vector optimization takes the following form.

**Proposition 3.5.** *With reference to a problem  $(\mathcal{P}_G)$ , let  $\bar{x} \in \mathcal{R} = S \cap G^{+1}(C)$ . Suppose that:*

- (i)  *$f$  is  $K$ -Lipschitz near  $\bar{x}$ , with constant  $\ell_f$  and  $e \in \text{int } K$ ;*
- (ii)  *$G$  is l.s.c. in a neighbourhood of  $\bar{x}$  and condition  $(\mathcal{CQ})$  is satisfied.*

*Then, for any  $\ell \geq \ell_f$ , every locally w-eff. solution to  $(\mathcal{P}_G)$  is also a locally w-eff. solution to problem*

$$(\mathcal{P}_{G,\ell}) \quad \text{Min}_K[f(x) + \ell\sigma^{-1}\nu_{G,C}(x)e] \quad \text{subject to } x \in S.$$

*For any  $\ell > \ell_f$ , every locally eff. solution to  $(\mathcal{P}_G)$  is a locally eff. solution to  $(\mathcal{P}_{G,\ell})$ .*

**Proof.** Since  $f$  is  $K$ -Lipschitz near  $\bar{x}$ ,  $\text{int } K$  is open and, under the above assumptions, according to Lemma 3.4,  $\sigma^{-1}\nu_{G,C}$  is a local error bound function for  $\mathcal{R}$ , then the first assertion in Proposition 3.3 can be invoked. This yields that  $\bar{x}$  is a local w-eff. solution to problem  $(\mathcal{P}_{G,\ell})$ , for any  $\ell \geq \ell_f$ .

Since  $\mathcal{R}$  is closed, in the case  $\bar{x} \in \mathcal{R}$  is a local eff. solution to  $(\mathcal{P}_G)$ , it suffices to apply the second assertion in Proposition 3.3, in order to conclude that  $\bar{x}$  is a local eff. solution to problem  $(\mathcal{P}_{G,\ell})$ , for any  $\ell > \ell_f$ .  $\square$

The constraint qualification  $(\mathcal{CQ})$  is expressed in terms of  $\nu_{G,C}$ . This function can be built by means of the problem data. Nevertheless, it would be useful to formulate conditions ensuring the validity of  $(\mathcal{CQ})$  directly on  $G$ . This can be done by exploiting the metric increase property, as introduced in [30].

**Definition 3.6** (Metrically  $C$ -increasing mapping). Let  $S \subseteq \mathbb{X}$  be a (nonempty) closed set and let  $C \subseteq \mathbb{Z}$  be a closed, convex cone, with  $C \neq \{\mathbf{0}\}$ . A set-valued mapping  $F : \mathbb{X} \rightrightarrows \mathbb{Z}$  between Banach spaces is said to be *metrically  $C$ -increasing* around  $\bar{x} \in \text{dom } F$ , relative to  $S$ , if there exist  $\delta > 0$  and  $\alpha > 1$  such that

$$\forall x \in B(\bar{x}, \delta) \cap S, \forall r \in (0, \delta) \quad \exists z \in B(x, r) \cap S : B(F(z), \alpha r) \subseteq B(F(x) + C, r). \quad (4)$$

The quantity

$$\text{inc}_C(F; S; \bar{x}) = \sup\{\alpha > 1 : \exists \delta > 0 \text{ for which (4) holds}\}$$

is called exact *exact bound of metric  $C$ -increase* of  $F$  around  $\bar{x}$ , relative to  $S$ .

Several examples of metrically increasing mappings, along with an infinitesimal criterion for detecting the occurrence of this property, are provided in [30]. The next proposition enlightens the role of the metric increase property as a constraint qualification condition.

**Proposition 3.7.** *Let  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$ ,  $S$  and  $C$  as in problem (SVT), and let  $\bar{x} \in \mathcal{R}$ . Suppose that:*

- (i)  $G$  is l.s.c. in a neighbourhood of  $\bar{x}$ ;
- (ii)  $G$  is metrically  $C$ -increasing around  $\bar{x}$ , relative to  $S$ .

*Then condition (CQ) holds true with  $\sigma = \alpha - 1$  and  $r = \delta$ , for any  $\alpha \in (1, \text{inc}_C(G; S; \bar{x}))$  and  $\delta$  as in (4).*

**Proof.** As already seen, by hypothesis (i) the function  $\nu_{G,C}$  is l.s.c. in  $B(\bar{x}, \delta_0)$ , for some  $\delta_0 > 0$ . According to hypothesis (ii), fixed  $\alpha \in (1, \text{inc}_C(G; S; \bar{x}))$ , there exists  $\delta_\alpha > 0$  such that (4) holds. Observe that the nature of the metric  $C$ -increase property around  $\bar{x}$  allows one to assume without loss of generality that  $\delta_\alpha < \delta_0$ .

Now, let us take an arbitrary  $x \in B(\bar{x}, \delta_\alpha) \cap S \cap [\nu_{G,C} > 0]$ . Since, under the current assumptions,  $\nu_{G,C}$  is l.s.c. at  $x \in B(\bar{x}, \delta_\alpha)$ , there exists  $\delta_x > 0$  such that  $\nu_{G,C}(z) > 0$  for every  $z \in B(x, \delta_x)$ . Take any  $r > 0$  such that  $r < \min\{\delta_\alpha, \delta_x\}$ . According to (4), there exists  $z_r \in B(x, r) \cap S$  such that

$$B(G(z_r), \alpha r) \subseteq B(G(x) + C, r). \quad (5)$$

Notice that it must be  $z_r \neq x$ . Indeed, since it is  $\nu_{G,C}(x) > 0$  (namely, it is  $\text{exc}(G(x), C) > 0$ ), if it were  $z_r = x$ , then by inclusion (5) and [30, Lemma 2.2], one would find

$$\begin{aligned} \nu_{G,C}(x) + \alpha r &= \text{exc}(B(G(x), \alpha r), C) = \text{exc}(B(G(z_r), \alpha r), C) \\ &\leq \text{exc}(B(G(x) + C, r), C) = \nu_{G,C}(x) + r, \end{aligned}$$

wherefrom  $\alpha \leq 1$ , in contrast with the fact that  $\alpha > 1$ . Furthermore, by applying once again inclusion (5) and taking into account that  $\nu_{G,C}(z_r) > 0$ , so [30, Lemma 2.2] still works, one obtains

$$\begin{aligned} \nu_{G,C}(z_r) &= \text{exc}(B(G(z_r), \alpha r), C) - \alpha r \leq \text{exc}(B(G(x) + C, r), C) - \alpha r \\ &= \text{exc}(G(x) + C, C) + r - \alpha r = \nu_{G,C}(x) + (1 - \alpha)r. \end{aligned}$$

As it is  $z_r \in B(x, r) \cap S$ , from the last inequality chain it follows

$$\nu_{G,C}(x) - \nu_{G,C}(z_r) \geq (\alpha - 1)r \geq (\alpha - 1)\|z_r - x\|.$$

This inequality says that  $x$  fails to be a local minimizer for  $\nu_{G,C}$  and therefore allows one to get the following estimate

$$|\nabla_S \nu_{G,C}|(x) = \limsup_{z \xrightarrow{S} x} \frac{\nu_{G,C}(x) - \nu_{G,C}(z)}{\|x - z\|} \geq \lim_{r \downarrow 0} \frac{\nu_{G,C}(x) - \nu_{G,C}(z_r)}{\|x - z_r\|} \geq \alpha - 1.$$

By arbitrariness of  $x \in B(\bar{x}, \delta_\alpha) \cap S \cap [\nu_{G,C} > 0]$ , the last inequalities show that condition (CQ) is satisfied with  $\sigma = \alpha - 1$  and  $r = \delta_\alpha$ , thereby completing the proof.  $\square$

On the base of the constraint system analysis exposed above, one is now in a position to formulate necessary weak efficiency condition for problems  $(\mathcal{P}_G)$ . To this aim, it remains to recall some further element of nonsmooth analysis.

Let  $\varphi : \mathbb{X} \rightarrow \mathbb{R} \cup \{\pm\infty\}$  be a function which is finite around  $\bar{x} \in \varphi^{-1}(\mathbb{R})$ . Following [22, Section 1.3.2], the set

$$\widehat{\partial}^+ \varphi(\bar{x}) = \left\{ x^* \in \mathbb{X}^* : \limsup_{x \rightarrow \bar{x}} \frac{\varphi(x) - \varphi(\bar{x}) - \langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\}$$

is called the Fréchet upper subdifferential of  $\varphi$  at  $\bar{x}$ . It is readily seen that, whenever  $\varphi$  is (Fréchet) differentiable at  $\bar{x}$ , then  $\widehat{\partial}^+ \varphi(\bar{x}) = \{D\varphi(\bar{x})\}$ , whereas in the case  $\varphi : \mathbb{X} \rightarrow \mathbb{R}$  is concave, the set  $\widehat{\partial}^+ \varphi(\bar{x})$  reduces to the superdifferential of  $\varphi$  at  $\bar{x}$ , in the sense of convex analysis.

**Remark 5.** The following variational description of the Fréchet upper subdifferential of  $\varphi$  at  $\bar{x}$  will be exploited in the sequel: for every  $x^* \in \widehat{\partial}^+ \varphi(\bar{x})$  there exists a function  $\varsigma : \mathbb{X} \rightarrow \mathbb{R}$ , Fréchet differentiable at  $\bar{x}$  and with  $\varphi(\bar{x}) = \varsigma(\bar{x})$ , such that  $\varphi(x) \leq \varsigma(x)$  for every  $x \in \mathbb{X}$  and  $D\varsigma(\bar{x}) = x^*$  (to get it, it suffices to remember that  $\widehat{\partial}^+ \varphi(\bar{x}) = -\widehat{\partial}(-\varphi)(\bar{x})$ , where  $\widehat{\partial}$  denotes the Fréchet subdifferential, and then apply [22, Theorem 1.88(i)]).

Given a mapping  $f : \mathbb{X} \rightarrow \mathbb{Y}$  between Banach spaces and  $\bar{x} \in \mathbb{X}$   $f'(\bar{x}; v)$  indicates the directional derivative of  $f$  at  $\bar{x}$ , in the direction  $v \in \mathbb{X}$ . If its directional derivative exists for every  $v \in \mathbb{X}$ ,  $f$  is said to be directionally differentiable at  $\bar{x}$ .

A first-order necessary condition for weak efficiency of solutions to  $(\mathcal{P}_G)$  can be stated as follows.

**Theorem 3.8** (Weak efficiency condition via penalization). *With reference to a problem  $(\mathcal{P}_G)$ , let  $\bar{x} \in \mathcal{R} = S \cap G^{+1}(C)$  be a locally w-eff. solution to  $(\mathcal{P}_G)$ . Suppose that:*

(i)  *$f$  is  $K$ -Lipschitz near  $\bar{x}$ , with constant  $\ell_f$  and  $e \in \text{int } K$ , and is directionally differentiable at  $\bar{x}$ ;*

(ii)  *$G$  is l.s.c. in a neighbourhood of  $\bar{x}$  and metrically  $C$ -increasing around  $\bar{x}$ , relative to  $S$ ;*

(iii)  *$\widehat{\partial}^+ \nu_{G,C}(\bar{x}) \neq \emptyset$ .*

*Then for any  $\alpha \in (1, \text{inc}_C(G; S; \bar{x}))$ ,  $\ell \geq \ell_f$  and  $x^* \in \widehat{\partial}^+ \nu_{G,C}(\bar{x})$  it must be*

$$f'(\bar{x}; v) + \frac{\ell}{\alpha - 1} \langle x^*, v \rangle e \notin -\text{int } K, \quad \forall v \in \mathbf{I}(S; \bar{x}). \quad (6)$$

**Proof.** Under the assumptions made, in the light of Proposition 3.7 the condition  $(\mathcal{CQ})$  is satisfied. Thus, it is possible to invoke Proposition 3.5, according to which  $\bar{x}$  turns out to be a w-eff. solution of problem  $(\mathcal{P}_{G,\ell})$ , for any  $\alpha \in (1, \text{inc}_C(G; S; \bar{x}))$  and  $\ell \geq \ell_f$ . This means that there exists  $\delta > 0$  such that

$$\left( f + \frac{\ell}{\alpha - 1} \nu_{G,C} e \right) (\mathbf{B}(\bar{x}, \delta) \cap S) \cap [f(\bar{x}) - \text{int } K] = \emptyset. \quad (7)$$

Take an arbitrary  $v \in \mathbf{I}(S; \bar{x}) \cap \mathbb{B}$ . By reducing the value of  $\delta > 0$  in (7) if needed, one can assume that  $\bar{x} + tv \in S$ , for all  $t \in (0, \delta)$ . Therefore, from the relation in (7) it

follows

$$\frac{f(\bar{x} + tv) - f(\bar{x})}{t} + \frac{\ell\nu_{G,C}(\bar{x} + tv)e}{(\alpha - 1)t} \in \mathbb{Y} \setminus (-\text{int } K), \quad \forall t \in (0, \delta). \quad (8)$$

Let  $x^*$  be an arbitrary element of  $\widehat{\partial}^+ \nu_{G,C}(\bar{x})$ . According to the characterization of upper Fréchet subgradients mentioned in Remark 5, there exists a Fréchet differentiable function  $\varsigma : \mathbb{X} \rightarrow \mathbb{R}$  such that  $\varsigma(\bar{x}) = \nu_{G,C}(\bar{x}) = 0$ ,  $\varsigma(x) \geq \nu_{G,C}(x)$  for every  $x \in \mathbb{X}$ , and  $D\varsigma(\bar{x}) = x^*$ . Therefore, one has

$$\varsigma(\bar{x} + tv) - \nu_{G,C}(\bar{x} + tv) \geq 0, \quad \forall t \in (0, +\infty),$$

whence

$$\frac{\ell[\varsigma(\bar{x} + tv) - \nu_{G,C}(\bar{x} + tv)]e}{(\alpha - 1)t} \in K, \quad \forall t \in (0, +\infty). \quad (9)$$

By combining (8) and (9) and observing that, for every  $y \in \mathbb{Y} \setminus (-\text{int } K)$  it holds  $y + K \subseteq \mathbb{Y} \setminus (-\text{int } K)$ , one obtains

$$\frac{f(\bar{x} + tv) - f(\bar{x})}{t} + \frac{\ell\varsigma(\bar{x} + tv)e}{(\alpha - 1)t} \in \mathbb{Y} \setminus (-\text{int } K), \quad \forall t \in (0, \delta).$$

By passing to the limit as  $t \downarrow 0$  in the last inclusion, while taking into account that the cone  $\mathbb{Y} \setminus (-\text{int } K)$  is closed and that  $f$  is directionally differentiable at  $\bar{x}$ , one achieves the relation in (6) for any  $v \in I(S; \bar{x}) \cap \mathbb{B}$ . Since the mapping  $v \mapsto f'(\bar{x}; v) + \frac{\ell}{\alpha - 1} \langle x^*, v \rangle e$  is positively homogeneous and  $\mathbb{Y} \setminus (-\text{int } K)$  is a cone, the validity of relation in (6) can be extended to the whole set  $I(S; \bar{x})$ . By arbitrariness of  $x^*$ , this reasoning completes the proof.  $\square$

Among the hypotheses of Theorem 3.8, the most involved is (iii), so it deserves some comment. In the next remark, some elements for discussion are provided in order to clarify the meaning of such an assumption.

**Remark 6.** According to its definition, the merit function  $\nu_{G,C}$  is nonnegative and, since it is  $\bar{x} \in G^{+1}(C)$  one has  $\nu_{G,C}(\bar{x}) = 0$ , so the hypothesis (iii) in Theorem 3.8 is about the nontriviality of the Fréchet upper subdifferential at a (global) minimizer. A systematic study of this tool of nonsmooth analysis (actually, not so often considered as its lower counterpart) and related optimality conditions for constrained minimization problems can be found in [21,23]. In particular, it was shown that, for given a function  $\varphi : \mathbb{X} \rightarrow \{\pm\infty\}$ , which is defined on an Asplund space and is locally Lipschitz around  $\bar{x}$ , the nonemptiness of  $\widehat{\partial}^+ \varphi(\bar{x})$  is automatic if  $\varphi$  is upper regular at  $\bar{x}$ , i.e.  $\widehat{\partial}^+ \varphi(\bar{x}) = \partial^+ \varphi(\bar{x})$ , where  $\partial^+ \varphi(\bar{x})$  denotes the limiting upper subdifferential of  $\varphi$  at  $\bar{x}$ , defined through the basic normals to the hypergraph of  $\varphi$  (see [22, Definition 1.78]). In such a circumstance, it holds

$$\partial_{\text{Cl}} \varphi(\bar{x}) = \text{cl}^* \widehat{\partial}^+ \varphi(\bar{x}),$$

where  $\partial_{\text{Cl}} \varphi(\bar{x})$  denotes the Clarke generalized gradient of  $\varphi$  at  $\bar{x}$  and  $\text{cl}^* A$  marks the closure of a set  $A$  with respect to the weak\* topology (see, for more details, [23,

Remark 4.5] and [23, Section 5.5.4]). Note that, as it is possible to check at once,  $\nu_{G,C}$  is locally Lipschitz around  $\bar{x}$  whenever  $G$  is Lipschitz continuous around  $\bar{x}$ .

The following example shows that the metric  $C$ -increase assumption on  $G$  plays an essential role in qualifying such constraints as set-valued inclusions.

**Example 3.9.** Let  $\mathbb{X} = \mathbb{Z} = \mathbb{R}$ ,  $\mathbb{Y} = \mathbb{R}^2$ ,  $K = \mathbb{R}_+^2$ ,  $C = (-\infty, 0]$  and  $S = \mathbb{R}$ . Let  $G : \mathbb{R} \rightrightarrows \mathbb{R}$  be defined by

$$G(x) = \{z \in \mathbb{R} : z \leq x^2\}.$$

By exploiting the continuity of the function  $x \mapsto x^2$  on  $\mathbb{R}$ , it is possible to prove that  $G$  is l.s.c. on  $\mathbb{R}$ . Let us show that  $G$  fails to be metrically  $(-\infty, 0]$ -increasing around  $\bar{x} = 0$ , relative to  $\mathbb{R}$ . According to Definition 3.6, such a property would require the existence of  $\delta > 0$  and  $\alpha > 1$  such that, taking  $x = 0$ , for  $r \in (0, \delta)$  there is  $u \in [-r, r]$  such that

$$(-\infty, u^2 + \alpha r] \subseteq (-\infty, r],$$

or equivalently

$$u^2 + \alpha r \leq r.$$

The last inequality can never be satisfied if  $\alpha > 1$  and  $r > 0$ , so the required  $u \in [-r, r]$  does not exist. In the present case, it is immediate to see that

$$\nu_{G,(-\infty,0]}(x) = \sup\{d(z, (-\infty, 0]) : z \leq x^2\} = x^2,$$

so  $\nu_{G,(-\infty,0]}$  turns out to be (Fréchet) differentiable at 0. Consequently, one has  $\widehat{\partial}^+ \nu_{G,(-\infty,0]}(0) = \{0\}$ .

With the above data, one finds

$$\mathcal{R} = G^{+1}((-\infty, 0]) \cap \mathbb{R}^2 = \{0\},$$

and hence, independently of the choice of the criterion  $f : \mathbb{R} \rightarrow \mathbb{R}^2$ , it results in  $\mathcal{WE}(\mathcal{P}_G) = \{0\}$ . Now, if taking  $f$  defined by  $f(x) = (x, x)$ , which is  $\mathbb{R}_+^2$ -Lipschitz on  $\mathbb{R}$ , with  $e = (\sqrt{2}/2)(1, 1)$ , and directionally differentiable at 0, with  $f'(0; v) = (v, v)$ , for any  $\alpha > 1$  and  $\ell \geq 0$  one obtains

$$f'(0; v) + \frac{\ell}{\alpha - 1} \langle x^*, v \rangle e = (v, v) + \frac{\ell}{\alpha - 1} \langle 0, v \rangle e = (v, v).$$

Thus the condition (6) can not be verified for every  $v \in I(\mathbb{R}; 0) = \mathbb{R}$ .

By introducing proper convexity/concavity assumptions on the problem data  $S$ ,  $f$  and  $G$ , it is possible to establish a first-order necessary weak efficiency condition in a scalarized form. To this aim, the next remark will be useful.

**Remark 7.** (i) It is readily seen that, whenever  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$  is  $C$ -bounded around a point  $\bar{x} \in \mathbb{X}$ , i.e. there exists  $\delta > 0$  such that  $G(x) \setminus C$  is bounded for every  $x \in B(\bar{x}, \delta)$ , then  $\bar{x} \in \text{int}(\text{dom } \nu_{G,C})$ .

(ii) whenever  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$  is  $C$ -concave on  $\mathbb{X}$  the function  $\nu_{G,C}$  is convex on  $\mathbb{X}$  (see, for instance, [31, Remark 4.14]).

**Theorem 3.10** (Weak efficiency condition via penalization under convexity). *With reference to a problem  $(\mathcal{P}_G)$ , let  $\bar{x} \in \mathcal{R} = S \cap G^{+1}(C)$  be a locally  $w$ -eff. solution to  $(\mathcal{P}_G)$ . Suppose that:*

(i)  $S$  is locally convex around  $\bar{x}$ ;

(ii)  $f$  is  $K$ -Lipschitz near  $\bar{x}$ , with constant  $\ell_f$  and  $e \in \text{int } K$ , and is directionally differentiable at  $\bar{x}$ , with  $f'(\bar{x}; \cdot) : \mathbb{X} \rightarrow \mathbb{Y}$  being  $K$ -sublinear;

(iii)  $G$  is l.s.c. in a neighbourhood of  $\bar{x}$  and metrically  $C$ -increasing around  $\bar{x}$ , relative to  $S$ ;

(iv)  $G$  is  $C$ -bounded around  $\bar{x}$  and Hausdorff u.s.c. at  $\bar{x}$ ;

(v)  $G$  is  $C$ -concave in  $\mathbb{X}$ .

Then, for any  $\alpha \in (1, \text{inc}_C(G; S; \bar{x}))$ ,  $\ell \geq \ell_f$  there exists  $y^* \in K^\ominus \setminus \{\mathbf{0}^*\}$  such that

$$\langle y^*, f'(\bar{x}; v) \rangle + \frac{\ell}{\alpha - 1} \langle y^*, \nu'_{G,C}(\bar{x}; v)e \rangle \geq 0, \quad \forall v \in I(S; \bar{x}). \quad (10)$$

**Proof.** Let us start with observing that, by virtue of the hypotheses of  $C$ -concavity,  $\nu_{G,C}$  is convex on  $\mathbb{X}$ . Moreover, by hypothesis (iv), it is  $\bar{x} \in \text{int}(\text{dom } \nu_{G,C})$ . Moreover, since  $G$  is Hausdorff u.s.c. at  $\bar{x}$ , function  $\nu_{G,C}$  is also u.s.c. at  $\bar{x}$  (see [30, Lemma 2.3(ii)]). So, remembering that it is also l.s.c. around  $\bar{x}$  on the account of hypothesis (iii),  $\nu_{G,C}$  turns out to be continuous and hence directionally differentiable at  $\bar{x}$  (remember [34, Theorem 2.4.9]). From Remark 2 it follows that the mapping  $v \mapsto \nu_{G,C}(v)e$  is  $K$ -sublinear on  $\mathbb{X}$ . As a sum of two  $K$ -sublinear mappings,  $f'(\bar{x}; \cdot) + \frac{\ell}{\alpha - 1} \nu'_{G,C}(\bar{x}; \cdot)e$  is  $K$ -sublinear on  $\mathbb{X}$ . On the other hand, since according to hypothesis (i)  $S$  is locally convex near  $\bar{x}$ , the cone  $I(S; \bar{x})$  is convex. It follows that the subset of  $\mathbb{Y}$ , given by

$$f'(\bar{x}; I(S; \bar{x})) + \frac{\ell}{\alpha - 1} \nu'_{G,C}(\bar{x}; I(S; \bar{x}))e + K,$$

is a convex cone as an image of a convex cone through a  $K$ -sublinear mapping.

Since  $\bar{x}$  is a local  $w$ -eff. solution of  $(\mathcal{P}_G)$ , by arguing as in the proof of Theorem 3.8, it is possible to show that

$$\left[ f'(\bar{x}; I(S; \bar{x})) + \frac{\ell}{\alpha - 1} \nu'_{G,C}(\bar{x}; I(S; \bar{x}))e \right] \cap (-\text{int } K) = \emptyset.$$

This entails that it holds also

$$\left[ f'(\bar{x}; I(S; \bar{x})) + \frac{\ell}{\alpha - 1} \nu'_{G,C}(\bar{x}; I(S; \bar{x}))e + K \right] \cap (-\text{int } K) = \emptyset.$$

In such a circumstance one can invoke the Eidelheit theorem (see, for instance, [34, Theorem 1.1.3]). It ensures the existence of  $y^* \in \mathbb{Y}^* \setminus \{\mathbf{0}^*\}$  and  $\gamma \in \mathbb{R}$ , such that

$$\langle y^*, f'(\bar{x}; v) \rangle + \frac{\ell}{\alpha - 1} \langle y^*, \nu'_{G,C}(\bar{x}; v)e \rangle \geq \gamma \geq \langle y^*, y \rangle, \quad (11)$$

$$\forall v \in I(S; \bar{x}), \quad \forall y \in \text{cl}(-\text{int } K) = -K.$$

Since, in particular, it holds

$$\langle y^*, f'(\bar{x}; \mathbf{0}) + \frac{\ell}{\alpha - 1} \nu'_{G,C}(\bar{x}; \mathbf{0})e \rangle = 0 \geq \gamma \geq 0 = \langle y^*, \mathbf{0} \rangle,$$

it follows that  $\gamma$  must be 0 and, by consequence, the second inequality in (11) gives  $y^* \in K^\ominus$ . This completes the proof.  $\square$

**Example 3.11.** Let  $\mathbb{X} = \mathbb{Y} = \mathbb{R}^2$ ,  $\mathbb{Z} = \mathbb{R}$ ,  $K = \mathbb{R}_+^2$  and  $C = (-\infty, 0]$ . Let  $G : \mathbb{R}^2 \rightrightarrows \mathbb{R}$  be defined by

$$G(x) = \{z \in \mathbb{R} : z \leq \varphi(x)\},$$

where  $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$  is given by  $\varphi(x) = \max\{x_1, x_2\}$ . Since  $\varphi$  is sublinear on  $\mathbb{R}^2$ ,  $G$  is a fan (see, for instance, [14, Section 2. Examples]) and, as such, it is a concave set-valued mapping (in particular,  $(-\infty, 0]$ -concave). Since  $\varphi$  is continuous on  $\mathbb{R}^2$ , it is readily seen that  $G$  is l.s.c. and Hausdorff u.s.c. on  $\mathbb{R}^2$ . Moreover, the set

$$G(x) \setminus (-\infty, 0] = \{z \in \mathbb{R} : 0 < z \leq \max\{x_1, x_2\}\} \subseteq [0, \|x\|]$$

is evidently bounded (possibly empty) for every  $x \in \mathbb{R}^2$ , so  $G$  is  $(-\infty, 0]$ -bounded on  $\mathbb{R}^2$ . Since for  $\hat{u} = -\frac{\sqrt{2}}{2}(1, 1) \in \mathbb{B}$  it is

$$G(\hat{u}) + \frac{\sqrt{2}}{2}[-1, 1] = \left\{z \in \mathbb{R} : z \leq -\frac{\sqrt{2}}{2}\right\} + \frac{\sqrt{2}}{2}[-1, 1] \subseteq (-\infty, 0],$$

one has that  $G(\hat{u}) + \frac{\sqrt{2}}{2}\mathbb{B} \subseteq (-\infty, 0]$ . According to [32, Proposition 2.15], this fact is sufficient to assert that the fan  $G$  is metrically  $(-\infty, 0]$ -increasing around each point  $x \in \mathbb{R}^2$ , relative to  $S = \mathbb{R}^2$ , with  $\text{inc}_{(-\infty, 0]}(G; \mathbb{R}^2; x) \geq \frac{\sqrt{2}}{2} + 1$ . In addition, a perusal of [32, Proposition 2.15] reveals that for a fan  $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$  it holds

$$\text{inc}_C(H; \mathbb{R}^n; x) = \sup\{\eta > 0 : \exists \hat{u} \in \mathbb{B} : H(\hat{u}) + \eta\mathbb{B} \subseteq C\} + 1.$$

As the infimum

$$\inf_{u \in \mathbb{B}} \varphi(u) = \inf_{\theta \in [0, 2\pi]} \max\{\cos \theta, \sin \theta\} = -\frac{\sqrt{2}}{2}$$

is attained at  $\theta = \frac{5}{2}\pi$ , one deduces that actually it holds  $\text{inc}_{(-\infty, 0]}(G; \mathbb{R}^2; x) = \frac{\sqrt{2}}{2} + 1$ .

All of this to show that  $G$  fulfils all hypotheses (iii)-(v) in Theorem 3.10. Clearly, one has

$$G^{+1}((-\infty, 0]) = \{x \in \mathbb{R}^2 : \max\{x_1, x_2\} \leq 0\} = -\mathbb{R}_+^2.$$

With these problem data, it results in

$$\nu_{G, (-\infty, 0]}(x) = \sup\{\text{dist}(z, (-\infty, 0]) : z \leq \max\{x_1, x_2\}\} = \max\{x_1, x_2, 0\}.$$

Consequently, one finds

$$\nu'_{G,(-\infty,0]}(x;v) = \begin{cases} 0, & \text{if } x \in -\text{int } \mathbb{R}_+^2, \\ v_2, & \text{if } x \in \mathbb{R}^2 \setminus (-\mathbb{R}_+^2), x_2 > x_1, \\ v_1, & \text{if } x \in \mathbb{R}^2 \setminus (-\mathbb{R}_+^2), x_2 < x_1, \\ \max\{v_1, v_2\}, & \text{if } x \in \mathbb{R}^2 \setminus (-\mathbb{R}_+^2), x_2 = x_1, \\ \max\{v_2, 0\}, & \text{if } x \in \mathbb{R}^2 \setminus (-\mathbb{R}_+^2), x_2 = 0, x_1 < 0, \\ \max\{v_1, 0\}, & \text{if } x \in \mathbb{R}^2 \setminus (-\mathbb{R}_+^2), x_1 = 0, x_2 < 0, \\ \max\{v_1, v_2, 0\} & \text{if } x = (0, 0). \end{cases} \quad (12)$$

Now, let  $S = \mathbb{R}^2$  and let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined as

$$f(x) = (-x_1, -x_2) = -\text{Id}_2 x,$$

where  $\text{Id}_2$  denotes the identity matrix  $2 \times 2$ . As a convex set,  $\mathbb{R}^2$  fulfils hypothesis (i) around each of its elements. It is possible to check that  $f$  is  $\mathbb{R}_+^2$ -Lipschitz with constant  $\ell_f = \sqrt{2}$  (due to the Euclidean structure of  $\mathbb{R}^2$ ) and  $e = \frac{\sqrt{2}}{2}(1, 1)$  and, as a linear mapping, it is directionally differentiable with

$$f'(x;v) = -\text{Id}_2 v, \quad \forall x \in \mathbb{R}^2. \quad (13)$$

Clearly, as a linear mapping  $f'(x; \cdot)$  is  $\mathbb{R}_+^2$ -sublinear, so also hypothesis (ii) in Theorem 3.10 is satisfied. Since  $\mathcal{R} = G^{+1}((-\infty, 0]) \cap \mathbb{R}^2 = -\mathbb{R}_+^2$  and  $f(\mathcal{R}) = \mathbb{R}_+^2$ , it is plain to see that for the resulting problem  $(\mathcal{P}_G)$  the set of all locally w-eff. solutions is given by

$$\mathcal{WE}(\mathcal{P}_G) = \{x \in -\mathbb{R}_+^2 : x_1 x_2 = 0\}.$$

Let us check that actually there exists  $y^* = (y_1, y_2) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$  satisfying condition (10) for every  $\bar{x} \in \mathcal{WE}(\mathcal{P}_G)$ . Take first  $\bar{x} = (0, 0) \in \mathcal{WE}(\mathcal{P}_G)$ . By taking into account formulae (13) and (12), the expression in (10), with  $\ell \geq \sqrt{2}$  and  $1 < \alpha < \frac{\sqrt{2}}{2} + 1$ , becomes

$$\begin{aligned} \langle y^*, f'((0, 0); v) \rangle + \frac{\ell}{\alpha - 1} \langle y^*, \nu'_{G,C}((0, 0); v)e \rangle &= \\ \langle y^*, -v \rangle + \frac{\ell}{\alpha - 1} \left\langle y^*, \max\{v_1, v_2, 0\} \cdot \frac{\sqrt{2}}{2}(1, 1) \right\rangle &= \\ -y_1 v_1 - y_2 v_2 + \frac{\ell}{\alpha - 1} \cdot \frac{\sqrt{2}}{2} \max\{v_1, v_2, 0\} (y_1 + y_2) &\geq \\ -y_1 v_1 - y_2 v_2 + \frac{1}{\alpha - 1} \max\{v_1, v_2, 0\} (y_1 + y_2). \end{aligned}$$

So, by taking  $y_1 = y_2 = \alpha - 1 > 0$ , one obtains

$$\langle y^*, f'((0, 0); v) \rangle + \frac{\ell}{\alpha - 1} \langle y^*, \nu'_{G,C}((0, 0); v)e \rangle \geq -(\alpha - 1)(v_1 + v_2) + 2 \max\{v_1, v_2, 0\}.$$

Observe that it holds

$$-(\alpha - 1)(v_1 + v_2) + 2 \max\{v_1, v_2, 0\} \geq 0, \quad \forall v \in I(\mathbb{R}^2; (0, 0)) = \mathbb{R}^2,$$



because if  $v_1 + v_2 < 0$ , then  $\alpha > 1$  implies  $-(\alpha - 1)(v_1 + v_2) > 0$ , whereas if  $v_1 + v_2 \geq 0$ , then  $\alpha < \frac{\sqrt{2}}{2} + 1$  implies

$$2 \max\{v_1, v_2, 0\} \geq v_1 + v_2 \geq \frac{\sqrt{2}}{2}(v_1 + v_2) \geq (\alpha - 1)(v_1 + v_2).$$

In the case  $\bar{x} = (\bar{x}_1, 0)$ , with  $\bar{x}_1 < 0$  (the case  $\bar{x} = (0, \bar{x}_2)$  with  $\bar{x}_2 < 0$  can be handled analogously, by symmetry), the expression in (10) becomes

$$\begin{aligned} \langle y^*, f'(\bar{x}; v) \rangle + \frac{\ell}{\alpha - 1} \langle y^*, \nu'_{G,C}(\bar{x}; v)e \rangle &= \\ \langle y^*, -v \rangle + \frac{\ell}{\alpha - 1} \left\langle y^*, \max\{v_2, 0\} \cdot \frac{\sqrt{2}}{2}(1, 1) \right\rangle &= \\ -y_1 v_1 - y_2 v_2 + \frac{1}{\alpha - 1} \max\{v_2, 0\}(y_1 + y_2). \end{aligned}$$

Thus, by choosing  $y_1 = 0$  and  $y_2 = \alpha - 1 > 0$ , one obtains

$$\begin{aligned} \langle y^*, f'(\bar{x}; v) \rangle + \frac{\ell}{\alpha - 1} \langle y^*, \nu'_{G,C}(\bar{x}; v)e \rangle &\geq -(\alpha - 1)v_2 + \max\{v_2, 0\} \geq 0, \quad (14) \\ \forall v \in \mathbf{I}(\mathbb{R}^2; \bar{x}) = \mathbb{R}^2. \end{aligned}$$

Indeed, if  $v = (v_1, v_2) \in \mathbb{R}^2$  is such that  $v_2 > 0$ , recalling that  $\alpha < \frac{\sqrt{2}}{2} + 1$ , one finds

$$\max\{v_2, 0\} = v_2 > \frac{\sqrt{2}}{2}v_2 > (\alpha - 1)v_2;$$

if  $v = (v_1, v_2) \in \mathbb{R}^2$  is such that  $v_2 \leq 0$ , recalling that  $\alpha > 1$  one sees that inequality (14) is trivially satisfied.

#### 4. Weak efficiency conditions via tangential approximations

Throughout this section, as a first-order approximation of set-valued mappings the notion of outer prederivative, introduced in [14], will be employed.

**Definition 4.1** (Outer prederivative). Let  $F : \mathbb{X} \rightrightarrows \mathbb{Z}$  be a set-valued mapping between Banach spaces and let  $\bar{x} \in \text{dom } F$ . A p.h. set-valued mapping  $H_F(\bar{x}; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Z}$  is said to be an *outer prederivative* of  $F$  at  $\bar{x}$  if for every  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$F(x) \subseteq F(\bar{x}) + H_F(\bar{x}; x - \bar{x}) + \epsilon \|x - \bar{x}\| \mathbb{B}, \quad \forall x \in \mathbf{B}(\bar{x}, \delta).$$

Extended discussions about this generalized differentiation concept can be found, for instance, in [13,14,24].

For subsequent considerations, it is worth observing that the notion of outer prederivative collapses to the notion of Bouligand-derivative (or B-derivative), when both  $F$  and  $H_F(\bar{x}; \cdot)$  are single-valued and  $H_F(\bar{x}; \cdot)$  is continuous. More precisely, following [25], a mapping  $f : \mathbb{X} \rightarrow \mathbb{Z}$  between Banach spaces is said to be *B-differentiable* at

$\bar{x} \in \mathbb{X}$  if there exists a p.h. and continuous mapping  $D_B f(\bar{x}; \cdot) : \mathbb{X} \rightarrow \mathbb{Z}$ , called the B-derivative of  $f$  at  $\bar{x}$ , such that

$$\lim_{x \rightarrow \bar{x}} \frac{f(x) - f(\bar{x}) - D_B f(\bar{x}; x - \bar{x})}{\|x - \bar{x}\|} = 0.$$

By exploiting as a constraint qualification the metric  $C$ -increase property of  $G$ , the following inner tangential approximation of  $\mathcal{R}$  has been established in [32], which is expressed in terms of outer prederivatives and tangent cones. Its proof was provided in a finite-dimensional setting, but a perusal of the involved arguments reveals that it can be extended without any modification to a Banach space setting.

**Proposition 4.2.** ([32, Theorem 3.1]) *Let  $G : \mathbb{X} \rightrightarrows \mathbb{Z}$ ,  $S$  and  $C$  as in problem (SVI), and let  $\bar{x} \in \mathcal{R} = S \cap G^{+1}(C)$ . Suppose that:*

- (i)  $G$  is l.s.c. in a neighbourhood of  $\bar{x}$ ;
- (ii)  $G$  is metrically  $C$ -increasing around  $\bar{x}$ , relative to  $S$ ;
- (iii)  $G$  admits  $H_G(\bar{x}; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Z}$  as an outer prederivative at  $\bar{x}$ .

Then it holds

$$H_G(\bar{x}; \cdot)^{+1}(C) \cap I_w(S; \bar{x}) \subseteq T(\mathcal{R}; \bar{x}). \quad (15)$$

If, in addition,

- (iv)  $H_G(\bar{x}; \cdot)$  is Lipschitz,

then the stronger inclusion holds

$$H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x}) \subseteq T(\mathcal{R}; \bar{x}). \quad (16)$$

Following the well-known Euler-Lagrange scheme for deriving necessary optimality conditions in the presence of constraints, from the above tangential approximation of the feasible region of  $(\mathcal{P}_G)$ , it is possible to obtain the below first-order weak efficiency condition.

**Theorem 4.3.** *With reference to a problem  $(\mathcal{P}_G)$ , let  $\bar{x} \in \mathcal{R}$  be a locally  $w$ -eff. solution. Suppose that:*

- (i)  $f$  is B-differentiable at  $\bar{x}$ ;
- (ii)  $G$  is l.s.c. in a neighbourhood of  $\bar{x}$  and is metrically  $C$ -increasing around  $\bar{x}$ , relative to  $S$ ;
- (iii)  $G$  admits  $H_G(\bar{x}; \cdot) : \mathbb{X} \rightrightarrows \mathbb{Z}$  as an outer prederivative at  $\bar{x}$ .

Then,

$$D_B f(\bar{x}; v) \notin -\text{int } K, \quad \forall v \in H_G(\bar{x}; \cdot)^{+1}(C) \cap I_w(S; \bar{x}). \quad (17)$$

If, in addition,

- (iv)  $D_B f(\bar{x}; \cdot) : \mathbb{X} \rightarrow \mathbb{Y}$  is  $K$ -convexlike on the set  $H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})$ ;
- (v)  $H_G(\bar{x}; \mathbf{0}) \subseteq C$ ;
- (vi)  $H_G(\bar{x}; \cdot)$  is Lipschitz,

there exists  $y^* \in K^\oplus \setminus \{\mathbf{0}^*\}$  such that

$$y^* \circ D_B f(\bar{x}; v) \geq 0, \quad \forall v \in H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x}). \quad (18)$$

In particular, whenever  $f$  is Fréchet differentiable at  $\bar{x}$ , it results in

$$-Df(\bar{x})^*y^* \in [H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})]^\ominus. \quad (19)$$

**Proof.** Upon hypotheses (ii) and (iii), the inner tangential approximation given by (15) can be employed. So, take an arbitrary  $v \in H_G(\bar{x}; \cdot)^{+1}(C) \cap I_w(S; \bar{x})$ . As it is also  $v \in T(\mathcal{R}; \bar{x})$ , there exist sequences  $(v_n)$ , with  $v_n \rightarrow v$ , and  $(t_n)$ , with  $t_n \downarrow 0$ , as  $n \rightarrow \infty$ , such that  $\bar{x} + t_nv_n \in \mathcal{R}$ . Since  $\bar{x}$  is a local w-eff. solution to  $(\mathcal{P}_G)$ , by recalling hypothesis (i), one obtains

$$D_Bf(\bar{x}; v_n) + \frac{o(\bar{x}; t_nv_n)}{t_n} = \frac{f(\bar{x} + t_nv_n) - f(\bar{x})}{t_n} \in \mathbb{Y} \setminus (-\text{int } K).$$

By passing to the limit as  $n \rightarrow \infty$ , taking into account that  $\mathbb{Y} \setminus (-\text{int } K)$  is a closed set and the mapping  $D_Bf(\bar{x}; \cdot)$  is continuous, one achieves the inequality in (17).

Upon the hypothesis (iv), the set  $D_Bf(\bar{x}; H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})) + K$  is a convex subset of  $\mathbb{Y}$ . By arguing as in the first part of the proof, one can show that

$$D_Bf(\bar{x}; v) \notin -\text{int } K, \quad \forall v \in H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x}),$$

which amounts to say

$$[D_Bf(\bar{x}; H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x}))] \cap (-\text{int } K) = \emptyset.$$

Notice that this implies

$$[D_Bf(\bar{x}; H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})) + K] \cap (-\text{int } K) = \emptyset.$$

By the Eidelheit theorem there exists  $y^* \in \mathbb{Y}^* \setminus \{\mathbf{0}^*\}$  and  $\gamma \in \mathbb{R}$  such that

$$\langle y^*, y \rangle \leq \gamma \leq \langle y^*, D_Bf(\bar{x}; v) \rangle, \quad (20)$$

$$\forall y \in \text{cl}(-\text{int } K) = -K, \quad \forall v \in H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x}).$$

Since owing to hypothesis (v) it is  $\mathbf{0} \in -K \cap H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})$ , according to the inequalities in (20) it must be  $\gamma = 0$ . Consequently, the first inequality in (20) gives  $y^* \in \mathbb{Y}^\oplus$ , whereas the second one yields (18). In the case of Fréchet differentiability of  $f$  at  $\bar{x}$ , inclusion (19) is a direct consequence of inequality (18). The proof is complete.  $\square$

**Remark 8.** The property of a mapping to be  $K$ -convexlike on a set  $D$  depends essentially on the set  $D$ . Notice that, if  $D_1 \subseteq D$ , a mapping  $K$ -convexlike on  $D$  may fail to be  $K$ -convexlike on  $D_1$ . Thus, hypothesis (iv) links crucially the behaviour of  $D_Bf(\bar{x}; \cdot)$  with the geometry of the set  $H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})$ . On the other hand, the  $K$ -sublinearity property is stable under convex restrictions, in the sense that if  $h : \mathbb{X} \rightarrow \mathbb{Y}$  is  $K$ -sublinear on a set  $D \subseteq \mathbb{X}$ , it still remains so on each convex subset  $D_1 \subseteq D$ . This fact makes it convenient to consider the following replacement of hypothesis (iv), with separate (but stricter) requirements on the involved problem data:

(iv')  $D_B f(\bar{x}; \cdot)$  is  $K$ -sublinear on  $\mathbb{X}$ ,  $H_G(\bar{x}; \cdot)$  is  $C$ -superlinear on  $\mathbb{X}$ , and  $S$  is locally convex near  $\bar{x}$ .

In such a circumstance,  $T(S; \bar{x})$  is a convex cone as well as  $H_G(\bar{x}; \cdot)^{+1}(C)$ . Since the set  $D_B f(\bar{x}; H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})) + K$  is convex,  $D_B f(\bar{x}; \cdot)$  turns out to be  $K$ -convexlike on the set  $H_G(\bar{x}; \cdot)^{+1}(C) \cap T(S; \bar{x})$ .

**Example 4.4.** Let  $\mathbb{X} = Y = \mathbb{R}^2$ ,  $\mathbb{Z} = \mathbb{R}$ ,  $K = \mathbb{R}_+^2$  and  $C = (-\infty, 0]$ . Let  $\Lambda_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $\Lambda_2 : \mathbb{R}^2 \rightarrow \mathbb{R}$  be linear mappings represented by the  $1 \times 2$  matrices

$$\Lambda_1 = (1 \quad 1) \quad \text{and} \quad \Lambda_2 = (-1 \quad 1),$$

respectively, and let  $G : \mathbb{R}^2 \rightrightarrows \mathbb{R}$  be defined as

$$G(x) = \{\Lambda x : \Lambda \in \mathcal{G}\},$$

where  $\mathcal{G} = \text{conv}\{\Lambda_1, \Lambda_2\} = \{t\Lambda_1 + (1-t)\Lambda_2 : t \in [0, 1]\}$ . In other words,  $G$  is the fan finitely generated by  $\text{conv}\{\Lambda_1, \Lambda_2\}$ . By consequence, one has

$$\begin{aligned} G(x) &= \{[t\Lambda_1 + (1-t)\Lambda_2]x : t \in [0, 1]\} \\ &= \{[(t \quad t) + (t-1 \quad 1-t)]x : t \in [0, 1]\} = \{(2t-1)x_1 + x_2 : t \in [0, 1]\}. \end{aligned}$$

It is clear that  $x \in G^{+1}((-\infty, 0])$  if and only if

$$\begin{aligned} x &\in \Lambda_1^{-1}((-\infty, 0]) \cap \Lambda_2^{-1}((-\infty, 0]) \\ &= \{x \in \mathbb{R}^2 : x_2 \leq -x_1\} \cap \{x \in \mathbb{R}^2 : x_2 \leq x_1\} = \{x \in \mathbb{R}^2 : x_2 \leq -|x_1|\}. \end{aligned}$$

Notice that, since  $G(x) = g(\mathcal{G}, x)$ , with  $g(\Lambda, x) = \Lambda x$ , and each  $g(\Lambda, \cdot)$  is continuous on  $\mathbb{R}^2$ , in the light of what was stated in Remark 4  $G$  is l.s.c. on  $\mathbb{R}^2$ . Moreover, since for  $\hat{u} = (0, -1)$  it is  $G(\hat{u}) = \{-1\}$  and therefore it holds

$$G(\hat{u}) + \mathbb{B} = \{-1\} + [-1, 1] = [-2, 0] \subseteq (-\infty, 0],$$

then according to [32, Proposition 2.15]  $G$  is metrically  $(-\infty, 0]$ -increasing around every  $x \in \mathbb{R}^2$ , relative to  $S = \mathbb{R}^2$ , with  $\text{inc}_C(G; \mathbb{R}^2; x) \geq 2$ . On the other hand, since, as a fan,  $G$  is concave and p.h., for every positive  $\epsilon$  it holds

$$G(x) \subseteq G(x_0) + G(x - x_0) \subseteq G(x_0) + G(x - x_0) + \epsilon \|x - x_0\| \mathbb{B}, \quad \forall x \in \mathbb{R}^2,$$

one sees that  $G$  admits  $H_G(x_0; \cdot) = G$  as an outer prederivative at each  $x_0 \in \mathbb{R}^2$ .

Let  $S = \mathbb{R}^2$  and let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by

$$f(x) = (x_2, |x_1|).$$

The reader should notice that  $f$  can be regarded as the composition of the clockwise rotation of the plane of an angle  $\pi/2$ , represented by

$$\text{rot}_{\pi/2}(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x,$$

and the mapping  $s : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $s(y) = (y_1, |y_2|)$ , namely

$$f(x) = (s \circ \text{rot}_{\pi/2})(x).$$

Since it is  $\mathcal{R} = \mathbb{R}^2 \cap G^{+1}((-\infty, 0]) = \{x \in \mathbb{R}^2 : x_2 \leq -|x_1|\}$  and

$$f(\mathcal{R}) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_2 \leq -y_1, y_1 \leq 0, y_2 \geq 0\},$$

for the resulting problem  $(\mathcal{P}_G)$  it turns out

$$\mathcal{WE}(\mathcal{P}_G) = \{x \in \mathcal{R} : x_1 = 0, x_2 \leq 0\}.$$

By employing the definition, it is possible to show that  $f$  is  $B$ -differentiable at each point  $\bar{x} \in \mathcal{WE}(\mathcal{P}_G)$  and it results in

$$D_B f(\bar{x}; v) = f(v), \quad \forall v \in \mathbb{R}^2.$$

Consistently with the thesis of Theorem 4.3, one obtains

$$D_B f(\bar{x}; v) = (v_2, |v_1|) \notin -\text{int } \mathbb{R}_+^2, \quad \forall v \in H_G(\bar{x}; \cdot)^{+1}((-\infty, 0]) \cap I_w(\mathbb{R}^2; \bar{x}) = \mathcal{R}.$$

Since the mapping  $D_B f(\bar{x}; \cdot)$  is also  $\mathbb{R}_+^2$ -sublinear (having both the components sublinear) and hence  $\mathbb{R}_+^2$ -convexlike on  $\mathbb{R}^2$ ,  $H_G(\bar{x}; (0, 0)) = \{0\}$  and  $H_G(\bar{x}; \cdot) : \mathbb{R}^2 \rightrightarrows \mathbb{R}$  is Lipschitz (remember Remark 3), as a fan finitely generated, then by taking  $y^* = (0, 1) \in \mathbb{R}_+^2 \setminus \{(0, 0)\}$  one finds

$$y^* \circ D_B f(\bar{x}; v) = |v_1| \geq 0, \quad \forall v \in H_G(\bar{x}; \cdot)^{+1}((-\infty, 0]) \cap I_w(\mathbb{R}^2; \bar{x}) = \mathcal{R},$$

in agreement with the condition expressed by formula (18).

The next result provides a refinement of Theorem 4.3, which can be established, under proper qualification conditions, by replacing general first-order approximations of the data with the local convexity of  $S$  and linear approximations of  $f$  and  $G$ .

**Theorem 4.5** (Multiplier rule via fans). *Let  $\bar{x} \in \mathcal{R}$  be a locally  $w$ -eff. solution to problem  $(\mathcal{P}_G)$ . Suppose that hypotheses (i)-(iii) are satisfied and, in addition, that:*

- (iv)  $S$  is locally convex near  $\bar{x}$ ;
- (v)  $f$  is Fréchet differentiable at  $\bar{x}$ ;
- (vi)  $H_G(\bar{x}; \cdot)$  is a fan finitely generated by  $\mathcal{G} = \text{conv} \{\Lambda_1, \dots, \Lambda_p\}$ ;
- (vii) the further qualification condition holds

$$\left( \bigcap_{i=1}^p \text{int } \Lambda_i^{-1}(C) \right) \cap \text{int } T(S; \bar{x}) \neq \emptyset. \quad (21)$$

Then there exist  $y^* \in K^\oplus \setminus \{\mathbf{0}^*\}$  and, for each  $i = 1, \dots, p$ ,  $x_i^* \in \mathbb{X}^*$  and sequences  $(z_{i,n}^*)_n$  in  $\mathbb{Z}^*$ , with  $z_{i,n}^* \in C^\ominus$  and  $\Lambda_i^* z_{i,n}^* \rightarrow x_i^*$ , such that

$$\mathbf{0}^* \in Df(\bar{x})^* y^* + \sum_{i=1}^p x_i^* + N(S; \bar{x}). \quad (22)$$

**Proof.** Observe that by hypothesis (v), it is  $H_G(\bar{x}; \mathbf{0}) = \{\mathbf{0}\} \subseteq C$ , so hypothesis (v) of Theorem 4.3 is fulfilled. As recalled in Remark 3, since the bundle generating  $H_G(\bar{x}; \cdot)$  is bounded according to hypothesis (vi),  $H_G(\bar{x}; \cdot)$  is Lipschitz. Moreover, it is readily seen that if  $H_G(\bar{x}; \cdot)$  is generated by  $\mathcal{G} = \text{conv} \{\Lambda_1, \dots, \Lambda_p\}$ , it holds

$$H_G(\bar{x}; \cdot)^{+1}(C) = \bigcap_{i=1}^p \Lambda_i^{-1}(C).$$

Since each element  $\Lambda_i^{-1}(C)$  in the above intersection is a convex cone as well as  $\text{T}(S; \bar{x})$  by hypothesis (iv), the set  $H_G(\bar{x}; \cdot)^{+1}(C) \cap \text{T}(S; \bar{x})$  turns out to be a convex cone. By hypothesis (v) it is  $\text{D}_B f(\bar{x}; \cdot) = \text{D}f(\bar{x})$ , so, as a linear mapping it is  $K$ -convexlike on  $H_G(\bar{x}; \cdot)^{+1}(C) \cap \text{T}(S; \bar{x})$ . One is therefore in a position to apply Theorem 4.3. Thus, there exists  $y^* \in K^\ominus \setminus \{\mathbf{0}^*\}$  such that

$$-\text{D}f(\bar{x})^* y^* \in \left[ \bigcap_{i=1}^p \Lambda_i^{-1}(C) \cap \text{T}(S; \bar{x}) \right]^\ominus.$$

By virtue of the qualification condition in hypothesis (vii), on account of the relations discussed in Remark 1, the last inclusion implies

$$-\text{D}f(\bar{x})^* y^* \in \sum_{i=1}^p (\Lambda_i^{-1}(C))^\ominus + \text{T}(S; \bar{x})^\ominus = \sum_{i=1}^p \text{cl} \Lambda_i^*(C^\ominus) + \text{N}(S; \bar{x}).$$

This means that there must exist  $x_i^* \in \text{cl} \Lambda_i^*(C^\ominus)$ , for every  $i = 1, \dots, p$ , such that

$$-\text{D}f(\bar{x})^* y^* \in \sum_{i=1}^p x_i^* + \text{N}(S; \bar{x}),$$

which immediately entails the existence of such sequences  $(z_{i,n}^*)_n$  in  $\mathbb{Z}^*$  as asserted in the thesis, thereby completing the proof.  $\square$

**Remark 9.** It is worth noting that, whenever  $\text{int} C \neq \emptyset$  and  $\bar{x} \in \text{int} S$ , the qualification condition in (21) is satisfied provided that the following Slater-type condition holds:

$$\exists x_0 \in \mathbb{X} : \Lambda_i x_0 \in \text{int} C, \quad \forall i = 1, \dots, p. \quad (23)$$

Indeed,  $\Lambda_i x_0 \in \text{int} C$  implies  $x_0 \in \Lambda_i^{-1}(\text{int} C) \subseteq \text{int} \Lambda_i^{-1}(C)$ , for every  $i = 1, \dots, p$ .

As one expects, the formulation of the multiplier rule expressed by (22) simplifies if specialized to a finite-dimensional space setting. This is done in the next result, where the adjoint operation (which can be viewed as a matrix transposition) is now denoted by the symbol  $\top$ .

**Corollary 4.6** (Weak Pareto efficiency condition in finite-dimensional spaces). *Let  $\bar{x} \in \mathcal{R}$  be a locally w-eff. solution to problem  $(\mathcal{P}_G)$ , with  $\mathbb{X} = \mathbb{R}^n$ ,  $\mathbb{Y} = \mathbb{R}^m$ ,  $\mathbb{Z} = \mathbb{R}^p$ ,  $K = \mathbb{R}_+^m$  and  $\text{int} C \neq \emptyset$ . Suppose that hypotheses (i)-(vii) of Theorem 4.5 are satisfied,*

along with condition (23). Then there exist  $v \in \mathbb{R}_+^m \setminus \{\mathbf{0}\}$  and  $c_i \in C^\ominus$ ,  $i = 1, \dots, p$ , such that

$$\mathbf{0} \in \text{Df}(\bar{x})^\top v + \sum_{i=1}^p \Lambda_i^\top c_i + \text{N}(S; \bar{x}). \quad (24)$$

If, in particular,  $\bar{x} \in \text{int } S$  hypothesis (vii) can be dropped out and it results in

$$\mathbf{0} = \text{Df}(\bar{x})^\top v + \sum_{i=1}^p \Lambda_i^\top c_i.$$

**Proof.** It suffices to apply Theorem 4.5 and to observe that, under condition (23),  $\Lambda_i x_0 \in \text{int } C$  implies  $x_0 \in \Lambda_i^{-1}(\text{int } C) = \Lambda_i^{-1}(\text{ri } C) \neq \emptyset$ . Thus, by taking into account what noted in Remark 1(ii), it is true that

$$[\Lambda_i^{-1}(C)]^\ominus = \Lambda_i^\top(C^\ominus), \quad \forall i = 1, \dots, p.$$

If  $\bar{x} \in \text{int } S$ , then it is  $\text{T}(S; \bar{x}) = \mathbb{R}^n$  and hence  $-\text{Df}(\bar{x})^* y^* \in [\bigcap_{i=1}^p \Lambda_i^{-1}(C)]^\ominus$ . Therefore, according to Remark 9, condition (23) makes hypothesis (vii) redundant.  $\square$

**Example 4.7.** Let  $\mathbb{X} = \mathbb{Y} = \mathbb{R}^2$ ,  $\mathbb{Z} = \mathbb{R}$ ,  $K = \mathbb{R}_+^2$  and  $C = (-\infty, 0]$ . Suppose that the set-valued mapping  $G : \mathbb{R}^2 \rightrightarrows \mathbb{R}$  and the set  $S$  defining the constraint system in  $(\mathcal{P}_G)$  are as in Example 4.4, namely with  $G(x) = g(\mathcal{G}, x)$ , where  $g(\Lambda, x) = \Lambda x$  and  $\mathcal{G} = \text{conv} \{\Lambda_1, \Lambda_2\}$ , and  $S = \mathbb{R}^2$ , in such a way that, as already seen,  $\mathcal{R} = \{x \in \mathbb{R}^2 : x_2 \leq -|x_1|\}$ . Suppose that  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined by

$$f(x) = (-(x_1 + x_2), -(x_1 + x_2)).$$

Since it results in

$$f(\mathcal{R}) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_2 = y_1, y_1 \geq 0\},$$

it is clear that for the problem  $(\mathcal{P}_G)$  under consideration it is  $\mathcal{WE}(\mathcal{P}_G) = \{(0, 0)\}$ . With these problem data,  $S$  is evidently locally convex near  $(0, 0)$  and  $f$  is Fréchet differentiable at  $(0, 0)$ , with

$$\text{Df}((0, 0)) = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}.$$

In Example 4.4 it has already been shown that  $G$  is l.s.c. on  $\mathbb{R}^2$ , it is globally metrically  $(-\infty, 0]$ -increasing relative to  $\mathbb{R}^2$ , and it admits as an outer prederivative the set-valued mapping  $H_G((0, 0); v) = G(v)$ . Notice that  $\text{int}(-\infty, 0] = (-\infty, 0) \neq \emptyset$ ,  $(0, 0) \in \text{int } S$ , and the Slater-type condition (23) is satisfied by  $x_0 = (0, -1)$ , inasmuch as

$$\Lambda_1 x_0 = -1 \in (-\infty, 0) \quad \text{and} \quad \Lambda_2 x_0 = -1 \in (-\infty, 0).$$

Thus the qualification condition (21) in hypothesis (vii) of Theorem 4.5 can be dropped out. Let us show that, consistently with the thesis of Corollary 4.6, there exist  $v \in$

$\mathbb{R}_+^2 \setminus \{(0,0)\}$  and  $c_1, c_2 \in (-\infty, 0]^\ominus = [0, +\infty)$  satisfying the multiplier rule in (24). With the above problem data, the condition in (24) becomes

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} v + \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} c_2 + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

This inclusion holds if and only if the following linear system in the unknown  $(c_1, c_2)$

$$\begin{cases} c_1 - c_2 = v_1 + v_2 \\ c_1 + c_2 = v_1 + v_2 \end{cases}$$

does have nonnegative solutions for some  $v \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ . In fact this happens to be true with  $c_1 = v_1 + v_2 > 0$  and  $c_2 = 0$ , for every  $v \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ .

The next example aims at showing that the condition established in Corollary 4.6 fails, in general, to be also sufficient for weak efficiency.

**Example 4.8.** Consider a problem  $(\mathcal{P}_G)$  defined by the same data as in Example 4.7, except for the criterion mapping  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ , which is now given by

$$f(x) = (x_1 + x_2, -(x_1 + x_2)^2).$$

Since in the present case one has

$$f(\mathcal{R}) = \{y = (y_1, y_2) \in \mathbb{R}^2 : y_2 = -y_1^2, y_1 \leq 0\},$$

it is readily seen that  $\mathcal{WE}(\mathcal{P}_G) = \emptyset$ . Nevertheless, for  $\bar{x} = (0,0) \in \mathcal{R}$  one finds

$$Df((0,0)) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

so condition in (24) becomes

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} v + \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} c_2 + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

which leads to the linear system

$$\begin{cases} c_1 - c_2 = -v_1 \\ c_1 + c_2 = -v_1. \end{cases}$$

Therefore inclusion (24) is satisfied for instance with  $c_1 = c_2 = 0$  and  $v = (0,1) \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ , even though  $\bar{x} = (0,0)$  is not a locally w-eff. solution of the problem under consideration.

On the other hand, if *ceteris paribus* the criterion mapping  $f$  is replaced by

$$f(x) = \text{Id}_2 x,$$



so that  $f(\mathcal{R}) = \mathcal{R}$ , still it is  $\mathcal{WE}(\mathcal{P}_G) = \emptyset$ . Consequently, the condition in (24) becomes

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v + \begin{pmatrix} 1 \\ 1 \end{pmatrix} c_1 + \begin{pmatrix} -1 \\ 1 \end{pmatrix} c_2 + \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$

which yields the linear system

$$\begin{cases} c_1 - c_2 = -v_1 \\ c_1 + c_2 = -v_2. \end{cases}$$

Since one finds as a unique solution of this system the pair

$$c_1 = -\frac{v_1 + v_2}{2} < 0, \quad c_2 = \frac{v_1 - v_2}{2},$$

in the last case none of the elements in  $\mathcal{R}$  passes the test expressed by the condition in (24), which reveals here to be effective.

As a terminal result, the multiplier rules formulated in Corollary 4.6 afford elements for an impact evaluation of the present approach. As a comment, let us point out the substantial differences appearing in comparison with similar results. Consider, for instance, the recent and reliable multiplier rule stated in [8, Theorem 3.1]. Thought less general in requiring the smoothness of  $f$  (but without any generalized convexity), Corollary 4.6 does not impose neither Lipschitz continuity assumptions (but surreptitiously it does on  $H_G(\bar{x}; \cdot)$  through hypothesis (vi)) nor generalized concavity on  $G$ . Besides, no preliminary assumption about convexity and compactness of  $\Omega$  is made.

On the other hand, the role played by the multipliers  $c_i$  is evidently less conventional: all of them refer to the same constraint  $G(x) \subseteq C$ . The reader should observe that their number is not linked to the dimension of the range space of  $G$  (each of them lives in  $\mathbb{R}^p$ ), but depends on the number of generators needed to represent  $H_G(\bar{x}; \cdot)$ . In other words, it is a parameter depending on the approximation tool used to express the rule, not an intrinsic constant of the constraint system.

Even with these features, the rule in Corollary 4.6 leads to deal with linear algebra tools, as illustrated in the above examples, with a consequent computational appeal.

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## References

- [1] Aubin J.-P., Frankowska H.: *Set-valued analysis*. Birkhuser Boston, Inc., Boston, MA, 1990.
- [2] Azé D., Corvellec J.-N.: *Nonlinear local error bounds via a change of metric*. J. Fixed Point Theory Appl. **16** (2014), no. 1–2, 351–372.
- [3] Bao T.Q., Le T.T., Tammer C., Tuan V.A.: *Duality in vector optimization with domination structures*. Appl. Set-Valued Anal. Optim. **1** (2019), no. 3, 319–335.
- [4] Ben-Tal A., Nemirovski A.: *Robust convex optimization*. Math. Oper. Res. **23** (1998), no. 4, 769–805.

- [5] Ben-Tal A., Nemirovski A.: *Robust optimization—methodology and applications*. Math. Program. **92** (2002), no. 3, Ser. B, 453–480.
- [6] Ben-Tal A., Ghaoui L.E., Nemirovski A.: *Robust optimization*. Princeton series in applied mathematics. Princeton University Press, Princeton, 2009.
- [7] Castellani M.: *Error bounds for set-valued maps*. Generalized convexity and optimization for economic and financial decisions, 121–135, Pitagora, Bologna, 1999.
- [8] Chen J., Köbis E., Yao J.-C.: *Optimality conditions and duality for robust nonsmooth multiobjective optimization problems with constraints*. J. Optim. Theory Appl. **181** (2019), no. 2, 411–436.
- [9] Chuong T.D.: *Optimality and duality for robust multiobjective optimization problems*. Nonlinear Anal. **134** (2016), 127–143.
- [10] Dauer J.P., Stadler W.: *A survey of vector optimization in infinite-dimensional spaces II*. J. Optim. Theory Appl. **51** (1986), no. 2, 205–241.
- [11] De Giorgi E., Marino A., Tosques M.: *Problems of evolution in metric spaces and maximal decreasing curves*. Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **68** (1980), no. 8, 180–187.
- [12] Frenk J.B., Kassay G.: *On classes of generalized convex functions, Gordan-Farkas type theorems, and Lagrangian duality*. J. Optim. Theory Appl. **102** (1999), no. 2, 315–343.
- [13] Gaydu M., Geoffroy M.H., Marcelin Y.: *Prederivatives of convex set-valued maps and applications to set optimization problems*. J. Global Optim. **64** (2016), no. 1, 141–158.
- [14] Ioffe A.D.: *Nonsmooth analysis: Differential calculus of nondifferentiable mappings*. Trans. Amer. Math. Soc. **266** (1981), no. 1, 1–56.
- [15] Jahan J.: *Vector optimization. Theory, applications, and extensions*. Springer-Verlag, Berlin, 2004.
- [16] Jahan J.: *General theorems of the alternative in variable ordering structures*. J. Nonlinear Var. Anal. **5** (2021), no. 1, 1–7.
- [17] Khan A.A., Tammer C., Zălinescu C.: *Set-valued optimization. An introduction with applications*. Springer, Heidelberg, 2015.
- [18] Kuroiwa D., Lee G.M.: *On robust multiobjective optimization*. Vietnam J. Math. **40** (2012), no. 2-3, 305–317.
- [19] Luc D.T.: *Theory of vector optimization*. Lecture Notes in Economics and Mathematical Systems, 319. Springer-Verlag, Berlin, 1989.
- [20] Mastroeni G., Pappalardo M., Raciti F.: *Some topics in vector optimization via image space analysis*. J. Nonlinear Var. Anal. **4** (2020), no. 1, 5–20.
- [21] Mordukhovich B.S.: *Necessary conditions in nonsmooth minimization via lower and upper subgradients*. Set-Valued Anal. **12** (2004), no. 1-2, 163–193.
- [22] Mordukhovich B.S.: *Variational analysis and generalized differentiation. I. Basic theory*. Springer, Berlin, 2006.
- [23] Mordukhovich B.S.: *Variational analysis and generalized differentiation. II. Applications*. Springer-Verlag, Berlin, 2006.
- [24] Pang C.H.J.: *Generalized differentiation with positively homogeneous maps: applications in set-valued analysis and metric regularity*. Math. Oper. Res. **36** (2011), no. 3, 377–397.
- [25] Robinson S.M.: *An implicit-function theorem for a class of nonsmooth functions*. Math. Oper. Res. **16** (1991), no. 2, 292–309.
- [26] Rockafellar R.T.: *Convex analysis*. Princeton University Press, Princeton, N.J. 1970.
- [27] Sawaragi Y., Nakayama, H., Tanino T.: *Theory of multiobjective optimization*. Mathematics in Science and Engineering, 176. Academic Press, Inc., Orlando, FL, 1985.
- [28] Schirotzek W.: *Nonsmooth analysis*. Universitext. Springer, Berlin, 2007.
- [29] Soyster A.L.: *Convex programming with set-inclusive constraints and applications to inexact linear programming*. Oper. Res. **21** (1973), 1154–1157.
- [30] Uderzo A.: *On some generalized equations with metrically  $C$ -increasing mappings: solvability and error bounds with applications to optimization*. Optimization **68** (2019), no. 1, 227–253.
- [31] Uderzo A.: *On differential properties of multifunctions defined implicitly by set-valued*

- inclusions*, to appear on Pure Appl. Funct. Anal.
- [32] Uderzo A.: *On tangential approximations of the solution set of set-valued inclusions*, to appear on J. Appl. Anal.
- [33] Ye J.J.: *The exact penalty principle*. Nonlinear Anal. **75** (2012), no. 3, 1642–1654.
- [34] Zălinescu C.: *Convex analysis in general vector spaces*, World Scientific Publishing Co., River Edge, NJ, 2002.